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Khovanov homotopy type, Burnside category and products

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We give a new construction of a Khovanov stable homotopy type, or spectrum. We show that this construction gives a space stably homotopy equivalent to the Khovanov spectra constructed by Lipshitz and Sarkar (J. Amer. Math. Soc. 27 (2014) 983–1042) and Hu, Kriz and Kriz (Topology Proc. 48 (2016) 327–360) and, as a corollary, that those two constructions give equivalent spectra. We show that the construction behaves well with respect to disjoint unions, connected sums and mirrors, verifying several of Lipshitz and Sarkar's conjectures. Finally, combining these results with Lipshitz and Sarkar's computations (J. Topol. 7 (2014) 817–848) and refined s–invariant (Duke Math. J. 163 (2014) 923–952), we obtain new results about the slice genera of certain knots.

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1 Introduction

In a sequence of revolutionary papers in the 1980s, Floer introduced a family of invariants in low-dimensional and symplectic topology, now known as Floer homologies [16; 17; 18]. While the definitions of these invariants appear to be a semi-infinite-dimensional version of Morse homology, unlike classical (finite- or infinite-dimensional) Morse homology, Floer homologies were not apparently isomorphic to the singular homologies of any natural space. In the 1990s, under appropriate hypotheses Cohen, Jones and Segal proposed a construction of a stable homotopy type whose singular homology was Floer homology [9]. Their construction builds a CW complex cell by cell, using a version of the Pontrjagin–Thom construction for manifolds with corners to define the attaching maps; the input data to this Pontrjagin–Thom construction is what Cohen, Jones and Segal call a *framed flow category*. Analytic difficulties mean that their proposal has not yet been carried out rigorously, though Manolescu constructed a stable homotopy refinement of the Seiberg–Witten Floer homology of rational homology 3–spheres by other techniques [41] (compare Lidman and Manolescu [33]).

Around the turn of the millennium, in a seminal paper Khovanov gave a bigraded homology theory whose graded Euler characteristic is the Jones polynomial [26]. The definition of this *Khovanov homology* is purely combinatorial, though equivalent or conjecturally equivalent definitions have been given using Floer homology — see Seidel and Smith [52] and also Abouzaid and Smith [1] — and algebraic geometry; see Cautis and Kamnitzer [8].

Inspired by Cohen, Jones and Segal's work and the relation with Floer homology, in [35] we introduced a stable homotopy refinement of Khovanov homology. That is, to each link diagram K we associated a spectrum $\mathcal{X}_{Kh}(K)$, with the following properties:

1. (1) The spectrum $\mathcal{X}_{Kh}(K)$ is a finite CW spectrum, that is, a formal desuspension of a finite CW complex.
2. (2) The spectrum $\mathcal{X}_{Kh}(K)$ comes with a wedge-sum decomposition $\mathcal{X}_{Kh}(K) = \bigvee_j \mathcal{X}^j_{Kh}(K)$.
3. (3) For each j, the cellular cochain complex of $\mathcal{X}^j_{Kh}(K)$ is isomorphic to the Khovanov complex $\mathcal{C}^{*,j}_{Kh}(K)$ in quantum grading j, via an isomorphism taking the standard generators for $C^*_{cell}(\mathcal{X}^j_{Kh}(K))$ to the standard generators for $\mathcal{C}^{*,j}_{Kh}(K)$.
4. (4) For each j, the stable homotopy type of $\mathcal{X}^j_{Kh}(K)$ is an invariant of the isotopy class of the link represented by the diagram K.

There is also a reduced version $\widetilde{\mathcal{X}}_{Kh}(K)$ of $\mathcal{X}_{Kh}(K)$, which satisfies properties (1)–(2) with $\mathcal{C}_{Kh}$ replaced by the reduced Khovanov complex $\widetilde{\mathcal{C}}_{Kh}$. The constructions of $\mathcal{X}_{Kh}(K)$ and $\widetilde{\mathcal{X}}_{Kh}(K)$ involve defining a framed flow category combinatorially, and then applying Cohen, Jones and Segal's Pontrjagin–Thom construction.

Largely independently, inspired by their homotopy-theoretic investigations of topological and conformal field theories, Hu, Kriz and Kriz gave another construction of a Khovanov stable homotopy type with the same basic properties [22]. Roughly, they turn the Khovanov cube into a functor from the cube category to the Burnside category of finite sets and correspondences. They then apply the Elmendorf–Mandell infinite loop space machine to obtain a functor from the cube to symmetric spectra, and then take an iterated mapping cone to obtain a spectrum.

These refinements give extra structure to the Khovanov homology groups. Because the Khovanov homology groups of K are cohomology groups of $\mathcal{X}_{Kh}(K)$, the mod-p cohomology groups possess Steenrod power operations such as the squares Sq^n at $p=2$. These refinements also allow one to apply generalized cohomology theories, such as topological K–theory, Morava K–theory and stable homotopy groups. Each of these has its own Atiyah–Hirzebruch spectral sequence determining a potentially new collection of knot invariants, and maps in the stable homotopy category give natural transformations relating these different invariants. In the philosophically related case of Seiberg–Witten Floer theory, for instance, equivariant KO–theory has been used to obtain interesting low-dimensional applications; see Manolescu [42] and Lin [34].

One goal of the present paper is to study the behavior of $\mathcal{X}_{Kh}(K)$ under disjoint unions and connected sums of links. In particular, we will prove:

Theorem 1 [35, Conjecture 10.3] *Let L_1 and L_2 be links, and $L_1 \sqcup L_2$ their disjoint union. Then*

$$\mathcal{X}^{j}_{Kh}(L_1 \sqcup L_2) \simeq \bigvee_{j_1+j_2=j} \mathcal{X}^{j_1}_{Kh}(L_1) \wedge \mathcal{X}^{j_2}_{Kh}(L_2). \tag{1.1}$$

Moreover, if we fix a basepoint p in L_1, not at a crossing, and consider the corresponding basepoint for $L_1 \sqcup L_2$, then

$$\widetilde{\mathcal{X}}^{j}_{Kh}(L_1 \sqcup L_2) \simeq \bigvee_{j_1+j_2=j} \widetilde{\mathcal{X}}^{j_1}_{Kh}(L_1) \wedge \mathcal{X}^{j_2}_{Kh}(L_2). \tag{1.2}$$

Theorem 2 [35, Conjecture 10.4] *Let L_1 and L_2 be based links and $L_1 \# L_2$ the connected sum of L_1 and L_2, where we take the connected sum near the basepoints.*

Then

$$\widetilde{\mathcal{X}}^{j}_{Kh}(L_1 \# L_2) \simeq \bigvee_{j_1+j_2=j} \widetilde{\mathcal{X}}^{j_1}_{Kh}(L_1) \wedge \widetilde{\mathcal{X}}^{j_2}_{Kh}(L_2). \tag{1.3}$$

(We also compute the unreduced Khovanov spectrum of a connected sum [35, Conjecture 10.6] as Theorem 10.)

These theorems, though unsurprising themselves, have some interesting corollaries:

Corollary 1.4 *For any n there exists an n–component link L_n such that the operation*

$$\mathrm{Sq}^n\colon Kh^{i,j}(L_n) \to Kh^{i+n,j}(L_n)$$

is nonzero for some $i, j \in \mathbb{Z}$. Similarly, there exists a knot K_n such that the operation

$$\mathrm{Sq}^n\colon \widetilde{Kh}^{i,j}(K_n) \to \widetilde{Kh}^{i+n,j}(K_n)$$

is nonzero for some $i, j \in \mathbb{Z}$. Further, for this knot, the operation

$$\mathrm{Sq}^n\colon Kh^{i,j}(K_n) \to Kh^{i+n,j}(K_n)$$

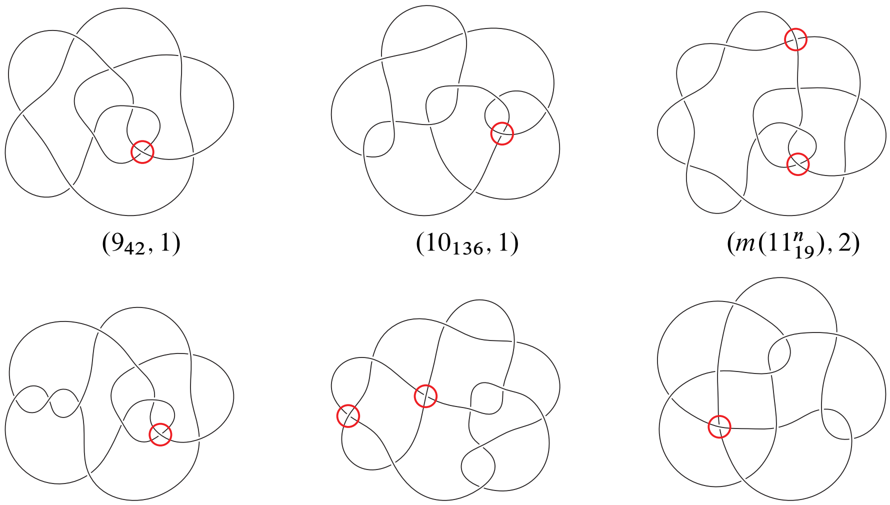
is also nonzero for some $i, j \in \mathbb{Z}$.

Corollary 1.5 *Let K be one of the knots 9_{42}, 10_{136}, $m(11^n_{19})$, $m(11^n_{20})$, 11^n_{70} or 11^n_{96}.* (Here m denotes the mirror.) *Let L be a knot which is the closure of a positive braid. Letting g_4 denote the four-ball genus, we have*

$$g_4(K \# L) = g_4(K) + g_4(L).$$

Remark 1.6 There is some confusion for nomenclature of knots regarding mirrors. See Figure 1 for our convention, and compare with the discussion around Table 1 on page 739. The value of $g_4(K)$ can be extracted from Figure 1 or Table 1, and $g_4(L)$ equals the genus of the Seifert surface obtained by applying Seifert's algorithm to the positive braid closure knot diagram for L; see Rasmussen [48, Theorem 4].

Corollary 1.5 is not implied by computations of Rasmussen's s–invariant or the Heegaard Floer τ–invariant or the signature; see Table 1. At least for 9_{42}, the result is not implied by the Heegaard Floer concordance invariant Υ; the Heegaard Floer d–invariant of $+1$ surgery (Krcatovich, personal communication 2014); or Hom, Rasmussen and Wu's ν^+–invariant [47; 21]. (These observations are not entirely independent.)



$(9_{42}, 1)$ $(10_{136}, 1)$ $(m(11^n_{19}), 2)$

$(m(11^n_{20}), 1)$ $(11^n_{70}, 2)$ $(11^n_{96}, 1)$

Figure 1: The knots appearing in Corollary 1.5. We have labeled the knot K by the pair $(K, g_4(K))$. The value of $g_4(K)$ is extracted from Knotinfo [38]; the knot diagrams have been produced using Knotilus [15]. Crossings giving a minimal unknotting for each knot are circled.

The construction of $\mathcal{X}_{Kh}(K)$ from [35] uses Cohen, Jones and Segal's notion of flow categories. To give a proof of Theorems 1 and 2 in this language seems tedious at best: it involves understanding the combinatorics of (broken) Morse flows on product manifolds, which turns out to be rather intricate.

Fortunately, as noted above, Hu, Kriz and Kriz gave another construction of a Khovanov stable homotopy type, and from their construction the behavior under connected sums and disjoint unions is clear. In this paper, we describe three additional constructions:

(1) A framing-free reformulation of the construction from [35] for a special family of flow categories, called cubical flow categories, in Section 3.

(2) A version similar to the construction from [22] as a homotopy colimit, but using the thickened cube instead of the Elmendorf–Mandell machinery, in Section 4.

(3) An intermediate object between the two in Section 5, using little k–cubes.

We then prove:

Theorem 3 *The Khovanov stable homotopy types constructed in Sections 3, 4 and 5, the Khovanov stable homotopy type constructed in* [22], *and the Khovanov stable homotopy type constructed in* [35] *are all stably homotopy equivalent.*

This theorem is proved in parts:

- Theorem 4 asserts that the cubical flow category realization (Section 3) agrees with the Cohen–Jones–Segal construction used in [35].
- Theorem 8 asserts that the cubical flow category realization (Section 3) agrees with the little k–cubes realization (Section 5).
- Theorem 7 asserts that the little k–cubes realization (Section 5) agrees with the thickening construction (Section 4).
- Theorem 9 asserts that the thickening construction (Section 4) agrees with the construction in [22].

Theorems 1 and 2 follow easily. Corollary 1.4 follows immediately from these theorems and computations in Lipshitz and Sarkar [37]. We also obtain, via a TQFT-style argument, that the Khovanov homotopy type of the mirror knot $m(K)$ is the Spanier–Whitehead dual to the Khovanov homotopy type of K (Theorem 11). Finally, Corollary 1.5 follows from these results, the refined s–invariant in Lipshitz and Sarkar [36], the computations in [37] and a brief further argument.

This paper is organized as follows. We collect some basic notation in Section 1.1. Section 2 has background on the cube category and the Khovanov construction, the latter of which is not needed again until Section 7 except in some examples. Other background appears at the beginning of the section in which it is first used. In Section 3, after recalling the basics of flow categories, we introduce a special class of them, cubical flow categories, which live over the cube, and give a reformulation of the Cohen–Jones–Segal realization for this class of flow categories ("cubical realization"). (The Khovanov flow category of [35] is a cubical flow category; this is a crucial tool in its construction.) In Section 4 we show that cubical flow categories are equivalent to 2–functors from the cube to the Burnside category, and give a different, choice-free way to realize such a functor. In Section 5 we give a smaller but less canonical way to realize a 2–functor from the cube to the Burnside category, and prove the two ways to realize such a functor are equivalent. Section 6 shows that the realization from Section 5 agrees with the cubical realization from Section 3. Section 7 is a brief interlude to summarize these results and recall the Khovanov homotopy type. Section 8 shows that these realizations agree with the Hu–Kriz–Kriz construction [22].

In Sections 9–11 we use these reformulated realizations to prove new properties of the Khovanov homotopy type. The realizations of the product (smash product) and disjoint union (wedge sum) of functors from the cube to the Burnside category have properties as one would expect; Section 9 uses these properties to study the Khovanov homotopy type of a disjoint union and connected sum, Theorems 1, 2 and 10, and verifies Corollary 1.4. Section 10 uses a TQFT-style argument suggested by the referee for [37] to deduce a formula for the Khovanov homotopy type of a mirror, verifying another conjecture from [35]. Finally, Section 11 gives an additivity property for the refined s–invariant introduced in [36] and obtains Corollary 1.5.

The appendix has a flow chart of how the different sections depend on each other, so a reader only interested in a particular result can choose the most efficient path to it.

Remark 1.7 It may be interesting to compare the homotopy colimit definition of the Khovanov homotopy type with Everitt and Turner [14]; but see also Everitt, Lipshitz, Sarkar and Turner [13].

1.1 Basic notation

The "cube" $\{0,1\}^n$ will appear in a number of contexts in this paper, as will some auxiliary notions related to it:

- There is a partial order on $\{0,1\}^n$ defined by $v \geq w$ if v is obtained from w by replacing some 0's by 1's. Define $v > w$ if $v \geq w$ and $v \neq w$, and define $\leq$ and $<$ in the corresponding ways. The maximum and minimum elements under this partial order are denoted by $\vec{1}$ and $\vec{0}$, respectively.
- We denote the Manhattan (or ℓ^1) norm on $\{0,1\}^n$ by $|v| = \sum_{i=1}^n v_i$.
- A *sign assignment* s on the cube consists of the following data: for every $u > v$ with $|u| - |v| = 1$, a choice of element $s_{u,v} \in \mathbb{F}_2$ such that for any $u > w$ with $|u| - |w| = 2$, we have

$$\sum_{\substack{v \\ u>v>w}} (s_{u,v} + s_{v,w}) = 1.$$

A number of categories will appear in this paper:

- The category Sets of finite sets and set maps.
- The Burnside (2–)category $\mathscr{B}$ of sets, correspondences and isomorphisms of correspondences (see Section 4.1).
- The cube category $\underline{2}^n = \{1 \to 0\}^n$ (see Section 2.1).
- The category $\mathsf{Top}_\bullet$ of (well-)based topological spaces.

- The subcategory $\mathsf{CW}_\bullet$ of $\mathsf{Top}_\bullet$ generated by based CW complexes and cellular maps.
- The category $\mathscr{S}$ of spectra. For concreteness, we can take the category of symmetric spectra in topological spaces; see Mandell, May, Schwede and Shipley [40].
- The category $R\text{–}\mathsf{Mod}$ of R–modules.
- The category Permu of permutative categories (see Section 8.4).
- The category $\mathsf{Cob}^{1+1}_{\mathrm{emb}}$ of oriented 1–manifolds embedded in S^2 and oriented cobordisms embedded in $[0,1]\times S^2$ (see Section 8.1).

Some other notation:

- Let $\mathbb{R}_+ = [0,\infty)$.
- Let $\mathbb{N} = \{0,1,2,\dots\}$ denote the set of nonnegative integers.

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2 Review of Khovanov homology

2.1 The cube category

Let $\underline{2}^1$ denote the category with two objects, denoted by 0 and 1, and a single non-identity morphism, from 1 to 0:

$$\underline{2}^1 = \{1 \to 0\}.$$

For $n \in \mathbb{Z}$ with $n > 1$ let $\underline{2}^n = \underline{2}^1 \times \underline{2}^{n-1}$. That is, $\underline{2}^n$ is the small category with object set $\{0,1\}^n$. Given objects $v, w \in \{0,1\}^n$ the morphism set $\mathrm{Hom}(v,w)$ is empty

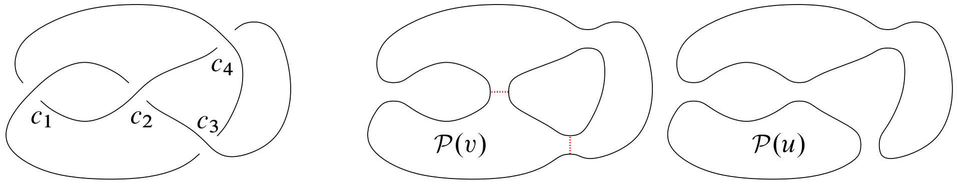


Figure 2: Some resolutions of a knot diagram. Left: A knot diagram for the figure-eight knot with crossings $c_1, \dots, c_4$. Right: The complete resolutions at the vertices $v = (1,0,0,0)$ and $u = (1,1,1,0)$, along with the embedded 1–handle cores representing the embedded cobordism between $\mathcal{P}(u)$ and $\mathcal{P}(v)$.

unless $v \geq w$ in the partial order induced by the relation $1 > 0$ on $\underline{2}^1$. If $v \geq w$, the set $\operatorname{Hom}(v, w)$ has a single element, which we will denote by $\varphi_{v,w}$.

The reader is warned that many authors refer to the opposite category $(\underline{2}^n)^{\mathrm{op}}$ as the cube category. (Our choice is made to agree with the grading conventions on Khovanov homology [26] and the Khovanov stable homotopy type [35].)

Often, we will view $\underline{2}^n$ as a 2–category with no nonidentity 2–morphisms. That is, for $f, g \in \operatorname{Hom}(v, w)$, we define $\operatorname{Hom}(f, g)$ to be empty unless $f = g$ and to have a single element if $f = g$.

2.2 The Khovanov construction

The material in the rest of this section is used in Examples 3.24 and 4.21 and Remark 4.22 and then is not used again until Section 7.

Khovanov homology, and several of its generalizations, are all constructed from the cube of resolutions of a link diagram. Let K be a link diagram in S^2 with n crossings, numbered $c_1, \dots, c_n$. Each of these crossings can be resolved locally in two different ways, called the 0–resolution and the 1–resolution (see for instance [26, Figure 14]). Therefore, to each $v \in \{0,1\}^n$ there is an associated complete resolution $\mathcal{P}(v)$ obtained by replacing the crossing c_i by its 0–resolution if $v_i = 0$ and its 1–resolution if $v_i = 1$. The complete resolution $\mathcal{P}(v)$ consists of a collection of disjoint circles in S^2.

For any $u \geq v \in \{0,1\}^n$, there is a cobordism embedded in $[0,1] \times S^2$ that connects the resolutions $\mathcal{P}(u)$ and $\mathcal{P}(v)$: to obtain $\mathcal{P}(u)$, one attaches embedded 1–handles to $\mathcal{P}(v)$ with cores certain arcs near the crossings c_i for all i with $u_i > v_i$. These arcs are illustrated in Figure 2. (See also [26, Figure 18]. This is also the first step in Bar-Natan's "picture world" approach to Khovanov homology in [4].) For the special case when $|u|-|v|=1$, the cobordism either *merges* two circles into one, or *splits* a single circle into two.

To construct the Khovanov complex, we apply a $(1+1)$–dimensional TQFT to this cube of cobordisms to obtain a commutative cube of abelian groups. Specifically, consider the rank-2 Frobenius algebras over $\mathbb{Z}[h,t]$ with basis $\{x_+, x_-\}$ and multiplication and comultiplication given by

$$x_+\otimes x_+ \xmapsto{m} x_+, \quad x_+\otimes x_- \xmapsto{m} x_-, \quad x_-\otimes x_+ \xmapsto{m} x_-, \quad x_-\otimes x_- \xmapsto{m} hx_- + tx_+,$$
$$x_+ \xmapsto{S} x_+\otimes x_- + x_-\otimes x_+ - hx_+\otimes x_+, \quad x_- \xmapsto{S} x_-\otimes x_- + tx_+\otimes x_+,$$

and the corresponding $(1+1)$–dimensional TQFT. Applying this TQFT to the cube of resolutions of K gives a commutative cube $A\colon (\underline{2}^n)^{\mathrm{op}} \to \mathbb{Z}[h,t]\text{–}\mathsf{Mod}$. Explicitly, given a vertex v, a *Khovanov generator over* v is a labeling of the circles in $\mathcal{P}(v)$ by elements of the set $\{x_+, x_-\}$. The module $A(v)$ is freely generated by the set of Khovanov generators $F(v)$ over v. For an edge of the cube $\varphi_{u,v}$ (where $u>v$ and $|u|-|v|=1$), $A(\varphi_{u,v})\colon A(v)\to A(u)$ is the multiplication or comultiplication above, depending on whether the cobordism from $\mathcal{P}(v)$ to $\mathcal{P}(u)$ is a merge or split, respectively.

Definition 2.1 Fix a sign assignment s on the cube (in the sense of Section 1.1). The chain complex $\mathcal{C}(K)$ is the totalization of A with respect to s. That is, the chain group is defined to be $\mathcal{C}(K)=\bigoplus_{v\in\{0,1\}^n} A(v)$ and the differential $\delta\colon \mathcal{C}(K)\to\mathcal{C}(K)$ is defined by stipulating the component $\delta_{u,v}$ of δ that maps from $A(v)$ to $A(u)$ to be

$$\delta_{u,v}=\begin{cases}(-1)^{s_{u,v}}A(\varphi^{\mathrm{op}}_{u,v}) & \text{if } u>v \text{ and } |u|-|v|=1,\\ 0 & \text{otherwise.}\end{cases}$$

The *Khovanov chain complex* $(\mathcal{C}_{Kh}(K),\delta_{Kh})$ is the specialization $h=t=0$. The homology of $\mathcal{C}_{Kh}(K)$ is the *Khovanov homology* $Kh(K)$.

(The Khovanov complex was introduced by Khovanov [26]. The specializations $(h,t)=(0,1)$ and $(h,t)=(1,0)$ were studied by Lee [31] and Bar-Natan [4], respectively; the case of general (h,t) was studied by Khovanov [29], Naot [43] and others.)

The homological grading of the summand $A(v)\subseteq\mathcal{C}(K)$ is $|v|-n_-$, where n_- denotes the number of negative crossings in the link diagram. There is additionally an internal grading, called the *quantum grading*, that persists throughout; the quantum grading of any Khovanov generator in $F(v)$ is

$$n-3n_-+|v|+\#\{\text{circles in } \mathcal{P}(v) \text{ labeled } x_+\}-\#\{\text{circles in } \mathcal{P}(v) \text{ labeled } x_-\}.$$

The quantum gradings of the formal variables (h,t) are $(-2,-4)$.

In the presence of a basepoint on the link diagram, and after setting $t=0$, there is also a reduced theory. In the reduced theory, for any $v\in\{0,1\}^n$, we only consider the Khovanov generators in $F(v)$ that label the pointed circle in $\mathcal{P}(v)$ as x_-, and shift quantum gradings up by 1. The reduced Khovanov chain complex and the reduced Khovanov homology are denoted by $\widetilde{\mathcal{C}}_{Kh}$ and $\widetilde{Kh}$, respectively. (Alternatively, we could have defined a reduced theory by setting $t=0$ and only considering the Khovanov generators that label the pointed circle as x_+, and shifting quantum gradings down by 1. When $h=t=0$, these two reduced theories are canonically isomorphic; see also [58].)

The Khovanov homology is an invariant of the link, not just of the link diagram, and the reduced Khovanov homology is an invariant of the pointed link. More generally, the chain homotopy type of the chain complex $\mathcal{C}(K)$ over $\mathbb{Z}[h,t]$ is a link invariant as well.

3 Cubical flow categories

The Khovanov stable homotopy type is defined using an auxiliary object, the Khovanov flow category. We review flow categories in Section 3.2, after first recalling some notions related to manifolds with corners, on which flow categories are based. The Khovanov flow category from [35] is defined as a kind of cover of another flow category, the cube flow category (Definition 3.16), which in turn is based on permutohedra, a family of polytopes reviewed in Section 3.3. After reviewing the cube flow category and some of its basic properties in Section 3.4, new material begins in Section 3.5, where we abstract the notion of a cover of the cube flow category into a cubical flow category (Definition 3.21). In Section 3.6 we give a slightly different notion of neat embeddings for cubical flow categories. Using this notion, in Section 3.7 we give a realization procedure for a cubical flow category, the cubical realization. In Section 3.8 we show that the cubical realization and the Cohen–Jones–Segal realization (reviewed in Section 3.2) give homotopy equivalent spaces.

3.1 Manifolds with corners and $\langle n\rangle$–manifolds

The original construction of the Khovanov stable homotopy type from [35] relies on Cohen, Jones and Segal's notion of flow categories [9], which in turn uses a particular notion of manifolds with corners, called $\langle n\rangle$–manifolds [25; 30]. For the reader's convenience, we review the relevant definitions here.

Definition 3.1 A *k–dimensional manifold with corners* is a topological space X together with an atlas $\{(U_\alpha, \phi_\alpha\colon U_\alpha \to (\mathbb{R}_+)^k)\}$ modeled on open subsets of $(\mathbb{R}_+)^k$ such that the transition functions are smooth. Given a point x in a chart (U,ϕ) let $c(x)$ be the number of coordinates in $\phi(x)$ which are 0; $c(x)$ is independent of the choice of chart. The *codimension-i boundary* of X is $\{x \in X \mid c(x) = i\}$. A k–dimensional manifold with corners X has a well-defined tangent space TX, which is an $\mathbb{R}^k$–plane bundle; a *Riemannian metric* on X means a Riemannian metric on TX.

Definition 3.2 A *facet* of X is the closure of a connected component of the codimension-1 boundary of X. (If X is a polytope then this agrees with the usual definition of facets.) A *multifacet* of X is a (possibly empty) union of disjoint facets of X. A manifold with corners X is a *multifaceted manifold* if every $x \in X$ belongs to exactly $c(x)$ facets of X. An *$\langle n\rangle$–manifold* is a multifaceted manifold X along with an ordered n–tuple $(\partial_1 X, \dots, \partial_n X)$ of multifacets of X such that $\bigcup_i \partial_i X = \partial X$ and, for all distinct i and j, $\partial_i X \cap \partial_j X$ is a multifacet of both $\partial_i X$ and $\partial_j X$. (The number n need not be the dimension of X.) See Laures [30] for more details. (Laures uses the terms "connected face", "face" and "manifold with faces" for "facet", "multifacet" and "multifaceted manifold", respectively. We have changed the terminology since "face" means something different for polytopes in Section 3.3.) Given a $\langle n\rangle$–manifold X and a vector $v \in \{0,1\}^n$ let $X(v) = \bigcap_{v_i=0} \partial_i X$, with the convention that $X(\vec{1}) = X$.

Example 3.3 An n–gon (polygon with n sides) is a multifaceted manifold if $n > 1$, while a 1–gon (disk with one corner on the boundary) is a manifold with corners but not a multifaceted manifold. Only the $2n$–gons can be made into $\langle 2\rangle$–manifolds, though $(2n{+}1)$–gons can be viewed as 2–dimensional $\langle 3\rangle$–manifolds. Of the Platonic solids only the tetrahedron, cube and dodecahedron are manifolds with corners, and all three are multifaceted manifolds. The cube can be made into a $\langle 3\rangle$–manifold by defining $\partial_1 X$ to be the front and back facets, $\partial_2 X$ to be the top and bottom facets, and $\partial_3 X$ to be the left and right facets. The tetrahedron and dodecahedron cannot be given the structure of $\langle 3\rangle$–manifolds, although both can be made into $\langle 4\rangle$–manifolds. An even more fundamental example is $(\mathbb{R}_+)^n$ itself, which is a $\langle n\rangle$–manifold by setting $\partial_i(\mathbb{R}_+)^n = \{v \in \mathbb{R}^n_+ \mid v_i = 0\}$. Similarly, $\mathbb{R}^N \times (\mathbb{R}^n_+)$ is an $(n{+}N)$–dimensional $\langle n\rangle$–manifold.

Construction 3.4 Given an $\langle n\rangle$–manifold X and an $\langle m\rangle$–manifold Y, the product $X\times Y$ inherits the structure of a $\langle n{+}m\rangle$–manifold, by declaring

$$\partial_i(X\times Y)=\begin{cases}(\partial_i X)\times Y & \text{if } 1\le i\le n,\\ X\times(\partial_{i-n}Y) & \text{if } n+1\le i\le n+m.\end{cases}$$

We end this subsection with the definition of neat embeddings.

Definition 3.5 Consider $\langle n\rangle$–manifolds X and Y and fix a Riemannian metric on Y. A *neat embedding* of X into Y is a smooth map $f\colon X\to Y$ such that:

- $f^{-1}(Y(v))=X(v)$ for all $v\in\{0,1\}^n$.
- $f|_{X(v)}\colon X(v)\to Y(v)$ is an embedding for each $v\in\{0,1\}^n$.
- For all $w<v\in\{0,1\}^n$, $f(X(v))$ is perpendicular to $Y(w)$ with respect to the Riemannian metric on Y, and in particular is transverse to $Y(w)$.

3.2 Flow categories

Next we recall some notions about flow categories, from Cohen, Jones and Segal [9] (see also [35, Section 3]), partly to fix terminology for this paper.

Definition 3.6 A *flow category* $\mathscr{C}$ is a topological category whose objects $\mathrm{Ob}(\mathscr{C})$ form a discrete space, equipped with a grading function $\mathrm{gr}\colon \mathrm{Ob}(\mathscr{C})\to\mathbb{Z}$, and whose morphism spaces satisfy the following conditions:

(FC-1) For any $x\in\mathrm{Ob}(\mathscr{C})$, $\mathrm{Hom}(x,x)=\{\mathrm{Id}\}$.

(FC-2) For distinct $x,y\in\mathrm{Ob}(\mathscr{C})$ with $\mathrm{gr}(x)-\mathrm{gr}(y)=k$, $\mathrm{Hom}(x,y)$ is a (possibly empty) compact $(k{-}1)$–dimensional $\langle k{-}1\rangle$–manifold; and

(FC-3) The composition maps combine to produce a diffeomorphism of $\langle k{-}2\rangle$–manifolds

$$\coprod_{\substack{z\in\mathrm{Ob}(\mathscr{C})\setminus\{x,y\}\\ \mathrm{gr}(z)-\mathrm{gr}(y)=i}}\mathrm{Hom}(z,y)\times\mathrm{Hom}(x,z)\cong\partial_i\,\mathrm{Hom}(x,y).$$

(In [9], it is not required that the space of objects be discrete. In Morse theory, this corresponds to allowing Morse–Bott functions.)

The identity morphisms in a flow category are somewhat special, and it is often convenient to ignore them. So, for objects x and y in $\mathscr{C}$, the *moduli space from x*

to y, $\mathcal{M}(x,y)$, is defined to be $\mathrm{Hom}(x,y)$ if $x \neq y$, and empty if $x = y$. (In Morse theory, this corresponds to the moduli space of nonconstant downwards gradient flows from x to y.)

For any flow category $\mathscr{C}$, let $\Sigma^k\mathscr{C}$ denote the flow category obtained by increasing the gradings of each object by k.

Definition 3.7 For each integer i, fix an integer $D_i \geq 0$, and let $\boldsymbol{D}$ denote this sequence. A *neat embedding* of a flow category $\mathscr{C}$ relative to $\boldsymbol{D}$ is a collection $\jmath_{x,y}$ of neat embeddings (with the standard Riemannian metric on the target space)

$$\jmath_{x,y}\colon \mathcal{M}(x,y) \hookrightarrow \mathbb{R}^{D_{\mathrm{gr}(y)}} \times \mathbb{R}_+ \times \mathbb{R}^{D_{\mathrm{gr}(y)+1}} \times \mathbb{R}_+ \times \cdots \times \mathbb{R}_+ \times \mathbb{R}^{D_{\mathrm{gr}(x)-1}}$$

of $\langle \mathrm{gr}(x) - \mathrm{gr}(y) - 1\rangle$–manifolds for all $x, y \in \mathrm{Ob}(\mathscr{C})$, subject to the following:

(1) For all integers i and j,

$$\coprod_{\substack{x,y\\ \mathrm{gr}(x)=i\\ \mathrm{gr}(y)=j}} \jmath_{x,y}\colon \coprod_{\substack{x,y\\ \mathrm{gr}(x)=i\\ \mathrm{gr}(y)=j}} \mathcal{M}(x,y) \to \mathbb{R}^{D_j} \times \mathbb{R}_+ \times \cdots \times \mathbb{R}_+ \times \mathbb{R}^{D_{i-1}}$$

is a neat embedding of $\langle i - j - 1\rangle$–manifolds.

(2) For all $x, y, z \in \mathrm{Ob}(\mathscr{C})$ and all points $(q, p) \in \mathcal{M}(y,z) \times \mathcal{M}(x,y)$,

$$\jmath_{x,z}(q \circ p) = (\jmath_{y,z}(q), 0, \jmath_{x,y}(p)).$$

A *coherent framing* for a neat embedding $\jmath$ is a collection of framings of the normal bundles $\nu_{\jmath_{x,y}}$ of $\jmath_{x,y}$ for all $x, y \in \mathrm{Ob}(\mathscr{C})$ such that for all $x, y, z \in \mathrm{Ob}(\mathscr{C})$, the product framing of $\nu_{\jmath_{y,z}} \times \nu_{\jmath_{x,y}}$ equals the pullback framing of $\circ^*\nu_{\jmath_{x,z}}$, where $\circ$ denotes composition.

Definition 3.8 [35, Definition 3.21] A *framed flow category* is a flow category $\mathscr{C}$, along with a coherent framing for some neat embedding of $\mathscr{C}$ (relative to some $\boldsymbol{D}$).

For a framed flow category, there is an *associated cochain complex* $C^*(\mathscr{C})$, defined as follows. The n^{th} chain group C^n is the $\mathbb{Z}$–module freely generated by the objects of $\mathscr{C}$ of grading n. The differential δ is of degree 1. For $x, y \in \mathrm{Ob}(\mathscr{C})$ with $\mathrm{gr}(x) - \mathrm{gr}(y) = 1$, the coefficient $\langle \delta y, x\rangle$ of x in $\delta(y)$ is the number of points in $\mathcal{M}(x,y)$, counted with sign. We say a framed flow category *refines* its associated chain complex.

Remark 3.9 In order to define the associated chain complex, one only needs the framing of the 0–dimensional moduli spaces; and in order to check that $\delta^2 = 0$, one only needs to ensure that the framing extends to the 1–dimensional moduli spaces.

To a framed flow category, Cohen, Jones and Segal associate a based CW complex $|\mathscr{C}|$ whose cells (except the basepoint) correspond to the objects of the flow category [9]. The following formulation of the Cohen–Jones–Segal construction is described in more detail in [35, Definition 3.24].

Definition 3.10 Let $\mathscr{C}$ be a framed flow category with a neat embedding $\jmath$ relative to some $\boldsymbol{D}$, and assume all objects of $\mathscr{C}$ have grading in $[B, A]$ for some fixed $A, B \in \mathbb{Z}$. Using the framing of $\nu_{\jmath_{x,y}}$, extend $\jmath_{x,y}$ to a map

$$\bar{\jmath}_{x,y}\colon \mathcal{M}(x,y)\times[-\delta,\delta]^{D_{\mathrm{gr}(y)}+\cdots+D_{\mathrm{gr}(x)-1}} \to \mathbb{R}^{D_{\mathrm{gr}(y)}}\times\mathbb{R}_+\times\cdots\times\mathbb{R}_+\times\mathbb{R}^{D_{\mathrm{gr}(x)-1}}.$$

Choose δ small enough and T large enough that the map $\coprod_{x,y\,|\,\mathrm{gr}(x)=i,\,\mathrm{gr}(y)=j}\bar{\jmath}_{x,y}$ is an embedding into $(-T,T)^{D_j}\times[0,T)\times\cdots\times[0,T)\times(-T,T)^{D_{i-1}}$ for all integers i and j. In the based CW complex $|\mathscr{C}|$, the cell associated to an object x of grading m is

$$\mathcal{C}(x)=\prod_{i=B}^{m-1}([0,T]\times[-T,T]^{D_i})\times[-\delta,\delta]^{D_m+\cdots+D_{A-1}}.$$

For any other object y with $\mathrm{gr}(y)=n<m$, the embedding $\bar{\jmath}_{x,y}$ identifies the product $\mathcal{C}(y)\times\mathcal{M}(x,y)$ with the subset of $\partial\mathcal{C}(x)$

$$\mathcal{C}_y(x)=\prod_{i=B}^{n-1}([0,T]\times[-T,T]^{D_i})\times\{0\}\times\mathrm{im}(\bar{\jmath}_{x,y})\times[-\delta,\delta]^{D_m+\cdots+D_{A-1}}.$$

The attaching map for $\mathcal{C}(x)$ sends $\mathcal{C}_y(x)\cong\mathcal{C}(y)\times\mathcal{M}(x,y)$ via the projection map to $\mathcal{C}(y)$, and sends $\partial\mathcal{C}(x)\setminus\bigcup_y\mathcal{C}_y(x)$ to the basepoint.

Lemma 3.11 [35, Lemma 3.25] *Definition 3.10 defines a CW complex, whose reduced cellular cochain complex is isomorphic (after shifting the gradings down by $D_B+\cdots+D_{A-1}-B$) to the chain complex associated to the framed flow category $\mathscr{C}$ from Definition 3.8. The isomorphism sends cells of $|\mathscr{C}|$ to the corresponding objects of $\mathscr{C}$.*

Definition 3.12 The *Cohen–Jones–Segal realization of $\mathscr{C}$* is the formal (de)suspension $\Sigma^{B-D_B-\cdots-D_{A-1}}|\mathscr{C}|$.

3.3 Permutohedra

We will be interested in a particular family of flow categories, in which the moduli spaces are unions of permutohedra. So, we recall some basic facts about permutohedra.

Before starting, let us fix some notation about polytopes, mostly following Ziegler [59]. Let $P \subset \mathbb{R}^n$ be a polytope or an $\mathcal{H}$–polyhedron (as described in [59, Definition 0.1]). If there is an affine half-space of $\mathbb{R}^n$ which contains P, then the intersection of its boundary with P is called a *face* of P; and if $\dim(P) = d$, then we declare the entire polytope P to be its unique d–dimensional face. This gives a CW complex structure on P, with the cells being the nonempty faces. The faces of dimension 0, 1 and $d-1$ are called vertices, edges and facets, respectively. A d–dimensional polytope is called *simple* if every vertex is contained in exactly d facets; simple polytopes are multifaceted manifolds.

For $\sigma \in \mathcal{S}_n$ a permutation, let $v_\sigma = (\sigma^{-1}(1), \dots, \sigma^{-1}(n)) \in \mathbb{R}^n$. The *$(n-1)$–dimensional permutohedron* Π_{n-1} is the convex hull in $\mathbb{R}^n$ of the $n!$ points v_σ [59, Example 0.10]. The permutohedron Π_{n-1} lies in the affine subspace $\mathbb{A}^{n-1} := \bigl\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_i x_i = \frac{1}{2}n(n+1)\bigr\}$ of $\mathbb{R}^n$. As its name suggests, Π_{n-1} is $(n-1)$–dimensional, and the v_σ are its vertices.

For each nonempty, proper subset S of $\{1, \dots, n\}$ of cardinality say k, let $H_S \subset \mathbb{A}^{n-1} \subset \mathbb{R}^n$ be the half-space $\bigl\{(x_1, \dots, x_n) \in \mathbb{A}^{n-1} \mid \sum_{i \in S} x_i \geq \frac{1}{2}k(k+1)\bigr\}$. The permutohedron Π_{n-1} can also be defined as the intersection of the $2^n - 2$ half-spaces H_S. In fact, the facets of Π_{n-1} are exactly the $F_S := \Pi_{n-1} \cap \partial H_S$.

The facets F_S are identified with products of lower-dimensional permutohedra:

Lemma 3.13 *Let $a_1 < a_2 < \dots < a_k$ be the elements in S, and let $b_1 < b_2 < \dots < b_{n-k}$ be the elements in $\{1, 2, \dots, n\} \setminus S$. Then the map $f_S \colon \mathbb{R}^n \to \mathbb{R}^k \times \mathbb{R}^{n-k}$,*

$$f_S(x_1, \dots, x_n) = ((x_{a_1}, \dots, x_{a_k}), (x_{b_1} - k, \dots, x_{b_{n-k}} - k)),$$

identifies the facet $F_S \subset \mathbb{R}^n$ with $\Pi_{k-1} \times \Pi_{n-1-k} \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$.

Proof It suffices to show that f_S takes the vertices of $F_S \subset \Pi_{n-1}$ to the vertices of $\Pi_{k-1} \times \Pi_{n-1-k}$. The vertices of F_S are the points $(x_1, \dots, x_n)$ such that $\{x_{a_1}, \dots, x_{a_k}\} = \{1, \dots, k\}$ and $\{x_{b_1}, \dots, x_{b_{n-k}}\} = \{k+1, \dots, n\}$. It is immediate that f_S takes these vertices bijectively to the vertices of $\Pi_{k-1} \times \Pi_{n-1-k}$. □

The permutohedron Π_{n-1} is simple, ie each vertex lies in exactly $n-1$ facets: v_σ lies in the facet $F_{\{\sigma(1), \dots, \sigma(k)\}}$ for each $1 \leq k < n$, and no others. Therefore, every d–dimensional face belongs to exactly $n-1-d$ facets, and the subsets corresponding to those facets are nested. Hence, d–dimensional faces correspond to sequences of $n-1-d$ nested proper, nonempty subsets of $\{1, \dots, n\}$. Further:

Lemma 3.14 *The space Π_{n-1} can be treated as an $\langle n-1\rangle$–manifold by declaring*

$$\partial_i \Pi_{n-1} = \bigcup_{S:|S|=i} F_S \quad \text{for } 1 \le i \le n-1.$$

Proof We must check:

(1) Every point x belongs to $c(x)$ facets.

(2) Each $\partial_i \Pi_{n-1}$ is a multifacet, that is, a disjoint union of facets.

(3) $\bigcup_i \partial_i \Pi_{n-1} = \partial \Pi_{n-1}$.

(4) For each $i \neq j$, $\partial_i \Pi_{n-1} \cap \partial_j \Pi_{n-1}$ is a multifacet of $\partial_i \Pi_{n-1}$ (and $\partial_j \Pi_{n-1}$).

Point (1) follows from the fact that Π_{n-1} is a simple polyhedron.

For point (2), we claim that if $|S| = |T| = i$ and $S \neq T$ then $F_S \cap F_T = \varnothing$; it follows that $\partial_i \Pi_{n-1}$ is the disjoint union of the facets F_S (with $|S| = i$). But if v_σ is a vertex in $F_S \cap F_T$, then

$$\begin{aligned}\sum_{j\in S} \sigma^{-1}(j) = \sum_{j\in T} \sigma^{-1}(j) = \tfrac{1}{2} i(i+1)& \\ &\iff \{\sigma^{-1}(j) \mid j \in S\} = \{\sigma^{-1}(j) \mid j \in T\} = \{1, \dots, i\} \\ &\iff S = T = \{\sigma(1), \dots, \sigma(i)\}.\end{aligned}$$

Point (3) is immediate from the definitions.

For point (4), suppose that $|S| = i$. After identifying F_S with $\Pi_{i-1} \times \Pi_{n-i-1}$ using Lemma 3.13, we get

$$F_S \cap \partial_j \Pi_{n-1} = \begin{cases} \Pi_{i-1} \times (\partial_{j-i} \Pi_{n-i-1}) & \text{if } i < j, \\ (\partial_j \Pi_{i-1}) \times \Pi_{n-i-1} & \text{if } i > j. \end{cases}$$

Therefore, $F_S \cap \partial_j \Pi_{n-1}$ is a disjoint union of facets of $F_S \cong \Pi_{i-1} \times \Pi_{n-i-1}$. Since the F_S for $|S| = i$ are disjoint, $\partial_i \Pi_{n-1} \cap \partial_j \Pi_{n-1}$ is a disjoint union of facets of $\partial_i \Pi_{n-1}$ as well. □

We will also use the following well-known cubical complex structure on Π_{n-1}. For any permutation $\sigma \in \mathcal{S}_n$, let C_σ be the convex hull of the barycenters of all the faces that contain the vertex v_σ.

Lemma 3.15 *Each C_σ is combinatorially equivalent to an $(n-1)$–dimensional cube, and these cubes form a cubical complex subdivision of Π_{n-1}.*

Proof We present the proof from Ovchinnikov [44, Section 3], for which he cites Ziegler. Consider the intersection of half-spaces

$$FC_\sigma = \bigcap_{S \mid v_\sigma \in F_S} H_S.$$

The space FC_σ is an $(n-1)$–dimensional cone with cone point v_σ, with $n-1$ facets (corresponding to the facets of Π_{n-1} that contain v_σ), and therefore is a simplicial cone. (For comparison with [44, Section 3], FC_σ is the dual of the facet cone of the dual of v_σ (in the face fan of the dual of Π_{n-1}).)

Next, consider the vertex cone of v_σ (in the normal fan of Π_{n-1}), VC_σ. By definition, the cone point of VC_σ is the barycenter of Π_{n-1}, and the edges of VC_σ are obtained by dropping perpendiculars from the barycenter to the facets of Π_{n-1} that contain v_σ. The cone VC_σ is an $(n-1)$–dimensional cone with $n-1$ edges, and, therefore, VC_σ is a simplicial cone. The d–dimensional faces of VC_σ correspond to the $(n-1-d)$–dimensional faces of Π_{n-1} that contain v_σ. Given corresponding faces f_{VC} of VC_σ and f of Π_{n-1}, f_{VC} is perpendicular to f and passes through the barycenter of f.

Therefore, C_σ is the intersection of the two simplicial cones FC_σ and VC_σ. Since the edges of VC_σ pass through the interiors of the facets of FC_σ, VC_σ and FC_σ intersect transversely, and, therefore, $VC_\sigma \cap FC_\sigma$ is combinatorially equivalent to a cube.

The facets of C_σ are of two types: the ones contained in FC_σ, which are not identified with the facets of any other cube and lie in the boundary of Π_{n-1}; and the ones contained in VC_σ, which are identified with facets of other cubes and lie in the interior of Π_{n-1}. Indeed the facets of the latter type correspond to the edges e of Π_{n-1} that contain v_σ: the facet corresponding to e is formed by taking the convex hull of the barycenters of all the faces of Π_{n-1} that contain e. With these identifications, it is clear that these cubes C_σ come together to form a cubical subdivision of the permutohedron. □

3.4 The cube flow category

We start by recalling the cube flow category from [35, Definition 4.1]. There we gave a definition in terms of Morse flows on $[0,1]^n$. Here, we give a more directly combinatorial definition in terms of permutohedra (see also Remark 3.18):

Definition 3.16 The objects of the cube flow category $\mathscr{C}_C(n)$ are the same as the objects of the cube category $\underline{2}^n$, ie tuples $u=(u_1,\dots,u_n)\in\{0,1\}^n$. The grading on the objects is defined by $\mathrm{gr}(u)=|u|=\sum_i u_i$.

The space $\mathcal{M}(u,v)$ is defined to be empty unless $u>v$. If $u>v$ and $\mathrm{gr}(u)-\mathrm{gr}(v)=k>0$ then we define $\mathcal{M}(u,v)=\Pi_{k-1}$, the $(k-1)$–dimensional permutohedron. Note that, by Lemma 3.14, $\mathcal{M}(u,v)$ is a $\langle k-1\rangle$–manifold.

The composition map $\mathcal{M}(v,w)\times\mathcal{M}(u,v)\to\mathcal{M}(u,w)$ is defined as follows. Assume that $u>v>w$ with $\mathrm{gr}(u)-\mathrm{gr}(v)=k$ and $\mathrm{gr}(v)-\mathrm{gr}(w)=l$. Suppose that $u_{a_1}=\cdots=u_{a_{k+l}}=1$ and $w_{a_1}=\cdots=w_{a_{k+l}}=0$ (ie the a_i are the coordinates in which u and w differ), where $a_1<a_2<\cdots<a_{k+l}$. Let S be the subset of $\{1,\dots,k+l\}$ such that $v_{a_s}=1$ for $s\in S$. (The set S has cardinality l.) By Lemma 3.13, there is a corresponding facet $F_S\subset\Pi_{k+l-1}$, and F_S is identified with $\Pi_{l-1}\times\Pi_{k-1}=\mathcal{M}(v,w)\times\mathcal{M}(u,v)$. The composition map is the corresponding inclusion map $\mathcal{M}(v,w)\times\mathcal{M}(u,v)=F_S\hookrightarrow\mathcal{M}(u,w)$.

