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Hyperbolicity and cubulability are preserved under elementary equivalence

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The following properties are preserved under elementary equivalence, among finitely generated groups: being hyperbolic (possibly with torsion), being hyperbolic and cubulable, and being a subgroup of a hyperbolic group. In other words, if a finitely generated group G has the same first-order theory as a group possessing one of the previous properties, then G enjoys that property as well.

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1 Introduction

Around 1945, Tarski asked whether all nonabelian free groups have the same first-order theory. This famous question remained open for six decades and was finally answered in the affirmative by Sela [26] and by Kharlampovich and Myasnikov [10]. Sela [27] then generalized his work and classified all finitely generated groups with the same first-order theory as a given torsion-free hyperbolic group. A consequence of Sela's classification is the following striking theorem; see [27, Theorem 7.10].

Theorem 1.1 (Sela) *A finitely generated group with the same first-order theory as a torsion-free hyperbolic group is itself torsion-free hyperbolic.*

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This result is particularly remarkable in view of the fact that hyperbolicity is defined in a purely geometric way. It is natural to ask whether Theorem 1.1 remains true if we allow torsion. We answer this question affirmatively.

Theorem 1.2 *A finitely generated group with the same first-order theory as a hyperbolic group is itself hyperbolic.*

In fact, the previous theorem is still valid if one only assumes that the groups satisfy the same ∀∃–sentences, that is, the same sentences of the form $\forall \boldsymbol{x}\, \exists \boldsymbol{y}\; \psi(\boldsymbol{x}, \boldsymbol{y})$, where $\boldsymbol{x}$ and $\boldsymbol{y}$ are two tuples of variables, and ψ is a quantifier-free formula in these variables. The set of such sentences satisfied by a group G is called the ∀∃ theory of G, denoted by $\mathrm{Th}_{\forall\exists}(G)$. Similarly, we denote by $\mathrm{Th}_{\forall}(G)$ the universal theory of G, ie the set of first-order sentences of the form $\forall \boldsymbol{x}\; \psi(\boldsymbol{x})$ that are true in G.

It is worth noting that, in Theorems 1.1 and 1.2, it is not sufficient to assume that the groups have the same universal theory. For instance, the class of finitely generated groups with the same universal theory as the free group F_2 coincides with the well-known class of nonabelian limit groups (see Sela [25]), also known as finitely generated fully residually free nonabelian groups (see Remeslennikov [21]), and this class contains some nonhyperbolic groups, such as $F_2 * \mathbb{Z}^2$. Note also that it is impossible to remove the assumption of finite generation. For example, Szmielew proved in [29] that $\mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Q}$ have the same first-order theory. Moreover, every hyperbolic group has the same first-order theory as groups of arbitrarily large cardinalities, by the Löwenheim–Skolem theorem.

We also show that the property of being a subgroup of a hyperbolic group is preserved under elementary equivalence, among finitely generated groups. More precisely, we prove the following theorem.

Theorem 1.3 *Let Γ be a group, possibly not finitely generated, that can be embedded into a hyperbolic group, and let G be a finitely generated group. If $\mathrm{Th}_{\forall\exists}(\Gamma) \subset \mathrm{Th}_{\forall\exists}(G)$, then G embeds into a hyperbolic group.*

A hyperbolic group is said to be *locally hyperbolic* if its finitely generated subgroups are hyperbolic. The class of locally hyperbolic groups is rich and includes, for instance, virtually free groups, fundamental groups of compact hyperbolic manifolds of dimension 2 or 3, lots of small cancellation groups (see McCammond and Wise [13; 14]), and lots of one-relator groups. For example, if S is a finite set and w is a nontrivial element in the free group $F(S)$, there exists an integer r_0 such that for every integer $r \geq r_0$, the group $G = \langle S \mid w^r = 1 \rangle$ is locally hyperbolic and cubulable; more precisely, G is hyperbolic for $r \geq 2$, cubulable for $r \geq 4$ and locally hyperbolic for $r \geq 3|w|_S$; see [13].

In Theorem 1.3, in the case where the group Γ embeds into a finitely generated locally hyperbolic group, we prove that G embeds into a locally hyperbolic group as well. In this case, the group G is itself hyperbolic.

