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Correction to the article Boundaries and automorphisms of hierarchically hyperbolic spaces

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We correct an error in Lemma 7.5 of our paper (Geom. Topol. 21 (2017) 3659–3758), affecting Theorem 7.1 and Theorem 9.15. Here, we prove Theorems 7.1 and 9.15 without the problematic lemma.

20F65, 20F67, 30F60

1 Introduction

We adopt the notation of [6]. We correct the proofs of Theorems 7.1 and 9.15 in [6]. The first says that in a hierarchically hyperbolic group $(G, \mathfrak{S})$, each $\mathbb{Z}$ subgroup is undistorted. The second says that G satisfies a Tits alternative. The proofs in [6] use an unstated hypothesis, discussed below. Here, we give proofs avoiding it.

The proofs in [6] use that, for each $U \in \mathfrak{S}$, the action of $\mathrm{Stab}_G(U)$ on $\mathcal{C}U$ factors through an acylindrical action. From Theorem 14.3 of Behrstock, Hagen and Sisto [2], one verifies this in the motivating examples — compact special groups and mapping class groups — but it does not hold in general; see Example 1.1. This property, called *hierarchical acylindricity* in [6, Definition 9.18], is hypothesised everywhere else it is used. The error is in Lemma 7.5, which does not hold without this assumption; see Example 1.1. To correct it:

- Theorem 7.1 needs a proof not using Lemma 7.5; this is done in Section 3.
- Theorem 9.15 needs a proof not using Lemma 7.5; this is done in Section 4. The statement is weaker than that in [6] in one way: we need the subgroup $H \leq G$ under consideration to be finitely generated.
- Theorem 9.20 is stated correctly in Section 9 of [6], but the special case in the introduction is missing the hierarchical acylindricity hypothesis.

Lemma 7.5 is not used elsewhere in [6].

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Example 1.1 Let T_1 and T_2 be infinite regular trees and let $G \leq \mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$ be a subgroup acting on $T_1 \times T_2$ freely and cocompactly, with the additional property that $G \hookrightarrow \mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2) \twoheadrightarrow \mathrm{Aut}(T_i)$ is nondiscrete for $i \in \{1,2\}$ (see Burger and Mozes [4], Janzen and Wise [9], Rattagi [11] and Wise [12]) and, moreover, for each n, there exists $g \in G$ such that g acts nontrivially on T_1 and T_2 but stabilises an arc of length n in T_2. Hence the action of G on T_i does not factor through an acylindrical action. Following [2, Section 8] shows that G admits a hierarchically hyperbolic group structure $(G, \mathfrak{S})$ where $\mathfrak{S}$ consists of T_1 and T_2 along with some elements whose associated hyperbolic spaces are bounded. The $\sqsubseteq$–maximal element of $\mathfrak{S}$ is associated to a bounded hyperbolic space, on which the action of G is vacuously acylindrical. The action of G on T_i does not factor through an acylindrical action. (The argument for Lemma 7.5 in [6] deals with the kernel of the action but neglects large point-stabilisers.)

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2 Some generalities

Fix a hierarchically hyperbolic space $(\mathcal{X}, \mathfrak{S})$, and let d be the metric. Assume $(\mathcal{X}, \mathsf{d})$ is a discrete geodesic space (see [1, Section 3]). We say $(\mathcal{X}, \mathsf{d})$ is *uniformly proper* if there exists $v\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that, for all $r \geq 0$, any ball in $\mathcal{X}$ of radius r has at most $v(r)$ points. As usual, for $x, y \in \mathcal{X}$ and $W \in \mathfrak{S}$, we write $\mathsf{d}_W(x,y)$ for $\mathsf{d}_W(\pi_W(x), \pi_W(y))$. Let E be the constant from [6, Definition 1.1].

2.1 Equivariant projections

Let $S \in \mathfrak{S}$ be the $\sqsubseteq$–maximal element, and let H be a group *acting* on $\mathcal{X}$ and acting by HHS automorphisms on $\mathfrak{S}$ (see [6, Defn. 1.11]). Necessarily, the action of H on $\mathcal{X}$ is by uniform quasi-isometries.

Remark 2.1 We have in mind the case where $\mathcal{X}$ is a hierarchically hyperbolic group with a word metric and H is a subgroup. In the parts of [6] where group actions are considered, Sections 7 and 9, we work in the setting of a group of HHS automorphisms *acting* on $\mathcal{X}$, although not necessarily an HHG. Recall that one only needs an actual action at the level of $\mathfrak{S}$ — hence a uniform quasiaction at the level of $\mathcal{X}$ — to induce an action on $\partial(\mathcal{X},\mathfrak{S})$. But since we will be working in the HHG setting in this note, and using results from [6, Section 9] that use an action on $\mathcal{X}$, we will restrict to that setting. This allows the following convenient perturbation of the HHS structure.

Fix $W\in\mathfrak{S}$. The coarse map $\pi_W\colon\mathcal{X}\to\mathcal{C}W$ has the property that $\pi_{gW}(gx)$ and $g(\pi_W(x))$ uniformly coarsely coincide for $g\in H, x\in\mathcal{X}$ (here $g\colon\mathcal{C}W\to\mathcal{C}gW$ is the isometry from the definition of an HHS automorphism). Let $\mathcal{C}$ contain one element of $\mathcal{X}$ from each H–orbit, and let $x\in\mathcal{X}$. Then $x=g\hat{x}$ for a unique $\hat{x}\in\mathcal{C}$, and g represents a unique coset of $\operatorname{Stab}_H(\hat{x})$. Let $\hat{\pi}_W(\hat{x})=\bigcup_{h\in\operatorname{Stab}_H(\hat{x})}h(\pi_{h^{-1}W}(\hat{x}))$. If $g\in\operatorname{Stab}_H(\hat{x})\cap\operatorname{Stab}_H(W)$, then

$$\begin{aligned}g\hat{\pi}_W(\hat{x})&=\bigcup_{gh\in\operatorname{Stab}_H(\hat{x})}gh(\pi_{h^{-1}W}(\hat{x}))\\&=\bigcup_{gh\in\operatorname{Stab}_H(\hat{x})}gh(\pi_{h^{-1}g^{-1}W}(\hat{x}))=\hat{\pi}_W(\hat{x})=\hat{\pi}_W(g\hat{x}).\end{aligned}$$

Hence, for arbitrary $g\in H$, the assignment $x\mapsto g(\hat{\pi}_{g^{-1}W}(\hat{x}))$ gives a well-defined coarse map $\hat{\pi}_W\colon\mathcal{X}\to\mathcal{C}W$. We redefine $\pi_W(g\hat{x})$ to be $g(\hat{\pi}_{g^{-1}W}(\hat{x}))$. This is a uniformly bounded perturbation of π_W, so (up to a single initial change in the constants), the HHS structure is unaffected. But, now, we have the following: given $x\in\mathcal{X}$, write $x=h\hat{x}$, where $\hat{x}$ is as above and $h\in H$. Let $g\in H$ and $W\in\mathfrak{S}$. Then $\pi_{gW}(gx)=\pi_{gW}(gh\hat{x})=gh\hat{\pi}_{(gh)^{-1}gW}(\hat{x})$, by (re)definition. So, $\pi_{gW}(gx)=gh\hat{\pi}_{h^{-1}W}(\hat{x})=g\pi_W(h\hat{x})=g\pi_W(x)$. In other words, $\pi_{gW}(gx)=g(\pi_W(x))$ for all $x\in\mathcal{X}$, $g\in H$ and $W\in\mathfrak{S}$.

For each $U\in\mathfrak{S}$ with $U\subsetneq W$ or $U\pitchfork W$, recall the uniformly bounded set $\rho^U_W\subset\mathcal{C}W$. For $g\in H$, the sets ρ^{gU}_{gW} and $g(\rho^U_W)$ uniformly coarsely coincide, by the definition of an HHS automorphism. At the expense of an initial change in the constants, we can modify ρ^U_W as above so that $\rho^{gU}_{gW}=g(\rho^U_W)$ for $g\in H$.

This means that we *can and shall assume* that for all $W\in\mathfrak{S}$, all $g\in H$, and all $U\in\mathfrak{S}$ with $U\pitchfork W$ or $U\subsetneq W$, we have $\rho^{gU}_{gW}=g(\rho^U_W)$. Also, we *can and shall assume* that $g(\pi_W(x))=\pi_{gW}(gx)$ for $x\in\mathcal{X}$. As explained in [6, Proposition 1.16] or [3, Remark 1.3], we *can and shall also assume* that $\pi_W\colon \mathcal{X} \to \mathcal{C}W$ is L–coarsely surjective for L independent of W. In the same discussion in [6], we mentioned that the latter assumption means that $\mathcal{C}W$ can be replaced with the union of geodesics starting and ending in $\pi_W(\mathcal{X})$. In particular, $\mathrm{Stab}_H(W)$ acts by isometries on $\mathcal{C}W$.

2.2 Induced actions on $(\boldsymbol{F}_W, \mathfrak{S}_W)$

Recall that for each $W \in \mathfrak{S}$, we have an HHS $(\boldsymbol{F}_W, \mathfrak{S}_W)$, where $\mathfrak{S}_W = \{U \sqsubseteq W\}$, so that the inclusion $\mathfrak{S}_W \hookrightarrow \mathfrak{S}$ induces a hieromorphism $(\boldsymbol{F}_W, \mathfrak{S}_W) \to (\mathcal{X}, \mathfrak{S})$ such that the induced maps on $\mathcal{C}U, U \sqsubseteq W$ are the identity. As in [6, Remark 1.14], we also have an action $\mathrm{Stab}_H(W) \to \mathrm{Aut}(\mathfrak{S}_W)$ by HHS automorphisms.

There are various slightly different ways to describe $\boldsymbol{F}_W$; for the sake of explicitness, we now fix one. First, let κ be a fixed constant provided by [3, Theorem 3.1]. Let $\boldsymbol{P}_W \subset \mathcal{X}$ be the standard product region associated to W, which we can choose to be $\mathrm{Stab}_H(W)$–invariant. As a set, define $\widehat{\boldsymbol{F}}_W$ to consist of exactly one point p_x for each distinct tuple $(\pi_U(x))_{U \in \mathfrak{S}_W}$ with $x \in \boldsymbol{P}_W$.