Lemma 3.17 *Definition 3.16 defines a flow category.*

Proof Conditions (FC-1) and (FC-2) of Definition 3.6 are immediate from the definitions and Lemma 3.14. For condition (FC-3), it is enough to recall from Lemma 3.14 that $\partial_l\Pi_{k+l-1}=\bigcup_{|S|=l}F_S$.

Finally, we need to check that this defines a category, or, in other words, that composition is associative. Towards this end, for any $u>v$ with $\mathrm{gr}(u)-\mathrm{gr}(v)=k>0$, it is convenient to treat $\mathcal{M}(u,v)=\Pi_{k-1}$ as a subset of $\prod_{i|u_i>v_i}\mathbb{R}$ instead of $\mathbb{R}^k$, where the two ambient spaces are identified by linearly ordering the coordinates $a_1<a_2<\cdots<a_k$ in which u and v differ. With this viewpoint, for $u>v>w$ with $\mathrm{gr}(u)-\mathrm{gr}(v)=k$ and $\mathrm{gr}(v)-\mathrm{gr}(w)=l$, the composition map $\Pi_{l-1}\times\Pi_{k-1}\to\Pi_{k+l-1}$ is induced from the map

$$\prod_{i|v_i>w_i}\mathbb{R}\times\prod_{i|u_i>v_i}\mathbb{R}\xrightarrow{+(\vec{0},\vec{l})}\prod_{i|v_i>w_i}\mathbb{R}\times\prod_{i|u_i>v_i}\mathbb{R}\xrightarrow{\cong}\prod_{i|u_i>w_i}\mathbb{R},$$

where the first map adds l to each of the coordinates of $\prod_{i|u_i>v_i}\mathbb{R}$ and the second map is coordinatewise identification. (See also Lemma 3.13.) Now, given $u>v>w>x$ with $\mathrm{gr}(u)-\mathrm{gr}(v)=k$, $\mathrm{gr}(v)-\mathrm{gr}(w)=l$ and $\mathrm{gr}(w)-\mathrm{gr}(x)=m$, the corresponding double compositions are

$$\begin{tikzcd}
\prod_{i|w_i>x_i}\mathbb{R}\times\prod_{i|v_i>w_i}\mathbb{R}\times\prod_{i|u_i>v_i}\mathbb{R} \arrow[r, "+(\vec{0},\vec{0},\vec{l})"] \arrow[d, "+(\vec{0},\vec{m},\vec{0})"] & \prod_{i|w_i>x_i}\mathbb{R}\times\prod_{i|u_i>w_i}\mathbb{R} \arrow[d, "+(\vec{0},\vec{m})"] \\
\prod_{i|v_i>x_i}\mathbb{R}\times\prod_{i|u_i>v_i}\mathbb{R} \arrow[r, "+(\vec{0},\vec{l}+\vec{m})"] & \prod_{i|u_i>x_i}\mathbb{R}
\end{tikzcd}$$

where we have suppressed the reshuffling of factors from the notation. So, both compositions are given by $+(\vec{0},\vec{m},\vec{l}+\vec{m})$ (and the same reshuffling of factors). □

Remark 3.18 We have not shown that the definition of the cube flow category from Definition 3.16 agrees with the Morse-theoretic definition from [35, Definition 4.1], as doing so would seem to require a nontrivial digression about the smooth structures on moduli spaces of Morse flows. For the purposes of this paper (and future work), note that the combinatorial definition used here works just as well for the construction of the Khovanov homotopy type in [35], and all the results stated in [35; 37; 36] remain true with this combinatorial definition. To be more precise, the only statements that involve the moduli spaces on the cube flow category coming from the Morse flows are [35, Lemmas 4.2–4.3], which are immediate for the combinatorial definition. Therefore, when we talk about the Khovanov homotopy type in this paper (and future work), we mean the homotopy type defined using the cube flow category from Definition 3.16.

Before moving on to the definition of cubical flow categories in Section 3.5, we digress a little to study the cubical complex structure from Lemma 3.15 on the permutohedron $\mathcal{M}(u,v)$.

Definition 3.19 For $u > v$ in $\{0,1\}^n$, define the space

$$M_{u,v} = \Bigl(\coprod_{\substack{m \in \{1,\dots,|u|-|v|\} \\ u=u^0>\dots>u^m=v}} [0,1]^{m-1} \Bigr) \Big/ {\sim},$$

where, for each chain $u = u^0 > \dots > u^m = v$, each $1 \le i \le m-1$ and each point $(t_1,\dots,t_{m-2})$ in the cube $[0,1]^{m-2}$ corresponding to the chain $u^0 > \dots > u^{i-1} > u^{i+1} > \dots > u^m$, the equivalence relation $\sim$ identifies

$$(t_1,\dots,t_{m-2})_{u=u^0>\dots>u^{i-1}>u^{i+1}>\dots>u^m=v} \sim (t_1,\dots,t_{i-1},1,t_i,\dots,t_{m-2})_{u=u^0>\dots>u^m=v}.$$

For $u > v > w$ in $\{0,1\}^n$, define a map $M_{v,w} \times M_{u,v} \to M_{u,w}$ by

$$((t_1,\dots,t_{\ell-1})_{v=v^0>\dots>v^\ell=w}, (s_1,\dots,s_{m-1})_{u=u^0>\dots>u^m=v}) \mapsto (s_1,\dots,s_{m-1},0,t_1,\dots,t_{\ell-1})_{u=u^0>\dots>u^m=v=v^0>\dots,v^\ell=w}.$$

Lemma 3.20 *There are homeomorphisms $h_{u,v}\colon M_{u,v} \xrightarrow{\cong} \mathcal{M}(u,v)$ from the spaces of Definition 3.19 to the permutohedra $\mathcal{M}(u,v)$ such that:*

(1) *The $h_{u,v}$ identify the cubes in the definition of $M_{u,v}$ with the cubes in the cubical complex structure on $\mathcal{M}(u,v)$ from Lemma 3.15 which are not entirely contained in $\partial\mathcal{M}(u,v)$.*

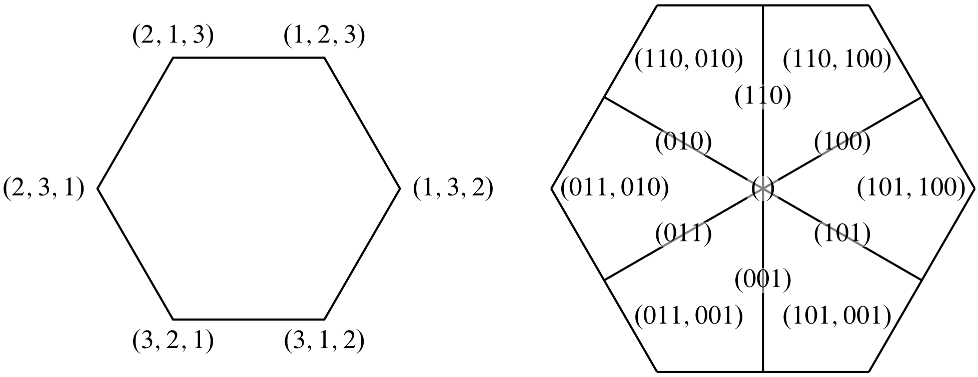


Figure 3: A cubical decomposition of the permutohedron. Left: The 2–dimensional permutohedron $\mathcal{M}(111, 000)$ with vertices labeled by the corresponding permutations of $(1, 2, 3)$. Right: The corresponding cubical complex $M_{111,000}$, where we have labeled the cube corresponding to a sequence $111 = u^0 > \cdots > u^m = 000$ by $(u^1, \dots, u^{m-1})$.

(2) *The* $h_{u,v}$ *identify the points in the cubes for* $M_{u,v}$ *where some coordinate is* 0 *with the points in* $\partial\mathcal{M}(u, v)$.

(3) *For any* $u > v > w$, *the following diagram commutes*:

$$\begin{tikzcd}
M_{v,w} \times M_{u,v} \arrow[r, "\cong", "h_{v,w} \times h_{u,v}"'] \arrow[d] & \mathcal{M}(v, w) \times \mathcal{M}(u, v) \arrow[d] \\
M_{u,w} \arrow[r, "\cong", "h_{u,w}"'] & \mathcal{M}(u, w)
\end{tikzcd}$$

Here, the left vertical arrow is the map from Definition 3.19, and the right vertical arrow is the composition in $\mathscr{C}_C(n)$.

(See Figure 3.)

Proof The chain $c = \{u = u^0 > \cdots > u^m = v\}$ in $\{0, 1\}^n$ corresponds to the $(|u|-|v|-m)$–dimensional face $F_c = \mathcal{M}(u^{m-1}, u^m) \times \cdots \times \mathcal{M}(u^0, u^1)$ of the permutohedron $\mathcal{M}(u, v)$. If we take the convex hull of the barycenters of all the faces of the permutohedron that contain F_c, we get an $(m-1)$–dimensional cube C_c which appears in the cubical complex structure from Lemma 3.15. (If F_c is a vertex of the permutohedron, or equivalently if c is a maximal chain, then the cube C_c is one of the C_σ from Lemma 3.15.) We will identify C_c with the cube $[0, 1]^{m-1}$ corresponding to c in $M_{u,v}$.

Let $t_1, \dots, t_{m-1}$ be the coordinates of $[0,1]^{m-1}$. As a first step, we identify the vertices of C_c and $[0,1]^{m-1}$. A vertex of C_c corresponds to a barycenter of some face containing F_c, which in turn corresponds to some subchain c' of c; the corresponding vertex in $[0,1]^{m-1}$ has $t_i = 0$ if $u^i \in c'$, and has $t_i = 1$ otherwise. The identification on the vertices leads to our desired identification as follows. Construct the simplicial complex subdivision of C_c (respectively $[0,1]^{m-1}$) by joining every face to the barycenter of $\mathcal{M}(u,v)$ (respectively the vertex $\vec{1} \in [0,1]^{m-1}$), and extend the identification linearly over each simplex.

It is fairly straightforward to check that such identifications induce a cubical complex homeomorphism between the cubical complex $M_{u,v}$ and the cubical complex structure on $\mathcal{M}(u,v)$, and these homeomorphisms satisfy the conditions of the lemma. Further details are left to the reader. □

3.5 Definition of a cubical flow category

Definition 3.21 A *cubical flow category* is a flow category $\mathscr{C}$ equipped with a grading-preserving functor $\mathfrak{f}\colon \Sigma^k\mathscr{C} \to \mathscr{C}_C(n)$ for some $k \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that for each $x, y \in \mathrm{Ob}(\mathscr{C})$, $\mathfrak{f}\colon \mathcal{M}(x,y) \to \mathcal{M}(\mathfrak{f}(x), \mathfrak{f}(y))$ is a (trivial) covering map.

Thus, if $\mathfrak{f}\colon \mathscr{C} \to \mathscr{C}_C(n)$ is a cubical flow category and $x, y \in \mathrm{Ob}(\mathscr{C})$ then $\mathrm{Hom}(x,y)$ is empty unless $\mathfrak{f}(x) \geq \mathfrak{f}(y)$. If $x = y$, then $\mathrm{Hom}(x,y) = \{\mathrm{Id}\}$, and if $\mathfrak{f}(x) = \mathfrak{f}(y)$ but $x \neq y$ then $\mathrm{Hom}(x,y)$ is empty. If $\mathfrak{f}(x) > \mathfrak{f}(y)$, then the moduli space $\mathcal{M}_{\mathscr{C}}(x,y) = \mathrm{Hom}(x,y)$ is a (possibly empty) disjoint union of permutohedra.

Convention 3.22 Sometimes we suppress the grading information if it is inessential to the discussion, and drop the grading shift Σ^k from the notation.

A framing of the cube flow category $\mathscr{C}_C(n)$ in the sense of Definition 3.8 induces a sign assignment s on the cube (see Section 1.1) as follows: for $u > v$ with $|u| - |v| = 1$, $s_{u,v} = 0$ if the point $\mathcal{M}(u,v)$ is framed positively, and $s_{u,v} = 1$ otherwise. Furthermore, every sign assignment on the cube is induced from an essentially unique framing of $\mathscr{C}_C(n)$; see [35, Section 4.2]. The chain complex associated to $\mathscr{C}_C(n)$, framed according to some sign assignment s, is defined as follows. The chain group is generated by the vertices of the cube $\{0,1\}^n$, and the differential is given by

$$\delta(v) = \sum_{\substack{u > v \\ |u| - |v| = 1}} (-1)^{s_{u,v}} u.$$

This is an acyclic chain complex, and is often referred to as the *cube chain complex*.

Furthermore, if $(\mathscr{C}, \mathfrak{f}\colon \mathscr{C} \to \mathscr{C}_C(n))$ is a cubical flow category then any sign assignment s on the cube induces a framing of the 0–dimensional moduli spaces in $\mathscr{C}$ as well: all the points in $\mathcal{M}_{\mathscr{C}}(x, y)$ are framed positively if $s_{\mathfrak{f}(x),\mathfrak{f}(y)} = 0$, and, otherwise, all the points in $\mathcal{M}_{\mathscr{C}}(x, y)$ are framed negatively. The pullback of the (essentially unique) framing on $\mathscr{C}_C(n)$ inducing s produces an essentially canonical extension of this framing to the entire cubical flow category $\mathscr{C}$. The chain complex associated to $\mathscr{C}$ for this framing has the following differential: for $x, y \in \mathrm{Ob}(\mathscr{C})$ with $\mathrm{gr}(x) - \mathrm{gr}(y) = 1$, the coefficient of x in δy is

$$\langle \delta(y), x\rangle = \begin{cases} (-1)^{s_{\mathfrak{f}(x),\mathfrak{f}(y)}} & \text{if } \mathfrak{f}(x) > \mathfrak{f}(y), \\ 0 & \text{otherwise.} \end{cases}$$

This chain complex only depends on the sign assignment s and not the entire framing of $\mathscr{C}_C(n)$.

Even though the definition of cubical flow categories seems fairly restrictive, there are many examples:

Example 3.23 Given any (finite) simplicial complex $S_\bullet$ with vertices $v_1, \dots, v_n$, there is a corresponding cubical flow category $(\mathscr{C}, \mathfrak{f}\colon \Sigma\mathscr{C} \to \mathscr{C}_C(n))$, defined as follows. The objects of $\mathscr{C}$ correspond to the simplices of $S_\bullet$, which in turn can be viewed as nonempty subsets of $\{v_1, \dots, v_n\}$. Given an object in $\mathscr{C}$, corresponding to a subset $T \subseteq \{v_1, \dots, v_n\}$, define $\mathfrak{f}(T)$ to be the vector in $\{0, 1\}^n$ whose i^{th} coordinate is 1 if and only if $v_i \in T$. Note that the map $\mathfrak{f}$ is injective on objects. Let $\mathscr{C}$ be the full subcategory of $\mathscr{C}_C(n)$ generated by the objects in the image of $\mathfrak{f}$. The chain complex associated to $\mathscr{C}$ is isomorphic to the simplicial chain complex for $S_\bullet$.

(One can think of the category $\mathscr{C}$ as coming from choosing a Morse function on each simplex in $S_\bullet$ with a unique interior maximum and no other interior critical points, and so that these Morse functions are compatible under restriction. The moduli spaces in $\mathscr{C}$ are then the corresponding Morse moduli spaces.)

Example 3.24 The Khovanov flow category [35, Definition 5.3] $\mathscr{C}_K(K)$ associated to a link diagram K (with n crossings $c_1, \dots, c_n$) is, by construction, a cubical flow category. For any $v \in \{0, 1\}^n$, the subset of $\mathrm{Ob}(\mathscr{C}_K(K))$ that maps to v are precisely the Khovanov generators over v (as defined in Section 2.2):

$$\mathfrak{f}^{-1}(v) = F(v).$$

For any $u > v \in \{0,1\}^n$ with $|u|-|v| = 1$ and any $x \in \mathfrak{f}^{-1}(u) = F(u)$ and $y \in \mathfrak{f}^{-1}(v) = F(v)$, the moduli space is

$$\mathcal{M}_{\mathscr{C}_K(K)}(x,y) = \begin{cases} \{\mathrm{pt}\} & \text{if } x \text{ appears in } \delta_{Kh}(y), \\ \varnothing & \text{otherwise.} \end{cases}$$

Therefore, the chain complex associated to $\mathscr{C}_K(K)$ is isomorphic to the Khovanov chain complex $\mathcal{C}_{Kh}(K)$ (from Definition 2.1).

3.6 Cubical neat embeddings

Consider the cube flow category $\mathscr{C}_C(n)$, and fix a tuple $\boldsymbol{d} = (d_0, \dots, d_{n-1}) \in \mathbb{N}^n$ and a real number $R > 0$. For any $u > v$ in $\mathrm{Ob}(\mathscr{C}_C(n)) = \{0,1\}^n$, let

$$E_{u,v} = \left[\prod_{i=|v|}^{|u|-1} (-R,R)^{d_i}\right] \times \mathcal{M}_{\mathscr{C}_C(n)}(u,v).$$

Equip $E_{u,v}$ with the Riemannian metric induced from the standard metric on the Euclidean space after viewing the permutohedron $\mathcal{M}_{\mathscr{C}_C(n)}(u,v)$ as a polyhedron in $\mathbb{R}^{|u|-|v|}$. For any $u > v > w$ in $\mathrm{Ob}(\mathscr{C}_C(n))$, there is a map $E_{v,w} \times E_{u,v} \to E_{u,w}$ given by

$$\begin{aligned} E_{v,w} \times E_{u,v} &= \prod_{i=|w|}^{|v|-1} (-R,R)^{d_i} \times \mathcal{M}_{\mathscr{C}_C(n)}(v,w) \times \prod_{i=|v|}^{|u|-1} (-R,R)^{d_i} \times \mathcal{M}_{\mathscr{C}_C(n)}(u,v) \\ &\cong \prod_{i=|w|}^{|u|-1} (-R,R)^{d_i} \times \mathcal{M}_{\mathscr{C}_C(n)}(v,w) \times \mathcal{M}_{\mathscr{C}_C(n)}(u,v) \\ &\qquad\qquad \xhookrightarrow{\mathrm{Id}\times\circ} \prod_{i=|w|}^{|u|-1} (-R,R)^{d_i} \times \mathcal{M}_{\mathscr{C}_C(n)}(u,w). \end{aligned}$$

Definition 3.25 Fix a cubical flow category $(\mathscr{C}, \mathfrak{f}\colon \Sigma^k \mathscr{C} \to \mathscr{C}_C(n))$. A *cubical neat embedding* ι of $(\mathscr{C}, \mathfrak{f})$ (or, more succinctly, of $\mathscr{C}$) relative to a tuple $\boldsymbol{d} = (d_0, \dots, d_{n-1}) \in \mathbb{N}^n$ consists of neat embeddings

$$\iota_{x,y}\colon \mathcal{M}_{\mathscr{C}}(x,y) \hookrightarrow E_{\mathfrak{f}(x),\mathfrak{f}(y)}$$

such that:

(1) For each $x, y \in \mathrm{Ob}(\mathscr{C})$, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{M}_{\mathscr{C}}(x,y) \arrow[r, "\iota_{x,y}"] \arrow[dr, "\mathfrak{f}"'] & E_{\mathfrak{f}(x),\mathfrak{f}(y)} \arrow[d, "\text{projection}"] \\
 & \mathcal{M}_{\mathscr{C}_C(n)}(\mathfrak{f}(x),\mathfrak{f}(y))
\end{tikzcd}$$

(2) For each $u, v \in \mathrm{Ob}(\mathscr{C}_C(n))$, the induced map

$$\coprod_{\substack{x,y\\ \mathfrak{f}(x)=u\\ \mathfrak{f}(y)=v}} \iota_{x,y}\colon \coprod_{\substack{x,y\\ \mathfrak{f}(x)=u\\ \mathfrak{f}(y)=v}} \mathcal{M}_{\mathscr{C}}(x,y) \to E_{u,v}$$

is a neat embedding.

(3) For each $x, y, z \in \mathrm{Ob}(\mathscr{C})$, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{M}_{\mathscr{C}}(y,z) \times \mathcal{M}_{\mathscr{C}}(x,y) \arrow[r] \arrow[d] & \mathcal{M}_{\mathscr{C}}(x,z) \arrow[d] \\
E_{\mathfrak{f}(y),\mathfrak{f}(z)} \times E_{\mathfrak{f}(x),\mathfrak{f}(y)} \arrow[r] & E_{\mathfrak{f}(x),\mathfrak{f}(z)}
\end{tikzcd}$$

In order to construct the cubical realization, we need to extend these embeddings $\iota_{x,y}$ to maps

$$\bar{\iota}_{x,y}\colon \left[\prod_{i=|\mathfrak{f}(y)|}^{|\mathfrak{f}(x)|-1} [-\epsilon,\epsilon]^{d_i}\right] \times \mathcal{M}_{\mathscr{C}}(x,y) \to E_{\mathfrak{f}(x),\mathfrak{f}(y)}$$

for some $\epsilon > 0$, so that the analogue of the diagram from condition (3) still commutes, and the extension of the map from condition (2) is still an embedding. One way to choose such a family of extensions would be to coherently frame the normal bundles of the embeddings $\iota_{x,y}$ (in a similar sense to Definition 3.7) and then use the construction from Definition 3.10. Instead, we will use the following explicit extension.

For any $x, y \in \mathrm{Ob}(\mathscr{C})$, let u and v denote $\mathfrak{f}(x)$ and $\mathfrak{f}(y)$, respectively, and let $\pi^R_{u,v}$ and $\pi^M_{u,v}$ denote the projections of $\left[\prod_{i=|v|}^{|u|-1}(-R,R)^{d_i}\right] \times \mathcal{M}_{\mathscr{C}_C(n)}(u,v)$ onto the two factors. Given sufficiently small $\epsilon > 0$, extend the embedding $\iota_{x,y}$ to a map

(3.26)
$$\bar{\iota}_{x,y}\colon \left[\prod_{i=|v|}^{|u|-1}[-\epsilon,\epsilon]^{d_i}\right] \times \mathcal{M}_{\mathscr{C}}(x,y) \to E_{u,v} = \left[\prod_{i=|v|}^{|u|-1}(-R,R)^{d_i}\right] \times \mathcal{M}_{\mathscr{C}_C(n)}(u,v),$$
$$(a,\gamma) \mapsto (a + \pi^R_{u,v}\iota_{x,y}(\gamma), \pi^M_{u,v}\iota_{x,y}(\gamma)).$$

The definition of $\bar{\iota}$ ensures that the analogue of the diagram from condition (3) still commutes. By condition (1), $\iota(\mathcal{M}_{\mathscr{C}}(x,y))$ is transverse to the fibers

$$\left[\prod_{i=|v|}^{|u|-1}(-R,R)^{d_i}\right]\times\{\gamma\}.$$

So, for ϵ sufficiently small, the extension of the map from condition (2) is still an embedding. We make this a requirement on ϵ:

Convention 3.27 When talking about extensions $\bar{\iota}_{x,y}$ of cubical neat embeddings, we will always assume that ϵ is chosen to be small in the sense that the induced map

$$\coprod_{\substack{x,y\\ \mathfrak{f}(x)=u\\ \mathfrak{f}(y)=v}}\bar{\iota}_{x,y}\colon \coprod_{\substack{x,y\\ \mathfrak{f}(x)=u\\ \mathfrak{f}(y)=v}}\left[\prod_{i=|v|}^{|u|-1}[-\epsilon,\epsilon]^{d_i}\right]\times\mathcal{M}_{\mathscr{C}}(x,y)\to E_{u,v}$$

is an embedding.

Remark 3.28 In [35], we used a framing of the normal bundles, rather than the kind of explicit extension above, to trivialize tubular neighborhoods. When identifying the cubical realization with the Cohen–Jones–Segal realization (see Section 3.8), we will need an isotopy between these two trivializations.

Let $V_0=\prod_{i=|v|}^{|u|-1}\mathbb{R}^{d_i}\times\{0\}\subset(TE_{u,v})|_{\iota_{x,y}(\mathcal{M}_{\mathscr{C}}(x,y))}$ and let V_1 be the normal bundle to $\iota_{x,y}\mathcal{M}_{\mathscr{C}}(x,y)$. Since $\pi^{M}_{u,v}\circ\iota_{x,y}$ is a covering map, the projection $d\pi^{R}_{u,v}\colon V_1\to V_0$ is an isomorphism. For $t\in[0,1]$ let $\pi_t\colon V_1\to(TE_{u,v})|_{\iota_{x,y}(\mathcal{M}_{\mathscr{C}}(x,y))}$ be $\pi_t=t\,\mathrm{Id}+(1-t)d\pi^{R}_{u,v}$ and let $V_t=\pi_t(V_1)$. The V_t are a 1–parameter family of subbundles connecting V_0 to V_1, and each V_t is a complement to $T(\iota_{x,y}\mathcal{M}_{\mathscr{C}}(x,y))$.

The bundle V_0 is a trivial bundle, and in particular framed. Pushing forward this framing by π_t gives a framing of each V_t. Exponentiating these framings gives a 1–parameter family of extensions $\bar{\iota}^{t}_{x,y}$ of $\iota_{x,y}$, each satisfying the analogue of condition (3). The framing $\bar{\iota}^{0}_{x,y}$ is our explicit extension $\bar{\iota}_{x,y}$ from (3.26), and $\bar{\iota}^{1}_{x,y}$ is an extension coming from coherently framing the normal bundles of $\iota_{x,y}$. Since each V_t is a complement to $T(\iota_{x,y}\mathcal{M}_{\mathscr{C}}(x,y))$, each $\bar{\iota}^{t}_{x,y}$ satisfies the analogue of condition (2) for some ϵ_t; compactness allows us to choose a uniform ϵ for which each $\bar{\iota}^{t}_{x,y}$ satisfies the analogue of condition (2). This produces the required isotopy between the extension from (3.26) and an extension coming from some framing of the normal bundle.

3.7 Cubical realization

Definition 3.29 Fix a cubical neat embedding ι of a cubical flow category

$$(\mathscr{C}, \mathfrak{f}\colon \Sigma^k\mathscr{C} \to \mathscr{C}_C(n)),$$

relative to a tuple $\boldsymbol{d} = (d_0, \dots, d_{n-1}) \in \mathbb{N}^d$, and fix $\epsilon > 0$ satisfying Convention 3.27. We build a CW complex $\|\mathscr{C}\| = \|\mathscr{C}\|_{\mathfrak{f},\iota}$ from this data as follows:

- The CW complex has one cell for each $x \in \mathrm{Ob}(\mathscr{C})$. Letting u denote $\mathfrak{f}(x)$, this cell is given by

$$\mathcal{C}(x) = \prod_{i=0}^{|u|-1} [-R, R]^{d_i} \times \prod_{i=|u|}^{n-1} [-\epsilon, \epsilon]^{d_i} \times \widetilde{\mathcal{M}}_{\mathscr{C}_C(n)}(u, \vec{0}),$$

where $\widetilde{\mathcal{M}}_{\mathscr{C}_C(n)}(u, \vec{0})$ is defined to be $[0,1] \times \mathcal{M}_{\mathscr{C}_C(n)}(u, \vec{0})$ if $u \neq \vec{0}$, or the point $\{0\}$ if $u = \vec{0}$.

- For any $x, y \in \mathrm{Ob}(\mathscr{C})$ with $\mathfrak{f}(x) = u > \mathfrak{f}(y) = v$, the cubical neat embedding ι furnishes an embedding

$$\begin{aligned}
&\mathcal{C}(y) \times \mathcal{M}_{\mathscr{C}}(x, y) \\
&= \prod_{i=0}^{|v|-1} [-R, R]^{d_i} \times \prod_{i=|v|}^{n-1} [-\epsilon, \epsilon]^{d_i} \times \widetilde{\mathcal{M}}_{\mathscr{C}_C(n)}(v, \vec{0}) \times \mathcal{M}_{\mathscr{C}}(x, y) \\
&\cong \prod_{i=0}^{|v|-1} [-R, R]^{d_i} \times \prod_{i=|u|}^{n-1} [-\epsilon, \epsilon]^{d_i} \times \widetilde{\mathcal{M}}_{\mathscr{C}_C(n)}(v, \vec{0}) \times \Bigl(\prod_{i=|v|}^{|u|-1} [-\epsilon, \epsilon]^{d_i} \times \mathcal{M}_{\mathscr{C}}(x, y) \Bigr) \\
&\xhookrightarrow{\mathrm{Id} \times \bar{\iota}_{x,y}} \prod_{i=0}^{|v|-1} [-R, R]^{d_i} \times \prod_{i=|u|}^{n-1} [-\epsilon, \epsilon]^{d_i} \times \widetilde{\mathcal{M}}_{\mathscr{C}_C(n)}(v, \vec{0}) \times \Bigl(\prod_{i=|v|}^{|u|-1} [-R, R]^{d_i} \times \mathcal{M}_{\mathscr{C}_C(n)}(u, v) \Bigr) \\
&\cong \prod_{i=0}^{|u|-1} [-R, R]^{d_i} \times \prod_{i=|u|}^{n-1} [-\epsilon, \epsilon]^{d_i} \times \widetilde{\mathcal{M}}_{\mathscr{C}_C(n)}(v, \vec{0}) \times \mathcal{M}_{\mathscr{C}_C(n)}(u, v) \\
&\hookrightarrow \prod_{i=0}^{|u|-1} [-R, R]^{d_i} \times \prod_{i=|u|}^{n-1} [-\epsilon, \epsilon]^{d_i} \times \partial(\widetilde{\mathcal{M}}_{\mathscr{C}_C(n)}(u, \vec{0})) \\
&\subset \partial\mathcal{C}(x).
\end{aligned}$$

Here, the second inclusion comes from the composition map if $v \neq \vec{0}$, or the inclusion $\{0\} \hookrightarrow [0,1]$ if $v = \vec{0}$. Let $\mathcal{C}_y(x) \subset \partial\mathcal{C}(x)$ denote the image of this embedding.

- The attaching map for $\mathcal{C}(x)$ sends $\mathcal{C}_y(x) \cong \mathcal{C}(y) \times \mathcal{M}_{\mathscr{C}}(x, y)$ by the projection map to $\mathcal{C}(y)$ and sends the complement of $\cup_y \mathcal{C}_y(x)$ in $\partial\mathcal{C}(x)$ to the basepoint.

The *cubical realization* of $(\mathscr{C},\mathfrak{f})$ is defined to be the formal desuspension $\mathcal{X}(\mathscr{C}) = \Sigma^{-(k+d_0+\cdots+d_{n-1})}\|\mathscr{C}\|$. (The desuspension ensures that the gradings of the objects in $\mathscr{C}$ agree with the dimensions of the corresponding cells in $\mathcal{X}(\mathscr{C})$.)

Lemma 3.30 *The attaching maps in the cubical realization are well defined.*

Proof As in the proof of [35, Lemma 3.25], we must show that for any $x, y, z \in \mathrm{Ob}(\mathscr{C})$ with $\mathrm{gr}(x) > \mathrm{gr}(y) > \mathrm{gr}(z)$, the dashed arrow in the following diagram exists such that the diagram commutes:

$$\begin{tikzcd}
\mathcal{C}_z(x)\cap\mathcal{C}_y(x) \arrow[r, hook] \arrow[d, hook] \arrow[ddrr, dashed] & \mathcal{C}_y(x) \arrow[r] & \mathcal{C}(y) \\
\mathcal{C}_z(x) \arrow[d] & & \partial\mathcal{C}(y) \arrow[u, hook] \\
\mathcal{C}(z) & & \mathcal{C}_z(y) \arrow[ll] \arrow[u, hook]
\end{tikzcd}$$

Let u, v and w denote $\mathfrak{f}(x)$, $\mathfrak{f}(y)$ and $\mathfrak{f}(z)$, respectively. We may assume $u > v > w$; otherwise, it is not hard to verify that $\mathcal{C}_z(x)$ and $\mathcal{C}_y(x)$ are disjoint. In a similar vein to Definition 3.29, let $\mathcal{C}_{y,z}(x) \subset \partial\mathcal{C}(x)$ be the image of the embedding

$$\begin{aligned}
&\mathcal{C}(z)\times\mathcal{M}_{\mathscr{C}}(y,z)\times\mathcal{M}_{\mathscr{C}}(x,y)\\
&= \prod_{i=0}^{|w|-1}[-R,R]^{d_i}\times\prod_{i=|w|}^{n-1}[-\epsilon,\epsilon]^{d_i}\times\widetilde{\mathcal{M}}_{\mathscr{C}_C(n)}(w,\vec{0})\times\mathcal{M}_{\mathscr{C}}(y,z)\times\mathcal{M}_{\mathscr{C}}(x,y)\\
&\cong \prod_{i=0}^{|w|-1}[-R,R]^{d_i}\times\prod_{i=|u|}^{n-1}[-\epsilon,\epsilon]^{d_i}\times\widetilde{\mathcal{M}}_{\mathscr{C}_C(n)}(w,\vec{0})\times\Bigl(\prod_{i=|w|}^{|v|-1}[-\epsilon,\epsilon]^{d_i}\times\mathcal{M}_{\mathscr{C}}(y,z)\Bigr)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\times\Bigl(\prod_{i=|v|}^{|u|-1}[-\epsilon,\epsilon]^{d_i}\times\mathcal{M}_{\mathscr{C}}(x,y)\Bigr)\\
&\xhookrightarrow{\mathrm{Id}\times\bar{\imath}_{y,z}\times\bar{\imath}_{x,y}} \prod_{i=0}^{|w|-1}[-R,R]^{d_i}\times\prod_{i=|u|}^{n-1}[-\epsilon,\epsilon]^{d_i}\times\widetilde{\mathcal{M}}_{\mathscr{C}_C(n)}(w,\vec{0})\\
&\qquad\qquad\times\Bigl(\prod_{i=|w|}^{|v|-1}[-R,R]^{d_i}\times\mathcal{M}_{\mathscr{C}_C(n)}(v,w)\Bigr)\times\Bigl(\prod_{i=|v|}^{|u|-1}[-R,R]^{d_i}\times\mathcal{M}_{\mathscr{C}_C(n)}(u,v)\Bigr)\\
&\cong \prod_{i=0}^{|u|-1}[-R,R]^{d_i}\times\prod_{i=|u|}^{n-1}[-\epsilon,\epsilon]^{d_i}\times\widetilde{\mathcal{M}}_{\mathscr{C}_C(n)}(w,\vec{0})\times\mathcal{M}_{\mathscr{C}_C(n)}(v,w)\times\mathcal{M}_{\mathscr{C}_C(n)}(u,v)\\
&\hookrightarrow \prod_{i=0}^{|u|-1}[-R,R]^{d_i}\times\prod_{i=|u|}^{n-1}[-\epsilon,\epsilon]^{d_i}\times\partial(\widetilde{\mathcal{M}}_{\mathscr{C}_C(n)}(u,\vec{0}))\subset\partial\mathcal{C}(x).
\end{aligned}$$

(As in Definition 3.29, the second inclusion usually comes from the composition map; the only special case is if $w = \vec{0}$, when it comes partly from the inclusion $\{0\} \hookrightarrow [0,1]$.)

We claim that $\mathcal{C}_{y,z}(x) = \mathcal{C}_z(x) \cap \mathcal{C}_y(x)$. The direction $\mathcal{C}_{y,z}(x) \subseteq \mathcal{C}_z(x) \cap \mathcal{C}_y(x)$ is immediate since the inclusion

$$\mathcal{C}(z) \times \mathcal{M}_{\mathscr{C}}(y,z) \times \mathcal{M}_{\mathscr{C}}(x,y) \xrightarrow{\cong} \mathcal{C}_{y,z}(x) \subset \partial\mathcal{C}(x)$$

factors in both of the following ways (using condition (3) of Definition 3.25):

$$\mathcal{C}(z) \times \mathcal{M}_{\mathscr{C}}(y,z) \times \mathcal{M}_{\mathscr{C}}(x,y) \xrightarrow{\mathrm{Id}\times\circ} \mathcal{C}(z) \times \mathcal{M}_{\mathscr{C}}(x,z) \xrightarrow{\cong} \mathcal{C}_z(x) \subset \partial\mathcal{C}(x),$$

$$\mathcal{C}(z) \times \mathcal{M}_{\mathscr{C}}(y,z) \times \mathcal{M}_{\mathscr{C}}(x,y) \xrightarrow{\cong} \mathcal{C}_z(y) \times \mathcal{M}_{\mathscr{C}}(x,y) \hookrightarrow \mathcal{C}(y) \times \mathcal{M}_{\mathscr{C}}(x,y) \xrightarrow{\cong} \mathcal{C}_y(x) \subset \partial\mathcal{C}(x).$$

The other direction requires more work. We will abuse notation slightly and identify points with their images under the composition map in $\mathscr{C}_C(n)$. View any point $p \in \mathcal{C}_z(x) \cap \mathcal{C}_y(x)$ as a point (p_1, p_2, p_3, p_4) in the following subspace of $\partial\mathcal{C}(x)$:

$$\left(\prod_{i=0}^{|w|-1}[-R,R]^{d_i} \times \prod_{i=|u|}^{n-1}[-\epsilon,\epsilon]^{d_i}\right) \times \prod_{i=|w|}^{|v|-1}[-R,R]^{d_i} \times \prod_{i=|v|}^{|u|-1}[-R,R]^{d_i} \times \partial(\widetilde{\mathcal{M}}_{\mathscr{C}_C(n)}(u,\vec{0})).$$

For p to lie in $\mathcal{C}_z(x)$, p_4 must lie in the subspace

$$\widetilde{\mathcal{M}}_{\mathscr{C}_C(n)}(w,\vec{0}) \times \mathcal{M}_{\mathscr{C}_C(n)}(u,w);$$

similarly, for p to lie in $\mathcal{C}_y(x)$, p_4 must lie in the subspace

$$\widetilde{\mathcal{M}}_{\mathscr{C}_C(n)}(v,\vec{0}) \times \mathcal{M}_{\mathscr{C}_C(n)}(u,v).$$

Therefore, p_4 must lie in the subspace $\widetilde{\mathcal{M}}_{\mathscr{C}_C(n)}(w,\vec{0}) \times \mathcal{M}_{\mathscr{C}_C(n)}(v,w) \times \mathcal{M}_{\mathscr{C}_C(n)}(u,v)$. Write p_4 also in component form as $(p_{4,1}, p_{4,2}, p_{4,3})$. Since p lies in $\mathcal{C}_y(x)$, we know $(p_3, p_{4,3}) \in \operatorname{im}(\bar{\iota}_{x,y})$, and since p lies in $\mathcal{C}_z(x)$, we know $(p_2, p_3, p_{4,2}, p_{4,3}) \in \operatorname{im}(\bar{\iota}_{x,z})$. Moreover, since ι is a cubical neat embedding (Definition 3.25),

$$\operatorname{im}(\bar{\iota}_{x,z}) \cap \left(\prod_{i=|w|}^{|v|-1}[-R,R]^{d_i} \times \mathcal{M}_{\mathscr{C}_C(n)}(v,w) \times \prod_{i=|v|}^{|u|-1}[-R,R]^{d_i} \times \mathcal{M}_{\mathscr{C}_C(n)}(u,v)\right) = \operatorname{im}\left(\coprod_{y' \mid \mathfrak{f}(y')=v} \bar{\iota}_{y',z} \times \bar{\iota}_{x,y'}\right),$$

and, therefore, there exists y' with $\mathfrak{f}(y') = v$ such that $(p_3, p_{4,3}) \in \operatorname{im}(\bar{\iota}_{x,y'})$ and $(p_2, p_{4,2}) \in \operatorname{im}(\bar{\iota}_{y',z})$. Condition (2) from Definition 3.25 (but with $\bar{\iota}$ instead of ι)

ensures that $y' = y$, and then it is straightforward to see that p lies in $\mathcal{C}_{y,z}(x)$. This completes the proof that $\mathcal{C}_{y,z}(x) = \mathcal{C}_z(x) \cap \mathcal{C}_y(x)$.

Define the dashed arrow from $\mathcal{C}_z(x) \cap \mathcal{C}_y(x) = \mathcal{C}_{y,z}(x) \cong \mathcal{C}(z) \times \mathcal{M}_{\mathscr{C}}(y,z) \times \mathcal{M}_{\mathscr{C}}(x,y)$ to $\mathcal{C}_z(y) \cong \mathcal{C}(z) \times \mathcal{M}_{\mathscr{C}}(y,z)$ to be the projection map. From the definition of $\mathcal{C}_{y,z}(x)$, it is easy to verify that the resulting diagram commutes. □

Proposition 3.31 *Up to stable homotopy equivalence, the cubical realization is independent of the choice of tuple* $\boldsymbol{d}$*. More precisely, let* ι *be a cubical neat embedding relative to* $\boldsymbol{d} = (d_0, \dots, d_{n-1})$ *and let* $\|\mathscr{C}\|$ *be the cubical realization corresponding to* ι*. Fix a tuple* $\boldsymbol{d}' = (d'_0, \dots, d'_{n-1})$ *with* $d'_i \geq d_i$ *for all* i*. There is an induced cubical neat embedding of* $\mathscr{C}$ *relative to* $\boldsymbol{d}'$*, gotten by identifying the space* $E_{u,v}$ *for* ι *with the subspace*

$$\prod_{i=|v|}^{|u|-1} (-R,R)^{d_i} \times \{0\}^{d'_i - d_i} \times \mathcal{M}_{\mathscr{C}_C(n)}(u,v)$$

of the space $E'_{u,v}$ *for* ι'*. Let* $\|\mathscr{C}\|'$ *be the cubical realization corresponding to* ι' *and let* $N = |\boldsymbol{d}'| - |\boldsymbol{d}| = \sum_{i=0}^{n-1} d'_i - \sum_{i=0}^{n-1} d_i$*. Then there is a homotopy equivalence*

$$\Sigma^N \|\mathscr{C}\| \simeq \|\mathscr{C}\|',$$

taking cells to the corresponding cells.

Proof The proof is the same as case (3) in the proof of [35, Lemma 3.27]. □

One can also show, by following the proofs of [35, Lemmas 3.25–3.27], that the stable homotopy type of $\|\mathscr{C}\|_\iota$ is independent of the cubical neat embedding ι and the parameters R and ϵ. Since this result also follows from Theorem 4, we do not give further details.

We conclude this section by returning to our simplicial complex example:

Proposition 3.32 *Let* $S_\bullet$ *be a simplicial complex with* n *vertices and let*

$$(\mathscr{C}, \mathfrak{f}\colon \Sigma\mathscr{C} \to \mathscr{C}_C(n))$$

be the corresponding cubical flow category, as in Example 3.23. Let $|S_\bullet|_+$ *denote the disjoint union of the geometric realization of* $S_\bullet$ *and a basepoint. Then there is a stable* *homotopy equivalence*

$$\mathcal{X}(\mathscr{C}) \simeq |S_\bullet|_+.$$

Moreover, the stable homotopy equivalence may be chosen so that it induces an isomorphism between the simplicial cochain complex of $S_\bullet$ and the reduced cellular cochain complex of $\mathcal{X}(\mathscr{C})$ (with the CW complex structure from Definition 3.29).

We leave the proof as a (somewhat involved) exercise to the reader.

3.8 Cubical realization agrees with the Cohen–Jones–Segal realization

The main aim of this subsection is to prove the following:

Theorem 4 *Let $(\mathscr{C}, \mathfrak{f}\colon \mathscr{C} \to \mathscr{C}_C(n))$ be a cubical flow category. For any cubical neat embedding ι relative to any tuple $\boldsymbol{d} = (d_0, \dots, d_{n-1})$, and parameters R and ϵ, the cubical realization $\mathcal{X}(\mathscr{C})$* (from Definition 3.29) *is stably homotopy equivalent to the Cohen–Jones–Segal realization* (from Section 3.2) *of $\mathscr{C}$ with framing induced from some framing of $\mathscr{C}_C(n)$. Furthermore, the stable homotopy equivalence sends cells to corresponding cells by degree ± 1 maps.*

Proof Using Proposition 3.31, we may assume that the d_i are sufficiently large that there is a neat embedding $\jmath$ of the cube flow category $\mathscr{C}_C(n)$ relative to $\boldsymbol{D} = (\dots, 0, \dots, 0, d_0, \dots, d_{n-1}, 0, \dots, 0, \dots)$ in the sense of Definition 3.7. Fix some framing of the cube flow category $\mathscr{C}_C(n)$, and construct the extension $\bar{\jmath}$ as in Definition 3.10. Let δ and T be the corresponding parameters. After scaling if necessary, we may assume $\delta = R$, and after increasing T if necessary, we may assume $T \geq 1$.

Note that $\bar{\jmath} \circ \iota$ is a neat embedding (in the sense of Definition 3.7) of our flow category $\mathscr{C}$ relative to $\boldsymbol{D}$; it has an extension $\bar{\jmath} \circ \bar{\iota}$ (again, as in Definition 3.10) with respect to the parameters ϵ and T, and $\bar{\jmath} \circ \bar{\iota}$ is isotopic to the extension coming from the framing of $\mathscr{C}$ induced from the framing of $\mathscr{C}_C(n)$; see also Remark 3.28.

Let $\|\mathscr{C}\|$ be the CW complex constructed from the cubical neat embedding ι and its extension $\bar{\iota}$ with respect to the parameters ϵ and R; and let $|\mathscr{C}|$ be the CW complex constructed from the neat embedding $\bar{\jmath} \circ \iota$ and its extension $\bar{\jmath} \circ \bar{\iota}$ with respect to the parameters ϵ and T. Cells of both $\|\mathscr{C}\|$ and $|\mathscr{C}|$ correspond to objects of $\mathscr{C}$, and hence to each other. We will produce a quotient map from $|\mathscr{C}|$ to $\|\mathscr{C}\|$ which will send each cell via a degree ± 1 map to the corresponding cell. It follows that the quotient map is a stable homotopy equivalence.

For any $x \in \mathrm{Ob}(\mathscr{C})$ with $\mathfrak{f}(x) = u \in \mathrm{Ob}(\mathscr{C}_C(n))$, the cell associated to x in $\|\mathscr{C}\|$ is

$$\mathcal{C}(x)' = \begin{cases} \prod_{i=0}^{|u|-1}[-R,R]^{d_i} \times \prod_{i=|u|}^{n-1}[-\epsilon,\epsilon]^{d_i} \times [0,1] \times \mathcal{M}_{\mathscr{C}_C(n)}(u,\vec{0}) & \text{if } u \neq \vec{0}, \\ \prod_{i=0}^{n-1}[-\epsilon,\epsilon]^{d_i} \times \{0\} & \text{if } u = \vec{0}, \end{cases}$$

while the cell associated to x in $|\mathscr{C}|$ is

$$\mathcal{C}(x) = \prod_{i=0}^{|u|-1}[-T,T]^{d_i} \times \prod_{i=|u|}^{n-1}[-\epsilon,\epsilon]^{d_i} \times \prod_{i=0}^{|u|-1}[0,T].$$

Let $[\mathcal{C}(x)]$ (respectively $[\mathcal{C}(x)']$) denote the image of $\mathcal{C}(x)$ (respectively $\mathcal{C}(x)'$) in $|\mathscr{C}|$ (respectively $\|\mathscr{C}\|$). We will define a map $\mathcal{C}(x) \to [\mathcal{C}(x)']$ and check that this induces a well-defined map $[\mathcal{C}(x)] \to [\mathcal{C}(x)']$.

If $u = \vec{0}$, identify $\mathcal{C}(x)' \cong \prod_{i=0}^{n-1}[-\epsilon,\epsilon]^{d_i} \cong \mathcal{C}(x)$. If $u \neq \vec{0}$, the embedding

$$\begin{aligned} \bar{\jmath}_{u,\vec{0}}\colon \mathcal{M}_{\mathscr{C}_C(n)}(u,\vec{0}) \times \prod_{i=0}^{|u|-1}[-R,R]^{d_i} &\hookrightarrow (-T,T)^{d_0} \times [0,T) \times \cdots \times [0,T) \times (-T,T)^{d_{|u|-1}} \\ &\cong \prod_{i=1}^{|u|-1}[0,T) \times \prod_{i=0}^{|u|-1}(-T,T)^{d_i} \end{aligned}$$

induces the codimension-zero embedding of $\mathcal{C}(x)'$ into $\mathcal{C}(x)$

$$\begin{aligned} \mathcal{C}(x)' &= \prod_{i=0}^{|u|-1}[-R,R]^{d_i} \times \prod_{i=|u|}^{n-1}[-\epsilon,\epsilon]^{d_i} \times [0,1] \times \mathcal{M}_{\mathscr{C}_C(n)}(u,\vec{0}) \\ &\cong \mathcal{M}_{\mathscr{C}_C(n)}(u,\vec{0}) \times \prod_{i=0}^{|u|-1}[-R,R]^{d_i} \times \prod_{i=|u|}^{n-1}[-\epsilon,\epsilon]^{d_i} \times [0,1] \\ &\xhookrightarrow{\bar{\jmath}_{u,\vec{0}} \times \mathrm{Id}} \prod_{i=1}^{|u|-1}[0,T] \times \prod_{i=0}^{|u|-1}[-T,T]^{d_i} \times \prod_{i=|u|}^{n-1}[-\epsilon,\epsilon]^{d_i} \times [0,1] \\ &\cong \prod_{i=0}^{|u|-1}[-T,T]^{d_i} \times \prod_{i=|u|}^{n-1}[-\epsilon,\epsilon]^{d_i} \times [0,1] \times \prod_{i=1}^{|u|-1}[0,T] \\ &\hookrightarrow \prod_{i=0}^{|u|-1}[-T,T]^{d_i} \times \prod_{i=|u|}^{n-1}[-\epsilon,\epsilon]^{d_i} \times \prod_{i=0}^{|u|-1}[0,T] \\ &= \mathcal{C}(x). \end{aligned}$$

(Here, the $\cong$ arrows correspond to the obvious reshuffling of the factors, and the second inclusion is induced from the inclusion $[0,1] \hookrightarrow [0,T]$.) In either case, map $\mathcal{C}(x)$ to $[\mathcal{C}(x)']$ by identifying the image of this embedding with $\mathcal{C}(x)'$, and quotienting everything else to the basepoint. To see that this gives a well-defined, continuous map from the CW complex $|\mathscr{C}|$ to the CW complex $\|\mathscr{C}\|$, we need to check that for any other $y \in \mathrm{Ob}(\mathscr{C})$ with $\mathfrak{f}(y) = v < u$, the following commutes:

$$\begin{tikzcd}
\mathcal{C}(y)' \times \mathcal{M}_{\mathscr{C}}(x,y) \arrow[r, hook] \arrow[d, hook] & \mathcal{C}(y) \times \mathcal{M}_{\mathscr{C}}(x,y) \arrow[d, hook] \\
\mathcal{C}(x)' \arrow[r, hook] & \mathcal{C}(x)
\end{tikzcd}$$

(The horizontal arrows are induced from the inclusions defined above. The right vertical arrow is the inclusion defined in Definition 3.10 for $\bar{\jmath} \circ \bar{\imath}$, while the left vertical arrow is the inclusion defined in Definition 3.29 for $\bar{\imath}$.)

