# Closed ideal planar curves

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We use a gradient flow to deform closed planar curves to curves with least variation of geodesic curvature in the $L^2$ sense. Given a smooth initial curve we show that the solution to the flow exists for all time and, provided the length of the evolving curve remains bounded, smoothly converges to a multiply covered circle. Moreover, we show that curves in any homotopy class with initially small $L^3\|k_s\|_2^2$ enjoy a uniform length bound under the flow, yielding the convergence result in these cases.

## 1 Overview

Let us define $E[\gamma]$ for a smooth closed planar curve $\gamma \colon \mathbb{S} \to \mathbb{R}^2$ by

$$E[\gamma] = \frac{1}{2} \int_\gamma k_s^2 \, ds.$$

Here, $k_s$ is the first arclength derivative of curvature. We note that applications of the energy $E$ appear in computer-aided design; see Harary and Tal [5].[1] Critical points for $E$ are termed *planar ideal curves*. These are the model one-dimensional case of *ideal submanifolds*, and their study is a preliminary proof-of-concept step in a larger program. We explain this broader perspective in Section 2, before moving into the details of the case we study here in Section 3.

Our primary interest is in the $L^2$–gradient flow for the functional $E$. In Section 3 we calculate the first variation of $E$, giving the resultant Euler–Lagrange operator and gradient flow. Next, Section 4 describes local existence for the flow. This is done by a standard method. Section 5 concerns the equilibrium set for the flow, proving that it consists only of multiply covered circles (we call these $\omega$–circles) and their rigid images. We move on to the analysis of the global behaviour of the flow in Section 6,

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[1] We thank Yann Bernard for providing us with this reference.

which identifies *length* as the quantity that influences asymptotic behaviour of the flow. The main result of Section 6 is global existence for arbitrary smooth initial data. This behaviour is reminiscent of the (unconstrained) elastic flow — see Dziuk, Kuwert and Schätzle [4] — and is in stark contrast with the curve diffusion flow — see Wheeler [12] — and its higher-order relatives; see Parkins and Wheeler [10].

In Section 7, we study the behaviour of the flow under a condition on the scale-invariant counterpart of $E$, namely (here $L$ is used to denote the length of $\gamma$)

$$(L^3 E)[\gamma].$$

While the flow exists for all time regardless of the initial energy, convergence is not straightforward. Since the rigidity result of Section 5 classifies all equilibria as circles, clearly initial data in the regular homotopy class of a lemniscate cannot converge. The key issue is in obtaining uniform upper and lower bounds for the length of the evolving family of curves. When length is straightforward to control, such as in the case of evolving families of curves with free boundary on parallel lines — see McCoy, Wheeler and Wu [9] — convergence can be obtained by a more standard, direct argument. In Section 7 we prove that smallness of $(L^3 E)[\gamma_0]$ allows us to estimate the scale-invariant energy by the square of the Euler–Lagrange operator in $L^2$. This allows us to first prove monotonicity of the scale-invariant energy, and eventually obtain uniform upper and lower bounds on length.

When full convergence of a gradient flow is difficult, adaptation of the classical Łojasiewicz–Simon gradient inequality is a powerful strategy. Relying on an observation due to Chill [2], this was successfully completed by Dall'Acqua, Pozzi and Spener [3] for the elastic flow. Here, we are able to avoid adapting this framework through further careful analysis of the Euler–Lagrange operator. In particular, we show that if the energy is small, it must decay exponentially fast. This is then enough to enact a standard argument to obtain full convergence of the flow. Apart from yielding exponential convergence, this further improves the resultant convergence statement by removing the need for a family of reparametrisations (compare with the convergence results in [3]). This is reminiscent of, for example, Huisken's original convergence result for the mean curvature flow [6]. This argument concludes Section 7.

We summarise the main results, Theorems 6.1 and 7.10, in the following statement:

**Theorem 1.1**   *Let* $\gamma \colon \mathbb{S} \times [0, T) \to \mathbb{R}^2$ *be the steepest descent* $L^2$*–gradient flow for the functional*

$$E[\gamma] = \frac{1}{2} \int_\gamma k_s^2 \, ds,$$

where $\gamma(\cdot, 0) = \gamma_0(\cdot)$ is smooth and $T$ is maximal, $T \in (0, \infty]$. Then:

(a)  The maximal time of existence is infinite $(T = \infty)$.

(b)  If the length $L[\gamma_t]$ is uniformly bounded along the flow, then $\gamma$ converges exponentially fast in the $C^\infty$–topology to a standard round $\omega$–circle, where $\omega = \frac{1}{2\pi} \int_{\gamma_0} k_0 \, ds_0$.

Furthermore, there exists a constant $\varepsilon_2 > 0$ depending only on $\omega$ such that if $\gamma_0 \colon \mathbb{S} \to \mathbb{R}^2$ satisfies

$$(L^3 E)[\gamma_0] < \varepsilon_2,$$

then length is uniformly bounded along the flow, and the convergence statement from (b) above holds.

## Acknowledgements

## 2  Ideal submanifolds

It is a classical pursuit to find geometric shapes that exhibit desirable properties. Let us take $X \colon M^n \to \mathbb{R}^{n+m}$ to be a smooth closed immersed submanifold of Euclidean space. Often, in the case of submanifold theory, desirable properties are determined by the *curvature* of the submanifold. For example, minimal submanifolds have vanishing mean curvature vector, and CMC submanifolds have parallel mean curvature vector. In the case of hypersurfaces $(m = 1)$, this means that the mean curvature scalar is constant.

Minimal hypersurfaces have a variational characterisation in terms of the $L^2$–gradient of the area functional

$$A[X] = \int_{M^n} d\mu,$$

where $d\mu$ is the area measure induced by $X$. For smooth hypersurfaces with constant mean curvature, there are several variational approaches. The first and most classical is to minimise $A(X)$ subject to a constraint on enclosed volume. In terms of gradient flows,

this perspective gives rise to the *volume-preserving mean curvature flow* (VPMCF); see [7].[2]

Another approach is to build the volume- or area-preservation property into the definition of the underlying Hilbert space, where the gradient is being taken. From this perspective, we consider not the $L^2$–gradient of $A$ with constraint but instead the $H^{-1}$–gradient of $A$. This gives rise to the *surface diffusion flow* (SDF); see [11]. The set of smooth equilibria set for both flows consists of CMC hypersurfaces, but the operators are very different. For VPMCF, the velocity is

$$H - \frac{1}{A[X]} \int_{M^n} H \, d\mu,$$

whereas for SDF the velocity is

$$\Delta H.$$

Both have zero average, which is why both preserve enclosed volume. However, one is nonlocal and second order, and the other is fourth order.

In this paper we propose a third option. The motivation for this choice is as follows. For VPMCF, we have a minimisation problem with a constraint. This results in a nonlocal operator. The SDF gives a local operator, but the variational problem is not in $L^2$; rather, it is in $H^{-1}$. Our proposal is to consider the functional

$$E[X] = \frac{1}{2} \int_{M^n} |\nabla H|^2 \, d\mu$$

and minimise $E$ in $L^2$. This is a variational problem in $L^2$ and the operator is local. Of course, nothing is free, and the resultant operator is now of sixth order. However we feel that, intuitively at least, this is at least as good a situation as VPMCF and SDF.

To test this intuition we need to check a few essential points:

- That the set of smooth equilibria consists only of CMC hypersurfaces.

- That the $L^2$–gradient flow around CMC hypersurfaces is stable.

The main results of the present paper are the confirmation of both of these points in the simplest case of $n = m = 1$.

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[2]Another natural approach is to fix area; see [8].

# 3 Planar ideal curves

Let us now give the details of the mathematical setting. Suppose $\gamma_0 \colon \mathbb{S} \to \mathbb{R}^2$ is a circle immersed regularly in the plane, and consider the energy

$$E[\gamma_0] = \frac{1}{2} \int_\gamma k_s^2 \, ds,$$

where $s$ denotes the Euclidean arc-length, $k = \langle \gamma_{ss}, \nu \rangle$ is the curvature, $\tau = (\tau_1, \tau_2) = \gamma_s$ is the tangent vector and $\nu = (-\tau_2, \tau_1)$ the unit normal vector along $\gamma$. Our convention here is that the normal vector points into the interior of $\gamma$.

Consider now a one-parameter family of curves $\gamma \colon \mathbb{S} \times [0, T) \to \mathbb{R}^2$ evolving with a purely normal velocity[3]

$$\partial_t \gamma = V \nu.$$

The commutator of the time and arc-length derivatives is given by

$$[\partial_t, \partial_s] = kV \partial_s,$$

and the measure $ds$ evolves by

$$\partial_t \, ds = -kV \, ds.$$

Since $[\partial_t, \partial_s]\gamma = kV\tau$, we find

$$\partial_t \tau = \partial_s(V\nu) + kV\tau = V_s \nu.$$

By the orthonormality of $\{\tau, \nu\}$, $\partial_t \nu = -V_s \tau$. Similarly, as $[\partial_t, \partial_s]\tau = k^2 V\nu$, we find

$$\partial_t \kappa = \partial_s(V_s \nu) + k^2 V\nu = (V_{ss} + k^2 V)\nu - V_s k\tau,$$

so that

$$\partial_t k = \partial_t \langle \kappa, \nu \rangle = V_{ss} + k^2 V.$$

Now $[\partial_t, \partial_s]k = Vkk_s$, so

$$\partial_t k_s = \partial_s(V_{ss} + k^2 V) + Vkk_s = V_{s^3} + V_s k^2 + 3Vkk_s.$$

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[3]This procedure can be carried out analogously with a tangential component to the velocity; this will carry through and then cancel out at appropriate moments in the derivation. This is because $E$ does not depend on parametrisation, and tangential terms correspond only to reparametrisation.