The restriction homomorphism $\mathrm{Stab}_H(W) \to \mathrm{Aut}(\mathfrak{S}_W)$ defines an action of $\mathrm{Stab}_H(W)$ on the set of tuples $(\pi_U(x))_{U \in \mathfrak{S}_W}$: for $V \in \mathfrak{S}_W$, the V–coordinate of $g \cdot (\pi_U(x))_U$ is $g(\pi_{g^{-1}V}(x)) = \pi_V(gx)$. Since $\boldsymbol{P}_W$ is $\mathrm{Stab}_H(W)$–invariant, this gives an action of $\mathrm{Stab}_H(W)$ on $\widehat{\boldsymbol{F}}_W$.

For each p_x, realisation [3, Theorem 3.1] provides $y \in \boldsymbol{P}_W$ such that $\mathsf{d}_U(p_x, y) \le \kappa$ when $U \sqsubseteq W$. Let $f(p_x) \subset \boldsymbol{P}_W$ be the (nonempty) set of all such y. Note that $f(p_x)$ is a uniformly hierarchically quasiconvex set, and π_U restricts on $f(p_x)$ to a uniformly coarse surjection for all $U \perp W$. Hence $\mathsf{d}_{\mathrm{haus}}(f(p_x), f(p_z))$ is finite for all $p_x, p_z \in \widehat{\boldsymbol{F}}_W$. (The two sets are "parallel" in the sense that the gate map from one to the other is a coarsely surjective hieromorphism which is the identity on each $\mathcal{C}U$ with $U \perp W$.)

Equip $\widehat{\boldsymbol{F}}_W$ with the pseudometric $\omega(p_x, p_z) = \mathsf{d}_{\mathrm{haus}}(f(p_x), f(p_z))$. Since $\mathcal{X}$ is discrete, $\omega(p_x, p_z) = 0$ only if $f(p_x) = f(p_z)$. Now, if $f(p_x) = f(p_z)$, then for all $y \in \boldsymbol{P}_W$ and $V \sqsubseteq W$, we have $\mathsf{d}_V(p_x, y) \le \kappa$ if and only if $\mathsf{d}_V(p_z, y) \le \kappa$. Hence, for any $g \in \mathrm{Stab}_H(W)$, we have by equivariance of projections that $\mathsf{d}_V(p_{gx}, gy) \le \kappa$ if and only if $\mathsf{d}_V(p_{gz}, gy) \le \kappa$, ie $f(p_{gx}) = f(p_{gz})$. Thus the action of $\mathrm{Stab}_W(H)$ on $\widehat{\boldsymbol{F}}_W$ descends to an action (not necessarily by isometries) on the metric quotient, which we call $\boldsymbol{F}_W$ and identify with the set of $f(p_x)$, equipped with the Hausdorff metric.

Fix $x_0 \in \boldsymbol{P}_W$. Define a map $h\colon \boldsymbol{F}_W \to \boldsymbol{P}_W$ by $f(p_x) \mapsto y(p_x)$, where $y(p_x) \in f(p_x)$ has the property that $\mathrm{d}_U(y, x_0) \leq \kappa$ whenever $U \perp W$; such a point is provided by realisation. By construction, this map is a quasi-isometric embedding whose image is uniformly hierarchically quasiconvex; indeed, $\pi_U\colon \boldsymbol{P}_W \to \mathcal{C}U$ restricts to a uniformly coarse surjection on the image of $\boldsymbol{F}_W$ when $U \sqsubseteq W$, and sends the image to a uniformly bounded set when $U \perp W$. Realisation then shows that the image is hierarchically quasiconvex. Hence $(\boldsymbol{F}_W, \mathfrak{S}_W)$ is a hierarchically hyperbolic space, and the above action of $\mathrm{Stab}_H(W)$ on $\boldsymbol{F}_W$ is an action by HHS automorphisms with equivariant projections.

In summary, we have two HHSs, $(\boldsymbol{F}_W, \mathfrak{S}_W)$ and $(h(\boldsymbol{F}_W), \mathfrak{S}_W)$. The group $\mathrm{Stab}_H(W)$ *acts* $\boldsymbol{F}_W$ by uniform quasi-isometries, and for all $x \in \boldsymbol{F}_W$ and $g \in \mathrm{Stab}_H(W)$ and $V \sqsubseteq W$, we have $g(\pi_V(x)) = \pi_V(gx)$.

On the other hand, $h(\boldsymbol{F}_W)$, equipped with the subspace metric inherited from $\mathcal{X}$, is uniformly proper if $\mathcal{X}$ is uniformly proper. In any case, the quasi-isometry $h\colon \boldsymbol{F}_W \to h(\boldsymbol{F}_W)$ is a hieromorphism, where the map at the level of index sets is the identity, and the map on each hyperbolic space is the identity. Conjugating elements of $\mathrm{Stab}_H(W)$ by h and a fixed quasi-inverse gives an action of $\mathrm{Stab}_H(W)$ on $(h(\boldsymbol{F}_W), \mathfrak{S}_W)$ by HHS automorphisms (at the level of $\mathfrak{S}_W$) which is a *quasiaction* by uniform quasi-isometries at the level of $h(\boldsymbol{F}_W)$.

When $\mathcal{X}$ is (uniformly) proper, $h(\boldsymbol{F}_W)$ is (uniformly) proper. Now, the HHS boundaries of $(\boldsymbol{F}_W, \mathfrak{S}_W)$ and $(h(\boldsymbol{F}_W), \mathfrak{S}_W)$ coincide, because the boundary is defined in terms of the index set, hyperbolic spaces and projections. In particular, if $\mathcal{X}$ is proper, then $\partial \boldsymbol{F}_W$ is compact, since it is homeomorphic to the boundary of the proper HHS $h(\boldsymbol{F}_W)$. This should perhaps have been made more explicit in [6]. It is used when one applies Proposition 9.2 of [6] to HHSs of the form $(\boldsymbol{F}_W, \mathfrak{S}_W)$, equipped with a $\mathrm{Stab}_H(W)$–action.

2.3 Hierarchy rays

We recall the notion of a *hierarchy ray*. Let $(\mathcal{X}, \mathfrak{S})$ be an HHS and let $x_0 \in \mathcal{X}$ and let $D \geq 0$. A (D, D)*–hierarchy ray* is a (D, D)–quasigeodesic $\gamma\colon \mathbb{N} \to \mathcal{X}$ such that $\pi_U \circ \gamma$ is an unparametrised (D, D)–quasigeodesic in $\mathcal{C}U$ for all $U \in \mathfrak{S}$.

The following lemma about hierarchy rays is stated exactly as we will use it in Theorem 3.1.

Lemma 2.2 *Let $(\mathcal{X},\mathfrak{S})$ be a proper hierarchically hyperbolic space. Then there exists $D\geq 0$ such that the following holds. Let $U_1,\dots,U_M\in\mathfrak{S}$ be pairwise orthogonal. For $i\leq M$, let $p_i\in\partial\mathcal{C}U_i$. Then there is a (D,D)–hierarchy ray γ in $\mathcal{X}$ such that for all $i\leq M$, the sequence $\pi_{U_i}\circ\gamma(n)$ converges to p_i.*

Moreover, if $V\in\mathfrak{S}$ satisfies $U_i\sqsubsetneq V$ or $U_i\pitchfork V$ for some i, or $U_i\perp V$ for all i, then $\operatorname{diam}(\pi_V\circ\gamma)\leq 10DE$.

Proof Fix a $(1,20E)$–quasigeodesic ray α_i in $\mathcal{C}U_i$ from $\pi_{U_i}(x_0)$ to p_i. Without loss of generality, we can assume, using partial realisation, that x_0 has the property that for all i and all V with $U_i\pitchfork V$ or $U_i\sqsubsetneq V$, the sets $\pi_V(x_0)$ and $\rho^{U_i}_V$ are E–close.

A sequence of points For each $N\in\mathbb{N}$, choose a point $x_N\in\mathcal{X}$ as follows. First, for each i, let b_{U_i} lie $100E$–close to α_i at distance at most $N+100E$ and at least $N-100E$ from $\pi_{U_i}(x_0)$. For each V such that either $V\pitchfork U_i$ or $U_i\sqsubsetneq V$ for some i, let $b_V=\rho^{U_i}_V$. For each V such that $V\perp U_i$ for all i, let $b_V=\pi_V(x_0)$.

Fix i and let $V\sqsubsetneq U_i$. Recall that the boundary point p_i projects to a bounded set $p_i(V)\in\mathcal{C}V$ as follows. Let β be a $(1,20E)$–quasigeodesic ray joining $\rho^V_{U_i}$ to p_i in $\mathcal{C}U_i$. Let $b(V)=\bigcup_{\beta'}\rho^{U_i}_V(\beta')$, where β' varies over the subrays of β avoiding the E–neighbourhood of $\rho^V_{U_i}$. Then $b(V)$ has diameter at most E by bounded geodesic image, and is coarsely independent of β.

Note that any subray α'_i of α_i avoiding the E–neighbourhood of $\rho^V_{U_i}$ has the property that $\rho^{U_i}_V(\alpha'_i)$ E–coarsely coincides with $b(V)$.

The tuple $(b_V)_{V\in\mathfrak{S}}$ is consistent. Indeed, let $U,V\in\mathfrak{S}$ be distinct. There are two cases:

- Suppose that $U\pitchfork V$. If $U\sqsubseteq U_i$ for some i, then ρ^U_V uniformly coarsely coincides with $\rho^{U_i}_V=b_V$, or $U,V\sqsubsetneq U_i$. In the latter case, choose a point in β (the ray in $\mathcal{C}U_i$ described above) far from $\rho^U_{U_i}$ and $\rho^V_{U_i}$. By partial realisation, this point has the form $\pi_{U_i}(y)$ for some $y\in\mathcal{X}$. By definition of b_U and consistency, b_U coarsely coincides with $\pi_U(y)$ and b_V coarsely coincides with $\pi_V(y)$. So, by consistency, either $\mathrm{d}_U(b_U,\rho^V_U)\leq E$ or the same holds with U and V reversed. We likewise have consistency if $V\sqsubseteq U_i$ for some i.

So suppose that $U\not\sqsubseteq U_i$ for all i and $V\not\sqsubseteq U_i$ for all i.

If $U,V\perp U_i$ for all i, then $b_U=\pi_U(x_0)$ and $b_V=\pi_V(x_0)$, so, by consistency of x_0, $\mathrm{d}_U(b_U,\rho^V_U)\leq E$ or the same holds with U and V reversed.