When $v = \vec{0}$, after removing the constant factor of $\prod_{i=|u|}^{n-1} [-\epsilon,\epsilon]^{d_i}$ and doing some consistent reshuffling, the diagram is

$$\begin{tikzcd}
\{0\} \times \prod\limits_{i=0}^{|u|-1} [-\epsilon,\epsilon]^{d_i} \times \mathcal{M}_{\mathscr{C}}(x,y) \arrow[r, hook, "\cong"] \arrow[d, hook, "(\kappa_1, \bar{\imath}_{x,y})"] & \prod\limits_{i=0}^{|u|-1} [-\epsilon,\epsilon]^{d_i} \times \mathcal{M}_{\mathscr{C}}(x,y) \arrow[d, hook, "(\kappa_3^0, \mathrm{Id}) \circ \bar{\jmath}_{u,\vec{0}} \circ \bar{\imath}_{x,y}"] \\
{[0,1]} \times \prod\limits_{i=0}^{|u|-1} [-R,R]^{d_i} \times \mathcal{M}_{\mathscr{C}_C(n)}(u,\vec{0}) \arrow[r, hook, "(\kappa_2, \bar{\jmath}_{u,\vec{0}})"] & \prod\limits_{i=0}^{|u|-1} [0,T] \times \prod\limits_{i=0}^{|u|-1} [-T,T]^{d_i}
\end{tikzcd}$$

where κ_1 is the inclusion $\{0\} \hookrightarrow [0,1]$, κ_2 is the inclusion $[0,1] \hookrightarrow [0,T]$, and κ_3^ℓ (for $0 \le \ell < |u|$) is the inclusion

$$\prod_{i=\ell+1}^{|u|-1} [0,T] \cong \{0\} \times \prod_{i=\ell+1}^{|u|-1} [0,T] \xhookrightarrow{(\kappa_2 \circ \kappa_1, \mathrm{Id})} \prod_{i=\ell}^{|u|-1} [0,T].$$

Therefore, the diagram commutes.

When $v \neq \vec{0}$, once again after removing the constant factor of $\prod_{i=|u|}^{n-1} [-\epsilon,\epsilon]^{d_i}$, and some consistent reshuffling, the diagram factors as:

$$\begin{tikzcd}
\begin{bmatrix} [0,1]\times\mathcal{M}_{\mathscr{C}_C(n)}(v,\vec{0})\times\prod_{i=0}^{|v|-1}[-R,R]^{d_i} \\ \times\prod_{i=|v|}^{|u|-1}[-\epsilon,\epsilon]^{d_i}\times\mathcal{M}_{\mathscr{C}}(x,y) \end{bmatrix} \arrow[r, hook, "{(\kappa_2,\bar{\jmath}_{v,\vec{0}},\mathrm{Id})}"] \arrow[d, hook, "{(\mathrm{Id},\bar{\iota}_{x,y})}"] & \begin{bmatrix} \prod_{i=0}^{|v|-1}[0,T]\times\prod_{i=0}^{|v|-1}[-T,T]^{d_i} \\ \times\prod_{i=|v|}^{|u|-1}[-\epsilon,\epsilon]^{d_i}\times\mathcal{M}_{\mathscr{C}}(x,y) \end{bmatrix} \arrow[d, hook, "{(\mathrm{Id},\bar{\jmath}_{u,\vec{0}}\circ\bar{\iota}_{x,y})}"] \\
\begin{bmatrix} [0,1]\times\mathcal{M}_{\mathscr{C}_C(n)}(v,\vec{0})\times\prod_{i=0}^{|u|-1}[-R,R]^{d_i} \\ \times\mathcal{M}_{\mathscr{C}_C(n)}(u,v) \end{bmatrix} \arrow[r, hook, "{(\kappa_2,\bar{\jmath}_{v,\vec{0}},\bar{\jmath}u,v)}"] \arrow[d, hook, "{(\mathrm{Id},\circ)\circ\rho}"] & \begin{bmatrix} \prod_{i=0}^{|v|-1}[0,T]\times\prod_{i=0}^{|u|-1}[-T,T]^{d_i} \\ \times\prod_{i=|v|+1}^{|u|-1}[0,T] \end{bmatrix} \arrow[d, hook, "{\rho\circ(\mathrm{Id},\kappa_3^{|v|})}"] \\
\begin{bmatrix} [0,1]\times\prod_{i=0}^{|u|-1}[-R,R]^{d_i}\times\mathcal{M}_{\mathscr{C}_C(n)}(u,\vec{0}) \end{bmatrix} \arrow[r, hook, "{(\kappa_2,\bar{\jmath}_{u,\vec{0}})}"] & \begin{bmatrix} \prod_{i=0}^{|u|-1}[0,T]\times\prod_{i=0}^{|u|-1}[-T,T]^{d_i} \end{bmatrix}
\end{tikzcd}$$

where ρ throughout denotes some (further) consistent reshuffling of factors, and κ_2 and κ_3^ℓ are from before. The top square commutes automatically; the bottom square commutes since condition (2) of Definition 3.7 holds for $\bar{\jmath}$ as well (because the extension was defined via coherent framings of the normal bundles of $\jmath$).

Therefore, we get a well-defined map from $|\mathscr{C}|$ to $\|\mathscr{C}\|$ which sends the cell $[\mathcal{C}(x)]$ to the corresponding cell $[\mathcal{C}(x)']$ by a degree ± 1 map. It is easy to check that the grading shifts match up, and, therefore, we get the required stable homotopy equivalence. □

4 Functors from the cube to the Burnside category and their realizations

In this section we give a reformulation of cubical flow categories, as 2–functors from the cube category to the Burnside category. We then give a choice-free way of realizing such a functor, in terms of a thickening construction. (A smaller but choice-dependent way to realize such a functor is given in Section 5.) The proof that this realization agrees with the cubical realization is deferred until Section 6. The main results of this section can be summarized in two theorems:

Theorem 5 *The data of a cubical flow category is equivalent to the data of a strictly unitary, lax 2–functor from the cube category to the Burnside category.*

This is stated more precisely and proved as Lemmas 4.18 and 4.20.

Theorem 6 *Given a functor F from the cube category to the Burnside category there is a canonically associated CW spectrum $|F|$, the realization of F.*

This is stated precisely as Construction 4.39 and Lemma 4.42. In Section 6, we show that this construction agrees with the cubical realization (a combination of Theorems 7 and 8). A feature of the realization procedure introduced in this section is that it behaves well with respect to products (Proposition 4.46).

We start by reviewing the Burnside (2–)category and the data required to define a functor from the cube to it, in Section 4.1. We then recall some properties of homotopy colimits, in Section 4.2, before turning to new material in Section 4.3, where we formulate precisely and prove Theorem 5. The thickening and realization procedure are given in Sections 4.4 and 4.5, while invariance under natural isomorphisms of 2–functors is proved in Section 4.6. Section 4.7 discusses the behavior of this realization procedure under certain kinds of products.

4.1 The Burnside category and 2–functors to it

Given sets X and Y, a *correspondence* (or *span*) from X to Y is a set A and maps $s\colon A \to X$ and $t\colon A \to Y$ (for *source* and *target*). Given a correspondence (A, s_A, t_A) from X to Y and (B, s_B, t_B) from Y to Z, the *composition* $(B, s_B, t_B) \circ (A, s_A, t_A)$ is the correspondence (C, s, t) from X to Z given by

$$C = B \times_Y A = \{(b,a) \in B \times A \mid t(a) = s(b)\}, \quad s(b,a) = s_A(a), \quad t(b,a) = t_B(b).$$

Given correspondences (A, s_A, t_A) and (B, s_B, t_B) from X to Y, a *morphism of correspondences* from (A, s_A, t_A) to (B, s_B, t_B) is a bijection of sets $f\colon A \to B$ which commutes with the source and target maps, ie such that $s_A = s_B \circ f$ and $t_A = t_B \circ f$. Composition (of set maps) makes the set of correspondences from X to Y into a category. Further, composition of correspondences makes

(sets, correspondences, morphisms of correspondences)

into a weak 2–category (bicategory in the language of [6]). By the *Burnside category* we mean the sub-2–category of finite sets and finite correspondences. We denote the Burnside category by $\mathscr{B}$. (More typically, one defines the Burnside category of a group G in terms of G–sets and G–equivariant correspondences; for us, G is the trivial group.)

Remark 4.1 Some authors refer to our Burnside category as the *Burnside* 2*–category*, and refer to the 1–category with objects sets and morphisms isomorphism classes of correspondences as the Burnside category.

We will typically drop the maps s and t from the notation, referring simply to a correspondence A from X to Y.

As mentioned above, the Burnside category is a weak 2–category: the identity and associativity axioms only hold up to natural isomorphism. That is, given a set X, the *identity correspondence* of X is simply the set X itself, with the identity map as source and target maps. Given another correspondence A from W to X there is a natural isomorphism

$$\lambda\colon X \times_X A \xrightarrow{\cong} A$$

defined by $\lambda(x, a) = a$. Similarly, given a correspondence B from X to Y there is a natural isomorphism

$$\rho\colon B \times_X X \xrightarrow{\cong} B$$

defined by $\rho(b, x) = b$. Finally, given correspondences A from W to X, B from X to Y and C from Y to Z there is a natural isomorphism

$$\alpha\colon (C \times_Y B) \times_X A \to C \times_Y (B \times_X A)$$

defined by $\alpha((c, b), a) = (c, (b, a))$.

This distinction between weak and strict 2–categories may seem superficial here, but the distinction between weak and strict 2–functors, to which we turn next, will be crucial.

Definition 4.2 Given (weak) 2–categories $\mathscr{C}$ and $\mathscr{D}$, a *lax* 2*–functor* $F\colon \mathscr{C} \to \mathscr{D}$ consists of the following data:

- For each object $x \in \mathrm{Ob}(\mathscr{C})$ an object $F(x) \in \mathrm{Ob}(\mathscr{D})$.
- For each pair of objects $x, y \in \mathrm{Ob}(\mathscr{C})$ a functor

$$F_{x,y}\colon \mathrm{Hom}(x, y) \to \mathrm{Hom}(F(x), F(y)).$$

- For each object $x \in \mathrm{Ob}(\mathscr{C})$ a 2–morphism $F_{\mathrm{Id}_x}\colon \mathrm{Id}_{F(x)} \to F_{x,x}(\mathrm{Id}_x)$.
- For any three objects $x, y, z \in \mathrm{Ob}(\mathscr{C})$ a natural transformation

$$F_{x,y,z}(\cdot,\cdot)\colon F_{y,z}(\cdot) \circ_1 F_{x,y}(\cdot) \to F_{x,z}(\cdot \circ_1 \cdot).$$

(Here, both $F_{y,z}(\,\cdot\,) \circ_1 F_{x,y}(\,\cdot\,)$ and $F_{x,z}(\,\cdot \circ_1 \cdot\,)$ are functors

$$\mathrm{Hom}(y,z) \times \mathrm{Hom}(x,y) \to \mathrm{Hom}(F(x),F(z));$$

$\circ_1$ denotes the 1–composition in $\mathscr{C}$ or $\mathscr{D}$, not the composition of functors.)

These data must satisfy the following compatibility conditions:

(Fn-1) For each pair of objects $x, y \in \mathrm{Ob}(\mathscr{C})$, the following diagrams commute:

$$\begin{tikzcd}
\mathrm{Id}_{F(y)} \circ_1 F_{x,y}(\,\cdot\,) \arrow[r, "\lambda"] \arrow[d, "F_{\mathrm{Id}_y} \circ_1 \mathrm{Id}"'] & F_{x,y}(\,\cdot\,) \\
F_{y,y}(\mathrm{Id}_y) \circ_1 F_{x,y}(\,\cdot\,) \arrow[r, "F_{x,y,y}"'] & F_{x,y}(\mathrm{Id}_y \circ_1 \cdot\,) \arrow[u, "F_{x,y}(\lambda)"']
\end{tikzcd}$$

$$\begin{tikzcd}
F_{x,y}(\,\cdot\,) \circ_1 \mathrm{Id}_{F(x)} \arrow[r, "\rho"] \arrow[d, "\mathrm{Id} \circ_1 F_{\mathrm{Id}_x}"'] & F_{x,y}(\,\cdot\,) \\
F_{x,y}(\,\cdot\,) \circ_1 F_{x,x}(\mathrm{Id}_x) \arrow[r, "F_{x,x,y}"'] & F_{x,y}(\,\cdot \circ_1 \mathrm{Id}_x) \arrow[u, "F_{x,y}(\rho)"']
\end{tikzcd}$$

(Here, λ is the natural isomorphism from $\mathrm{Id}_y \circ_1 \cdot$ to the identity functor of $\mathscr{C}$ or $\mathscr{D}$, as appropriate, and ρ is the natural isomorphism from $\cdot \circ_1 \mathrm{Id}_x$ to the identity functor of $\mathscr{C}$ or $\mathscr{D}$.)

(Fn-2) For each quadruple of objects $x, y, z, w \in \mathrm{Ob}(\mathscr{C})$, the following diagram commutes:

$$\begin{tikzcd}
(F_{y,z}(\,\cdot\,) \circ_1 F_{x,y}(\,\cdot\,)) \circ_1 F_{w,x}(\,\cdot\,) \arrow[r, "\alpha"] \arrow[d, "F_{x,y,z}(\cdot,\cdot) \circ_1 \cdot"'] & F_{y,z}(\,\cdot\,) \circ_1 (F_{x,y}(\,\cdot\,) \circ_1 F_{w,x}(\,\cdot\,)) \arrow[d, "\cdot \circ_1 F_{w,x,y}(\cdot,\cdot)"] \\
F_{x,z}(\,\cdot \circ_1 \cdot\,) \circ_1 F_{w,x}(\,\cdot\,) \arrow[d, "F_{w,x,z}(\cdot \circ_1 \cdot, \cdot)"'] & F_{y,z}(\,\cdot\,) \circ_1 F_{w,y}(\,\cdot \circ_1 \cdot\,) \arrow[d, "F_{w,y,z}(\cdot, \cdot \circ_1 \cdot)"] \\
F_{w,z}((\,\cdot \circ_1 \cdot\,) \circ_1 \cdot\,) \arrow[r, "F_{w,z}(\alpha)"'] & F_{w,z}(\,\cdot \circ_1 (\,\cdot \circ_1 \cdot\,))
\end{tikzcd}$$

(Here, α is the natural isomorphism from $((\,\cdot \circ_1 \cdot\,) \circ_1 \cdot\,)$ to $(\,\cdot \circ_1 (\,\cdot \circ_1 \cdot\,))$, the two different orders of triple compositions, associated to $\mathscr{C}$ or $\mathscr{D}$, as appropriate.)

(See for instance [6, Definition 4.1 and Remark 4.2], where lax 2–functors are called homomorphisms.)

We will often drop the subscript from F: given an element $f \in \mathrm{Hom}_{\mathscr{C}}(x,y)$ and a lax 2–functor $F\colon \mathscr{C} \to \mathscr{D}$ we will often write $F(f)$ for $F_{x,y}(f)$.

Definition 4.3 [6, Remark 4.2] We call a lax 2–functor $F\colon \mathscr{C} \to \mathscr{D}$ *strictly unitary* if for all objects $x \in \mathrm{Ob}(\mathscr{C})$, $F_{x,x}(\mathrm{Id}_x) = \mathrm{Id}_{F(x)}$ and F_{Id_x} is the identity 2–morphism.

As mentioned above, we will be mainly interested in 2–functors from $\underline{2}^n$ to $\mathscr{B}$, which moreover will be strictly unitary. In this case, Definition 4.2 simplifies substantially:

Lemma 4.4 *A strictly unitary, lax 2–functor $F\colon \underline{2}^n \to \mathscr{B}$ is determined by the following data:*

- *For each object $v \in \mathrm{Ob}(\underline{2}^n) = \{0,1\}^n$ the set $X_v = F(v)$.*
- *For each pair of objects $v, w \in \mathrm{Ob}(\underline{2}^n)$ such that $v > w$, a correspondence $A_{v,w} = F(\varphi_{v,w})$ from X_v to X_w.*
- *For each triple of objects $u, v, w \in \mathrm{Ob}(\underline{2}^n)$ such that $u > v > w$, a bijection $F_{u,v,w}\colon A_{v,w} \times_{X_v} A_{u,v} \to A_{u,w}$.*

These data satisfy the compatibility condition:

(CF-1) *For any $u, v, w, x \in \mathrm{Ob}(\underline{2}^n)$ with $u > v > w > x$, the following diagram commutes:*

$$\begin{tikzcd}
A_{w,x} \times_{X_w} A_{v,w} \times_{X_v} A_{u,v} \arrow[r, "\mathrm{Id}\times F_{u,v,w}"] \arrow[d, "F_{v,w,x}\times\mathrm{Id}"'] & A_{w,x} \times_{X_w} A_{u,w} \arrow[d, "F_{u,w,x}"] \\
A_{v,x} \times_{X_v} A_{u,v} \arrow[r, "F_{u,v,x}"'] & A_{u,x}
\end{tikzcd}$$

(We have suppressed some inessential parentheses.)

Moreover, any collection of data satisfying this compatibility condition determines a strictly unitary, lax 2–functor $F\colon \underline{2}^n \to \mathscr{B}$.

Proof Since F is strictly unitary, we have $F(\varphi_{v,v}) = \mathrm{Id}_{X_v}$ (which is X_v, viewed as a correspondence from itself to itself). If $v \not\geq w$ then $\mathrm{Hom}(v,w) = \varnothing$, so $F(\varphi_{v,w})$ is the unique functor from the empty category. Thus, the $F(\varphi_{v,w})$ are entirely specified by the correspondences $A_{v,w} = F_{v,w}(\varphi_{v,w})$ with $v > w$. Next, the source $\mathrm{Hom}(v,w) \times \mathrm{Hom}(u,v)$ of $F_{u,v,w}$ is nonempty if and only if $u \geq v \geq w$, in which case $\mathrm{Hom}(v,w) \times \mathrm{Hom}(u,v)$ consists of the single element $(\varphi_{v,w}, \varphi_{u,v})$. Since $\underline{2}^n$

is a strict 2–category and F is strictly unitary, condition (Fn-1) is equivalent to the statement that $F_{v,v,w}\colon A_{v,w}\times_{X_v}X_v\to A_{v,w}$ is the canonical isomorphism ρ, and $F_{v,w,w}\colon X_w\times_{X_w}A_{v,w}\to A_{v,w}$ is the canonical isomorphism λ. So, F is determined by the specified data. Condition (Fn-2) is equivalent to condition (CF-1); in condition (CF-1) we have abused notation to identify the two sides of the top row of condition (Fn-2), and the bottom arrow in condition (Fn-2) is an equality because $\underline{2}^n$ is a strict 2–category and F is strictly unitary. The result follows. □

Lemma 4.5 *Up to natural isomorphism, a strictly unitary, lax* 2*–functor* $F\colon\underline{2}^n\to\mathscr{B}$ *is determined by the sets* $F(v)$*, the correspondences* $F(\varphi_{v,w})$ *with* $v>w$ *and* $|v|-|w|=1$*, and the maps* $F^{-1}_{u,v',w}\circ F_{u,v,w}\colon F(\varphi_{v,w})\circ F(\varphi_{u,v})\to F(\varphi_{v',w})\circ F(\varphi_{u,v'})$ *with* $u>v,v'>w$ *and* $|u|-|w|=2$*.*

Proof This follows from the observations that:

- Any morphism in $\underline{2}^n$ is a composition of maps associated to edges, so $F(\varphi_{v,w})$ is determined for all $v\geq w$.
- Any two (directed) edge paths from v to w in $\underline{2}^n$ are related by a sequence of swaps across 2–dimensional faces, so $F_{u,v,w}$ is determined for all $u\geq v\geq w$.

Further details are left to the reader. □

4.2 Homotopy colimits and homotopy coherent diagrams

The realization of a 2–functor $\underline{2}^n\to\mathscr{B}$ in this section and Section 5 will involve an iterated mapping cone, which can be described as a homotopy colimit. So, we review briefly the notion of homotopy colimits here.

Given a diagram $F\colon\mathscr{D}\to\mathsf{Top}_\bullet$ of based topological spaces, the *homotopy colimit* of F, $\operatorname{hocolim}F=\operatorname{hocolim}_{\mathscr{D}}F$, is another based topological space. Similarly, if $F\colon\mathscr{D}\to\mathscr{S}$ is a diagram of spectra then we can again form the homotopy colimit of F, $\operatorname{hocolim}F$, which is a spectrum. We will give a construction in a slightly more general setting presently, but first we note some key properties of the homotopy colimit, all of which hold for both diagrams of spaces and diagrams of spectra:

(ho-1) Suppose that $F,G\colon\mathscr{C}\to\mathsf{Top}_\bullet$ are diagrams and $\eta\colon F\to G$ is a natural transformation. Then η induces a map $\operatorname{hocolim}\eta\colon\operatorname{hocolim}F\to\operatorname{hocolim}G$. If $\eta(c)$ is a stable homotopy equivalence for each $c\in\operatorname{Ob}(\mathscr{C})$ then $\operatorname{hocolim}\eta$ is a stable homotopy equivalence as well.

(ho-2) Suppose that $F, G\colon \mathscr{C} \to \mathsf{Top}_\bullet$ are diagrams and $F \vee G\colon \mathscr{C} \to \mathsf{Top}_\bullet$ is the diagram obtained by taking their wedge sum, $(F \vee G)(v) = F(v) \vee G(v)$. Then the natural map $\operatorname{hocolim} F \vee G \to \operatorname{hocolim} F \vee \operatorname{hocolim} G$ is an equivalence.

(ho-3) Suppose that $F\colon \mathscr{C} \to \mathsf{Top}_\bullet$ and $G\colon \mathscr{D} \to \mathsf{Top}_\bullet$. Then there is an induced functor $F \wedge G\colon \mathscr{C} \times \mathscr{D} \to \mathsf{Top}_\bullet$, with $(F \wedge G)(v \times w) = F(v) \wedge G(w)$, and a natural weak equivalence $(\operatorname{hocolim} F) \wedge (\operatorname{hocolim} G) \to \operatorname{hocolim}(F \wedge G)$.

(ho-4) Let $G\colon \mathscr{C} \to \mathscr{D}$ be a map of diagrams (ie a functor between small categories). Given $d \in \mathrm{Ob}(\mathscr{D})$, the *undercategory* of d has objects

$$\{(c, f) \mid c \in \mathscr{C},\ f\colon d \to G(c)\},$$

and $\mathrm{Hom}((c, f), (c', f')) = \{g\colon c \to c' \mid f' = G(g) \circ f\}$. Let $d \downarrow G$ denote the undercategory of d. The functor G is called *homotopy cofinal* if for each $d \in \mathrm{Ob}(\mathscr{D})$, $d \downarrow G$ has contractible nerve.

Now, let $F\colon \mathscr{D} \to \mathsf{Top}_\bullet$ be a diagram. Then there is an induced functor $F \circ G\colon \mathscr{C} \to \mathsf{Top}_\bullet$. Suppose that G is homotopy cofinal. Then

$$\operatorname{hocolim} F \circ G \simeq \operatorname{hocolim} F.$$

(In the case of homotopy limits, this is [7, Cofinality Theorem XI.9.2]. The homotopy colimit can be characterized by knowing that the mapping space *Map*(hocolim F, Z) is equivalent to the homotopy limit of the diagram of spaces *Map*(F, Z).)

In Section 5, we will need to talk about homotopy colimits of diagrams which are only homotopy-commutative, but where the homotopies are part of the data, and are coherent up to higher homotopies (also part of the data). The rest of this subsection focuses on one formulation of this generalization.

We start with an appropriate notion of diagrams:

Definition 4.6 [57, Definition 2.3] A *homotopy coherent diagram in* $\mathsf{Top}_\bullet$ consists of:

- A small category $\mathscr{C}$.
- For each $x \in \mathrm{Ob}(\mathscr{C})$ a space $F(x) \in \mathsf{Top}_\bullet$.
- For each $n \geq 1$ and each sequence

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} x_n$$

of composable morphisms in $\mathscr{C}$ a continuous map

$$F(f_n,\dots,f_1)\colon [0,1]^{n-1}\times F(x_0)\to F(x_n)$$

with $F(f_n,\dots,f_1)([0,1]^{n-1}\times\{*\})=*$, the basepoint in $F(x_n)$.

Letting $(t_1,\dots,t_{n-1})$ denote points in $[0,1]^{n-1}$, these maps F are required to satisfy the conditions

$$\textbf{(4.7)}\quad F(f_n,\dots,f_1)(t_1,\dots,t_{n-1})$$
$$=\begin{cases} F(f_n,\dots,f_2)(t_2,\dots,t_{n-1}) & \text{if } f_1=\mathrm{Id},\\ F(f_n,\dots,f_{i+1},f_{i-1},\dots,f_1)(t_1,\dots,t_{i-1}\cdot t_i,\dots,t_{n-1}) & \\ & \text{if } f_i=\mathrm{Id},\ 1<i<n,\\ F(f_{n-1},\dots,f_1)(t_1,\dots,t_{n-2}) & \text{if } f_n=\mathrm{Id},\\ [F(f_n,\dots,f_{i+1})(t_{i+1},\dots,t_{n-1})]\circ[F(f_i,\dots,f_1)(t_1,\dots,t_{i-1})] & \\ & \text{if } t_i=0,\\ F(f_n,\dots,f_{i+1}\circ f_i,\dots,f_1)(t_1,\dots,t_{i-1},t_{i+1},\dots,t_{n-1}) & \\ & \text{if } t_i=1.\end{cases}$$

(A homotopy coherent diagram is what Vogt [57] calls an $h\mathscr{C}$–diagram. We are restricting to the case that his topological category $\mathscr{C}$ is discrete.)

We will abuse notation and denote a homotopy coherent diagram as above by $F\colon\mathscr{C}\to\mathsf{Top}_\bullet$. It will be clear from context when we mean a commutative diagram or a homotopy coherent diagram.

Example 4.8 Any commutative diagram $F\colon\mathscr{C}\to\mathsf{Top}_\bullet$ can be viewed as a homotopy coherent diagram by defining

$$F(f_n,\dots,f_1)(t_1,\dots,t_{n-1})=F(f_n\circ\cdots\circ f_1).$$

Remark 4.9 Associated to an ordinary category $\mathscr{C}$, there is a simplicially enriched category $\mathfrak{C}[\mathscr{C}]$, introduced by Leitch [32] and further developed by many authors (eg [10; 11; 39]) such that homotopy coherent diagrams $\mathscr{C}\to\mathsf{Top}_\bullet$ are the same (up to the replacement of the continuous function $t_{i-1}\cdot t_i$ by an equivalent piecewise linear one) as simplicial functors from $\mathfrak{C}[\mathscr{C}]$ to spaces. Lemma 3.20 essentially proves that the cube flow category $\mathscr{C}_C(n)$ is the topological category obtained from $\mathfrak{C}[\underline{2}^n]$ by taking the realization of each morphism set. So, homotopy coherent $\underline{2}^n$–diagrams give functors out of $\mathscr{C}_C(n)$.

Definition 4.10 [57, Paragraph (5.10)] Given a homotopy coherent diagram $F\colon \mathscr{C} \to \mathsf{Top}_\bullet$, the *homotopy colimit* of F is defined by

$$\text{hocolim}\, F = \{*\} \amalg \coprod_{n \geq 0} \coprod_{x_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} x_n} [0,1]^n \times F(x_0)/{\sim}, \tag{4.11}$$

where the second coproduct is over n–tuples of composable morphisms in $\mathscr{C}$ and the case $n = 0$ corresponds to the objects $x_0 \in \mathrm{Ob}(\mathscr{C})$. Letting $(t_1, \dots, t_n)$ denote points in $[0,1]^n$ and p a point in $F(x_0)$, the equivalence relation $\sim$ is given by

$$\begin{aligned} &(f_n, \dots, f_1; t_1, \dots, t_n; p) \\ &\sim \begin{cases} (f_n, \dots, f_2; t_2, \dots, t_n; p) & \text{if } f_1 = \mathrm{Id}, \\ (f_n, \dots, f_{i+1}, f_{i-1}, \dots, f_1; t_1, \dots, t_{i-1}t_i, \dots, t_n; p) & \text{if } f_i = \mathrm{Id},\ i > 1, \\ (f_n, \dots, f_{i+1}; t_{i+1}, \dots, t_n; F(f_i, \dots, f_1)(t_1, \dots, t_{i-1}, p)) & \text{if } t_i = 0, \\ (f_n, \dots, f_{i+1} \circ f_i, \dots, f_1; t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n; p) & \text{if } t_i = 1,\ i < n, \\ (f_{n-1}, \dots, f_1; t_1, \dots, t_{n-1}; p) & \text{if } t_n = 1, \\ * & \text{if } p = *, \end{cases} \end{aligned}$$

where $*$ denotes the basepoint.

Observation 4.12 The first three cases in the compatibility condition (4.7) for a homotopy coherent diagram imply that F is determined by its values on sequences of nonidentity morphisms (and on objects). If one restricts to only nonidentity morphisms, however, the compatibility condition for F becomes more complicated. In the special case that $\mathscr{C}$ has no isomorphisms except for identity maps, however, the compatibility condition for sequences $(f_n, \dots, f_1)$ of nonidentity morphisms is simply the last two cases of (4.7).

Similarly, the first two relations in the definition of $\sim$ mean that we can write

$$\text{hocolim}\, F = \{*\} \amalg \coprod_{n \geq 0} \coprod_{\substack{x_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} x_n \\ f_i \neq \mathrm{Id} \text{ for } i \in \{1, \dots, n\}}} [0,1]^n \times F(x_0)/{\sim'}$$

for some equivalence relation $\sim'$, the difference being that we consider only nonidentity morphisms when $n > 0$. The equivalence relation $\sim'$ is more complicated than $\sim$. In the special case that $\mathscr{C}$ has no isomorphisms except for identity maps, $\sim'$ is simply given by the last four cases of the definition of $\sim$.

In the homotopy coherent diagrams and homotopy colimits considered in this paper, the categories $\mathscr{C}$ will have no nonidentity isomorphisms, and so we will work with these smaller formulations.

There is a notion of a morphism between homotopy coherent diagrams (h–morphisms [57, Definition 2.7]) $F, G\colon \mathscr{C} \to \mathsf{Top}_\bullet$, which relaxes the notion of a morphism (natural transformation) between diagrams. In particular, a morphism $F \to G$ of homotopy coherent diagrams includes the data of maps $F(v) \to G(v)$ for each $v \in \mathrm{Ob}(\mathscr{C})$; we call these maps the *underlying maps* of the morphism. There is also the notion of a (simplicial) homotopy of morphisms [57, Definition 2.7], and hence the notion of a homotopy equivalence of homotopy coherent diagrams. A special case is that any morphism of homotopy coherent diagrams whose underlying maps are homotopy equivalences is a homotopy equivalence of diagrams [57, Proposition 4.6]. Further:

Proposition 4.13 [57, Theorem 5.12] *If $F, G\colon \mathscr{C} \to \mathsf{Top}_\bullet$ are homotopy equivalent diagrams then* $\operatorname{hocolim} F \simeq \operatorname{hocolim} G$.

There is also a rectification result, that any homotopy coherent diagram can be made coherent:

Proposition 4.14 [57, Theorem 5.6] *Given any homotopy coherent diagram $F\colon \mathscr{C} \to \mathsf{Top}_\bullet$ there is an honest diagram $G\colon \mathscr{C} \to \mathsf{Top}_\bullet$ which is homotopy equivalent to F.*

Finally:

Proposition 4.15 [57, Section 8] *If $F\colon \mathscr{C} \to \mathsf{Top}_\bullet$ is an honest diagram then the homotopy colimits of F, viewed as an honest diagram and as a homotopy coherent diagram, are homotopy equivalent.*

Corollary 4.16 *Properties* (ho-1)–(ho-4), *or their obvious analogues, hold for homotopy colimits of homotopy coherent diagrams.*

4.3 Cubical flow categories are functors from the cube to the Burnside category

Construction 4.17 Fix a cubical flow category $\mathfrak{f}\colon \mathscr{C} \to \mathscr{C}_C(n)$. We will construct a strictly unitary, lax 2–functor $F\colon \underline{2}^n \to \mathscr{B}$, as follows. By Lemma 4.4, it suffices to define the sets X_v ($v \in \{0,1\}^n$), correspondences $A_{v,w}$ ($v > w \in \{0,1\}^n$) and isomorphisms of correspondences $F_{u,v,w}$ ($u > v > w \in \{0,1\}^n$). We do so as follows:

- Given $v \in \{0,1\}^n$, define $F(v) = \mathfrak{f}^{-1}(v)$.
- Given $v > w$, define $A_{v,w}$ to be the set of path components of
$$\coprod_{\substack{x \in \mathfrak{f}^{-1}(v) \\ y \in \mathfrak{f}^{-1}(w)}} \operatorname{Hom}(x,y).$$
We will write $\pi_0(X)$ for the set of path components of X. Then the source (respectively target) map $s\colon A_{v,w} \to X_v$ (respectively $t\colon A_{v,w} \to X_w$) is defined by $s\big(\pi_0(\operatorname{Hom}(x,y))\big) = x$ $\big($respectively $t\big(\pi_0(\operatorname{Hom}(x,y))\big) = y\big)$.
- Given $u > v > w$ the composition map in $\mathscr{C}$ induces a continuous map
$$\circ\colon \Bigg(\coprod_{\substack{y \in \mathfrak{f}^{-1}(v) \\ z \in \mathfrak{f}^{-1}(w)}} \operatorname{Hom}(y,z)\Bigg) \times_{\mathfrak{f}^{-1}(v)} \Bigg(\coprod_{\substack{x \in \mathfrak{f}^{-1}(u) \\ y \in \mathfrak{f}^{-1}(v)}} \operatorname{Hom}(x,y)\Bigg) \to \coprod_{\substack{x \in \mathfrak{f}^{-1}(u) \\ z \in \mathfrak{f}^{-1}(w)}} \operatorname{Hom}(x,z).$$
Taking path components gives a map $A_{v,w} \times_{X_v} A_{u,v} \to A_{u,w}$, which we define to be $F_{u,v,w}$.

Lemma 4.18 *Construction 4.17 defines a strictly unitary, lax 2–functor.*

Proof By Lemma 4.4, we only need to check the compatibility condition (CF-1), which is immediate from associativity of composition in $\mathscr{C}$. □

Construction 4.19 Fix a strictly unitary, lax 2–functor $F\colon \underline{2}^n \to \mathscr{B}$. We will construct a cubical flow category $\mathfrak{f}\colon \mathscr{C} \to \mathscr{C}_C(n)$, as follows:

- $\operatorname{Ob}(\mathscr{C}) = \coprod_{v \in \{0,1\}^n} F(v)$. The functor $\mathfrak{f}$ sends an object $x \in F(v)$ to v.
- For any object x, $\operatorname{Hom}(x,x)$ consists of the identity morphism.
- Given objects x and y, with $v = \mathfrak{f}(x) > \mathfrak{f}(y) = w$, consider the set
$$B_{x,y} = s^{-1}(x) \cap t^{-1}(y) \subset A_{v,w} = F(\varphi_{v,w}).$$
Define $\operatorname{Hom}(x,y) = B_{x,y} \times \mathcal{M}_{\mathscr{C}_C(n)}(v,w)$. The map
$$\mathfrak{f}\colon \operatorname{Hom}(x,y) \to \operatorname{Hom}(\mathfrak{f}(x), \mathfrak{f}(y))$$
is projection to the permutohedron $\mathcal{M}_{\mathscr{C}_C(n)}(v,w)$.

- Given objects x, y and z with $\mathfrak{f}(x) > \mathfrak{f}(y) > \mathfrak{f}(z)$ define the composition map $\mathrm{Hom}(y,z) \times \mathrm{Hom}(x,y) \to \mathrm{Hom}(x,z)$ as follows. Let $u = \mathfrak{f}(x)$, $v = \mathfrak{f}(y)$ and $w = \mathfrak{f}(z)$. The 2–functor includes a map $F_{u,v,w}\colon A_{v,w} \times_{F(v)} A_{u,v} \to A_{u,w}$. The composition map in $\mathscr{C}_C(n)$ gives a map $\circ\colon \mathcal{M}_{\mathscr{C}_C(n)}(v,w) \times \mathcal{M}_{\mathscr{C}_C(n)}(u,v) \to \mathcal{M}_{\mathscr{C}_C(n)}(u,w)$. Define the composition map in $\mathscr{C}$ to be
$$F_{u,v,w} \times \circ\colon (B_{y,z} \times \mathcal{M}_{\mathscr{C}_C(n)}(v,w)) \times (B_{x,y} \times \mathcal{M}_{\mathscr{C}_C(n)}(u,v)) \\ \to (B_{x,z} \times \mathcal{M}_{\mathscr{C}_C(n)}(u,w)).$$
(That is, we apply $F_{u,v,w}$ to the B–factors and $\circ$ to the $\mathcal{M}$–factors.)

Lemma 4.20 *Construction 4.19 defines a cubical flow category.*

Proof The proof is similar to the proof of Lemma 3.17, and is left to the reader. □

It is straightforward to verify that Constructions 4.17 and 4.19 are inverse to each other, in a sense which does not seem worth spelling out precisely.

Example 4.21 Applying Construction 4.17 to the Khovanov flow category from Example 3.24 yields a functor $F_{Kh}\colon \underline{2}^n \to \mathscr{B}$. For any $v \in \{0,1\}^n$, the set $F_{Kh}(v)$ consists of the Khovanov generators over v, denoted by $F(v)$ in Section 2.2. For $u > v \in \{0,1\}^n$ with $|u| - |v| = 1$, and for $x \in F_{Kh}(u)$ and $y \in F_{Kh}(v)$, the set
$$B_{x,y} = s^{-1}(x) \cap t^{-1}(y) \subseteq A_{u,v} = F_{Kh}(\varphi_{u,v})$$
consists of one element if x appears in $\delta_{Kh}(y)$ (see Definition 2.1), and is empty otherwise. The maps $F_{u,v,w}$ when $|u| - |w| = 2$ are defined using the *ladybug matching* [35, Section 5.4]; see also Section 8.1.

Remark 4.22 In Section 2.2 we discussed a generalization of the Khovanov theory that works over the ring $\mathbb{Z}[h,t]$: a functor from $(\underline{2}^n)^{\mathrm{op}}$ to the category of (graded) $\mathbb{Z}[h,t]$–modules. Setting $h = t = 0$ recovers Khovanov's original functor to $\mathbb{Z}$–Mod [26], setting $(h,t) = (0,1)$ gives the theory studied by Lee [31], and setting $(h,t) = (1,0)$ gives a theory introduced by Bar-Natan [4].

There is a functor $(\mathscr{B})^{\mathrm{op}} \to \mathbb{Z}$–$\mathsf{Mod}$ given as follows: to a set X, associate the abelian group $\mathbb{Z}\langle X\rangle$ freely generated by the elements of X; to a correspondence (A,s,t) from X to Y, associate the map $\mathbb{Z}\langle Y\rangle \to \mathbb{Z}\langle X\rangle$,

$$y \mapsto \sum_{x \in X} \#\bigl|\{a \in A \mid s(a) = x,\ t(a) = y\}\bigr| x. \tag{4.23}$$

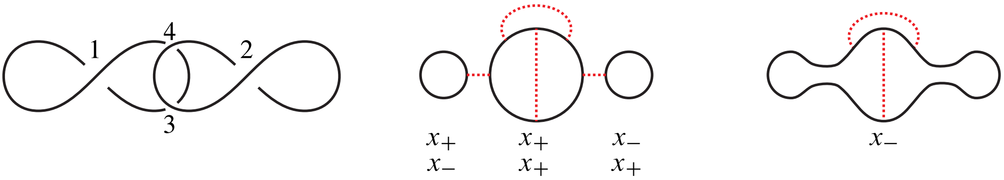


Figure 4: An example showing that the Lee complex does not come from a functor $\underline{2}^n \to \mathscr{B}$ that extends F_{Kh}. Left: A particular diagram for the two-component unlink, and an ordering of its crossings. Center: The corresponding resolution configuration (the $\vec{0}$–resolution with dashed lines recording the crossings) and two labelings of this resolution. Both labelings are in the image of $F(\varphi_{1100,0000})$ of the labeling x_- of the circle in the $(1,1,0,0)$–resolution (right). Further, the two labelings give incompatible restrictions on the map $F^{-1}_{(1111,1110,1100)} \circ F_{(1111,1101,1100)}$ associated to the subcube $(1,1,*,*)$.

Example 4.21 lifts Khovanov's functor $(\underline{2}^n)^{\mathrm{op}} \to \mathbb{Z}\text{–Mod}$ to the functor $F_{Kh}\colon \underline{2}^n \to \mathscr{B}$. It is natural to ask whether any other specializations of h and t comes from a strictly unitary, lax 2–functor $\underline{2}^n \to \mathscr{B}$. Any candidate must have $h = 0$ and $t \in \mathbb{N}$, since the coefficients in (4.23) need to be positive and integral. The special case $(h,t) = (0,1)$ (ie Lee's theory) does not come from any functor $\underline{2}^n \to \mathscr{B}$ for arbitrary link diagrams extending the functor F_{Kh}, as can be seen by considering the diagram in Figure 4. The question of whether there is such an extension for $h = 0$ and $t > 1$ is, as far as we know, open.

4.4 The thickened diagram

Fix a small category $\mathscr{D}$, which we regard as a strict 2–category whose only 2–morphisms are identity maps. Fix also a strictly unitary, lax 2–functor $F\colon \mathscr{D} \to \mathscr{B}$, ie a $\mathscr{D}$–diagram in $\mathscr{B}$. In this section, we will associate to $(\mathscr{D}, F)$ a new (1–)category $\widehat{\mathscr{D}}$ and, for each $k \geq 1$, an (honest) functor $\widehat{F}_k\colon \widehat{\mathscr{D}} \to \mathsf{Top}_\bullet$, the category of based topological spaces. There will also be natural transformations $\Sigma \circ \widehat{F}_k \to \widehat{F}_{k+1}$ (where Σ denotes suspension), so that we get an induced diagram $\widehat{F}\colon \widehat{\mathscr{D}} \to \mathscr{S}$, the category of symmetric spectra. To realize a functor from the cube category to the Burnside category we will apply this construction and then take an iterated mapping cone; see Section 4.5.

We start by defining $\widehat{\mathscr{D}}$:

Definition 4.24 Let $\mathscr{D}$ be a small category. The *thickening* of $\mathscr{D}$ is the small category $\widehat{\mathscr{D}}$ defined as follows:

- The objects of $\widehat{\mathscr{D}}$ are composable pairs of morphisms $u \xrightarrow{f} v \xrightarrow{g} w$ in $\mathscr{D}$.

- The morphisms in $\widehat{\mathscr{D}}$ are commutative diagrams: given composable pairs $u \xrightarrow{f} v \xrightarrow{g} w$ and $u' \xrightarrow{f'} v' \xrightarrow{g'} w'$,

  $$\operatorname{Hom}((f,g),(f',g')) = \{(\alpha\colon u \to u', \beta\colon v' \to v, \gamma\colon w \to w') \mid \mathcal{D} \text{ commutes}\},$$

  where $\mathcal{D}$ is the diagram

  $$\begin{tikzcd} u \arrow[r, "f"] \arrow[d, "\alpha"] & v \arrow[r, "g"] & w \arrow[d, "\gamma"] \\ u' \arrow[r, "f'"] & v' \arrow[r, "g'"] \arrow[u, "\beta"] & w' \end{tikzcd}$$

  (Note the direction of the middle vertical arrow.)

- Composition of morphisms is given by stacking diagrams vertically:

  $$(\alpha',\beta',\gamma') \circ (\alpha,\beta,\gamma) = (\alpha' \circ \alpha, \beta \circ \beta', \gamma' \circ \gamma).$$

Example 4.25 For $\mathscr{D} = \underline{2}^1$, $\widehat{\mathscr{D}}$ has four objects: $1 \to 1 \to 1$, $1 \to 1 \to 0$, $1 \to 0 \to 0$ and $0 \to 0 \to 0$. There are unique morphisms

$$(1 \to 1 \to 1) \to (1 \to 1 \to 0) \leftarrow (1 \to 0 \to 0) \to (0 \to 0 \to 0).$$

(Again, note the direction of the middle arrow.)

The thickening operation respects products:

Lemma 4.26 *Given small categories $\mathscr{C}$ and $\mathscr{D}$, there is an isomorphism $q\colon \widehat{\mathscr{C}} \times \widehat{\mathscr{D}} \xrightarrow{\cong} \widehat{\mathscr{C} \times \mathscr{D}}$ given on objects by*

$$q\big([u_C \xrightarrow{f_C} v_C \xrightarrow{g_C} w_C] \times [u_D \xrightarrow{f_D} v_D \xrightarrow{g_D} w_D]\big) = [u_C \times u_D \xrightarrow{f_C \times f_D} v_C \times v_D \xrightarrow{g_C \times g_D} w_C \times w_D].$$

Proof This is immediate from the definitions. □

Next we define $\widehat{F}_k$.

Definition 4.27 Let $\mathscr{D}$ be a small category, which we regard as a 2–category with only identity 2–morphisms and let $F\colon \mathscr{D} \to \mathscr{B}$ be a strictly unitary, lax 2–functor. Given an integer $k \geq 0$ we define a new functor $\widehat{F}_k\colon \widehat{\mathscr{D}} \to \mathsf{Top}_\bullet$ (of 1–categories) as follows.

On objects, let

$$\widehat{F}_k(u \xrightarrow{f} v \xrightarrow{g} w) = \bigvee_{a \in F(f)} \prod_{\substack{b \in F(g) \\ s(b)=t(a)}} S^k.$$

(Here, $s\colon F(g) \to F(v)$ and $t\colon F(f) \to F(v)$ are the source and target maps of the correspondences $F(g)$ and $F(f)$, respectively.)

To define $\widehat{F}_k$ on morphisms fix a commutative diagram

(4.28)

$$\begin{tikzcd} u \arrow[r, "f"] \arrow[d, "\alpha"] & v \arrow[r, "g"] & w \arrow[d, "\gamma"] \\ u' \arrow[r, "f'"] & v' \arrow[r, "g'"] \arrow[u, "\beta"] & w' \end{tikzcd}$$

We must construct a map

(4.29)

$$\bigvee_{a \in F(f)} \prod_{\substack{b \in F(g) \\ s(b)=t(a)}} S^k \to \bigvee_{a' \in F(f')} \prod_{\substack{b' \in F(g') \\ s(b')=t(a')}} S^k.$$

It suffices to construct this map one a at a time, so fix $a \in F(f)$. The maps $F_{u,u',v'}$ and $F_{u,v',v}$ induce a bijection

$$F(f) \cong F(\beta) \times_{F(v')} F(f') \times_{F(u')} F(\alpha);$$

let $(y, a', x) \in F(\beta) \times_{F(v')} F(f') \times_{F(u')} F(\alpha)$ be the triple corresponding to a. The map $\widehat{F}_k$ will send $\prod_{b \in F(g),\, s(b)=t(a)} S^k$, the summand corresponding to a, to $\prod_{b' \in F(g'),\, s(b')=t(a')} S^k$, the summand corresponding to a'.

Next, the maps $F_{v',v,w}$ and $F_{v',w,w'}$ induce a bijection

$$F(g') \cong F(\gamma) \times_{F(w)} F(g) \times_{F(v)} F(\beta).$$

Consider the map

(4.30)

$$\prod_{\substack{b \in F(g) \\ s(b)=t(a)}} S^k \xrightarrow{\prod_b \Delta_b} \prod_{\substack{b \in F(g) \\ s(b)=t(a)}} \prod_{\substack{b'=(z,\tilde{b},\tilde{y}) \in F(g') \\ \tilde{b}=b \\ \tilde{y}=y}} S^k \cong \prod_{\substack{b' \in F(g') \\ b'=(z,b,y) \\ s(b)=t(a)}} S^k,$$

where Δ_b is the diagonal map $S^k \to \prod_{b'} S^k$. Notice that

$$\{b' \in F(g') \mid b' = (z, b, y),\ s(b) = t(a)\}$$

$$\begin{tikzcd}
 & & & F(f) \arrow[dl] \arrow[dr] & & & & F(g') \arrow[dl] \arrow[dr] & & & \\
 & & {} \arrow[dl] \arrow[dr] & & {} \arrow[dl] \arrow[dr] & & {} \arrow[dl] \arrow[dr] & & {} \arrow[dl] \arrow[dr] & & \\
 & F(\alpha) \arrow[dl] \arrow[dr] & & F(f') \arrow[dl] \arrow[dr] & & F(\beta) \arrow[dl] \arrow[dr] & & F(g) \arrow[dl] \arrow[dr] & & F(\gamma) \arrow[dl] \arrow[dr] & \\
F(u) & & F(u') & & F(v') & & F(v) & & F(w) & & F(w')
\end{tikzcd}$$

Figure 5: Constructing the functor from the thickened diagram. In the first row of arrows, leftwards arrows are source maps, and rightwards arrows are target maps. All squares are fiber products. To define the map (4.30), fix $a \in F(f)$ and $b \in F(g)$. The element a specifies $a' \in F(f')$ and $y \in F(\beta)$. We then consider the elements $b' \in F(g')$ which map down to y and b. The diagonal map Δ_b corresponds to this set. Any such b' has source equal to the target of a'.

is a subset of $\{b' \in F(g') \mid s(b') = t(a')\}$ since $s(b') = s(y) = t(a')$. We can extend the map (4.30) to a map

$$\prod_{\substack{b \in F(g) \\ s(b)=t(a)}} S^k \to \prod_{\substack{b' \in F(g') \\ s(b')=t(a')}} S^k$$

by mapping to the basepoint in the remaining factors. This is the desired map.

