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Appearance of stable minimal spheres along the Ricci flow in positive scalar curvature

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We construct spherical space forms $(S^3/\Gamma, g)$ with positive scalar curvature and containing no stable embedded minimal surfaces such that the following happens along the Ricci flow starting at $(S^3/\Gamma, g)$: a stable embedded minimal 2–sphere appears and a nontrivial singularity occurs. We also give in dimension 3 a general construction of Type I neckpinching and clarify the relationship between stable spheres and nontrivial Type I singularities of the Ricci flow. Some symmetry assumptions prevent the appearance of stable spheres, and this has consequences on the types of singularities which can occur for metrics with these symmetries.

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For quotients of the spheres of dimension 2 and 3, endowed with an arbitrary metric, the Ricci flow eventually makes the metric converge to a round metric. In dimension 2, it was proved by Hamilton [25] and B Chow [12] that any initial metric evolves smoothly under the Ricci flow until a trivial singularity, where the whole surface disappears at a point and after rescaling becomes asymptotically round. The situation is much more complicated in dimension 3 because nontrivial singularities can occur. In a series of papers, Perelman [37; 39; 38], was able to analyze and control the singularities by a surgery process initially proposed by Hamilton which enables continuation of the flow. One simple consequence of this breakthrough is that for quotients of the 3–sphere, after a finite number of surgeries, the manifold disappears in finite time, and also becomes asymptotically round. From a related point of view, if the initial metric is already known to be round enough, then it becomes even more so during the flow: this is the theorem of Hamilton [24] which states that if a closed 3–manifold has positive Ricci curvature then this property is preserved and after rescaling the metric converges smoothly to a round metric. Besides, it is well known that the positivity condition $\mathrm{Ric} > 0$ prevents the existence of two-sided closed stable minimal surfaces. Hence it seems natural to expect that the absence of such stable minimal surfaces would also be preserved along the flow.

The study of stable minimal surfaces in the context of the Ricci flow is relevant for understanding singularity formation. For instance the heuristic picture for the Ricci

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flow on a 3–sphere is that a nondegenerate singularity which is nontrivial should be a neckpinching, and thus there should be small stable minimal spheres just before the singularity time. To our knowledge, the only rigorously proved examples of initial metrics on the 3–sphere eventually producing a nontrivial singularity (see Angenent and Knopf [2; 3]) contain a stable minimal sphere. Thus one might hope to avoid nontrivial singularities if the initial metric does not contain stable minimal surfaces.

It will be enough for us to focus on the case where the scalar curvature is positive. This condition $R > 0$ is considerably weaker than $\mathrm{Ric} > 0$ but nevertheless conveys an idea of roundness and is preserved along the flow. Notice that if $R > 0$, any two-sided oriented closed stable minimal surface is a 2–sphere. Let us reformulate the two previous questions:

Q1 Suppose that (M, g) is a closed oriented 3–manifold with positive scalar curvature and containing no stable minimal spheres. Can a stable minimal sphere appear along the Ricci flow starting at (M, g)?

Q2 Let (M, g) be as in **Q1**. Can a nontrivial singularity occur along the Ricci flow starting at (M, g)?

It turns out that the answer to both questions is yes and the counterexamples are the subject of our main theorem (see Theorems 14 and 17):

Theorem 1 *There exists a metric g on S^3 with positive scalar curvature such that*

(i) *(S^3, g) contains no stable minimal 2–spheres,*

(ii) *a stable minimal 2–sphere appears along the Ricci flow starting at (S^3, g),*

(iii) *a nontrivial singularity occurs in finite time.*

These counterexamples show that, even when $R > 0$, the absence of stable spheres at the beginning cannot prevent nontrivial singularities. It means that the analysis of Bamler [4; 5; 6; 7; 8] to get finitely many surgeries is necessary and cannot be replaced by a geometric argument using stable spheres. Actually the appearance of stable geodesics is also true for some 2–spheres (see Theorem 13), but of course we cannot impose a curvature positivity condition in that case since it would be preserved by the flow and this would prevent the existence of stable closed geodesics.

Several authors previously studied the evolution of minimal surfaces along the Ricci flow in different contexts. Hamilton was the first to use that interaction for stable geodesics or minimal surfaces; see Hamilton [26] and Chow [12]. Colding and Minicozzi [16] exploited it with branched immersed minimal spheres for bounding the extinction time of some 3–manifolds. Marques and Neves [32] proved rigidity results for min–max minimal surfaces in some 3–manifolds using similar methods. Let us point out how **Q1** is related to [32, Theorem 1.3]. Assuming $\mathrm{Ric} > 0$ and $R \geq 6$ on a quotient S^3/Γ and by controlling the evolution of a min–max width along the Ricci flow, Marques and Neves are able to produce a small area minimal surface in the initial metric. While in [44] we proved using a different method that this result remains true without any assumption on the Ricci curvature, it would be desirable to understand to what extent their combination of Ricci flow and min–max theory can be realized when Ric is not necessarily positive. The reason for the assumption $\mathrm{Ric} > 0$ in [32] is twofold. First they are making use of Hamilton's theorem so that they do not have to deal with surgeries. The second and most serious reason for this assumption is the following: as recalled previously, it excludes the existence of two-sided stable minimal surfaces so in particular it enables the construction of optimal sweepouts from a given unstable two-sided minimal surface. The examples that we construct to answer **Q1** suggest that combining min–max theory with the Ricci flow when $R > 0$ is not as natural as in the more restrictive case $\mathrm{Ric} > 0$.

After answering **Q1** and **Q2**, we clarify the link between small stable spheres and Type I singularities in dimension 3, without curvature assumptions (see Theorem 22):

Theorem 2 *If a nontrivial Type I singularity occurs at time T, then there are stable immersed minimal spheres with embedded image near time T whose area decreases linearly to zero, and a local converse holds true.*

Moreover, we construct in Proposition 21 general examples of Type I neckpinching by joining any two closed 3–manifolds with a sufficiently thin neck, which generalizes in dimension 3 the rotationally symmetric metrics on S^{n+1} constructed by Angenent and Knopf [2].

Finally, one can ensure that no stable spheres appear if the initial metric is symmetric enough (see Theorem 24):

Theorem 3 *Let M be a closed connected oriented 3–manifold with a d–dimensional Lie group of isometries acting on M such that $d > 1$, or $d = 1$ and the action is free. Then along the Ricci flow, no stable immersed minimal spheres with embedded image can appear if there were none at the beginning.*

In fact, there is a nonfree S^1–action on the examples constructed previously to answer **Q1**. Hence these examples essentially have a maximal amount of symmetry among 3–manifolds such that new stable spheres appear along the Ricci flow. Since we have seen that stable spheres are linked to nontrivial Type I singularities, we get as a corollary (see Corollary 27):

Corollary 4 *Suppose that M is as in the previous theorem. If M is not rotationally symmetric and if a singularity occurs along the Ricci flow, then it is a Type I trivial singularity.*

The paper is organized as follows. In Section 1, some preliminaries on the Ricci flow and min–max theory are presented. In Section 2, which constitutes the main part of the article, we construct certain "thin hooks", enabling us to give explicit examples that answer simultaneously **Q1** and **Q2**. In Section 3, after showing a general procedure to get Type I neckpinching, we prove a relation between stable spheres and nontrivial Type I singularities. Finally, we propose in Section 4 optimal symmetry assumptions preventing the appearance of new stable spheres, and we derive a corollary concerning singularities which can possibly occur.

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1 Preliminaries

1.1 Ricci flow, singularities, canonical neighborhoods and geometric limits

Let (M, g) be a closed oriented Riemannian 3–manifold. A standard Ricci flow starting at (M, g), defined on $[0, T)$, is a smooth solution of

$$\begin{cases} \dfrac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}_{g(t)}, & t \in [0, T), \\ g(0) = g. \end{cases}$$

The flow is said to develop a singularity at time T if the norm of the curvature tensor becomes unbounded as $t \to T$. The singularity is said to be

- *trivial* when
$$\{x \in M : \lim_{t \to T} |\mathrm{Rm}(x,t)| = \infty\} = M,$$
- a *Type I* singularity when there is a constant $\overline{C}$ such that for all $t \in [0, T)$,
$$\sup_M |\mathrm{Rm}_{g(t)}| \le \frac{\overline{C}}{T-t},$$
- a *Type II* singularity when
$$\limsup_{t \uparrow T} \sup_M |\mathrm{Rm}_{g(t)}|(T-t) = \infty.$$

As Perelman showed, the regions where the scalar curvature is large are modeled by the so-called canonical neighborhoods. In this subsection we will explain some of their properties. We refer the reader to B Kleiner and J Lott [29], J Morgan and G Tian [34] and H-D Cao and X-P Zhu [9] for more details. Our presentation will follow [34]. First let us recall the definition of (C,ϵ)–canonical neighborhoods. Fix two positive constants C and ϵ. An open neighborhood U of $x \in (M, g(t))$ is a strong (C,ϵ)–canonical neighborhood if one of the following holds (see [34, Section 8 in Chapter 9 and Definition 14.18]):

(i) U is a strong ϵ–neck in (M,g) centered at x.

(ii) U is a (C,ϵ)–cap in (M,g) whose core contains x.

(iii) U is a C–component of (M,g) satisfying Condition (8) of [34, Definition 9.72].

(iv) U is an ϵ–round component of (M,g).

A strong ϵ–neck centered at $x \in (M, g(t))$ is a submanifold $N \subset M$ and a diffeomorphism $\overline{\psi}_N\colon S^2 \times (-1/\epsilon, 1/\epsilon) \to N$ with $x \in \overline{\psi}_N(S^2 \times \{0\})$ such that $t - R(x,t)^{-1} \ge 0$ and the evolving metric $R(x,t)\overline{\psi}^*\big(g(t + s/R(x,t))\big)$ with $-1 < s \le 0$ is ϵ–close in the $C^{[1/\epsilon]}$–topology to the evolving cylindrical metric $ds^2 + d\theta^2$ with $-1 < s \le 0$, where $d\theta^2$ denotes the round metric of scalar curvature $1/(1-s)$ on S^2. A (C,ϵ)–cap is a noncompact submanifold $\mathcal{C} \subset M$ diffeomorphic to a 3–ball or $\mathbb{R}P^3$ minus a ball, with a neck $N \subset \mathcal{C}$ such that $\overline{Y} = \mathcal{C} \backslash N$ is a compact submanifold. The boundary $\partial \overline{Y}$ of the so-called core (the interior of $\mathcal{C} \backslash N$) is required to be the central sphere of a strong ϵ–neck in $\mathcal{C}$. After rescaling the metric to have $R(x) = 1$ at some point x in the cap, the diameter, volume and scalar curvature ratios at any two points are bounded by C. A C–component is a compact manifold diffeomorphic to S^3 or $\mathbb{R}P^3$, of positive sectional curvature and of bounded geometry controlled by C after rescaling (we added Condition (8) of [34, Definition 9.72] since the next theorem is actually true with this definition). An ϵ–round component is a compact connected manifold such that, after rescaling to make $R(x)=1$ at some point $x\in M$, the metric is close in the $C^{[1/\epsilon]}$–topology to a round metric. The definition of strong (C,ϵ)–canonical neighborhoods is hence scale invariant.