If $U_i \sqsubseteq U$ or $U_i \pitchfork U$ for some i, then $b_U = \rho_U^{U_i}$, which is E–close to $\pi_U(x_0)$. If $U_i \pitchfork V$ or $U_i \sqsubsetneq V$, then b_V coarsely coincides with $\pi_U(x_0)$ for the same reason, and so, by consistency, b_V coarsely coincides with ρ_V^U or the same holds with U and V reversed. The final possibility is that $V \perp U_i$, in which case $b_V = \rho_V^{U_j}$ for some $j \neq i$. So, again, b_V coarsely coincides with $\pi_V(x_0)$ and we conclude as before.

- If $U \sqsubsetneq V$, the argument is almost identical, except we use consistency for nesting instead of consistency for transversality.

Now apply realisation to obtain $x_N \in \mathcal{X}$ such that $\mathsf{d}_V(x_N, b_V) \le E$ for all $V \in \mathfrak{S}$.

Construction of γ As shown in [3], for each N there is a (D_0, D_0)–hierarchy path γ_N joining x_0 to x_N, where D_0 is a constant depending only on the HHS structure. Since $\mathcal{X}$ is proper and $\mathsf{d}_{\mathcal{X}}(x_0, x_N) \to \infty$ as $N \to \infty$, the paths γ converge uniformly on compact sets to a path $\gamma\colon \mathbb{N} \to \mathcal{X}$, which is a (D_1, D_1)–quasigeodesic for some D_1 depending only on D_0 and $\mathcal{X}$.

Projections of γ For $V \in \mathfrak{S}$ such that $V \pitchfork U_i$ or $U_i \sqsubsetneq V$, we have that $\pi_V \circ \gamma_N$ has image a set of diameter at most $10ED$ for all N, so the same is true of $\pi_V \circ \gamma$. The same holds if $V \perp U_i$ for all i.

For each i, the path $\pi_{U_i} \circ \gamma_N$ is an unparametrised (D_0, D_0)–quasigeodesic from $\pi_{U_i}(x_0)$ to $\pi_{U_i}(x_N)$. So, $\pi_{U_i} \circ \gamma$ is an unparametrised (D_2, D_2)–quasigeodesic that coarsely coincides with α_i, where D_2 depends on D_0 and E. (This is because each x_N lies uniformly close to α_i and $\mathsf{d}_{U_i}(x_0, x_N) \to \infty$ as $N \to \infty$.)

Finally, if $V \sqsubsetneq U_i$ for some i, then for all sufficiently large N, the path $\pi_V \circ \gamma_N$ is an unparametrised quasigeodesic from $\pi_V(x_0)$ to a point E–close to $p(V)$, so the same holds for $\pi_V \circ \gamma$. Thus γ is a (D, D)–hierarchy path, where $D = D(E, D_0)$. We saw above that $\pi_{U_i} \circ \gamma$ coarsely coincides with α_i, so $\pi_{U_i} \circ \gamma(n) \to p_i$, as required. □

3 Cyclic subgroups are undistorted

We now prove [6, Theorem 7.1].

Theorem 3.1 [6, Theorem 7.1] *Let $(G, \mathfrak{S})$ be a hierarchically hyperbolic group. Let $g \in G$. Then $\langle g \rangle$ is undistorted in G. Moreover, for any $U \in \mathrm{Big}(g)$, the action of g^t on $\mathcal{C}U$ is loxodromic, where t is any nonzero multiple of $|\mathrm{Big}(g)|!$. In particular, $\pi_U(\langle g \rangle)$ is a quasi-isometrically embedded copy of $\mathbb{Z}$ in $\mathcal{C}U$.*

Proof If g has finite order, the first assertion is obvious and the second holds vacuously since $\mathrm{Big}(g) = \varnothing$. So, suppose that g has infinite order. Then $\mathrm{Big}(g) \neq \varnothing$ and g^t fixes each element of $\mathrm{Big}(g)$, where t is as in the statement. The "moreover" assertion implies that $\langle g \rangle$ is undistorted, since each π_U is coarsely Lipschitz.

We now prove the "moreover" assertion. We may assume that the HHS structure is normalised, and that the projections are equivariant in the sense of Section 2.1. For convenience, we assume that g fixes each element of $\mathrm{Big}(g)$; this is achieved by replacing g by g^t.

Let $\mathrm{Big}(g) = \{U_1, \dots, U_k, \dots, U_M\}$, where $k \geq 0$ is such that $\langle g \rangle$ is loxodromic on $\mathcal{C}U_i$ if and only if $i \leq k$. For $i > k$, we have that $\langle g \rangle$ is parabolic on $\mathcal{C}U_i$, fixing a point $\chi_i \in \partial \mathcal{C}U_i$.

If $k = M$, we are done, so suppose $k < M$. (A priori, we allow the possibility that $k = 0$.)

Orbits of $\langle g \rangle$ Fix $x_0 \in G$; for convenience, we choose x_0, using partial realisation, so that $\pi_V(x_0)$ is E–close to $\rho_V^{U_i}$ whenever $U_i \pitchfork V$ or $U_i \subsetneq V$. For each V such that $U_i \subsetneq V$ or $U_i \pitchfork V$ for some i, the orbit $\langle g \rangle \cdot x_0$ projects to a set in $\mathcal{C}V$ that (by consistency) $10E$–coarsely coincides with $\rho_V^{U_i}$, and thus has diameter at most $100E$.

For $i > k$, let α_i be a $(1, 20E)$–quasigeodesic ray joining $\pi_{U_i}(x_0)$ to χ_i, where $\chi_i \in \partial \mathcal{C}U_i$ is the unique point fixed by g.

By Proposition 6.6 of [6], there exists $D(g)$ such that $\pi_V(\langle g \rangle \cdot x_0)$ has diameter at most $D(g)$ whenever $V \subsetneq U_i$ for some i, or $V \perp U_i$ for all i.

Next, we bound the projections of the rays α_i to nested domains:

Claim 1 *Let $i > k$ and let $V \subsetneq U_i$. Let $\chi_i(V)$ be the projection of χ_i on $\mathcal{C}V$. Then there exists $R_0 = R_0(g, x_0)$ such that $\mathrm{d}_V(x_0, \chi_i(V)) \leq R_0$ for all such V.*

Proof Suppose that $V \subsetneq U_i$ satisfies $\mathrm{d}_V(x_0, \chi_i(V)) > 100E$.

Then, by bounded geodesic image and consistency, $\rho_{U_i}^V$ lies $100E$–close to α_i. Using the fact that g acts parabolically on $\mathcal{C}U_i$, we can choose $n > 0$ such that the ray $g^n \alpha_i$ enters the $100E$–neighbourhood of α_i at a point p that is $100E$–close to a point $p' \in \alpha_i$ with $\mathrm{d}_{U_i}(p', \rho_{U_i}^V) > 1000E$. Hence $\mathrm{d}_V(g^n x_0, \chi_i(V)) \leq E$, by consistency and bounded geodesic image. But this implies that $\mathrm{d}_V(x_0, \chi_i(V)) \leq D(g) + E$, so the claim holds with $R_0 = \max\{D(g) + E, 100E\}$. ◁

In the next claim, we choose a sequence (z_N) of points in G that project near the basepoint except in various $\mathcal{C}U_i$ with $i > k$; in such $\mathcal{C}U_i$, the z_N project far out along α_i.

Claim 2 *There exists a constant $\ell = \ell(G, \mathfrak{S})$ and a sequence (z_N) of points in G such that*:

(1) $\mathsf{d}_V(x_0, z_N) \le \ell$ *whenever* $V \pitchfork U_i$ *or* $U_i \subsetneq V$ *for some* $i \le M$, *or* $V \perp U_i$ *for all* $i > k$.

(2) $\mathsf{d}_{U_i}(x_0, z_N) \le \ell$ *for* $i \le k$.

(3) $\mathsf{d}_{U_i}(x_0, z_N) > N$ *for* $i > k$, *and* $\pi_{U_i}(z_N)$ *is* $100DE$*–close to* α_i *for* $i > k$.

(4) *If* $V \subsetneq U_i$ *for some* $i > k$, *then* $\mathsf{d}_V(x_0, z_N) \le R_0 + \ell$.

Proof Let $\gamma\colon \mathbb{N} \to G$ be a (D, D)–hierarchy ray as provided by Lemma 2.2, applied to the points $\chi_i \in \partial\mathcal{C}U_i$ with $i > k$. Choose $z_N = \gamma(N)$. By passing to a subsequence, we can assume that $\{z_N\}$ satisfies the third assertion. The first and second hold by Lemma 2.2, where ℓ depends on D and E, and hence only on the HHS structure. Finally, if $V \subsetneq U_i$, then $\pi_V \circ \gamma$ lies ℓ–close to the geodesic in $\mathcal{C}V$ from $\pi_V(x_0)$ to $\chi_i(V)$, so $\mathsf{d}_V(x_0, z_N) \le \ell + R_0$ by Claim 1. ◁

We now show that each z_N is moved a bounded distance in G by the element g.

Claim 3 *There exists $R_1 \ge 0$, independent of N, such that* $\mathsf{d}_G(gz_N, z_N) \le R_1$ *for all* N.

Proof Let $V \in \mathfrak{S}$. We bound $\mathsf{d}_V(z_N, gz_N)$ as follows:

- If $U_i \subsetneq V$ or $U_i \pitchfork V$ for some i, then $\langle g\rangle \cdot x_0$ and $\langle g\rangle \cdot z_N$ have projections to $\mathcal{C}V$ that uniformly coarsely coincide, since they both coarsely coincide with $\rho_V^{U_i}$. Hence $\mathsf{d}_V(z_N, gz_N) \le 100\ell$ in this case.
- If $V = U_i$ for $i > k$, then $\mathsf{d}_V(z_N, gz_N) \le 100E\,\mathsf{d}_V(x_0, gx_0) + 100E + 2\ell$, by considering a $(1, 20E)$–quasigeodesic ideal triangle with vertices x_0, gx_0 and χ_i and sides α_i, $g\alpha_i$ and a geodesic from $\pi_V(x_0)$ to $g\pi_V(x_0)$. If $V = U_i$ for $i \le k$, then, since $\pi_V(z_N)$ and $\pi_V(x_0)$ are E–close, we reach the same conclusion.
- If $V \perp U_i$ for all i, then the same is true of $g^{-1}V$, since g stabilises each U_i. So, $\pi_{g^{-1}V}(z_N)$ is ℓ–close to $\pi_{g^{-1}V}(x_0)$, and the same is true with V replacing $g^{-1}V$. Hence $\pi_V(gx_0)$ and $\pi_V(gz_N)$ are ℓ–close, so $\mathsf{d}_V(z_N, gz_N) \le 2\ell + D(g)$.