(It can be helpful to think of this argument diagrammatically; see Figure 5.)

Lemma 4.31 *Definition 4.27 defines a functor $\widehat{F}_k$ whose values have natural actions of the symmetric group $\mathcal{S}_k$.*

We omit the proof, which is straightforward, albeit a bit elaborate.

Example 4.32 Consider the functor $F\colon \underline{2}^1 \to \mathscr{B}$ given by $F(1) = \{x, y\}$, $F(0) = \{z, w\}$ and $F(\varphi_{1,0}) = \{a, b, c, d\}$ with $s(a) = x$, $s(b) = s(c) = s(d) = y$, $t(a) = t(b) = t(c) = z$ and $t(d) = w$. Graphically, F is given by

$$\begin{tikzcd}
1: & x \arrow[d, "a"] & & y \arrow[dll, "{b,c}"'] \arrow[d, "d"] \\
0: & z & & w
\end{tikzcd}$$

Recall the thickening $\widehat{\underline{2}^1}$ from Example 4.25. The induced diagram $\widehat{F}_k\colon \widehat{\underline{2}^1} \to \mathsf{Top}_\bullet$ is given by

$$S^k_{x,x} \vee S^k_{y,y} \xrightarrow{\mathrm{Id} \vee \Delta} (S^k_{x,a}) \vee (S^k_{y,b} \times S^k_{y,c} \times S^k_{y,d}) \leftarrow S^k_{a,z} \vee S^k_{b,z} \vee S^k_{c,z} \vee S^k_{d,w} \to S^k_{z,z} \vee S^k_{w,w}.$$

(Here, for instance, the sphere $S^k_{x,a}$ corresponds to the pair $(x,a)\in F(\varphi_{1,1})\times F(\varphi_{1,0})$ over the object $1\to 1\to 0$.) We claim that the second map is the inclusion of the k–skeleton. To see this, note that it corresponds to the diagram

$$\begin{tikzcd}
1 \arrow[r, "\varphi_{1,0}"] \arrow[d, "\varphi_{1,1}"] & 0 \arrow[r, "\varphi_{0,0}"] & 0 \arrow[d, "\varphi_{0,0}"] \\
1 \arrow[r, "\varphi_{1,1}"] & 1 \arrow[r, "\varphi_{1,0}"] \arrow[u, "\varphi_{1,0}"] & 0
\end{tikzcd}$$

The map decomposes along wedge sums. For b, for instance, we get the map $S^k_{b,z}\to(S^k_{y,b}\times S^k_{y,c}\times S^k_{y,d})$ as follows:

$$\prod_{\{z\}} S^k = \Biggl[\prod_{\substack{p\in F(\varphi_{0,0})\\ s(p)=t(b)}} S^k\Biggr] \xrightarrow{\prod\Delta} \Biggl[\prod_{\substack{q'\in F(\varphi_{1,0})\\ q'=(r,p,b)\in F(\varphi_{0,0})\times F(\varphi_{0,0})\times F(\varphi_{1,0})\\ s(p)=t(b)}} S^k\Biggr] = \prod_{\{b\}} S^k \hookrightarrow \prod_{\{b,c,d\}} S^k.$$

Similarly, the third map sends $S^k_{a,z}$, $S^k_{b,z}$ and $S^k_{c,z}$ by the identity map to $S^k_{z,z}$ and $S^k_{d,w}$ by the identity map to $S^k_{w,w}$.

Finally, we show that the $\widehat{F}_k$ induce a functor $\widehat{F}\colon \widehat{\mathscr{D}}\to\mathscr{S}$. Given a based space X, identify $S^n\wedge X$ as $[0,1]^n\times X/(\partial[0,1]^n\times X\cup[0,1]^n\times\{*\})$, where $*$ is the basepoint in X. Given $n\geq 0$ and based spaces X_i, there is a canonical map

$$\sigma^n\colon S^n\wedge\prod_i X_i\to\prod_i S^n\wedge X_i,\qquad (y,x_1,\dots,x_n)\mapsto((y,x_1),\dots,(y,x_n)).$$

Given $k,n\geq 0$, define a map

$$\textbf{(4.33)}\qquad \eta(u\xrightarrow{f}v\xrightarrow{g}w)\colon S^n\wedge\widehat{F}_k(u\xrightarrow{f}v\xrightarrow{g}w)\to\widehat{F}_{n+k}(u\xrightarrow{f}v\xrightarrow{g}w)$$

as the composition

$$\textbf{(4.34)}\quad S^n\wedge\widehat{F}_k(u\xrightarrow{f}v\xrightarrow{g}w)=S^n\wedge\Biggl(\bigvee\nolimits_{a\in F(f)}\prod_{\substack{b\in F(g)\\ s(b)=t(a)}} S^k\Biggr)\cong\bigvee\nolimits_{a\in F(f)} S^n\wedge\prod_{\substack{b\in F(g)\\ s(b)=t(a)}} S^k$$
$$\xrightarrow{\bigvee_{a\in F(f)}\sigma^n}\bigvee\nolimits_{a\in F(f)}\prod_{\substack{b\in F(g)\\ s(b)=t(a)}} S^{n+k}=\widehat{F}_{k+n}(u\xrightarrow{f}v\xrightarrow{g}w).$$

Lemma 4.35 *The maps η are $\mathcal{S}_n \times \mathcal{S}_k$–equivariant and form a natural transformation, in the sense that for each $u \xrightarrow{f} v \xrightarrow{g} w$ the square*

$$\begin{tikzcd}[ampersand replacement=\&]
S^n \wedge \widehat{F}_k(u \xrightarrow{f} v \xrightarrow{g} w) \arrow[r, "S^n \wedge \widehat{F}_k\left(\begin{smallmatrix} u \& \to \& v \& \to \& w \\ \downarrow \& \& \uparrow \& \& \downarrow \\ u' \& \to \& v' \& \to \& w' \end{smallmatrix}\right)"] \arrow[d, "\eta(u \xrightarrow{f} v \xrightarrow{g} w)"'] \& S^n \wedge \widehat{F}_k(u' \xrightarrow{f'} v' \xrightarrow{g'} w') \arrow[d, "\eta(u' \xrightarrow{f'} v' \xrightarrow{g'} w')"] \\
\widehat{F}_{k+n}(u \xrightarrow{f} v \xrightarrow{g} w) \arrow[r, "\widehat{F}_{k+n}\left(\begin{smallmatrix} u \& \to \& v \& \to \& w \\ \downarrow \& \& \uparrow \& \& \downarrow \\ u' \& \to \& v' \& \to \& w' \end{smallmatrix}\right)"'] \& \widehat{F}_{k+n}(u' \xrightarrow{f'} v' \xrightarrow{g'} w')
\end{tikzcd}$$

commutes. Further, each of the maps $\eta(u \xrightarrow{f} v \xrightarrow{g} w)$ induces an isomorphism on π_i for $0 \le i \le 2k-2$.

Proof The proof is straightforward, and is left to the reader. □

Corollary 4.36 *The functors $\widehat{F}_k$ and natural transformations η assemble to define a diagram of symmetric spectra $\widehat{F}\colon \widehat{\mathscr{D}} \to \mathscr{S}$.*

Remark 4.37 Lemma 4.35 amounts to a verification that we can more concisely express $\widehat{F}$ within the category of spectra itself by the formula

$$\widehat{F}(u \xrightarrow{f} v \xrightarrow{g} w) = \bigvee_{a \in F(f)} \prod_{\substack{b \in F(g) \\ s(b)=t(a)}} \mathbb{S}.$$

The wedge and product play the roles of coproduct and product in this category.

Remark 4.38 The construction of the thickened functor $\widehat{F}_k\colon \widehat{\mathscr{D}} \to \mathsf{Top}_\bullet$ from a functor $F\colon \mathscr{D} \to \mathscr{B}$ would work with any based space X_k in place of the sphere S^k: the construction only uses properties of the wedge sum and product operations. Further, if the spaces X_k form a sequential spectrum then there is a corresponding thickened diagram in spectra. However, this construction is scarcely more general than the one we gave: in spectra, the diagram coming from (X_k) is naturally equivalent to the smash product of the spectrum X and the diagram obtained from the sphere spectrum (S^k).

4.5 The realization

Construction 4.39 Let $\widehat{\underline{2}^n_+}$ be the category obtained from $\widehat{\underline{2}^n}$ by adding a new object $*$ and a unique morphism $(u \to v \to w) \to *$ from each vertex $(u \to v \to w)$ of $\widehat{\underline{2}^n}$ with $w \neq \vec{0}$.

Given a functor $F\colon \underline{2}^n \to \mathscr{B}$, define $\widehat{F}^+\colon \widehat{\underline{2}^n_+} \to \mathscr{S}$ by $\widehat{F}^+|_{\widehat{\underline{2}^n}} = \widehat{F}$ and $\widehat{F}^+(*) = \{\mathrm{pt}\}$, a single point.

Let $|F|$ be the homotopy colimit of $\widehat{F}^+$. We call $|F|$ the *realization of F*.

Example 4.40 Continuing with Example 4.32, we have

$$|F| = \operatorname{hocolim}\left(\begin{tikzcd} \mathbb{S}_{x,x} \vee \mathbb{S}_{y,y} \arrow[r] \arrow[d] & (\mathbb{S}_{x,a}) \vee (\mathbb{S}_{y,b} \times \mathbb{S}_{y,c} \times \mathbb{S}_{y,d}) & \mathbb{S}_{a,z} \vee \mathbb{S}_{b,z} \vee \mathbb{S}_{c,z} \vee \mathbb{S}_{d,w} \arrow[l] \arrow[d] \\ \{\mathrm{pt}\} & & \mathbb{S}_{z,z} \vee \mathbb{S}_{w,w} \end{tikzcd}\right),$$

where $\mathbb{S}$ denotes the sphere spectrum.

Instead of $\widehat{\underline{2}^n_+}$, it will be convenient, sometimes, to work with the larger enlargement $\widehat{\underline{2}^n_\dagger} = (\widehat{\underline{2}^1_+})^n$ of $\widehat{\underline{2}^n}$. Given a functor $F\colon \underline{2}^n \to \mathscr{B}$ extend $\widehat{F}$ to a functor $\widehat{F}^\dagger\colon \widehat{\underline{2}^n_\dagger} \to \mathscr{S}$ by setting $\widehat{F}^\dagger|_{\widehat{\underline{2}^n}} = \widehat{F}$ and $\widehat{F}^\dagger(d) = \{\mathrm{pt}\}$ if $d \notin \mathrm{Ob}(\widehat{\underline{2}^n})$.

Lemma 4.41 *For any functor $F\colon \underline{2}^n \to \mathscr{B}$ there is a stable homotopy equivalence* $\operatorname{hocolim} \widehat{F}^+ \simeq \operatorname{hocolim} \widehat{F}^\dagger$.

Proof Consider the functor $G\colon \widehat{\underline{2}^n_\dagger} \to \widehat{\underline{2}^n_+}$ which is the identity on $\widehat{\underline{2}^n}$ and sends all objects not in $\widehat{\underline{2}^n}$ to $*$. We claim that G is homotopy cofinal. To see this, we divide the computation of the undercategories into three cases:

(a) The undercategory $* \downarrow G$ of $*$ is the full subcategory of $\widehat{\underline{2}^n_\dagger}$ spanned by the objects not in $\widehat{\underline{2}^n}$. The nerve of this category is homeomorphic to
$$\{\vec{x} = (x_1, \dots, x_n) \in [0,1]^n \mid x_i = 0 \text{ for some } i\},$$
which is contractible.

(b) For any object $d = (u \to v \to \vec{0})$ of $\widehat{\underline{2}^n}$, the undercategory $d \downarrow G$ of d is the full subcategory of $\widehat{\underline{2}^n}$ of objects d' for which there is a map $d \to d'$. The object d is an initial object for this subcategory, so the nerve is contractible.

(c) Fix an object $d = (u \to v \to w)$ of $\widehat{\underline{2}^n}$ with $w \neq \vec{0}$. The undercategory $d \downarrow G$ of d is the full subcategory of $\widehat{\underline{2}^n_\dagger}$ spanned by the objects d' not in $\widehat{\underline{2}^n}$ and the objects d' in $\widehat{\underline{2}^n}$ for which there is a map $d \to d'$. Let $\mathscr{D}$ be the full subcategory of $\widehat{\underline{2}^n_\dagger}$ spanned by the objects d' not in $\widehat{\underline{2}^n}$ and let $\mathscr{E}$ be the full subcategory of $\widehat{\underline{2}^n_\dagger}$ consisting of objects d' for which there is a morphism $d \to d'$. The categories $\mathscr{D}$ and $\mathscr{E}$ are each downwards closed in $d \downarrow G$, ie there are no morphisms out of $\mathscr{D}$ or $\mathscr{E}$. Thus, the nerve of $d \downarrow G$ is the union of the nerves of $\mathscr{D}$ and $\mathscr{E}$, glued along the nerve of $\mathscr{D} \cap \mathscr{E}$. We already saw in part (a) that the nerve of $\mathscr{D}$ is contractible. The category $\mathscr{E}$ has an initial object, d, and hence the nerve of $\mathscr{E}$ is contractible. Finally, the realization of the category $\mathscr{D} \cap \mathscr{E}$ is similar to the realization of $\mathscr{D}$ (ie a union of coordinate hyperplanes), and so contractible. It follows that the realization of $d \downarrow G$ is contractible.

Thus, the functor G is homotopy cofinal. It is immediate from the definitions that $\widehat{F}^{\dagger} = \widehat{F}^{+} \circ G$, so the result follows from property (ho-4) of homotopy colimits (Section 4.2). □

4.6 An invariance property of the realization

Lemma 4.42 *If $F, G\colon \underline{2}^n \to \mathscr{B}$ are naturally isomorphic 2–functors then the realizations of F and G are stably homotopy equivalent.*

Proof A natural isomorphism T from F to G specifies

- a bijection $T_v\colon F(v) \to G(v)$ for each $v \in \{0,1\}^n$, and
- a bijection $T_{v,w}\colon F(\varphi_{v,w}) \to G(\varphi_{v,w})$ for each $v > w \in \{0,1\}^n$

satisfying the conditions that

$$\begin{tikzcd}
& F(\varphi_{v,w}) \arrow[rrr, "T_{v,w}"] \arrow[dl, "s"'] \arrow[dr, "t"] & & & G(\varphi_{v,w}) \arrow[dl, "s"'] \arrow[dr, "t"] & \\
F(v) \arrow[rrr, bend right, "T_v"'] & & F(w) \arrow[rrr, bend right, "T_w"'] & G(v) & & G(w)
\end{tikzcd}$$

and

$$\begin{tikzcd}
F(\varphi_{v,w}) \times_{F(v)} F(\varphi_{u,v}) \arrow[r, "F_{u,v,w}"] \arrow[d, "T_{v,w} \times T_{u,v}"] & F(\varphi_{u,w}) \arrow[d, "T_{u,w}"] \\
G(\varphi_{v,w}) \times_{G(v)} G(\varphi_{u,v}) \arrow[r, "G_{u,v,w}"'] & G(\varphi_{u,w})
\end{tikzcd}$$

commute. (See [19, Section I,2.4] and note that an isomorphism in $\mathscr{B}$ between two sets induces a bijection between them.) Given a natural transformation T, there is a corresponding map of diagrams defined as follows. Given $u > v > w$ we want a map

$$\bigvee_{a \in F(\varphi_{u,v})} \prod_{\substack{b \in F(\varphi_{v,w}) \\ s(b)=t(a)}} \mathbb{S} \to \bigvee_{a \in G(\varphi_{u,v})} \prod_{\substack{b \in G(\varphi_{v,w}) \\ s(b)=t(a)}} \mathbb{S}.$$

This map sends the wedge summand corresponding to $a \in F(\varphi_{u,v})$ to the wedge summand corresponding to $T_{u,v}(a)$, and sends the factor corresponding to $b \in F(\varphi_{v,w})$ to the factor corresponding to $T_{v,w}(b)$. It is straightforward to verify that this gives a map of diagrams, which is by definition an isomorphism. It follows that the map induces a stable homotopy equivalence of homotopy colimits. (In fact, if we worked with diagrams of S^k's, the map would be a homeomorphism.) $\square$

4.7 Products and realization

In the language of flow categories, the product is a rather complicated object. In the language of functors from the cube to the Burnside category, however, the product is quite simple:

Definition 4.43 There is a map $\times\colon \mathscr{B}\times\mathscr{B} \to \mathscr{B}$ which sends a pair of objects (X, Y) to their direct product $X \times Y$, a pair of correspondences $((A, s_A, t_A), (B, s_B, t_B))$ to $(A \times B, s_A \times s_B, t_A \times t_B)$, and a pair of isomorphisms $(F\colon A \to A',\ G\colon B \to B')$ of correspondences to the isomorphism $F \times G\colon A \times B \to A' \times B'$. Given functors $F\colon \underline{2}^m \to \mathscr{B}$ and $G\colon \underline{2}^n \to \mathscr{B}$, we define the *product* $F \times G$ of F and G to be the composition

$$\underline{2}^{m+n} \xrightarrow{\cong} \underline{2}^m \times \underline{2}^n \xrightarrow{(F,G)} \mathscr{B}\times\mathscr{B} \xrightarrow{\times} \mathscr{B}.$$

Explicitly, $(F \times G)\colon \underline{2}^{m+n} \to \mathscr{B}$ is given as follows. Identify $\{0,1\}^m \times \{0,1\}^n$ with $\{0,1\}^{m+n}$. Then:

- For $(u_1, u_2) \in \{0,1\}^m \times \{0,1\}^n = \{0,1\}^{m+n}$ define
$$(F \times G)(u_1, u_2) = F(u_1) \times G(u_2).$$
- For $(u_1, u_2) > (v_1, v_2) \in \{0,1\}^{m+n}$ define $(F \times G)(\varphi_{(u_1,u_2),(v_1,v_2)})$ to be the correspondence
$$\begin{tikzcd} & F(\varphi_{u_1,v_1}) \times G(\varphi_{u_2,v_2}) \arrow[dl, "s_F \times s_G"'] \arrow[dr, "t_F \times t_G"] & \\ F(u_1) \times G(u_2) & & F(v_1) \times G(v_2) \end{tikzcd}$$
- Define $(F \times G)_{(u_1,u_2),(v_1,v_2),(w_1,w_2)}$ for $(u_1, u_2) > (v_1, v_2) > (w_1, w_2) \in \{0,1\}^{m+n}$ by
$$(F \times G)_{(u_1,u_2),(v_1,v_2),(w_1,w_2)}(x, y) = (F_{u_1,v_1,w_1}(x), G_{v_1,v_2,v_3}(y)).$$

Lemma 4.44 *Definition 4.43 specifies a strictly unitary, lax 2–functor.*

Proof This is immediate. □

Note that smash products distribute across wedge sums. Moreover, while $X\wedge(Y\times Z)$ is not homotopy equivalent to $(X\wedge Y)\times(X\wedge Z)$, there is a natural map $X\wedge(Y\times Z)\to(X\wedge Y)\times(X\wedge Z)$ defined by $(x,y,z)\mapsto((x,y),(x,z))$. This generalizes to a map

$$p\colon\Bigl(\prod_{a\in A}X_a\Bigr)\wedge\Bigl(\prod_{b\in B}Y_b\Bigr)\to\prod_{(a,b)\in A\times B}X_a\wedge Y_b.$$

The map p is natural in both factors.

Given functors $F\colon\mathscr{C}\to\mathsf{Top}_\bullet$ and $G\colon\mathscr{D}\to\mathsf{Top}_\bullet$ we can take the smash product of F and G to obtain a functor $F\wedge G\colon\mathscr{C}\times\mathscr{D}\to\mathsf{Top}_\bullet$ (compare (ho-3)). At a vertex (c,d) of $\mathscr{C}\times\mathscr{D}$, $(F\wedge G)(c,d)=F(c)\wedge G(d)$.

Lemma 4.45 *The thickening construction from Section 4.4 respects products, in the following sense. Fix functors $F\colon\underline{2}^m\to\mathscr{B}$ and $G\colon\underline{2}^n\to\mathscr{B}$. Let $F\times G\colon\underline{2}^m\times\underline{2}^n\to\mathscr{B}$ be as in Definition 4.43. Let $q\colon\widehat{\mathscr{C}}\times\widehat{\mathscr{D}}\to\widehat{\mathscr{C}\times\mathscr{D}}$ be the isomorphism from Lemma 4.26. Then, the map p induces a natural transformation from $\widehat{F}_k\wedge\widehat{G}_l$ to $(\widehat{F\times G})_{k+l}\circ q$ such that on vertices the natural transformation is a weak homotopy equivalence up to dimension $k+l+\min(k,l)-1$. These natural transformations respect the spectrum structure, and so induce maps of diagrams $\widehat{F}\wedge\widehat{G}\to(\widehat{F\times G})\circ q$ such that the map at each vertex is a stable homotopy equivalence.*

Proof This is straightforward from the definitions. To illustrate, we describe the natural transformation at the level of vertices. At a vertex $[u_C\xrightarrow{f_C}v_C\xrightarrow{g_C}w_C]\times[u_D\xrightarrow{f_D}v_D\xrightarrow{g_D}w_D]$ we have

$$\begin{aligned}&\widehat{F}(u_C\xrightarrow{f_C}v_C\xrightarrow{g_C}w_C)\wedge\widehat{G}(u_D\xrightarrow{f_D}v_D\xrightarrow{g_D}w_D)\\&\qquad=\Bigl(\bigvee\nolimits_{a_C\in F(f_C)}\prod_{\substack{b_C\in F(g_C)\\ s(b_C)=t(a_C)}}S^k\Bigr)\wedge\Bigl(\bigvee\nolimits_{a_D\in G(f_D)}\prod_{\substack{b_D\in G(g_D)\\ s(b_D)=t(a_D)}}S^l\Bigr)\end{aligned}$$

while

$$\begin{aligned}&(\widehat{F\times G})_{k+l}\bigl(u_C\times u_D\xrightarrow{f_C\times f_D}v_C\times v_D\xrightarrow{g_C\times g_D}w_C\times w_D\bigr)\\&\qquad=\bigvee\nolimits_{\substack{a_C\in F(f_C)\\ a_D\in G(f_D)}}\prod_{\substack{b_C\in F(g_C)\\ b_D\in G(g_D)\\ s(b_C)=t(a_C)\\ s(b_D)=t(a_D)}}S^{k+l}.\end{aligned}$$

The map p sends the first to the second in an obvious way, and is an equivalence up to dimension $k+l+\min(k,l)-1$. Moreover, this map is $(\mathcal{S}_k \times \mathcal{S}_l)$–equivariant and respects the structure maps of the spectrum. $\square$

Proposition 4.46 *Given functors* $F\colon \underline{2}^n \to \mathscr{B}$ *and* $G\colon \underline{2}^m \to \mathscr{B}$*, we have* $|F \times G| \simeq |F| \wedge |G|$*.*

Proof Let $\simeq$ denote stable homotopy equivalence. By Lemma 4.41,

$$|F| \simeq \operatorname{hocolim} \widehat{F}^{\dagger}, \quad |G| \simeq \operatorname{hocolim} \widehat{G}^{\dagger}, \quad |F \times G| \simeq \operatorname{hocolim}(\widehat{F \times G})^{\dagger}.$$

From the definitions, it follows that the natural transformation from $\widehat{F} \wedge \widehat{G}$ to $(\widehat{F \times G})$ of Lemma 4.45 extends to a natural transformation from $\widehat{F}^{\dagger} \wedge \widehat{G}^{\dagger}$ to $(\widehat{F \times G})^{\dagger}$, which is a stable homotopy equivalence on objects. By point (ho-1) in Section 4.2,

$$\operatorname{hocolim}(\widehat{F \times G})^{\dagger} \simeq \operatorname{hocolim}(\widehat{F}^{\dagger} \wedge \widehat{G}^{\dagger}).$$

By point (ho-3) in Section 4.2,

$$\operatorname{hocolim}(\widehat{F}^{\dagger} \wedge \widehat{G}^{\dagger}) \simeq (\operatorname{hocolim} \widehat{F}^{\dagger}) \wedge (\operatorname{hocolim} \widehat{G}^{\dagger}).$$

The result follows. $\square$

An even simpler operation is disjoint union:

Definition 4.47 There is a map $\amalg\colon \mathscr{B} \times \mathscr{B} \to \mathscr{B}$ which sends a pair of objects (X, Y) to their disjoint union $X \amalg Y$, a pair of correspondences $((A, s_A, t_A), (B, s_B, t_B))$ to $(A \amalg B, s_A \amalg s_B, t_A \amalg t_B)$, and a pair of isomorphisms of correspondences

$$(F\colon A \to A', G\colon B \to B')$$

to the isomorphism $F \amalg G\colon A \amalg B \to A' \amalg B'$. Given functors $F, G\colon \underline{2}^n \to \mathscr{B}$, we define the *disjoint union* $F \amalg G$ of F and G to be the composition

$$\underline{2}^n \xrightarrow{\Delta} \underline{2}^n \times \underline{2}^n \xrightarrow{(F,G)} \mathscr{B} \times \mathscr{B} \xrightarrow{\amalg} \mathscr{B},$$

where the first arrow is the diagonal map.

Explicitly, $(F \amalg G)\colon \underline{2}^n \to \mathscr{B}$ is given as follows:

- For $v \in \{0,1\}^n$, $(F \amalg G)(v) = F(v) \amalg G(v)$.

- For $v > w \in \{0,1\}^n$, $(F \sqcup G)_{v,w}(\varphi_{v,w})$ is defined to be the correspondence

$$\begin{tikzcd}
& F(\varphi_{v,w}) \sqcup G(\varphi_{v,w}) \arrow[dl, "s_F \sqcup s_G"'] \arrow[dr, "t_F \sqcup t_G"] & \\
F(v) \sqcup G(v) & & F(w) \sqcup G(w)
\end{tikzcd}$$

- For $u > v > w \in \{0,1\}^n$, $(F \sqcup G)_{u,v,w}$ is defined by the commutative diagram

$$\begin{tikzcd}
{[F(\varphi_{v,w}) \sqcup G(\varphi_{v,w})] \times_{F(v) \sqcup G(v)} [F(\varphi_{u,v}) \sqcup G(\varphi_{u,v})]} \arrow[r, "(F \sqcup G)_{u,v,w}"] \arrow[d, "\cong"'] & F(\varphi_{u,w}) \sqcup G(\varphi_{u,w}) \\
{[F(\varphi_{v,w}) \times_{F(v)} F(\varphi_{u,v})] \sqcup [G(\varphi_{v,w}) \times_{G(v)} G(\varphi_{u,v})]} \arrow[ur, "F_{u,v,w} \sqcup G_{u,v,w}"'] &
\end{tikzcd}$$

  where the vertical arrow is the obvious bijection.

Lemma 4.48 *Definition 4.47 specifies a strictly unitary, lax 2–functor.*

Proof This is immediate from the definitions. □

Given diagrams $F, G\colon \mathscr{C} \to \mathsf{Top}_\bullet$ there is an induced diagram $F \vee G\colon \mathscr{C} \to \mathsf{Top}_\bullet$ with $(F \vee G)(v) = F(v) \vee G(v)$ and $(F \vee G)(f) = F(f) \vee G(f)$.

Lemma 4.49 *The thickening construction respects disjoint unions in the sense that given $F, G\colon \mathscr{C} \to \mathscr{B}$, $(F \mathbin{\widehat{\sqcup}} G)_k \cong \widehat{F}_k \vee \widehat{G}_k$.*

Proof Again, this is immediate from the definitions. □

Proposition 4.50 *Given functors $F, G\colon \underline{2}^n \to \mathscr{B}$, $|F \sqcup G| \simeq |F| \vee |G|$.*

Proof Lemma 4.49 extends immediately to the statement that $F \mathbin{\widehat{\sqcup}} G^{+} \cong \widehat{F}^{+} \vee \widehat{G}^{+}$, and homotopy colimit commutes with wedge sum (point (ho-2) in Section 4.2). □

5 Building a smaller cube from little box maps

In this section we show that the realization of a functor $\underline{2}^n \to \mathscr{B}$ can be understood in terms of a smaller diagram.

Definition 5.1 Let $\underline{2}^n_+$ be the result of adding a single object, which we will denote by $*$, to $\underline{2}^n$ and declaring that

$$\mathrm{Hom}(v,*)=\begin{cases}\text{one element } \varphi_{v,*} & \text{if } v\neq\vec{0},\\ \varnothing & \text{if } v=\vec{0},\end{cases}\qquad \mathrm{Hom}(*,v)=\begin{cases}\{\mathrm{Id}\} & \text{if } v=*,\\ \varnothing & \text{if } v\neq *.\end{cases}$$

In other words, $\underline{2}^n_+$ is the result of replacing the terminal object in $\underline{2}^n$ by two copies of itself, with no maps between them.

The goal of this section is to prove:

Theorem 7 *Fix a strictly unitary, lax 2–functor $F\colon \underline{2}^n\to\mathscr{B}$. Then there is a homotopy coherent diagram $\widetilde{F}^+\colon \underline{2}^n_+\to\mathscr{S}$ such that:*

1. *For each $v\in\mathrm{Ob}(\underline{2}^n)$, $\widetilde{F}^+(v)=\bigvee_{a\in F(v)}\mathbb{S}$.*
2. *$\widetilde{F}^+(*)$ is a single point.*
3. *$\mathrm{hocolim}\,\widetilde{F}^+\simeq|F|$.*

The diagram $\widetilde{F}^+$ is a special case of a more general construction, which we give in Section 5.2. We specialize to $\widetilde{F}^+$ in Section 5.3, and prove that this homotopy colimit is equivalent to $|F|$ in Section 5.4. Before turning to these constructions, we introduce a particular class of maps between spheres in Section 5.1.

5.1 Box maps

Definition 5.2 By a *box in $\mathbb{R}^k$* we mean a subset of the form $B=[a_1,b_1]\times\cdots\times[a_k,b_k]$ for some $a_1,\dots,b_k$ with $a_i<b_i$. Given a box B in $\mathbb{R}^k$, a *subbox B'* of B is a box B' in $\mathbb{R}^k$ such that $B'\subset B$.

Definition 5.3 Given a box B in $\mathbb{R}^k$, let $E(B,\ell)$ be the space of ordered ℓ–tuples of disjoint boxes in B, topologized as a subspace of $(\mathbb{R}^{2k})^\ell$.

Given a correspondence $(A,s\colon A\to X,t\colon A\to Y)$, consider a collection $\{B_x\mid x\in X\}$ of distinct copies of the standard unit box. Let $F(\{B_x\},s,t)$ be the space of collections $\{B_a\subset B_{s(a)}\mid a\in A\}$ of subboxes labeled by elements of A, satisfying the condition that $B_a\cap B_{a'}=\varnothing$ if $a\neq a'$ and $s(a)=s(a')$ and $t(a)=t(a')$. See Figure 6.

Let $E(\{B_x\},s)\subset F(\{B_x\},s,t)$ be the space of collections $\{B_a\subset B_{s(a)}\mid a\in A\}$ satisfying the stronger condition that $B_a\cap B_{a'}=\varnothing$ if $a\neq a'$ and $s(a)=s(a')$. Again, see Figure 6.

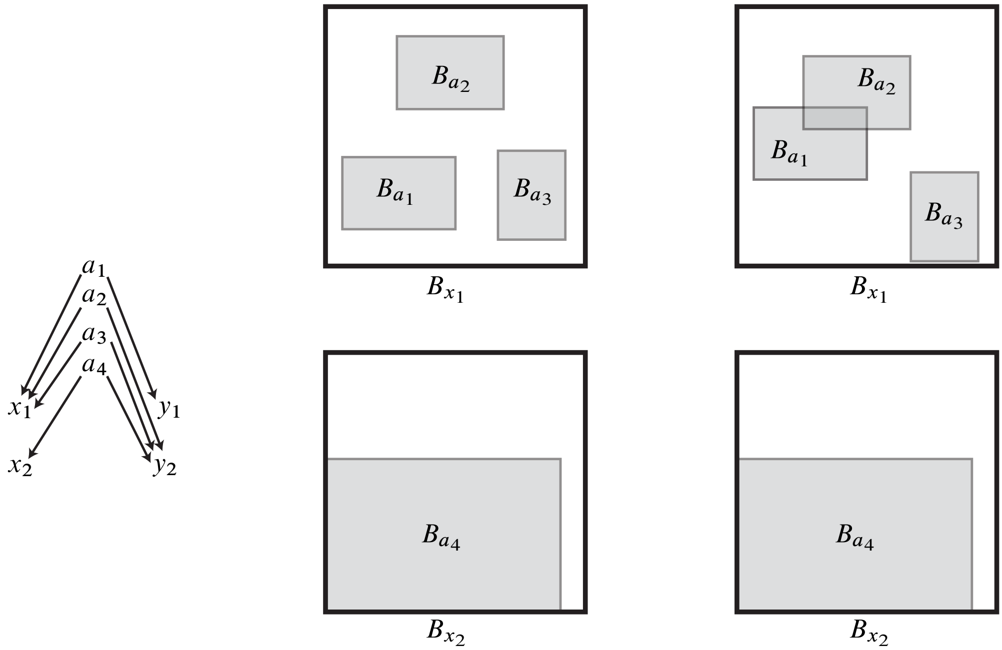


Figure 6: Spaces of boxes. Left: A correspondence A from a 2–element set X to another 2–element set Y. Center: A point in $E(\{B_x\}, s)$. Right: A point in $F(\{B_x\}, s, t)$ which is not in $E(\{B_x\}, s)$. The boxes B_{a_i} are shaded.

Example 5.4 If A is an ℓ–element correspondence and X has one element x then $E(\{B_x\}, s) = E(B_x, \ell)$.

Definition 5.5 Fix once and for all an identification $S^k = [0,1]^k/\partial$. When we view S^k as a pointed space, we assume that this identification identifies the basepoint and $\partial[0,1]^k$. For any box B in $\mathbb{R}^k$, there is a standard homeomorphism $B \to [0,1]^k$, given by $(x_1, \dots, x_k) \mapsto ((x_1 - a_1)/(b_1 - a_1), \dots, (x_k - a_k)/(b_k - a_k))$, and hence an induced identification $S^k \cong B/\partial B$.

Given $\{B_1, \dots, B_\ell\} \in E(B, \ell)$, the associated *basic box map* is the composition

$$\textbf{(5.6)}\quad S^k = B/\partial B \to B/(B \setminus (\mathring{B}_1 \cup \dots \cup \mathring{B}_\ell)) = \bigvee_{a=1}^{\ell} B_a/\partial B_a = \bigvee_{a=1}^{\ell} S^k \to S^k,$$

where the last map is the identity on each summand (so the composition has degree ℓ). This defines a continuous map $E(B, \ell) \to \mathit{Map}(S^k, S^k)$.

Remark 5.7 The map $E(B, \ell) \to \mathit{Map}(S^k, S^k)$ is the composition of the coaction of the little k–cubes operad with the fold map.

Definition 5.8 Given a correspondence A from X to Y and a collection of boxes $e = \{B_a \subset B_{s(a)} \mid a \in A\} \in E(\{B_x\}, s)$, there is an induced map

$$\Phi(e, A)\colon \bigvee_{x \in X} S^k \to \bigvee_{y \in Y} S^k \tag{5.9}$$

defined by

$$\Phi_{e,A}|_{S^k_x}\colon S^k_x = B_x/\partial B_x \to B_x \Big/ \Big(B_x \setminus \Big(\bigcup_{\substack{a \in A \\ s(a)=x}} \mathring{B}_a \Big)\Big) \tag{5.10}$$
$$= \bigvee_{\substack{a \in A \\ s(a)=x}} B_a/\partial B_a = \bigvee_{\substack{a \in A \\ s(a)=x}} S^k_a \to \bigvee_{y \in Y} S^k_y,$$

where the last map sends S^k_a by the identity map to $S^k_{t(a)}$. This defines a continuous map $\Phi(\cdot, A)\colon E(\{B_x\}, s) \to \mathit{Map}\big(\bigvee_{x \in X} S^k, \bigvee_{y \in Y} S^k\big)$. We call a map of the form $\Phi(e, A)$ for some $e \in E(\{B_x\}, s)$ a *disjoint box map*, and say it *refines the correspondence* (A, s, t).

Example 5.11 The basic box maps are exactly the disjoint box maps where X and Y each consist of a single element.

Definition 5.12 Given a correspondence A from X to Y and a collection of boxes $e = \{B_a \subset B_{s(a)} \mid a \in A\} \in F(\{B_x\}, s, t)$, there is an induced map

$$\bigvee_{x \in X} S^k \to \prod_{y \in Y} S^k \tag{5.13}$$

whose component sending S^k_x to S^k_y is the basic box map associated to the collection $(B_x, \{B_a \mid s(a) = x,\ t(a) = y\})$. We call a map arising this way an *overlapping box map*, and say it *refines the correspondence* (A, s, t).

By a *box map* we mean either a disjoint box map or an overlapping box map.

Example 5.14 Given $e \in E(\{B_x\}, s) \subset F(\{B_x\}, s, t)$, the overlapping box map associated to e is obtained by composing the disjoint box map $\Phi(e, A)$ with the standard inclusion $\bigvee_{y \in Y} S^k \hookrightarrow \prod_{y \in Y} S^k$.

Remark 5.15 We have not defined, and will not need, box maps from products, just box maps to products.

Example 5.16 The diagonal map $S^k \to \prod_{i=1}^m S^k$ is a box map: take $X = \{x\}$ to have a single element, A and Y to have m elements each, t to be a bijection, and each box B_a to be all of B_x.

Lemma 5.17 *A composition of box maps is a box map.*

Proof Given composable box maps F and G, the preimages under F of the boxes for G are the boxes for $G \circ F$. □

The value to us of these constructions comes from the following:

Lemma 5.18 *If B is k–dimensional then the space $E(B,\ell)$ is $(k-2)$–connected. More generally, $E(\{B_x\},s)$ and $F(\{B_x\},s,t)$ are $(k-2)$–connected.*

Proof The space $E(B,\ell)$ is homotopy equivalent to the ordered configuration space of ℓ points in the interior of B, ie $B^\ell \setminus \Delta$, where Δ is the fat diagonal. Since Δ is a finite union of smooth submanifolds of codimension k, the result for $E(B,\ell)$ follows. Next, $E(\{B_x\},s) \cong \prod_{x\in X} E([0,1]^k, |s^{-1}(x)|)$ is a product of $(k-2)$–connected spaces, and hence is $(k-2)$–connected. Finally,

$$F(\{B_x\},s,t) \cong \prod_{(x,y)\in X\times Y} E\big([0,1]^k, |s^{-1}(x)\cap t^{-1}(y)|\big),$$

and so again is $(k-2)$–connected. □

Informally, Lemma 5.18 says that the space of box maps is highly connected. (For this statement to be correct, we must think of a box map as not just the map but also the labeling of the boxes.) Indeed, we will show next that the space of box maps is also highly connected in relative terms.

Definition 5.19 For any set map $s\colon A \to X$, define $E^\circ(\{B_x\},s)$ to be the subspace of $E(\{B_x\},s)$ where we require the box B_a to lie in the interior of the box $B_{s(a)}$ for all $a \in A$.

If $A_0 \subset A$ is a subset and $s_0\colon A_0 \to X$ is the restriction of s then there is a map $E^\circ(\{B_x\},s) \to E^\circ(\{B_x\},s_0)$ gotten by forgetting the boxes labeled by $A \setminus A_0$.

Lemma 5.20 *Fix a set map $s\colon A \to X$ and a subset $A_0 \subset A$. Let $s_0 = s|_{A_0}$. If the boxes $\{B_x\}$ are k–dimensional, then for any $i \le k-1$ and any commutative diagram*

$$\begin{tikzcd} S^{i-1} \arrow[r] \arrow[d, hook] & E^\circ(\{B_x\},s) \arrow[d] \\ D^i \arrow[r] & E^\circ(\{B_x\},s_0) \end{tikzcd}$$

there exists a lift $D^i \to E^\circ(\{B_x\},s)$ making the diagram commute.

Proof By induction, we may assume that $A \setminus A_0$ consists of a single element, say a_1, and let $x_1 = s(a_1)$ and $\ell = |s_0^{-1}(x_1)|$. The projection $\pi\colon E^\circ(\{B_x\}, s) \to E^\circ(\{B_x\}, s_0)$ is seen to be a fiber bundle by the following argument.

Let $\pi_\bullet\colon E_\bullet \to E^\circ(\{B_x\}, s_0)$ be the fiber bundle where the fiber over a point is the complement of the ℓ boxes in the interior of B_{x_1}. It is easy to see that $\pi_\bullet$ is a fiber bundle as follows. For $z \in E^\circ(\{B_x\}, s_0)$, fix a minimal triangulation of the complement of the ℓ boxes in z. This triangulation persists in some small neighborhood U of z in $E^\circ(\{B_x\}, s_0)$, and induces a PL homeomorphism between $\pi_\bullet^{-1}(z)$ and $\pi_\bullet^{-1}(z')$ for any $z' \in U$.

Now, for any $z \in E^\circ(\{B_x\}, s_0)$, construct a coordinate chart on $\pi^{-1}(z)$ by the following variables: the center C of the box B_{a_1} viewed as a point in $\pi_\bullet^{-1}(z)$; the "aspect ratio" R of the box B_{a_1}, presented as a $(k-1)$–tuple of ratios of the k side-lengths of B_{a_1}; and a proportion $P \in (0,1)$ of the volume of the box B_{a_1} relative to the volume of the largest box with the same center and same aspect ratio that lies in B_{x_1} in the complement of the interiors of other ℓ boxes. This identifies $\pi^{-1}(z)$ with $\pi_\bullet^{-1}(z) \times (0,\infty)^{k-1} \times (0,1)$, and the identification holds for small open sets around z. But $\pi_\bullet$ is a fiber bundle, and, therefore, so is π.

The fiber over each point is homeomorphic to the space of boxes in the complement of ℓ disjoint boxes in the interior of B_{x_1}. The fiber, being homotopy equivalent to the complement of ℓ points in $\mathbb{R}^k$, is $(k-2)$–connected, so the statement follows. $\square$

5.2 Refining diagrams via box maps

Definition 5.21 Fix a small category $\mathscr{D}$ and a strictly unitary, lax 2–functor $F\colon \mathscr{D} \to \mathscr{B}$ (ie a $\mathscr{D}$–diagram in $\mathscr{B}$). A *k–dimensional spatial refinement* of F is a homotopy coherent diagram $\widetilde{F}_k\colon \mathscr{D} \to \mathsf{Top}_\bullet$ such that

- for any $u \in \mathrm{Ob}(\mathscr{D})$, $\widetilde{F}_k(u) = \bigvee_{x \in F(u)} S^k = \big(\coprod_{x \in F(u)} B_x\big)/\partial$;
- for any $u, v \in \mathrm{Ob}(\mathscr{D})$ and $f\colon u \to v$, $\widetilde{F}_k(f)$ is a disjoint box map which refines the correspondence $F(f)$ from $F(u)$ to $F(v)$ (see Section 5.1); and, more generally,
- for any sequence of morphisms
$$u_0 \xrightarrow{f_1} u_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} u_n$$
in $\mathscr{D}$,
$$\widetilde{F}_k(f_n, \dots, f_1)\colon [0,1]^{n-1} \to \mathit{Map}\big(\textstyle\bigvee_{x \in F(u_0)} S^k, \bigvee_{x \in F(u_n)} S^k\big)$$

is a family of box maps induced by a map

$$[0,1]^{n-1} \to E(\{B_x\}_{x\in F(u_0)}, s_{F(f_n\circ\cdots\circ f_0)})$$

(refining the correspondence $F(f_n\circ\cdots\circ f_1)\cong F(f_n)\times_{F(u_{n-1})}\cdots\times_{F(u_1)}F(f_1)$ from $F(u_0)$ to $F(u_n)$).

Proposition 5.22 *Let $\mathscr{D}$ be a small category in which every sequence of composable nonidentity morphisms has length at most n. Fix a $\mathscr{D}$–diagram F in $\mathscr{B}$.*

1. *If $k\geq n$ then there is a k–dimensional spatial refinement of F.*
2. *If $k\geq n+1$ then any two k–dimensional spatial refinements of F are homotopic (as homotopy coherent diagrams).*
3. *If $\widetilde{F}_k$ is a k–dimensional spatial refinement of F then the result of suspending each $\widetilde{F}_k(u)$ and $\widetilde{F}_k(f_n,\dots,f_1)(\vec{t})$ gives a $(k+1)$–dimensional spatial refinement of F.*

Proof We start with point (1). Given $u\in\mathrm{Ob}(\mathscr{D})$, define $\widetilde{F}_k(u)=\bigvee_{x\in F(u)}S^k$; write the S^k–summand corresponding to x as B_x/∂, where B_x is a box in $\mathbb{R}^k$ (eg $B_x=[0,1]^k$). Next, by Observation 4.12 it suffices to consider only nonidentity morphisms. For each nonidentity morphism $f\colon u\to v$ in $\mathscr{D}$ choose a disjoint box map which refines the correspondence $F(f)$. Let $e_f\in E(\{B_x\mid x\in F(u)\}, s_{F(f)})$ be the collection of little boxes corresponding to $F(f)$.

We have now defined $\widetilde{F}_k$ on vertices and arrows. The diagram does not commute, so it remains to define the coherence homotopies associated to sequences of composable morphisms. We will build these inductively. As a warm up, we spell out the first case carefully before proceeding to the general case.

Fix a composable pair of morphisms $u\xrightarrow{f}v\xrightarrow{g}w$ in $\mathscr{D}$. There are two points in $E\big(\{B_x\mid x\in F(u)\}, s_{F(g\circ f)}\big)$ associated to (g,f). One is the point $e_{g\circ f}$. The other is defined as follows. The point e_g corresponds to a collection of boxes $\mathcal{B}_g$ in $\{B_y\mid y\in F(v)\}$, labeled by elements of $F(g)$. The inverse image $\Phi(e_f,F(f))^{-1}(\mathcal{B}_g)$ of these boxes is a collection of boxes in $\{B_x\mid x\in F(u)\}$. These $\Phi(e_f,F(f))^{-1}(\mathcal{B}_g)$ inherit a labeling by elements of $F(g)\times_{F(v)}F(f)\cong_{F_{u,v,w}}F(g\circ f)$. This labeling makes $\Phi(e_f,F(f))^{-1}(\mathcal{B}_g)$ into a second point in $E\big(\{B_x\mid x\in F(u)\}, s_{F(g\circ f)}\big)$, which by abuse of notation we will call $e_g\circ e_f$ (see also Lemma 5.17).

By Lemma 5.18, since $k\geq 2$ (or else we would not have a composable pair (g,f)), the space $E\big(\{B_x\mid x\in F(u)\}, s_{F(g\circ f)}\big)$ is connected, so we can find a path from $e_{g\circ f}$

to $e_g \circ e_f$. Fix such a path, and call it $e_{g,f}\colon [0,1] \to E\bigl(\{B_x \mid x \in F(u)\}, s_{F(g\circ f)}\bigr)$. Then $e_{g,f}$ defines a homotopy $\Phi(e_{g,f}, F(g\circ f))$ from $\Phi(e_g, F(g)) \circ \Phi(e_f, F(f))$ to $\Phi(e_{g\circ f}, F(g\circ f))$.

More generally, suppose that for any sequence $v_0 \xrightarrow{f_1} \cdots \xrightarrow{f_\ell} v_\ell$ of nonidentity morphisms we have chosen a map $e_{f_\ell,\dots,f_1}\colon [0,1]^{\ell-1} \to E\bigl(\{B_x \mid x \in F(v_0)\}, s_{F(f_\ell\circ\cdots\circ f_1)}\bigr)$, and these maps are compatible in the following sense. Let $(t_1,\dots,t_{\ell-1})$ be the coordinates on $[0,1]^{\ell-1}$. Then, for any $1 \le i \le \ell-1$, we require that

$$\begin{aligned} \textbf{(5.23)}\quad & e_{f_\ell,\dots,f_1}(t_1,\dots,t_{i-1},0,t_{i+1},\dots,t_{\ell-1}) \\ &\qquad = e_{f_\ell,\dots,f_i}(t_{i+1},\dots,t_{\ell-1}) \circ e_{f_{i-1},\dots,f_1}(t_1,\dots,t_{i-1}), \\ & e_{f_\ell,\dots,f_1}(t_1,\dots,t_{i-1},1,t_{i+1},\dots,t_{\ell-1}) \\ &\qquad = e_{f_\ell,\dots,f_i\circ f_{i-1},\dots,f_1}(t_1,\dots,t_{i-1},t_{i+1},\dots,t_{\ell-1}). \end{aligned}$$

Then, given $v_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{\ell+1}} v_{\ell+1}$, there is a map

$$S^{\ell-1} = \partial([0,1]^\ell) \to E\bigl(\{B_x \mid x \in F(v_0)\}, s_{F(f_{\ell+1}\circ\cdots\circ f_1)}\bigr)$$

defined by

$$\begin{aligned} \textbf{(5.24)}\quad & (t_1,\dots,t_{i-1},0,t_{i+1},\dots,t_\ell) \\ &\qquad \mapsto e_{f_{\ell+1},\dots,f_{i+1}}(t_{i+1},\dots,t_\ell) \circ e_{f_i,\dots,f_1}(t_1,\dots,t_{i-1}), \\ & (t_1,\dots,t_{i-1},1,t_{i+1},\dots,t_\ell) \\ &\qquad \mapsto e_{f_{\ell+1},\dots,f_{i+1}\circ f_i,\dots,f_1}(t_1,\dots,t_{i-1},t_{i+1},\dots,t_\ell). \end{aligned}$$

The inductive hypothesis implies that this map is continuous. Since $k \ge \ell+1$, by Lemma 5.18, the space $E\bigl(\{B_x \mid x \in F(v_0)\}, s_{F(f_{\ell+1}\circ\cdots\circ f_1)}\bigr)$ is $(\ell-1)$–connected, so the map (5.24) extends to a map $[0,1]^\ell \to E\bigl(\{B_x \mid x \in F(v_0)\}, s_{F(f_{\ell+1}\circ\cdots\circ f_1)}\bigr)$. Define $e_{f_{\ell+1},\dots,f_1}$ to be any such extension.