We say that a (standard) Ricci flow $(M,g(t))_{t\in[a,b)}$ satisfies the (C,ϵ)–canonical neighborhood assumption with parameter r if every point $(x,t)\in M\times[a,b)$ with $R_{g(t)}(x)\geq r^{-2}$ has a (C,ϵ)–canonical neighborhood. When ϵ and $1/C$ are small enough, one has the following canonical neighborhood theorem (see for instance [34, Chapter 9, Chapter 17 and Theorem 15.9]):

Theorem 5 *Let $T>0$. Then there exists an $r_0>0$ depending only on T such that the following holds. Suppose that (M,g) is a closed oriented 3–manifold endowed with a normalized metric, ie for all $x\in M$,*

$$\max_M|\mathrm{Rm}(x,0)|_{g(0)}\leq 1 \quad \textit{and} \quad \mathrm{vol}_{g(0)}\,B(x,0,1)\geq \tfrac{1}{2}\omega,$$

where ω is the volume of the unit ball in $\mathbb{R}^3$. Assume the Ricci flow $(M_t,g(t))_{t\in[0,t_1)}$ is well defined until a time $t_1\leq T$. Then $(M_t,g(t))_{t\in[0,t_1)}$ satisfies the strong (C,ϵ)–canonical neighborhood assumption with parameter r_0.

The relationship between the Type I/II classification and the canonical neighborhoods was given in [18]: a singularity at time T is of Type II if and only if there is a sequence (x_k,t_k) with $x_k\in M$ and $t_k\to T$ such that the scalar curvature at (x_k,t_k) goes to infinity and (x_k,t_k) is contained in a (C,ϵ)–cap diffeomorphic to a 3–ball (it corresponds to item iv in [18, Proposition 1.3]). This geometric characterization of Type II singularities will be useful. We note that the other kind of (C,ϵ)–caps, those diffeomorphic to $\mathbb{R}P^3$ minus a point, have a double cover which is a strong ϵ–neck.

The scalar curvature evolves according to

$$\frac{\partial R}{\partial t}=\Delta R+2|\,\mathrm{Ric}\,|^2.$$

Thus when x is in a strong ϵ–neck at time t or a (C,ϵ)–cap diffeomorphic to $\mathbb{R}P^3$ minus a point, there is a positive constant C_1 such that

$$\frac{1}{C_1}R(x,t)^2\leq\frac{\partial R(x,t)}{\partial t}\leq C_1R(x,t)^2, \tag{1}$$

and there is a positive constant C_2 such that whenever x is in a strong canonical neighborhood,

$$\left|\frac{\partial R(x,t)}{\partial t}\right| \le C_2 R(x,t)^2. \tag{2}$$

Along the Ricci flow, as the scalar curvature gets large it controls the whole curvature tensor. Let $(M, g(t))_{t\in[0,T)}$ be a Ricci flow such that for all $x \in M$ the smallest eigenvalue of $\mathrm{Rm}(x,0)$, denoted by $\nu(x,0)$, is at least -1. Set $X(x,t) = \max(-\nu(x,t), 0)$. Then Ivey [28] and Hamilton [27] showed the following "pinching towards positive" property:

Theorem 6 *We have the following properties:*

(i) $R(x,t) \ge -6/(4t+1)$.

(ii) *For all* (x,t) *for which* $0 < X(x,t)$,

$$R(x,t) \ge 2X(x,t)(\log X(x,t) + \log(1+t) - 3).$$

The notion of geometric convergence [34, Chapter 5] describes the convergence of based Ricci flows, and can be extended to any time interval (ie to intervals not of the form $(-T, 0]$). We will need the following convergence property of the Ricci flow.

Lemma 7 *Let* $T > 0$, *and let* (M_k, g_k) *be a sequence of closed normalized 3–manifolds. Suppose that for any sequence of points* $\mathfrak{s} = \{x_k\}_k$ *where* $x_k \in M_k$, *the following holds. Subsequentially, the sequence of based manifolds* (M_k, g_k, x_k) *converges geometrically to a complete based manifold* $(M^{\mathfrak{s}}_\infty, g^{\mathfrak{s}}_\infty, x_\infty)$ *such that*

(i) *the Ricci flow* $(M^{\mathfrak{s}}_\infty, g^{\mathfrak{s}}_\infty(t))$ *with initial metric* $g^{\mathfrak{s}}_\infty(0) = g^{\mathfrak{s}}_\infty$ *exists, is unique, and is defined for* $0 \le t < T$,

(ii) *for each* $t < T$ *there is a constant* $C_0 = C_0(t)$ *independent of the sequence* $\mathfrak{s}$ *so that the norm of the curvature tensor of* $(M^{\mathfrak{s}}_\infty, g^{\mathfrak{s}}_\infty(t'))$ *is bounded by* C_0 *for all* $t' \le t$.

Then for any sequence $\mathfrak{s} = \{x_k\}$ *with* $x_k \in M_k$, *the sequence of based Ricci flows* $(M_k, g_k(t), x_k)$ *starting at* $g_k(0) = g_k$ *subsequentially converges geometrically to* $(M^{\mathfrak{s}}_\infty, g^{\mathfrak{s}}_\infty(t), x_\infty)$ *on* $[0,T)$.

Proof Define

$$\tau = \sup\bigl\{t \in [0,T] : \exists C(t) > 0\ \forall t' \in [0,t]\ \limsup_k \max_{M_k} |\mathrm{Rm}(\cdot, t')| \le C(t)\bigr\}.$$

By (7.4a) and (7.4b) in [13] and the argument in [14, Lemma 6.1], we check that τ

is positive and that for any integer m, there is a positive time t_m for which $|\nabla^j \mathrm{Rm}|$ $(0 \leq j \leq m)$ are bounded on $[0, t_m]$ uniformly in k. It follows from Shi's derivative estimates (see [14, Chapter 6] and [34, Chapter 5] for instance) that for all $\mathfrak{s} = \{x_k\}$ with $x_k \in M_k$, $(M_k, g_k(t), x_k)$ subsequentially converges geometrically to $(M_\infty^{\mathfrak{s}}, g_\infty^{\mathfrak{s}}(t), x_\infty)$ on $[0, \tau)$ because of the first item in the assumptions. Hence it remains to show $\tau = T$. Suppose by contradiction that $\tau < T$. Then, by Theorem 5 and (2), for all $C' > 0$ there is a $\delta > 0$ such that there are subsequences $M_{k(l)}$ and $x_l \in M_{k(l)}$ with the following property: the curvature at $(x_l, \tau - \delta)$ in $(M_{k(l)}, g_{k(l)}(\tau - \delta))$ has norm larger than C'. But by the geometric convergence on $[0, \tau)$ that was just explained and the second item in the assumptions, it is absurd when $C' > C_0(\tau)$. Thus $\tau = T$ and the lemma is proved. □

1.2 Some min–max theory

In this subsection, we present a variation of the min–max theorem in the continuous setting as described by De Lellis and Tasnady in [17].

Let (M^{n+1}, g) be a closed Riemannian manifold. In what follows, the topological boundary of a subset of M will be denoted by ∂. Consider two open subsets X and N of M possibly with smooth boundaries such that $(X \cup \partial X) \subset N$. The notation for the m–dimensional Hausdorff measure will be $\mathcal{H}^m$. Take $a < b$ and $k \in \mathbb{N}$.

Definition 8 A family of $\mathcal{H}^n$–measurable closed subsets $\{\Gamma_t\}_{t\in[a,b]^k}$ in N with finite $\mathcal{H}^n$–measure is called a generalized smooth family if

- for each t there is a finite subset $P_t \subset N$ such that $\Gamma_t \cap N$ is a smooth hypersurface in $N \backslash P_t$,
- $t \mapsto \mathcal{H}^n(\Gamma_t)$ is continuous and $t \mapsto \Gamma_t$ is continuous in the Hausdorff topology,
- for each t_0, $\Gamma_t \to \Gamma_{t_0}$ smoothly in any compact $U \Subset N \backslash P_{t_0}$ as $t \to t_0$.

A generalized smooth family $\{\Sigma_t\}_{t\in[a,b]}$ is called a continuous sweepout in N associated to X if there exists a family of open subsets $\{\Omega_t\}_{t\in[a,b]}$ of N such that

(i) $(\Omega_t \cup \partial\Omega_t) \subset N$ for all $t \in [a, b]$,

(ii) $(\Sigma_t \backslash \partial\Omega_t) \subset P_t$ for any $t \in [a, b]$,

(iii) $\mathcal{H}^{n+1}(\Omega_t \backslash \Omega_s) + \mathcal{H}^{n+1}(\Omega_s \backslash \Omega_t) \to 0$ as $s \to t \in [a, b]$,

(iv) $\Omega_a = X$ and $\Omega_b = \varnothing$.

Still following [17], we define a notion of homotopy equivalence:

Definition 9 Two continuous sweepouts $\{\Sigma_t^1\}_{t\in[a,b]}$ and $\{\Sigma_t^2\}_{t\in[a,b]}$ associated with X are homotopic if

- there is a generalized smooth family $\{\Gamma_{(s,t)}\}_{(s,t)\in[a,b]^2}$ such that $\Gamma_{(a,t)} = \Sigma_t^1$ and $\Gamma_{(b,t)} = \Sigma_t^2$ for all $t \in [a,b]$,
- $\Gamma_{(s,t)} \subset N$ for $t \in [a,b]$ and there exists a small $\alpha > 0$ such that $\Gamma_{(s,t)} = \Gamma_{(a,t)}$ for $(s,t) \in [a,b]\times[a,a+\alpha]$.

A family Λ of continuous sweepouts in N associated to X is said to be homotopically closed if it contains the homotopy class of each of its element.

If Λ is a homotopically closed family of continuous sweepouts in N associated with X, the width of Λ in N is defined as the min–max quantity

$$W(N,\partial N,\Lambda) = \inf_{\{\Sigma_t\}\in\Lambda} \max_t \mathcal{H}^n(\Sigma_t).$$

A sequence $\{\{\Sigma_t^k\}_{t\in[a,b]}\}_{k\in\mathbb{N}} \subset \Lambda$ is called a minimizing sequence if

$$\max_t \mathcal{H}^n(\Sigma_t^k) \to W(N,\partial N,\Lambda) \quad \text{as } k\to\infty.$$

A sequence of slices $\{\Sigma_{t_k}^k\}_{k\in\mathbb{N}}$ is called a min–max sequence if

$$\mathcal{H}^n(\Sigma_{t_k}^k) \to W(N,\partial N,\Lambda) \quad \text{as } k\to\infty.$$

The following theorem is a slight extension of [45, Theorem 2.7]. It roughly says that if the starting slice of sweepouts belonging to a homotopically closed family Λ has n–volume less than the width of Λ, then the min–max theorem still holds as long as all the sweepouts are contained in an open set with mean convex boundary. Note that "mean convex" can be generalized to "piecewise smooth mean convex" (see [44]).

Theorem 10 *Let (M,g) be a closed $(n+1)$–manifold with $2 \le n \le 6$, and let N and X be open subsets of M. Suppose that $\partial X \neq \varnothing$ and that $(X\cup\partial X)\subset N$. When $\partial N \neq \varnothing$, assume that ∂N is mean convex. Then for any homotopically closed family Λ of sweepouts in N associated with X such that*

$$W(N,\partial N,\Lambda) > \mathcal{H}^n(\partial X),$$

there exists a min–max sequence $\{\Sigma_{t_n}^n\}$ of Λ converging in the varifold sense to an embedded minimal hypersurface Σ (possibly disconnected) contained in N. Moreover the n–volume of Σ, if counted with multiplicities, is equal to $W(N,\partial N,\Lambda)$.