- The remaining case is where $V \subsetneq U_i$ for some i. But for all such V, we have $\mathsf{d}_V(x_0, z_N) \le R_0 + \ell$ by Claim 1. Now, if $V \subsetneq U_i$, then $g^{-1}V \subsetneq g^{-1}U_i = U_i$, so $\mathsf{d}_{g^{-1}V}(z_N, x_0) \le R_0 + \ell$. Hence $\mathsf{d}_V(gz_N, gx_0) \le R_0 + \ell$. This gives $\mathsf{d}_V(z_N, gz_N) \le 2R_0 + 2\ell + D(g)$.

Hence, by the uniqueness axiom, there exists R_1 depending only on $(G, \mathfrak{S})$ and $D(g)$ such that $\mathsf{d}_G(z_N, gz_N) \le R_1$ for all N. $\triangleleft$

We are now ready to conclude, using the following strategy: we will produce a set $\{h_N\}_{N\in\mathbb{N}}$ of elements of G, all conjugate to g, such that each h_N moves x_0 a distance at most R_1. Hence the number of such h_N will be bounded in terms of R_1. On the other hand, we will choose these h_N in such a way that, as N increases, it takes larger and larger powers of h_N to move x_0 a given distance, contradicting that there are only boundedly many h_N.

The finite set $\{h_N\}$ For each N, choose $k_N \in G$ so that $k_N z_N = x_0$. Let $h_N = k_N g k_N^{-1}$. By equivariance of projections, $\mathsf{d}_V(h_N x_0, x_0) = \mathsf{d}_{k_n^{-1}V}(z_N, gz_N)$ for all $V \in \mathfrak{S}$. So, by uniqueness, $\mathsf{d}_G(x_0, h_N x_0) \le R_1$ for all N. Since d_G is uniformly proper, there is a constant $N_0 = N_0(R_1)$ such that $|\{h_N\}| \le N_0$.

Distinguishing $\{h_N\}$ For each N, let $f(N)$ be the minimum $s \in \mathbb{N}$ such that, for all $W \in \mathrm{Big}(h_N)$, we have $\mathsf{d}_W(h_N^s x_0, x_0) > 1000E\ell$. Since $f(N)$ depends only on h_N, and there are finitely many distinct h_N, we have $\sup_N f(N) < \infty$.

On the other hand, fix h_N and fix $i > k$. Then, since $U_i \in \mathrm{Big}(g)$, we have $k_N U_i \in \mathrm{Big}(h_N)$. Now, for any $s \in \mathbb{N}$, there exists $N = N(s)$ such that the following holds, by parabolicity of g and our choice of z_N: $\mathsf{d}_{U_i}(z_N, g^t z_N) \le 1000E\ell$ for $t \in \{0, \dots, s\}$. Hence $\mathsf{d}_{k_N U_i}(x_0, h_N^t x_0) \le 1000E\ell$ for such t, so $f(N) > s$. Thus $\sup_N f(N) = \infty$, a contradiction.

We conclude that $k = M$, as required. $\square$

4 The Tits alternative

Now we provide a corrected proof of the Tits alternative for hierarchically hyperbolic groups. We proceed roughly as in [6], with a change to avoid [6, Lemma 7.5]. We retain our assumption that projections are equivariant.

Theorem 4.1 [6, Theorem 9.15] *Let $(G, \mathfrak{S})$ be a hierarchically hyperbolic group and let $H \leq G$ be a finitely generated subgroup. Then either H contains a nonabelian free group, or H is virtually $\mathbb{Z}^\ell$ for some ℓ at most the complexity of $\mathfrak{S}$.*

Proof Assume H is infinite, for otherwise we are done.

The hull of H For each $W \in \mathfrak{S}$, let $K_W \subset \mathcal{C}W$ be the union of all geodesics in $\mathcal{C}W$ that start and end in $\pi_W(H)$. Then K_W is a uniformly quasiconvex subset of $\mathcal{C}W$, and if $A \leq H$ fixes W, then the action of A on $\mathcal{C}W$ preserves K_W. Let k_0 be the quasiconvexity constant for the various K_W.

Let $\mathcal{X}$ be the *hull* of H in G, in the sense of [3, Definition 6.1]. By [3, Lemma 6.2], $\mathcal{X}$ is hierarchically quasiconvex in G, and is hence an HHS, where the index set is $\mathfrak{S}$ and the hyperbolic space associated to each $U \in \mathfrak{S}$ is K_U. (See [3, Remark 5.7] or [6, Proposition 1.16] for an explanation of the hierarchically hyperbolic structure.) Equivariance of the projections and the definition of the hull implies that $\mathcal{X}$ is H–invariant. The projection $\mathcal{X} \to K_U$ coincides with the projection π_U.

Since $\mathcal{X} \subset G$, the space $\mathcal{X}$ is uniformly proper, and the action is essential in the sense of Section 8 of [6]. If H fixes $U \in \mathfrak{S}$, and H has bounded orbits in $\mathcal{C}U$, then $\partial K_U = \varnothing$.

First, we find pairwise-orthogonal domains U_i, each invariant under a finite-index subgroup $\widehat{H} \leq H$, where $\widehat{H}$ has unbounded orbits. The rest of the proof will then involve an analysis of the action of $\widehat{H}$ on these domains:

Claim 4 *There exists a finite-index subgroup $\widehat{H} \leq H$ and pairwise orthogonal $U_1, \dots, U_M \in \mathfrak{S}$ such that all of the following hold*:

- *$\widehat{H}$ fixes each U_i and has unbounded orbits in $\mathcal{C}U_i$.*
- *For all i, if $W \sqsubsetneq U_i$ or $W \pitchfork U_i$, then $\widehat{H} \cdot W \neq W$.*
- *If $W \perp U_i$ for all i, and $|H \cdot W| < \infty$, then $\pi_W(H)$ has finite diameter.*

Proof We have two cases:

Fixed boundary points case If H fixes some $p \in \partial\mathcal{X}$, then H has a finite-index subgroup H' fixing each $U \in \mathrm{Supp}(p)$, and H' fixes a point in each ∂K_U. In particular, $\partial K_U \neq \varnothing$, so H' has unbounded orbits in K_U.

No fixed boundary points case Suppose that no finite-index subgroup of H fixes a point in $\partial\mathcal{X}$. Choose $U \in \mathfrak{S}$ to be $\sqsubseteq$–minimal with the property that H has a finite-index subgroup H' with $H' \cdot U = U$.

As in Section 2, $(\boldsymbol{F}_U, \mathfrak{S}_U)$ is an HHS with a (possibly nonproper) action of H' by HHS automorphisms. Now, since $\boldsymbol{F}_U$ is quasi-isometric to a proper HHS (via the quasi-isometric embedding $\boldsymbol{F}_U \to \mathcal{X}$ from Section 2), $\partial \boldsymbol{F}_U$ is compact. Moreover, $\mathfrak{S}_U$ is countable since $\mathfrak{S}_U \subset \mathfrak{S}$ and $\mathfrak{S}$ is G–finite. Also, H' cannot fix a point in $\partial \boldsymbol{F}_U$ without also fixing a point in $\partial \mathcal{X}$.

At this point, we would like to apply [6, Proposition 9.2], which warrants some care. As a space equipped with an H'–action, $\boldsymbol{F}_U$ need not be proper (recall that when we quasi-isometrically embed it in $\mathcal{X}$ to make it proper, the action becomes a quasiaction at the level of the space, while the proof of [6, Proposition 9.2] is phrased in terms of actions). However, in the proof of [6, Proposition 9.2], properness of the space is used only in two ways. First, it is needed to ensure $\partial \boldsymbol{F}_U$ is compact, which we checked above. Second, it is used to allow us to assume that $\boldsymbol{F}_U$ is countable, as in [6, Remark 9.6]; this plays a role in Lemma 9.8 of [6].

We achieve this here as follows. First, let $\boldsymbol{P}_U$ be the associated standard product region, which we can choose to be H'–invariant. Since it is a subset of the finitely generated group G, $\boldsymbol{P}_U$ is countable. Now, given $x, y \in \boldsymbol{P}_U$, add an edge joining x to y whenever $\mathrm{d}_W(x, y) \leq 100E$ for all $W \perp U$. Then $\boldsymbol{F}_U$ can be modified within its hieromorphism class to be $\boldsymbol{P}_U$, equipped with the resulting graph metric. We still have an action of H' on $\boldsymbol{F}_U$ by HHS automorphisms (by uniform quasi-isometries at the level of $\boldsymbol{F}_U$), and this coincides with the original action at the level of $\mathfrak{S}_U$. Also, the action of H' on $\boldsymbol{F}_U$ is free (but not proper).

Now we apply [6, Proposition 9.2] to conclude that either H' has bounded orbits in $\boldsymbol{F}_U$ or there exists $h \in H'$ acting loxodromically on K_U. The third option, that H' has a finite orbit in $\mathfrak{S}_U - \{U\}$, is ruled out by minimality of U.

Bounded orbits Since H' stabilises $\boldsymbol{P}_U \subset G$ and acts properly on G, and since H' is infinite, H' has unbounded orbits in either $\boldsymbol{F}_U$ or in $\boldsymbol{E}_U$. In either case, applying [6, Proposition 9.2] to $\boldsymbol{F}_U$ or $\boldsymbol{E}_U$ (as above), and replacing H' if necessary by a finite-index subgroup, there exists a $\sqsubseteq$–minimal U' satisfying $H'U' = U'$ and that some $h \in H'$ is loxodromic on $K_{U'}$.