Theorem 1.4 *Let Γ be a group, possibly not finitely generated, that can be embedded into a finitely generated locally hyperbolic group, and let G be a finitely generated group. If $\mathrm{Th}_{\forall\exists}(\Gamma) \subset \mathrm{Th}_{\forall\exists}(G)$, then G is hyperbolic.*

Recall that a group is called cubulable if it admits a proper and cocompact action by isometries on a locally finite CAT(0) cube complex (see Sageev [22] for an introduction). Groups that are both hyperbolic and cubulable have remarkable properties; they play a crucial role in the proof of the virtually Haken conjecture (see Agol [1]). By using results of Agol, Haglund and Wise, we prove the following result.

Theorem 1.5 *Let Γ be a hyperbolic group and G a finitely generated group. Suppose that $\mathrm{Th}_{\forall\exists}(\Gamma) = \mathrm{Th}_{\forall\exists}(G)$. Then Γ is cubulable if and only if G is cubulable.*

Strategy We now describe the strategy of proof of Theorems 1.2 and 1.3. The proofs of these results rely crucially on the *shortening argument* proved by Sela [27] for torsion-free hyperbolic groups and subsequently generalized by Reinfeldt and Weidmann [20] to hyperbolic groups with torsion.

Theorem 1.6 (Sela; Reinfeldt and Weidmann) *Let Γ be a hyperbolic group, and let G be a one-ended finitely generated group. There exists a finite set $F \subset G \setminus \{1\}$ such that for every noninjective homomorphism $\phi \in \mathrm{Hom}(G, \Gamma)$, there exists an automorphism σ of G such that $\ker(\phi \circ \sigma) \cap F \neq \varnothing$.*

We refer the reader to Theorem 2.2 for a more general version of the result stated above.

First, we shall outline a proof of Theorems 1.2 and 1.3 in the particular case where G is one-ended and finitely presented, and $\mathrm{Out}(G)$ is trivial.

Claim 1.7 *Let Γ be a hyperbolic group, and let G be a one-ended finitely presented group. Suppose that $\mathrm{Th}_{\forall}(\Gamma) \subset \mathrm{Th}_{\forall}(G)$. If $\mathrm{Out}(G)$ is trivial, then G embeds into Γ. Suppose moreover that Γ is one-ended and $\mathrm{Out}(\Gamma)$ is trivial. Then Γ and G are isomorphic.*

Proof of Claim 1.7 Let $G = \langle s_1, \dots, s_n \mid \Sigma(s_1, \dots, s_n) = 1 \rangle$ be a finite presentation of G, where Σ stands for a finite system of equations in n variables. Note that there is a one-to-one correspondence between $\mathrm{Hom}(G, \Gamma)$ and the set of n–tuples $(\gamma_1, \dots, \gamma_n) \in \Gamma^n$ such that $\Sigma(\gamma_1, \dots, \gamma_n) = 1$. Let $F = \{g_1, \dots, g_p\}$ be the finite set given by Theorem 1.6. Every element g_k can be written as a word $w_k(s_1, \dots, s_n)$. Assume towards a contradiction that G does not embed into Γ. Then every homomorphism from G to Γ is noninjective and thus kills an element of F by Theorem 1.6. In other words, the group Γ satisfies the following first-order sentence:

$$\forall x_1 \dots \forall x_n \;\; (\Sigma(x_1, \dots, x_n) = 1) \Rightarrow ((w_1(x_1, \dots, x_n) = 1) \vee \cdots \vee (w_k(x_1, \dots, x_n) = 1)).$$

Since $\mathrm{Th}_{\forall}(\Gamma) \subset \mathrm{Th}_{\forall}(G)$, this sentence is true in G also. Taking $x_1 = s_1, \dots, x_n = s_n$, the previous sentence means that an element of F is trivial. This is a contradiction. Thus, G embeds into Γ.