Now, (5.9) gives a map

$$\Phi(e_{v_0,\dots,v_{\ell+1}}, F(f_{\ell+1}\circ\cdots\circ f_1))\colon [0,1]^\ell \times \textstyle\bigvee_{x\in F(v_0)} S^k \to \bigvee_{x\in F(v_{\ell+1})} S^k.$$

It follows from the compatibility conditions (5.23) that these maps define a homotopy coherent diagram.

Next, for point (2), fix spatial refinements $\widetilde{F}_k$ and $\widetilde{F}'_k$ of F. Consider the category $\underline{2}^1 \times \mathscr{D}$. It suffices to define a homotopy coherent diagram $G\colon \underline{2}^1 \times \mathscr{D} \to \mathsf{Top}_\bullet$ so that $G|_{\{0\}\times\mathscr{D}} = \widetilde{F}_k$, $G|_{\{1\}\times\mathscr{D}} = \widetilde{F}'_k$, and, for any $u \in \mathrm{Ob}(\mathscr{D})$, $G(\varphi_{1,0} \times \mathrm{Id}_u)$ is a homotopy equivalence [57, Proposition 4.6]. To define G, note that $G|_{\{0\}\times\mathscr{D}}$ and $G|_{\{1\}\times\mathscr{D}}$ are already specified. Let $G(\varphi_{1,0}\times \mathrm{Id}_u)$ be the identity map. More generally, define (somewhat arbitrarily) $G(\varphi_{1,0}\times g)=\widetilde{F}_k(g)$. It follows from the fact that both $\widetilde{F}_k$ and $\widetilde{F}'_k$ refine F that the resulting diagram G is homotopy commutative. Extend G to a homotopy coherent diagram inductively, as in the proof of point (1).

Finally, point (3) is immediate from the definitions. □

5.3 A coherent cube of box maps

Definition 5.25 Given a strictly unitary, lax 2–functor $F\colon \underline{2}^n\to\mathscr{B}$ let $\widetilde{F}_k\colon \underline{2}^n\to\mathsf{Top}_\bullet$ be a spatial refinement of F. Let $\widetilde{F}_k^+\colon \underline{2}^n_+\to\mathsf{Top}_\bullet$ be the diagram obtained from $\widetilde{F}_k$ by defining $\widetilde{F}_k^+(*)$ to be a single point. Let $\widetilde{F}^+$ be the diagram obtained from $\widetilde{F}_k^+$ by replacing each vertex $\widetilde{F}_k^+(u)$ with $\Sigma^{-k}(\Sigma^\infty\widetilde{F}_k^+(u))$, the k–fold formal desuspension of its suspension spectrum.

Corollary 5.26 *Up to stable homotopy equivalence, the spectrum* hocolim $\widetilde{F}^+$ *depends only on the functor F. In fact, for any $k>n$, the homotopy type of* hocolim $\widetilde{F}_k^+$ *is independent of the choices in its construction.*

Proof This is immediate from Proposition 5.22, together with the fact that the homotopy colimits of homotopy equivalent homotopy coherent diagrams are homotopy equivalent (Theorem 5.12 of [57], quoted as Proposition 4.13). □

As in Section 4.5, we can also work with a larger enlargement $\underline{2}^n_\dagger=(*\leftarrow 1\to 0)^{\times n}$ of $\underline{2}^n$. Extend $\widetilde{F}_k$ to a functor $\widetilde{F}_k^\dagger\colon \underline{2}^n_\dagger\to\mathsf{Top}_\bullet$ by setting $\widetilde{F}_k^\dagger|_{\underline{2}^n}=\widetilde{F}_k$ and $\widetilde{F}_k^\dagger(v)=\{\mathrm{pt}\}$ if v is an object which is not in $\underline{2}^n$, ie if some coordinate of v is $*$.

Lemma 5.27 *For any functor $F\colon \underline{2}^n\to\mathscr{B}$ and any spatial refinement $\widetilde{F}_k$ of F there is a stable homotopy equivalence* hocolim $\widetilde{F}_k^+\simeq$ hocolim $\widetilde{F}_k^\dagger$.

Proof The proof is similar to but easier than the proof of Lemma 4.41, and is left to the reader. □

5.4 The realizations of the small cube and big cube agree

Before proving Theorem 7, we introduce an auxiliary category, the arrow category of $\underline{2}^n$, and study its relationship with $\underline{2}^n$ and $\widehat{\underline{2}^n}$.

Definition 5.28 Given a small category $\mathscr{C}$, *the arrow category of* $\mathscr{C}$, which we denote by $\mathsf{Arr}(\mathscr{C})$, has $\mathrm{Ob}(\mathsf{Arr}(\mathscr{C})) = \bigcup_{u,v\in\mathrm{Ob}(\mathscr{C})} \mathrm{Hom}(u,v)$ the set of morphisms in $\mathscr{C}$. Given objects $f\colon u \to v$ and $g\colon w \to x$ in the arrow category, $\mathrm{Hom}(f,g)$ consists of pairs $(\alpha\colon u \to w, \beta\colon v \to x)$ such that

(5.29)
$$\begin{tikzcd}
u \arrow[r, "f"] \arrow[d, "\alpha"'] & v \arrow[d, "\beta"] \\
w \arrow[r, "g"'] & x
\end{tikzcd}$$

commutes. Maps compose in the obvious way: $(\gamma,\delta)\circ(\alpha,\beta) = (\gamma\circ\alpha, \delta\circ\beta)$.

There is a functor $A\colon \mathscr{C} \to \mathsf{Arr}(\mathscr{C})$ defined by

$$A(u) = \mathrm{Id}_u, \qquad A(f\colon u \to v) = \begin{tikzcd}
u \arrow[r, "\mathrm{Id}_u"] \arrow[d, "f"] & u \arrow[d, "f"] \\
v \arrow[r, "\mathrm{Id}_v"] & v
\end{tikzcd}$$

There is also a functor $B\colon \widehat{\mathscr{C}} \to \mathsf{Arr}(\mathscr{C})$ defined by

$$B(u \xrightarrow{f} v \xrightarrow{g} w) = g \circ f,$$

$$B\left(\begin{tikzcd}
u \arrow[r, "f"] \arrow[d, "\alpha"] & v \arrow[r, "g"] & w \arrow[d, "\gamma"] \\
u' \arrow[r, "f'"] & v' \arrow[r, "g'"] \arrow[u, "\beta"] & w'
\end{tikzcd}\right) = \begin{tikzcd}
u \arrow[r, "g\circ f"] \arrow[d, "\alpha"'] & w \arrow[d, "\gamma"] \\
u' \arrow[r, "g'\circ f'"'] & w'
\end{tikzcd}$$

In the special case of the cube category,

$$\mathsf{Arr}(\underline{2}^1) = (\varphi_{1,1} \to \varphi_{1,0} \to \varphi_{0,0}), \quad \mathsf{Arr}(\underline{2}^n) = (\mathsf{Arr}(\underline{2}^1))^n.$$

We will need a version with extra objects added, analogous to $\underline{2}^n_\dagger$:

Definition 5.30 Let

$$\mathsf{Arr}(\underline{2}^1)_\dagger = (* \leftarrow \varphi_{1,1} \to \varphi_{1,0} \to \varphi_{0,0}), \quad \mathsf{Arr}(\underline{2}^n)_\dagger = (\mathsf{Arr}(\underline{2}^1)_\dagger)^n.$$

The functors A and B have extensions

$$A_\dagger\colon \underline{2}^n_\dagger \to \mathsf{Arr}(\underline{2}^n)_\dagger, \quad B_\dagger\colon \widehat{\underline{2}^n_\dagger} \to \mathsf{Arr}(\underline{2}^n)_\dagger.$$

These are products of the 1–dimensional case, which is given by

$$\begin{tikzcd}
* \arrow[d, dashed] & 1 \arrow[l] \arrow[rr] \arrow[d, dashed] & & 0 \arrow[d, dashed] \\
* & \varphi_{1,1} \arrow[l] \arrow[r] & \varphi_{1,0} \arrow[r] & \varphi_{0,0}
\end{tikzcd}$$

$$\begin{tikzcd}
* \arrow[d, dotted] & 111 \arrow[l] \arrow[r] \arrow[d, dotted] & 110 \arrow[dr, dotted] & & 100 \arrow[ll] \arrow[r] \arrow[dl, dotted] & 000 \arrow[d, dotted] \\
* & \varphi_{1,1} \arrow[l] \arrow[rr] & & \varphi_{1,0} \arrow[rr] & & \varphi_{0,0}
\end{tikzcd}$$

The dashed arrows denote $A_\dagger$, and the dotted arrows denote $B_\dagger$.

To relate various diagrams, we will need to know $A_\dagger$ and $B_\dagger$ are homotopy cofinal:

Lemma 5.31 *The functor $A_\dagger$ is homotopy cofinal.*

Proof Recall from point (ho-4) in Section 4.2 that homotopy cofinality means that each undercategory $d \downarrow A_\dagger$ has contractible nerve. Since taking undercategories commutes with taking products, it suffices to verify the one-dimensional case. This verification is straightforward, and is left to the reader. □

Lemma 5.32 *The functor $B_\dagger$ is homotopy cofinal.*

Proof As in the proof of Lemma 5.31, it suffices to verify the 1–dimensional case, which is straightforward. □

Next we see that for any small category $\mathscr{C}$, any functor $F\colon \mathscr{C} \to \mathscr{B}$ lifts to a functor $\vec{F}\colon \mathsf{Arr}(\mathscr{C}) \to \mathscr{B}$.

Definition 5.33 Given a functor $F\colon \mathscr{C} \to \mathscr{B}$, define a functor $\vec{F}\colon \mathsf{Arr}(\mathscr{C}) \to \mathscr{B}$ as follows. For $f \in \mathrm{Ob}(\mathsf{Arr}(\mathscr{C}))$, $F(f)$ is a correspondence, and in particular a set; define $\vec{F}(f) = F(f)$. This defines $\vec{F}$ on $\mathrm{Ob}(\mathsf{Arr}(\mathscr{C}))$. Given a diagram as in formula (5.29), define $\vec{F}(\alpha,\beta) = F(g \circ \alpha)$ (which is exactly the same as $F(\beta \circ f)$, but merely in bijection with $F(g) \circ F(\alpha)$ and $F(\beta) \circ F(f)$). The source and target maps are given by

$$\begin{tikzcd}
F(\beta) \times_{F(v)} F(f) \arrow[d] & \vec{F}(\alpha,\beta) \arrow[l, "F^{-1}_{u,v,x}"'] \arrow[r, "F^{-1}_{u,w,x}"] & F(g) \times_{F(w)} F(\alpha) \arrow[d] \\
\vec{F}(f) = F(f) & & F(g) = \vec{F}(g)
\end{tikzcd}$$

For any pair of composable morphisms

$$\begin{tikzcd}
u \arrow[r, "f"] \arrow[d, "\alpha"'] & v \arrow[d, "\beta"] \\
w \arrow[r, "g"] \arrow[d, "\gamma"'] & x \arrow[d, "\delta"] \\
y \arrow[r, "h"'] & z
\end{tikzcd}$$

$\vec{F}_{f,g,h}$ should specify an isomorphism from

$$\vec{F}(\gamma,\delta)\times_{F(g)}\vec{F}(\alpha,\beta)=F(\delta\circ g)\times_{F(g)}F(g\circ\alpha)$$

to $F(\delta\circ g\circ\alpha)=F(\gamma\circ\alpha,\delta\circ\beta)$. Define this isomorphism to be the composition of the isomorphisms

$$\begin{aligned}
F(\delta\circ g)\times_{F(g)}F(g\circ\alpha) &\cong F(\delta)\times_{F(x)}F(g)\times_{F(g)}F(g)\times_{F(w)}F(\alpha)\\
&\cong F(\delta)\times_{F(x)}F(g)\times_{F(w)}F(\alpha)\\
&\cong F(\delta\circ g\circ\alpha).
\end{aligned}$$

Lemma 5.34 *These maps make $\vec{F}$ into a strictly unitary, lax 2–functor, and $\vec{F}\circ A=F$.*

We leave the proof as an exercise to the reader.

Lemma 5.35 *In $\mathsf{Arr}(\underline{2}^n)_\dagger$, any sequence of composable, nonidentity morphisms has length at most $2n$.*

Proof This is immediate from the definitions. □

Corollary 5.36 *If $k\geq 2n$ then any strictly unitary, lax 2–functor $\vec{F}\colon \mathsf{Arr}(\underline{2}^n)\to\mathscr{B}$ admits a k–dimensional spatial refinement $\overset{\rightrightarrows}{F}\colon \mathsf{Arr}(\underline{2}^n)\to\mathsf{Top}_\bullet$.*

Proof This is immediate from Lemma 5.35 and Proposition 5.22. □

Lemma 5.37 *Fix $k>2n$. Given $F\colon \underline{2}^n\to\mathscr{B}$, let $\overset{\rightrightarrows}{F}$ be a k–dimensional spatial refinement of $\vec{F}$ and consider the homotopy coherent diagram $\overset{\rightrightarrows}{F}\circ B\colon \widehat{\underline{2}^n}\to\mathsf{Top}_\bullet$. There is a morphism of homotopy coherent diagrams $G_k\colon \overset{\rightrightarrows}{F}\circ B\to\widehat{F}_k$ such that on each object the underlying map induces an isomorphism on H_i for $i\leq 2k-1$.*

Proof Recall that a morphism from $\overset{\rightrightarrows}{F} \circ B$ to $\widehat{F}_k$ is a diagram over $\underline{2}^1 \times \widehat{\underline{2}^n}$ whose restriction to $\{1\} \times \widehat{\underline{2}^n}$ is $\overset{\rightrightarrows}{F} \circ B$ and whose restriction to $\{0\} \times \widehat{\underline{2}^n}$ is $\widehat{F}_k$. We will build such a diagram inductively, using box maps from wedges to products.

On $\{0\} \times \widehat{\underline{2}^n}$ and $\{1\} \times \widehat{\underline{2}^n}$, G is already specified. Notice that for each object $(u \xrightarrow{\varphi_{u,v}} v \xrightarrow{\varphi_{v,w}} w) \in \mathrm{Ob}(\widehat{\underline{2}^n})$, the space $(\overset{\rightrightarrows}{F} \circ B)(u \to v \to w) = \bigvee_{a \in F(\varphi_{v,w} \circ \varphi_{u,v})} S^k$ is the k–skeleton of $\widehat{F}_k(u \to v \to w) = \bigvee_{a \in F(\varphi_{u,v})} \prod_{b \in F(\varphi_{v,w}),\, s(b)=t(a)} S^k$. For each arrow of the form $\varphi_{1,0} \times \mathrm{Id}_{u \to v \to w}$, define $G(\varphi_{1,0} \times \mathrm{Id}_{u \to v \to w})$ to be the inclusion of the k–skeleton (which is an isomorphism on H_i for $i \leq 2k-1$). More generally, given a morphism (α, β, γ) as in formula (4.28), define $G(\varphi_{1,0} \times (\alpha, \beta, \gamma))$ to be the composition $\widehat{F}_k(\alpha, \beta, \gamma) \circ G(\varphi_{1,0} \times \mathrm{Id}_{u \to v \to w})$. (Factoring in the other order would work just as well, though it would give a different map.)

The result is a homotopy commutative diagram G, so that the restriction to $\{0\} \times \widehat{\underline{2}^n}$ is commutative and the restriction to $\{1\} \times \widehat{\underline{2}^n}$ is homotopy coherent. By construction, the maps on the 1–side are disjoint box maps. The maps associated to the edges $\varphi_{1,0} \times \mathrm{Id}_{u \to v \to w}$ from the 1–side to the 0–side are wedge sums of overlapping box maps: the inclusion

$$\bigvee_{x \in F(\varphi_{v,w} \circ \varphi_{u,v})} S^k = \bigvee_{a \in F(\varphi_{u,v})} \bigvee_{\substack{b \in F(\varphi_{v,w}) \\ s(b)=t(a)}} S^k \hookrightarrow \bigvee_{a \in F(\varphi_{u,v})} \prod_{\substack{b \in F(\varphi_{v,w}) \\ s(b)=t(a)}} S^k$$

restricts to $\bigvee_{b \in F(\varphi_{v,w}),\, s(b)=t(a)} S^k$ as the overlapping box map associated to the correspondence $\bigl(\{b \in F(\varphi_{v,w}) \mid s(b)=t(a)\}, \mathrm{Id}, \mathrm{Id}\bigr)$ from $\{b \in F(\varphi_{v,w}) \mid s(b)=t(a)\}$ to itself where each subbox is the whole box; see Example 5.14. More succinctly, the map $S^k_{(a,b)} \to \prod_{b' \in F(\varphi_{v,w}),\, s(b')=t(a)} S^k$ is the box map corresponding to the 1–element correspondence from $\{(a,b)\}$ to $b \in \{b' \in F(\varphi_{v,w}),\ s(b')=t(a)\}$, where the subbox $B_{(a,b)} \subset B_b$ is equal to the whole box B_b.

To extend the diagram constructed so far to an entire homotopy coherent diagram, we need to define families of maps

$$\bigvee_{x \in F(\varphi_{u,w})} S^k \to \bigvee_{a \in F(\varphi_{u',v'})} \prod_{\substack{b \in F(\varphi_{v',w'}) \\ s(b)=t(a)}} S^k$$

corresponding to sequences of morphism starting at $(1, u \to v \to w)$ and ending at $(0, u' \to v' \to w')$. Observe that for there to be such a sequence we must have $u \geq u'$ and $w \geq w'$. We choose these extensions inductively, maintaining the following restrictions:

(X-1) Fix a morphism $(u \to v \to w) \to (u' \to v' \to w')$ in $\widehat{\underline{2}^n}$. Let $x \in F(\varphi_{u,w})$. Decompose x as

$$(x_1, x_2, x_3, x_4) \in F(\varphi_{u,u'}) \times_{F(u')} F(\varphi_{u',v'}) \times_{F(v')} F(\varphi_{v',v}) \times_{F(v)} F(\varphi_{v,w}).$$

Then $G((u \to v \to w) \to (u' \to v' \to w'))$ sends the sphere associated to x to the wedge summand of

$$\bigvee_{a \in F(\varphi_{u',v'})} \prod_{\substack{b \in F(\varphi_{v',w'}) \\ s(b)=t(a)}} S^k$$

corresponding to x_2.

(X-2) The map

$$h_x\colon S^k \to \prod_{\substack{b \in F(\varphi_{v',w'}) \\ s(b)=t(x_2)}} S^k$$

associated to x in the previous condition is an overlapping box map.

For the inductive step, we need to show that

(Y-1) if we postcompose a map satisfying properties (X-1)–(X-2) with the map in formula (4.29), we obtain another map satisfying (X-1)–(X-2); and

(Y-2) if we precompose a map satisfying properties (X-1)–(X-2) with one of the box maps used to define G on the 1–side, we obtain another map satisfying (X-1)–(X-2).

Then, Lemma 5.18 implies we can find the desired family of maps to continue the induction.

Statement (Y-2) is clear. For (Y-1), let h_x be as in condition (X-2) for $x \in F(\varphi_{u,w})$. Given a map $g\colon (u' \to v' \to w') \to (u'' \to v'' \to w'')$, the map $\widehat{F}_k(g)$ sends the wedge summand corresponding to $x_2 \in F(\varphi_{u',v'})$ to the summand corresponding to x_2', where x_2 corresponds to

$$(x_1', x_2', x_3') \in F(\varphi_{u',u''}) \times_{F(u'')} F(\varphi_{u'',v''}) \times_{F(v'')} F(\varphi_{v'',v'}).$$

Since x decomposes as $(x_1, x_1', x_2', x_3', x_3, x_4)$, property (X-1) is satisfied. Further, the map

$$\widehat{F}_k(g)\colon \prod_{\substack{b \in F(\varphi_{v',w'}) \\ s(b)=t(x_2)}} S^k \to \prod_{\substack{b' \in F(\varphi_{v'',w''}) \\ s(b')=t(x_2')}} S^k$$

sends S^k_b to $\prod_{b'=(x'_3,b,y)} S^k_{b'} = \prod_{y|s(y)=t(b)} S^k_{(x'_3,b,y)}$ by the diagonal map. If we replace each box for h labeled by c with $|\{y \in F(\varphi_{w',w''}) \mid s(y) = t(b)\}|$ boxes, labeled by the corresponding $b' = (x'_3, b, y)$, then the resulting box map is $\widehat{F}_k(g) \circ h$. In particular, we have verified that $\widehat{F}_k(g) \circ h$ is a box map, as desired.

The following diagram may be helpful in keeping track of the source and targets of the various elements in the previous paragraph:

$$\begin{tikzcd}
u \arrow[r] \arrow[rr, bend left, "x"] \arrow[d, "x_1"'] & v \arrow[r, "x_4"'] & w \arrow[d] \\
u' \arrow[r, "x_2"'] \arrow[d, "x'_1"'] & v' \arrow[u, "x_3"'] \arrow[r, "b"'] & w' \arrow[d, "y"] \\
u'' \arrow[r, "x'_2"'] & v'' \arrow[u, "x'_3"'] \arrow[r, "b'"'] & w''
\end{tikzcd}$$

The letter labeling each arrow $\to$ is the element of $F(\to)$ under consideration.

As noted above, Lemma 5.18 and induction now complete the proof. □

Corollary 5.38 *There is a stable homotopy equivalence*

$$\operatorname{hocolim}(\vec{\vec{F}} \circ B)^\dagger \simeq \operatorname{hocolim} \widehat{F}^\dagger.$$

Proof The morphism of diagrams G_k from Lemma 5.37 extends uniquely to a morphism of thickened diagrams $G^\dagger_k\colon (\vec{\vec{F}} \circ B)^\dagger \to \widehat{F}^\dagger_k$. Further, the diagram $\vec{\vec{F}}$ and the morphisms $G^\dagger_k$ can be chosen so that $G^\dagger_{k+1}$ is the suspension of $G^\dagger_k$. It follows that there is an induced map of diagrams of spectra $G^\dagger$, and the underlying maps of $G^\dagger$ are equivalences. □

Proof of Theorem 7 Let $\vec{F}\colon \mathsf{Arr}(\underline{2}^n) \to \mathscr{B}$ be the functor from Lemma 5.34. By Corollary 5.36 there is a spatial refinement $\vec{\vec{F}}$ of $\vec{F}$. The composition $\widetilde{F} = \vec{\vec{F}} \circ A$ is a spatial refinement of F. We will show that the corresponding diagram $\widetilde{F}^+\colon \underline{2}^n_+ \to \mathscr{S}$ satisfies the conditions of the theorem. Indeed, all of the conditions except that the homotopy colimit is $|F|$ are immediate. We compute the homotopy colimit.

By Lemma 5.27,

$$\operatorname*{hocolim}_{\underline{2}^n_+} \widetilde{F}^+ \simeq \operatorname*{hocolim}_{\underline{2}^n_\dagger} \widetilde{F}^\dagger.$$

By Lemmas 5.31 and 5.32 and property (ho-4) of homotopy colimits,

$$\operatorname*{hocolim}_{\underline{2}^n_\dagger} \widetilde{F}^\dagger \simeq \operatorname*{hocolim}_{\mathsf{Arr}(\underline{2}^n)_\dagger} \vec{\vec{F}}^\dagger \simeq \operatorname*{hocolim}_{\widehat{\underline{2}^n_\dagger}} \vec{\vec{F}}^\dagger \circ B^\dagger.$$

By Corollary 5.38, there is a homotopy equivalence

$$\operatorname*{hocolim}_{\widehat{\underline{2}^n}_{\dagger}} \overset{\rightsquigarrow}{\widehat{F}}{}^{\dagger} \circ B^{\dagger} \simeq \operatorname*{hocolim}_{\widehat{\underline{2}^n}_{\dagger}} \widehat{F}^{\dagger}.$$

By Lemma 4.41,

$$\operatorname*{hocolim}_{\widehat{\underline{2}^n}_{\dagger}} \widehat{F}^{\dagger} \simeq \operatorname*{hocolim}_{\widehat{\underline{2}^n}_{+}} \widehat{F}^{+} = |F|,$$

proving the result. □

6 A CW complex structure on the realization of the small cube

In this section, we prove that the realization in terms of little cubes (Section 5) is stably homotopy equivalent to the cubical realization (Section 3). We start by studying the cell structure on the little cubes realization:

Proposition 6.1 *Let* $F\colon \underline{2}^n \to \mathscr{B}$ *be a strictly unitary, lax* 2*–functor and* $\widetilde{F}_k\colon \underline{2}^n \to \mathsf{Top}_{\bullet}$ *a spatial refinement of* F (Definition 5.21). *Then the homotopy colimit of* $\widetilde{F}_k^{+}$ *carries a CW complex structure whose cells except the basepoint correspond to the elements of the set* $\coprod_{u\in\{0,1\}^n} F(u)$. *Further, the equivalences* $\Sigma\operatorname{hocolim}\widetilde{F}_k^{+} \simeq \operatorname{hocolim}\widetilde{F}_{k+1}^{+}$ *can be chosen to be cellular, so* $\operatorname{hocolim}\widetilde{F}^{+}$ *inherits the structure of a CW spectrum* (in the sense of [2, Section III.2]).

Proof Per Observation 4.12, when taking the homotopy colimit we may (and will) consider only chains of nonidentity arrows.

For $u \in \operatorname{Ob}(\underline{2}^n)$ and $x \in F(u)$, let B_x be the box that is associated to x during the construction of $\widetilde{F}_k^{+}$; that is, $B_x/\partial B_x$ is the S^k–summand corresponding to x in $\widetilde{F}_k^{+}(u) = \bigvee_{x\in F(u)} S^k$. Following Definition 4.10 (and with $\sim$ denoting the same equivalence relation), we can write the homotopy colimit as

$$\begin{aligned}
&\operatorname{hocolim}\widetilde{F}_k^{+}\\
&\quad= \Biggl(\{*\} \amalg \coprod_{u\in\{0,1\}^n} \Biggl(\coprod_{m\geq 0} \coprod_{\substack{u=u^0\xrightarrow{f_1}\cdots\xrightarrow{f_m}u^m\\ u^i\in\{0,1\}^n\cup\{*\}\\ f_i\neq\mathrm{Id}}} [0,1]^m\Biggr) \times \bigl(\textstyle\bigvee_{x\in F(u)} B_x/\partial B_x\bigr)\Biggr)\Big/\sim\\
&\quad= \Biggl(\Biggl[\{*\} \amalg \coprod_{u\in\{0,1\}^n} \Biggl(\coprod_{m\geq 0} \coprod_{\substack{u=u^0\xrightarrow{f_1}\cdots\xrightarrow{f_m}u^m\\ u^i\in\{0,1\}^n\cup\{*\}\\ f_i\neq\mathrm{Id}}} [0,1]^m\Biggr)\Big/\sim_1\Biggr] \times \Biggl(\coprod_{x\in F(u)} B_x\Biggr)\Biggr)\Big/\sim_2.
\end{aligned}$$

where we have broken up the identification $\sim$ into a two-step identification $\sim_1$ and $\sim_2$, defined as

$$\begin{aligned}&(f_m,\dots,f_1;t_1,\dots,t_m)\\ &\qquad\sim_1\begin{cases}(f_m,\dots,f_{i+1}\circ f_i,\dots,f_1;t_1,\dots,t_{i-1},t_{i+1},\dots,t_m) & \text{if } t_i=1,\ i<m,\\ (f_{m-1},\dots,f_1;t_1,\dots,t_{m-1}) & \text{if } t_m=1,\end{cases}\\ &(f_m,\dots,f_1;t_1,\dots,t_m;y)\\ &\qquad\sim_2\begin{cases}(f_m,\dots,f_{i+1};t_{i+1},\dots,t_m;\widetilde{F}^+_k(f_i,\dots,f_1)(t_1,\dots,t_{i-1})(y)) & \text{if } t_i=0,\\ * & \text{if } y\in\partial B_x.\end{cases}\end{aligned}$$

Now fix $u\in\{0,1\}^n$, and let us study the cubical complex

$$M_u:=\Bigl(\coprod_{m\ge 0}\ \coprod_{\substack{u=u^0\xrightarrow{f_1}\cdots\xrightarrow{f_m}u^m\\ u^i\in\{0,1\}^n\cup\{*\}\\ f_i\ne\mathrm{Id}}}[0,1]^m\Bigr)\Big/\!\sim_1. \tag{6.2}$$

If $u=\vec{0}$, then M_u is a single point, which we write as $\{0\}$ for reasons that will soon be apparent. When $u\ne\vec{0}$, we divide the chains $u=u^0\xrightarrow{f_1}\cdots\xrightarrow{f_m}u^m$ into two types: the ones ending at $\vec{0}$ or $*$, and the ones ending at neither. In the first case, when $u^m\in\{\vec{0},*\}$, the facet $[0,1]^{m-1}\times\{1\}$ is identified with the cube $[0,1]^{m-1}$ coming from the subchain $u=u^0\xrightarrow{f_1}\cdots\xrightarrow{f_{m-1}}u^{m-1}$. Therefore, we can write

$$\begin{aligned}M_u&=\Biggl(\Bigl(\coprod_{m\ge1}\ \coprod_{\substack{u=u^0\xrightarrow{f_1}\cdots\xrightarrow{f_m}\vec{0}\\ f_i\ne\mathrm{Id}}}[0,1]^{m-1}\times[0,1]\Bigr)\amalg\Bigl(\coprod_{m\ge1}\ \coprod_{\substack{u=u^0\xrightarrow{f_1}\cdots\xrightarrow{f_m}*\\ f_i\ne\mathrm{Id}}}[0,1]^{m-1}\times[0,1]\Bigr)\\ &\qquad\amalg\Bigl(\coprod_{m\ge1}\ \coprod_{\substack{u=u^0>\cdots>u^{m-1}\\ u^i\in\{0,1\}^n\setminus\{\vec{0}\}}}[0,1]^{m-1}\Bigr)\Biggr)\Big/\!\sim_1\\ &\cong\Biggl(\Bigl(\coprod_{m\ge1}\ \coprod_{\substack{u=u^0\xrightarrow{f_1}\cdots\xrightarrow{f_m}\vec{0}\\ f_i\ne\mathrm{Id}}}[0,1]^{m-1}\times[0,1]\Bigr)\amalg\Bigl(\coprod_{m\ge1}\ \coprod_{\substack{u=u^0\xrightarrow{f_1}\cdots\xrightarrow{f_m}*\\ f_i\ne\mathrm{Id}}}[0,1]^{m-1}\times[1,2]\Bigr)\\ &\qquad\amalg\Bigl(\coprod_{m\ge1}\ \coprod_{\substack{u=u^0>\cdots>u^{m-1}\\ u^i\in\{0,1\}^n\setminus\{\vec{0}\}}}[0,1]^{m-1}\Bigr)\Biggr)\Big/\!\sim_1\\ &=\Bigl(\coprod_{m\ge1}\ \coprod_{\substack{u=u^0>\cdots>u^{m-1}\\ u^i\in\{0,1\}^n\setminus\{\vec{0}\}}}[0,1]^{m-1}\Bigr)\Big/\!\sim_1\times[0,2],\end{aligned}$$

where the second identification is via the linear map $[0,1] \to [1,2]$ that sends 0 to 2 and 1 to 1. This quotient space is just $M_{u,\vec{0}} \times [0,2]$, where $M_{u,\vec{0}} \cong \mathcal{M}_{\mathscr{C}_C(n)}(u,\vec{0})$ is the cubical complex from Definition 3.19.

Therefore, we can write

$$\operatorname{hocolim} \widetilde{F}_k^+ = \Bigl(\{*\} \amalg \Bigl[\coprod_{u \in \{0,1\}^n \setminus \{\vec{0}\}} \coprod_{x \in F(u)} \mathcal{M}_{\mathscr{C}_C(n)}(u,\vec{0}) \times [0,2] \times B_x\Bigr] \amalg \Bigl[\coprod_{x \in F(\vec{0})} \{0\} \times B_x\Bigr]\Bigr) \Big/ \sim_2.$$

This gives the required CW complex on the homotopy colimit, where the cell corresponding to x is

$$\mathcal{C}(x) = \begin{cases} \mathcal{M}_{\mathscr{C}_C(n)}(u,\vec{0}) \times [0,2] \times B_x & \text{if } u \neq \vec{0}, \\ \{0\} \times B_x & \text{if } u = \vec{0}. \end{cases}$$

The identification $\sim_2$ glues the boundary $\partial \mathcal{C}(x)$ to lower-dimensional cells. Specifically, everything over ∂B_x is identified with the basepoint $*$. If $u \neq \vec{0}$, everything over $\{2\} \subset \partial([0,2])$ is identified with $*$ as well, and everything over $\{0\} \subset \partial([0,2])$ is identified with cells corresponding to $u = \vec{0}$. Finally, points on $\partial \mathcal{M}_{\mathscr{C}_C(n)}(u,\vec{0})$ correspond to points on some cube $[0,1]^l \subset \mathcal{M}_{\mathscr{C}_C(n)}(u,\vec{0})$ where some coordinate is 0 (Lemma 3.20(2)), and such points are identified by $\sim_2$ to points of $\mathcal{M}_{\mathscr{C}_C(n)}(u',\vec{0})$ for some $u > u' > \vec{0}$.

The fact that the suspension maps are cellular is trivial if we choose compatible spatial refinements. Specifically, take the boxes used to define $\widetilde{F}_{k+1}^+$ to be $[0,1]$ times the boxes used to define $\widetilde{F}_k^+$. □

It is not hard to show that if $\mathscr{C}$ is the cubical flow category corresponding to F (from Construction 4.19) then its associated chain complex (from Definition 3.8) is isomorphic to the reduced cellular cochain complex of the above CW complex, via an isomorphism sending the objects of $\mathscr{C}$ to the corresponding cells in the CW complex. We will not prove this now, since it follows from Theorem 8.

Theorem 8 *Let* $(\mathscr{C}, \mathfrak{f}\colon \mathscr{C} \to \mathscr{C}_C(n))$ *be a cubical flow category, and let* $F\colon \underline{2}^n \to \mathscr{B}$ *be the corresponding functor* (Construction 4.17). *Then the cubical realization of* $\mathscr{C}$ (Definition 3.29) *is stably homotopy equivalent to the realization of* F *as the homotopy colimit of the homotopy coherent diagram* $\widetilde{F}^+\colon \underline{2}_+^n \to \mathscr{S}$ *from Theorem 7; and the homotopy equivalence sends the cells in the CW complex structure on the homotopy* *colimit from Proposition 6.1 to the corresponding cells in the cubical realization of $\mathscr{C}$ via maps of degree* ± 1.

Proof Fix a cubical neat embedding ι of $\mathscr{C}$ relative to $\boldsymbol{d} = (d_0, \dots, d_{n-1})$ and let $k = \sum_i d_i$. Let ϵ and R be the parameters from Section 3.6, and let

$$\bar{\iota}_{x,y}\colon \prod_{i=|\mathfrak{f}(y)|}^{|\mathfrak{f}(x)|-1} [-\epsilon, \epsilon]^{d_i} \times \mathcal{M}_{\mathscr{C}}(x, y) \to \left[\prod_{i=|\mathfrak{f}(y)|}^{|\mathfrak{f}(x)|-1} [-R, R]^{d_i}\right] \times \mathcal{M}_{\mathscr{C}_C(n)}(\mathfrak{f}(x), \mathfrak{f}(y))$$

be the extension of ι from (3.26). Recall that given $u \in \operatorname{Ob}(\underline{2}^n)$ and $x \in F(u)$ the cubical realization $\|\mathscr{C}\|$ has a corresponding cell

$$\mathcal{C}(x) = \begin{cases} \prod_{i=0}^{|u|-1}[-R, R]^{d_i} \times \prod_{i=|u|}^{n-1}[-\epsilon, \epsilon]^{d_i} \times [0, 1] \times \mathcal{M}_{\mathscr{C}_C(n)}(u, \vec{0}) & \text{if } u \neq \vec{0}, \\ \prod_{i=0}^{n-1}[-\epsilon, \epsilon]^{d_i} \times \{0\} & \text{if } u = \vec{0}. \end{cases}$$

The strategy of the proof is to use the cubical neat embedding to build a particular spatial refinement $\widetilde{F}_k$ of F and construct a map from hocolim $\widetilde{F}_k^+$, with the CW structure from Proposition 6.1, to the cubical realization $\|\mathscr{C}\|$ of the cubical flow category $\mathscr{C}$ associated to F, that sends cells to cells by degree ± 1 maps, and hence is a stable homotopy equivalence.

The diagram $\widetilde{F}_k$ is defined as follows. The box associated to $x \in F(u)$ is

$$B_x = \prod_{i=0}^{|u|-1} [-R, R]^{d_i} \times \prod_{i=|u|}^{n-1} [-\epsilon, \epsilon]^{d_i}.$$

Next, consider a sequence of composable nonidentity morphisms $u = u^0 \xrightarrow{f_1} \cdots \xrightarrow{f_m} u^m = v$ in $\underline{2}^n$. We will define

$$\widetilde{F}_k^+(f_m, \dots, f_1) = \Phi(e_{t_1,\dots,t_{m-1}}, F(\varphi_{u,v}))\colon [0, 1]^{m-1} \times \widetilde{F}_k^+(u) \to \widetilde{F}_k^+(v)$$

for an appropriate family of boxes $e_{t_1,\dots,t_{m-1}}\colon [0, 1]^{m-1} \to E(\{B_x \mid x \in F(u)\}, s_{F(\varphi_{u,v})})$. In other words, if for $\gamma \in F(\varphi_{u,v})$ we write $B_\gamma = \prod_{i=0}^{|v|-1}[-R, R]^{d_i} \times \prod_{i=|v|}^{n-1}[-\epsilon, \epsilon]^{d_i}$ then $e_{t_1,\dots,t_{m-1}}$ is a $[0, 1]^{m-1}$–parameter family of embeddings (as disjoint subboxes)

$$\coprod_{\substack{\gamma \\ s(\gamma)=x}} B_\gamma \hookrightarrow B_x \quad \text{for all } x \in F(u).$$

To define $e_{t_1,\dots,t_{m-1}}$, fix $\gamma \in F(\varphi_{u,v})$ with $s(\gamma) = x$, and let $y = t(\gamma)$. Let υ_γ be the section of the covering map $\mathcal{M}_{\mathscr{C}}(x, y) \to \mathcal{M}_{\mathscr{C}_C(n)}(u, v)$ whose image is the path component corresponding to γ. Consider the map

$$\begin{aligned}
&\mathcal{M}_{\mathscr{C}_C(n)}(u,v)\times B_\gamma\\
&\quad=\mathcal{M}_{\mathscr{C}_C(n)}(u,v)\times\prod_{i=0}^{|v|-1}[-R,R]^{d_i}\times\prod_{i=|v|}^{n-1}[-\epsilon,\epsilon]^{d_i}\\
&\quad\xhookrightarrow{(\upsilon_\gamma,\mathrm{Id})}\mathcal{M}_{\mathscr{C}}(x,y)\times\prod_{i=0}^{|v|-1}[-R,R]^{d_i}\times\prod_{i=|v|}^{n-1}[-\epsilon,\epsilon]^{d_i}\\
&\quad\cong\prod_{i=0}^{|v|-1}[-R,R]^{d_i}\times\Bigl(\prod_{i=|v|}^{|u|-1}[-\epsilon,\epsilon]^{d_i}\times\mathcal{M}_{\mathscr{C}}(x,y)\Bigr)\times\prod_{i=|u|}^{n-1}[-\epsilon,\epsilon]^{d_i}\\
&\quad\xhookrightarrow{(\mathrm{Id},\bar{\iota}_{x,y},\mathrm{Id})}\prod_{i=0}^{|v|-1}[-R,R]^{d_i}\times\Bigl(\prod_{i=|v|}^{|u|-1}[-R,R]^{d_i}\times\mathcal{M}_{\mathscr{C}_C(n)}(u,v)\Bigr)\times\prod_{i=|u|}^{n-1}[-\epsilon,\epsilon]^{d_i}\\
&\quad\overset{(\mathrm{Id},\pi^R,\mathrm{Id})}{\twoheadrightarrow}\prod_{i=0}^{|v|-1}[-R,R]^{d_i}\times\prod_{i=|v|}^{|u|-1}[-R,R]^{d_i}\times\prod_{i=|u|}^{n-1}[-\epsilon,\epsilon]^{d_i}\\
&\quad\cong\prod_{i=0}^{|u|-1}[-R,R]^{d_i}\times\prod_{i=|u|}^{n-1}[-\epsilon,\epsilon]^{d_i}=B_x
\end{aligned}$$

and the induced map

$$\mathcal{M}_{\mathscr{C}_C(n)}(u,v)\times\coprod_{\substack{\gamma\in F(\varphi_{u,v})\\ s(\gamma)=x}}B_\gamma\to B_x. \tag{6.3}$$

It follows from the definition of cubical neat embeddings and the formula for $\bar{\iota}_{x,y}$ that for any point pt $\in\mathcal{M}_{\mathscr{C}_C(n)}(u,v)$, the restriction $\{\mathrm{pt}\}\times\coprod_{\gamma\in F(\varphi_{u,v})|s(\gamma)=x}B_\gamma\to B_x$ is an inclusion of disjoint subboxes. Therefore, we may view the map from (6.3) as an $\mathcal{M}_{\mathscr{C}_C(n)}(u,v)$–parameter family of subboxes $\coprod_{\gamma|s(\gamma)=x}B_\gamma\subset B_x$.

The chain $u=u^0>\cdots>u^m=v$ corresponds to some cube $[0,1]^{m-1}$ in the cubical complex $M_{u,v}$ from Definition 3.19, which via Lemma 3.20 is identified with some cube $[0,1]^{m-1}\subset\mathcal{M}_{\mathscr{C}_C(n)}(u,v)$. Restrict the map from formula (6.3) to $[0,1]^{m-1}\times\coprod_{\gamma|s(\gamma)=x}B_\gamma$ to obtain the required $[0,1]^{m-1}$–parameter family of subboxes $\coprod_{\gamma|s(\gamma)=x}B_\gamma\subset B_x$.

We check that $\widetilde{F}_k$ is indeed a homotopy coherent diagram. For any sequence of composable nonidentity morphisms $u=u^0\xrightarrow{f_1}\cdots\xrightarrow{f_m}u^m=v$ in $\underline{2}^n$, we need to show that

$$\begin{aligned}&\widetilde{F}_k(f_m,\dots,f_1)(t_1,\dots,t_{i-1},1,t_{i+1},\dots,t_{m-1})\\&\qquad\qquad=\widetilde{F}_k(f_m,\dots,f_{i+1}\circ f_i,\dots,f_1)(t_1,\dots,t_{i-1},t_{i+1}\dots,t_{m-1}),\\&\widetilde{F}_k(f_m,\dots,f_1)(t_1,\dots,t_{i-1},0,t_{i+1},\dots,t_{m-1})\\&\qquad\qquad=\widetilde{F}_k(f_m,\dots,f_{i+1})(t_{i+1},\dots,t_{m-1})\circ\widetilde{F}_k(f_i,\dots,f_1)(t_1,\dots,t_{i-1}).\end{aligned}$$

The first equation is immediate from Definition 3.19 since the facet of $[0,1]^{m-1}$ which has $t_i=1$ is identified with the cube $[0,1]^{m-2}$ coming from the sequence of composable morphisms $u=u^0\xrightarrow{f_1}\cdots\xrightarrow{f_{i-1}}u^{i-1}\xrightarrow{f_{i+1}\circ f_i}u^{i+1}\xrightarrow{f_{i+2}}\cdots\xrightarrow{f_m}u^m=v$. The second equation follows from Lemma 3.20(3) since the facet of $[0,1]^{m-1}$ that has $t_i=0$ lies in the facet $\mathcal{M}_{\mathscr{C}_C(n)}(u^i,v)\times\mathcal{M}_{\mathscr{C}_C(n)}(u,u^i)$ of $\mathcal{M}_{\mathscr{C}_C(n)}(u,v)$ and is identified with the product $[0,1]^{m-i-1}\times[0,1]^{i-1}$, coming from the sequences $u^i\xrightarrow{f_{i+1}}\cdots\xrightarrow{f_m}u^m=v$ and $u^0\xrightarrow{f_1}\cdots\xrightarrow{f_i}u^i$, respectively. Since the maps $\widetilde{F}_k^+$ were defined via cubical neat embeddings that satisfied Definition 3.25(3), the second equation holds.

Finally, we construct the desired cellular map from $\operatorname{hocolim}\widetilde{F}_k^+$ to $\|\mathscr{C}\|$.

For any $u\in\mathrm{Ob}(\underline{2}^n)$ and any $x\in F(u)$, the cell associated to x in $\operatorname{hocolim}\widetilde{F}_k^+$ is

$$\mathcal{C}(x)'=\begin{cases}\mathcal{M}_{\mathscr{C}_C(n)}(u,\vec{0})\times[0,2]\times B_x & \text{if } u\neq\vec{0},\\ \{0\}\times B_x & \text{if } u=\vec{0},\end{cases}$$

while the cell associated to x in $\|\mathscr{C}\|$ is

$$\mathcal{C}(x)=\begin{cases}\mathcal{M}_{\mathscr{C}_C(n)}(u,\vec{0})\times[0,1]\times B_x & \text{if } u\neq\vec{0},\\ \{0\}\times B_x & \text{if } u=\vec{0}.\end{cases}$$

Map $\mathcal{C}(x)'$ to $\mathcal{C}(x)$ by the quotient map $[0,2]\to[0,2]/[1,2]\cong[0,1]$, and the identity map on all other factors. This map certainly has degree ±1 on each cell. To check that it produces a well-defined map on CW complexes, we must check that it commutes with the attaching maps. Everything over ∂B_x was quotiented to the basepoint on either side. If $u=\vec{0}$, this is the only identification on either side. If $u\neq\vec{0}$, everything over $\{2\}\subset[0,2]$ was quotiented to the basepoint for $\mathcal{C}(x)'$, while everything over $\{1\}\subset[0,1]$ was quotiented to the basepoint for $\mathcal{C}(x)$. Therefore, we only need to consider $u\neq\vec{0}$ and concentrate on the attaching maps on the portion of the boundary of $\mathcal{C}(x)$ (respectively $\mathcal{C}(x)'$) that lives over $\partial\mathcal{M}_{\mathscr{C}_C(n)}(u,\vec{0})$ or $\{0\}\subset\partial([0,1])$ (respectively $\{0\}\subset\partial([0,2])$).

Consider the subcomplex

$$\widetilde{M}_u := \Biggl(\coprod_{\substack{u=u^0>\cdots>u^m \\ u^i \in \{0,1\}^n}} [0,1]^m \Biggr) \Big/ \sim_1 \;\cong \mathcal{M}_{\mathscr{C}_C(n)}(u, \vec{0}) \times [0,1]$$

of the cubical complex $M_u \cong \mathcal{M}_{\mathscr{C}_C(n)}(u, \vec{0}) \times [0,2]$ from equation (6.2) in the proof of Proposition 6.1. We are interested in the part of $\partial \mathcal{C}(x)'$ that lives over the following subset N_u of $\partial \widetilde{M}_u$:

$$\begin{aligned} N_u &:= \bigl(\partial \mathcal{M}_{\mathscr{C}_C(n)}(u, \vec{0}) \times [0,1]\bigr) \cup \bigl(\mathcal{M}_{\mathscr{C}_C(n)}(u, \vec{0}) \times \{0\}\bigr) \\ &= \Biggl(\coprod_{u=u^0>\cdots>u^m} \coprod_{i=1}^{m} [0,1]^{i-1} \times \{0\} \times [0,1]^{m-i} \Biggr) \Big/ \sim_1 . \end{aligned}$$

Let $p \in \partial \mathcal{C}(x)'$ be some point living over N_u, and write $p = (p_1, p_2)$, where $p_1 \in N_u$ and $p_2 \in B_x$. Assume p_1 lies in the cube $[0,1]^m$ corresponding to some chain $u = u^0 > \cdots > u^m$. Let $p_{1,1}, \dots, p_{1,m}$ be the coordinates of p_1 as a point in the cube, and assume $p_{1,\ell} = 0$. Let ϕ denote the restriction of $\widetilde{F}_k(\varphi_{u^{\ell-1},u_\ell}, \dots, \varphi_{u^0,u^1})$ to $[0,1]^{\ell-1} \times (B_x/\partial B_x)$. Under the CW complex attaching map (denoted by $\sim_2$ in the proof of Proposition 6.1), p is attached to the point

$$\bigl((p_{1,\ell+1}, \dots, p_{1,m}), \phi((p_{1,1}, \dots, p_{1,\ell-1}), p_2)\bigr) \in [0,1]^{m-\ell} \times \textstyle\bigvee_{y \in F(u^\ell)} (B_y/\partial B_y),$$

where $[0,1]^{m-\ell}$ is the cube in $\widetilde{M}_{u^\ell}$ corresponding to the chain $u^\ell > \cdots > u^m$. The map ϕ is constructed as a $[0,1]^{m-\ell}$–parameter family of box maps. Therefore, the attaching map glues p to the basepoint $*$ unless p_2 lies in the interior of one of the subboxes at the point $(p_{1,1}, \dots, p_{1,\ell-1})$ in the family; and if p_2 lies in the interior of some box B_{γ_0} then p is glued to the point

$$q := ((p_{1,\ell+1}, \dots, p_{1,m}), q_2) \in [0,1]^{m-\ell} \times B_{\gamma_0} \subset \widetilde{M}_{u^\ell} \times B_{\gamma_0} \subset \mathcal{C}(y_0)',$$

where $y_0 = t(\gamma_0)$ and $q_2 := \phi((p_{1,1}, \dots, p_{1,\ell-1}), p_2) \in B_{\gamma_0}$.