Pairwise-orthogonal domains The above shows that there exists U such that $H \cdot U$ is finite, and there exists a finite-index subgroup H' such that $H' \cdot U = U$, and H' has unbounded orbits in K_U, and U is $\sqsubseteq$–minimal with these properties. Suppose that $V \in \mathfrak{S} - \{U\}$ is another such domain, fixed by a finite-index subgroup H''. Then U and V are not $\sqsubseteq$–related, by minimality. If $U \pitchfork V$, then the H''–orbit of the bounded set ρ_V^U in $\mathcal{C}V$ is unbounded, contradicting that $H'' \cap H'$ fixes U. Thus $U \perp V$. Hence there exist pairwise-orthogonal $U_1, \dots, U_M$ and $\widehat{H} \leq_{\text{f.i.}} H$ such that:

- $\widehat{H}$ fixes each U_i and has unbounded orbits in $\mathcal{C}U_i$.
- If $W \subsetneq U_i$ or $W \pitchfork U_i$, then $\widehat{H} \cdot W \neq W$.
- If $W \perp U_i$ for all i and $|H \cdot W| < \infty$, then $\operatorname{diam}(\pi_W(H)) < \infty$.

This completes the proof of the claim. ◁

Loxodromic domains and horocyclic domains Given $1 \leq i \leq M$, consider the action of $\widehat{H}$ on the hyperbolic space K_{U_i}. By Claim 4, this action has unbounded orbits. So, up to relabeling, there exists $k \in \{0, \dots, M\}$ such that $\widehat{H}$ contains a loxodromic isometry of K_{U_i} if and only if $i \leq k$, and, if $i > k$, then the action of $\widehat{H}$ on K_{U_i} is *horocyclic*, ie the orbit is unbounded, but there is no loxodromic isometry. In the horocylic case, Theorem 3.1 implies that each cyclic subgroup of $\widehat{H}$ has a bounded orbit in K_{U_i} when $i > k$. We call $U_1, \dots, U_k$ the *loxodromic domains* and $U_{k+1}, \dots, U_M$ the *horocyclic domains*.

Ruling out general type loxodromic domains Let $i \leq k$. For each i, let $h_i \in \widehat{H}$ act loxodromically on K_{U_i}, and let $p_i^{\pm} \in \partial K_{U_i}$ be fixed by h_i. We now reduce to the case where there are no general-type actions on the K_{U_i}:

Claim 5 *Either $\widehat{H}$, and hence H, contains a nonabelian free subgroup, or the following holds (up to passing to an index-2 subgroup of $\widehat{H}$): for $1 \leq i \leq k$, there exists $p_i \in \{p_i^{\pm}\}$ with $\widehat{H} \cdot p_i = p_i$.*

Proof Fix $i \leq k$. The action of $\widehat{H}$ on $\mathcal{C}U_i$ is thus either *lineal* (exactly two fixed boundary points), or *focal* (exactly one fixed boundary point), or of *general type* (no global fixed point). In the general-type case, $\widehat{H}$ contains a nonabelian free group [8, Section 8.2.F].

In the lineal case, the endpoints of h_i in $\partial \mathcal{C}U_i$ are fixed by the action of $\widehat{H}$, as required. In the focal case (where, by definition, any two loxodromics have dependent axes), one of the two endpoints of h_i must be fixed by $\widehat{H}$. This proves the claim. ◁

Bounding projections of $\widehat{H}$ In view of Claim 5, assume from now on that $\widehat{H}$ fixes some $p_i \in \partial K_{U_i}$, which is the endpoint of an axis of a loxodromic $h_i \in \widehat{H}$, for all $i \leq k$. Fix $a \in \mathcal{X}$ so that $a \in \boldsymbol{P}_{U_i}$ for $1 \leq i \leq M$. In particular, for all i and all $W \in \mathfrak{S}$ such that $W \pitchfork U_i$ or $U_i \subsetneq W$, the set $\pi_W(\widehat{H} \cdot a)$ uniformly coarsely coincides with $\rho_W^{U_i}$.

In the next two claims, we bound the diameter of the image of $\widehat{H}\cdot a$ in all domains V except when $V \sqsubsetneq U_i$ for some horocyclic domain U_i.

Claim 6 *There exists $R_0 \in \mathbb{R}$ such that the following holds. Suppose that $W \in \mathfrak{S}$ satisfies one of*

- *$W \perp U_i$ for all $1 \le i \le M$;*
- *$W \pitchfork U_i$ for some $i \le M$;*
- *$U_i \sqsubsetneq W$ for some $i \le M$.*

Then $\operatorname{diam}(\pi_W(\widehat{H}\cdot a)) \le R_0$. Hence, after enlarging R_0 uniformly, $\operatorname{diam}(K_W) \le R_0$.

Proof First suppose that either $W \pitchfork U_i$ or $U_i \sqsubsetneq W$. Then $\rho_W^{U_i}$ is a bounded subset of $\mathcal{C}W$, and $\pi_W(\widehat{H}\cdot a)$ uniformly coarse coincides with $\rho_W^{U_i}$. Hence there exists R_0, independent of W, such that $\operatorname{diam}(\pi_W(\widehat{H}\cdot a)) \le R_0$ for all such W.

Next, suppose that $W \in \mathfrak{S}$ satisfies $W \perp U_i$ for all i. We will consider the action of $\widehat{H}$ on a hierarchically hyperbolic space which is the "orthogonal complement" of the U_i.

Let $\mathfrak{S}^{\perp}_{\{U_i\}}$ be the set of V such that $V \perp U_i$ for all i.

For each i, let $\boldsymbol{P}_{U_i}$ be the standard product region associated to U_i. Let $\boldsymbol{P} = \bigcap_i \boldsymbol{P}_{U_i}$. By equivariance of projection, $\boldsymbol{P}$ is $\widehat{H}$–invariant. Choose $x_1 \in \boldsymbol{P}$, and let $\boldsymbol{E}$ be the set of $x \in \boldsymbol{P}$ such that $\mathsf{d}_V(x, x_1) > 100E$ implies $V \perp U_i$ for all i. Then $\boldsymbol{E}$ is hierarchically quasiconvex in $\mathcal{X}$. Hence, by [3, Proposition 5.6], $(\boldsymbol{E}, \mathfrak{S})$ is a hierarchically hyperbolic space, and $\boldsymbol{E}$ is proper. Note that $\pi_V(\boldsymbol{E}) = K_V$ if $V \in \mathfrak{S}^{\perp}_{U_i}$. Otherwise, $\pi_V(\boldsymbol{E})$ has uniformly bounded diameter. So, by normalising, $(\boldsymbol{E}, \mathfrak{S})$ is a hierarchically hyperbolic space where the hyperbolic space associated to $V \in \mathfrak{S}$ is unbounded if and only if $V \in \mathfrak{S}^{\perp}_{\{U_i\}}$. Now, $\widehat{H}$ acts on the HHS structure by hieromorphisms, so, as in the proof of Claim 4, we can modify $\boldsymbol{E}$ in its hieromorphism class so that $\widehat{H}$ acts on $\boldsymbol{E}$.

Suppose that $\widehat{H}$ has unbounded orbits in $\boldsymbol{E}$. Then, applying [6, Proposition 9.2] as in Claim 4, we see that some $V \in \mathfrak{T}$ has unbounded $\widehat{H}$–orbits in $\mathcal{C}V$. Now, V cannot be orthogonal to all U_i, by Claim 4. But, by construction, $\pi_V(\boldsymbol{E})$ is bounded for any other V. So, $\widehat{H}$ has bounded orbits in $\boldsymbol{E}$, and thus $\pi_W(\widehat{H}\cdot a)$ is bounded by a constant depending only on $\widehat{H}$, $\{U_i\}$, a and the HHS constants. This proves the claim. ◁

Claim 7 *There exists $R_1 \in \mathbb{R}$ such that the following holds. Let $W \in \mathfrak{S}$ and suppose that $W \sqsubsetneq U_i$ for some $i \le k$. Then $\operatorname{diam}(\pi_W(\widehat{H}\cdot a)) \le R_1$. Hence, after enlarging R_1 uniformly, $\operatorname{diam}(K_W) \le R_1$.*

The U_i in Claim 7 has the property that the action of $\widehat{H}$ on K_{U_i} is either lineal or focal.

Proof Recall that there is a point $p_i \in \partial K_{U_i}$ — the attracting fixed point of h_i, say — such that p_i is fixed by all of $\widehat{H}$. By Lemma 6.6 of [6], there exists $D(h_i)$, independent of W, such that $\mathsf{diam}(\pi_W(\langle h_i\rangle \cdot a)) \leq D(h_i)$. For any $g \in \widehat{H}$, we have $\mathsf{diam}(\pi_W(g\langle h_i\rangle \cdot a)) = \mathsf{diam}(\pi_{g^{-1}W}(\langle h_i\rangle \cdot a)) \leq D(h_i)$, since $g^{-1}W \sqsubsetneq U_i$ (because g fixes U_i).

Now let $g \in \widehat{H}$. Let $T = \mathsf{d}_W(a, h_i a)$. Since $gp_i = p_i$, we can choose $n \in \mathbb{Z}$ so that $gh_i^n a$ lies at distance at most $100(E+T)$ from some $h_i^m a$, where m satisfies $\mathsf{d}_{U_i}(h_i^m a, \rho^W_{U_i}) > 10^9(E+T)$. Hence, by bounded geodesic image,

$$\mathsf{d}_W(h_i^m a, gh_i^n a) \leq E.$$

From above, $\mathsf{d}_W(ga, gh_i^n a) \leq D(h_i)$. So, by the triangle inequality, $\mathsf{d}_W(ga, h_i^m a) \leq 2E + D(h_i)$. Finally, $\mathsf{d}_W(h_i^m a, a) \leq D(h_i)$ since $\mathsf{diam}(\pi_W(\langle h_i\rangle \cdot a)) \leq D(h_i)$. Another application of the triangle inequality gives $\mathsf{d}_W(ga, a) \leq 2E + 2D(h_i)$. Hence the diameter of $\pi_W(\widehat{H} \cdot a)$ is at most $2E + 2\max_i D(h_i)$, as required. $\triangleleft$

Choosing a hierarchy ray For each $i > k$, let α_i be a uniform quasigeodesic joining $\pi_{U_i}(a)$ to the unique point $\chi_i \in \partial \mathcal{C} U_i$ fixed by $\widehat{H}$. Define α_i analogously when $i \leq k$ and U_i is a focal domain for $\widehat{H}$.

For each such i and each $V \sqsubsetneq U_i$, let $\chi_i(V) \in \mathcal{C} V$ be the projection of χ_i to K_V.

We now construct a hierarchy ray γ that will serve a very similar purpose to the hierarchy ray in the proof of Theorem 3.1.