The evolution of the functional $E$ can now be calculated:

$$\frac{d}{dt}\frac{1}{2}\int_\gamma k_s^2\,ds = \int_\gamma k_s(V_{s^3} + V_sk^2 + 3Vkk_s)\,ds - \int_\gamma Vkk_s^2\,ds$$

$$= \int_\gamma V\big[-k_{s^4} - k_{ss}k^2 - 2kk_s^2 + 3kk_s^2 - \tfrac{1}{2}kk_s^2\big]\,ds$$

$$= \int_\gamma V\big[-k_{s^4} - k_{ss}k^2 + \tfrac{1}{2}k_s^2k\big]\,ds.$$

Above we have used the notation $V_{s^3} = V_{sss}$ and $k_{s^4} = k_{ssss}$. Now, for the flow $\gamma$ to be the steepest descent gradient flow of $E$ in $L^2$, we must have $E' = -\|\gamma_t\|_2^2$, that is, we require

$$V = k_{s^4} + k_{ss}k^2 - \tfrac{1}{2}k_s^2k.$$

We therefore have the flow

(1)                     $$\partial_t\gamma = \big(k_{s^4} + k_{ss}k^2 - \tfrac{1}{2}k_s^2k\big)v.$$

# 4  Local existence and uniqueness

By writing the solution locally in time as a graph over the initial data and using classical PDE theory, we obtain the following local well-posedness result. This procedure is carried out in detail in [1] for a more general class of equations than we consider here.

**Theorem 4.1**  *Let $\gamma_0\colon \mathbb{S}\to\mathbb{R}^2$ be a closed immersed curve of class $C^{6,\alpha}$. There exists a unique smooth maximal family $\gamma\colon \mathbb{S}\times[0,T)\to\mathbb{R}^2$ with $T\in(0,\infty]$ of immersed curves such that $\gamma(s,0) = \gamma_0(s)$ and*

$$\partial_t\gamma = \big(k_{s^4} + k_{ss}k^2 - \tfrac{1}{2}k_s^2k\big)v.$$

*Furthermore, if $T < \infty$, then the quantity*

$$Q[\gamma_t] = L[\gamma_t] + \int_\gamma k_{s^5}^2\,ds$$

*is unbounded as $t\to T$.*

**Remark**  The regularity condition and blowup criterion of Theorem 4.1 are not optimal. One natural hypothesis would be that the flow exists uniquely for initial data of class $W^{3,2}$. Then, the energy controls automatically the high derivative term in the $W^{3,2}$–norm, whereas the lowest-order term is controlled by the length. We identify in Sections 6 and 7 estimates on length as the critical ingredient for convergence of the flow.

**Remark**   If the quantity $Q[\gamma_t]$ remains uniformly bounded and $T < \infty$ is maximal, then, by the standard Sobolev inequality, $\gamma$ is uniformly bounded in $C^{6,\alpha}$, and we may assert that the flow converges as $t \to T$ to a $C^{6,\alpha}$ limiting curve. This then allows us to apply again the short-time existence theorem, contradicting the maximality of $T$.

# 5   Rigidity for closed ideal curves

Let $\gamma$ be a closed curve satisfying

$$\mathcal{K}[\gamma] := k_{s^4} + k_{ss}k^2 - \tfrac{1}{2}k_s^2 k = 0,$$

that is, a stationary solution to the $L^2$–gradient flow of $E$. Recall that such curves are called *ideal*. In this section we prove:

**Theorem 5.1**   *Suppose $\gamma \colon \mathbb{S} \to \mathbb{R}^2$ is a smooth closed ideal curve. Set*

$$\omega := \frac{1}{2\pi} \int_\gamma k \, ds$$

*to be the winding number of $\gamma$.*

*Then $\gamma(\mathbb{S}) = \mathbb{S}_r(x)$, that is, $\gamma$ is a standard round $\omega$–circle.*

**Proof**   Since $\mathcal{K}[\gamma] = 0$ we find by a standard argument that $\gamma$ is analytic and

$$
\begin{aligned}
(k_{s^3}^2)_s + 2k_{s^3}k_{ss}k^2 - k_{s^3}k_s^2 k &= (k_{s^3}^2 + k_{ss}^2 k^2)_s - 2k_{ss}^2 k_s k - k_{s^3}k_s^2 k \\
&= (k_{s^3}^2 + k_{ss}^2 k^2 - k_{ss}k_s^2 k)_s - 2k_{ss}^2 k_s k + 2k_{ss}^2 k_s k + k_{ss}k_s^3 \\
&= 0,
\end{aligned}
$$

which implies

$$\left(k_{s^3}^2 + k_{ss}^2 k^2 + \tfrac{1}{4}k_s^4 - k_{ss}k_s^2 k\right)_s = 0.$$

Therefore there exists a $C \in \mathbb{R}$ such that for each $s \in [0, L]$,

$$Q(s) = k_{s^3}^2 + k_{ss}^2 k^2 + \tfrac{1}{4}k_s^4 - k_{ss}k_s^2 k = C.$$

Let $s_0 \in [0, L]$ be a point where $k_s(s_0) = 0$ (note that $k$ is a smooth periodic function). Then

$$
(2) \qquad Q(s_0) = (k_{s^3}^2 + k_{ss}^2 k^2)(s_0) = C \geq 0.
$$

We have two cases:

**Case 1** $(C = 0)$ In this case integration yields

$$\int_\gamma \left[ k_{s^3}^2 + k_{ss}^2 k^2 + \tfrac{7}{12} k_s^4 \right] ds = 0.$$

We conclude that $k_s$ is constant, which, together with the closedness of $\gamma$, implies the result.

**Case 2** $(C > 0)$ Consider the rescaling $\eta = \rho\gamma$. We calculate, on $\eta$,

$$Q(s) = k_{s^3}^2 + k_{ss}^2 k^2 + \tfrac{1}{4} k_s^4 - k_{ss} k_s^2 k = C\rho^{-8}.$$

Choosing $\rho = C^{1/8}$ we find

(3) $$Q(s) = k_{s^3}^2 + k_{ss}^2 k^2 + \tfrac{1}{4} k_s^4 - k_{ss} k_s^2 k = 1.$$

Now we write $Q(s) = M^2(s) + N^2(s)$, where

$$M(s) = k_{s^3} \quad \text{and} \quad N(s) = k_{ss} k - \tfrac{1}{2} k_s^2.$$

Equation (3) implies that there exists a $\phi \colon [0, L] \to \mathbb{R}$ such that

$$M(s) = \cos\phi(s) \quad \text{and} \quad N(s) = \sin\phi(s).$$

Since

$$N_s = \phi_s \cos\phi = k_{s^3} k = kM$$

and

$$M_s = k_{s^4} = -\phi_s \sin\phi = -\phi_s N$$

we have, by analyticity, $\phi_s = k$.

Since the integral of $k$ is $2\omega\pi$, we have

$$\phi(s + nL) = \phi(s) + 2\omega n\pi,$$

where $\omega$ is the winding number of $\gamma$. Let $\tau(s) = (x_s, y_s)$, where $\gamma = (x, y)$. Then $x_s^2 + y_s^2 = 1$ and again we find a function $\theta \colon [0, L] \to \mathbb{R}$ such that

$$\tau(s) = (\cos\theta(s), \sin\theta(s)).$$

This function $\theta$ is (up to translation) the standard notion of tangential angle. As $\tau_s = k\nu = \theta_s(s)(-\sin\theta, \cos\theta)$ and $\nu(s) = (-\sin\theta, \cos\theta)$, we must also have that $\theta_s = k$ and so

$$\theta(s) = \phi(s) + \theta_0$$

for some $\theta_0 \in \mathbb{R}$. This implies that

$$\tau(s) = \big(\cos(\phi(s) + \theta_0), \sin(\phi(s) + \theta_0)\big)$$
$$= (\cos\phi(s)\cos\theta_0 - \sin\phi(s)\sin\theta_0, \sin\phi(s)\cos\theta_0 + \cos\phi(s)\sin\theta_0).$$

In particular,

$$y_s = N(s)\cos\theta_0 + M(s)\sin\theta_0.$$

By closedness, we have

$$0 = \int_\gamma y_s \, ds = \cos\theta_0 \int_\gamma N(s) \, ds + \sin\theta_0 \int_\gamma M(s) \, ds.$$

Now

$$\int_\gamma M(s) \, ds = \int_\gamma k_{s^3} \, ds = 0$$

and

$$\int_\gamma N(s) \, ds = -\frac{3}{2} \int_\gamma k_s^2 \, ds.$$

Therefore we find

$$\cos\theta_0 \int_\gamma k_s^2 \, ds = 0.$$

Either $k_s = 0$ and we again conclude the result, or $\cos\theta_0 = 0$. In the latter case, we note that $\sin\theta_0 \neq 0$ and calculate

$$0 = \int_\gamma x_s \, ds = \tfrac{3}{2}\sin\theta_0 \int_\gamma k_s^2 \, ds.$$

This again implies $k_s = 0$ and so we are finished. $\qquad\square$

# 6 Global existence for arbitrary initial data

Along an ideal curve flow, the functional $\|k_s\|_2^2$ is monotone decreasing. Inflating the curve by scaling $\gamma \mapsto \rho\gamma$ for $\rho \to \infty$ decreases the energy to zero, regardless of $\gamma$. This is why the flow tends to enlarge the initial data. In fact, the flow may enlarge the initial data without end. We prove in this section that the only kind of blowup that can occur along an ideal curve flow is that length becomes unbounded, and that furthermore this cannot happen in finite time; implying that the ideal curve flow with smooth initial data exists for all time.