We need to check that p, now viewed as a point in $\partial \mathcal{C}(x)$, is also glued to q, now viewed as a point in $\mathcal{C}(y_0)$, in the CW complex $\|\mathscr{C}\|$. In the construction of $\|\mathscr{C}\|$ (Definition 3.29), we extended the cubical neat embedding

$$\iota_{x,y_0} \colon \mathcal{M}_{\mathscr{C}}(x, y_0) \hookrightarrow \mathcal{M}_{\mathscr{C}_C(n)}(u^\ell, \vec{0}) \times \prod_{i=|u^\ell|}^{|u|-1} (-R, R)^{d_i}$$

to an embedding

$$\bar{\imath}_{x,y_0}\colon \mathcal{M}_{\mathscr{C}}(x,y_0)\times\prod_{i=|u^\ell|}^{|u|-1}[-\epsilon,\epsilon]^{d_i}\hookrightarrow \mathcal{M}_{\mathscr{C}_C(n)}(u^\ell,\vec{0})\times\prod_{i=|u^\ell|}^{|u|-1}[-R,R]^{d_i},$$

and used it to define embeddings $\imath\colon \mathcal{M}_{\mathscr{C}}(x,y_0)\times B_{y_0}\hookrightarrow \mathcal{M}_{\mathscr{C}_C(n)}(u,u^\ell)\times B_x$ and $\jmath\colon \mathcal{M}_{\mathscr{C}}(x,y_0)\times \mathcal{C}(y_0)\hookrightarrow \partial\mathcal{C}(x)$. (Note that $\widetilde{\mathcal{M}}_{\mathscr{C}_C(n)}(u^\ell,\vec{0})=\widetilde{M}_{u^\ell}$.)

For any path component γ of $\mathcal{M}_{\mathscr{C}}(x,y_0)$, let $\imath_\gamma$ and $\jmath_\gamma$ denote the restrictions $\imath|_{\gamma\times B_{y_0}}$ and $\jmath|_{\gamma\times\mathcal{C}(y_0)}$; and as before, let υ_γ denote the section of $\mathcal{M}_{\mathscr{C}}(x,y_0)\to\mathcal{M}_{\mathscr{C}_C(n)}(u,u^\ell)$ whose image is γ. Since $\imath_{x,y_0}$ satisfies Definition 3.25(1), and its extension $\bar{\imath}_{x,y_0}$ was defined via (3.26), there exists a map $\mu_\gamma\colon B_{y_0}\to\mathcal{M}_{\mathscr{C}_C(n)}(u,u^\ell)\times B_x$ such that $\imath_\gamma(\upsilon_\gamma(a),b)=(a,\mu_\gamma(a,b))$ for all $(a,b)\in\mathcal{M}_{\mathscr{C}_C(n)}(u,u^\ell)\times B_{y_0}$.

Let $q_1=(p_{1,\ell+1},\dots,p_{1,m})\in[0,1]^{m-\ell}$ and $q'=(p_{1,1},\dots,p_{1,\ell-1})\in[0,1]^{\ell-1}$; treat q' as a point in $\mathcal{M}_{\mathscr{C}_C(n)}(u,u^\ell)$, after viewing the cube $[0,1]^{\ell-1}$ as the cube corresponding to the chain $u=u^0>\dots>u^\ell$ in the cubical complex structure on $\mathcal{M}_{\mathscr{C}_C(n)}(u,u^\ell)$ from Lemma 3.20. Let κ be the inclusion map $\mathcal{M}_{\mathscr{C}_C(n)}(u,u^\ell)\times\widetilde{M}_{u^\ell}\times B_x\hookrightarrow\mathcal{C}(x)$. Since $\widetilde{F}_k(\varphi_{u^{\ell-1},u_\ell},\dots,\varphi_{u^0,u^1})$ (used to define ϕ) is defined via (6.3) using the cubical neat embeddings $\imath_{x,y_0}$ (which are used to define the maps $\imath_\gamma$, and, consequently, μ_γ), p_2 is in the interior of the box B_{y_0}, and $\phi(q',p_2)=q_2$, it follows that $\mu_{\gamma_0}(q',q_2)=p_2$. Therefore,

$$\begin{aligned}\jmath_{\gamma_0}(\upsilon_\gamma(q'),q)&=\jmath_{\gamma_0}(\upsilon_\gamma(q'),q_1,q_2)=\kappa(q',q_1,\mu_{\gamma_0}(q',q_2))\\&=\kappa(q',q_1,p_2)=\kappa((p_{1,1},\dots,p_{1,\ell-1}),(p_{1,\ell+1},\dots,p_{1,m}),p_2)\\&=((p_{1,1},\dots,p_{1,\ell-1},0,p_{1,\ell+1},\dots,p_{1,m}),p_2)=(p_1,p_2)=p.\end{aligned}$$

The last line is justified by the fact that the cubical complex structures on $\mathcal{M}_{\mathscr{C}_C(n)}(u,u^\ell)$ and $\widetilde{M}_{u^\ell}$ respect the product structure on facets of $\widetilde{M}_u=\mathcal{M}_{\mathscr{C}_C(n)}(u,\vec{0})\times[0,1]$ (Lemma 3.20(3)). Therefore, p is glued to q in the CW complex $\|\mathscr{C}\|$ as well. □

7 The Khovanov homotopy type

We pause briefly to review where we stand. We have introduced a special kind of flow categories, cubical flow categories (Section 3), and shown that the data of a cubical flow category is equivalent to a strictly unitary, lax 2–functor from the cube $\underline{2}^n$ to the Burnside 2–category (Section 4). Given a cubical flow category (or functor from the cube to the Burnside category) we have four ways of realizing the functor as a spectrum:

- The original Cohen–Jones–Segal realization (Section 3.2).
- The cubical realization, a modification of the Cohen–Jones–Segal construction taking into account the map to the cube (Section 3.7).
- Thickening the diagram, producing a canonical diagram in spectra, and taking the iterated mapping cone (Section 4).
- Using the “little box” construction to produce a homotopy coherent cube in spectra, and then taking the iterated mapping cone (Section 5.3).

Moreover, Theorems 4, 7 and 8 together imply that, up to stable homotopy equivalence, these realizations all agree.

For the rest of the paper, we turn to a particular cubical flow category: the Khovanov flow category constructed in [35] (see also Section 2.2 and Examples 3.24 and 4.21).

Definition 7.1 Given an oriented link diagram K, let $F_{Kh}(K)\colon \underline{2}^n \to \mathscr{B}$ be the Khovanov functor constructed in Example 4.21. Let $|F_{Kh}(K)|$ be the result of applying Construction 4.39 to $F_{Kh}(K)$ and let $\mathcal{X}_{Kh}(K) = \Sigma^{-n_-}|F_{Kh}(K)|$, where n_- is the number of negative crossings in K.

Theorems 4, 7 and 8 imply that, up to stable homotopy equivalence, $\mathcal{X}_{Kh}(K)$ is the same as applying any of the other three realization constructions to $F_{Kh}(K)$. A more direct description of $F_{Kh}(K)$ was given by Hu, Kriz and Kriz [22]; see Section 8.1.

Similar constructions can be carried out for the reduced Khovanov flow category:

Definition 7.2 Let $F_{\widetilde{Kh}}(K)\colon \underline{2}^n \to \mathscr{B}$ be the functor corresponding to the reduced Khovanov flow category from [35, Section 8], via Construction 4.17. Let $\widetilde{\mathcal{X}}_{Kh}(K)$ be the corresponding spectrum, obtained by applying Construction 4.39 to $F_{\widetilde{Kh}}(K)$ and desuspending n_- times.

8 Relationship with Hu–Kriz–Kriz

The goal of this section is to prove:

Theorem 9 *Fix a link diagram K. Let $M(K)$ be the homotopy type associated to K by Hu, Kriz and Kriz* [22, Theorem 5.4]. *Then $\mathcal{X}_{Kh}(K)$ is stably homotopy equivalent to $M(K)$.*

The Hu–Kriz–Kriz construction has four steps:

(1) Construct a 2–functor $\underline{2}^n \to \mathscr{B}$. (The category $\underline{2}^n$ is denoted by I^n in [22], and the category $\mathscr{B}$ is denoted by $\mathcal{S}_2$.)

(2) Use the Elmendorf–Mandell machine [12] to turn the 2–functor $\underline{2}^n \to \mathscr{B}$ into an A_∞–functor $B_2(\underline{2}^n)' \to \mathscr{S}$, where $B_2(\underline{2}^n)'$ is an auxiliary category which we review below.

(3) Use Elmendorf and Mandell's rectification result [12, Theorem 1.4] to lift this composition to a strict functor $\underline{2}^n \to \mathscr{S}$.

(4) Expand $\underline{2}^n$ to a category $\mathcal{I}$, analogous to the expansion of $\underline{2}^n$ to $\underline{2}^n_\dagger$, extend the functor $\underline{2}^n \to \mathscr{S}$ to a functor $\mathcal{I} \to \mathscr{S}$, and take the homotopy colimit.

We will prove that the two constructions agree, step-by-step.

8.1 The functors from the cube to the Burnside category agree

Hu, Kriz and Kriz's construction of the Khovanov homotopy type uses the following auxiliary category:

Definition 8.1 The $(1+1)$*–dimensional embedded cobordism category* $\mathsf{Cob}^{1+1}_{\mathrm{emb}}$ is the 2–category defined as follows. The objects of $\mathsf{Cob}^{1+1}_{\mathrm{emb}}$ are oriented 1–manifolds C embedded in S^2 along with a 2–coloring of the components of $S^2 \setminus C$, by the colors "white" and "black", such that if $B(S^2 \setminus C)$ denotes the closure of the black region, then C is the oriented boundary of $B(S^2 \setminus C)$. The morphisms from C_1 to C_0 are oriented cobordisms Σ embedded in $[0,1] \times S^2$ satisfying $\Sigma \cap (\{i\} \times S^2) = \{i\} \times C_i$ for $i \in \{0,1\}$, along with a 2–coloring of $([0,1] \times S^2) \setminus \Sigma$, so that if $B(([0,1] \times S^2) \setminus \Sigma)$ denotes the closure of the black region, then Σ is oriented as the boundary of $B(([0,1] \times S^2) \setminus \Sigma)$. The 2–morphisms are isotopy classes of isotopies of cobordisms relative boundary.

The Hu–Kriz–Kriz functor $\underline{2}^n \to \mathscr{B}$ is constructed in two steps [22, Section 5], which we give as Constructions 8.2 and 8.3.

Construction 8.2 [22, Section 4.3] Given a link diagram L in S^2 with n crossings $c_1, \dots, c_n$, along with a checkerboard coloring of the link diagram, we construct a lax 2–functor from $\underline{2}^n$ to $\mathsf{Cob}^{1+1}_{\mathrm{emb}}$. This functor was partially described in Section 2.2: to $v \in \{0,1\}^n$, associate the complete resolution $\mathcal{P}(v)$ which is a collection of disjoint circles in S^2. The checkerboard coloring for L induces a 2–coloring of the complement of these circles; orient the circles as the boundary of the black region. To $u > v \in \{0,1\}^n$, associate the embedded cobordism $\Sigma \subset [0,1] \times S^2$, which is a product cobordism outside a neighborhood of the crossings where u and v differ, and has a saddle for each such crossing. We declare the cobordism to be running from $\mathcal{P}(u) = \Sigma \cap (\{1\} \times S^2)$ to $\mathcal{P}(v) = \Sigma \cap (\{0\} \times S^2)$. Up to isotopy, Σ is independent of the order of the saddles, and in fact the isotopies changing the order of saddles are themselves well defined up to isotopy. Thus, this construction gives a lax 2–functor $\underline{2}^n \to \mathsf{Cob}^{1+1}_{\mathrm{emb}}$.

Construction 8.3 [22, Section 3.4] There is a lax 2–functor $\mathcal{L}\colon \mathsf{Cob}^{1+1}_{\mathrm{emb}} \to \mathscr{B}$ defined as follows. On objects, $\mathcal{L}$ sends a 1–manifold $C \subset S^2$ to the set of all possible labelings of the components of C by elements of $\{x_+, x_-\}$, $\mathcal{L}(C) = \prod_{C_i \in \pi_0(C)} \{x_+, x_-\}$. The value of $\mathcal{L}$ on morphisms is more complicated. For any embedded cobordism Σ and any connected component Σ_0 of Σ, consider the 2–coloring of $([0,1]\times S^2)\setminus\Sigma_0$ that agrees with the given 2–coloring of $([0,1]\times S^2)\setminus\Sigma$ near Σ_0, and let $B(([0,1]\times S^2)\setminus\Sigma_0))$ denote the closure of the black region. Observe that

$$H_1(([0,1]\times S^2)\setminus\Sigma_0)/H_1((\{0,1\}\times S^2)\setminus\partial\Sigma_0) \cong \mathbb{Z}^{2g(\Sigma_0)},$$
$$H_1(B(([0,1]\times S^2)\setminus\Sigma_0))/H_1(B((\{0,1\}\times S^2)\setminus\partial\Sigma_0)) \cong \mathbb{Z}^{g(\Sigma_0)}.$$

A *valid labeling* of a cobordism $\Sigma \subset [0,1]\times S^2$ consists of

- a labeling of each boundary component of Σ by x_+ or x_-, and
- a labeling of each genus 1 component Σ_0 of Σ by α or $-\alpha$, where $\{\pm\alpha\}$ are the generators of $H_1\big(B(([0,1]\times S^2)\setminus\Sigma_0)\big)/H_1\big(B((\{0,1\}\times S^2)\setminus\partial\Sigma_0)\big) \cong \mathbb{Z}$,

such that:

- Each connected component of Σ has genus 0 or 1.
- For each genus 0 connected component of Σ, the number of boundary components in $\{0\}\times S^2$ labeled x_- plus the number of boundary components in $\{1\}\times S^2$ labeled x_+ is 1.
- For each genus 1 connected component of Σ, all boundary components in $\{0\}\times S^2$ are labeled x_+ and all boundary components in $\{1\}\times S^2$ are labeled x_-.

(See [22, Formula (12)].) Define $\mathcal{L}(\Sigma)$ to be the set of valid labelings of Σ. The source and target maps of $\mathcal{L}(\Sigma)$ send a labeling of Σ to the induced labeling of the boundary components.

For Σ_0 and Σ_1 composable cobordisms in $\left[0,\frac{1}{2}\right]\times S^2$ and $\left[\frac{1}{2},1\right]\times S^2$, respectively, the composition 2–isomorphism $\mathcal{L}(\Sigma_0)\circ\mathcal{L}(\Sigma_1) \xrightarrow{\cong} \mathcal{L}(\Sigma_0\circ\Sigma_1)$ is obvious except for the situation when one gets a cobordism $\Sigma \subset [0,1]\times S^2$ with a genus 1 component by stacking two genus 0 cobordisms. The composition map decomposes as a product over connected components of Σ, so for simplicity assume that Σ is connected. Given valid labelings on Σ_0 and Σ_1 that agree on $(\partial\Sigma_0)\cap(\partial\Sigma_1)\subset\{\frac{1}{2}\}\times S^2$, we want to construct a valid labeling on Σ, which amounts to labeling Σ by α or $-\alpha$. It follows from the labeling conditions that there is a unique component C of $(\partial\Sigma_0)\cap(\partial\Sigma_1)$ that is nonseparating in Σ and is labeled x_+. Orient C as the boundary of the black region, and let C_b and C_w be the push-offs of C into the black and the white regions, respectively. One of C_b and C_w is a generator of $H_1(([0,1]\times S^2)\setminus\Sigma)/H_1((\{0,1\}\times S^2)\setminus\partial\Sigma)\cong\mathbb{Z}^2$ and the other one is zero. If C_b is the generator, label Σ by $[C]$. If C_w is the generator, let D be a curve on Σ, oriented so that the algebraic intersection number is $D\cdot C=1$, and label Σ by $[D]$. It is clear that these composition 2–isomorphisms satisfy the coherence condition (Fn-2).

Remark 8.4 It is not hard to see that choosing the other checkerboard coloring on the link diagram yields a naturally isomorphic functor $\underline{2}^n\to\mathscr{B}$. On the other hand, one could have required $D\cdot C$ to be -1 instead of 1; this would have produced a different functor. This global choice is essentially the choice of ladybug matching from [35, Section 5.4].

Lemma 8.5 *Hu, Kriz and Kriz's* 2*–functor* $F_{HKK}\colon\underline{2}^n\to\mathscr{B}$ [22] *is naturally isomorphic to the* 2*–functor* F_{Kh} *constructed in Example 4.21 by applying Construction 4.17 to the Khovanov flow category from* [35].

Proof The functors F_{HKK} and F_{Kh} are identical on objects. By Lemma 4.5, we only need to show that they agree on the edges and that the composition 2–isomorphisms for the functors agree on the 2–dimensional faces of the cube.

For $u>v\in\{0,1\}^n$ with $|u|-|v|=1$, the corresponding embedded cobordism is a merge or split. Let F denote either F_{HKK} or F_{Kh}. It is straightforward from the definitions that for any $x\in F(u)$ and $y\in F(v)$, $s^{-1}(x)\cap t^{-1}(y)\subset F(\varphi_{u,v})$ is empty if x does not appear in the Khovanov differential of y, and consists of one point otherwise, so in particular there is a unique bijection $F_{HKK}(\varphi_{u,v})\cong F_{Kh}(\varphi_{u,v})$.

Now consider $u>w\in\{0,1\}^n$ with $|u|-|w|=2$, and let v and v' be the two intermediate vertices. The composition 2–isomorphisms for F_{HKK} and F_{Kh} produce bijections between $F(\varphi_{v,w})\circ F(\varphi_{u,v})$ and $F(\varphi_{v',w})\circ F(\varphi_{u,v'})$. We want to show that these two bijections are the same.

Fix $x\in F(u)$ and $z\in F(w)$, and consider $s^{-1}(x)\cap t^{-1}(z)\subset F(\varphi_{u,w})$. This set can have 0, 1, or 2 elements. The only nontrivial case to check is when the set has 2

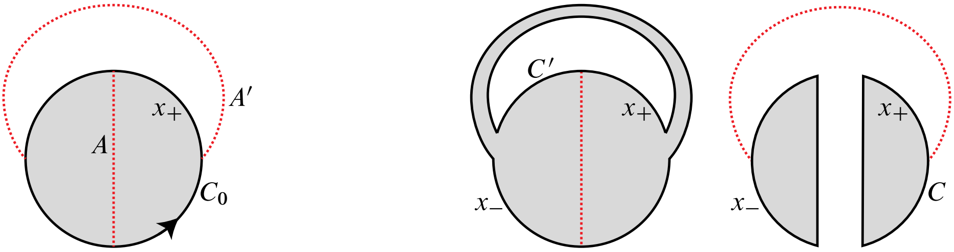


Figure 7: The ladybug matching. Left: The ladybug configuration. Right: Two intermediate configurations, obtained by attaching embedded 1–handles along the two arcs. Black regions are indicated by dark shading.

elements, which occurs precisely when x and z are related by a ladybug configuration (Figure 7).

Therefore, assume x and z are related by a ladybug configuration, and without loss of generality assume that the corresponding embedded genus 1 cobordism Σ is connected. The composition 2–isomorphisms for F are unchanged under isotopy in S^2: this is [35, Lemma 5.8] for F_{Kh}, and is immediate from the definition for F_{HKK}. Therefore, we may further assume that the ladybug configuration is as shown in Figure 7, in the following sense. The circle C_0 is the complete resolution at w, and it is labeled x_+ by z; and the circle C_1, the complete resolution at u, is obtained by attaching embedded 1–handles along the arcs A and A', and C_1 is labeled x_- by x. The embedded cobordism $\Sigma \subset [0,1] \times S^2$ connects C_1 to C_0, that is, $\Sigma \cap (\{i\} \times S^2) = \{i\} \times C_i$. Further, the black region contains the arc A. Consider the labelings of the circles at the complete resolutions at v and v' shown in Figure 7; the corresponding generators $y \in F(v)$ and $y' \in F(v')$ are matched by the ladybug matching for F_{Kh} [35, Figure 5.1b].

To show that y and y' are also matched by F_{HKK}, let C (respectively C') be the circle that is labeled x_+ by y (respectively y'); therefore, C (respectively C') is a homology generator of the white (respectively black) component of the complement of Σ. Isotope the cobordism Σ inside $[0,1] \times S^2$ relative boundary so that both the saddles occur at $\left\{\frac{1}{2}\right\} \times S^2$. Isotope the curves C and C' on Σ so that they intersect transversally at one point, C lies in $\left[0, \frac{1}{2}\right] \times S^2$ and C' lies in $\left[\frac{1}{2}, 1\right] \times S^2$. Therefore, C and C' intersect at the saddle point p corresponding to the arc A. The portion of the cobordism Σ near p is shown in Figure 8. Let $\vec{n}$ be the normal vector to Σ at p pointing away from the black region, and let $\vec{t}$ and $\vec{t}'$ be the tangent vectors to C and C' at p. It is clear from Figure 8 that $[\vec{n}, \vec{t}', \vec{t}]$ constitutes a positive basis; consequently in Σ, oriented as the boundary of the black region, the intersection number $C' \cdot C = 1$. Therefore,

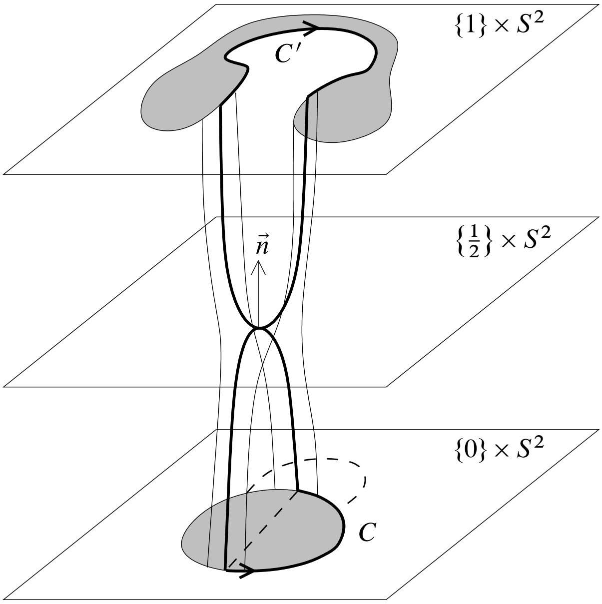


Figure 8: The embedded cobordism Σ. The cobordism connects C_0 at the bottom to C_1 at the top. The portion of the cobordism near the saddle point p, the oriented curves C and C', and the normal vector $\vec{n}$ are shown.

for both y and y', the surface Σ is labeled by $[C']$, and thus y and y' are matched by F_{HKK}. $\square$

8.2 Iterated mapping cones

The easiest part of the identification is to see that the Hu–Kriz–Kriz notion of iterated mapping cone agrees with ours. Specifically, to take the iterated mapping cone of their functor $K(F_{HKK})\colon \underline{2}^n \to \mathscr{S}$ (see Sections 8.1 and 8.4) they enlarge $\underline{2}^n$ slightly to a category $\mathcal{I}$, and extend F_{HKK} to a functor $\widetilde{F}_{HKK}\colon \mathcal{I} \to \mathscr{S}$ by declaring that the new vertices in $\mathcal{I}$ map to $\{*\}$, a one-point space [22, Section 5.2]. We observe that their enlargement is the same as $\underline{2}^n_{\dagger}$:

Lemma 8.6 *There is an isomorphism $\mathcal{I} \cong \underline{2}^n_{\dagger}$ which commutes with the inclusions of $\underline{2}^n$:*

$$\begin{tikzcd}
\mathcal{I} \arrow[rr, "\cong"] & & \underline{2}^n_{\dagger} \\
& \underline{2}^n \arrow[ul, hook] \arrow[ur, hook] &
\end{tikzcd}$$

Proof The category $\mathcal{I}$ has, as objects, pairs $(J \subseteq \{1, \dots, n\},\ \phi\colon J \to \{0,1\})$, and there is a morphism $\phi \to \psi$ (which is unique) if and only if ϕ is a restriction of ψ. The cube $\underline{2}^n$ sits in $\mathcal{I}$ as the subcategory $\{(J,\phi) \mid 0 \notin \operatorname{im}(\phi)\}$. Recall that $\underline{2}^n_{\dagger} = (\underline{2}^1_{+})^n$, and $\operatorname{Ob}(\underline{2}^1_{+}) = \{0, 1, *\}$. Given an object $o = (v_1, \dots, v_n) \in \underline{2}^n_{\dagger}$, define $J = \{i \mid v_i \in \{0, *\}\}$ and

$$\phi(i) = \begin{cases} 0 & \text{if } v_i = *, \\ 1 & \text{if } v_i = 0. \end{cases}$$

With this dictionary, the rest of the verification is straightforward. □

8.3 Another kind of homotopy coherent diagram

Hu, Kriz and Kriz use a slightly different notion from Vogt of homotopy coherent diagrams (Section 4.2), which is defined in two steps:

Definition 8.7 Given a small category $\mathscr{C}$, let $\mathscr{C}'$ be the 2–category with the same objects as $\mathscr{C}$,

$$\operatorname{Hom}_{\mathscr{C}'}(x,y) = \coprod_{x=x_0,x_1,\dots,x_n=y} \operatorname{Hom}_{\mathscr{C}}(x_{n-1},x_n) \times \operatorname{Hom}_{\mathscr{C}}(x_{n-2},x_{n-1}) \times \cdots \times \operatorname{Hom}_{\mathscr{C}}(x_0,x_1),$$

the set of finite sequences of composable morphisms starting at x and ending at y, and a unique 2–morphism from $(f_n, \dots, f_1)$ to $(g_m, \dots, g_1)$ whenever $f_n \circ \cdots \circ f_1 = g_m \circ \cdots \circ g_1$ (compare [22, Section 4.1]). Composition is given by concatenation of sequences. There is a projection $\mathscr{C}' \to \mathscr{C}$, where we view $\mathscr{C}$ as a 2–category with only identity 2–morphisms, which sends $(f_n, \dots, f_1)$ to $f_n \circ \cdots \circ f_1$.

Definition 8.8 Given a 2–category $\mathscr{D}$, we can form a simplicially enriched category $B_2(\mathscr{D})$ by replacing each Hom category in $\mathscr{D}$ by the realization of its nerve.

Combining Definitions 8.7 and 8.8, the analogue of a homotopy coherent diagram in [22] is a (simplicially enriched) functor $B_2(\mathscr{C}') \to \mathscr{S}$, or more generally an A_∞–functor $B_2(\mathscr{C}') \to \mathscr{S}$. Note that there is a projection $\Pi\colon B_2(\mathscr{C}') \to \mathscr{C}$ induced by the projection $\mathscr{C}' \to \mathscr{C}$ (and the triviality that $B_2(\mathscr{C}) = \mathscr{C}$). So, given a functor $F\colon \mathscr{C} \to \mathscr{S}$ (say) there is an induced (simplicially enriched) functor $B_2(F') = F \circ \Pi\colon B_2(\mathscr{C}') \to \mathscr{S}$.

Lemma 8.9 *For any small category $\mathscr{C}$, the projection map $\Pi\colon B_2(\mathscr{C}') \to \mathscr{C}$ is a homotopy equivalence on each* Hom *space.*

Proof The category $\mathrm{Hom}_{\mathscr{C}'}(x,y)$ decomposes as a disjoint union of subcategories

$$\mathrm{Hom}_{\mathscr{C}'}(x,y) = \coprod_{f \in \mathrm{Hom}(x,y)} \{(f_n,\dots,f_1) \mid f_n \circ \cdots \circ f_1 = f\},$$

and for each of these subcategories, every object is initial. □

The notion of homotopy colimits extends easily to functors from simplicially enriched categories: in formula (4.11), say, one replaces the disjoint union over sequences of composable morphisms with the disjoint union of products $\coprod_{x_0,\dots,x_n} \mathrm{Hom}(x_{n-1},x_n) \times \cdots \times \mathrm{Hom}(x_0,x_1) \times [0,1]^n \times F(x_0)$, and quotients by the same equivalence relation from Definition 4.10. The properties of homotopy colimits stated in Section 4.2 extend without change to diagrams from simplicially enriched categories.

Lemma 8.10 *Let $F\colon \mathscr{C} \to \mathscr{S}$ be a diagram from a small category $\mathscr{C}$. Then there is a homotopy equivalence* $\mathrm{hocolim}_{\mathscr{C}} F \simeq \mathrm{hocolim}_{B_2(\mathscr{C}')} B_2(F')$.

Proof By definition $B_2(F') = F \circ \Pi$. By Lemma 8.9, the projection Π is a quasi-equivalence, and in particular homotopy cofinal, so the result follows from property (ho-4) of homotopy colimits. □

8.4 The Elmendorff–Mandell machine

A permutative category is a strictly unital, strictly associative, symmetric monoidal category (see eg [12, Definition 3.1]). Let Permu denote the (strict) 2–category of permutative categories [12, Definition 3.2]. The Elmendorff–Mandell machine [12] is a functor

$$K\colon B_2(\mathsf{Permu}) \to \mathscr{S}$$

from (B_2 of) the category of permutative categories to spectra. (Constructions of this type go back to Segal [51].) Rather than explain how the machine works, we list the properties we will need:

(EM-1) Given a permutative category $\mathscr{C}$ and an object x in $\mathscr{C}$, there is an induced map $K(x)\colon \mathbb{S} \to K(\mathscr{C})$. Further, this is natural in the sense that given a functor of permutative categories $F\colon \mathscr{C} \to \mathscr{D}$ the following diagram commutes:

$$\begin{tikzcd}
\mathbb{S} \arrow[d, "K(x)"'] \arrow[dr, "K(F(x))"] & \\
K(\mathscr{C}) \arrow[r, "K(F)"'] & K(\mathscr{D})
\end{tikzcd}$$

(EM-2) The Cartesian product is both the categorical product and coproduct in the category of permutative categories.

(EM-3) Given permutative categories $\mathscr{C}$ and $\mathscr{D}$, $K(\mathscr{C}\times\mathscr{D}) = K(\mathscr{C})\times K(\mathscr{D})$.

(EM-4) The category Sets of finite sets, with disjoint union, is equivalent to a permutative category [23]; to keep the exposition clear we will continue to use the name Sets for this category. The map $\mathbb{S}\to K(\mathsf{Sets})$ induced by a 1–element set and property (EM-1) is a stable homotopy equivalence. (This is a version of the Barratt–Priddy–Quillen theorem.)

(EM-5) Given a 2–category $\mathscr{C}$ in which all 2–morphisms are isomorphisms and a strict 2–functor $F\colon \mathscr{C}\to\mathsf{Permu}$, there is an induced (simplicially enriched) functor $K(B_2(F))\colon B_2(\mathscr{C})\to\mathscr{S}$.

Construction 8.11 Given a set X, we can consider the category $\prod_{x\in X}\mathsf{Sets}$. Given a correspondence (C,s,t) from X to Y, there is an induced functor $\prod_{x\in X}\mathsf{Sets}\to\prod_{y\in Y}\mathsf{Sets}$ which sends

$$(A_x)_{x\in X}\mapsto\Bigl(\bigcup_{x\in X}(s^{-1}(x)\cap t^{-1}(y))\times A_x\Bigr)_{y\in Y}.$$

(Note that the union operation in the above formula is actually a disjoint union.) An isomorphism between correspondences (C,s,t) and (C',s',t') can be viewed as simply a relabeling of the elements of C; and this relabeling induces a natural isomorphism between the two functors.

Lemma 8.12 *Construction 8.11 defines a lax 2–functor $\mathscr{B}\to\mathsf{Permu}$.*

Sketch of proof Verify that the category $\prod_{x\in X}\mathsf{Sets}$ is naturally isomorphic to the category Sets/X, and that under this identification the functor induced by C is naturally isomorphic to the functor $A\mapsto C\times_X A$. □

Remark 8.13 Using properties (EM-3) and (EM-4),

$$K\Bigl(\prod_{x\in X}\mathsf{Sets}\Bigr)\simeq\prod_{x\in X}K(\mathsf{Sets})\simeq\prod_{x\in X}\mathbb{S}.$$

With respect to this decomposition, however, the map $\prod_{x\in X}\mathbb{S}\to\prod_{y\in Y}\mathbb{S}$ induced by a correspondence $C\colon X\to Y$ is not geometrically obvious.

$$\begin{tikzcd}
\operatorname{hocolim} K(B_2(F'))^\dagger \arrow[r, leftrightarrow, "(1)"] & \operatorname{hocolim} K(B_2((\vec{F}\circ B)'))^\dagger \arrow[r, leftrightarrow, "(5)"] & \operatorname{hocolim} K(B_2(G_p'))^\dagger \arrow[d, leftrightarrow, "(4)"] \\
\operatorname{hocolim} \widehat{F}^\dagger \arrow[r, leftrightarrow, "(2)"] & \operatorname{hocolim} G^\dagger \arrow[r, leftrightarrow, "(3)"] & \operatorname{hocolim} K(G_p)^\dagger
\end{tikzcd}$$

Figure 9: The steps in the proof of Proposition 8.15. Each arrow is labeled by the step showing it is an equivalence.

Construction 8.14 Given a category $\mathscr{C}$, viewed as a 2–category with only identity 2–morphisms, and a strictly unitary, lax 2–functor $F\colon \mathscr{C} \to \mathscr{B}$, there is an induced strict 2–functor $F'\colon \mathscr{C}' \to \mathscr{B}$, where $\mathscr{C}'$ is as in Definition 8.7. Composing with the $\prod_{x\in(-)} \mathsf{Sets}$ construction gives a strict 2–functor $\mathscr{C}' \to \mathsf{Permu}$, which we will still denote by F', with $F'(u) = \prod_{x\in F(u)} \mathsf{Sets}$. Finally, by (EM-5), the K–theory functor gives a functor $K(B_2(F'))\colon B_2(\mathscr{C}') \to \mathscr{S}$.

Hu, Kriz and Kriz apply this construction to the Khovanov functor $F_{HKK}\colon \underline{2}^n \to \mathscr{B}$, and then take the iterated mapping cone as in Section 8.2 to obtain their Khovanov stable homotopy type.

Proposition 8.15 *Given a strictly unitary, lax 2–functor $F\colon \underline{2}^n \to \mathscr{B}$, there is a stable homotopy equivalence between the Hu–Kriz–Kriz realization* $\operatorname{hocolim} K(B_2(F'))^\dagger$ *and the realization* $\operatorname{hocolim} \widehat{F}^\dagger$ *from Section 4.5.*

Proof To identify the Hu–Kriz–Kriz construction and the thickening construction from Section 4, we use a sequence of intermediate diagrams, somewhat in the spirit of Section 5.4. (See Figure 9 for an overview.)

(1) By Lemma 5.34, we can lift the functor $F\colon \underline{2}^n \to \mathscr{B}$ to a functor $\vec{F}\colon \mathsf{Arr}(\underline{2}^n) \to \mathscr{B}$, so that $F = \vec{F}\circ A$. Composing with the composition map $B\colon \widehat{\underline{2}^n} \to \mathsf{Arr}(\underline{2}^n)$ gives a functor $\vec{F}\circ B\colon \widehat{\underline{2}^n} \to \mathscr{B}$.

There is a homotopy equivalence $\operatorname{hocolim} K(B_2(F'))^\dagger \simeq \operatorname{hocolim} K(B_2((\vec{F}\circ B)'))^\dagger$; the argument is similar to the one in Section 5.4. In the commutative diagram

$$\begin{tikzcd}
B_2(\underline{2}^{n\prime})_\dagger \arrow[r] \arrow[d, "\simeq"'] & B_2(\mathsf{Arr}(\underline{2}^n)')_\dagger \arrow[d, "\simeq"] & B_2((\widehat{\underline{2}^n})')_\dagger \arrow[l] \arrow[d, "\simeq"] \\
\underline{2}^n_\dagger \arrow[r, "A_\dagger"'] & \mathsf{Arr}(\underline{2}^n)_\dagger & \widehat{\underline{2}^n}_\dagger \arrow[l, "B_\dagger"]
\end{tikzcd}$$

the vertical arrows are quasiequivalences by Lemma 8.9, and $A_\dagger$ and $B_\dagger$ are homotopy cofinal by Lemmas 5.31 and 5.32, so the functors in the top row are also homotopy cofinal. Hence, by property (ho-4) of homotopy colimits, $\operatorname{hocolim} K(B_2(F'))^\dagger \simeq \operatorname{hocolim} K(B_2(\vec{F}'))^\dagger \simeq \operatorname{hocolim} K(B_2((\vec{F} \circ B)'))^\dagger$.

(2) Recall that $\widehat{F}$ sends an object $(u \xrightarrow{f} v \xrightarrow{g} w)$ to $\bigvee_{a \in F(f)} \prod_{b \in F(g), s(b)=t(a)} \mathbb{S}$. Further, the definition of $\widehat{F}$ uses only the universal properties of product and coproduct, so for any spectrum X we could define a functor $\widehat{F}_X \colon \widehat{\underline{2}^n} \to \mathscr{S}$ with

$$\widehat{F}_X(u \xrightarrow{f} v \xrightarrow{g} w) = \bigvee_{a \in F(f)} \prod_{\substack{b \in F(g)\\ s(b)=t(a)}} X$$

(see Remark 4.38). Moreover, this thickening procedure is clearly natural in X. Taking the special case $X = K(\mathsf{Sets})$ and using property (EM-4) gives a diagram $G \colon \widehat{\underline{2}^n} \to \mathscr{S}$ with

$$G(u \xrightarrow{f} v \xrightarrow{g} w) = \bigvee_{a \in F(f)} \prod_{\substack{b \in F(g)\\ s(b)=t(a)}} K(\mathsf{Sets}),$$

and a natural transformation of diagrams $\widehat{F} \to G$ which is an equivalence on objects. In particular, $\operatorname{hocolim} \widehat{F}^\dagger \simeq \operatorname{hocolim} G^\dagger$.

(3) There is also a functor $G_p \colon \widehat{\underline{2}^n} \to \mathsf{Permu}$ defined similarly to $\widehat{F}$, using the product and coproduct on Permu in place of the product and wedge sum of spaces, and using Sets in place of $\mathbb{S}$. That is,

$$G_p(u \xrightarrow{f} v \xrightarrow{g} w) = \coprod_{a \in F(f)} \prod_{\substack{b \in F(g)\\ s(b)=t(a)}} \mathsf{Sets}.$$

Recall from (EM-2) that finite products and coproducts of permutative categories are given by the Cartesian product. Thus, using property (EM-3),

$$K(G_p)(u \xrightarrow{f} v \xrightarrow{g} w) \simeq \prod_{a \in F(f)} \prod_{\substack{b \in F(g)\\ s(b)=t(a)}} K(\mathsf{Sets}).$$

(Note that since $\widehat{\underline{2}^n}$ has only identity 2–morphisms, $B_2(\widehat{\underline{2}^n})$ can be identified with $\widehat{\underline{2}^n}$, and, therefore, we have written $K(G_p)$ instead of $K(B_2(G_p))$.)

Using the stable homotopy equivalence between wedge sum and product, we get a natural transformation of diagrams from G to $K(G_p)$ which is an equivalence on objects. In particular, $\operatorname{hocolim} G^\dagger \simeq \operatorname{hocolim} K(G_p)^\dagger$.

(4) By Lemma 8.10, there is a homotopy equivalence

$$\operatorname{hocolim} K(G_p)^\dagger \simeq \operatorname{hocolim} B_2(K(G_p)')^\dagger .$$

However, since G_p is a strict functor from a 1–category to Permu, $B_2(K(G_p)') = K(G_p) \circ \Pi = K(B_2(G_p'))$, where $\Pi\colon B_2((\widehat{\underline{2}^n})') \to \widehat{\underline{2}^n}$ is the projection; therefore, $\operatorname{hocolim} K(G_p)^\dagger \simeq \operatorname{hocolim} K(B_2(G_p'))^\dagger$.

(5) To finish the identification, there are isomorphisms H from G_p' to $(\vec{F} \circ B)'$. On the objects, the natural transformation will induce the equivalence from

$$G_p'(u \xrightarrow{f} v \xrightarrow{g} w) = G_p(u \xrightarrow{f} v \xrightarrow{g} w) = \coprod_{a \in F(f)} \prod_{\substack{b \in F(g) \\ s(b)=t(a)}} \mathsf{Sets}$$

to

$$(\vec{F} \circ B)'(u \xrightarrow{f} v \xrightarrow{g} w) = (\vec{F} \circ B)(u \xrightarrow{f} v \xrightarrow{g} w) = \prod_{c \in F(g \circ f)} \mathsf{Sets}$$

which comes from the natural bijection $F(g \circ f) \to F(g) \times_{F(v)} F(f)$ between their indexing sets, and the identification of coproducts and products in Permu with Cartesian products. Because Cartesian product of sets is not strictly associative, and because of the 2–categorical nature of the identification between coproducts and products in Permu, this does not strictly define a strict natural isomorphism $G_p' \to (\vec{F} \circ B)'$. Instead, it defines a pseudonatural equivalence: for a map $\varphi\colon (u \xrightarrow{f} v \xrightarrow{g} w) \to (u' \xrightarrow{f'} v' \xrightarrow{g'} w')$ in $\widehat{\underline{2}^n}$, there is a natural isomorphism of functors

$$(\vec{F} \circ B)'(\varphi) \circ H(u \xrightarrow{f} v \xrightarrow{g} w) \cong H(u' \xrightarrow{f'} v' \xrightarrow{g'} w') \circ G_p'(\varphi)$$

which respects composition in $\widehat{\underline{2}^n}$.

We define $\mathscr{E}$ to be the category with objects 1 and 0 and a unique morphism between any pair of objects. Let $\mathscr{D} = \mathscr{E} \times \widehat{\underline{2}^n_\dagger}$, and let $\mathscr{D}_0$ and $\mathscr{D}_1$ be the full subcategories of $\mathscr{D}$ spanned by the objects of $\{0\} \times \widehat{\underline{2}^n_\dagger}$ and $\{1\} \times \widehat{\underline{2}^n_\dagger}$, respectively. Note that the projection $\mathscr{D} \to \widehat{\underline{2}^n_\dagger}$ is an equivalence, and restricts to isomorphisms from both $\mathscr{D}_0$ and $\mathscr{D}_1$ to $\widehat{\underline{2}^n_\dagger}$.

The pseudonatural equivalence above defines a lax 2–functor, also denoted by H, from $\mathscr{D}$ to Permu whose restriction to $\{1\} \times \widehat{\underline{2}^n}$ is G_p' and whose restriction to $\{0\} \times \widehat{\underline{2}^n}$ is $(\vec{F} \circ B)'$. Therefore, $K(B_2(H))$ restricts to $K(B_2(G_p'))^\dagger$ and $K(B_2(\vec{F} \circ B)')^\dagger$, and since both inclusions $\mathscr{D}_0 \to \mathscr{D}$ and $\mathscr{D}_1 \to \mathscr{D}$ are homotopy cofinal,

$$\operatorname{hocolim} K(B_2((\vec{F} \circ B)'))^\dagger \simeq \operatorname{hocolim} K(B_2(H)) \simeq \operatorname{hocolim} K(B_2(G_p'))^\dagger .$$

Putting these all together (see Figure 9), in conjunction with Lemma 8.6, the Hu–Kriz–Kriz realization $\operatorname{hocolim} K(B_2(F'))^\dagger$ agrees with our thickening construction $\operatorname{hocolim} \widehat{F}^\dagger$. □

8.5 Proof that the Khovanov homotopy types agree

Proof of Theorem 9 Lemma 8.5 identifies the 2–functors $\underline{2}^n \to \mathscr{B}$ used in this paper and [22]. Proposition 8.15 identifies Hu, Kriz and Kriz's realization of this functor with ours. □

9 Khovanov homotopy type of a disjoint union and connected sum

In this section we prove Theorems 1, 2 and 10. For the first two theorems, we merely need to show that the functor associated to a disjoint union (respectively connect sum) of links is the product of the functors of the individual links:

Proposition 9.1 *Let L_1 and L_2 be link diagrams, and let $L_1 \sqcup L_2$ be their disjoint union. Order the crossings in $L_1 \sqcup L_2$ so that all of the crossings in L_1 come before all of the crossings in L_2. Then*

$$F^j_{Kh}(L_1 \sqcup L_2) \cong \coprod_{j_1+j_2=j} F^{j_1}_{Kh}(L_1) \times F^{j_2}_{Kh}(L_2),$$

where $\times$ denotes the product of functors (Definition 4.43), *$\coprod$ denotes the disjoint union of functors* (Definition 4.47) *and $\cong$ denotes natural isomorphism of 2–functors. If we fix a basepoint on L_1 then*

$$F^j_{\widetilde{Kh}}(L_1 \sqcup L_2) \cong \coprod_{j_1+j_2=j} F^{j_1}_{\widetilde{Kh}}(L_1) \times F^{j_2}_{Kh}(L_2).$$

If we fix basepoints on L_1 and L_2 and let $L_1 \# L_2$ denote the connected sum (*at the basepoints*), *then*

$$F^j_{\widetilde{Kh}}(L_1 \# L_2) \cong \coprod_{j_1+j_2=j} F^{j_1}_{\widetilde{Kh}}(L_1) \times F^{j_2}_{\widetilde{Kh}}(L_2).$$

Proof We will prove the first statement; the proofs of the other two are similar. Let n_i be the number of crossings in L_i. To keep notation simple, write $F = F^j_{Kh}(L_1 \sqcup L_2)$,

$X_v = F(v)$, $A_{v,w} = F(\varphi_{v,w})$, $G = \coprod_{j_1+j_2=j} F^{j_1}_{Kh}(L_1) \times F^{j_2}_{Kh}(L_2)$, $Y_v = G(v)$ and $B_{v,w} = G(\varphi_{v,w})$.

By Lemma 4.5, it suffices to construct bijections $\phi_v\colon X_v \xrightarrow{\cong} Y_v$ and $\psi_{v,w}\colon A_{v,w} \xrightarrow{\cong} B_{v,w}$ for all $v > w$ with $|v| - |w| = 1$ so that $\psi_{v,w}$ respects the source and target maps and for any $u > w$ with $|u| - |w| = 2$, the following diagram commutes:

(9.2)
$$\begin{tikzcd}
A_{v,w} \times_{X_v} A_{u,v} \arrow[r, "\psi_{v,w}\times\psi_{u,v}"] \arrow[d, "F_{u,v,w}"] & B_{v,w} \times_{Y_v} B_{u,v} \arrow[d, "G_{u,v,w}"] \\
A_{u,w} & B_{u,w} \\
A_{v',w} \times_{X_{v'}} A_{u,v'} \arrow[r, "\psi_{v',w}\times\psi_{u,v'}"] \arrow[u, "F_{u,v',w}"] & B_{v',w} \times_{Y_{v'}} B_{u,v'} \arrow[u, "G_{u,v',w}"]
\end{tikzcd}$$

Here, v and v' are the two vertices such that $u > v, v' > w$. Note that all arrows in this diagram are isomorphisms.

The map ϕ_v is the canonical identification between Khovanov generators for $L_1 \sqcup L_2$ and pairs of a Khovanov generator for L_1 and a Khovanov generator for L_2. There is a unique map $\psi_{v,w}\colon A_{v,w} \to B_{v,w}$ for $v > w$ with $|v| - |w| = 1$ which commutes with the source and target maps, because:

(1) Given $x_v \in X_v$ and $x_w \in X_w$, $s^{-1}(x_v) \cap t^{-1}(x_w) \subset A_{v,w}$ is either empty, if x_v does not occur in the Khovanov differential of x_w, or consists of a single point, if x_v does occur in the Khovanov differential of x_w. Similar statements hold for Y_v and $B_{v,w}$. It follows that if $\psi_{v,w}$ exists then it is unique.

(2) The canonical identification of Khovanov generators does, in fact, give a chain map. So, by the observations in the previous point, the map $\psi_{v,w}$ does exist.

Except in one case, the same argument shows that the diagram (9.2) commutes: typically, for each $x_u \in X_u$ and $x_w \in X_w$ (with $u > w$ and $|u| - |w| = 2$), $s^{-1}(x_u) \cap t^{-1}(x_w) \subset A_{u,w}$ is either empty or has a single element. The exceptional case is the case of a ladybug configuration, as in [35, Section 5.4] (see also Figure 7). In the ladybug case, either both crossings under consideration lie in L_1 or both crossings lie in L_2, from which it follows easily that the diagram commutes. (This is immediate for the present case when we are considering the disjoint union $L_1 \sqcup L_2$; the connect-sum case $L_1 \# L_2$ is also fairly obvious.) □

We are now ready to prove Theorems 1 and 2, which we recall for the reader's convenience:

Theorem 1 *Let L_1 and L_2 be links, and $L_1 \sqcup L_2$ their disjoint union. Then*

$$\mathcal{X}^j_{Kh}(L_1 \sqcup L_2) \simeq \textstyle\bigvee_{j_1+j_2=j} \mathcal{X}^{j_1}_{Kh}(L_1) \wedge \mathcal{X}^{j_2}_{Kh}(L_2). \tag{1.1}$$

Moreover, if we fix a basepoint p in L_1, not at a crossing, and consider the corresponding basepoint for $L_1 \sqcup L_2$, then

$$\widetilde{\mathcal{X}}^j_{Kh}(L_1 \sqcup L_2) \simeq \textstyle\bigvee_{j_1+j_2=j} \widetilde{\mathcal{X}}^{j_1}_{Kh}(L_1) \wedge \mathcal{X}^{j_2}_{Kh}(L_2). \tag{1.2}$$

Theorem 2 *Let L_1 and L_2 be based links and $L_1 \# L_2$ the connected sum of L_1 and L_2, where we take the connected sum near the basepoints. Then*

$$\widetilde{\mathcal{X}}^j_{Kh}(L_1 \# L_2) \simeq \textstyle\bigvee_{j_1+j_2=j} \widetilde{\mathcal{X}}^{j_1}_{Kh}(L_1) \wedge \widetilde{\mathcal{X}}^{j_2}_{Kh}(L_2). \tag{1.3}$$

Proof of Theorems 1 and 2 We will prove formula (1.1); the proofs of formulas (1.2) and (1.3) are essentially the same.