Proof We essentially reproduce the proof of [45, Theorem 2.7]. Recall that theorem is an application to higher dimensions of an idea in [32], where the authors construct a vector field $\boldsymbol{V}$ in N whose support is contained in a small neighborhood of ∂N so that the corresponding flow is area-decreasing. Thanks to this flow, they show that Proposition 4.1 in [15] still holds. What we modify here is that, in the proof of this proposition, we restrict ourselves to the set $\mathfrak{X}$ of varifolds whose mass is bounded above by $4W(N,\partial N,\Lambda)$ and also bounded below by $\mathcal{H}^n(\partial X)+\epsilon$, where $0<\epsilon<W(N,\partial N,\Lambda)-\mathcal{H}^n(\partial X)$. In this way the starting slice remains fixed. More precisely, let $\mathcal{V}_\infty$ be the set of stationary varifolds contained in $\mathfrak{X}$. By choosing a sufficiently fine locally finite covering of $\mathfrak{X}\setminus\mathcal{V}_\infty$, we construct for each varifold V of mass less than $4W(N,\partial N,\Lambda)$ an ambient isotopy $\{\Psi_V(s,\cdot)\}_{s\in[0,1]}$ satisfying the properties listed in Step 3 of the proof of [15, Proposition 4.1] if $V\in\mathfrak{X}$ but such that $\Psi_V(s,\cdot)=\mathrm{Id}$ for all $s\in[0,1]$ if the mass of V is less than $\mathcal{H}^n(\partial X)+\frac{1}{2}\epsilon$. Finally, by modifying $\{\Psi_V(s,\cdot)\}_{s\in[0,1]}$ with the vector field $\boldsymbol{V}$ if necessary, we can deform a minimizing sequence $\{\{\Sigma_t^k\}_{t\in[0,1]}\}$ into another minimizing sequence $\{\{\widetilde{\Sigma}_t^k\}_{t\in[0,1]}\}$ such that all $\widetilde{\Sigma}_t^k$ with area larger than $\mathcal{H}^n(\partial X)+\epsilon$ lie at bounded distance from ∂N. Then the end of the proof remains unchanged compared to [45]. □

Remark 11 If $n=1$, the following elementary version of Theorem 10 will be useful. Suppose that N and X are diffeomorphic to the unit disk D in $\mathbb{R}^2$, define $\{c_t\}_{t\in[0,1]}$ as the smooth sweepout of X obtained by the foliation $\{x\in\mathbb{R}^2:\|x\|_{\mathrm{eucl}}=t\}_{t\in[0,1]}$ of *D*, where $\|\cdot\|_{\mathrm{eucl}}$ is the Euclidean norm in $\mathbb{R}^2$. Let $\mathcal{C}$ be the space of smooth curves endowed with the C^∞–topology. Let Λ be the homotopically closed family of sweepouts $\{\tilde{c}_t\}_{t\in[0,1]}\subset\mathcal{C}$ that continuously isotope to $\{c_t\}_{t\in[0,1]}$ in N and such that $\tilde{c}_0=c_0$. Define $W(N,\partial N,\Lambda)$ as for the higher-dimensional case. If ∂N is convex and

$$W(N,\partial N,\Lambda)>\mathcal{H}^1(\partial X),$$

then there is a simple closed geodesic in N of length $W(N,\partial N,\Lambda)$. This can be proved using the mean curvature flow $\{\Phi(s,\cdot)\}$ where s is the time parameter. Define $\theta\colon\mathbb{R}\times\mathcal{C}\to\mathbb{R}$ by

$$\theta(s,c)=\sup\bigl\{s'\in[0,s]:\Phi(s',c)\text{ has length at least }\tfrac{1}{2}(W(N,\partial N,\Lambda)+\mathcal{H}^1(\partial X))\bigr\},$$

where we use the convention $\sup\varnothing=0$. Then given a minimizing sequence of sweepouts $\{\{c_t^n\}_{t\in[0,1]}\}$, we consider the new tightened sequence $\bigl\{\{\Phi(\theta(s,c_t^n),c_t^n)\}_{t\in[0,1]}\bigr\}$ for each $s\geq 0$. By the maximum principle, the new sweepouts are entirely contained in N. Letting $s \to \infty$, any min–max sequence converges subsequentially to a simple closed geodesic inside N (see [20]).

2 Appearance of stable spheres and nontrivial singularities

2.1 Construction of thin hooks

We will construct a family of $(n+1)$–dimensional closed manifolds by defining embedded hypersurfaces in $\mathbb{R}^{n+2}$ and using the metric induced by the Euclidean metric. As we will see, they look like hook-shaped $(n+1)$–spheres, whose one branch is slightly swollen. The properties of these hooks will be useful to prove the two appearance theorems stated in the next subsection.

Consider a curve μ: $[0,1] \to \mathbb{R}^2$ such that

$$\mu(s) = \begin{cases} (1,s) & \text{for } s \in \left[0, \frac{1}{6}\right], \\ \left(\cos(s\pi), \frac{1}{6} + \sin(s\pi)\right) & \text{for } s \in \left[\frac{1}{3}, \frac{2}{3}\right], \\ (-1, 1-s) & \text{for } s \in \left[\frac{5}{6}, 1\right], \end{cases}$$

and μ is chosen on $\left[\frac{1}{6}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{5}{6}\right]$ so that it is a smooth curve. For each integer $L > 0$, consider the smooth curve γ_L: $[0,4] \to \mathbb{R}^2$ defined by

$$\gamma_L(s) = \begin{cases} (1, (s-1)L - 1) & \text{for } s \in [0,1), \\ (1, s-2) & \text{for } s \in [1,2), \\ \mu(s-2) & \text{for } s \in [2,3], \\ (-1, (3-s)L) & \text{for } s \in [3,4]. \end{cases}$$

It will be convenient to introduce the following function f: $\left[-\frac{1}{2}, \frac{1}{2}\right] \to \mathbb{R}$:

$$f(x) = \exp\left(1 + \frac{1}{4x^2 - 1}\right) \quad \text{for all } x \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right].$$

Let $d_0 > 0$ be smaller than half the focal radius of the curve μ. Now, we identify $\mathbb{R}^2$ with $\mathbb{R}^2 \times \{0\}$ in $\mathbb{R}^{n+2}$. At each point $p \in \gamma_L$, denote by $H[p]$ the normal hyperplane to γ_L at p. For each $L > 1$ and $\bar{\epsilon} = (\epsilon_1, \epsilon_2) \in \left(0, \frac{1}{4}\right)^2$, we choose a function $\phi[L,\bar{\epsilon}]$: $[0,4] \to \mathbb{R}^+$ such that

- $\phi[L,\bar{\epsilon}](s) = 1 + \epsilon_1 f(s - 1.5)$ for $s \in [1+\epsilon_2, 2-\epsilon_2]$,
- $\phi[L,\bar{\epsilon}](1/L) = \phi[L,\bar{\epsilon}](4 - 1/L) = 1$,
- $\phi[L,\bar{\epsilon}]$ is increasing on $[0, 1.5]$ and decreasing on $[1.5, 4]$,
- $\phi[L,\bar{\epsilon}]$ is strictly concave on $[0,1]$.

Define

$$\Gamma[L,\bar{\epsilon}] = \{x \in H[\gamma_L(s)] : d(x,\gamma_L(s)) = d_0\phi[L,\bar{\epsilon}](s) \text{ for } s \in [0,4]\}.$$

We can further impose that $\phi[L,\bar{\epsilon}]$ satisfy the following properties:

- $\Gamma[L,\bar{\epsilon}]$ is a smooth closed hypersurface.
- For each $L > 1$ and $\epsilon_1 \in \left(0,\frac{1}{4}\right)$ fixed, $\Gamma[L,\bar{\epsilon}]$ converges smoothly to a hypersurface $\Gamma[L,(\epsilon_1,0)]$ and $\phi[L,\bar{\epsilon}]$ converges uniformly to a function $\phi[L,(\epsilon_1,0)]$ when $\epsilon_2 \to 0$.
- The domains $\Delta[L,\epsilon_1] := \Gamma[L,(\epsilon_1,0)] \cap \{(x_1,\dots,x_{n+2}) : x_2 < -L\}$ are all isometric to each other for $L > 1$ and $\epsilon_1 \in \left(0,\frac{1}{4}\right)$, and they have positive sectional curvature.

In the following lemma, we list some useful properties of the manifold $\Gamma[L,\bar{\epsilon}]$ for any $n \geq 1$:

Lemma 12 (i) *If $n > 1$ and if d_0, ϵ_1 and ϵ_2 are small enough, $\Gamma[L,\bar{\epsilon}]$ has (arbitrarily large) positive scalar curvature bounded below by a positive constant independent of L.*

(ii) *The manifold $\Gamma[L,\bar{\epsilon}]$ has positive sectional curvature on*

$$\Gamma[L,\bar{\epsilon}] \cap \{(x_1,\dots,x_{n+2}) : x_1 > 0 \text{ and } x_2 < -1\}$$

and on the open neighborhood of

$$\Gamma[L,\bar{\epsilon}] \cap \left\{(x_1,\dots,x_{n+2}) : x_1 > 0 \text{ and } x_2 = -\tfrac{1}{2}\right\}$$

consisting of all points in $\Gamma[L,\bar{\epsilon}]$ at distance less than $\widetilde{\delta} > 0$ from the above set, where $\widetilde{\delta}$ is independent of L and $\bar{\epsilon}$.

(iii) *Let $Z^a = \{x \in H[\gamma_L(a)] : d(x,\gamma_L(a)) = d_0\}$ and consider Z^2 and $Z^{2.5}$ as hypersurfaces in $\Gamma[L,(\epsilon_1,0)]$. Then*

$$-\int_{Z^{2.5}} (R - \mathrm{Ric}(\nu,\nu)) > -\int_{Z^2} (R - \mathrm{Ric}(\nu,\nu)),$$

where ν denotes a unit normal on these hypersurfaces and R (resp. Ric) is the scalar curvature of $\Gamma[L,(\epsilon_1,0)]$ (resp. its Ricci curvature) endowed with the metric induced by $\mathbb{R}^{n+2}$.

Proof As $\epsilon_2 \to 0$, $\Gamma[L, \bar{\epsilon}]$ converges to $\Gamma[L, (\epsilon_1, 0)]$, and as $\epsilon_1 \to 0$, $\Gamma[L, (\epsilon_1, 0)]$ converges to a manifold called Γ_L. Hence to prove point (i), it is enough to show that for d_0 small enough, Γ_L has arbitrarily large positive scalar curvature. Since the scalar curvature on $\Gamma_L \cap \{(x_1, \dots, x_{n+2}) : x_2 < 0\}$ is positive and arbitrarily large as d_0 goes to 0, we only have to study

$$\Gamma_L \cap \{(x_1, \dots, x_{n+2}) : x_2 \geq 0\}.$$

But the desired property is clear since when d_0 goes to zero, the above subset converges after rescaling to a subset of a neck $S^n \times \mathbb{R}$ endowed with the product of a round metric and the standard metric on $\mathbb{R}$.

Point (ii) follows readily from the concavity of the function $\phi[L, \bar{\epsilon}]$ at the corresponding values.

The last point can be checked by computing the curvature for warped products; see [36, Chapter 7, Corollary 43] for instance (which holds for one-dimensional fibers). Indeed locally around Z^2 and $Z^{2.5}$, the metric is a warped product metric with base a round n–sphere of sectional curvature d_0^{-2} and with fiber [0, 1]. Let $f_w > 0$ be the warping function for $Z^{2.5}$, the warping function for Z^2 being constant. On one hand

$$\int_{Z^2} (R - \mathrm{Ric}(\nu, \nu)) = \int_{Z^2} \frac{n(n-1)}{d_0^2},$$

while, on the other hand,

$$\begin{aligned}\int_{Z^{2.5}} (R - \mathrm{Ric}(\nu, \nu)) &= \int_{Z^{2.5}} \left(\frac{n(n-1)}{d_0^2} - \frac{\Delta f_w}{f_w} \right) \\ &= \int_{Z^2} \frac{n(n-1)}{d_0^2} - \int_{Z^{2.5}} \frac{|\nabla f_w|^2}{f_w^2} \\ &< \int_{Z^2} (R - \mathrm{Ric}(\nu, \nu)).\end{aligned}$$

□

2.2 Appearance of stable geodesics and stable spheres

In this subsection, we will use "stable sphere" (resp. "stable geodesic") to denote a closed stable embedded minimal 2–sphere (resp. a simple closed stable geodesic).