**Theorem 6.1** *Let $\gamma_0\colon \mathbb{S} \to \mathbb{R}^2$ be a smooth immersed curve, and $\gamma\colon \mathbb{S} \times [0, T) \to \mathbb{R}^2$ be the ideal curve flow with $\gamma_0$ as initial data. Then $T = \infty$, and if there exists an $L_0 \in \mathbb{R}$ such that $L(t) \leq L_0$ for all $t \in [0, T)$, $\gamma$ converges exponentially fast in the $C^\infty$–topology to a standard round $\omega$–circle.*

In this section we focus on the $T = \infty$ part of Theorem 6.1, leaving the convergence result for Section 7. There the ideal curve flow is studied under a condition on

$$\text{(4)} \qquad\qquad \tfrac{1}{2} L^3(t) \|k_s\|_2^2(t) = (L^3 E)[\gamma_t].$$

We prove there exists a constant $\varepsilon_2 > 0$, depending only on $\omega$, such that $(L^3 E)[\gamma_t] < \varepsilon_2$ implies convergence to an $\omega$–circle (Theorem 7.10).

In the present section, we wish to work without any condition on $L^3 E$ at $t = 0$. However, note that, since $E[\gamma]$ is decreasing, if at any time $L^3(t_0) < \varepsilon_2 / E[\gamma_0]$, then the smallness condition would be satisfied at $t_0$, and the results of the next section (and in particular Theorem 7.10) would apply. Therefore we assume that this is not the case for the remainder of this section, that is, that the estimate

$$\text{(5)} \qquad\qquad L^3(t) \geq \frac{\varepsilon_2}{E[\gamma_0]}$$

holds.

The local existence result, Theorem 4.1, states that if $T < \infty$, then at least one of length or $\|k_{s^5}\|_2^2$ become unbounded in finite time. We will show below that neither one of these possibilities can occur.

We begin by noting the following scale-invariant curvature estimate:

**Lemma 6.2** *For any immersed curve $\gamma\colon \mathbb{S} \to \mathbb{R}^2$ we have the estimate*

$$L\|k\|_\infty \leq \sqrt{L^3 \|k_s\|_2^2} + 2\omega\pi.$$

*Here $\omega$ is the winding number of $\gamma$.*

**Proof** We calculate

$$k = k - \bar{k} + \bar{k} \leq \int_\gamma |k_s| \, ds + \frac{2\omega\pi}{L}.$$

Taking a supremum and using the Hölder inequality, we find

$$\|k\|_\infty \leq \frac{1}{L}\left(\sqrt{L^3 \|k_s\|_2^2} + 2\omega\pi\right). \qquad \square$$

**Remark** We do not need the hypothesis (5) for Lemma 6.2 to hold. This will be useful in the next section.

Now we show that length cannot become unbounded in finite time.

**Lemma 6.3** *Along any ideal curve flow* $\gamma\colon \mathbb{S} \times [0, T) \to \mathbb{R}^2$ *we have*

$$L[\gamma_t] \le \big(L[\gamma_0]\exp(E[\gamma_0])\big)e^{c_0 t},$$

*where* $c_0 = c_0(\omega, E[\gamma_0])$.

**Proof** We calculate and estimate, using Lemma 6.2,

$$\frac{d}{dt}L = -\int_\gamma k\mathcal{K}\,ds \le \|k\|_2\|\mathcal{K}\|_2 \le \Big(\frac{2\omega\pi}{\sqrt{L}} + \sqrt{2}E^{1/2}[\gamma_0]L\Big)\|\mathcal{K}\|_2.$$

Now, (5) implies

$$L^{-1} \le \Big(\frac{4E[\gamma_0]}{\varepsilon_0}\Big)^{1/3} := c_L.$$

Therefore,

$$\begin{aligned}
\frac{d}{dt}\log L &\le L^{-1/2}\Big(\frac{2\omega\pi}{L} + \sqrt{2}E^{1/2}[\gamma_0]\Big)\|\mathcal{K}\|_2 \\
&\le \sqrt{c_L}(2\omega\pi c_L + \sqrt{2}E^{1/2}[\gamma_0])\|\mathcal{K}\|_2 \\
&\le \tfrac{1}{4}c_L(\sqrt{2}\omega\pi c_L + E^{1/2}[\gamma_0])^2 + 2\|\mathcal{K}\|_2^2.
\end{aligned}$$

Let us set $c_0 := \tfrac{1}{4}c_L(\sqrt{2}\omega\pi c_L + E^{1/2}[\gamma_0])^2$. Integrating and using

$$(6) \qquad -2\int_0^t \|\mathcal{K}\|_2^2\,d\tau = \int_0^t \frac{d}{d\tau}\|k_s\|_2^2\,d\tau = \|k_s\|_2^2(t) - \|k_s\|_2^2(0),$$

we find

$$\log L[\gamma_t] \le \log L[\gamma_0] + c_0 t + E[\gamma_0],$$

or

$$L[\gamma_t] \le \big(L[\gamma_0]\exp(E[\gamma_0])\big)e^{c_0 t}. \qquad\qquad \square$$

Let us now prove global existence, that is, $T = \infty$. Although we only need to show that $\|k_{s^5}\|_2^2 \le C(t_0)$ on $[0, t_0]$ for any $t_0 < \infty$, what we shall actually prove is that *all* derivatives of curvature are uniformly bounded on any bounded time interval.

**Theorem 6.4** *Along any ideal curve flow* $\gamma\colon \mathbb{S} \times [0, T) \to \mathbb{R}^2$ *we have, for all* $l \in \mathbb{N}$,

$$\|k_{s^l}\|_\infty \le C(l),$$

*where* $C(l)$ *a constant depending only on* $l$, $T$, $\omega$ *and* $E[\gamma_0]$.

**Proof** We first need to identify the structure of the evolution for $k_{s^l}$. Recall the commutator equation

$$[\partial_t, \partial_s] = k\mathcal{K}\partial_s.$$

Recall also the base case,

$$\partial_t k = \mathcal{K}_{ss} + k^2\mathcal{K}.$$

We calculate the additional terms that arise when moving up an order,

$$\partial_t k_{s^{l+1}} = [\partial_t, \partial_s]k_{s^l} + \partial_s(\partial_t k_{s^l}) = k\mathcal{K}k_{s^{l+1}} + \partial_s(\partial_t k_{s^l}).$$

Now

$$\mathcal{K} = k_{s^4} + P_3^2(k),$$

where we use $P_i^j(u)$ to mean a linear combination of terms each consisting of $j$ arc-length derivatives on $i$ copies of $u$. Thus we have

$$\partial_t k = (k_{s^6} + P_3^4(k)) + (k^2 k_{s^4} + P_5^2(k)) = k_{s^6} + P_3^4(k) + P_5^2(k),$$

$$\partial_t k_s = k_{s^7} + P_3^5(k) + P_5^3(k) + kk_s\mathcal{K} = k_{s^7} + P_3^5(k) + P_5^3(k)$$

and, by induction,

$$\partial_t k_{s^l} = k_{s^{l+6}} + P_3^{4+l}(k) + P_5^{2+l}(k).$$

The inductive step follows (given the inductive hypothesis) by noticing that

$$\begin{aligned}
\partial_t k_{s^{l+1}} &= \partial_s(\partial_t k_{s^l}) + k\mathcal{K}k_{s^{l+1}} \\
&= k_{s^{l+7}} + P_3^{5+l}(k) + P_5^{3+l}(k) + \left(kk_{s^{l+1}}(k_{s^4} + P_3^2(k))\right) \\
&= k_{s^{l+7}} + P_3^{5+l}(k) + P_5^{3+l}(k).
\end{aligned}$$