At this stage, we cannot conclude that G is hyperbolic. However, if Γ is one-ended and $\mathrm{Out}(\Gamma)$ is trivial, then we can prove in the same way as in the previous paragraph that Γ embeds into G by observing that Theorem 1.6 is still valid for subgroups of hyperbolic groups (see Corollary 2.4). Since one-ended hyperbolic groups are co-Hopfian (see Sela [24] for torsion-free hyperbolic groups and Moioli [15], based on Delzant [6], for hyperbolic groups with torsion), the groups Γ and G are isomorphic. □

In the proof of the claim, the hypothesis that G is finitely presented can be replaced by the hypothesis that G is finitely generated, using the fact that hyperbolic groups are equationally noetherian [27; 20], meaning that every infinite system of equations in finitely many variables Σ is equivalent to a finite subsystem of Σ.

Note that the proof of Claim 1.7 adapts easily to the case where $\mathrm{Out}(G)$ is finite. If $\mathrm{Out}(G)$ is infinite, the proof does not work, because we cannot fully express Theorem 1.6 by means of first-order logic as we did in the proof of the claim, since the expressive power of first-order logic is not sufficient to describe precomposition by an automorphism. However, by employing a strategy comparable to that used by Perin [19], one can express some fragments of Theorem 1.6 and establish the following dichotomy.

Proposition 1.8 *Let Γ be a hyperbolic group and G a one-ended finitely generated group such that $\mathrm{Th}_{\forall\exists}(\Gamma) \subset \mathrm{Th}_{\forall\exists}(G)$. Then either*

- *G embeds into Γ, or*
- *G splits as a particular graph of groups called a **quasifloor** over a group G_1* (see Definition 5.2).

Quasifloors generalize *hyperbolic floors* of *hyperbolic towers* in the sense of Sela and Perin (see Section 5 for further details). Without torsion, a quasifloor is equivalent to a hyperbolic floor (see Proposition 5.5).

By using Proposition 1.8, we shall sketch a proof of (a particular case of) Theorem 1.3. Let Γ be a hyperbolic group, and let G be a one-ended finitely generated group such that $\mathrm{Th}_{\forall\exists}(\Gamma) \subset \mathrm{Th}_{\forall\exists}(G)$. We aim to prove that G can be embedded into a hyperbolic group.

For now, suppose that the group G is torsion-free. If the second alternative of the dichotomy in Proposition 1.8 holds, then G is a hyperbolic floor over a group G_1. In particular, G_1 is a proper subgroup of G, and there exists a retraction from G onto G_1. Moreover, G_1 satisfies the following key property, which is an easy consequence of the combination theorem of Bestvina and Feighn [4]: if G_1 embeds into Γ, then G embeds into a hyperbolic group. In order to prove that G embeds into a hyperbolic group, one wants to iterate the previous argument with G_1 instead of G. However, $\mathrm{Th}_{\forall\exists}(\Gamma)$ is not contained in $\mathrm{Th}_{\forall\exists}(G_1)$ and G_1 is not one-ended in general. We can nevertheless refine Proposition 1.8 and prove that G_1 satisfies the same dichotomy as G. By iterating this argument, we get a sequence of groups $(G_n)_{n\in\mathbb{N}}$ such that for every integer n, the group G_n is a hyperbolic floor over G_{n+1}. In particular, G_{n+1} is a proper subgroup of G_n, and G_n retracts onto G_{n+1}. It follows from the descending chain condition for Γ–limit groups, Theorem 2.8, that this sequence of groups is necessarily finite. Note that the descending chain condition applies since G_{n+1} is a quotient of G_n and $\mathrm{Th}_{\exists}(G_n)$ is contained in $\mathrm{Th}_{\exists}(\Gamma)$ (indeed, $\mathrm{Th}_{\exists}(G_n)$ is contained in $\mathrm{Th}_{\exists}(G)$ because G_n is a subgroup of G, and $\mathrm{Th}_{\exists}(G)$ is contained in $\mathrm{Th}_{\exists}(\Gamma)$ since $\mathrm{Th}_{\forall\exists}(\Gamma) \subset \mathrm{Th}_{\forall\exists}(G)$). Let G_m denote the last group of the sequence. The group G is a hyperbolic tower over G_m since, for every integer n, the group G_n is a hyperbolic floor over G_{n+1}. As a consequence of the dichotomy, G_m embeds into the hyperbolic group Γ. It follows that G embeds into a hyperbolic group. This concludes the proof of Theorem 1.3 in the torsion-free case (assuming G is one-ended).