Lemma 2.2 provides a (D, D)–hierarchy ray $\gamma\colon \mathbb{N} \to \mathcal{X}$ such that $\pi_V \circ \gamma$ has image contained in the $100E$–neighbourhood of $\pi_V(a)$ unless $V \sqsubseteq U_i$ for some $i > k$ or $V \sqsubseteq U_i$ for some $i \leq k$ such that U_i is focal. Also, for each such i, we have that $\pi_{U_i} \circ \gamma(n)$ converges to χ_i (and in fact lies D–close to α_i) as $n \to \infty$.

Strategy for the rest of the proof The rest of the proof will proceed as follows. First, we will use horofunctions coming from the lineal and focal domains to construct a coarsely Lipschitz map $\beta\colon \widehat{H} \to \mathbb{R}^k$. By studying a sequence of points sampled from γ, we will prove that β is a proper map — this is the most technical part of the argument. From this, we will deduce that $\widehat{H}$ has polynomial growth, and is therefore virtually nilpotent; an application of Theorem 3.1 will then imply that $\widehat{H}$ is virtually abelian.

The map β For each $i \le k$ (ie each U_i lineal or focal), define $\beta_i\colon \hat{H} \to \mathbb{R}$ by

$$\beta_i(g) = \limsup_{n\to\infty}\bigl(\mathsf{d}_{U_i}(a,\gamma(n)) - \mathsf{d}_{U_i}(ga,\gamma(n))\bigr).$$

By Corollary 4.8 of [10], each β_i is a quasimorphism of defect $16E$. Let $\beta(g) = (\beta_i(g))_{i=1}^k \in \mathbb{R}^k$. This defines a map $\beta\colon \hat{H} \to \mathbb{R}^k$ such that $\|\beta(gh) - (\beta(g)+\beta(h))\|_1 \le 16kE$ for all $g, h \in \hat{H}$. (For simplicity, we equip $\mathbb{R}^k$ with the ℓ_1–metric.)

Let $\mathsf{d}_{\hat{H}}$ be a word metric on $\hat{H}$ with respect to some finite generating set.

Claim 8 *There exists $C \ge 1$ such that $\beta\colon (\hat{H}, \mathsf{d}_{\hat{H}}) \to \mathbb{R}^k$ is (C,C)–coarsely Lipschitz.*

Proof Fix $g, h \in \hat{H}$. Then

$$\|\beta(g) - \beta(h)\|_1 = \sum_{i=1}^k |\beta_i(g) - \beta_i(h)|.$$

Each $\beta_i\colon (\hat{H}\cdot\pi_{U_i}(a), \mathsf{d}_{U_i}) \to \mathbb{R}$ is the restriction of a horofunction, and hence coarsely Lipschitz. Thus, $\|\beta(g) - \beta(h)\|_1 \le C_1 \sum_{i=1}^k \mathsf{d}_{U_i}(ga, ha) + C_1$ for some $C_1 \ge 1$. So, by the distance formula, and for some K_0 depending on C_1 and the distance formula constants, $\|\beta(g)-\beta(h)\|_1 \le K_0 \mathsf{d}_G(ga, ha) + K_0 = K_0|a^{-1}(h^{-1}g)a|_G + K_0$. Hence there exists C, depending only on the word metrics $\mathsf{d}_{\hat{H}}$ and d_G and the (fixed) basepoint $a \in G$, such that $\|\beta(g) - \beta(h)\|_1 \le C\mathsf{d}_{\hat{H}}(g,h) + C$. So, β is coarsely Lipschitz. $\triangleleft$

The map β is proper Let $\mathcal{R} \subset \hat{H}$ be a finite subset, and let $r = \max_{g\in\mathcal{R}} \|\beta(g)\|_1$. Let $L = L(\mathcal{X}, \mathfrak{S})$ be a natural number to be determined (independently of $\mathcal{R}$).

Remark 4.2 (plan of the rest of the proof) Our goal is to choose points $a_n \in \mathcal{X}$ with the following property: a definite proportion of the elements of $\mathcal{R}$ move a_n at most a bounded distance in G. This distance is bounded in terms of $\mathcal{R}$, and the "definite proportion", $1/L$, depends only on the ambient HHS structure. Morally, we choose the a_n to be an unbounded sequence of points along the hierarchy ray γ, but the actual choice of a_n is slightly more complicated, to enable us to handle projections to domains $V \subsetneq U_i$ where U_i is horocyclic.

From this, we will conclude that the map β is proper. From that, it will follow (Claim 11) that H has polynomial growth and is therefore virtually nilpotent. Then, an application of Theorem 3.1 will complete the proof that H is virtually abelian.

The reader should note that the point a_n is constructed (in terms of $\mathcal{R}$) in Construction 4.3, and that this construction relies on Claim 9; the construction is completed in the text following the proof of that claim. Following that, in Claim 10, we actually show that β is proper.

Having sketched the plan, we now resume the proof.

Let $i > k$. For each $g \in \mathcal{R}$, the subgroup $\langle g \rangle$ has bounded orbits in $\mathcal{C}U_i$. Indeed, this is clear if g has finite order. Otherwise, by Theorem 3.1, if $U_i \in \mathrm{Big}(g)$, then g is loxodromic on $\mathcal{C}U_i$, but since $i > k$, the action of $\widehat{H}$ on $\mathcal{C}U_i$ is horocyclic.

Thus there exists $n_0 \in \mathbb{N}$ such that for all $i > k$, and each of the finitely many $g \in \mathcal{R}$, we have $\mathsf{d}_{U_i}(\gamma(n), g\gamma(n)) \le 100(D+E)$ for all $n \ge n_0$, because $\widehat{H}$ fixes χ_i and γ is a hierarchy ray whose projection to $\mathcal{C}U_i$ is a quasigeodesic ray fellow-traveling α_i at distance D.

Moreover, by choosing n_0 sufficiently large (in terms of $\mathcal{R}$), we have the following: for all $i > k$, all $n \ge n_0$, and all $V \sqsubsetneq U_i$ such that $\mathsf{d}_{U_i}(\rho^V_{U_i}, \gamma(n)) \le 10^6(D+E)$, the set $\rho^V_{U_i}$ is E–far from any geodesic joining two points in $\{\pi_{U_i}(g^m a) : g \in \mathcal{R},\ m \in \mathbb{Z}\}$. Hence, by consistency and bounded geodesic image, we have $\mathsf{diam}\big(\bigcup_{g \in \mathcal{R}, m \in \mathbb{Z}} \pi_V(g^m a)\big) \le E$.

We can also choose n_0 sufficiently large that for all $n \ge n_0$ and all $i \le k$, we have $\mathsf{d}_{U_i}(g\gamma(n), \gamma(n)) \le 100DEr$, because on such $\mathcal{C}U_i$, the action of $\widehat{H}$ is lineal or focal and $|\beta_i(g)| \le r$ for all such i and all $g \in \mathcal{R}$.

Fix $n \ge n_0$. Fix $n_1 > n$ such that $\mathsf{d}_{U_i}(\gamma(n), \gamma(n_1)) > 10^9(D+E)$ for all i such that U_i is horocyclic or focal. (In other words, $\gamma(n_1)$ is much further along γ than $\gamma(n)$.)

Let $\mathcal{V}$ be the set of $V \in \mathfrak{S}$ satisfying each of the following conditions:

- $V \sqsubsetneq U_i$ for some $i > k$;
- $\mathsf{d}_V(a, \gamma(n_1)) > 10^5(D+E)$;
- for the unique i with $V \sqsubsetneq U_i$, we have $\mathsf{d}_{U_i}(\rho^V_{U_i}, \gamma(n)) \le 200(D+E)$;
- V is not properly nested in any $W \sqsubsetneq U_i$ with $\mathsf{d}_W(a, \gamma(n_1)) > 10^5(D+E)$.

Construction 4.3 (the "test point" a_n) If $\mathcal{V} = \varnothing$, let $a_n = \gamma(n)$.

Otherwise, define a_n as follows. Since $\mathsf{d}_V(a, \gamma(n_1)) > 10^5(D+E)$ for each $V \in \mathcal{V}$, the set $\mathcal{V}$ is finite (by, for example, the distance formula). The $\sqsubseteq$–maximality assumption guarantees that any two elements of $\mathcal{V}$ are either $\prec$–comparable or orthogonal. As in

[3, Section 2], we equip $\mathcal{V}$ with a partial ordering $\prec$ coming from the points $a, \gamma(n_1)$: if $V, W \in \mathcal{V}$, then $V \prec W$ if $V \pitchfork W$ and ρ^W_V is E–close to $\pi_V(\gamma(n_1))$. Let $V_1, \dots, V_s$ be the $\prec$–maximal elements of $\mathcal{V}$. Since they are pairwise orthogonal, s is bounded by the complexity of $\mathfrak{S}$.

Claim 9 *There exists a natural number $L = L(\mathcal{X}, \mathfrak{S}) \geq 1$ such that the following holds: there exists $g_1 \in \mathcal{R}$ and $\mathcal{R}' \subseteq \mathcal{R}$ such that*

- $|\mathcal{R}'| > |\mathcal{R}|/L$;
- *for all V_j with $j \leq s$, we have $gV_j = g_1 V_j$ whenever $g \in \mathcal{R}'$.*

Proof Let U_i be horocyclic, with the property that $V_1 \subsetneq U_i$. Then $\rho^{V_1}_{U_i}$ is $200(D+E)$–close to $\pi_{U_i}(\gamma(n))$, since $V_1 \in \mathcal{V}$. Hence, $\mathsf{d}_{U_i}(g\rho^{V_1}_{U_i}, \gamma(n)) \leq 500(D+E)$ for all $g \in \mathcal{R}$.