Theorem 13 *There exists a 2–sphere (M, g) such that*

(i) *(M, g) does not contain stable geodesics,*

(ii) *a stable geodesic appears along the Ricci flow starting at (M, g).*

Theorem 14 *Let M be a spherical space form S^3/Γ which is endowed with a metric g of positive scalar curvature. Suppose that (M,g) does not contain any stable sphere or embedded minimal $\mathbb{R}P^2$ with stable oriented double cover. Then for any point $p\in M$ and radius $r>0$, there is a metric $\tilde{g}$ on M coinciding with g outside $B_g(p,r)$ such that*

(i) *$\tilde{g}$ has positive scalar curvature,*

(ii) *$(M,\tilde{g})$ does not contain any stable sphere or embedded minimal $\mathbb{R}P^2$ with stable oriented double cover,*

(iii) *a stable sphere appears along the Ricci flow starting at $(M,\tilde{g})$.*

In the case where M is two-dimensional, we clearly cannot assume its Gauss curvature to be positive at time 0 since this property will be preserved along the Ricci flow and this will prevent the existence of stable geodesics.

From now on, we assume d_0, ϵ_1 and ϵ_2 are small enough that by Lemma 12(i), $R>0$ on $\Gamma[L,\bar{\epsilon}]$ when $n=2$. To prove Theorem 13, we will need the following lemma.

Lemma 15 *Let $n=1$. There exists a positive constant C_0 such that if $L>C_0$ and $\epsilon_1<1/C_0$, then for all ϵ_2 sufficiently small the surface $\Gamma[L,\bar{\epsilon}]$ contains no stable geodesics.*

Proof Suppose by contradiction that there are two families

$$\{L_k\}_{k\in\mathbb{N}} \quad \text{and} \quad \{\bar{\epsilon}_{k,l}=(\epsilon_{1,k},\epsilon_{2,k,l})\}_{(k,l)\in\mathbb{N}^2}$$

such that

$$\begin{aligned} L_k &\to \infty \quad \text{as } k\to\infty,\\ \epsilon_{1,k} &\to 0 \quad \text{as } k\to\infty,\\ \epsilon_{2,k,l} &\to 0 \quad \text{as } l\to\infty \quad \text{for all } k, \end{aligned}$$

and a simple closed stable geodesic $S_{k,l}$ in $\Gamma[L_k,\bar{\epsilon}_{k,l}]$ for all $(k,l)\in\mathbb{N}^2$. We orient a curve in $\Gamma[L_k,\bar{\epsilon}_{k,l}]$ of the form

$$Z^s_{k,l}=\{x\in H[\gamma_{L_k}(s)] : d(x,\gamma_{L_k}(s))=d_0\phi[L_k,\bar{\epsilon}_{k,l}](s)\},$$

where $s\in(0,4)$, by imposing that the outward normal ν satisfy $\langle\nu,\gamma'_{L_k}(s)\rangle>0$. By construction, $\{Z^s_{k,l}\}_{s\in(1.5,4)}$ (resp. $\{Z^s_{k,l}\}_{s\in(0,1.5)}$) is a foliation of

$$A^+_{k,l}=\Gamma[L_k,\bar{\epsilon}_{k,l}]\cap\bigl\{(x_1,x_2,x_3) : x_1<0 \text{ or } x_2>-\tfrac{1}{2}\bigr\}$$
$$\bigl(\text{resp. } A^-_{k,l}=\Gamma[L_k,\bar{\epsilon}_{k,l}]\cap\bigl\{(x_1,x_2,x_3) : x_1>0 \text{ and } x_2<-\tfrac{1}{2}\bigr\}\bigr)$$

by concave (resp. convex) curves. Hence by the maximum principle, $S_{k,l}$ cannot be entirely contained in $A^+_{k,l}$ or in $A^-_{k,l}$. In other words, $S_{k,l}$ must intersect the central curve $Z^{1.5}_{k,l}$.

For a point $p \in \mathbb{R}^2$, we denote by $x_2(p)$ its second coordinate. Let $p_{k,l}$ be a point of $S_{k,l}$ such that $x_2(p_{k,l}) = \min_{p \in S_{k,l}} x_2(p)$. We already know by the previous paragraph that $x_2(p_{k,l}) \le -\frac{1}{2}$. By extracting a subsequence in k and then in l for each n, one can distinguish two situations:

(i) There is a constant $\kappa_0 > 0$ independent of k and l such that $x_2(p_{k,l}) < -L_k - \kappa_0$.

(ii) $\liminf_{k\to\infty} [\inf_l (x_2(p_{k,l}) + L_k)] \ge 0$.

Recall the notation

$$\Delta[L_k, \epsilon_{1,k}] = \Gamma[L_k, (\epsilon_{1,k}, 0)] \cap \{(x_1, x_2, x_3) : x_2 < -L_k\}.$$

Suppose by contradiction that (i) holds. Using the limit surfaces $\Gamma[L_k, (\epsilon_{1,k}, 0)]$ and the fact that the $\Delta[L_k, \epsilon_{1,k}]$ have positive Gauss curvature K and are isometric, we infer that there is a constant $\kappa_2 > 0$ (independent of k) such that for all k,

$$\limsup_{l\to\infty} \int_{\widetilde{S}_{k,l}} K > \kappa_2,$$

where $\widetilde{S}_{k,l} = S_{k,l} \cap \Delta[L_k, \epsilon_{1,k}]$. Since

$$\Gamma[L_k, \bar{\epsilon}_{k,l}] \cap \{(x_1, x_2, x_3) : x_1 > 0 \text{ and } x_2 < -1\}$$

has positive Gauss curvature and since we can choose k so that the length of

$$S_{k,l} \cap \{(x_1, x_2, x_3) : x_1 > 0 \text{ and } x_2 \in (-L_k, -1)\}$$

is arbitrarily large, we can find a function ϕ on $S_{k,l}$ having a support included in $S_{k,l} \cap \{(x_1, x_2, x_3) : x_1 > 0 \text{ and } x_2 < -1\}$ such that

$$\int_{S_{k,l}} (|\nabla\phi|^2 - K\phi^2) < 0$$

for a k sufficiently large and l large in comparison. This contradicts the stability of the geodesic $S_{k,l}$.

We have to rule out situation (ii) by using the embeddedness of $S_{k,l}$. Let us show that the length of $S_{k,l}$ is necessarily bounded, for example by $6\pi d_0$, for k large and l large in comparison. Consider the subset $I_{k,l}$ of $S_{k,l}$ consisting of all the points in $S_{k,l}$ at distance less than $3\pi d_0$ to $p_{k,l}$, where the intrinsic distance of $S_{k,l}$ is used. Let $s_{k,l}$ be such that $x_2(\gamma_{L_k}(s_{k,l})) = x_2(p_{k,l})$. Since the tangent vector of $S_{k,l}$ at $p_{k,l}$ is orthogonal to $(0,1,0)$, and because of the geometry of the limit surfaces $\Gamma[L_k,(\epsilon_{1,k},0)]$, the subset $I_{k,l}$ is an embedded multivalued graph with small gradient in $\Gamma[L_k,\bar{\epsilon}_{k,l}]$ over $Z_{k,l}^{s_{k,l}}$ for k and l large. But this is close to a standard circle of radius d_0 for k and l large so situation (ii) is possible only if $I_{k,l}$ actually contains the whole geodesic $S_{k,l}$ and is a one-valued graph. Now that we have bounded the length of $S_{k,l}$ independently of l for each k large, and since each $S_{k,l}$ intersects $Z_{k,l}^{1.5}$, we can extract a subsequence in l converging with multiplicity one to a stable geodesic S_k in $\Gamma[L_k,(\epsilon_{1,k},0)]$ of length less than $6\pi d_0$. The sequence $\{S_k\}$ in turn converges subsequentially in $\mathbb{R}^3$ to

$$Z^{1.5} = \{x \in H[\gamma_L(1.5)] : d(x,\gamma_L(1.5)) = d_0\},$$

because $\Gamma[L_k,(\epsilon_{1,k},0)] \cap \{(x_1,x_2,x_3) : x_1 > 0 \text{ and } -1 < x_2 < 0\}$ becomes cylindrical as $k \to \infty$. This is a contradiction with the stability assumption since in a neighborhood of $Z_{k,l}^{1.5}$ independent of (k,l) (see Lemma 12(ii)), the sectional curvature of $\Gamma[L_k,\bar{\epsilon}_{k,l}]$ is positive. □

The next lemma is true for $1 \le n \le 6$. We fix $\epsilon_1 \in \left(0,\frac{1}{4}\right)$ and $L > 1$. Let $\delta > 0$ and define

$$Y^{\epsilon_2} = \Gamma[L,(\epsilon_1,\epsilon_2)] \cap \{(x_1,\dots,x_{n+2}) : x_1 < 0 \text{ or } x_2 > -\delta\}.$$

We choose $\delta \in \left(0,\frac{1}{2}\right)$ so that the boundaries ∂Y^{ϵ_2} are isometric and convex for all $0 < \epsilon_2 < \delta$. Define also

$$X^{\epsilon_2} = \Gamma[L,(\epsilon_1,\epsilon_2)] \cap \{(x_1,\dots,x_{n+2}) : x_1 < 0 \text{ or } x_2 > 0\}.$$

Similarly, we define Y and X by replacing $\Gamma[L,(\epsilon_1,\epsilon_2)]$ by $\Gamma[L,(\epsilon_1,0)]$ in the above formulas and we write $Y^0 := Y$ and $X^0 := X$. Now for $\epsilon_2 \in [0,\delta)$, suppose that Y^{ϵ_2} is isometrically embedded in a closed $(n+1)$–manifold N^{ϵ_2}, in such a way that N^{ϵ_2} converges to N^0 as $\epsilon_2 \to 0$. Let $\{N_t^{\epsilon_2}\}_{t\in[0,T)}$ be a solution of the Ricci flow starting at N^{ϵ_2} defined on a time interval $[0,T)$. If V is a subset of N^{ϵ_2}, let V_t denote the Ricci flow at time t starting at V obtained by restriction of the original Ricci flow solution on N^{ϵ_2}. By abuse of notation, we view Y^{ϵ_2} as a subset of N^{ϵ_2} in Lemma 16.

In the proof, we will consider currents and varifolds in the closure $\overline{Y}$ which is isometrically embedded in $\mathbb{R}^{n+2}$. If U is an open subset of $\overline{Y}$, the corresponding $(n+1)$–dimensional current will be called $[|U|]$ and if C is an integral current, $|C|$ will be the name of the integer rectifiable varifold it determines by forgetting its

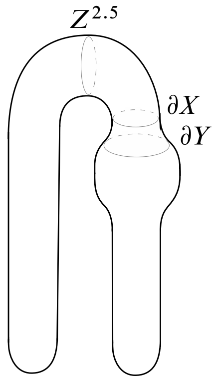

Figure 1: When ϵ_1 is small and L large, $\Gamma[L,(\epsilon_1,0)]$ looks like a thin hook.

orientation. If $k \in [|0, n+1|]$, the Grassmannian of k–planes in $\mathbb{R}^{n+2}$ is denoted by $\mathrm{Gr}(k, n+2)$ and its restriction to Y is denoted by $\mathrm{Gr}(k, n+2, Y)$. The Hausdorff measure $\mathcal{H}^k$ of a subset of Y_t is computed using the metric on Y_t.