Unfortunately, quasifloors do not have the same good features as hyperbolic floors. For every integer n, the group G_n (obtained as above by iterating the dichotomy of Proposition 1.8) splits over finite groups as a graph of groups all of whose vertex groups embed into G, but G_n is not a subgroup of G in general. Consequently, $\mathrm{Th}_{\exists}(G_n)$ is not contained in $\mathrm{Th}_{\exists}(\Gamma)$ a priori, as shown by Example 1.9 below. Moreover, G_{n+1} is not a quotient of G_n. These pathologies make more complicated the termination of the iterative construction described in the previous paragraph, since we cannot use the descending condition directly. We solve part of these problems by proving that all hyperbolic groups can be embedded into torsion-saturated hyperbolic groups (see Theorem 1.10), a class of groups we introduce in Section 4.

A new phenomenon arising from the presence of torsion In the case of torsion-free groups, performing an HNN extension over a finite group is the same as doing a free product with $\mathbb{Z}$, and every torsion-free nonelementary hyperbolic group G has the same universal theory as $G * \mathbb{Z}$ (see Proposition 2.21), and even the same first-order theory, as a consequence of the work of Sela [27]. By contrast, in the presence of torsion, performing an HNN extension over a finite, even trivial, subgroup may modify the universal theory of a hyperbolic group. Let us consider the following simple example.

Example 1.9 Let $G = F_2 \times \mathbb{Z}/2\mathbb{Z}$. Then the sentence $\forall x\,\forall y\,(x^2 = 1) \Rightarrow (xy = yx)$ is satisfied by G, but not by $G * \mathbb{Z}$.

This example shows that, in general, the class of groups with the same universal theory as a given hyperbolic group with torsion is not closed under HNN extensions and amalgamated products over finite groups. In Section 4, we deal with this problem by proving the following result, which is crucial for proving that the iterative construction outlined above eventually terminates.

Theorem 1.10 *Every hyperbolic group Γ can be embedded into a torsion-saturated hyperbolic group $\overline{\Gamma}$, ie a hyperbolic group such that the class of $\overline{\Gamma}$–limit groups is closed under amalgamated free products and HNN extensions over finite groups.*

The hyperbolic group $\overline{\Gamma}$ is obtained from Γ by performing finitely many HNN extensions over finite groups (see Theorem 4.8).

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2 Preliminaries

In this section we recall some facts and definitions about the elementary theory of groups, Γ–limit groups, K–CSA groups and JSJ decompositions and prove some results that are useful in the sequel, and whose proofs are independent of the main body of the paper.

2.1 Notation

Let $n \geq 1$ be an integer. We use the notation $[\![1,n]\!]$ to denote the set of integers k such that $1 \leq k \leq n$.

Let G be a group and g an element of G. We write $\mathrm{ad}(g)$ for the inner automorphism $x \in G \mapsto gxg^{-1}$.

2.2 The elementary theory of groups

For detailed background, we refer the reader to [12]. A *first-order formula* in the language of groups is a finite formula using the following symbols: $\forall$, $\exists$, $=$, $\wedge$, $\vee$, $\Rightarrow$, $\neq$, 1 (standing for the identity element), $^{-1}$ (standing for the inverse), $\cdot$ (standing for the group multiplication) and variables $x, y, g, z, \ldots$ which are to be interpreted as elements of a group. A variable is *free* if it is not bound by any quantifier $\forall$ or $\exists$. A *first-order sentence* is a first-order formula without free variables.

Given a formula $\psi(x_1, \ldots, x_n)$ with $n \geq 0$ free variables, and n elements $g_1, \ldots, g_n$ of a group G, we say that $\psi(g_1, \ldots, g_n)$ is satisfied by G if its interpretation is true in G. This is denoted by $G \models \psi(g_1, \ldots, g_n)$. The *elementary theory* of a group G, denoted by $\mathrm{Th}(G)$, is the collection of all sentences which are true in G. The *universal-existential* theory of G, denoted by $\mathrm{Th}_{\forall\exists}(G)$, is the collection of sentences true in G of the form