We next show that $\mathsf{d}_{gV_1}(a, \gamma(n_1)) > 10^5(D+E) - 2E$ for all $g \in \mathcal{R}$. Indeed, we have $\mathsf{d}_{V_1}(a, \gamma(n_1)) \geq 10^5(D+E)$. We also have that $\mathsf{d}_{U_i}(\gamma(n), \rho^{V_1}_{U_i}) \leq 200(D+E)$, so $\mathsf{d}_{U_i}(g\gamma(n), \rho^{gV_1}_{U_i}) \leq 200(D+E)$. By the triangle inequality and the fact that $\mathsf{d}_{U_i}(g\gamma(n), \gamma(n)) \leq 100(D+E)$, we have $\mathsf{d}_{U_i}(\gamma(n), \rho^{gV_1}_{U_i}) \leq 300(D+E)$. Now, $\mathsf{d}_{U_i}(\gamma(n), \gamma(n_1)) > 10^9(D+E)$, so it follows that $\rho^{gV_1}_{U_i}$ is not E–close to any geodesic in $\mathcal{C}U_i$ from $\gamma(n_1)$ to $g\gamma(n_1)$ — such geodesics have length at most $100(D+E)$. Moreover, since $\mathsf{d}_{U_i}(ga, a)$ is bounded in terms of $\mathcal{R}$, we could have made our choice of n_0 sufficiently large compared to $\mathsf{d}_{U_i}(ga, a)$ so as to ensure that $\rho^{gV_1}_{U_i}$ is not E–close to any geodesic from a to ga. Hence, by bounded geodesic image and consistency, $\mathsf{d}_{gV_1}(a, ga) \leq E$ and $\mathsf{d}_{gV_1}(\gamma(n_1), g\gamma(n_1)) \leq E$. Finally, $\mathsf{d}_{gV_1}(ga, g\gamma(n_1)) \geq 10^5(D+E)$, so, by the triangle inequality, $\mathsf{d}_{gV_1}(a, \gamma(n_1)) > 10^5(D+E) - 2E$.

Choose $x, y \in \gamma$ such that $\mathsf{d}_{U_i}(x, y) \leq 10^6(D+E)$, and x and y lie D–close to points on α_i on opposite sides of a point D–close to $\pi_{U_i}(\gamma(n))$. Moreover, we choose x and y so that $\mathsf{d}_{U_i}(x, a) \geq 10^4(D+E)$ and $\mathsf{d}_{U_i}(y, \gamma(n)) \geq 10^4(D+E)$.

Then, provided we choose n_0 sufficiently large (in terms of D and E and $\mathcal{R}$), we have that $\mathsf{d}_{U_i}(x, gx) \leq 100(D+E)$ and $\mathsf{d}_{U_i}(y, gy) \leq 100(D+E)$ for all $g \in \mathcal{R}$. Hence, for each $g \in \mathcal{R}$, we have that $\pi_{gV_1}(gx)$ and $\pi_{gV_1}(a)$ are E–close, and the same is true with x replaced by y and a replaced by $\gamma(n_1)$. Thus $\mathsf{d}_{gV_1}(a, \gamma(n_1)) > 100E$ for all $g \in \mathcal{R}$.

By [3, Lemma 2.5], there exists C, depending only on D and E, such that $|\{gV_1 : g \in \mathcal{R}\}|$ has at most C distinct elements. Indeed, if not, then, by the aforementioned lemma, some gV_1 would be properly nested into some $W \sqsubseteq U_i$ with $\mathsf{d}_W(x, y) > 10^7(D+E)$

and hence $\mathsf{d}_W(a,\gamma(n_1)) > 10^6(D+E)$. If $V \sqsubsetneq g^{-1}W \sqsubsetneq U_i$, then $\mathsf{d}_{g^{-1}W}(a,\gamma(n_1)) > 10^5(D+E)$, contradicting that $V_1 \in \mathcal{V}$. If $W = U_i$, then we would have $\mathsf{d}_{U_i}(x,y) > 10^7(D+E)$, another contradiction. Hence we have $\mathcal{R}_1 \subset \mathcal{R}$ such that $gV_1 = hV_1$ for all $g,h \in \mathcal{R}_1$, and $|\mathcal{R}_1| \geq |\mathcal{R}|/C$.

Now apply the same argument, with $\mathcal{R}$ replaced by $\mathcal{R}_1$ and V_1 replaced by V_2. Continuing in this way, we find a subset $\mathcal{R}' \subset \mathcal{R}$ such that $gV_j = hV_j$ for all $j \leq s$ and $g,h \in \mathcal{R}'$, and $|\mathcal{R}'| \geq |\mathcal{R}|/C^s$. Take s_0 to be the complexity of $\mathfrak{S}$ and let $L = C^{s_0}$. $\triangleleft$

Now let g_1 and $\mathcal{R}'$ be as in Claim 9. Let $\boldsymbol{P} = \bigcap_{j=1}^s \boldsymbol{P}_{V_j}$ be the $\bigcap_{j=1}^s \mathrm{Stab}_{\hat{H}}(V_j)$–invariant standard product region in $\mathcal{X}$ associated to $V_1,\dots,V_s$. Then $\boldsymbol{P}$ is uniformly hierarchically quasiconvex, and there is a coarse *gate map* $\mathfrak{g}_{\boldsymbol{P}}$ sending points in $\mathcal{X}$ to uniformly bounded sets in $\boldsymbol{P}$ such that the following holds for all $x \in \mathcal{X}$:

- $\pi_V(\mathfrak{g}_{\boldsymbol{P}}(x))$ is $10E$–close to $\rho_V^{V_j}$ whenever $V_j \pitchfork V$ or $V_j \sqsubsetneq V$ for some $j \leq s$.
- $\pi_V(\mathfrak{g}_{\boldsymbol{P}}(x))$ is $10E$–close to $\pi_V(x)$ whenever $V \sqsubseteq V_j$ for some j or $V \perp V_j$ for all j.
- For all $h \in \hat{H}$, we have $h \cdot \mathfrak{g}_{\boldsymbol{P}}(x) = \mathfrak{g}_{h\boldsymbol{P}}(hx)$, because of equivariance of projections.

We define a_n as follows. Let $\boldsymbol{F}$ be the hierarchically quasiconvex subset of $g_1\boldsymbol{P}$ consisting of those points x such that $\pi_V(x)$ is $100E$–close to $\pi_V(\gamma(n))$ whenever $V \sqsubseteq U_i$ and U_i is either lineal, or focal, or is horocylic but has the property that $V_j \not\sqsubseteq U_i$ for all $j \leq s$. Let $a_n = \mathfrak{g}_{\boldsymbol{F}}(\gamma(n_1))$. So, $\mathsf{d}_W\big(a_n, \mathfrak{g}_{g_1\boldsymbol{P}}(\gamma(n_1))\big) \leq 10E$ if $W \sqsubseteq U_i$, where U_i is a horocyclic domain and some $V_j \sqsubsetneq U_i$.

Claim 10 *There exists $M(r)$, depending only on $\mathcal{X}, \mathfrak{S}$ and $r = \max_{g\in\mathcal{R}} \|\beta(g)\|_1$, such that the following holds: Let $I(r)$ be the cardinality of the $M(r)$–ball in (G, d_G) about 1. Then $|\mathcal{R}| \leq L \cdot I(r)$.*

Proof We continue to let g_1, $\mathcal{R}'$, the V_j and $\boldsymbol{P}$ be as above. We will first produce $M(r)$ such that one of the following holds:

(i) $\mathsf{d}_G(a_n, ga_n) \leq M(r)$ for all $g \in \mathcal{R}'$. In this case, $\mathsf{d}_G(1, a_n^{-1}ga_n) \leq M(r)$ for all $g \in \mathcal{R}'$, and hence $|\{a_n^{-1}ga_n : g \in \mathcal{R}'\}| = |\mathcal{R}'| \leq I(r)$. So, by Claim 9, $|\mathcal{R}| \leq L \cdot I(r)$, as required.

(ii) $\mathsf{d}_G(a_n, gg_1^{-1}a_n) \leq M(r)$ for all $g \in \mathcal{R}'$. In this case, $\mathsf{d}_G(1, a_n^{-1}gg_1^{-1}a_n) \leq M(r)$ for all $g \in \mathcal{R}'$, and hence $|\{a_n^{-1}gg_1^{-1}a_n : g \in \mathcal{R}'\}| = |\{gg_1^{-1} : g \in \mathcal{R}'\}| = |\mathcal{R}'| \leq I(r)$. Thus, by Claim 9, $|\mathcal{R}| \leq L \cdot I(r)$, as required.

So, it remains to show that either item (i) or item (ii) holds; these correspond to the cases where $\mathcal{V}=\varnothing$ and $\mathcal{V}\neq\varnothing$.

The case $\mathcal{V}=\varnothing$ First, suppose that $\mathcal{V}=\varnothing$, so that $a_n=\gamma(n)$. Let $V\in\mathfrak{S}$ and let $g\in\mathcal{R}'$. If V is not nested into any U_i with $i\leq M$, then $\mathsf{d}_V(a_n,ga_n)\leq R_0$ by Claim 6. If $V\sqsubsetneq U_i$ for some U_i that is lineal or focal, then $\mathsf{d}_V(a_n,ga_n)\leq R_1$ by Claim 7. If $V=U_i$ for some U_i lineal or focal, then $\mathsf{d}_V(a_n,ga_n)\leq 100DEr$, by our choice of n. If $V=U_i$ for U_i horocyclic, then $\mathsf{d}_{U_i}(a_n,ga_n)\leq 100(D+E)$, by our choice of n.

Suppose that $V\sqsubsetneq U_i$ for some U_i horocyclic. First observe that $\pi_V(a_n)$ lies D–close to a geodesic α from $\pi_V(a)$ to $\pi_V(\gamma(n_1))$, and $\pi_{g^{-1}V}(a_n)$ lies D–close to a geodesic from $\pi_{g^{-1}V}(a)$ to $\pi_{g^{-1}V}(\gamma(n_1))$. So $\pi_V(a_n)$ and $\pi_V(ga_n)$ both lie $(D+2E)$–close to α. If $\mathsf{d}_V(a,\gamma(n_1))\leq 10^6(D+E)$, this implies that $\mathsf{d}_V(a_n,ga_n)\leq 10^7(D+E)$.

On the other hand, if $\mathsf{d}_V(a,\gamma(n_1))>10^6(D+E)$, then, since $\mathcal{V}=\varnothing$, there exists a $\sqsubseteq$–maximal W (in U_i) such that $V\sqsubseteq W\sqsubsetneq U_i$ and $\mathsf{d}_W(a,\gamma(n_1))>10^5(D+E)$, and W satisfies $\mathsf{d}_{U_i}(\rho^W_{U_i},\gamma(n))>200(D+E)$. So, $\mathsf{d}_{U_i}(\rho^V_{U_i},\gamma(n))\geq 199(D+E)$. Now, any K_{U_i}–geodesic from $\pi_{U_i}(a_n)=\pi_{U_i}(\gamma(n))$ to $\pi_{U_i}(ga_n)=\pi_{U_i}(g\gamma(n))$ has length at most $100(D+E)$, so $\rho^V_{U_i}$ cannot lie E–close to such a geodesic. Hence, by consistency and bounded geodesic image, we have $\mathsf{d}_V(a_n,ga_n)\leq E$.