Lemma 16 *Suppose $1 \le n \le 6$ and let $\epsilon_1 \in \left(0, \frac{1}{4}\right)$. There exists a positive constant C_1 such that if $L > C_1$, then the following holds. For each $\epsilon_2 \in [0,\delta)$ small enough, there is a positive time $t_0 = t_0(\epsilon_2)$ such that $Y_{t_0}^{\epsilon_2}$ contains an embedded stable minimal hypersurface. Moreover, t_0 can be chosen so that*

$$t_0(\epsilon_2) \to 0 \quad \text{as } \epsilon_2 \to 0.$$

Proof From the proof, it will be clear that $\lim_{\epsilon_2 \to 0} t_0(\epsilon_2) = 0$.

The boundary ∂Y^{ϵ_2} is convex with respect to the outward normal. Let Λ^{ϵ_2} be a sweepout in $Y_t^{\epsilon_2}$ associated with $X_t^{\epsilon_2}$ (see Definition 8). We will show that when ϵ_2 is small enough, for a positive time t_0 such that $\partial Y_{t_0}^{\epsilon_2}$ is still convex, we have:

$$W(Y_{t_0}^{\epsilon_2}, \partial Y_{t_0}^{\epsilon_2}, \Lambda_{t_0}^{\epsilon_2}) > \mathcal{H}^n(\partial X_{t_0}^{\epsilon_2}). \tag{3}$$

Applying Theorem 10 and Remark 11, we get an embedded minimal hypersurface S^{ϵ_2} in $Y_{t_0}^{\epsilon_2}$. If it is stable then the lemma is verified. If S^{ϵ_2} is not stable, then by minimizing its area in the connected open subset of $Y_{t_0}^{\epsilon_2} \backslash S^{\epsilon_2}$ whose boundary contains $\partial Y_{t_0}^{\epsilon_2}$, we get an embedded stable hypersurface and the lemma is also verified in that case.

Hence to complete the proof, it remains to show (3). Actually since the Ricci flow depends smoothly on the initial data, it is enough to check that if Λ is the sweepout in Y associated with X, then for all small positive times t_0, ∂Y_{t_0} is convex and

$$W(Y_{t_0}, \partial Y_{t_0}, \Lambda_{t_0}) > \mathcal{H}^n(\partial X_{t_0}).$$

For small times τ, ∂Y_τ remains convex and subsequently we will only consider such small times. Assume by contradiction that for all small $\tau > 0$, $W(Y_\tau, \partial Y_\tau, \Lambda_\tau) = \mathcal{H}^n(\partial X_\tau)$, and for each small $\tau > 0$ let us choose a continuous sweepout $\{\Sigma_s^\tau\}_{s\in[0,1]}$ such that

$$\max_s \mathcal{H}^n(\Sigma_s^\tau) \le \mathcal{H}^n(\partial X_\tau) + \epsilon(\tau), \tag{4}$$

where $\epsilon(\tau)$ is an arbitrary positive function converging to 0 as τ goes to 0 to be determined later. Let $\{\Omega_s^\tau\}_s$ be the family of open subsets of Y_τ associated with $\{\Sigma_s^\tau\}_s$ by Definition 8. For $a \in [2, 4-1/L]$, denote by U^a the subset of Y whose boundary (in N^0) is Z^a, where $Z^a = \{x \in H[\gamma_L(a)] : d(x, \gamma_L(a)) = d_0\}$. Let $\{\tau_k\}$ and $\{s_k\}$ be two sequences such that $\tau_k \to 0$ and

$$\mathcal{H}^{n+1}(\Omega_{s_k}^{\tau_k}) = \mathcal{H}^{n+1}(U^{2.5}).$$

We denote by V^k the subset of Y such that $V_{\tau_k}^k = \Omega_{s_k}^{\tau_k}$ (ie the open set of N^0 which becomes $\Omega_{s_k}^{\tau_k}$ at time τ_k). By [19], we can choose $\{\tau_k\}$ and $\{s_k\}$ so that $\partial[|V^k|]$ converges to an integral current $C = \partial[|V^\infty|]$ in the flat topology of $\overline{Y}$, where $\mathcal{H}^{n+1}(V^\infty) = \mathcal{H}^{n+1}(U^{2.5})$. We observe that (4) implies

$$\boldsymbol{M}(C) \le \omega_n d_0^n = \boldsymbol{M}\big(\partial[|U^{2.5}|]\big) \tag{5}$$

where ω_n is the volume of the n–dimensional unit round sphere.

Claim *If L was chosen large enough, then for $b \in [2,3]$, $\partial[|U^b|]$ is the unique area-minimizing current among the currents $C' = \partial[|U'|]$ such that*

$$\mathcal{H}^{n+1}(U') = \mathcal{H}^{n+1}(U^b),$$

where U' is an open subset relatively compact in Y, with a rectifiable boundary.

Let us prove this claim. Denote by $\hat{a}$ the function defined on $Y \backslash U^{4-1/L}$ such that $\hat{a}(x) = a$ if $\gamma_L(a)$ is the nearest point of γ_L to x. Define the projection $\hat{p}$: $Y \backslash U^{4-1/L} \to Z^{4-1/L}$ such that $\hat{p}^{-1}(y)$ is exactly the line in Y orthogonal to every Z^a and beginning at $y \in Z^{4-1/L}$. This projection enjoys the useful property of being area-decreasing in the sense that if R is a connected rectifiable boundary then

$$\mathcal{H}^n(\hat{p}(R)) \le \mathcal{H}^n(R)$$

with equality if and only if R is included in a certain Z^a. Let b, C' and U' be as above. To prove the claim, first note that when the support of C' is contained in $Y\backslash U^{4-1/L}$, we have $\boldsymbol{M}(C') \geq \boldsymbol{M}\big(\partial[|U^b|]\big)$ with equality if and only if $C' = \partial[|U^b|]$. Indeed, we can project C' on $Z^{4-1/L}$ and get the current $\hat{p}_\sharp(C')$. By the constancy theorem, it is an integer multiple of $\partial[|U^{4-1/L}|]$. If it is nonzero then $\boldsymbol{M}(C') \geq \boldsymbol{M}\big(\partial[|U^b|]\big)$ with equality only when $C' = \partial[|U^b|]$. If it is zero then the varifold $\hat{p}_\sharp(|C'|)$ has mass at least twice $\mathcal{H}^n(\hat{p}(\partial U'))$, which has to be larger than $\frac{1}{2}\mathcal{H}^n(Z^{4-1/L})$ for large L: this is because if $A \subset Z^{4-1/L}$ has n–volume at most $\frac{1}{2}\mathcal{H}^n(Z^{4-1/L})$, then $\hat{p}^{-1}(A)$ has $(n+1)$–volume strictly less than $\mathcal{H}^{n+1}(U^b)$ (for L large). When the support of C' is not contained in $U^{4-1/L}$, then by the coarea formula, there is a constant κ independent of L and an $a \in [3, 3.5]$ which depends on L such that $\operatorname{spt}(C') \cap Z^a$ is rectifiable,

$$\boldsymbol{M}(\langle C', \hat{a}, a\rangle) \leq \kappa/L \quad \text{and} \quad \boldsymbol{M}\big(\langle [|Y\backslash U'|], \hat{a}, a\rangle\big) \leq \kappa/L,$$

where the notation for slicing is the same as in [43, Chapter 2, Section 28]. Consider the current $\hat{C} = \partial[|U^a \cup U'|]$. In fact, for L large enough,

$$\boldsymbol{M}(\hat{C}) < \boldsymbol{M}(C'). \tag{6}$$

Indeed, by the monotonicity formula for minimal submanifolds, if L is large then any area-minimizing hypersurface in Y with boundary $\langle C, \hat{a}, a\rangle$ must be contained in $Y\backslash U^{4-1/L}$ and so is equal to $\langle Y\backslash U', \hat{a}, a\rangle$ by the constancy theorem. Since $\operatorname{spt}\hat{C} \subset Y\backslash U^{4-1/L}$ and $\mathcal{H}^{n+1}(U^a \cup U') \geq \frac{1}{2}\mathcal{H}^{n+1}(Y)$ for large L, the previous argument shows that $\boldsymbol{M}(\hat{C}) \geq \boldsymbol{M}\big(\partial[|U^b|]\big)$. But then C' has a bigger mass than $\partial[|U^b|]$ by (6) as wished, and the claim is verified.

Consequently for L large enough, (5) implies that the limit C is actually $\partial[|U^{2.5}|]$ and that as $k \to \infty$,

$$\boldsymbol{M}\big(\partial[|V^k|]\big) \to \boldsymbol{M}\big(\partial[|U^{2.5}|]\big).$$

By [40, (18)(f) in Section 2.1], the sequence of varifolds $\big|\partial[|V^k|]\big|$ converges subsequentially to $\big|\partial[|U^{2.5}|]\big|$. Applying the definition of varifolds convergence to the function which sends $(x, H) \in \operatorname{Gr}(n, n+2, Y)$ to $-R + \operatorname{Ric}(\nu, \nu)$ where ν is a unit vector orthogonal to H in $T_x\Gamma$, we have

$$\lim_{k\to\infty} \int_{\partial V^k} (-R + \operatorname{Ric}(\nu,\nu)) = \int_{Z^{2.5}} (-R + \operatorname{Ric}(\nu,\nu)),$$

which exactly means

$$\lim_{k\to\infty} \frac{\partial}{\partial t}\Big|_{t=0} \mathcal{H}^n(\partial V_t^k) = \frac{\partial}{\partial t}\Big|_{t=0} \mathcal{H}^n(Z_t^{2.5}). \tag{7}$$

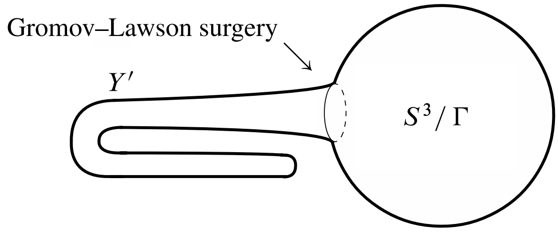

Figure 2: Part of a thin hook, Y', is glued to $S^3/\,\Gamma$ via the Gromov–Lawson procedure.