Fix a diagram for $L_1 \sqcup L_2$ so that there are no crossings between L_1 and L_2. Order the crossings in $L_1 \sqcup L_2$ so that all of the crossings in L_1 come before all of the crossings in L_2. By Proposition 9.1,

$$F^j_{Kh}(L_1 \sqcup L_2) \cong \coprod_{j_1+j_2=j} F^{j_1}_{Kh}(L_1) \times F^{j_2}_{Kh}(L_2).$$

By Lemma 4.42, naturally isomorphic functors have stably homotopy equivalent realizations. By Propositions 4.46 and 4.50, the realization of $\coprod_{j_1+j_2=j} F^{j_1}_{Kh}(L_1) \times F^{j_2}_{Kh}(L_2)$ is $\bigvee_{j_1+j_2=j} \mathcal{X}^{j_1}_{Kh}(K_1) \wedge \mathcal{X}^{j_2}_{Kh}(L_2)$. □

These results quickly imply Corollary 1.4, which we also recall:

Corollary 1.4 *For any n there exists a link L_n such that the operation*

$$\mathrm{Sq}^n\colon Kh^{i,j}(L_n) \to Kh^{i+n,j}(L_n)$$

is nonzero for some $i, j \in \mathbb{Z}$. Similarly, there exists a knot K_n such that the operation

$$\mathrm{Sq}^n\colon \widetilde{Kh}^{i,j}(K_n) \to \widetilde{Kh}^{i+n,j}(K_n)$$

is nonzero for some $i, j \in \mathbb{Z}$. Further, for this knot, the operation

$$\mathrm{Sq}^n\colon \widetilde{Kh}^{i,j}(K_n) \to \widetilde{Kh}^{i+n,j}(K_n)$$

is also nonzero for some $i, j \in \mathbb{Z}$.

Proof For the first statement, consider the disjoint union L_n of n copies of the left-handed trefoil T. It follows from [35, Proposition 9.2] that

$$\mathcal{X}_{Kh}(T) \simeq \Sigma^{-3}\mathbb{S} \vee \Sigma^{-2}\mathbb{S} \vee \mathbb{S} \vee \mathbb{S} \vee \Sigma^{-4}\mathbb{R}\mathrm{P}^2$$

(compare [35, Example 9.4]). So, by Theorem 1,

$$\mathcal{X}_{Kh}(L_n) \simeq \Sigma^{-4n}(\overbrace{\mathbb{R}\mathrm{P}^2 \wedge \cdots \wedge \mathbb{R}\mathrm{P}^2}^{n \text{ copies}}) \vee Y$$

for some space Y. It follows from the Cartan formula that

$$\mathrm{Sq}^n\colon H^n((\mathbb{R}\mathrm{P}^2)^{\wedge n}; \mathbb{Z}/2\mathbb{Z}) \to H^{2n}((\mathbb{R}\mathrm{P}^2)^{\wedge n}; \mathbb{Z}/2\mathbb{Z})$$

is nontrivial. This proves the first part of the result.

For the second part of the result, let K be the knot 15^n_{41127} and let K_n be the connect sum of n copies of K. According to the calculation in [53, Figure 6], $\widetilde{Kh}^{-2,0}(K;\mathbb{Z}) \cong \mathbb{Z}$, $\widetilde{Kh}^{0,0}(K;\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ and $\widetilde{Kh}^{i,0}(K;\mathbb{Z}) = 0$ for $i \neq -2, 0$. In particular, there is a class $\alpha \in \widetilde{Kh}^{-1,0}(K;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ such that $\mathrm{Sq}^1(\alpha)$ is nonzero and $\mathrm{Sq}^i(\alpha) = 0$ for $i > 1$. So, it follows from Theorem 2 and the Cartan formula that for the class $\beta = \alpha \wedge \cdots \wedge \alpha \in \widetilde{Kh}^{-n,0}(K_n)$, $\mathrm{Sq}^n(\beta)$ is nontrivial.

Finally, we argue that Sq^n is nonvanishing on $Kh^{*,*}(K_n)$, as well. To see this, recall from [35, Theorem 3] that there is a cofibration sequence

$$\widetilde{\mathcal{X}}^{j-1}_{Kh}(K_n) \to \mathcal{X}^{j}_{Kh}(K_n) \to \widetilde{\mathcal{X}}^{j+1}_{Kh}(K_n)$$

inducing the long exact sequence

$$\cdots \to \widetilde{Kh}^{i,j+1}(K_n) \to Kh^{i,j}(K_n) \to \widetilde{Kh}^{i,j-1}(K_n) \to \widetilde{Kh}^{i+1,j+1}(K_n) \to \cdots.$$

Further, for coefficients in $\mathbb{Z}/2\mathbb{Z}$ — see[45, Proposition 1.7] —

$$\dim(Kh^{i,j}(K_n;\mathbb{Z}/2\mathbb{Z})) = \dim(\widetilde{Kh}^{i,j-1}(K_n;\mathbb{Z}/2\mathbb{Z})) + \dim(\widetilde{Kh}^{i,j+1}(K_n;\mathbb{Z}/2\mathbb{Z})),$$

so the map $Kh^{i,j}(K_n;\mathbb{Z}/2\mathbb{Z}) \to \widetilde{Kh}^{i,j-1}(K_n;\mathbb{Z}/2\mathbb{Z})$ is surjective. In particular, there is a class $\gamma \in Kh^{-n,1}(K_n;\mathbb{Z}/2\mathbb{Z})$ which maps to $\beta \in \widetilde{Kh}^{-n,0}(K_n;\mathbb{Z}/2\mathbb{Z})$. By naturality of Sq^n, it follows that $\mathrm{Sq}^n(\gamma)$ maps to $\mathrm{Sq}^n(\beta) \neq 0$, so $\mathrm{Sq}^n(\gamma) \neq 0$, as desired. □

Finally, we turn to the unreduced Khovanov homology of a connected sum.

Definition 9.3 Consider the Khovanov homotopy type associated to the unknot, $\mathcal{X}_{Kh}(U) = \mathbb{S} \vee \mathbb{S}$, which is the suspension spectrum of $S^0 \vee S^0 = \{*, p_-, p_+\}$. The spectrum $\mathbb{H}^1 := \mathcal{X}_{Kh}(U)$ has a product $\mu\colon \mathbb{H}^1 \wedge \mathbb{H}^1 \to \mathbb{H}^1$ induced by the map of spaces $(S^0 \vee S^0) \wedge (S^0 \vee S^0) \to (S^0 \vee S^0)$ given by

$$p_- \wedge p_- \mapsto p_-, \quad p_+ \wedge p_- \mapsto p_+, \quad p_- \wedge p_+ \mapsto p_+, \quad p_+ \wedge p_+ \mapsto *.$$

Lemma 9.4 *The operation μ makes $\mathbb{H}^1$ into a ring spectrum.*

Proof This is immediate from the definitions. □

Remark 9.5 The map on reduced cohomology induced by μ is the split map $Kh(U) \to Kh(U) \otimes Kh(U)$. (The generators of $Kh(U)$ are x_- corresponding to p_- and x_+ corresponding to p_+.)

Remark 9.6 The notation $\mathbb{H}^1$ is chosen to be reminiscent of the first of Khovanov's arc algebras H^n.

Next, fix a link diagram K with n crossings and a basepoint $p \in K$. We make $\mathcal{X}_{Kh}(K)$ into a module spectrum over $\mathbb{H}^1$. We will use the box map realization from Section 5, applied to the functor $F_{Kh}\colon \underline{2}^n \to \mathscr{B}$, using a particular choice of spatial refinement, which we need to specify. The resulting CW complex is described in Proposition 6.1, producing a finite CW spectrum $\mathcal{X}_{Kh}(K)$. Before specifying the spatial refinement, we recall the definitions of the reduced Khovanov functors $F_{\pm Kh}\colon \underline{2}^n \to \mathscr{B}$, to fix notation:

Notation 9.7 For $u \in \{0,1\}^n$, define $F_{+Kh}(u)$ (respectively $F_{-Kh}(u)$) to be the subset of $F_{Kh}(u)$ where the circle in the complete resolution $\mathcal{P}(u)$ containing the basepoint is labeled x_+ (respectively x_-). For $u > v \in \{0,1\}^n$, define the correspondence from $F_{+Kh}(u)$ to $F_{+Kh}(v)$ (respectively from $F_{-Kh}(u)$ to $F_{-Kh}(v)$) to be the subset $s^{-1}(F_{+Kh}(u)) \cap t^{-1}(F_{+Kh}(v))$ (respectively $s^{-1}(F_{-Kh}(u)) \cap t^{-1}(F_{-Kh}(v))$) of the correspondence from $F_{Kh}(u)$ to $F_{Kh}(v)$. It is straightforward from the definition of the Khovanov differential (Section 2.2) that this produces well-defined functors $F_{\pm Kh}\colon \underline{2}^n \to \mathscr{B}$. Furthermore, the map from $F_{+Kh}(u)$ to $F_{-Kh}(u)$ which relabels the pointed circle in $\mathcal{P}(u)$ from x_+ to x_- induces a natural isomorphism from F_{+Kh} to F_{-Kh}; we often write $F_{\widetilde{Kh}}$ to denote either functor. Finally, for any $u \in \{0,1\}^n$, $F_{Kh}(u) = F_{+Kh}(u) \amalg F_{-Kh}(u)$; and for any $u > v \in \{0,1\}^n$, the correspondence $F_{Kh}(\varphi_{u,v})$ from $F_{Kh}(u)$ to $F_{Kh}(v)$ is the disjoint union of the correspondence $F_{+Kh}(\varphi_{u,v})$ from $F_{+Kh}(u)$ to $F_{+Kh}(v)$, the correspondence $F_{-Kh}(\varphi_{u,v})$ from $F_{-Kh}(u)$ to $F_{-Kh}(v)$, and some correspondence from $F_{-Kh}(u)$ to $F_{+Kh}(v)$.

Definition 9.8 A spatial refinement $\widetilde{F}_{Kh}$ of F_{Kh} induces spatial refinements $\widetilde{F}_{+Kh}$ of F_{+Kh} and $\widetilde{F}_{-Kh}$ of F_{-Kh}. We call $\widetilde{F}_{Kh}$ a *pointed spatial refinement* if, with respect to the natural isomorphism between F_{+Kh} and F_{-Kh}, the boxes used to define $\widetilde{F}_{+Kh}$ and $\widetilde{F}_{-Kh}$ are identical. In this case, there is an induced natural isomorphism between $\widetilde{F}_{+Kh}$ and $\widetilde{F}_{-Kh}$. For pointed spatial refinements, the CW complexes $\operatorname{hocolim}(\widetilde{F}^{+}_{+Kh})$ and $\operatorname{hocolim}(\widetilde{F}^{+}_{-Kh})$ are canonically isomorphic.

Lemma 9.9 *Each pointed link diagram (K,p) admits a pointed spatial refinement $\widetilde{F}_{Kh}$.*

Proof We construct a spatial refinement $\widetilde{F}_{Kh}$ of F_{Kh} in several steps. First construct a spatial refinement $\widetilde{F}_{-Kh}$ of F_{-Kh} with the additional restriction that the box maps come from the subspaces $E^\circ(\{B_x\},s)$ of $E(\{B_x\},s)$, that is, the subboxes are contained in the interiors of the bigger boxes. Then use the natural isomorphism between F_{+Kh} and F_{-Kh} to get a spatial refinement $\widetilde{F}_{+Kh}$ of F_{+Kh}. Finally, extend $\widetilde{F}_{+Kh}$ and $\widetilde{F}_{-Kh}$ to construct a spatial refinement $\widetilde{F}_{Kh}$ of F_{Kh}, following the inductive argument in the proof of Proposition 5.22(1). For the induction step, fix a length-ℓ sequence $v_0 \to \cdots \to v_\ell$ of nonidentity morphisms in $\underline{2}^n$. There is a correspondence $F_{Kh}(\varphi_{v_0,v_\ell})$ and a subset $F_{+Kh}(\varphi_{v_0,v_\ell}) \sqcup F_{-Kh}(\varphi_{v_0,v_\ell})$; let s be the source map of the correspondence $F_{Kh}(\varphi_{v_0,v_\ell})$ and s' the restriction of s to $F_{+Kh}(\varphi_{v_0,v_\ell}) \sqcup F_{-Kh}(\varphi_{v_0,v_\ell})$. Induction and $\widetilde{F}_{+Kh}$ and $\widetilde{F}_{-Kh}$ give a diagram

$$\begin{tikzcd}
\partial([0,1]^{\ell-1}) \arrow[r] \arrow[d] & E^\circ(\{B_x\},s) \arrow[d] \\
{[0,1]^{\ell-1}} \arrow[r] & E^\circ(\{B_x\},s')
\end{tikzcd}$$

where the right-hand vertical map forgets the boxes labeled by elements of

$$F_{Kh}(\varphi_{v_0,v_\ell}) \setminus (F_{+Kh}(\varphi_{v_0,v_\ell}) \sqcup F_{-Kh}(\varphi_{v_0,v_\ell})).$$

The inductive step is to construct a lift $[0,1]^{\ell-1} \to E^\circ(\{B_x\},s)$ making the diagram commute. Lemma 5.20 guarantees the existence of such a lift. Thus, induction implies that F_{Kh} has a spatial refinement extending $\widetilde{F}_{+Kh}$ and $\widetilde{F}_{-Kh}$. □

Construction 9.10 Given a pointed spatial refinement, define a map

$$\Psi\colon \operatorname{hocolim}(\widetilde{F}^{+}_{Kh}) \to \operatorname{hocolim}(\widetilde{F}^{+}_{Kh})$$

as follows: $\operatorname{hocolim}(\widetilde{F}^{+}_{+Kh})$ is a subcomplex of $\operatorname{hocolim}(\widetilde{F}^{+}_{Kh})$ and $\operatorname{hocolim}(\widetilde{F}^{+}_{-Kh})$ is the corresponding quotient complex; define Ψ to be the composition

$$\operatorname{hocolim}(\widetilde{F}^{+}_{Kh}) \twoheadrightarrow \operatorname{hocolim}(\widetilde{F}^{+}_{-Kh}) \xrightarrow{\cong} \operatorname{hocolim}(\widetilde{F}^{+}_{+Kh}) \hookrightarrow \operatorname{hocolim}(\widetilde{F}^{+}_{Kh}),$$

where the first map is the quotient map, the second map is the canonical isomorphism and the third map is the subcomplex inclusion. Note that Ψ is a cellular map. The induced map on $\mathcal{C}_{Kh}(K)$, the reduced cellular cochain complex of $\operatorname{hocolim}(\widetilde{F}^+_{Kh})$, sends generators that label the pointed circle by x_- to zero and on the rest of the Khovanov generators relabels the pointed circle from x_+ to x_-.

Now we are ready to define the $\mathbb{H}^1$–module structure on $\mathcal{X}_{Kh}(K)$.

Definition 9.11 Define a map $\mathcal{X}_{Kh}(K) \wedge \mathbb{H}^1 \to \mathcal{X}_{Kh}(K)$ induced the following map of spaces:

$$\operatorname{hocolim}(\widetilde{F}^+_{Kh}) \wedge \{*, p_-, p_+\} = (\operatorname{hocolim}(\widetilde{F}^+_{Kh}) \times \{p_-\}) \vee (\operatorname{hocolim}(\widetilde{F}^+_{Kh}) \times \{p_+\}) \to \operatorname{hocolim}(\widetilde{F}^+_{Kh}). \tag{9.12}$$

On the first summand, the map $\operatorname{hocolim}(\widetilde{F}^+_{Kh}) \times \{p_-\} \to \operatorname{hocolim}(\widetilde{F}^+_{Kh})$ is projection to the first factor. On the second summand, the map is projection to the first factor composed with the map $\Psi\colon \operatorname{hocolim}(\widetilde{F}^+_{Kh}) \to \operatorname{hocolim}(\widetilde{F}^+_{Kh})$ defined above.

Lemma 9.13 *Definition 9.11 makes $\mathcal{X}_{Kh}(K)$ into a module spectrum over $\mathbb{H}^1$.*

Proof This follows from the fact that $\Psi \circ \Psi$ sends all of $\operatorname{hocolim}(\widetilde{F}^+_{Kh})$ to the basepoint. □

The ring spectrum $\mathbb{H}^1$ is commutative, so we can view the action of $\mathbb{H}^1$ on $\mathcal{X}_{Kh}(K)$ as either a left or a right action.

Note that the induced map on the reduced cellular cochain complexes associated to the map (9.12) is the split map $\mathcal{C}_{Kh}(K) \to \mathcal{C}_{Kh}(K) \otimes Kh(U)$.

Proposition 9.14 *Up to weak equivalence of $\mathbb{H}^1$–module spectra, $\mathcal{X}_{Kh}(K)$ is an invariant of pointed links. That is, if (K, p) and (K', p') are pointed link diagrams representing isotopic pointed links, then there exist $\mathbb{H}^1$–module spectra $\mathcal{X}_{Kh}(K) = X_0$, $X_1, \dots, X_{\ell-1}$, $X_\ell = \mathcal{X}_{Kh}(K')$, and, for any adjacent pair X_i and X_{i+1}, either a map $X_i \to X_{i+1}$ or a map $X_{i+1} \to X_i$, which is both an $\mathbb{H}^1$–module map and a weak equivalence.*

Proof First, observe that the $\mathbb{H}^1$–module structure is independent of the choice of box maps; the proof is essentially the same as Proposition 5.22(2), but using Lemma 5.20 instead of Lemma 5.18.

Next we show that, up to weak equivalence, this $\mathbb{H}^1$–module spectrum is invariant under Reidemeister moves. By a standard argument [27, Section 3], we only need to consider Reidemeister moves that do not cross the marked point p. We follow the framework from [35, Section 6]. Let K_0 and K_1 (with n_0 and n_1 crossings, respectively) be pointed link diagrams related by any of the three Reidemeister moves of [35, Figure 6.1], and assume $n_0 < n_1$. The usual proof of invariance of Khovanov homology shows that $\mathcal{C}_{Kh}(K_0)$ can be identified with a subquotient complex of $\mathcal{C}_{Kh}(K_1)$, inducing a (two-step) zigzag of isomorphisms connecting $Kh(K_0)$ and $Kh(K_1)$ (see also the proofs of [35, Propositions 6.2–6.4]). Indeed, there is a particular vertex $w \in \{0,1\}^{n_1-n_0}$ such that for every $u \in \{0,1\}^{n_0}$, $F_{Kh}(K_0)(u)$ is identified with a certain subset $S_u \subseteq F_{Kh}(K_1)((u,w))$, and, for every $u > v \in \{0,1\}^{n_0}$, the correspondence $F_{Kh}(K_0)(\varphi_{u,v})$ is identified with the subset $s^{-1}(S_u) \cap t^{-1}(S_v) \subseteq F_{Kh}(K_1)(\varphi_{(u,w),(v,w)})$. Furthermore, these identifications identify $F_{+Kh}(K_0)(u)$ with $S_u \cap F_{+Kh}(K_1)((u,w))$ and, consequently, $F_{-Kh}(K_0)(u)$ with $S_u \cap F_{-Kh}(K_1)((u,w))$.

Construct the $\mathbb{H}^1$–module spectrum $\mathcal{X}_{Kh}(K_1)$ using some pointed spatial refinement $\widetilde{F}_{Kh}(K_1)$ for K_1. Restricting to the subsets S_u and the correspondences between them, we get a pointed spatial refinement $\widetilde{F}_{Kh}(K_0)$ for K_0, which we use to construct the $\mathbb{H}^1$–module spectrum $\mathcal{X}_{Kh}(K_0)$. With the CW complex structures from Proposition 6.1, $\operatorname{hocolim}(\widetilde{F}^+_{Kh}(K_0))$ can be identified with a subquotient complex of $\operatorname{hocolim}(\widetilde{F}^+_{Kh}(K_1))$, leading to a two-step zigzag of maps connecting $\mathcal{X}_{Kh}(K_1)$ and $\mathcal{X}_{Kh}(K_2)$. Since the Reidemeister moves do not cross the basepoint, it is immediate from the definitions of these sub- and quotient complexes that the maps are $\mathbb{H}^1$–equivariant. Since they also induce isomorphisms on homology, they are weak equivalences. □

For the rest of this section, fix a link diagram for $K_1 \sqcup K_2$, which is a disjoint union of link diagrams for K_1 and K_2, with n_1 and n_2 crossings, respectively, and fix basepoints p_i on K_i so that the two basepoints are next to one another (ie on the boundary of the same component of $S^2 \setminus (K_1 \sqcup K_2)$).

Recall:

Definition 9.15 The *(derived) tensor product* of the module spectra $\mathcal{X}_{Kh}(K_1)$ and $\mathcal{X}_{Kh}(K_2)$ is the homotopy colimit of the diagram

$$\begin{aligned}\textbf{(9.16)}\qquad \mathcal{X}_{Kh}(K_1) \wedge \mathcal{X}_{Kh}(K_2) &\leftleftarrows \mathcal{X}_{Kh}(K_1) \wedge \mathbb{H}^1 \wedge \mathcal{X}_{Kh}(K_2) \\ &\mathrel{\substack{\leftarrow\\ \leftarrow\\ \leftarrow}} \mathcal{X}_{Kh}(K_1) \wedge \mathbb{H}^1 \wedge \mathbb{H}^1 \wedge \mathcal{X}_{Kh}(K_2) \mathrel{\substack{\leftarrow\\ \leftarrow\\ \leftarrow\\ \leftarrow}} \cdots,\end{aligned}$$

where the maps are all possible ways of applying μ to a pair of consecutive factors. To be more precise, let Δ_{inj} be the category with one object $\underline{\boldsymbol{n}} = \{0, \dots, n-1\}$ for each positive integer n and $\mathrm{Hom}(\underline{\boldsymbol{m}}, \underline{\boldsymbol{n}})$ the set of order-preserving injections $\{0, \dots, m-1\} \to \{0, \dots, n-1\}$; for $n > 0$ and $0 \le i \le n$, let $f_{\underline{\boldsymbol{n}},i} \in \mathrm{Hom}_{\Delta_{inj}}(\underline{\boldsymbol{n}}, \underline{\boldsymbol{n+1}})$ be the morphism $\underline{\boldsymbol{n}} \to \underline{\boldsymbol{n+1}}$ whose image is $\underline{\boldsymbol{n+1}} \setminus \{i\}$. (The category Δ_{inj} is the subcategory of the simplex category generated by the face maps, and the $f_{\underline{\boldsymbol{n}},i}$ are the face maps themselves.) Then the diagram (9.16) can be treated as a (strict) functor $F_\otimes$ from $\Delta_{inj}^{\mathrm{op}}$ to $\mathscr{CW}$, the category of CW spectra. On objects, $F_\otimes(\underline{\boldsymbol{n}}) = \mathcal{X}_{Kh}(K_1) \wedge \bigl(\bigwedge_{i=1}^{n-1} \mathbb{H}^1\bigr) \wedge \mathcal{X}_{Kh}(K_2)$. On morphisms, $F_\otimes(f_{\underline{\boldsymbol{n}},i}^{\mathrm{op}})$ is the map

$$\mathcal{X}_{Kh}(K_1) \wedge \bigl(\textstyle\bigwedge_{i=1}^{n} \mathbb{H}^1\bigr) \wedge \mathcal{X}_{Kh}(K_2) \to \mathcal{X}_{Kh}(K_1) \wedge \bigl(\textstyle\bigwedge_{i=1}^{n-1} \mathbb{H}^1\bigr) \wedge \mathcal{X}_{Kh}(K_2)$$

gotten by applying μ to the $(i+1)^{\mathrm{st}}$ pair of consecutive factors. Let

$$\mathcal{X}_{Kh}(K_1) \otimes_{\mathbb{H}^1} \mathcal{X}_{Kh}(K_2) := \mathrm{hocolim}(F_\otimes)$$

denote the derived tensor product of $\mathcal{X}_{Kh}(K_1)$ and $\mathcal{X}_{Kh}(K_2)$.

Theorem 10 *There is a stable homotopy equivalence*

$$\mathcal{X}_{Kh}(K_1 \# K_2) \simeq \mathcal{X}_{Kh}(K_1) \otimes_{\mathbb{H}^1} \mathcal{X}_{Kh}(K_2).$$

The proof of Theorem 10 involves three components. First, we show how the functor $F_{Kh}(K_1 \# K_2)$ is determined by the functors $F_{Kh}(K_1)$ and $F_{Kh}(K_2)$; this is Lemmas 9.18 and 9.20, which take a little work but are purely combinatorial. Second, in Lemmas 9.21 and 9.23, we prove that Theorem 10 holds at the level of cellular cochains, for an appropriate CW complex structure on $\mathcal{X}_{Kh}(K_1) \otimes_{\mathbb{H}^1} \mathcal{X}_{Kh}(K_2)$. This is essentially immediate from Segal's construction of homotopy colimits and the connected sum theorem for the Khovanov chain complex. Third, using the description of $F_{Kh}(K_1 \# K_2)$ in terms of $F_{Kh}(K_1)$ and $F_{Kh}(K_2)$ and carefully chosen spatial refinements, we produce a (strict) map from the diagram (9.16) to $\mathcal{X}_{Kh}(K_1 \# K_2)$, inducing the desired map of cellular cochains. This argument is Lemmas 9.24 and 9.25. From these three steps, Theorem 10 follows easily.

We begin by reconstructing the functor $F_{Kh}(K_1 \# K_2)$ from the functor

$$F_{Kh}(K_1 \sqcup K_2) \colon \underline{2}^{n_1+n_2} \to \mathscr{B}.$$

Definition 9.17 For $a, b \in \{*, +, -\}$, let $F_{abKh}(K_1 \sqcup K_2) \colon \underline{2}^{n_1+n_2} \to \mathscr{B}$ denote the functor where we only consider the Khovanov generators that label the circle containing p_1 by x_a if $a \in \{+, -\}$ and label the circle containing p_2 by x_b if $b \in \{+, -\}$, and we restrict the correspondences correspondingly. (If a or b is $*$, we make no restriction on the label of the corresponding circle.)

Lemma 9.18 *For $v \in \{0,1\}^{n_1+n_2}$, the map $F_{+-Kh}(K_1 \sqcup K_2)(v) \to F_{-+Kh}(K_1 \sqcup K_2)(v)$ that interchanges the labelings of the two pointed circles in $\mathcal{P}(v)$ induces an isomorphism $F_{+-Kh}(K_1 \sqcup K_2) \to F_{-+Kh}(K_1 \sqcup K_2)$.*

Proof The isomorphism from Proposition 9.1 identifies either functor with the product $F_{\widetilde{Kh}}(K_1) \times F_{\widetilde{Kh}}(K_2)$. The given map is the composition $F_{+-Kh}(K_1 \sqcup K_2) \cong F_{\widetilde{Kh}}(K_1) \times F_{\widetilde{Kh}}(K_2) \cong F_{-+Kh}(K_1 \sqcup K_2)$. □

Notation 9.19 Let $F_{+/+Kh}(K_1 \sqcup K_2)$ denote the functor $\underline{2}^{n_1+n_2} \to \mathscr{B}$, where we only consider the Khovanov generators that label at least one of the two pointed circles by x_-, and we restrict the correspondences correspondingly. (The notation $F_{+/+Kh}$ is the mnemonic "not $++$".) That is, for all $u \in \{0,1\}^{n_1+n_2}$, $F_{+/+Kh}(K_1 \sqcup K_2) = F_{--Kh}(K_1 \sqcup K_2)(u) \sqcup F_{+-Kh}(K_1 \sqcup K_2)(u) \sqcup F_{-+Kh}(K_1 \sqcup K_2)(u)$; and for all $u > v \in \{0,1\}^{n_1+n_2}$, the correspondence $F_{+/+Kh}(K_1 \sqcup K_2)(\varphi_{u,v})$ is the disjoint union of the correspondences $F_{--Kh}(K_1 \sqcup K_2)(\varphi_{u,v})$, $F_{+-Kh}(K_1 \sqcup K_2)(\varphi_{u,v})$, $F_{-+Kh}(K_1 \sqcup K_2)(\varphi_{u,v})$, some correspondence $P_{u,v}$ from $F_{--Kh}(K_1 \sqcup K_2)(u)$ to $F_{+-Kh}(K_1 \sqcup K_2)(v)$, and some correspondence $Q_{u,v}$ from $F_{--Kh}(K_1 \sqcup K_2)(u)$ to $F_{-+Kh}(K_1 \sqcup K_2)(v)$. Let F be the functor obtained from $F_{+/+Kh}(K_1 \sqcup K_2)$ by identifying $F_{+-Kh}(K_1 \sqcup K_2)$ and $F_{-+Kh}(K_1 \sqcup K_2)$ by the isomorphism from Lemma 9.18. That is, for all $u \in \{0,1\}^{n_1+n_2}$,

$$F(u) = (F_{--Kh}(u) \sqcup F_{+-Kh}(u) \sqcup F_{-+Kh}(u))/(F_{+-Kh}(u) = F_{-+Kh}(u));$$

and for all $u > v \in \{0,1\}^{n_1+n_2}$,

$$\begin{aligned} F(\varphi_{u,v}) = (F_{--Kh}(\varphi_{u,v}) \sqcup F_{+-Kh}(\varphi_{u,v}) \sqcup F_{-+Kh}(\varphi_{u,v}) \sqcup P_{u,v} \sqcup Q_{u,v})/(F_{+-Kh}(\varphi_{u,v}) \\ = F_{-+Kh}(\varphi_{u,v})). \end{aligned}$$

Lemma 9.20 *The functor F constructed above is isomorphic to $F_{Kh}(K_1 \# K_2)$ via the following map: for all $u \in \{0,1\}^{n_1+n_2}$, the isomorphism sends $x \in F(u)$ to $y \in F_{Kh}(K_1 \# K_2)(u)$ where y labels the connect-sum circle by x_- if and only if x labels both the pointed circles by x_-, and x and y label all the circles that are disjoint from the connect-sum region identically.*

Proof The proof is similar to the proof of Proposition 9.1. To keep the notation similar, let $X_v = F(v)$, $A_{u,v} = F(\varphi_{u,v})$, $G = F_{Kh}(K_1 \# K_2)$, $Y_v = G(v)$ and

$B_{u,v} = G(\varphi_{u,v})$. The bijections $\phi_v\colon X_v \xrightarrow{\cong} Y_v$ are already provided to us. To construct bijections $\psi_{u,v}\colon A_{u,v} \xrightarrow{\cong} B_{u,v}$, for all $u > v$ with $|u|-|v| = 1$, we need to check the conditions (1) and (2) of the proof of Proposition 9.1.

For any $u \in \{0,1\}^{n_1+n_2}$ and any $z_u \in F_{+/+Kh}(K_1 \sqcup K_2)(u)$, let $\pi(z_u)$ denote the image of z_u in X_u; and for any $x_u \in X_u$, let $\iota^1(x_u)$ (respectively $\iota^2(x_u)$) denote the preimage of x_u in $F_{*-Kh}(K_1 \sqcup K_2)(u)$ (respectively $F_{-*Kh}(K_1 \sqcup K_2)(u)$). Then, for any $u > v$ with $|u|-|v| = 1$, $z_u \in F_{+/+Kh}(K_1 \sqcup K_2)(u)$ and $x_v \in X_v$, one of the two subsets $s^{-1}(z_u) \cap t^{-1}(\iota^1(x_v)) \subseteq F_{*-Kh}(K_1 \sqcup K_2)(\varphi_{u,v})$ and $s^{-1}(z_u) \cap t^{-1}(\iota^2(x_v)) \subseteq F_{-*Kh}(K_1 \sqcup K_2)(\varphi_{u,v})$ is empty, and the other one is canonically identified with the subset $s^{-1}(\pi(z_u)) \cap t^{-1}(x_v) \subseteq A_{u,v}$. This follows from the fact that the correspondences in $F_{+/+Kh}(K_1 \sqcup K_2)$ preserve two quantum gradings, the one coming from K_1 and the one coming from K_2; however, the double quantum gradings of $\iota^1(x_v)$ and $\iota^2(x_v)$ are different, and, therefore, at least one of $s^{-1}(z_u) \cap t^{-1}(\iota^1(x_v))$ and $s^{-1}(z_u) \cap t^{-1}(\iota^2(x_v))$ is empty.

From this observation, condition (1) is immediate. Condition (2) follows from additionally noting that the composition $F_{Kh}(K_1 \sqcup K_2) \to X \xrightarrow{\phi} Y$ induces the cobordism map $\mathcal{C}_{Kh}(K_1 \# K_2) \to \mathcal{C}_{Kh}(K_1 \sqcup K_2)$ associated to splitting at the connect-sum region, which is a chain map.

Finally, we need to check that diagram (9.2) commutes for all $u > w$ with $|u|-|w| = 2$. Using the observation in the previous paragraph, this follows from the same arguments as in the proof of Proposition 9.1. □

Next we observe that Theorem 10 holds at the level of cellular cochain complexes.

Lemma 9.21 *There exists a CW complex structure on $\mathcal{X}_{Kh}(K_1) \otimes_{\mathbb{H}^1} \mathcal{X}_{Kh}(K_2)$ such that the reduced cellular cochain complex is the chain complex*

(9.22) $$\mathcal{C}_{Kh}(K_1) \otimes \mathcal{C}_{Kh}(K_2) \to \mathcal{C}_{Kh}(K_1) \otimes Kh(U) \otimes \mathcal{C}_{Kh}(K_2) \\ \to \mathcal{C}_{Kh}(K_1) \otimes Kh(U) \otimes Kh(U) \otimes \mathcal{C}_{Kh}(K_2) \to \cdots$$

with the differential given by

$$d(x_0 \otimes \cdots \otimes x_n) = \delta(x_0) \otimes \cdots \otimes x_i \otimes \cdots \otimes x_n + x_0 \otimes \cdots \otimes x_i \otimes \cdots \otimes \delta(x_n) \\ + \sum_{i=0}^{n} (-1)^{i+n} x_0 \otimes \cdots \otimes x_{i-1} \otimes S(x_i) \otimes x_{i+1} \otimes \cdots \otimes x_n,$$

where S denotes either the Khovanov Frobenius algebra comultiplication map $Kh(U) \to Kh(U) \otimes Kh(U)$ or the cobordism map $\mathcal{C}_{Kh}(K_1) \to \mathcal{C}_{Kh}(K_1) \otimes Kh(U)$ (respectively

$\mathcal{C}_{Kh}(K_2) \to Kh(U) \otimes \mathcal{C}_{Kh}(K_2)$) for splitting off a trivial unknot at p_1 (respectively p_2), and δ denotes the Khovanov differential on $\mathcal{C}_{Kh}(K_1)$ and $\mathcal{C}_{Kh}(K_2)$.

Proof If $F\colon \Delta_{inj}^{\mathrm{op}} \to \mathsf{CW}_\bullet$ is a functor, Segal [51, Appendix A] defines its geometric realization as

$$\|F\| = \left(\{*\} \sqcup \coprod_{n=1}^{\infty} \Delta^{n-1} \times F(\underline{\boldsymbol{n}})\right)\Big/\sim$$

with $(f_*(\zeta), a) \sim (\zeta, F(f^{\mathrm{op}})(a))$ for all $\zeta \in \Delta^{m-1}$, $a \in F(\underline{\boldsymbol{n}})$ and $f \in \mathrm{Hom}_{\Delta_{inj}}(\underline{\boldsymbol{m}}, \underline{\boldsymbol{n}})$, and $\Delta^{n-1} \times \{*\} \sim *$ for all n. (The map $f_*\colon \Delta^{m-1} \to \Delta^{n-1}$ is the face inclusion corresponding to f.) If Δ_+^{n-1} denotes the disjoint union of Δ^{n-1} and a basepoint, then we can rewrite

$$\|F\| = \left(\textstyle\bigvee_n \Delta_+^{n-1} \wedge F(\underline{\boldsymbol{n}})\right)/\sim$$

with just the relation $(f_*(\zeta), a) \sim (\zeta, F(f^{\mathrm{op}})(a))$ for all $\zeta \in \Delta^{m-1}$, $a \in F(\underline{\boldsymbol{n}})$ and $f \in \mathrm{Hom}_{\Delta_{inj}}(\underline{\boldsymbol{m}}, \underline{\boldsymbol{n}})$.

Let $F_{\otimes,\ell}\colon \Delta_{inj}^{\mathrm{op}} \to \mathsf{CW}_\bullet$ be the functor obtained from $F_\otimes\colon \Delta_{inj}^{\mathrm{op}} \to \mathscr{CW}$ by looking at the ℓ^{th} spaces in the spectra. Define the geometric realization $\|F_\otimes\|$ as a CW spectrum whose ℓ^{th} space is $\|F_\otimes\|_\ell = \|F_{\otimes,\ell}\|$ with the structure map

$$\begin{aligned}\Sigma\|F_\otimes\|_\ell \cong \Sigma\left(\textstyle\bigvee_n \Delta_+^{n-1} \wedge F_{\otimes,\ell}(\underline{\boldsymbol{n}})\right)/\sim\, = \left(\textstyle\bigvee_n \Delta_+^{n-1} \wedge \Sigma F_{\otimes,\ell}(\underline{\boldsymbol{n}})\right)/\sim \\ \to \left(\textstyle\bigvee_n \Delta_+^{n-1} \wedge F_{\otimes,\ell+1}(\underline{\boldsymbol{n}})\right)/\sim\, = \|F_\otimes\|_{\ell+1},\end{aligned}$$

where the middle arrow is induced by the structure maps $\Sigma(F_\otimes(n))_\ell \to (F_\otimes(n))_{\ell+1}$. Equipped with the natural CW complex structure [51, Proposition A.1(i)], the reduced cellular cochain complex of $\|F_\otimes\|$ is easily seen to be the one from formula (9.22).

To identify $\|F_\otimes\|$ with $\mathrm{hocolim}(F_\otimes)$, use the construction of homotopy colimits of the strict functor $F_\otimes$ via simplices, instead of cubes; Vogt [57, Corollary 8.5] shows the two definitions agree. Using the simplicial model for the homotopy colimit, it is well known that $\mathrm{hocolim}(F_\otimes)$ is the barycentric subdivision of $\|F_\otimes\|$. (See also [51, Proposition A.3], where $|\mathrm{simp}(\cdot)|$ plays the role of this space.) □

Lemma 9.23 *The cobordism map $S\colon \mathcal{C}_{Kh}(K_1 \# K_2) \to \mathcal{C}_{Kh}(K_1) \otimes \mathcal{C}_{Kh}(K_2)$ associated to splitting at the connected sum region induces a quasi-isomorphism*

$$\begin{tikzcd}
\mathcal{C}_{Kh}(K_1\#K_2) \arrow[d, "S"] & & & \\
\mathcal{C}_{Kh}(K_1)\otimes\mathcal{C}_{Kh}(K_2) \arrow[r] & \mathcal{C}_{Kh}(K_1)\otimes Kh(U)\otimes\mathcal{C}_{Kh}(K_2) \arrow[r] & \mathcal{C}_{Kh}(K_1)\otimes Kh(U)\otimes Kh(U)\otimes\mathcal{C}_{Kh}(K_2) \arrow[r] & \cdots
\end{tikzcd}$$

from $\mathcal{C}_{Kh}(K_1 \# K_2)$ to the chain complex from formula (9.22).

Proof The Khovanov complex $\mathcal{C}_{Kh}(K_1 \# K_2)$ is the cotensor product of $\mathcal{C}_{Kh}(K_1)$ and $\mathcal{C}_{Kh}(K_2)$ as comodules over $Kh(U)$ [35, Lemma 10.5], while formula (9.22) is the derived cotensor product of $\mathcal{C}_{Kh}(K_1)$ and $\mathcal{C}_{Kh}(K_2)$. Thus, the statement presumably follows from the fact that $\mathcal{C}_{Kh}(K_1)$ and $\mathcal{C}_{Kh}(K_2)$ are coflat. Rather than going down this rabbit hole, dualize the complex (9.22) over $\mathbb{Z}$, which exchanges the split map S and the merge map m, the (derived) cotensor product and the (derived) tensor product, and $\mathcal{C}_{Kh}(K)$ and $\mathcal{C}_{Kh}(m(K))$. (The last assertion is [26, Proposition 32].) The result then follows from the fact that $\mathcal{C}_{Kh}(m(K_i))$ is free as a $Kh(U)$–module and Khovanov's connected sum theorem [27, Proposition 3.3]. □

We turn to the third part of the argument, constructing compatible spatial refinements for $F_{Kh}(K_1 \# K_2)$ and $F_{Kh}(K_1 \sqcup U \sqcup \cdots \sqcup U \sqcup K_2)$.

Lemma 9.24 *Consider any spatial refinement* $\widetilde{F}_{+/+Kh}(K_1 \sqcup K_2)$ *of* $F_{+/+Kh}(K_1 \sqcup K_2)$ *whose induced spatial refinements* $\widetilde{F}_{-+Kh}(K_1 \sqcup K_2)$ *and* $\widetilde{F}_{+-Kh}(K_1 \sqcup K_2)$ *of*

$$F_{-+Kh}(K_1 \sqcup K_2) \cong F_{+-Kh}(K_1 \sqcup K_2)$$

agree. Then, identifying $\widetilde{F}_{-+Kh}(K_1 \sqcup K_2)$ *and* $\widetilde{F}_{+-Kh}(K_1 \sqcup K_2)$ *produces a spatial refinement* $\widetilde{F}_{Kh}(K_1 \# K_2)$ *of* $F_{Kh}(K_1 \# K_2)$.

Proof It is immediate from the definitions that identifying $\widetilde{F}_{-+Kh}(K_1 \sqcup K_2)$ and $\widetilde{F}_{+-Kh}(K_1 \sqcup K_2)$ produces a spatial refinement of the functor F above. The isomorphism from Lemma 9.20 then produces the spatial refinement $\widetilde{F}_{Kh}(K_1 \# K_2)$ of $F_{Kh}(K_1 \# K_2)$. □

Let $\overline{\Delta}_{inj} = \Delta_{inj} \cup \underline{\mathbf{0}}$ be the category obtained by adding an object $\underline{\mathbf{0}} = \varnothing$ to Δ_{inj} and a unique morphism $\underline{\mathbf{0}} \to \underline{\boldsymbol{n}}$ for each n; let $f_{\underline{\mathbf{0}},0}$ denote the unique morphism from $\underline{\mathbf{0}}$ to $\underline{\mathbf{1}}$. We will extend diagram (9.16) to construct a functor $\overline{\Delta}_{inj}^{\mathrm{op}} \to \mathscr{CW}$.

Lemma 9.25 *There exists a functor* $\overline{F}_{\times} \colon \overline{\Delta}_{inj}^{\mathrm{op}} \to \mathscr{CW}$ *satisfying the following*:

(1) $\overline{F}_{\times}(\underline{\mathbf{0}}) = \mathcal{X}_{Kh}(K_1 \# K_2)$ *and* $\overline{F}_{\times}(\underline{\boldsymbol{n}}) = \mathcal{X}_{Kh}(K_1 \sqcup K_2) \wedge \left(\bigwedge_{i=1}^{n-1} \mathbb{H}^1\right)$ *for all* $n > 0$.

(2) *Let* $F_{\times}$ *denote the restriction* $\overline{F}_{\times}|_{\Delta_{inj}^{\mathrm{op}}}$. *Then there is a natural transformation* η *from the functor* $F_{\otimes}$ *of diagram (9.16) to* $F_{\times}$, *so that for all* $n > 0$, $\eta_{\underline{\boldsymbol{n}}} \colon F_{\otimes}(\underline{\boldsymbol{n}}) \to F_{\times}(\underline{\boldsymbol{n}})$ *sends each cell in* $\mathcal{X}_{Kh}(K_1) \wedge \left(\bigwedge_{i=1}^{n-1} \mathbb{H}^1\right) \wedge \mathcal{X}_{Kh}(K_2)$ *to the corresponding cell in* $\mathcal{X}_{Kh}(K_1 \sqcup K_2) \wedge \left(\bigwedge_{i=1}^{n-1} \mathbb{H}^1\right)$ *by a degree* 1 *map.*

(3) *$\overline{F}_\times(f^{\mathrm{op}}_{\underline{\mathbf{0}},0})$ is a map $\mathcal{X}_{Kh}(K_1 \sqcup K_2) \to \mathcal{X}_{Kh}(K_1 \# K_2)$ such that the induced map on reduced cellular cochains is the cobordism map*

$$S\colon \mathcal{C}_{Kh}(K_1 \# K_2) \to \mathcal{C}_{Kh}(K_1 \sqcup K_2)$$

induced by splitting at the connected sum region.

Proof During the construction of $\overline{F}_\times$, we will use spatial refinements for $K_1 \sqcup K_2$ that are pointed spatial refinements with respect to both p_1 and p_2. We will call such spatial refinements *doubly pointed spatial refinements*. Doubly pointed spatial refinements are spatial refinements that agree on $F_{*+Kh}(K_1 \sqcup K_2)$ and $F_{*-Kh}(K_1 \sqcup K_2)$, and also on $F_{+*Kh}(K_1 \sqcup K_2)$ and $F_{-*Kh}(K_1 \sqcup K_2)$ (and, therefore, agree on $F_{++Kh}(K_1 \sqcup K_2)$, $F_{+-Kh}(K_1 \sqcup K_2)$, $F_{-+Kh}(K_1 \sqcup K_2)$ and $F_{--Kh}(K_1 \sqcup K_2)$). The CW spectrum $\mathcal{X}_{Kh}(K_1 \sqcup K_2)$ constructed using any such doubly pointed spatial refinement can be viewed as a strict bimodule over $\mathbb{H}^1$, with the two actions coming from the two basepoints p_1 and p_2. Therefore, we can construct a strict functor $G\colon \Delta^{\mathrm{op}}_{inj} \to \mathsf{CW}_\bullet$ by declaring $G(\underline{\boldsymbol{n}}) = \mathcal{X}_{Kh}(K_1 \sqcup K_2) \wedge \big(\bigwedge_{i=1}^{n-1} \mathbb{H}^1\big)$ and by defining the map $G(f^{\mathrm{op}}_{\underline{\boldsymbol{n}},i})\colon \mathcal{X}_{Kh}(K_1 \sqcup K_2) \wedge \big(\bigwedge_{i=1}^{n} \mathbb{H}^1\big) \to \mathcal{X}_{Kh}(K_1 \sqcup K_2) \wedge \big(\bigwedge_{i=1}^{n-1} \mathbb{H}^1\big)$ to be the ring multiplication map applied to the i^{th} pair of consecutive $\mathbb{H}^1$–factors if $0 < i < n$, and the bimodule map coming from p_1 (respectively p_2) using the first (respectively last) $\mathbb{H}^1$–factor if $i = 0$ (respectively n).

Now we are in a position to construct $F_\times$. The construction proceeds in several stages.

(F-1) We start with pointed spatial refinements $\widetilde{F}_{Kh}(K_1)$ and $\widetilde{F}_{Kh}(K_2)$ of $F_{Kh}(K_1)$ and $F_{Kh}(K_2)$ (using k_1–dimensional and k_2–dimensional boxes with $k_1 + k_2 = k$). Since $F_{Kh}(K_1 \sqcup K_2) = F_{Kh}(K_1) \times F_{Kh}(K_2)$ (Proposition 9.1), $\widetilde{F}_{Kh}(K_1 \sqcup K_2) := \widetilde{F}_{Kh}(K_1) \wedge \widetilde{F}_{Kh}(K_2)$ (see (ho-3)) is a doubly pointed spatial refinement for $K_1 \sqcup K_2$.

(F-2) Define $F_\times\colon \Delta^{\mathrm{op}}_{inj} \to \mathscr{CW}$ to be the functor associated to this doubly pointed spatial refinement. This automatically satisfies the second part of Lemma 9.25(1). To relate $F_\otimes$ and $F_\times$, observe that for all $n > 0$, $F_\times(\underline{\boldsymbol{n}}) = F_\times(\underline{\mathbf{1}}) \wedge \big(\bigwedge_{i=1}^{n-1} \mathbb{H}^1\big)$ and $F_\otimes(\underline{\boldsymbol{n}})$ is canonically isomorphic to $F_\otimes(\underline{\mathbf{1}}) \wedge \big(\bigwedge_{i=1}^{n-1} \mathbb{H}^1\big)$ (preserving the order of the $\mathbb{H}^1$–factors); therefore, it is enough to relate $F_\times(\underline{\mathbf{1}})$, which is a formal desuspension of the suspension spectrum of $\operatorname{hocolim}\big((\widetilde{F}_{Kh}(K_1) \wedge \widetilde{F}_{Kh}(K_2))^+\big)$, and $F_\otimes(\underline{\mathbf{1}})$, which is a formal desuspension of the suspension spectrum of $\operatorname{hocolim}\big((\widetilde{F}_{Kh}(K_1))^+\big) \wedge \operatorname{hocolim}\big((\widetilde{F}_{Kh}(K_2))^+\big)$. However, the former is easily seen to be a quotient of the latter, with the quotient map sending each cell by a degree 1 map to the corresponding cell. This proves Lemma 9.25(2).

(F-3) The doubly pointed spatial refinement $\widetilde{F}_{Kh}(K_1 \sqcup K_2)$ induces a spatial refinement $\widetilde{F}_{+/+Kh}(K_1 \sqcup K_2)$ of $F_{+/+Kh}(K_1 \sqcup K_2)$; and its induced spatial refinements $\widetilde{F}_{-+Kh}(K_1 \sqcup K_2)$ and $\widetilde{F}_{+-Kh}(K_1 \sqcup K_2)$ of $F_{-+Kh}(K_1 \sqcup K_2)$ and $F_{+-Kh}(K_1 \sqcup K_2)$ agree (with $F_{-+Kh}(K_1 \sqcup K_2)$ and $F_{+-Kh}(K_1 \sqcup K_2)$ identified by Lemma 9.18). Therefore, by Lemma 9.24, we get a (pointed) spatial refinement $\widetilde{F}_{Kh}(K_1 \# K_2)$ for $K_1 \# K_2$ (with the basepoint chosen on either of the two strands near the connect-sum region).