We have shown that $\mathsf{d}_V(a_n,ga_n)\leq\max\{10^7(D+E),R_0,R_1,100DEr\}$ for all $V\in\mathfrak{S}$, so, by the uniqueness axiom, there exists $M(r)$, depending on r but not $\mathcal{R}$, such that $\mathsf{d}_G(a_n,ga_n)\leq M(r)$ for all $g\in\mathcal{R}$ (and hence all $g\in\mathcal{R}'$). This proves that (i) holds.

The case $\mathcal{V}\neq\varnothing$ For each $g\in\mathcal{R}'$, we have by Claim 9 that $g\boldsymbol{P}=g_1\boldsymbol{P}$. Letting $h=g_1^{-1}g$, we thus have $h\boldsymbol{P}=\boldsymbol{P}$ and, by the same claim, $hV_j=V_j$ for all j.

Let $\omega=gg_1^{-1}$. Let $V\in\mathfrak{S}$. We wish to bound $\mathsf{d}_V(a_n,\omega a_n)$ uniformly, to establish (ii).

There are several cases:

- If $V=U_i$ for some U_i horocyclic, then either $\pi_{U_i}(a_n)$ is $10E$–close to $\pi_{U_i}(\gamma(n))$, or $\pi_{U_i}(a_n)$ is $20E$–close to some $\rho^{g_1V_j}_{U_i}$. In either case, we have $\mathsf{d}_{U_i}(a_n,g^{-1}a_n)\leq 10^3(D+E)$. The same holds with g_1^{-1} replacing g^{-1} since $g_1\in\mathcal{R}'$. So, by the triangle inequality, $\mathsf{d}_{U_i}(a_n,\omega a_n)=\mathsf{d}_{U_i}(g_1^{-1}a_n,g^{-1}a_n)\leq 2\cdot 10^3(D+E)$.
- If $V=U_i$ for some U_i lineal or focal, then $\mathsf{d}_V(a_n,g^{-1}a_n)\leq 100DEr$. The same holds for $\mathsf{d}_V(a_n,g_1^{-1}a_n)$, so $\mathsf{d}_V(a_n,\omega a_n)\leq 200DEr$.
- If $V\sqsubsetneq U_i$ for some U_i lineal or focal, then $\mathsf{d}_V(a_n,\omega a_n)\leq R_1$, since, by Claim 7, $\mathsf{diam}(K_V)\leq R_1$.

- If V is not nested in any U_i, then K_V has diameter at most R_0 by Claim 6, so $\mathsf{d}_V(a_n, \omega a_n) \le R_0$.

- The remaining case is where $i > k$ (so U_i is horocyclic) and $V \sqsubsetneq U_i$. There are two subcases:

– Suppose that no $V_j \sqsubsetneq U_i$. Then $\pi_V(a_n)$ is $100E$–close to the $\pi_V(\gamma(n))$. Moreover, $\pi_V(\omega a_n) = \omega(\pi_{\omega^{-1}V}(a_n))$. Since there is no $V_j \sqsubsetneq U_i$ and $\omega^{-1}V \sqsubsetneq U_i$, we have $\pi_{\omega^{-1}V}(a_n)$ is $100E$–close to $\pi_{\omega^{-1}V}(\gamma(n))$, so $\pi_V(\omega a_n)$ is $100E$–close to $\pi_V(\omega\gamma(n))$. Exactly as in the "$\mathcal{V} = \varnothing$" discussion, we obtain a uniform bound on $\mathsf{d}_V(a_n, \omega a_n)$, except that we use the bound $\mathsf{d}_{U_i}(a_n, \omega a_n) \le 200(D+E)$ coming from the corresponding bounds of $100(D+E)$ for g^{-1} and g_1^{-1}, along with the triangle inequality.

– Suppose that some $V_j \sqsubsetneq U_i$. By definition, since $V \sqsubsetneq U_i$, we have that $\pi_V(a_n)$ is $10E$–close to $\pi_V\big(\mathfrak{g}_{g_1\boldsymbol{P}}(\gamma(n_1))\big)$.

Let $\sim$ denote $\sqsubseteq$, $\sqsubsetneq$, $\perp$ or $\pitchfork$. Suppose that $V \sim g_1 V_j$. Then $g^{-1}V \sim g^{-1}g_1 V_j = h^{-1}V_j = V_j$. So $g_1 g^{-1} V = \omega^{-1} V \sim g_1 V_j$.

We now further divide into subcases, according to how V and $g_1 V_j$ are related:

 * If $V \pitchfork g_1 V_j$ or $g_1 V_j \sqsubsetneq V$, then $\omega^{-1}V \pitchfork g_1 V_j$ or $g_1 V_j \sqsubsetneq \omega^{-1}V$. In this case, we have that $\pi_V(a_n)$ uniformly coarsely coincides with $\rho_V^{g_1 V_j}$. Similarly, $\pi_{\omega^{-1}V}(a_n)$ uniformly coarsely coincides with $\rho_{\omega^{-1}V}^{g_1 V_j}$. So, $\pi_V(\omega a_n)$ uniformly coarsely coincides with $\rho_V^{\omega g_1 V_j}$. Now, $\omega g_1 V_j = g g_1^{-1} g_1 V_j = g V_j = g_1 V_j$, so $\pi_V(a_n)$ and $\pi_V(\omega a_n)$ uniformly coarsely coincide.

 * If $V \sqsubseteq g_1 V_j$ or $V \perp g_1 V_j$, then $\omega^{-1}V \sqsubseteq g_1 V_j$ or $\omega^{-1}V \perp g_1 V_j$. By the definition of the gate, $\pi_V(a_n)$ is $10E$–close to $\pi_V(\gamma(n_1))$ and $\pi_{\omega^{-1}V}(a_n)$ is $10E$–close to $\pi_{\omega^{-1}V}(\gamma(n_1))$. So, $\pi_V(\omega a_n)$ is $10E$–close to $\pi_V(\omega\gamma(n_1))$. Now, $\rho_{U_i}^V$ uniformly coarsely coincides with $\rho_{U_i}^{V_j}$, which is $200(D+E)$–close to $\gamma(n)$, and hence $\mathsf{d}_{U_i}(\rho_{U_i}^V, \gamma(n_1)) > 10^8(D+E)$. So, since
 $$\mathsf{d}_{U_i}(\omega\gamma(n_1), \gamma(n_1)) \le 200(D+E)$$
 (by the corresponding bounds on how far g and g_1 move $\pi_{U_i}(\gamma(n_1))$ and the triangle inequality), we can apply consistency and bounded geodesic image to bound $\mathsf{d}_V(\omega\gamma(n_1), \gamma(n_1))$. This gives the required uniform bound on $\mathsf{d}_V(a_n, \omega a_n)$.

Thus, if $\mathcal{V} \neq \varnothing$, we have proved item (ii). This completes the proof of the claim. ◁

Claim 11 *The group $\widehat{H}$ is virtually nilpotent.*

Proof Fix $r \geq 0$. Let $\mathcal{B}$ be a ball of radius r in $(\widehat{H}, \mathrm{d}_{\widehat{H}})$. Then $\beta(\mathcal{B})$ is contained in a ball of radius $Cr + C$ in $\mathbb{R}^k$, by Claim 8. There exist r_0 and C_0, depending only on k, such that each point in $\beta(\mathcal{B})$ lies at distance at most r_0 from one of at most $C_0(Cr + C)^k$ points in $\mathbb{Z}^k \subset \mathbb{R}^k$.

Let $t \in \mathbb{Z}^k$ be such a lattice point and suppose that $g_1, \dots, g_s \in \widehat{H}$ satisfy $\|\beta(g_j) - t\|_1 \leq r_0$ for all $j \leq s$, so $\|\beta(g_j) - \beta(g_1)\|_1 \leq 2r_0$ for all j. Then

$$\|\beta(g_j g_1^{-1}) - (\beta(g_j) - \beta(g_1)) - (\beta(g_1) + \beta(g_1^{-1}))\|_1 \leq 16Ek,$$

ie $\|\beta(g_j g_1^{-1})\|_1 \leq 2r_0 + 32Ek$. Hence $s \leq L \cdot I(2r_0 + 32Ek)$, by Claim 10, from which we get $|\mathcal{B}| \leq L \cdot I(2r_0 + 32Ek) \cdot C_0 \cdot (Cr + C)^k$. So $\widehat{H}$ has polynomial growth, and is thus virtually nilpotent [7]. ◁

Conclusion For each i, let $\beta_i' \colon \widehat{H} \to \mathbb{R}$ be $\beta_i'(g) = \lim_{n \to \infty} \beta_i(g)/n$. Then, for each $g \in \widehat{H}$ that is loxodromic on some K_{U_i}, we have $\beta_i'(g) > 0$ (see eg [10, Proposition 4.9]). Moreover, for each infinite-order $g \in \widehat{H}$, we have by Theorem 3.1 that g is loxodromic on each element of $\mathrm{Big}(g)$. Now, since β restricts to a proper map on $\langle g \rangle$, we have that $\langle g \rangle$ has unbounded orbits on some U_i with $i \leq k$, and thus $\beta_i'(g) > 0$.

Let $\ddot{H}$ be a finite-index nilpotent subgroup of $\widehat{H}$ as provided by Claim 11. By passing if necessary to a further finite-index subgroup, we can assume that $\ddot{H}$ is torsion-free (eg [5, Corollary 13.81]). If $\ddot{H}$ is abelian, we are done. Otherwise, by eg [5, Lemma 14.15], $\ddot{H}$ contains an element g such that $\langle g \rangle$ is distorted in $\ddot{H}$. It follows that $\beta_i'(g) = 0$. This contradicts the above discussion. Hence $\ddot{H}$ is free abelian; let ℓ be its rank. Since the growth of $\ddot{H}$ was bounded in Claim 11 by a polynomial of degree k, we have that $\ell \leq k$. Hence H is virtually $\mathbb{Z}^\ell$ for some ℓ bounded by the complexity of $\mathfrak{S}$. □

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