$$\forall x_1 \ldots \forall x_m \ \exists y_1 \ldots \exists y_n \ \ \psi(x_1, \ldots, x_m, y_1, \ldots, y_n),$$

where $m, n \geq 1$ and ψ is a quantifier-free formula with $m+n$ free variables. In the same way, we define the *universal theory* of G, denoted by $\mathrm{Th}_{\forall}(G)$, and its *existential theory* $\mathrm{Th}_{\exists}(G)$. Note that $\mathrm{Th}_{\exists}(G)$ is contained in $\mathrm{Th}_{\exists}(\Gamma)$ if and only if $\mathrm{Th}_{\forall}(\Gamma)$ is contained in $\mathrm{Th}_{\forall}(G)$.

2.3 Γ–limit groups

Let Γ and G be two groups. We say that G is *fully residually* Γ if, for every finite subset $F \subset G$, there exists a homomorphism $\phi\colon G \to \Gamma$ whose restriction to F is injective. If G is countable, then G is fully residually Γ if and only if there exists a sequence $(\phi_n)_{n\in\mathbb{N}}$ of homomorphisms from G to Γ such that for every nontrivial element $g \in G$, its image $\phi_n(g)$ is nontrivial for every n large enough. Such a sequence is called a *discriminating sequence.*

Sela [27] introduced Γ–*limit groups* in order to study $\mathrm{Hom}(G,\Gamma)$, where Γ stands for a torsion-free hyperbolic group, and G stands for a finitely generated group. Sela proved that the class of Γ–limit groups coincides with the class of finitely generated groups that are fully residually Γ [27, Proposition 1.18]. Reinfeldt and Weidmann [20] generalized this result in the case where Γ is hyperbolic, possibly with torsion.

The following easy proposition builds a bridge between group theory and first-order logic. Recall that a group Γ is *equationally noetherian* if every infinite system of equations in uniformly finitely many variables Σ is equivalent to a finite subsystem of Σ.

Proposition 2.1 *Let G and Γ be finitely generated groups. Suppose that G is finitely presented or that Γ is equationally noetherian. Then G is a Γ–limit group if and only if* $\mathrm{Th}_{\exists}(G) \subset \mathrm{Th}_{\exists}(\Gamma)$.

Proof Suppose that G is fully residually Γ. Let $(\phi_n\colon G \to \Gamma)_{n\in\mathbb{N}}$ be a discriminating sequence. Let $\Sigma(x_1,\dots,x_m) = 1 \wedge \psi(x_1,\dots,x_m) \neq 1$ be a system of equations and inequations. Suppose that this system has a solution $(g_1,\dots,g_m)$ in G^m. Then, for n large enough, $\psi(\phi_n(g_1),\dots,\phi_n(g_m)) = \phi_n(\psi(g_1,\dots,g_m))$ is nontrivial. Thus, $(\phi_n(g_1),\dots,\phi_n(g_m))$ is a solution of the previous system in Γ^m.

Conversely, suppose that $\mathrm{Th}_{\exists}(G) \subset \mathrm{Th}_{\exists}(\Gamma)$. Let $G = \langle s_1,\dots,s_k \mid \Sigma(s_1,\dots,s_k) = 1\rangle$ be a presentation of G, where Σ stands for a system of equations in k variables, possibly infinite. If G is finitely presentable, one can suppose that the system Σ is finite. If Γ is equationally noetherian, Σ is equivalent in Γ to a finite subsystem of Σ, so one can assume without loss of generality that Σ is finite. Let $S = \{g_1,\dots,g_r\}$ be a finite subset of $G \setminus \{1\}$. Every element g_i can be written as a word $w_i(s_1,\dots,s_k)$. We shall find a homomorphism $\phi\colon G \to \Gamma$ such that $\ker(\phi) \cap S = \varnothing$. Observe that the following sentence, θ, is satisfied by G, since the identity of G kills no element of S:

$$\exists x_1 \dots \exists x_k \;\; \Sigma(x_1,\dots,x_k) = 1 \wedge \textstyle\bigwedge_{1\le i\le r} w_i(x_1,\dots,x_k) \neq 1.$$