Now we calculate

$$\frac{d}{dt}\int_\gamma (k_{s^l})^2\, ds = 2\int_\gamma k_{s^l}(k_{s^{l+6}} + P_3^{4+l}(k) + P_5^{2+l}(k))\, ds + \int_\gamma k_{s^l}^2 k(k_{s^4} + P_3^2(k))\, ds$$

$$= -2\int_\gamma k_{s^{l+3}}^2\, ds + 2\int_\gamma k_{s^l}(P_3^{4+l}(k) + P_5^{2+l}(k))\, ds.$$

We now apply Proposition 2.5 of Dziuk, Kuwert and Schätzle [4] in combination with the length and curvature bounds derived above, yielding the estimates

$$\int_\gamma k_{s^l} P_3^{4+l}(k)\, ds \le \delta\|k_{s^{l+3}}\|_2^2 + c(\delta, T, \omega, E[\gamma_0], \|k\|_\infty)$$

and

$$\int_\gamma k_{s^l} P_5^{2+l}(k)\, ds \le \delta\|k_{s^{l+3}}\|_2^2 + c(\delta, T, \omega, E[\gamma_0], \|k\|_\infty).$$

Choosing $\delta = \frac{1}{4}$, we find

(7)
$$\frac{d}{dt} \int_\gamma k_{s^l}^2 \, ds + \int_\gamma k_{s^{l+3}}^2 \, ds \leq C.$$

This implies that $\|k_{s^l}\|_2^2$ is uniformly bounded on all bounded time intervals, yielding the result. □

Now we have global existence by a standard argument.

**Theorem 6.5** *The ideal curve flow $\gamma \colon \mathbb{S} \times [0, T) \to \mathbb{R}^2$ with $\gamma_0$ as initial data exists for all time ($T = \infty$).*

**Proof** Let us suppose $T < \infty$ (and $T$ is maximal). In this proof we use $C(l)$ to denote a constant only depending on $l$, $\gamma_0$ and $T$. We use the same symbol $C(l)$ to denote possibly different constants throughout the proof.

The main tool is Theorem 6.4. This implies, for all $l \in \mathbb{N}$,

$$\|\partial_s^l \mathcal{K}\|_\infty \leq C(l).$$

Set $v = |\partial_u \gamma|$, where $u$ is the initial space parameter before reparametrisation by arc length. From the evolution equation, $v$ satisfies

$$\partial_t v = -k\mathcal{K}v,$$

so that $v$ is uniformly bounded from above and below on any bounded time interval. We observe that for any function $\phi \colon \mathbb{S} \to \mathbb{R}$ we have

(8)
$$\partial_u^l \phi = v^l \partial_s^l \phi + P^l(v, \dots, \partial_u^{l-1} v, \phi, \dots, \partial_s^{l-1} \phi),$$

where $P^l$ is a polynomial. Using this for $\phi = k\mathcal{K}$, we see that

$$\|\partial_u^l (k\mathcal{K})\|_\infty \leq C(l).$$

By differentiating the ODE for $v$ we have, for $\psi_l = \partial_u^l v$,

$$\partial_t \psi_l + k\mathcal{K}\psi_l \leq C(l),$$

which gives

$$\|\psi_l\|_\infty \leq C(l).$$

The evolution equation $\partial_t \gamma = \mathcal{K}v$, and the equations $|\partial_s \gamma| = 1$ and $\partial_s^2 \gamma = kv$ imply

$$\|\partial_s^l \gamma\|_\infty \leq C(l),$$

where $l \in \mathbb{N}_0$. These uniform estimates imply $\gamma$ extends smoothly to $\mathbb{S} \times [0, T]$, and, by short-time existence, beyond $T$, contradicting the maximality of $T$. □

**Proof of Theorem 6.1** Theorem 6.5 is the $T = \infty$ part of Theorem 6.1. The remaining part of Theorem 6.1 that is to be established is the convergence statement under the assumption that length is bounded.

The proof consists of two steps:

(1) If the length is uniformly bounded, then there is a time $t_0$ such that

$$L^3(t)\|k_s\|_2^2(t) < \varepsilon_2 \quad \text{for all } t > t_0,$$

where $\varepsilon_2$ is a constant depending only on $\omega$.

(2) Apply Theorem 7.10 to the flow $\gamma \colon \mathbb{S} \times [t_0 + 1, \infty) \to \mathbb{R}^2$.

Then we will have proved

length bounded $\implies$ $T = \infty$, and $\gamma$ converges exponentially fast in the $C^\infty$–topology to a standard round $\omega$–circle,

which is precisely the second part of Theorem 6.1.

**Proof of Step 1** Our estimate exploits the rigidity proof from Section 5. We claim that

(9) $$E \leq CL^{3/2}\|\mathcal{K}\|_2,$$

where $C = C(\omega)$.

To prove (9), we work briefly in the complex plane. In this setting, we identify $\gamma(s) = (x(s), y(s))$ with $s \mapsto x(s) + iy(s)$. Let $M$ and $N$ be as in Section 5, and set $Q \colon \mathbb{S} \times [0, T) \to \mathbb{R}^2$ to be the complex-valued function

$$Q = (M + iN)e^{-i\theta},$$

where $\theta$ is the angle of the unit tangent vector to $\gamma$, that is,

$$e^{i\theta} = \cos\theta + i\sin\theta = x' + iy'.$$

Calculating,

$$Q' = (M' + iN' - ik(M + iN))e^{-i\theta} = (M' + kN + i(N' - kM))e^{-i\theta} = \mathcal{K}e^{-i\theta}.$$

This gives

$$M + iN = Q(x' + iy').$$

Integrating, we find (with $\bar{Q}$ the average of $Q$)

$$-\tfrac{3}{2}iE = \int_\gamma (M + iN)\,ds = \int_\gamma (Q - \bar{Q})(x' + iy')\,ds + \bar{Q}\int_\gamma (x' + iy')\,ds,$$

which implies

$$E \leq CL\|Q - \bar{Q}\|_\infty \leq CL\int_\gamma |\mathcal{K}|\,ds \leq CL^{3/2}\|\mathcal{K}\|_2,$$

as required.

The estimate (9) implies controlled decay of $E$, since (recall length is uniformly bounded here) then

$$\frac{d}{dt}\int_\gamma k_s^2\,ds = -2\int_\gamma |\mathcal{K}|^2\,ds \leq -2CL_0^{-3/2}\left(\int_\gamma k_s^2\,ds\right)^2,$$

implying that

$$E(t) \leq \frac{E[\gamma_0]}{1 + 2CL_0^{-3/2}E[\gamma_0]t}.$$

Therefore we can estimate the waiting time $t_0$ by (for example)

$$t_0 \leq \frac{L_0^{3/2}}{2CE[\gamma_0]}\left(\frac{E[\gamma_0]}{\varepsilon_2}\right) = \frac{L_0^{3/2}}{2C\varepsilon_2}.$$

This finishes the proof of Step 1.

**Proof of Step 2** Setting $\varepsilon_2$ to be that from the hypothesis of Theorem 7.10, the flow $\gamma\colon \mathbb{S}\times[t_0 + 1, \infty) \to \mathbb{R}^2$ satisfies the hypothesis of Theorem 7.10 by Step 1 above. Then, applying Theorem 7.10, we obtain that the flow $\gamma\colon \mathbb{S}\times[t_0 + 1, \infty) \to \mathbb{R}^2$ converges exponentially fast in the $C^\infty$–topology to a standard round $\omega$–circle. This then implies the same statement for the flow $\gamma\colon \mathbb{S}\times[t_0 + 1, \infty) \to \mathbb{R}^2$, which finishes the proof. $\qquad\square$

# 7 Stability of $\omega$–circles

Our primary goal in this section is to use the hypothesis

(E) $$L^3(t)\|k_s\|_2^2(t) < \varepsilon$$

at $t = 0$ to control evolving length.

While it may seem at first more natural to seek a condition only on the energy itself, a scaling argument gives that this is vacuous for any curve $\gamma: \mathbb{S} \to \mathbb{R}^2$ that has enough regularity so that $E[\gamma]$ makes sense. Briefly, suppose that we impose the restriction

$$E[\gamma_0] < \varepsilon,$$

instead of (E).

Take $\gamma: \mathbb{S} \to \mathbb{R}^2$ to be any curve for which $E[\gamma]$ makes sense. Then, for $\rho > 0$ a constant, the new curve $\gamma_\rho = \rho\gamma$ satisfies

$$E[\gamma_\rho] = \rho^{-3} E[\gamma].$$

Scaling the original curve $\gamma$ by the factor $\rho = (\varepsilon/(2E[\gamma]))^{1/3}$ ensures

$$E[\gamma_\rho] = \tfrac{1}{2}\varepsilon < \varepsilon.$$

Furthermore, the flow generated by $\gamma_\rho$ will only differ from that generated by $\gamma$ by parabolic scaling, and so will have the same qualitative behaviour.