(F-4) We use $\widetilde{F}_{Kh}(K_1 \# K_2)$ to construct the CW spectrum $\overline{F}_\times(\underline{\mathbf{0}}) = \mathcal{X}_{Kh}(K_1 \# K_2)$; this automatically satisfies the first part of Lemma 9.25(1).

(F-5) The space $\operatorname{hocolim}\big((\widetilde{F}_{+/+Kh}(K_1 \sqcup K_2))^+\big)$ is a quotient complex of the space $\operatorname{hocolim}\big((\widetilde{F}_{Kh}(K_1 \sqcup K_2))^+\big)$; it has two subcomplexes $\operatorname{hocolim}\big((\widetilde{F}_{-+Kh}(K_1 \sqcup K_2))^+\big)$ and $\operatorname{hocolim}\big((\widetilde{F}_{+-Kh}(K_1 \sqcup K_2))^+\big)$ that have a canonical isomorphism between them; and the coequalizer is canonically identified with the space $\operatorname{hocolim}\big((\widetilde{F}_{Kh}(K_1 \# K_2))^+\big)$. That is, we have a diagram

$$\begin{tikzcd}
& \operatorname{hocolim}\big((\widetilde{F}_{-+Kh}(K_1 \sqcup K_2))^+\big) \arrow[dr, hook] & \cong & \operatorname{hocolim}\big((\widetilde{F}_{+-Kh}(K_1 \sqcup K_2))^+\big) \arrow[dl, hook'] \\
\operatorname{hocolim}\big((\widetilde{F}_{Kh}(K_1 \sqcup K_2))^+\big) \arrow[rr, two heads] & & \operatorname{hocolim}\big((\widetilde{F}_{+/+Kh}(K_1 \sqcup K_2))^+\big) \arrow[d, two heads] & \\
& & \operatorname{hocolim}\big((\widetilde{F}_{Kh}(K_1 \# K_2))^+\big) &
\end{tikzcd}$$

Let $\overline{F}_\times(f^{\mathrm{op}}_{\underline{\mathbf{0}},0})\colon \overline{F}_\times(\underline{\mathbf{1}}) = \mathcal{X}_{Kh}(K_1 \sqcup K_2) \to \mathcal{X}_{Kh}(K_1 \# K_2) = \overline{F}_\times(\underline{\mathbf{0}})$ be the map induced from the composition $\operatorname{hocolim}\big((\widetilde{F}_{Kh}(K_1 \sqcup K_2))^+\big) \to \operatorname{hocolim}\big((\widetilde{F}_{Kh}(K_1 \# K_2))^+\big)$. This map is a cellular map sending the cells in $\mathcal{X}_{Kh}(K_1 \sqcup K_2)$ to the corresponding cells in $\mathcal{X}_{Kh}(K_1 \# K_2)$ by degree 1 maps (with the correspondence described in Lemma 9.20). Therefore, this map satisfies Lemma 9.25(3).

(F-6) We have to define the map $\overline{F}_\times(\underline{\mathbf{2}}) \to \overline{F}_\times(\underline{\mathbf{0}})$ to be both $\overline{F}_\times(f^{\mathrm{op}}_{\underline{\mathbf{0}},0}) \circ \overline{F}_\times(f^{\mathrm{op}}_{\underline{\mathbf{1}},0})$ and $\overline{F}_\times(f^{\mathrm{op}}_{\underline{\mathbf{0}},0}) \circ \overline{F}_\times(f^{\mathrm{op}}_{\underline{\mathbf{1}},1})$; so we merely need to check that the latter two maps agree. The two maps $\overline{F}_\times(f^{\mathrm{op}}_{\underline{\mathbf{1}},0})$ and $\overline{F}_\times(f^{\mathrm{op}}_{\underline{\mathbf{1}},1})$ are both induced from maps of spaces

$$\begin{aligned}
&\operatorname{hocolim}\big((\widetilde{F}_{Kh}(K_1 \sqcup K_2))^+\big) \wedge \{*, p_-, p_+\} \\
&\quad = \Big(\operatorname{hocolim}\big((\widetilde{F}_{Kh}(K_1 \sqcup K_2))^+\big) \times \{p_-\}\Big) \vee \Big(\operatorname{hocolim}\big((\widetilde{F}_{Kh}(K_1 \sqcup K_2))^+\big) \times \{p_+\}\Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \to \operatorname{hocolim}\big((\widetilde{F}_{Kh}(K_1 \sqcup K_2))^+\big).
\end{aligned}$$

The two maps agree on the first summand: both are the projection

$$\operatorname{hocolim}\bigl((\widetilde{F}_{Kh}(K_1 \sqcup K_2))^+\bigr) \times \{p_-\} \to \operatorname{hocolim}\bigl((\widetilde{F}_{Kh}(K_1 \sqcup K_2))^+\bigr).$$

On the second summand, the two maps are compositions of the projection map $\operatorname{hocolim}\bigl((\widetilde{F}_{Kh}(K_1 \sqcup K_2))^+\bigr) \times \{p_+\} \to \operatorname{hocolim}\bigl((\widetilde{F}_{Kh}(K_1 \sqcup K_2))^+\bigr)$ with the following two maps:

$$\begin{tikzcd}
 & \operatorname{hocolim}\bigl((\widetilde{F}_{Kh}(K_1 \sqcup K_2))^+\bigr) \arrow[dl, two heads] \arrow[dr, two heads] & \\
\operatorname{hocolim}\bigl((\widetilde{F}_{-*Kh}(K_1 \sqcup K_2))^+\bigr) \arrow[d, "\cong"] & & \operatorname{hocolim}\bigl((\widetilde{F}_{*-Kh}(K_1 \sqcup K_2))^+\bigr) \arrow[d, "\cong"'] \\
\operatorname{hocolim}\bigl((\widetilde{F}_{+*Kh}(K_1 \sqcup K_2))^+\bigr) \arrow[dr, hook] & & \operatorname{hocolim}\bigl((\widetilde{F}_{*+Kh}(K_1 \sqcup K_2))^+\bigr) \arrow[dl, hook'] \\
 & \operatorname{hocolim}\bigl((\widetilde{F}_{Kh}(K_1 \sqcup K_2))^+\bigr) &
\end{tikzcd}$$

Therefore, after composing with the map

$$\operatorname{hocolim}\bigl((\widetilde{F}_{Kh}(K_1 \sqcup K_2))^+\bigr) \to \operatorname{hocolim}\bigl((\widetilde{F}_{Kh}(K_1 \# K_2))^+\bigr)$$

from step (F-5) that induces $\overline{F}_\times(f^{\mathrm{op}}_{\underline{\mathbf{0}},\mathbf{0}})$, we get the two maps

$$\begin{tikzcd}
 & \operatorname{hocolim}\bigl((\widetilde{F}_{Kh}(K_1 \sqcup K_2))^+\bigr) \arrow[d, two heads] & \\
 & \operatorname{hocolim}\bigl((\widetilde{F}_{--Kh}(K_1 \sqcup K_2))^+\bigr) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\
\operatorname{hocolim}\bigl((\widetilde{F}_{+-Kh}(K_1 \sqcup K_2))^+\bigr) \arrow[dr, hook] & & \operatorname{hocolim}\bigl((\widetilde{F}_{-+Kh}(K_1 \sqcup K_2))^+\bigr) \arrow[dl, hook'] \\
 & \operatorname{hocolim}\bigl((\widetilde{F}_{+/+Kh}(K_1 \sqcup K_2))^+\bigr) \arrow[d, two heads] & \\
 & \operatorname{hocolim}\bigl((\widetilde{F}_{Kh}(K_1 \# K_2))^+\bigr) &
\end{tikzcd}$$

(The first vertical map is the quotient map to $\operatorname{hocolim}\bigl((\widetilde{F}_{--Kh}(K_1 \sqcup K_2))^+\bigr)$, as the composition sends the rest of $\operatorname{hocolim}\bigl((\widetilde{F}_{Kh}(K_1 \sqcup K_2))^+\bigr)$ to the basepoint.) However, since $\operatorname{hocolim}\bigl((\widetilde{F}_{Kh}(K_1 \# K_2))^+\bigr)$ is the coequalizer (see step (F-5)), these maps agree.

(F-7) The n different morphisms $\underline{\boldsymbol{n}} \to \underline{\boldsymbol{1}}$ in $\bar{\Delta}^{\mathrm{op}}_{inj}$ induce the following two maps $\bar{F}_\times(\underline{\boldsymbol{n}}) \to \bar{F}_\times(\underline{\boldsymbol{1}})$: first apply the ring multiplication on the $\mathbb{H}^1$–factors to get a map

$$\phi\colon \bar{F}_\times(\underline{\boldsymbol{n}}) = \mathcal{X}_{Kh}(K_1 \sqcup K_2) \wedge \bigl(\bigwedge_{i=1}^{n-1} \mathbb{H}^1\bigr) \to \mathcal{X}_{Kh}(K_1 \sqcup K_2) \wedge \mathbb{H}^1 = \bar{F}_\times(\underline{\boldsymbol{2}}),$$

and then compose with the two maps $\bar{F}_\times(f^{\mathrm{op}}_{\underline{\boldsymbol{1}},0})$ and $\bar{F}_\times(f^{\mathrm{op}}_{\underline{\boldsymbol{1}},1})$. We have already defined the map $\bar{F}_\times(\underline{\boldsymbol{2}}) \to \bar{F}_\times(\underline{\boldsymbol{0}})$; define the map $\bar{F}_\times(\underline{\boldsymbol{n}}) \to \bar{F}_\times(\underline{\boldsymbol{0}})$ to be its composition with ϕ.

The functor $\bar{F}_\times$ thus constructed satisfies the conditions of the lemma (see steps (F-2), (F-4) and (F-5)), thereby concluding the proof. □

Remark 9.26 Steps (F-5) and (F-6) in the proof of Lemma 9.25 actually show that with the specific choices made in the proof, $\mathcal{X}_{Kh}(K_1 \# K_2)$ is the ordinary tensor product of $\mathcal{X}_{Kh}(K_1)$ and $\mathcal{X}_{Kh}(K_2)$ over $\mathbb{H}^1$. However, these spectra and their module structures are only defined up to weak equivalence (see Proposition 9.14); and the ordinary tensor product is not invariant under such equivalences, while the derived tensor product is.

Proof of Theorem 10 Let $F_\otimes\colon \Delta^{\mathrm{op}}_{inj} \to \mathscr{CW}$ be the functor from diagram (9.16), and let $\bar{F}_\times\colon \bar{\Delta}^{\mathrm{op}}_{inj} \to \mathscr{CW}$ be the functor constructed in Lemma 9.25. The derived tensor product $\mathcal{X}_{Kh}(K_1) \otimes_{\mathbb{H}^1} \mathcal{X}_{Kh}(K_2)$ is defined to be $\operatorname{hocolim}(F_\otimes)$. The natural transformation $\eta\colon F_\otimes \to F_\times = \bar{F}_\times|_{\Delta^{\mathrm{op}}_{inj}}$ from Lemma 9.25(2) is a stable homotopy equivalence on each object; therefore, by property (ho-1) in Section 4.2, $\operatorname{hocolim}(F_\otimes) \simeq \operatorname{hocolim}(F_\times)$. Since $\underline{\boldsymbol{0}}$ is a terminal object in $\bar{\Delta}^{\mathrm{op}}_{inj}$, the functor $\bar{F}_\times$ induces a map

$$\mathcal{X}_{Kh}(K_1) \otimes_{\mathbb{H}^1} \mathcal{X}_{Kh}(K_2) = \operatorname{hocolim}(F_\times) \to \bar{F}_\times(\underline{\boldsymbol{0}}) = \mathcal{X}_{Kh}(K_1 \# K_2).$$

Using Lemmas 9.21 and 9.25(2), we can view $\operatorname{hocolim}(F_\times)$ as a CW spectrum whose reduced cellular cochain complex is the complex from formula (9.22). Lemma 9.25(3) implies that the map $\operatorname{hocolim}(F_\times) \to \bar{F}_\times(\underline{\boldsymbol{0}})$ induces the quasi-isomorphism from Lemma 9.23 at the level of reduced cellular cochain complexes. Therefore, the map is a stable homotopy equivalence. □

10 Khovanov homotopy type of a mirror

Given a spectrum X, let $X^\vee$ denote the Spanier–Whitehead dual of X, which is the internal function object parametrizing maps $X \to \mathbb{S}$. If X is a finite spectrum, the spectrum $X^\vee$ is also finite, and is characterized by the existence of map of spectra

$$\mu\colon X \wedge X^\vee \to \mathbb{S}$$

such that the slant product map

$$\textbf{(10.1)} \qquad \mu^*(\gamma)/\cdot\colon \widetilde{H}_*(X) \to \widetilde{H}^*(X^\vee)$$

is an isomorphism. (Here, $\gamma \in \widetilde{H}^0(\mathbb{S})$ is the fundamental class.) We will say that μ *witnesses* S*–duality between* X *and* $X^\vee$. See Switzer [54, Proposition 14.37] and Adams [2, Section III.9] for more details.

Theorem 11 [35, Conjecture 10.1] *Let* L *be a link and let* $m(L)$ *denote the mirror of* L*. Then*

$$\mathcal{X}^j_{Kh}(m(L)) \simeq \mathcal{X}^{-j}_{Kh}(L)^\vee.$$

Before proving the theorem we recall a reformulation of the Spanier–Whitehead duality criterion, after some homological algebra.

Lemma 10.2 *Let* C_* *and* D_* *be finitely generated chain complexes of abelian groups and let* $f\colon C_* \to D_*$ *be a chain map. Suppose that for any field* k*, the induced map* $f \otimes \mathrm{Id}_k\colon C_* \otimes k \to D_* \otimes k$ *is a quasi-isomorphism. Then* f *is a quasi-isomorphism.*

Proof (compare [20, Corollary 3.A.7]) It suffices to prove that the mapping cone $\mathrm{Cone}(f)$ of f is acyclic. Observe that $\mathrm{Cone}(f \otimes \mathrm{Id}_k) = \mathrm{Cone}(f) \otimes k$, and by assumption $\mathrm{Cone}(f \otimes \mathrm{Id}_k)$ is acyclic for any field k. It follows from the universal coefficient theorem that if $E_* \otimes \mathbb{F}_p$ is acyclic for all primes p then E_* is acyclic. □

Lemma 10.3 *Let* C_* *and* D_* *be finitely generated chain complexes of free abelian groups and* $F\colon C_* \otimes D_* \to \mathbb{Z}$ *a chain map. Write* $D^* = \mathrm{Hom}(D_*, \mathbb{Z})$ *for the dual complex to* D_**. Then the following conditions are equivalent*:

(1) *The map* $C_* \to D^*$ *induced by* F *is a quasi-isomorphism.*

(2) *For any field* k*, the map* $H_*((C_* \otimes k)) \otimes H_*((D_* \otimes k)) \to k$ *induced by* F *is a perfect pairing.*

Proof (1)$\Longrightarrow$(2) Tensoring with k, the map $f\colon C_* \otimes k \to D^* \otimes k = \mathrm{Hom}(D_*, k)$ induced by F is a quasi-isomorphism. Suppose α is a nontrivial element of $H_*(C_* \otimes k)$. Then $f_*(\alpha) \neq 0 \in H_*(D^* \otimes k) = \mathrm{Hom}(H_*(D_* \otimes k), k)$. Let $\beta \in H_*(D_* \otimes k)$ be an element on which $f_*(\alpha)$ evaluates nontrivially. Then $F_*(\alpha \otimes \beta) = f_*(\alpha)(\beta) \neq 0$.

Similarly, if $\beta \neq 0 \in H_*(D_* \otimes k)$ then there is an element $b \in H_*(D^* \otimes k)$ such that $b(\beta) \neq 0$. Since f is a quasi-isomorphism there is an $\alpha \in H_*(C_* \otimes k)$ such that $b = f(\alpha)$. Then $F_*(\alpha \otimes \beta) = f_*(\alpha)(\beta) = b(\beta) \neq 0$.

(2)$\Longrightarrow$(1) We start by checking that for any field k, the map $f\colon C_*\otimes k\to D^*\otimes k$ is a quasi-isomorphism. To this end, suppose $\alpha\neq 0\in H_*(C_*\otimes k)$. Then there exists $\beta\in H_*(D_*\otimes k)$ such that $F_*(\alpha\otimes\beta)\neq 0\in k$. Then $f_*(\alpha)(\beta)=F_*(\alpha\otimes\beta)\neq 0$, so $f_*(\alpha)\neq 0$. Thus, f_* is injective. Next, given any element $b\in H_*(D^*\otimes k)=\operatorname{Hom}(H_*(D_*\otimes k),k)$, since F_* is a perfect pairing there is an element $\alpha\in H_*(C_*\otimes k)$ such that $b(\cdot)=F_*(\alpha\otimes\cdot)$. Since $F_*(\alpha\otimes\cdot)=f_*(\alpha)(\cdot)$, we get $b=f_*(\alpha)$. So, f_* is surjective.

Thus, the map $f\otimes\mathrm{Id}_k\colon C_*\otimes k\to D^*\otimes k$ is a quasi-isomorphism for any field k. So, by Lemma 10.2, the map $f\colon C_*\to D^*$ induced by F is a quasi-isomorphism, as desired. □

Proposition 10.4 *Let X and Y be finite CW complexes and let $\mu\colon X\wedge Y\to S^n$ be a continuous map. Then μ witnesses S–duality between X and $\Sigma^{-n}Y$ if and only if for every field k and integer i, the induced map*

$$\widetilde{H}_i(X;k)\otimes\widetilde{H}_{n-i}(Y;k)\to\widetilde{H}_n(S^n;k)=k$$

is a perfect pairing.

Proof Let $C_*^{cell}(X)$ denote the reduced cellular chain complex of X and $C^*_{cell}(X)$ the reduced cellular cochain complex. The slant product map of formula (10.1) is the map on homology induced by

$$C_*^{cell}(X)\to C^{n-*}_{cell}(Y)=\operatorname{Hom}(C^{cell}_{n-*}(Y),\mathbb{Z}),\quad x\mapsto\big(y\mapsto\gamma(\mu_\#(x\wedge y))\big).$$

In other words, this is the map induced by the pairing

$$C_*^{cell}(X)\otimes C^{cell}_{n-*}(Y)\to C^{cell}_n(S^n)=\mathbb{Z},\quad x\otimes y\mapsto\gamma(\mu_\#(x\wedge y)).$$

So, the result is immediate from Lemma 10.3. □

The other ingredients in the proof of Theorem 11 are some facts about functoriality of Khovanov homology and the Khovanov stable homotopy type.

Suppose that $F\subset[0,1]\times\mathbb{R}^3$ is a link cobordism from L_0 to L_1. Associated to F is a map $\Phi_F\colon Kh(L_0)\to Kh(L_1)$, well-defined up to multiplication by ± 1; see [24; 28; 4]. The following properties of Φ are immediate from the definition:

(1) If F and F' are isotopic link cobordisms (relative boundary) then $\Phi_F=\pm\Phi_{F'}$.

(2) If $F=[0,1]\times L$ is the identity cobordism from L to L then

$$\Phi_F=\pm\,\mathrm{Id}\colon Kh(L)\to Kh(L).$$

(3) If F_1 is a cobordism from L_0 to L_1 and F_2 is a cobordism from L_1 to L_2, and $F_2 \circ F_1$ denotes the composition of F_2 and F_1, then

$$\Phi_{F_2 \circ F_1} = \pm \Phi_{F_2} \circ \Phi_{F_1}.$$

(4) If F is a cobordism from L_0 to L_1 and F' is a cobordism from L'_0 to L'_1, and $F \sqcup F'$ denotes the disjoint union of F and F', which is a cobordism from $L_0 \sqcup L'_0$ to $L_1 \sqcup L'_1$, then, for any field k,

$$\Phi_{F \sqcup F'} = \pm \Phi_F \otimes \Phi_{F'}\colon Kh(L_0;k) \otimes Kh(L'_0;k) = Kh(L_0 \sqcup L'_0;k) \\ \to Kh(L_1 \sqcup L'_1;k) = Kh(L_1;k) \otimes Kh(L'_1;k).$$

Given a link L there is a canonical, genus 0 cobordism F from $L \sqcup m(L)$ to the empty link.

Proposition 10.5 *For any field k, the map*

$$\Phi_F\colon Kh(L;k) \otimes Kh(m(L);k) = Kh(L \sqcup m(L);k) \to Kh(\varnothing;k) = k$$

associated to the canonical cobordism from $L \sqcup m(L)$ to the empty link is a perfect pairing.

Proof This follows from properties (1), (2), (3) and (4) above via the usual snake-straightening argument in topological field theory (see for instance [46, Lecture 7]). □

As noted above, the cohomology groups of $\mathcal{X}_{Kh}(L)$ are the Khovanov homology of L. Since we are viewing Khovanov homology as covariant in the cobordism, it is more convenient to work with the homology groups of $\mathcal{X}_{Kh}(L)$. These can be understood as follows:

Lemma 10.6 *Let L be a link diagram. Then the cellular chain complex for $\mathcal{X}_{Kh}(L)$ agrees with the Khovanov complex for $m(L)$. In particular,*

$$\widetilde{H}_i(\mathcal{X}^j_{Kh}(L)) = Kh^{-i,-j}(m(L)).$$

Proof This is immediate from the definitions. To wit, $\mathcal{X}_{Kh}(L)$ can be constructed as a CW complex whose reduced cellular cochain complex $C^*_{cell}(\mathcal{X}_{Kh}(L))$ is isomorphic to the Khovanov complex $\mathcal{C}_{Kh}(L)$ from Section 2.2. Therefore, the reduced cellular chain complex $C^{cell}_*(\mathcal{X}_{Kh}(L))$ is isomorphic to the dual complex $\mathrm{Hom}(\mathcal{C}_{Kh}(L), \mathbb{Z})$. However, the dual complex is isomorphic to $\mathcal{C}_{Kh}(m(L))$ [26, Proposition 32]. In the language of Section 2.2, this isomorphism takes a Khovanov generator in $F_L(v)$ to a Khovanov generator in $F_{m(L)}(\vec{1}-v)$ by changing the labels on the circles of $\mathcal{P}_L(v)=\mathcal{P}_{m(L)}(\vec{1}-v)$ from x_+ to x_- and vice versa. The gradings work out, and we get $\widetilde{H}_i(\mathcal{X}^j_{Kh}(L))=Kh^{-i,-j}(m(L))$. □

Functoriality for the Khovanov spectrum has not yet been verified, but in [36] we did associate maps to elementary cobordisms:

Proposition 10.7 *Let L_1 and L_2 be links in $\mathbb{R}^3$ and F a cobordism from L_1 to L_2. Then there is a map of spectra*

$$\Psi_{m(F)}\colon \mathcal{X}_{Kh}(m(L_1))\to\mathcal{X}_{Kh}(m(L_2))$$

such that the induced map

$$\Psi_{m(F),*}\colon \widetilde{H}_*\big(\mathcal{X}_{Kh}(m(L_1))\big)=Kh(L_1)\to Kh(L_2)=\widetilde{H}_*\big(\mathcal{X}_{Kh}(m(L_2))\big)$$

agrees with the cobordism map Φ_F up to sign.

Proof The corresponding statement in cohomology is immediate from [36, Proposition 3.4, Lemma 3.6 and Lemma 3.7]; the homology statement comes from dualizing [36, Proposition 3.4 and Lemmas 3.6–3.7] (see Lemma 10.6). □

With these ingredients, we are now ready to verify the mirror theorem.

Proof of Theorem 11 The following argument was suggested to us by the referee for [37].

By Proposition 10.4, it suffices to construct a map $\mathcal{X}_{Kh}(m(L))\wedge\mathcal{X}_{Kh}(L)\to\mathbb{S}$ so that for any field k, the induced map

$$\widetilde{H}_i\big(\mathcal{X}_{Kh}(m(L));k\big)\otimes\widetilde{H}_{-i}(\mathcal{X}_{Kh}(L);k)\to\widetilde{H}_0(\mathbb{S})=k$$

is a perfect pairing. By Proposition 10.5, applying Proposition 10.7 to the canonical cobordism from $L\sqcup m(L)$ to the empty link gives such a map. □

Remark 10.8 Theorem 11 (perhaps) gives an obstruction to knots being amphichiral: if K is amphichiral then $\mathcal{X}^j_{Kh}(K)$ is Spanier–Whitehead dual to $\mathcal{X}^{-j}_{Kh}(K)$. Using KnotKit [50] (and the technique in [37]), it is possible to verify that this obstruction does not give any additional restrictions on amphichirality for prime knots up to 15 crossings, beyond those implied by Khovanov homology itself. Indeed, the only

Khovanov homology-symmetric knots with 15 or fewer crossings for which Sq^2 is nonvanishing are 14^n_{8440}, 14^n_{9732}, 14^n_{21794}, 14^n_{22073} and 15^n_{139717}. In each of these cases, the non-Moore space summands of $\mathcal{X}_{Kh}$ are copies of (various suspensions and desuspensions of) $\mathbb{R}P^5/\mathbb{R}P^2$ and $\mathbb{R}P^4/\mathbb{R}P^1$, and these summands (with appropriate grading shifts) are exchanged by $(i,j) \mapsto (-i,-j)$.

It remains open whether the stable homotopy type gives nontrivial restrictions for larger prime knots, although we expect that it does. As a proof of concept, look at [49, Table 1] and consider the link $L = 14^n_{11196} \sqcup m(14^n_{17546})$. Its Khovanov homology is symmetric since Khovanov homologies of 14^n_{11196} and 14^n_{17546} are isomorphic (as bigraded groups); the homology of either knot is supported on three adjacent diagonals, the only torsion present is 2–torsion, and the lowest and highest quantum gradings where 2–torsion appears are -11 and 5. However, 14^n_{17546} has no Sq^2, but 14^n_{11196} has a single Sq^2 in quantum grading -1. Using the Cartan formula and Theorem 1, the lowest and highest quantum gradings where $Kh(L)$ has nontrivial Sq^3 are -6 and 10, and, therefore, its Khovanov homotopy type is not symmetric. (Of course this is not our recommended proof for showing L is achiral!)

11 Applications

In this section we give an application of the Künneth theorem for the Khovanov stable homotopy type to knot concordance. We begin with some background, continuing from Section 2.2. Recall that Rasmussen [48] used the Lee deformation of the Khovanov complex [31] (the specialization $(h,t) = (0,1)$ of Definition 2.1) to define a concordance invariant $s^{\mathbb{Q}}_K \in 2\mathbb{Z}$. As the notation suggests, to define $s^{\mathbb{Q}}$, Rasmussen used Khovanov homology with coefficients in $\mathbb{Q}$, though any field of characteristic different from 2 would work as well. Bar-Natan [4] gave an analogue $\mathcal{C}_{BN}$ of the Lee deformation that also works over $\mathbb{F}_2$ (see also [43; 55]). The Bar-Natan complex $\mathcal{C}_{BN}$ is the specialization $(h,t) = (1,0)$ of the universal complex $\mathcal{C}$ of Definition 2.1. Let $s_K = s^{\mathbb{F}_2}_K$ denote the corresponding Rasmussen-type invariant; see also [36, Section 2.2].

Since the formal variable h has quantum grading -2, the differential in the Bar-Natan chain complex either preserves the quantum grading or increases it. Let $\mathcal{F}_q\mathcal{C}_{BN}$ denote the subcomplex of $\mathcal{C}_{BN}$ supported in quantum grading q or higher. This defines a filtration on the Bar-Natan complex, and the associated graded object $\bigoplus_q \mathcal{F}_q\mathcal{C}_{BN}/\mathcal{F}_{q+2}\mathcal{C}_{BN}$ is the Khovanov chain complex $\mathcal{C}_{Kh}$.

The second Steenrod square Sq^2 for the Khovanov spectrum $\mathcal{X}_{Kh}(K)$ gives a map

$$\mathrm{Sq}^2\colon Kh^{i,j}(K;\mathbb{F}_2) \to Kh^{i+2,j}(K;\mathbb{F}_2)$$

for each i and j. Using this, in [36] we constructed a refined s–invariant $s_+^{\mathrm{Sq}^2}(K)$, as follows:

Definition 11.1 [36, Definition 1.2 and Lemma 4.2] Consider configurations of the following form:

(11.2)

$$\begin{tikzcd}
\langle \tilde{a}, \tilde{b} \rangle \arrow[r] \arrow[d, hook] & \langle \hat{a}, \hat{b} \rangle \arrow[d, hook] & \langle a, b \rangle \arrow[l] \arrow[r] \arrow[d, hook] & \langle \bar{a}, \bar{b} \rangle \arrow[d, equal] \\
Kh^{-2,s_K-1}(K;\mathbb{F}_2) \arrow[r, "\mathrm{Sq}^2"] & Kh^{0,s_K-1}(K;\mathbb{F}_2) & H_0(\mathcal{F}_{s_K-1}\mathcal{C}_{BN}(K);\mathbb{F}_2) \arrow[l] \arrow[r] & H_0(\mathcal{C}_{BN}(K);\mathbb{F}_2)
\end{tikzcd}$$

Define

$$s_+^{\mathrm{Sq}^2}(K) = \begin{cases} s_K & \text{if there does not exist such a configuration,} \\ s_K + 2 & \text{otherwise.} \end{cases}$$

Theorem 12 [36, Theorem 1] *The integer $s_+^{\mathrm{Sq}^2}(K)$ is a concordance invariant, and*

$$2g_4(K) \geq |s_+^{\mathrm{Sq}^2}(K)|$$

where $g_4(K)$ is the 4–ball genus of K.

Lemma 11.3 *Let K and L be two knots such that the following holds*:

1. $s_+^{\mathrm{Sq}^2}(K) = s_K + 2$.
2. $Kh^{0,s_L+1}(L;\mathbb{F}_2) \xrightarrow{\mathrm{Sq}^2} Kh^{2,s_L+1}(L;\mathbb{F}_2)$ *is the zero map.*
3. *Either the map* $Kh^{0,s_L+1}(L;\mathbb{F}_2) \xrightarrow{\mathrm{Sq}^1} Kh^{1,s_L+1}(L;\mathbb{F}_2)$ *is the zero map or the map* $Kh^{-2,s_K-1}(K;\mathbb{F}_2) \xrightarrow{\mathrm{Sq}^1} Kh^{-1,s_K-1}(K;\mathbb{F}_2)$ *is the zero map.*

Then $s_+^{\mathrm{Sq}^2}(K \# L) = s_{K\#L} + 2 = s_K + s_L + 2$.

Proof Consider the saddle cobordism from $K \sqcup L$ to $K\#L$. Choose some orientations o_K, o_L and $o_{K\#L}$, of the knots K, L and $K \# L$, which are coherent with respect to the saddle cobordism (see Figure 10). Let $\{\pm o_K\}$, $\{\pm o_L\}$ and $\{\pm o_{K\#L}\}$ denote the sets of orientations of K, L and $K \# L$. For any orientation o there is a corresponding generator $g(o)$ of $H_0(\mathcal{C}_{BN};\mathbb{F}_2)$ [31, Theorem 4.2; 56, Theorem 1]. We will use the following facts:

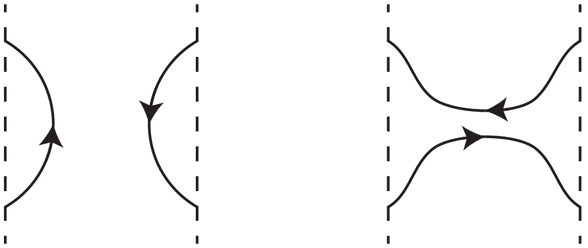


Figure 10: Oriented connected sum. Pieces of two knots and their connected sum, oriented coherently with the obvious saddle cobordism, are shown.

(1) $\mathcal{C}_{BN}(K \sqcup L)$ is canonically identified with $\mathcal{C}_{BN}(K) \otimes \mathcal{C}_{BN}(L)$. The induced identification on the associated graded objects, $Kh(K \sqcup L) \cong Kh(K) \otimes Kh(L)$, is the one induced from the equivalence from Theorem 1. (This is immediate from the definitions.)

(2) The sets $\{g(\pm o_K)\}$, $\{g(\pm o_L)\}$, $\{g(\pm o_{K\#L})\}$ and $\{g(\pm o_K) \otimes g(\pm o_L)\}$ form bases of $H_0(\mathcal{C}_{BN}(K))$, $H_0(\mathcal{C}_{BN}(L))$, $H_0(\mathcal{C}_{BN}(K \# L))$ and $H_0(\mathcal{C}_{BN}(K \sqcup L))$, respectively [31, Section 4.4.3; 56, Theorem 1].

(3) $g(o_L) + g(-o_L)$ has a cycle representative in $\mathcal{F}_{s_L+1}\mathcal{C}_{BN}(L)$ [36, Proposition 2.6].

(4) The saddle cobordism map $\mathcal{C}_{BN}(K \sqcup L) \to \mathcal{C}_{BN}(K \# L)$ preserves the homological grading and either increases the quantum grading or decreases it by exactly one. The induced map on Kh commutes with the map Sq^2 [36, Theorem 4]. The induced map on $H_0(\mathcal{C}_{BN}; \mathbb{F}_2)$ is the following:

$$\begin{aligned} g(o_K) \otimes g(o_L) &\mapsto g(o_{K\#L}), & g(-o_K) \otimes g(o_L) &\mapsto 0, \\ g(o_K) \otimes g(-o_L) &\mapsto 0, & g(-o_K) \otimes g(-o_L) &\mapsto g(-o_{K\#L}). \end{aligned}$$

(This follows from the same argument as [48, Proposition 4.1], since Turner's change of basis diagonalizes the Bar-Natan Frobenius algebra.)

Since $s_+^{\mathrm{Sq}^2}(K) = s_K + 2$, there is a configuration as in (11.2). Since $\{g(o_K), g(-o_K)\}$ also form a basis for $H_0(\mathcal{C}_{BN}(K); \mathbb{F}_2)$ (fact (2)), after performing a change of basis if necessary, we may assume that $\overline{a} = g(o_K)$ and $\overline{b} = g(-o_K)$.

Using fact (3), choose some configuration of the form

$$\begin{tikzcd} \langle \hat{c} \rangle \arrow[d, hook] & \langle c \rangle \arrow[l] \arrow[r] \arrow[d, hook] & \langle g(o_L) + g(-o_L) \rangle \arrow[d, hook] \\ Kh^{0,s_L+1}(L; \mathbb{F}_2) & H_0(\mathcal{F}_{s_L+1}\mathcal{C}_{BN}(L); \mathbb{F}_2) \arrow[l] \arrow[r] & H_0(\mathcal{C}_{BN}(L); \mathbb{F}_2) \end{tikzcd}$$

Combining the two configurations and using the identification from fact (1), we get the configuration (transposed to fit)

$$\begin{tikzcd}
\langle \widetilde{a} \otimes \widehat{c}, \widetilde{b} \otimes \widehat{c} \rangle \arrow[r, hook] \arrow[d] & Kh^{-2,s_K+s_L}(K \sqcup L; \mathbb{F}_2) \arrow[d, "\mathrm{Sq}^2"'] \\
\langle \widehat{a} \otimes \widehat{c}, \widehat{b} \otimes \widehat{c} \rangle \arrow[r, hook] & Kh^{0,s_K+s_L}(K \sqcup L; \mathbb{F}_2) \\
\langle a \otimes c, b \otimes c \rangle \arrow[r, hook] \arrow[u] \arrow[d] & H_0(\mathcal{F}_{s_K+s_L}\mathcal{C}_{BN}(K \sqcup L); \mathbb{F}_2) \arrow[u] \arrow[d] \\
\begin{array}{l} \langle g(o_K) \otimes (g(o_L) + g(-o_L)), \\ \quad g(-o_K) \otimes (g(o_L) + g(-o_L)) \rangle \end{array} \arrow[r, hook] & H_0(\mathcal{C}_{BN}(K \sqcup L); \mathbb{F}_2)
\end{tikzcd}$$

We should justify the topmost square; that is, assuming $\mathrm{Sq}^2(\widetilde{a}) = \widehat{a}$ and $\mathrm{Sq}^2(\widetilde{b}) = \widehat{b}$, we need to show that $\mathrm{Sq}^2(\widetilde{a} \otimes \widehat{c}) = \widehat{a} \otimes \widehat{c}$ and $\mathrm{Sq}^2(\widetilde{b} \otimes \widehat{c}) = \widehat{b} \otimes \widehat{c}$. Since the identification $Kh(K \sqcup L) \cong Kh(K) \otimes Kh(L)$ is induced from the identification $\mathcal{X}_{Kh}(K \sqcup L) \cong \mathcal{X}_{Kh}(K) \wedge \mathcal{X}_{Kh}(L)$ of Theorem 1 — see fact (1) —

$$\mathrm{Sq}^2(\widetilde{a} \otimes \widehat{c}) = \mathrm{Sq}^2(\widetilde{a}) \otimes \widehat{c} + \mathrm{Sq}^1(\widetilde{a}) \otimes \mathrm{Sq}^1(\widehat{c}) + \widetilde{a} \otimes \mathrm{Sq}^2(\widehat{c}) = \widehat{a} \otimes \widehat{c}.$$

The first equality is the Cartan formula; the second uses the lemma's hypotheses. Similarly, $\mathrm{Sq}^2(\widetilde{b} \otimes \widehat{c}) = \widehat{b} \otimes \widehat{c}$.

Now consider the image of this configuration under the saddle cobordism map. Using fact (4), we get a configuration

$$\begin{tikzcd}
\langle \widetilde{p}, \widetilde{q} \rangle \arrow[r, hook] \arrow[d] & Kh^{-2,s_{K\#L}-1}(K \# L; \mathbb{F}_2) \arrow[d, "\mathrm{Sq}^2"'] \\
\langle \widehat{p}, \widehat{q} \rangle \arrow[r, hook] & Kh^{0,s_{K\#L}-1}(K \# L; \mathbb{F}_2) \\
\langle p, q \rangle \arrow[r, hook] \arrow[u] \arrow[d] & H_0(\mathcal{F}_{s_{K\#L}-1}\mathcal{C}_{BN}(K \# L); \mathbb{F}_2) \arrow[u] \arrow[d] \\
\langle g(o_{K\#L}), g(-o_{K\#L}) \rangle \arrow[r, hook] & H_0(\mathcal{C}_{BN}(K \# L); \mathbb{F}_2)
\end{tikzcd}$$

By fact (2), $\langle g(o_{K\#L}), g(-o_{K\#L}) \rangle = H_0(\mathcal{C}_{BN}(K \# L); \mathbb{F}_2)$; therefore, $s_+^{\mathrm{Sq}^2}(K \# L) = s_{K\#L} + 2$. □

K	$\sigma(K)$	s_K	$\tau(K)$	$s_+^{\mathrm{Sq}^2}(K)$	$g_4(K)$	$s_K^{\mathbb{Q}}$	$u(K)$
9_{42}	-2	0	0	2	1	0	1
10_{136}	-2	0	0	2	1	0	1
$m(11^n_{19})$	-4	2	1	4	2	2	2
$m(11^n_{20})$	-2	0	0	2	1	0	1
11^n_{70}	-4	2	1	4	2	2	2
11^n_{96}	-2	0	0	2	1	0	1

Table 1

We are almost ready to prove Corollary 1.5. First, we tabulate some invariants of the knots K that appear in its statement.

There is some confusion in the nomenclature of knots regarding mirrors, stemming primarily from the fact that Gauss codes do not distinguish between a knot and its mirror. See Figure 1 for our convention. We have followed the convention from the Knot Atlas [5], using their planar diagram presentations (which do detect chirality). The knots in Figure 1 are produced in Knotilus [15] using Gauss codes from the Knot Atlas; however, for 11^n_{70}, our knot diagram differs from the one produced by Knotilus (as well as the one in the Knot Atlas) by mirroring. Alternatively, one can deduce our convention from the value of the signature $\sigma(K)$ in Table 1 (which, for us, is negative for positive knots).

The values of $s_K = s_K^{\mathbb{F}_2}$ and $s_+^{\mathrm{Sq}^2}(K)$ are imported from [37]; s_K can also be computed independently by Knotkit [50]. The values of the four-ball genus $g_4(K)$ and the unknotting number $u(K)$ are extracted from Knotinfo [38]. The absolute values of the signature $\sigma(K)$ and $s_K^{\mathbb{Q}}$ are extracted from [5] and the absolute value of Ozsváth and Szabó's invariant $\tau(K)$ come from [3]. We deduce the signs of $\sigma(K)$, $s_K^{\mathbb{Q}}$ and $\tau(K)$ by noticing that all the unknotting crossings in Figure 1 are positive, and changing a positive crossing to a negative one decreases $-\frac{1}{2}\sigma(K)$, $\frac{1}{2}s^{\mathbb{Q}}$ and $\tau(K)$ by 0 or 1. We list $\tau(K)$, $s_K^{\mathbb{Q}}$ and $u(K)$ purely for the reader's interest.

Finally, we deduce Corollary 1.5, which, to recall, was the following:

Corollary 1.5 *Let K be one of the knots 9_{42}, 10_{136}, $m(11^n_{19})$, $m(11^n_{20})$, 11^n_{70} or 11^n_{96}.* (Here m denotes the mirror.) *Let L be a knot which is the closure of a positive braid. Letting g_4 denote the four-ball genus, we have*

$$g_4(K \# L) = g_4(K) + g_4(L).$$

Proof Certainly $g_4(K\# L) \leq g_4(K) + g_4(L)$, so we only need to show $g_4(K\# L) \geq g_4(K) + g_4(L)$.

First consider the knot $K \in \{9_{42}, 10_{136}, m(11^n_{19}), m(11^n_{20}), 11^n_{70}, 11^n_{96}\}$. In all cases $s^{\mathrm{Sq}^2}_{+}(K) = s_K + 2 = 2g_4(K)$ (see Table 1).

Next consider the knot L. Draw L as the closure of a positive braid with n crossings and m strands. We will classify the generators of the Khovanov chain complex $\mathcal{C}_{Kh}(L)$ in quantum grading $n+2-m$ or less.

Let v be some vertex in the cube of resolutions for L. Assume there are c_v circles in the resolution at v. The smallest quantum grading over v is achieved by the Khovanov generator which labels all the c_v circles by x_-. The value of this smallest quantum grading is $n + |v| - c_v$.

Since all the crossings of L are positive, the resolution at the zero vertex $\vec{0}$ is the oriented resolution; therefore $c_{\vec{0}} = m$. If u is some vertex of weight 1, ie $|u| = 1$, then the resolution at u is obtained from the oriented resolution by a merge; therefore $c_u = m - 1$. The resolution at any other vertex v is obtained from some weight 1 vertex by $|v| - 1$ merges or splits; therefore $c_v \leq m - 1 + |v| - 1 = m + |v| - 2$, with equality holding if and only if the resolution at v can be obtained from some weight 1 resolution by splits only.

It follows that the minimum quantum grading over any vertex $v \neq \vec{0}$ is at least $n-m+2$; and over $\vec{0}$, the minimum quantum grading is $n - m$, which is attained only by the Khovanov generator that labels all the m circles by x_-. Observe that this is enough to compute the values of s_L, $g_4(L)$ and $g(L)$: $s_L \geq n - m + 1$, and Seifert's algorithm yields a surface of genus $\frac{1}{2}(n - m + 1)$; therefore, the inequality

$$n - m + 1 \leq s_L \leq 2g_4(L) \leq 2g(L) \leq n - m + 1$$

leads to the equality

$$s_L = 2g_4(L) = 2g(L) = n - m + 1.$$

Next, we compute the Khovanov homology in these quantum gradings. We have $Kh^{*,q}(L;\mathbb{Z}) = 0$ for all $q < n - m$ and $Kh^{*,n-m}(L;\mathbb{Z})$ is $\mathbb{Z}$ supported in homological grading zero. However, our interest lies in quantum grading $n - m + 2$; we show that $Kh^{*,n-m+2}(L;\mathbb{Z})$ is $\mathbb{Z}$ supported in homological grading zero as well.

Number the m strands in the braid diagram from left to right. For $1 \leq i < m$, let n_i be the number of crossings in the diagram between the i^{th} and the $(i+1)^{\text{st}}$ strands, ie the number of times σ_i occurs in the braid word. (Then $n = \sum_i n_i$, and, since L is a knot, $n_i \geq 1$ for all i.) Number the n_i crossings from top to bottom. For $1 \leq i \leq m$, let $\boldsymbol{x}_i = (\vec{0}, x_i)$ be the Khovanov generator where x_i labels the i^{th} circle in the oriented resolution by x_+, and the rest by x_-. For $1 \leq i < m$ and $\varnothing \neq J \subseteq \{1, \dots, n_i\}$, let $u_{i,J}$ be the weight $|J|$ vertex in the cube of resolutions where the 1–resolution is taken only at the crossings that appear in J between the i^{th} and $(i+1)^{\text{st}}$ strands; and let $\boldsymbol{y}_{i,J}$ be the Khovanov generator living over $u_{i,J}$ where all the circles are labeled by x_-. From the discussion above it is clear that the Khovanov chain group $\mathcal{C}_{Kh}^{*,n+2-m}(L)$ is generated by these generators $\boldsymbol{x}_i$ and $\boldsymbol{y}_{i,J}$. The differential is fairly straightforward:

$$\delta \boldsymbol{x}_i = \begin{cases} \sum_{j=1}^{n_1} \boldsymbol{y}_{1,\{j\}} & \text{if } i = 1, \\ \sum_{j=1}^{n_i} \boldsymbol{y}_{i,\{j\}} + \sum_{j=1}^{n_{i-1}} \boldsymbol{y}_{i-1,\{j\}} & \text{if } 1 < i < m, \\ \sum_{j=1}^{n_{m-1}} \boldsymbol{y}_{m-1,\{j\}} & \text{if } i = m, \end{cases} \qquad \delta \boldsymbol{y}_{i,J} = \sum_{\substack{J' \supset J \\ |J' \setminus J| = 1}} \pm \boldsymbol{y}_{i,J'}.$$

Here we are using the standard sign assignment on the cube, and the signs are determined by the ordering of the n crossings.

Using the change of basis replacing $\boldsymbol{x}_k$ by $\sum_{i=1}^{k} (-1)^i \boldsymbol{x}_i$ for all $1 \leq k \leq m$, it is easy to see that the chain complex $\mathcal{C}_{Kh}^{*,n+2-m}(L)$ is isomorphic to the direct sum of cube complexes

$$\mathcal{C}_{Kh}^{*,n+2-m}(L; \mathbb{Z}) \cong \mathbb{Z} \oplus \left(\bigoplus_{i=1}^{m-1} \left(\bigotimes_{j=1}^{n_i} (\mathbb{Z} \to \mathbb{Z}) \right) \right).$$

Therefore, $Kh^{*,n+2-m}(L; \mathbb{Z}) \cong \mathbb{Z}$, supported in homological grading zero (and is generated by the cycle $\sum_{i=1}^{m} (-1)^i \boldsymbol{x}_i$).

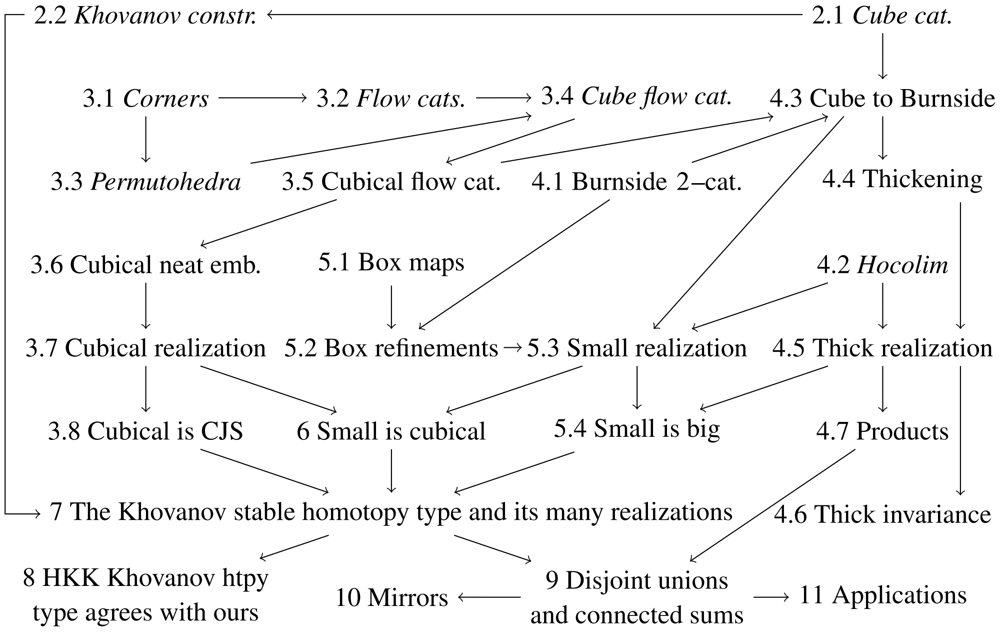
As an immediate consequence, we have that both the maps

$$\mathrm{Sq}^1\colon Kh^{0,s_L+1}(L; \mathbb{F}_2) \to Kh^{1,s_L+1}(L; \mathbb{F}_2) = 0,$$
$$\mathrm{Sq}^2\colon Kh^{0,s_L+1}(L; \mathbb{F}_2) \to Kh^{2,s_L+1}(L; \mathbb{F}_2) = 0$$

vanish. Therefore, K and L satisfy the conditions of Lemma 11.3, so

$$2g_4(K \# L) \geq s_+^{\mathrm{Sq}^2}(K \# L) = s_K + s_L + 2 = 2g_4(K) + 2g_4(L). \qquad \square$$

Appendix Flow chart of sections



Background material is in italics. Superficial dependencies (eg for examples or small pieces of notation) have been omitted.

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