To contradict inequality (4), we write the following Taylor expansions near $t=0$:

$$\mathcal{H}^n(\partial V_t^k) = \mathcal{H}^n(\partial V^k) + t\cdot \frac{\partial}{\partial t}\Big|_{t=0}\mathcal{H}^n(\partial V_t^k) + t^2\cdot\varphi_k(t),$$
$$\mathcal{H}^n(\partial X_t) = \mathcal{H}^n(\partial X) + t\cdot \frac{\partial}{\partial t}\Big|_{t=0}\mathcal{H}^n(\partial X_t) + t^2\cdot\phi(t),$$

where φ_k and ϕ are functions bounded independently of k near $t=0$. By Lemma 12(iii),

$$\frac{\partial}{\partial t}\Big|_{t=0}\mathcal{H}^n(Z_t^{2.5}) > \frac{\partial}{\partial t}\Big|_{t=0}\mathcal{H}^n(\partial X_t). \tag{8}$$

Besides, the previous claim implies

$$\mathcal{H}^n(\partial V^k) \ge \mathcal{H}^n(Z^{2.5}) = \mathcal{H}^n(\partial X). \tag{9}$$

Hence, recalling that

$$\partial V_{\tau_k}^k = \partial\Omega_{s_k}^{\tau_k} = \Sigma_{s_k}^{\tau_k},$$

we combine (7), (8) and (9) and the Taylor expansions to conclude for k large that

$$\mathcal{H}^n(\Sigma_{s_k}^{\tau_k}) - \mathcal{H}^n(\partial X_{\tau_k}) > \tfrac{1}{2}\tau_k\Big(\frac{\partial}{\partial t}\Big|_{t=0}\mathcal{H}^n(Z_t^{2.5}) - \frac{\partial}{\partial t}\Big|_{t=0}\mathcal{H}^n(\partial X_t)\Big).$$

This is indeed the desired contradiction since the function $\epsilon(\cdot)$ in (4) could converge arbitrarily fast to 0, and this ends the proof. □

Proof of Theorems 13 and 14 Theorem 13 follows from Lemmas 15 and 16, by taking $M=\Gamma[L,(\epsilon_1,\epsilon_2)] = N^{\epsilon_2}$ with $\epsilon_1>0$, $\epsilon_2>0$ and $1/L$ sufficiently small.

To prove Theorem 14, let (M,g) be as in the statement. Choose any $p\in M$ and $r>0$ smaller than the injectivity radius of (M,g) so that $\partial B_g(p,s)$ is convex whenever $0<s<r$. Let $r_0>0$ be smaller than r. Let ϵ_2 be positive, and consider a scaled-down version of Y^{ϵ_2} (as defined just before Lemma 16) that we call Y' and glue it to M

around p by applying the Gromov–Lawson construction in $B_g(p,r_0)$ (see Figure 2). We can modify the size of r_0, δ, ϵ_2 and Y'. Analyzing how the forementioned construction is defined in [21] and taking the previous parameters small enough, we see that it can be done so as to get a new metric $\widetilde{g}$ on M satisfying:

- $\widetilde{g}$ coincides with g outside $B_g(p,r_0)$.
- Y' is isometrically embedded in $(B_g(p,r_0),\widetilde{g})$.
- $(B_g(p,r),\widetilde{g})$ is foliated by convex spheres.
- The scalar curvature of $(M,\widetilde{g})$ is bounded below by a positive constant independent of the parameters as they go to zero.

If the parameters are all small enough, then $(M,\widetilde{g})$ contains no stable embedded minimal 2–sphere or minimal $\mathbb{R}P^2$ with stable oriented double cover. Otherwise we could take the limit (subsequentially by [41]) and get a nontrivial oriented stable embedded minimal surface S in $(M\backslash\{p\},g)$ with finite area by the fourth item above (see [32, Proposition A.1] for instance). By curvature estimates for stable surfaces [41], the embedding is then proper, S has finite Euler characteristic and the singularity at p is removable (see [30, Lemma 2.5; 11, Proposition C.1; 23] for instance). Hence the closure $\overline{S}$ is a smooth stable minimal 2–sphere or a minimal $\mathbb{R}P^2$ with stable oriented double cover in (M,g), contradicting our assumption. Finally, Lemma 16 ensures that a stable sphere appears along the Ricci flow starting at $(M,\widetilde{g})$ provided $1/L$ and ϵ_2 are small enough. □

2.3 Appearance of nontrivial singularities

Theorem 17 *Let M be a closed 3–manifold satisfying the hypotheses of Theorem 14. Then for any point $p\in M$ and radius $r>0$, there is a metric $\widehat{g}$ on M coinciding with g outside $B_g(p,r)$ such that*

(i) *$\widehat{g}$ has positive scalar curvature,*

(ii) *$(M,\widehat{g})$ does not contain any stable sphere or embedded minimal $\mathbb{R}P^2$ with stable oriented double cover,*

(iii) *a nontrivial singularity occurs along the Ricci flow starting at $(M,\widehat{g})$.*

For the appearance of stable spheres described in the previous subsection, a first-order argument on the evolution of the thin hooks was enough. However we now want to study the long-time behavior, and to prove that a nontrivial singularity occurs, we need to modify the metric of the hooks. We twist and stretch the bent part as follows. For each $\bar{\epsilon}$ and L, using the previous notation, we consider the subsets

$$Z^a[L,\bar{\epsilon}] = \{s \in H[\gamma_L(a)] : d(x,\gamma_L(a)) = d_0\phi[L,\bar{\epsilon}](a)\} \quad \text{for all } a \in [2,3].$$

Let β: $[2,3] \to [0,1]$ be a bump function equal to zero in a neighborhood of $\{2,3\}$ and equal to one in $\left[2+\frac{1}{12}, 3-\frac{1}{12}\right]$. Denote by g_{eucl} the Euclidean metric in $\mathbb{R}^{n+2}$. For any $x \in Z^a[L,\bar{\epsilon}]$ (with $a \in [2,3]$), let $V(x)$ be a unit vector based at x tangent to $\Gamma[L,\bar{\epsilon}]$ but normal to the hypersurface $Z^a[L,\bar{\epsilon}]$, in the metric induced by g_{eucl}. Then for any stretching factor $L_{\mathrm{st}} \geq 0$, we define a new metric $g(L_{\mathrm{st}})$ on $\Gamma[L,\bar{\epsilon}]$ such that it only differs from the metric induced by g_{eucl} in $\bigcup_{a\in[2,3]} Z^a[L,\bar{\epsilon}]$ and for all $x \in \bigcup_{a\in[2,3]} Z^a[L,\bar{\epsilon}]$ and $u,v \in T_x\Gamma[L,\bar{\epsilon}] \subset T_x\mathbb{R}^{n+2}$,

$$g(L_{\mathrm{st}})(u,v) = g_{\mathrm{eucl}}(u,v) + \beta(a)L_{\mathrm{st}}g_{\mathrm{eucl}}(u,V(x))g_{\mathrm{eucl}}(v,V(x)).$$

Hence the modified metric $g(L_{\mathrm{st}})$ is similar to the metric induced by g_{eucl}, but strongly twisted and stretched between Z^2 and Z^3 when L_{st} is large. The choice of β guarantees that when L_{st} goes to infinity and ϵ_2 goes to zero, with the other parameters fixed,

- $(\Gamma[L,\bar{\epsilon}], g(L_{\mathrm{st}}))$ converges locally around $Z^{2+\frac{1}{12}}$ to $S^2 \times \mathbb{R}$ endowed with the product metric $d_0^2 h_1 + d\theta^2$, where h_1 is the round metric of Gauss curvature 1,
- $(\Gamma[L,\bar{\epsilon}], g(L_{\mathrm{st}}))$ converges locally around $Z^{2.5}$ to a warped product metric g_{tw} on $S^2 \times \mathbb{R}$ different from a product metric.

Note nevertheless that in the second limit any slice $S^2 \times \{\theta\}$ is also endowed with the round metric $d_0^2 h_1$. The Ricci flow for warped product metrics on $S^2 \times \mathbb{R}$ which are $\mathbb{R}$–invariant (and hence with base S^2) has a well-controlled behavior and was studied in [31]. For a metric g on a 3–manifold, let $T_{\mathrm{ext}}(g) \in (0,\infty)$ be its extinction time when well defined: when it exists it is defined as the time where a trivial singularity occurs. The following is a key lemma explaining why we consider these twisted hooks.

Lemma 18 *Let g_{inv} be an $\mathbb{R}$–invariant warped product metric on $S^2 \times \mathbb{R}$. Suppose that g_{inv} is not a product metric and that any slice $S^2 \times \{\theta\} \subset S^2 \times \mathbb{R}$ has area $4\pi d_0^2$ computed with g_{inv}. Then*

$$T_{\mathrm{ext}}(g_{\mathrm{inv}}) > T_{\mathrm{ext}}(d_0^2 h_1 + d\theta^2).$$

Proof Note that this lemma is the long-time counterpart of Lemma 12(iii), which is a first-order property. The proof is essentially the same computation.

Let $\{g_{\mathrm{inv}}(t)\}_{t\in[0,T_{\mathrm{ext}}(g_{\mathrm{inv}}))}$ be the maximal solution starting at g_{inv}. We observe that the slices $S^2\times\{\theta\}$ remain totally geodesic for all times. Hence by the Gauss equation, their area $A(t)$ evolves according to

$$\frac{dA}{dt}=-\int_{S^2\times\{\theta\}}(R-\mathrm{Ric}(\nu,\nu))=-8\pi-\int_{S^2\times\{\theta\}}\mathrm{Ric}(\nu,\nu), \tag{10}$$

where ν is a unit normal. Suppose now that $g_{\mathrm{inv}}(t)=k(t)+e^{2u(t)}d\theta^2$, where $k(t)$ and $u(t)$ are respectively a metric and a function on S^2. Then according to [31, (2.4)], the integral in the right-hand side of (10) is equal to

$$\int_{S^2}-|\nabla u(t)|^2\,d\mathrm{vol}_{k(t)}$$

where ∇ and $|\cdot|$ are computed using $k(t)$. Since $u(0)$ is not constant by hypothesis, $u(t)$ remains so and we obtain $dA/dt>-8\pi$. Since the analogue derivative for a product metric is equal to -8π, and since the extinction time coincides with the time when the area of the slices $S^2\times\{\theta\}$ converges to 0, we conclude that

$$T_{\mathrm{ext}}(g_{\mathrm{inv}})>T_{\mathrm{ext}}(d_0^2h_1+d\theta^2). \qquad \square$$

Heuristically, to make a singularity appear, we will choose the stretching factor L_{st} very large so that there are two regions evolving locally like two $\mathbb{R}$–invariant $S^2\times\mathbb{R}$, one of them being endowed with a product metric and separating the other one from a large region (to which we glued the twisted hook). Since the previous lemma suggests that the neck $S^2\times\mathbb{R}$ with a product metric should disappear first while the other regions stay large, a nontrivial singularity should occur. Let us make this reasoning rigorous with the following lemma.

Lemma 19 *There exists a constant $\widehat{C}>0$ and a time $\widehat{T}>0$ such that the following holds. Let $(N,g(t))$ for $0\le t\le t_1$ be a solution of the Ricci flow, and assume that the initial metric $g(0)$ is normalized and that N is a closed oriented connected 3–manifold. Suppose at t_1 that $x\in N$ is in the center of a strong ϵ–neck. Suppose that the center sphere of this strong ϵ–neck separates N into two components N_1 and N_2 such that there are $x_i\in N_i$ $(i=1,2)$ with*

$$R(x,t_1)>\widehat{C}\big(1+|R(x_i,t_1)|\big),$$

where $R(\cdot,t_1)$ is the scalar curvature function at time t_1. Then, along the Ricci flow starting at $(N,g(0))$, a nontrivial singularity occurs before time $t_1+\widehat{T}$.

Proof We can suppose that $R(x,t_1) \geq r_0^{-2}$, where r_0 comes from the canonical neighborhood theorem (Theorem 5). Consider the Ricci flow defined on a maximal time interval $[0,T)$ where $t_1 < T \leq \infty$; we want to show that a nontrivial singularity occurs at some t_2 larger than t_1. By definition of strong canonical neighborhoods, and by (1) and (2), since ϵ is small, there is a positive constant C_1 only depending on ϵ such that

(i) $\partial R(x,t)/\partial t \geq R(x,t)^2/C_1$ as long as x is in a strong ϵ–neck,

(ii) either $R(x_i,t) \leq r_0^{-2}$ or $\partial R(x_i,t)/\partial t \leq C_2 R(x_i,t)^2$.