Since $\mathrm{Th}_{\exists}(G) \subset \mathrm{Th}_{\exists}(\Gamma)$, this sentence is satisfied by Γ as well. Let $\gamma_1,\dots,\gamma_k$ be the elements of Γ given by the interpretation of the sentence θ in Γ. One defines a homomorphism $\phi\colon G \to \Gamma$ by sending s_i to γ_i. This homomorphism satisfies $\phi(g_i) = w_i(\gamma_1,\dots,\gamma_k) \neq 1$, for every $1 \le i \le r$. This concludes the proof. □

The proofs of our main results rely essentially on the *shortening argument*, proved by Sela for torsion-free hyperbolic groups in [27] and later generalized by Reinfeldt and Weidmann to the case of hyperbolic groups possibly with torsion in [20].

Given a hyperbolic group Γ and a Γ–limit group G, the *modular group* $\mathrm{Mod}(G)$ is a subgroup of $\mathrm{Aut}(G)$ that will be defined in Section 2.6.5, by means of the JSJ decomposition of G over virtually cyclic groups (with infinite center).

Theorem 2.2 [27; 20, Theorem 4.2] *Let Γ be a hyperbolic group and let G be a one-ended finitely generated group. There exist nontrivial elements $g_1, \dots, g_k$ of G such that for every noninjective homomorphism $\phi\colon G \to \Gamma$, there exist a modular automorphism $\sigma \in \mathrm{Mod}(G)$ and an integer $1 \leq \ell \leq k$ such that $(\phi \circ \sigma)(g_\ell) = 1$.*

Remark 2.3 In the case where G is not a Γ–limit group, the result is obvious.

One easily sees that Theorem 2.2 remains true if one only assumes that Γ is a subgroup of a hyperbolic group.

Corollary 2.4 *Let Γ be a group that embeds into a hyperbolic group, and let G be a one-ended finitely generated group. There exist nontrivial elements $g_1, \dots, g_k$ of G such that for every noninjective homomorphism $\phi\colon G \to \Gamma$, there exist a modular automorphism $\sigma \in \mathrm{Mod}(G)$ and an integer $1 \leq \ell \leq k$ such that $(\phi \circ \sigma)(g_\ell) = 1$.*

The following result is not explicitly stated by Reinfeldt and Weidmann in [20], but can be derived from their arguments.

Theorem 2.5 (Sela, Reinfeldt–Weidmann) *Let Γ be a hyperbolic group and let G be a one-ended finitely generated group. Suppose that G embeds into Γ. Then there exists a finite set $\{\phi_1, \dots, \phi_\ell\}$ of monomorphisms from G into Γ such that for every monomorphism $\phi\colon G \to \Gamma$, there exist a modular automorphism $\sigma \in \mathrm{Mod}(G)$, an integer $1 \leq \ell \leq k$ and an element $\gamma \in \Gamma$ such that*

$$\phi = \mathrm{ad}(\gamma) \circ \phi_\ell \circ \sigma.$$

Remark 2.6 In contrast with Theorem 2.2, Theorem 2.5 does not extend to the case where Γ is a subgroup of a hyperbolic group.

Theorem 2.7 [27; 20, Corollary 6.13] *Hyperbolic groups are equationally noetherian.*

The following result is known as the *descending chain condition* for Γ–limit groups, with Γ hyperbolic. It is an easy consequence of the equational noetherianity of hyperbolic groups.

Theorem 2.8 [27; 20, Corollary 5.3] *Let Γ be a hyperbolic group. Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of Γ–limit groups. If $(\phi_n\colon G_n \twoheadrightarrow G_{n+1})_{n\in\mathbb{N}}$ is a sequence of epimorphisms, then ϕ_n is an isomorphism for n sufficiently large.*

Theorem 2.9 [27; 20, Proposition 6.2] *Let Γ be a hyperbolic group and G a Γ–limit group. Then every abelian subgroup of G is finitely generated.*

2.4 K–CSA groups

A group is said to be CSA if its maximal abelian subgroups are malnormal. It is well known that every torsion-free hyperbolic group is CSA. However, it is not true anymore in the presence of torsion. To overcome this problem, Guirardel and Levitt defined K–CSA groups in [7, Definition 9.7].