While the ideal curve flow always exists for all time, it does not always converge as $t \to \infty$. If it did so, then the limit must satisfy $\mathcal{K} = 0$, which means by Theorem 5.1 that the limit is an $\omega$–circle. But there is no circle for $\omega = 0$, which means that in the topology of a lemniscate we have no convergence.[4] Therefore, the condition that we impose must not be vacuous and satisfiable by simply rescaling the initial data. This is our motivation behind considering (E).

Note that it remains an interesting open question if there exist initial data not in the topology of a lemniscate that gives rise to nonconvergence. From the theorem that we are aiming to prove in this section (Theorem 7.10), any such example must have scale-invariant energy $L^3 \|k_s\|_2^2 \geq \varepsilon_2$.

To begin, we prove some preparatory estimates.

**Lemma 7.1** *Let* $\mathcal{P} = \int_\gamma k_{s^4}^2 \, ds$ *and*

$$\mathcal{K}_0 = k_{s^4} + \left(\frac{2\pi\omega}{L}\right)^2 k_{ss}.$$

*Then there exists a constant* $C_\omega$ *depending only on* $\omega$ *such that*

$$\int_\gamma \mathcal{K}_0^2 \, ds \geq C_\omega \mathcal{P} - 4^5 \omega^8 \pi^8 L^{-3} E^2.$$

---

[4]By the results in this article, there must not be a length bound along any such flow, and so we expect the only possible convergence statement to be that under some initial conditions the flow is asymptotically expanding.

**Proof** Consider the Fourier series for $k$,

$$k = \sum_p a_p \exp\left(i \frac{2\pi}{L} ps\right).$$

Then

$$\mathcal{K}_0 = \sum_p a_p \left[\left(\frac{4\pi^2}{L^2} p^2\right)^2 - \frac{4\pi^2}{L^2} \omega^2 \frac{4\pi^2}{L^2} p^2\right] \exp\left(i \frac{2\pi}{L} ps\right)$$

$$= \sum_p a_p \left(\frac{4\pi^2}{L^2}\right)^2 p^2 (p^2 - \omega^2) \exp\left(i \frac{2\pi}{L} ps\right).$$

This implies, by the Plancherel theorem,

$$\int_\gamma \mathcal{K}_0^2 \, ds = \sum_p |a_p|^2 \left(\frac{4\pi^2}{L^2}\right)^4 p^4 (p^2 - \omega^2)^2 L.$$

We calculate

$$a_{\pm\omega} = \frac{1}{L} \int_\gamma k \exp\left(\pm i \frac{2\pi}{L} \omega s\right) ds = \frac{1}{L} \int_\gamma \left(k - \frac{2\pi\omega}{L}\right) \exp\left(\pm i \frac{2\pi}{L} \omega s\right) ds.$$

This implies

$$|a_{\pm\omega}| \le \frac{1}{L} \int_\gamma \left|k - \frac{2\pi\omega}{L}\right| \left|\exp\left(\pm i \frac{2\pi}{L} \omega s\right) - \exp(\pm i \theta)\right| ds$$

$$+ \frac{1}{L} \left|\int_\gamma \left(k - \frac{2\pi\omega}{L}\right) \exp(\pm i \theta) \, ds\right|.$$

In the above we have again used $\theta$ to denote the tangential angle.

Noting that

$$\int_\gamma \left(k - \frac{2\pi\omega}{L}\right) \exp(\pm i \theta) \, ds = \int_\gamma \theta_s \exp(\pm i \theta) \, ds - \frac{2\pi\omega}{L} \int_\gamma \tau \, ds = 0$$

and

$$\left|\frac{2\pi}{L} \omega s - \theta\right| \le \int_0^s \left|\frac{d}{ds}\left(\frac{2\pi}{L} \omega s - \theta\right)\right| ds$$

$$\le \int_0^s \left|\frac{2\pi}{L} \omega - k\right| ds \le L \int_\gamma |k_s| \, ds \le \sqrt{2} L^{3/2} E^{1/2},$$

we obtain

$$|a_{\pm\omega}| \leq \frac{1}{L}\int_\gamma \left|k - \frac{2\pi\omega}{L}\right|\left|\exp\left(\pm i\frac{2\pi}{L}\omega s\right) - \exp(\pm i\theta)\right| ds$$

$$\leq \frac{1}{L}\int_\gamma |k_s|\,ds \int_\gamma \left|\left(\pm i\frac{2\pi}{L}\omega s\right) - (\pm i\theta)\right| ds$$

$$\leq L\left(\int_\gamma |k_s|\,ds\right)^2$$

$$\leq 2L^2 E.$$

Now

$$\mathcal{P} = \sum_p \left(\frac{4\pi^2}{L^2}\right)^4 p^8 |a_p|^2 L,$$

so we have

$$\int_\gamma \mathcal{K}_0^2\,ds = \sum_p |a_p|^2 \left(\frac{4\pi^2}{L^2}\right)^4 p^4 (p^2 - \omega^2)^2 L$$

$$= \sum_{|p|\neq\omega} |a_p|^2 \left(\frac{4\pi^2}{L^2}\right)^4 p^4 (p^2 - \omega^2)^2 L$$

$$\geq \sum_{|p|\neq\omega,0} |a_p|^2 \left(\frac{4\pi^2}{L^2}\right)^4 p^8 \left(1 - \frac{\omega^2}{p^2}\right)^2 L.$$

We define $C_\omega$ by

$$\left(1 - \frac{\omega^2}{p^2}\right)^2 \geq \min\left\{\left(1 - \frac{\omega^2}{(\omega-1)^2}\right)^2, \left(1 - \frac{\omega^2}{(\omega+1)^2}\right)^2\right\} := C_\omega.$$

Then

$$\int_\gamma \mathcal{K}_0^2\,ds \geq C_\omega \mathcal{P} - \left(\frac{4\pi^2}{L^2}\right)^4 \omega^8 (|a_{+\omega}|^2 + |a_{-\omega}|^2) L$$

$$\geq C_\omega \mathcal{P} - 4\omega^8 \left(\frac{4\pi^2}{L^2}\right)^4 L^5 E^2$$

$$\geq C_\omega \mathcal{P} - 4^5 \omega^8 \pi^8 L^{-3} E^2,$$

as required.                                                                    □

**Proposition 7.2** Let $\gamma_0\colon \mathbb{S} \to \mathbb{R}^2$ be a smooth immersed curve. Then there exist constants $\hat{C}_\omega$ and $\varepsilon_0 > 0$ depending only on $\omega$ such that

$$(L^3 E)[\gamma] < \varepsilon_0 \implies \int_\gamma \mathcal{K}^2\,ds \geq \hat{C}_\omega L^{-6} E.$$

**Proof** Using Lemma 7.1,

$$
(10) \qquad \int_\gamma \mathcal{K}^2 \, ds = \int_\gamma \mathcal{K}_0^2 \, ds + 2 \int_\gamma \mathcal{K}_0 (\mathcal{K} - \mathcal{K}_0) \, ds + \int_\gamma (\mathcal{K} - \mathcal{K}_0)^2 \, ds
$$

$$
\geq \frac{1}{2} \int_\gamma \mathcal{K}_0^2 \, ds - \int_\gamma (\mathcal{K} - \mathcal{K}_0)^2 \, ds
$$

$$
\geq \tfrac{1}{2} C_\omega \mathcal{P} - 2^9 \omega^8 \pi^8 L^{-3} E^2 - \int_\gamma (\mathcal{K} - \mathcal{K}_0)^2 \, ds.
$$

Now $\mathcal{K} - \mathcal{K}_0 = (k^2 - (2\pi\omega/L)^2)k_{ss} - \frac{1}{2} k k_s^2$, so

$$
\int_\gamma (\mathcal{K} - \mathcal{K}_0)^2 \, ds \leq 2 \int_\gamma \Big(k^2 - \Big(\frac{2\pi\omega}{L}\Big)^2\Big)^2 k_{ss}^2 \, ds + \frac{1}{2} \int_\gamma k^2 k_s^4 \, ds.
$$

Now the curvature bound (Lemma 6.2) yields

$$
(11) \quad \int_\gamma (\mathcal{K} - \mathcal{K}_0)^2 \, ds \leq 2\Big(\Big(\sqrt{2LE} + \frac{2\omega\pi}{L}\Big) + \Big(\frac{2\omega\pi}{L}\Big)\Big)^2 \Big(\int_\gamma |k_s| \, ds\Big)^2 \int_\gamma k_{ss}^2 \, ds
$$

$$
+ \frac{1}{2} \Big(\sqrt{2LE} + \frac{2\omega\pi}{L}\Big)^2 \int_\gamma k_s^4 \, ds
$$

$$
\leq 4\sqrt{2}(\sqrt{2L^3 E} + 4\omega\pi)^2 \frac{E}{L} \int_\gamma k_{ss}^2 \, ds
$$

$$
+ \tfrac{1}{2}(\sqrt{2L^3 E} + 2\omega\pi)^2 L^{-2} \int_\gamma k_s^4 \, ds.
$$