The second item means that there is a time $t_3 > t_1$ such that if the flow runs into a trivial singularity, then it does not occur before t_3. If the condition in the first item is verified as long as the classical Ricci flow is defined then $R(x,t)$ goes to infinity before a time t_4. Choose $\widehat{C}$ large enough that $t_4 < t_3$. Suppose by contradiction that the point x ceases to be in a strong ϵ–neck at time $t' \in [t_1,T)$. Then (x,t') is in one of the following canonical neighborhoods:

- a (C,ϵ)–cap (where in particular the scalar curvature is comparable at every point),
- a C–component,
- an ϵ–round component.

Since t' is the first time after t_1 such that x is not in a strong ϵ–neck, (x,t') is actually in a (C,ϵ)–cap. Either x_1 or x_2 is also in this cap. Now if $\widehat{C}$ is large enough, then by (2) again each $R(x_i,t')$ cannot be comparable to $R(x,t')$, so this is a contradiction. Hence either the scalar curvature $R(x,t)$ goes to infinity before t_4 or a singularity happens elsewhere before t_4. Because $t_4 < t_3$, this singularity is not trivial. Taking $\widehat{T} = t_4$ finishes the proof. □

Proof of Theorem 17 First, we glue a small twisted hook to M around a point p as in the proof of Theorem 14. If ϵ_1, ϵ_2, $1/L$ and the size of the twisted hook are sufficiently small, then the new metric $\tilde{g}$ on M does not contain any stable sphere or minimal $\mathbb{R}P^2$ with stable oriented double cover and has positive scalar curvature. We take care of rescaling the new metric so that it becomes normalized. Let (M_k, g_k) be a sequence of such rescalings, where the parameters ϵ_1, ϵ_2 and $1/L$ and the size of the hook go to 0. It is also possible to guarantee that for any sequence $\mathfrak{s} = \{x_k\}$ with $x_k \in M_k$, the based manifolds (M_k, g_k, x_k) converge to one of the following geometric limits:

(a) The flat $\mathbb{R}^3$ (corresponding to points x_k not near the hook).

(b) A rotationally symmetric noncompact 3–manifold with two ends, one being a standard product metric on $S^2 \times [0, \infty)$ with scalar curvature 1 and the other one being a flat $\mathbb{R}^3 \backslash B(0,1)$ (corresponding to x_k near the part where the hook is glued).

(c) A warped product on $S^2 \times \mathbb{R}$ with base a round S^2 with scalar curvature 1 (corresponding to x_k inside the hook far from the tip).

(d) An $\mathbb{R}^3$ endowed with the standard initial metric (see [34, Chapter 12]) (corresponding to x_k near the tip of the hook).

Let T be the maximum of the maximal times for which the Ricci flows starting at one of these four metrics are smoothly defined. By [10] and [42], the hypotheses of Lemma 7 are satisfied. Notice that if $x_k \in Z^{2+\frac{1}{12}}$ (resp. $Z^{2.5}$) for all k then the geometric limit is a product metric (resp. nontrivial warped product metric) on $S^2 \times \mathbb{R}$, whose life span under the Ricci flow is equal (resp. strictly longer) than that of the standard initial metric by [34, Theorem 12.5] (resp. Lemma 18).

Suppose by contradiction that no nontrivial singularity occurs along the Ricci flow starting at $(M, \tilde{g})$. Two cases are a priori possible: $T = 1$, the life span of the standard initial metric, or $T < 1$. The latter situation corresponds to the maximum of the scalar curvature being reached around the gluing part near T, namely it means that T is the maximal existence time for the second geometric limit in the previous list. Note that since this Ricci flow is rotationally symmetric with two ends, the only canonical neighborhood that can appear is a strong ϵ–neck. By the above remarks and Lemma 7, in both cases one finds $\delta > 0$ so that for all k large, the Ricci flows $(M_k, g_k(t))$ have no singularity until at least $\hat{t} := T - \delta$, the time at which for some $q, q_1 \in M_k$, and for any $q_2 \in Z^{2.5}$:

- $R(q, \hat{t}) > \hat{C}(1 + R(q_i, \hat{t}))$ for $i = 1, 2$. ($\hat{C}$ is the constant in Lemma 19.)
- q is in a strong ϵ–neck whose central sphere separates q_1 and q_2.

Actually, q_1 is chosen to be a point of M far from p where the gluing is realized in the original metric g. The hypothesis of Lemma 19 are satisfied and a nontrivial singularity occurs, which contradicts our assumption that only a trivial singularity occurs. □

Remark 20 (i) In the proof of Theorem 14, we used hooks with a stretching factor $L_{\mathrm{st}} = 0$ for simplicity. However, it is not difficult to check that $\bar{\epsilon}$ and L can be chosen so that for any stretching factor L_{st}, Lemma 16 remains true. Hence putting

Theorems 14 and 17 together, we conclude that there are 3–manifolds with positive scalar curvature such that along the Ricci flow, a stable sphere appears and some time later, a nontrivial singularity occurs. In particular Theorem 1 is proved.

(ii) Although according to Theorem 17, a nontrivial singularity occurs in certain cases, it does not provide information on where it happens: intuitively one expects the singularity to occur at the neck with a product metric or at the tip of the twisted hook or at both places, depending on the shape of the tip.

3 Stable spheres and Type I singularities

In [2], examples of rotationally symmetric S^{n+1} developing a Type I neckpinching are constructed. Actually, in dimension 3, this is part of a much more general fact. By joining any two oriented 3–manifolds with a thin neck, we obtain initial data which will produce a nontrivial Type I singularity under the Ricci flow.

Proposition 21 *Let (M_1, g_1) and (M_2, g_2) be two closed oriented 3–manifolds. For any pair of points $p_i \in M_i$ $(i = 1, 2)$, radius $\hat{r} > 0$ small enough, length $l \geq 0$ and $\delta > 0$, there exists a metric g on the connected sum $M = M_1 \# M_2$ such that:*

- (i) *There is a subset $N \subset M$ diffeomorphic to $S^2 \times (0, 1)$, so that $M \backslash N$ is isometric to $(M_1 \backslash B(p_1, \hat{r})) \cup (M_2 \backslash B(p_2, \hat{r}))$.*
- (ii) *M is δ–close in the Hausdorff–Gromov distance to the union of M_1 and M_2 and a curve of length l joining p_1 to p_2,*
- (iii) *A nontrivial singularity of Type I occurs along the Ricci flow starting at (M, g).*

Proof As previously the proof is a limiting argument. We can glue an arbitrarily thin neck joining M_1 and M_2 so that M is δ–close in the Hausdorff–Gromov distance to the union of M_1 and M_2 and a curve of length l joining p_1 to p_2. We can ensure that this gluing is done locally around p_i, which does not affect the original metric in $(M_1 \backslash B(p_1, \hat{r})) \cup (M_2 \backslash B(p_2, \hat{r}))$. Let h_k be a sequence of metrics corresponding to thinner and thinner such necks. Let Q_k be the maximum of the scalar curvature on (M, h_k), which we assume is achieved at the middle of the neck. Denote by $\tilde{h}_k$ the rescaling $Q_k h_k$, and let $\tilde{h}_k(t)$ for $0 \leq t \leq T_k$ be a maximal solution for the Ricci flow. We choose the sequence of metrics so that for any sequence of points $x_k \in M$, the rescalings $(M, \tilde{h}_k(0), x_k)$ subsequentially converge geometrically to either a flat $\mathbb{R}^3$ or a product metric on $S^2 \times \mathbb{R}$ or a limit of type (b) described in the proof of Theorem 17. Then by Lemma 7, for k large, there is a point x which was in the neck at time 0 that is in a strong ϵ–neck with arbitrarily large scalar curvature (in particular at least r_0^{-2}) at a certain time t' independent of k. Notice that the rescalings at points in $(M_1\backslash B(p_1,\hat{r}))\cup(M_2\backslash B(p_2,\hat{r}))$ converge geometrically to a static flat $\mathbb{R}^3$. Hence by Lemma 19, for every large k a nontrivial singularity occurs at time $T_k\in(t',t'+\hat{T})$.

Claim *If k is large enough, a (C,ϵ)–cap with scalar curvature at least $2r_0^{-2}$ cannot appear during the Ricci flow $(M,\tilde{h}_k(t))$ for $0\le t\le T_k$.*

Suppose the claim to be true. Then the singularity is of Type I according to [18]. The theorem is thus proved modulo the claim.

To verify the claim, let us consider a sequence $\{(M,\tilde{h}_{k(l)}(t))\}_l$ of counterexamples. For each l, let t_l (resp. s_l) be the infimum of the times at which there is a (C,ϵ)–cap with scalar curvature at least $2r_0^{-2}$ (resp. r_0^{-2}), for the metric $\tilde{h}_{k(l)}(t_l)$. We can suppose that $T_{k(l)}$ (resp. $t_{k(l)}$, s_l) converges to T_∞ (resp. t_∞, s_∞). Actually we have $s_\infty<t_\infty\le T_\infty$. Indeed note that the only kinds of canonical neighborhoods with large scalar curvature that can appear are ϵ–necks which are diffeomorphic to $S^2\times(0,1)$ and (C,ϵ)–caps which are diffeomorphic to a ball or $\mathbb{R}P^3$ minus a point. For this reason, there is a (C,ϵ)–cap at time t_l with scalar curvature at least $2r_0^{-2}$ and by tracking this region we can go back in time to find a (C,ϵ)–cap with scalar curvature at least r_0^{-2} at time $t_l-\delta$. In view of the derivative estimate (2) this delta can be chosen independent of l, and we get $s_\infty<t_\infty$ as desired. Next we pick p_l to be a point in a (C,ϵ)–cap at time s_l. By definition of t_l and by (1), the curvature tensor is uniformly bounded on $\left[0,\frac{1}{2}(s_l+t_l)\right]$. Recall that the pointed zero-time time slices $(M,\tilde{h}_{k(l)}(0),p_l)$ converge to a limit $(M_\infty,\tilde{h}_\infty(0),p_\infty)$ with bounded curvature, so the flow starting at this limit exists and is unique [42; 10]. Let $S>0$ such that $(M_\infty,\tilde{h}_\infty(t),p_\infty)$ is maximally defined on $[0,S)$. By construction this limit flow is rotationally symmetric noncompact with two ends when nonflat, so the only canonical neighborhoods with large curvature which could appear are strong ϵ–necks. Actually by Lemma 7, $S\ge\frac{1}{2}(s_\infty+t_\infty)$. Indeed otherwise for l large and t'' close to S there should be an arbitrarily thin ϵ–neck for the metric $\tilde{h}_{k(l)}(t'')$ but then (1) and (2) would contradict $\frac{1}{2}(s_l+t_l)>S$. So $S\ge\frac{1}{2}(s_\infty+t_\infty)$ and by Lemma 7 again, $(M_\infty,\tilde{h}_\infty(s_\infty))$ should then contain a (C,ϵ)–cap, which is impossible and our claim is proved. □

From the proof of Proposition 21, it can be shown for the examples where a Type I singularity appears at some time t_1 that for all $t\in[0,t_1)$ there is an embedded stable minimal sphere $S(t)$ whose area goes to 0 as $t \to t_1$. These spheres correspond to the neckpinching. One can wonder if this is a general phenomenon. The next theorem confirms that indeed small stable spheres or $\mathbb{R}P^2$ with stable oriented double cover are closely related to Type I singularities. When a minimal surface is an embedded stable sphere or an embedded $\mathbb{R}P^2$ with stable oriented double cover, we will call it a *stable immersed sphere with embedded image*.