Definition 2.10 Let $K>0$. A group G is called a K–CSA group if the following conditions hold:

- Every finite subgroup of G has order bounded by above by K (hence, an element g has infinite order if and only if $g^{K!}\neq e$).
- Every element g of infinite order is contained in a unique maximal virtually abelian subgroup of G, denoted by $M(g)$. Moreover $M(g)$ is K–virtually torsion-free abelian (ie $M(g)$ has a torsion-free abelian subgroup of index less than K).
- $M(g)$ is equal to its normalizer.

We recall some useful facts about K–CSA groups.

Proposition 2.11 *Every hyperbolic group is K–CSA for some $K>0$.*

Proposition 2.12 [7, Lemma 9.8] *Let G be a K–CSA group.*

(1) *If $g,h\in G$ have infinite order, the following conditions are equivalent:*
   (a) $M(g)=M(h)$.
   (b) $g^{K!}$ *and* $h^{K!}$ *commute.*
   (c) $\langle g,h\rangle$ *is virtually abelian.*

(2) *Let H be an infinite virtually abelian subgroup of G. Then H is contained in a unique maximal virtually abelian subgroup of G, denoted by $M(H)$. This group is almost malnormal: if $gM(H)g^{-1}\cap M(H)$ is infinite, then g belongs to $M(H)$. Moreover, for every element h of H of infinite order, $M(H)=M(h)$.*

Proposition 2.13 *Let G be a K–CSA group and g an element of G of infinite order. The subgroup $M(g)$ is definable without quantifiers with respect to g. In other words, there exists a first-order formula $\psi_K(x,y)$ without quantifiers such that*

$$M(g)=\{h\in G\mid \psi_K(h,g)\}.$$

Proof First, note that

$$M(g) = \{h \in G \mid \langle g, h\rangle \text{ is } K\text{–virtually torsion-free abelian}\}.$$

Indeed, if $\langle g, h\rangle$ is K–virtually abelian, then $\langle g, h\rangle \subset M(g)$ by maximality of $M(g)$. Conversely, if $h \in M(g)$ then $\langle g, h\rangle$ is a subgroup of $M(g)$, which is K–virtually torsion-free abelian, so $\langle g, h\rangle$ is K–virtually torsion-free abelian. We now prove that there exists a first-order formula $\psi_K(x, y)$ with two free variables such that $\langle g, h\rangle$ is K–virtually torsion-free abelian if and only if $\psi_K(g, h)$ is true in G. Let $\pi\colon F_2 = \langle x, y\rangle \to G$ be the epimorphism sending x to g and y to h. If A is a subgroup of $\langle g, h\rangle$ of index less than K, there exists a subgroup B of $\langle x, y\rangle$ of index less than K such that $A = \pi(B)$. Denote by $H_1, \dots, H_n$ the n subgroups of F_2 of index $\leq K$. For each $1 \leq i \leq n$, let $(w_{i,j}(x, y))_{1\leq j\leq n_i}$ be a finite generating set of H_i. We can define $\psi_K(g, h)$ by

$$\psi_K(g, h) = \textstyle\bigvee_{i=1}^{n} \bigwedge_{k=1}^{n_i} \bigwedge_{\ell=1}^{n_i} [w_{i,k}(g, h), w_{i,\ell}(g, h)] = 1. \qquad \square$$

Proposition 2.14 [7, Proposition 9.9] *The property K–CSA is defined by a set of universal formulas.*

Since every hyperbolic group is K–CSA for some $K > 0$ (see Proposition 2.11), the following proposition holds.

Proposition 2.15 [7, Corollary 9.10] *Let Ω be a hyperbolic group. There exists a constant $K > 0$ such that every group satisfying $\mathrm{Th}_{\exists}(G) \subset \mathrm{Th}_{\exists}(\Omega)$ is K–CSA.*

We now prove that a group with the same $\forall\exists$–theory as a subgroup of a hyperbolic group does not contain $\mathbb{Z}^2$ as a subgroup (see Corollary 2.18 below). First, let us recall the following well-known result.