Now the Gagliardo–Nirenberg Sobolev inequality yields constants $C_3$ and $C_4$ depending only on $\omega$ such that the inequalities

$$
(12) \quad C_1 \int_\gamma k_{ss}^2 \, ds \leq C_1 C_3 \Big(\int_\gamma k_s^2 \, ds\Big)^{2/3} \Big(\int_\gamma k_{s^4}^2 \, ds\Big)^{1/3} \leq C_1 C_3 (2E)^{2/3} \mathcal{P}^{1/3}
$$

$$
\leq \frac{L C_\omega}{8E} \mathcal{P} + C(\omega) C_1^{3/2} C_3^{3/2} L^{-1/2} E^{3/2},
$$

$$
C_2 \int_\gamma k_s^4 \, ds \leq C_2 C_4 L^2 \Big(\int_\gamma k_s^2 \, ds\Big)^{3/2} \Big(\int_\gamma k_{s^4}^2 \, ds\Big)^{1/2} \leq C_2 C_4 L^2 (2E)^{3/2} \mathcal{P}^{1/2}
$$

$$
\leq \tfrac{1}{8} L^2 C_\omega \mathcal{P} + C(\omega) C_2^2 C_4^2 E^3 L^2
$$

hold. Using these with $C_1 = 4\sqrt{2}(\sqrt{2L^3 E} + 4\omega\pi)^2$ and $C_2 = \frac{1}{2}(\sqrt{2L^3 E} + 2\omega\pi)^2$ in combination with (11) above yields

$$
\int_\gamma (\mathcal{K} - \mathcal{K}_0)^2 \, ds \leq \tfrac{1}{4} C_\omega \mathcal{P} + C(\omega)[((L^3 E)^{3/2} + \omega^3)(L^{-3/2} E^{5/2}) + ((L^3 E)^2 + \omega^4) E^3],
$$

where $C(\omega)$ is a constant depending only on $\omega$. Plugging this into (10) we find

$$\int_\gamma \mathcal{K}^2 \, ds$$

$$\geq \tfrac{1}{2} C_\omega \mathcal{P} - 2^9 \omega^8 \pi^8 L^{-3} E^2 - \int_\gamma (\mathcal{K} - \mathcal{K}_0)^2 \, ds$$

$$\geq \tfrac{1}{4} C_\omega \mathcal{P} - C(\omega)[L^{-3} E^2 + ((L^3 E)^{3/2} + \omega^3)(L^{-3/2} E^{5/2}) + ((L^3 E)^2 + \omega^4) E^3]$$

$$\geq \tfrac{1}{4} C_\omega \left(\frac{4\omega^2 \pi^2}{L^2}\right)^3 E$$

$$\quad - C(\omega)[L^{-3} E^2 + ((L^3 E)^{3/2} + \omega^3)(L^{-3/2} E^{5/2}) + ((L^3 E)^2 + \omega^4) E^3].$$

This implies

$$L^9 \int_\gamma \mathcal{K}^2 \, ds \geq a(L^3 E) - b(L^3 E)^2 - c(L^3 E)^{5/2} - d(L^3 E)^3 - e(L^3 E)^4 - f(L^3 E)^5,$$

where $a$, $b$, $c$, $d$, $e$ and $f$ are constants that depend only on $\omega$. Therefore, for $L^3 E$ small enough (depending only on $\omega$), we have

$$L^9 \int_\gamma \mathcal{K}^2 \, ds \geq \tfrac{1}{2} a L^3 E,$$

as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The first consequence of Proposition 7.2 is preservation and exponential improvement of the scale-invariant smallness condition.

**Proposition 7.3** *There exist constants $\varepsilon_1$, $\varepsilon_2$ and $C_0$ depending only on $\omega$ (with $0 < \varepsilon_2 \leq \varepsilon_1 < \varepsilon_0$ and $C_0 > 0$) such that the following statements hold. Let $\gamma_0 \colon \mathbb{S} \to \mathbb{R}^2$ be a smooth immersed curve satisfying*

$$(L^3 E)[\gamma_0] < \varepsilon_1$$

*and $\gamma \colon \mathbb{S} \times [0, \infty) \to \mathbb{R}^2$ be the ideal curve flow with $\gamma_0$ as initial data. Then*

$$(L^3 E)[\gamma_t] \leq (L^3 E)[\gamma_0].$$

*Furthermore, if $(L^3 E)[\gamma_0] < \varepsilon_2$ then for any $t \in [0, \infty)$ we have the estimate*

$$\int_0^t L^3 \|\mathcal{K}\|_2^2 \, d\tau \leq C_0 [(L^3 E)[\gamma_0] - (L^3 E)[\gamma_t]].$$

**Proof** We calculate and estimate to find

$$(L^3 E)' = -L^3 \int_\gamma \mathcal{K}^2 \, ds + 3(L^3 E) \frac{(-\|k_{ss}\|_2^2 + \frac{7}{2}\|k k_s\|_2^2)}{L}$$

$$\leq -L^3 \int_\gamma \mathcal{K}^2 \, ds + \tfrac{21}{2}(L^3 E)L^{-1}\|k k_s\|_2^2.$$

First, since $\varepsilon_1 < \varepsilon_0$, the hypothesis of Proposition 7.2 holds for a maximal time interval $[0, \delta)$. The curvature estimate (Lemma 6.2) and Proposition 7.2 imply that, on this time interval,

$$\|k k_s\|_2^2 \leq L^{-2}(\sqrt{2L^3 E} + 2\omega\pi)^2 \left( \frac{2}{\widehat{C}_\omega} L^6 \int_\gamma \mathcal{K}^2 \, ds \right).$$

Combining with our first estimate, we find

$$(L^3 E)' \leq L^3 \int_\gamma \mathcal{K}^2 \, ds \left( -1 + \frac{21}{\widehat{C}_\omega}(L^3 E)L^{-3}(\sqrt{2L^3 E} + 2\omega\pi)^2 L^3 \right)$$

$$\leq L^3 \int_\gamma \mathcal{K}^2 \, ds \left( -1 + \frac{21}{\widehat{C}_\omega}(L^3 E)(\sqrt{2\varepsilon_0} + 2\omega\pi)^2 \right).$$

Therefore, by assuming that

$$(L^3 E)(0) < \min\left\{ \varepsilon_0, \frac{\widehat{C}_\omega}{21(\sqrt{2\varepsilon_0} + 2\omega\pi)^2} \right\} := \varepsilon_1,$$

we see that $(L^3 E)' \leq 0$, showing that $\delta = T$ and preserving the hypothesis $(L^3 E) < \varepsilon_1$ for all time.

To see the second statement, take $(L^3 E)(0) < \min\{\varepsilon_0, 1/2C\} := \varepsilon_2$, and apply Proposition 7.2 one more time, to see

$$(L^3 E)' \leq -\tfrac{1}{2}CL^3 \int_\gamma \mathcal{K}^2 \, ds \leq -\frac{1}{C_0} L^3 \|\mathcal{K}\|_2^2,$$

implying the result by integration. $\qquad\square$

We are now able to use Proposition 7.3 to establish an a priori estimate on length, the crucial ingredient needed to obtain our convergence result.

**Proposition 7.4** Let $\gamma_0 \colon \mathbb{S} \to \mathbb{R}^2$ be a smooth immersed curve satisfying

$$(L^3 E)[\gamma_0] < \varepsilon_2,$$

*where $\varepsilon_2$ is as in Proposition 7.3, and $\gamma\colon \mathbb{S}\times[0,\infty)\to\mathbb{R}^2$ be the ideal curve flow with $\gamma_0$ as initial data. Then*

$$L[\gamma_t] \le L[\gamma_0]\exp(C_1(L^3 E)[\gamma_0]),$$

*where $C_1$ is a constant depending only on $\omega$.*

**Proof**   We estimate the evolution of length using first the curvature estimate, Lemma 6.2:

$$(\log L)' = -\frac{\|k_{ss}\|_2^2}{L} + \frac{7}{2L}\|kk_s\|_2^2 \le C(\omega)L^{-3}E.$$

Then Proposition 7.2 implies

$$(\log L)' \le C(\omega)L^3\|\mathcal{K}\|_2^2,$$

which, after application of the estimate in Proposition 7.3, yields

$$\log L[\gamma_t] \le \log L[\gamma_0] + C(\omega)(L^3 E)[\gamma_0],$$

which implies the claimed a priori estimate for length.                              □

With a uniform upper bound for length in hand, we are able to conclude exponential decay of the energy, and therefore the scale-invariant energy also.

**Corollary 7.5**   *Let $\gamma_0\colon \mathbb{S}\to\mathbb{R}^2$ be a smooth immersed curve satisfying*

$$(L^3 E)[\gamma_0] < \varepsilon_2,$$

*where $\varepsilon_2$ is as in Proposition 7.3, and $\gamma\colon \mathbb{S}\times[0,\infty)\to\mathbb{R}^2$ be the ideal curve flow with $\gamma_0$ as initial data. Then there exists a constant $C_2$ (depending only on $\omega$) such that*

$$\int_\gamma \mathcal{K}^2\,ds \ge C_2 E,$$

*and, in particular,*

$$E[\gamma_t] \le E[\gamma_0]e^{-C_2 t}.$$

**Proof**   Proposition 7.2 implies

$$E'[\gamma_t] = -\int_\gamma \mathcal{K}^2\,ds \le -C_2 E[\gamma_t],$$

from which the claim immediately follows.                              □

We may now use our estimates to bound length uniformly from below.