Theorem 22 *Let M be an oriented closed connected 3–manifold. Consider a Ricci flow $(M, g(t))$ for $0 \le t < T$ and suppose that there is a nontrivial Type I singularity at time T. Then, for all time t close to T, $(M, g(t))$ contains a stable immersed sphere with embedded image $S(t)$ such that*

$$C'(T-t) \le \mathcal{H}^2(S(t)) \le C''(T-t),$$

where C' and C'' are constants independent of t.

Conversely, suppose that there is a sequence of times s_k converging to T and a sequence of stable immersed spheres with embedded image S_k in $(M, g(s_k))$. Suppose also that the area of S_k goes to zero and

$$\mathcal{H}^2(S_k) \ge C'(T-s_k),$$

where C' is a constant independent of k. Then there is a singularity at time T which is locally of Type I in the sense that for all $A > 0$ there exists a $\overline{C} = \overline{C}(C', A, C, \epsilon)$ such that

$$\sup\bigl\{|\mathrm{Rm}(x, s_k)| : (\max_{S_k} R) \cdot d(x, S_k)^2 \le A\bigr\} \le \frac{\overline{C}}{T-s_k} \quad \text{for all } k > \overline{C}.$$

Proof Without loss of generality we assume $(M, g(0))$ to be normalized. Suppose that $(M, g(t))$ for $0 \le t < T$ develops a Type I singularity at T. Since the singularity is nontrivial, for all times close to T, say for $t \in (t_0, T)$, there is a constant $A > 0$ independent of t such that the points of scalar curvature larger than A are in strong ϵ–necks or in (C, ϵ)–caps diffeomorphic to $\mathbb{R}P^3$ minus a point according to [18]. By [34, Proposition A.21], this means that at time $t \in [t_0, T)$ two situations can happen:

- M is covered by the previous canonical neighborhoods and is diffeomorphic to $S^2 \times S^1$ or $\mathbb{R}P^3 \# \mathbb{R}P^3$. The existence of a stable immersed sphere with embedded image $S(t)$ is obtained by γ–reduction [33]; furthermore, $S(t)$ has area going to zero as t goes to T.

- $(M, g(t))$ contains $T(t)$, an ϵ–tube or a C–capped ϵ–tube (diffeomorphic to $\mathbb{R}P^3$ minus a point), whose curvature at the end(s) is at most A but which contains points whose scalar curvature goes to infinity as t approaches T.

In the second case, choose a sphere $Z(t)$ in $T(t)$, which is the central sphere of a strong ϵ–neck and has area going to zero if t is close to T. We can try to minimize its area in $T(t)$ because the boundary component(s) of $T(t)$ have large area in comparison. Actually $Z(t)$ is homologically nontrivial in $T(t)$ and one cannot reduce its area to zero by isotopies. By deforming slightly the boundaries of $T(t)$ to make them strictly mean convex, we can use γ–reduction again to find a stable immersed sphere with embedded image $S(t)$. For times close to T, this minimal surface is far from the boundaries where we deformed the metric by the monotonicity formula and the geometry of the necks, so it is in fact minimal for the original metric $g(t)$ and

$$\mathcal{H}^2(S(t)) \leq \mathcal{H}^2(Z(t)) \to 0 \quad \text{as } t \to T.$$

Choose $S(t)$ to be of least area among stable immersed spheres with embedded image at time t close to T. Notice that the scalar curvature on $S(t)$ is comparable everywhere to the maximum of the scalar curvature on $(M, g(t))$ by the choice of $S(t)$ and the canonical neighborhood theorem. But it is known that for a Type I singularity the scalar curvature blows up in $1/(T-t)$ hence the area of $S(t)$ decreases to zero linearly and the first part of the theorem is proved.

For the second part, we can argue as follows. Let $p_k \in S_k$ be a point where the scalar curvature achieves its minimum on S_k. Then, by [34, Theorem 11.19] and the monotonicity formula, since the area of S_k converges to 0, $R(p_k, s_k)$ goes to infinity. By the area upper bound (depending on the scalar curvature) [32, Proposition A.1] and curvature bound for stable spheres [41], by the classification of canonical neighborhoods and their properties, for k large, S_k has to be a sphere or $\mathbb{R}P^2$ entirely contained in a strong ϵ–neck or in a (C, ϵ)–cap diffeomorphic to $\mathbb{R}P^3$ minus a point. The area bound from below for S_k implies that the scalar curvature on S_k is smaller than $C''/(T-s_k)$ for a certain constant C''. The conclusion now follows from the "bounded curvature at bounded distance" property [34, Chapter 10]. □

Remark 23 From the proof of the previous theorem, it becomes clear that when there is a sequence of stable immersed spheres with embedded image S_k at times s_k going to T, with area converging to 0, then the minimum of the scalar curvature on these spheres, $\min_{S_k} R$, goes to infinity and $\max_{S_k} R / \min_{S_k} R$ is bounded.

4 Symmetry and nonappearance of stable spheres

In this section, we study under which symmetry assumptions one can rule out the appearance of stable immersed spheres with embedded image along the Ricci flow. In the case of a finite group G acting effectively by isometries on a 3–manifold, there is a point p which is fixed only by the identity, and one can glue disjoint thin hooks at the images of p under the elements of G, in an equivariant way. This gives a G–invariant metric for which stable spheres appear along the Ricci flow. Hence, we will only focus on positive-dimensional compact Lie groups. Consider (M, g), an oriented connected closed 3–manifold on which a d–dimensional compact Lie group G of isometries acts effectively. Assume that

(i) $d > 1$, or

(ii) $d = 1$ and the action is free.

We will say that (M, g) (as above) is rotationally symmetric if a subgroup G_0 of G is isomorphic to SO(3) and there is a G_0–invariant 2–sphere or $\mathbb{R}P^2$ embedded in (M, g). This amounts to saying that a cover of (M, g) is a warped product $I \times S^2$ with fiber S^2, where $I = \mathbb{R}$ or $I = [0, 1]$ (the warped product is then degenerate at 0 and 1). In that case, M is diffeomorphic to S^3, $\mathbb{R}P^3$, $\mathbb{R}P^3 \# \mathbb{R}P^3$ or $S^2 \times S^1$.

Theorem 24 *Let (M, g) be as above. Suppose that it contains no stable immersed spheres with embedded image. Then, along the Ricci flow starting at (M, g), stable immersed spheres with embedded image cannot appear.*

Proof Let $(M, g(t))$ for $0 \le t \le T$ with $g(0) = g$ be a solution of the Ricci flow and suppose by contradiction that there is a stable immersed sphere with embedded image $S(t_1)$ in $(M, g(t_1))$. By uniqueness of the Ricci flow, G still acts by isometries on $(M, g(t_1))$. For all $V \in \mathfrak{g}$ a vector in the Lie algebra of G, we define $\phi_V(s)$ to be the 1–parameter family of diffeomorphisms of G generated by the left-invariant vector field corresponding to V. Note that for any $V \in \mathfrak{g}$, the projection of

$$\frac{d(\phi_V(s).x)}{ds} \in T_x M, \quad x \in S(t_1),$$

on the normal bundle of $S(t_1)$ is a Jacobi field J_V. By stability either it is identically zero or it does not vanish. Let us show that J_V has to be zero. In the case where $S(t_1)$ is an $\mathbb{R}P^2$ it is clear since its normal bundle is nontrivial (M is oriented). If $S(t_1)$ is an embedded sphere and $J_V \neq 0$, then in a neighborhood of $S(t_1)$, $\{\phi_V(s).(S(t_1))\}_{s\in[0,s_0]}$ foliates one side of $S(t_1)$ as long as $\phi_V(s_0).(S(t_1))$ does not touch $\phi_V(0).(S(t_1))$ from the other side. When it does so at s_0, by minimality, the two surfaces coincide: $\phi_V(0).(S(t_1)) = \phi_V(s_0).(S(t_1))$. Since such an $s_0 > 0$ exists in the case where J_v is not identically zero, we deduce by connectedness and orientability of M that M is an $S^2 \times S^1$, a contradiction since for topological reasons it always contains a stable sphere.

We just proved that for all $x \in S(t_1)$, the vector $X = d(\phi_V(s).x)/ds$ is tangent to the sphere $S(t_1)$ for all $V \in \mathfrak{g}$. This means that G acts on $S(t_1)$. Since any compact 1–dimensional group of isometries acting on a 2–sphere or $\mathbb{R}P^2$ fixes a point, G is of dimension $d > 1$, so case (ii) is proved. For case (i), since G is of dimension greater than 1 and acts effectively by isometries on a 2–sphere, the connected component G_0 containing Id is isomorphic to the rotation group $\mathrm{SO}(3)$. In other words, M is rotationally symmetric and then the nonappearance of stable spheres along the Ricci flow is reduced to an ODE argument. By [1, Theorem A], stable spheres invariant under G_0 cannot appear if there were none at the beginning and we can check that any stable sphere, if it exists, is G_0–invariant. The assumption that a stable sphere appears is thus absurd. The situation for $\mathbb{R}P^2$ with stable oriented double cover is similar. □

Remark 25 (i) The 3–dimensional (twisted) hooks defined in Section 2 have an effective S^1–action which is not free, so according to Theorem 24 these examples where stable spheres appear have in some sense a maximal amount of symmetry.

(ii) A byproduct of the proof of Theorem 24 is that if (M, g) (as above) contains a stable immersed sphere with embedded image S and if (M, g) is not rotationally symmetric, then it is an $S^2 \times S^1$ foliated by stable spheres which are images of S under a family of isometries.

Lemma 26 *Let (M, g) be as above. If a Type II singularity occurs then (M, g) is a rotationally symmetric sphere or $\mathbb{R}P^3$.*

Proof Let t_1 be the time of a Type II singularity. By [18], just before t_1, there is a region of high scalar curvature which is a (C, ϵ)–cap diffeomorphic to a 3–ball. By [35, Lemma 14.3.11, Proposition 14.3.12], the action of H is equivariant to a linear action and there is a fixed point. Consequently, H cannot be 1–dimensional by our assumption on G, and M is a rotationally symmetric sphere or $\mathbb{R}P^3$. □

Because of the link between Type I singularities and stable spheres described in Section 3, we readily obtain the following corollary.

Corollary 27 *Let (M, g) be as above. The following hold along the Ricci flow:*

(i) *When M is a rotationally symmetric 3–sphere and does not contain stable spheres, no nontrivial Type I singularity occurs.*

(ii) *When M is a rotationally symmetric $\mathbb{R}P^3$ and does not contain stable immersed spheres with embedded image, no nontrivial Type I singularity occurs.*

(iii) *When M is rotationally symmetric and neither a 3–sphere nor an $\mathbb{R}P^3$, no Type II singularity occurs.*

(iv) *When M is not rotationally symmetric and a singularity occurs, it is a Type I trivial singularity.*

Proof The first item comes from Theorems 22 and 24, the second item is proved in the same way considering a double cover. Lemma 26 yields the third item. For the fourth item, a singularity must be of Type I by Lemma 26 and [18]. Let T be a time of singularity. Suppose that the singularity is nontrivial, then by Theorem 22 and Remark 25(ii), $(M, g(t))$ is an $S^2 \times S^1$ foliated by small spheres for all t close to T. The curvature blows up everywhere in that case (see Remark 23), contradicting our assumption and the corollary is verified. □

A question still left unanswered is whether a Type II singularity can appear in the case of item (i). In item (iii), the other kinds of singularities can occur. The first item was proved in [22] for all dimensions. In the case of a free S^1–action, it was already suggested in [31, Remark 2.6] to combine the singularity analysis with the symmetry.

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