**Lemma 7.6** Let $\gamma_0\colon \mathbb{S} \to \mathbb{R}^2$ be a smooth immersed curve satisfying

$$(L^3 E)[\gamma_0] < \varepsilon_2,$$

where $\varepsilon_2$ is as in Proposition 7.3, and $\gamma\colon \mathbb{S} \times [0, \infty) \to \mathbb{R}^2$ be the ideal curve flow with $\gamma_0$ as initial data. Then

$$L[\gamma_t] \geq L[\gamma_0] \exp(-C_3),$$

where $C_3$ is a constant depending only on $\omega$ and $E[\gamma_0]$.

**Proof**  We calculate

$$(13) \qquad \log\Big(\frac{1}{L}\Big)' = \frac{\|k_{ss}\|_2^2}{L} - \frac{7}{2L}\|kk_s\|_2^2 \leq L^{-1}\|k_{ss}\|_2^2.$$

Now, invoking estimate (12) and its successors in the proof of Proposition 7.2, we find

$$L^{-1}\int_\gamma k_{ss}^2 \, ds \leq \frac{L^3}{16\omega^4\pi^4}\mathcal{P}$$

and

$$C_\omega\mathcal{P} \leq C(\omega)\Big(\int_\gamma \mathcal{K}^2 \, ds + L^{-3}E + \int_\gamma (\mathcal{K}-\mathcal{K}_0)^2 \, ds\Big)$$

$$\leq \tfrac{1}{2}C_\omega\mathcal{P} + C(\omega)\Big(\int_\gamma \mathcal{K}^2 \, ds + L^{-3}E$$

$$+ [((L^3 E)^{3/2} + \omega^3)(L^{-3/2}E^{5/2}) + ((L^3 E)^2 + \omega^4)E^3]\Big),$$

so that absorbing yields

$$L^3\mathcal{P} \leq C(\omega)\Big(L^3\int_\gamma \mathcal{K}^2 \, ds + E$$

$$+ [((L^3 E)^{3/2} + \omega^3)(L^{3/2}E^{5/2}) + ((L^3 E)^2 + \omega^4)L^3 E^3]\Big).$$

The estimate in Proposition 7.3 and the exponential decay of $E$ (Corollary 7.5) implies then that

$$\int_0^\infty L^3\mathcal{P}\,dt \leq C(\omega, E[\gamma_0]).$$

We may then integrate (13) to find

$$\log\Big(\frac{1}{L[\gamma_t]}\Big) \leq \log\Big(\frac{1}{L[\gamma_0]}\Big) + C_3,$$

which implies the result. $\qquad\square$

**Remark** Combined with the discussion around (5), this estimate shows that length is bounded uniformly away from zero along any ideal curve flow.

This means we have a uniform length bound. We now have global existence by Theorem 6.1. In fact, since length is uniformly bounded from above and below, we have the following uniform version of Theorem 6.4:

**Corollary 7.7** *Let* $\gamma_0 \colon \mathbb{S} \to \mathbb{R}^2$ *be a smooth immersed curve satisfying*

$$(L^3 E)[\gamma_0] < \varepsilon_2,$$

*where* $\varepsilon_2$ *is as in Proposition 7.3, and* $\gamma \colon \mathbb{S} \times [0, \infty) \to \mathbb{R}^2$ *be the ideal curve flow with* $\gamma_0$ *as initial data. We have, for all* $l \in \mathbb{N}_0$,

$$\|k_{s^l}\|_\infty \leq C(l),$$

*where* $C(l)$ *is a constant depending only on* $l$ *and* $\gamma_0$.

**Proof** First, the estimate for $l = 0$ follows from Lemma 6.2 and the uniform bound for $L$ from below (Lemma 7.6).

So, let us assume $l > 0$. In the proof of Theorem 6.4, we used Proposition 2.5 from Dziuk, Kuwert and Schätzle [4]. The time dependence here was a result of using the exponential-in-time estimate for length, Lemma 6.3. In that proof we also used the standing assumption that length was uniformly bounded from below. This means we have the estimate (7) for all $t \in [0, \infty)$ with a uniform constant $C$ on the right depending only on $\omega$, $E[\gamma_0]$ and $L[\gamma_0]$. That is,

$$\frac{d}{dt} \int_\gamma k_{s^l}^2 \, ds + \int_\gamma k_{s^{l+3}}^2 \, ds \leq C_l.$$

This implies (with the Poincaré inequality)

(14) $$\frac{d}{dt} \int_\gamma k_{s^l}^2 \, ds \leq C_l - \left(\frac{L^2}{4\omega^2 \pi^2}\right)^3 \int_\gamma k_{s^l}^2 \, ds.$$

We assume that $C_l > \|k_{s^l}\|_2^2|_{t=0}$ (if not, replace it by this constant). Estimate (14) implies

(15) $$\int_\gamma k_{s^l}^2 \, ds \leq \left(\frac{4\omega^2 \pi^2}{L^2}\right)^3 C_l.$$

To see this, note that (15) is initially true, and if $\|k_{s^l}\|_2^2$ grew to ever attain the value on the right-hand side of (15) at $t = t_0$, the estimate (14) would imply that $(\|k_{s^l}\|_2^2)'(t_0) \leq 0$. Therefore $\|k_{s^l}\|_2^2$ can never exceed the value on the right-hand side of (15).

Finally, from (15) we see that

$$\|k_{s^l}\|_\infty^2 \leq L \int_\gamma k_{s^{l+1}}^2 \, ds \leq C(l, \omega, E[\gamma_0], L[\gamma_0]),$$

as required. □

Global existence and the uniform estimates on length imply that $L(t)$ converges along a subsequence of times $\{t_j\}$ with $t_j \to \infty$. We can use any of the uniform $L^1$–in-time functions identified earlier to obtain convergence along possibly a further subsequence to a standard round $\omega$–circle parametrised by $\gamma_\infty$. Here $\gamma_\infty$ is not just the limit as $t \to \infty$ but includes possible one-off composition with a tangential diffeomorphism.

**Remark**  Although we work toward full convergence, one should be careful to claim uniqueness of the parametrisation $\gamma_\infty$. In geometric problems, this is typically false. For example, taking initial data for the flow to be $\gamma_0(\theta) = \gamma_\infty(\theta + \frac{\pi}{2})$ (a rotation of the circle $\gamma_\infty$, whatever it might be) will produce a stationary flow that remains a fixed distance from $\gamma_\infty$ in all $C^k$–norms.

This is why it is necessary to include at least implicitly a one-time reparametrisation to obtain full convergence. In our statement, we do this by saying that we have full convergence for each flow with given initial data to an $\omega$–circle, but do not pick out a specific parametrisation for this $\omega$–circle. From a larger perspective, an interesting open question is how to determine in general properties of the limit from the initial data, for example the centre of the limit or its radius.

Our approach now is to prove that we have convergence of every derivative of the parametrisation by directly integrating and differentiating the evolution equation. We begin by using exponential decay of $E$ plus uniform estimates for the curvature to obtain, by interpolation, exponential decay of all derivatives of curvature.

**Corollary 7.8**  *Let $\gamma_0 \colon \mathbb{S} \to \mathbb{R}^2$ be a smooth immersed curve satisfying*

$$(L^3 E)[\gamma_0] < \varepsilon_2,$$

where $\varepsilon_2$ is as in Proposition 7.3, and $\gamma\colon \mathbb{S}\times[0,\infty)\to\mathbb{R}^2$ be the ideal curve flow with $\gamma_0$ as initial data. Then, for all $l\in\mathbb{N}$,

$$\|k_{s^l}\|_\infty \le C(l)e^{-C_2/2t},$$

where $C(l)$ is a constant depending only on $l$, $\omega$, $E[\gamma_0]$ and $L[\gamma_0]$, and $C_2$ is as in Corollary 7.5.

**Proof**   We estimate

$$\|k_{s^l}\|_\infty^2 \le CL^2\|k_{s^{l+1}}\|_2^2 \le CL^2\left(\int_\gamma k_s^2\,ds\right)^{1/2}\left(\int_\gamma k_{s^{2l+1}}^2\,ds\right)^{1/2}.$$

Now the uniform curvature estimates (Corollary 7.7) and uniform length estimates (Proposition 7.4 and Lemma 7.6) imply

$$\|k_{s^l}\|_\infty^2 \le C(l)\sqrt{E},$$

from which the result follows.                                                 $\square$

We can use the control above to obtain uniform bounds for all derivatives of the evolving family $\gamma$.

**Proposition 7.9**   Let $\gamma_0\colon \mathbb{S}\to\mathbb{R}^2$ be a smooth immersed curve satisfying

$$(L^3E)[\gamma_0] < \varepsilon_2,$$

where $\varepsilon_2$ is as in Proposition 7.3, and $\gamma\colon \mathbb{S}\times[0,\infty)\to\mathbb{R}^2$ be the ideal curve flow with $\gamma_0$ as initial data. Then, for all $l\in\mathbb{N}_0$,

$$\|\partial_{u^l}\gamma\|_\infty \le C'(l) + \sum_{p=0}^l\|\partial_{s^l}\gamma_0\|_\infty,$$

where $C'(l)$ is a constant depending only on $l$, $\omega$, $E[\gamma_0]$ and $L[\gamma_0]$, and $C_2$ is as in Corollary 7.5.

**Proof**   We claim that, for $l\in\mathbb{N}_0$,

$$(16)\qquad\qquad \partial_t\partial_{s^l}\gamma = (\nu+)_{p=0}^l(P_{1+l-p}^{4+p}(k) + P_{3+l-p}^{2+p}(k)).$$

We prove this by induction. First, note that $\mathcal{K} = P_1^4(k) + P_3^2(k)$, so the equation above holds for $l = 0$. For the inductive step, we differentiate to find

$$\partial_t \partial_{s^{l+1}} \gamma = [\partial_t, \partial_s] \partial_{s^l} \gamma + \partial_s(\partial_t \partial_{s^l} \gamma)$$

$$= k\mathcal{K}\partial_{s^{l-1}}\tau + \partial_s \left[ (\nu + \tau) \sum_{p=0}^{l} (P_{1+l-p}^{4+p}(k) + P_{3+l-p}^{2+p}(k)) \right]$$

$$= k\mathcal{K}\left( \nu \sum_{p+q=l-1} P_p^q(k) + \tau \sum_{p+q=l-2} kP_p^q(k) \right)$$

$$+ (\nu + \tau) \sum_{p=0}^{l} (P_{2+l-p}^{4+p}(k) + P_{4+l-p}^{2+p}(k))$$

$$+ (\nu + \tau) \sum_{p=0}^{l} (P_{1+l-p}^{4+p+1}(k) + P_{3+l-p}^{2+p+1}(k))$$

$$= \nu \sum_{p+q=l-1} (P_{p+2}^{q+4}(k) + P_{p+4}^{q+2}(k)) + \tau \sum_{p+q=l-2} (P_{p+3}^{q+4}(k) + P_{p+5}^{q+2}(k))$$

$$+ (\nu + \tau) \sum_{p=0}^{l+1} (P_{1+l-p}^{4+p}(k) + P_{3+l-p}^{2+p}(k))$$

$$= (\nu + \tau) \sum_{p=0}^{l+1} (P_{1+l-p}^{4+p}(k) + P_{3+l-p}^{2+p}(k)),$$

as required.

Integrating (16) and using Corollary 7.8, we find

$$\|\partial_{s^l}\gamma\|_\infty \leq \|\partial_{s^l}\gamma_0\|_\infty + C(l) \int_0^t e^{-C_2/2t'} \, dt' \leq \|\partial_{s^l}\gamma_0\|_\infty + C'(l).$$

Converting arc-length derivatives back to the given derivatives on the parametrisation by (8), we find

$$\|\partial_{u^l}\gamma\|_\infty \leq C''(l) + \sum_{p=0}^{l} \|\partial_{s^l}\gamma_0\|_\infty,$$

as required. $\qquad\square$

We may now deduce full convergence. The full result is as follows:

**Theorem 7.10** *Let $\gamma_0\colon \mathbb{S} \to \mathbb{R}^2$ be a smooth immersed curve satisfying*

$$(L^3 E)[\gamma_0] < \varepsilon_2,$$

*where $\varepsilon_2$ is as in Proposition 7.3, and $\gamma\colon \mathbb{S} \times [0,\infty) \to \mathbb{R}^2$ be the ideal curve flow with $\gamma_0$ as initial data. Then $\gamma$ converges exponentially fast in the $C^\infty$–topology to a standard round $\omega$–circle.*

**Proof** We use Theorem A.1 to conclude full convergence of the flow.

We take $(N^n, h) = (\mathbb{R}^2, g^{\mathbb{R}^2})$, $M^m = \mathbb{S}^1$, $f = \gamma$ and $F = \mathcal{K}$. Uniform boundedness of $\gamma$ and all its derivatives (Proposition 7.9) implies the first hypothesis of Theorem A.1 is satisfied. For the second, we note that (using Corollary 7.8 and Proposition 7.4)

$$\int_\gamma \mathcal{K}^2 \, ds \le C e^{-C_2 t}.$$

This implies

$$\int_0^T \left( \int_\gamma \mathcal{K}^2 \, ds \right)^{1/2} dt \le C \int_0^T e^{-C_2/2 t} \, dt \le \widehat{C},$$

where $\widehat{C}$ is a constant depending only on $\omega$, $E[\gamma_0]$ and $L[\gamma_0]$.

Finally, we consider the third hypothesis. Uniform boundedness of all derivatives of $\gamma$ (Proposition 7.9) yields that for any sequence $t_j \to \infty$, the $C^\infty$–norm of $\gamma(t_j, \cdot)$ is uniformly bounded. We have exponential decay of the energy, and so $E[\gamma(t_j, \cdot)] \to 0$, which implies that a subsequence $\gamma(t_{j_k}, \cdot)$ converges to an $\omega$–circle in the $C^\infty$–topology. Of course, $\omega$–circles are smooth, and so the third hypothesis is satisfied.

Therefore we apply Theorem A.1 to conclude full convergence of the flow. $\qquad\square$

**Remark** This theorem implies the following geometric inequality. Clearly, perturbations of any $\omega$–circle satisfy $(L^3 E)[\gamma] < \varepsilon_2$. Theorem 7.10 implies that if $\gamma\colon \mathbb{S} \to \mathbb{R}^2$ is in the regular homotopy class of a lemniscate, then

$$(L^3 E)[\gamma] \ge \varepsilon_2.$$

If this were not the case, then Theorem 7.10 would imply that the curve $\gamma$ is diffeomorphic to an $\omega$–circle, which is impossible.

# Appendix   A convergence result

In this appendix we briefly prove that bounded flows whose velocity decays in a certain weak sense have unique limits.

**Theorem A.1**   *Let $(N^n, h)$ be an $n$–dimensional Riemannian manifold and $M^m$ be an $m$–dimensional manifold with $n > m$. Suppose $f: M^m \times [0, \infty) \to N^n$ is a one-parameter family of smooth isometric immersions satisfying*

$$\partial_t f = F.$$

*Suppose furthermore that*:

- ***Uniform bounds***   *We have the estimates*

  $$\int_M |f|^2 \, d\mu \le c_1 \quad \text{and} \quad \int_M |H|^2 |f|^4 \, d\mu \le c_2$$

  *for time-independent constants $c_1$ and $c_2$.*

- ***$L^1$–$L^2$ velocity***   *The $L^2$–norm of the velocity is uniformly $L^1$ in time, that is,*

  $$\int_0^T \left( \int_M |F|^2 \, d\mu \right)^{1/2} dt \le c_3$$

  *for a constant $c_3$ that does not depend on $T$.*

- ***Subconvergence***   *There exists a smooth immersion $f_\infty: M^m \to N^n$ and a sequence $\{t_j\} \subset [0, \infty)$ with $t_j \to \infty$ such that $f(\cdot, t_j) \xrightarrow{C^\infty} f_\infty$.*

*Then $f$ converges to $f_\infty$.*

**Proof**   Suppose there exists a sequence $\{s_j\} \subset [0, \infty)$ with $s_j \to \infty$ such that $f(\cdot, s_j) \xrightarrow{C^\infty} \tilde{f} \neq f_\infty$. Consider the functional

$$\mathbb{G}[f] = \int_M |f - f_\infty|^2 \, d\mu.$$

Since $\tilde{f}$ and $f_\infty$ are smooth, it follows that

(17) $$\lim_{j \to \infty} \mathbb{G}[f(\cdot, s_j)] \neq 0.$$

We compute

$$\left|\frac{d}{dt}\mathbb{G}\right| \leq \left|2\int_M |F||f-f_\infty|(1+|H||f-f_\infty|)\,d\mu\right|$$

$$\leq \|F\|_2\left[\int_M |f-f_\infty|^2(1+|H||f-f_\infty|)^2\,d\mu\right]^{1/2}$$

$$\leq c\|F\|_2\left[\int_M |f-f_\infty|^2 + |H|^2|f-f_\infty|^4\,d\mu\right]^{1/2}$$

$$\leq c\|F\|_2\left[\|f\|_2^2 + \|f_\infty\|_2^2 + \||H||f|^2\|_2^2 + \||H||f_\infty|^2\|_2^2\right]^{1/2}$$

$$\leq c\|F\|_2,$$

by hypothesis. This is in contradiction with (17), since then

$$|\mathbb{G}[f(\cdot,s_j)] - \mathbb{G}[f(\cdot,t_j)]| \leq c\int_{\min\{s_j,t_j\}}^{\infty} \|F\|_2\,dt \to 0.$$

Therefore there cannot exist such a sequence $\{s_j\}$, and the theorem is proved. $\qquad\square$

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