Lemma 2.16 *There exists a $\forall\exists$–sentence ψ such that if G is a finitely generated torsion-free abelian group, $G \models \psi$ if and only if G is cyclic.*

Proof Since $\mathbb{Z}^n/2\mathbb{Z}^n$ has 2^n elements, the pigeonhole principle shows that the following $\forall\exists$–sentence is satisfied by $\mathbb{Z}^n$ if and only if $n \leq 1$:

$$\forall x_1\ \forall x_2\ \forall x_3\ \exists x_4\ \ (x_1 = x_2x_4^2) \vee (x_1 = x_3x_4^2) \vee (x_2 = x_3x_4^2). \qquad \square$$

Proposition 2.17 *Let Ω be a hyperbolic group, let Γ be a subgroup of Ω, and let G be a finitely generated group such that $\mathrm{Th}_{\forall\exists}(\Gamma) \subset \mathrm{Th}_{\forall\exists}(G)$. If g is an element of G of infinite order, then $M(g)$ is virtually cyclic.*

Proof First, note that $\mathrm{Th}_{\exists}(G) \subset \mathrm{Th}_{\exists}(\Gamma) \subset \mathrm{Th}_{\exists}(\Omega)$. By Proposition 2.15, the groups Ω, Γ and G are K–CSA for some K. Let g be an element of G of infinite order. Since the group $M(g)$ is K–virtually torsion-free abelian, it has a normal torsion-free abelian subgroup N of index dividing $K!$. For every element h of $M(g)$ the element $h^{K!}$ belongs to N. Denote by $N(g)$ the subgroup of $M(g)$ generated by $\{h^{K!} \mid h \in M(g)\}$. This is a subgroup of N, so it is torsion-free abelian.

It is enough to show that $N(g)$ is cyclic, because then we will be able to conclude that $M(g)$ is virtually cyclic. Indeed, if $N(g)$ is cyclic, then so is $\langle x^{K!}, y^{K!}\rangle$ for every $x, y \in M(g)$. As a consequence there is no pair of elements of $M(g)$ generating a subgroup isomorphic to $\mathbb{Z}^2$. Since $M(g)$ is virtually abelian and finitely generated by Theorem 2.9, it is virtually cyclic.

For every integer $\ell \geq 1$, let $N_\ell(g) = \{h_1^{K!} \cdots h_\ell^{K!} \mid h_1, \ldots, h_\ell \in M(g)\}$. Since $N(g)$ is finitely generated, there exist an integer r and some elements $g_1, \ldots, g_r$ of $M(g)$ such that $N(g)$ is generated by $\{g_1^{K!}, \ldots, g_r^{K!}\}$. We claim that $N(g) = N_r(g)$. In order to see this, note that, since $N(g)$ is abelian, every element $h \in N(g)$ can be written

$$h = (g_1^{K!})^{n_1} \cdots (g_r^{K!})^{n_r} = (g_1^{n_1})^{K!} \cdots (g_r^{n_r})^{K!},$$

where $n_1, \ldots, n_r$ lie in $\mathbb{Z}$. This proves that $N(g) \subset N_r(g)$, and the reverse inclusion is immediate. Then, recall that there exists a first-order formula without quantifiers $\psi(x, y)$ such that $M(g) = \{h \in G \mid \psi(h, g)\}$ (see Proposition 2.13). Hence

$$N_r(g) = \bigl\{h \in G \mid \exists h_1 \ldots \exists h_r \; (h = h_1^{K!} \cdots h_r^{K!} \wedge \psi(h_1, g) \wedge \ldots \wedge \psi(h_r, g))\bigr\}.$$

It remains to prove that $N_r(g)$ is cyclic. Recall that, by Lemma 2.16, a finitely generated torsion-free abelian group is cyclic if and only if it satisfies

$$\forall x_1 \; \forall x_2 \; \forall x_3 \; \exists x_4 \; \theta(x_1, x_2, x_3, x_4),$$

where $\theta(x_1, x_2, x_3, x_4)$ is $(x_1 = x_2 x_4^2) \vee (x_1 = x_3 x_4^2) \vee (x_2 = x_3 x_4^2)$. Since Ω is a hyperbolic group and $\Gamma \subset \Omega$, every torsion-free abelian subgroup of Γ is cyclic and thus satisfies the previous sentence. We can write a $\forall\exists$–sentence χ_r satisfied by Γ, with this interpretation: for every element γ of Γ of infinite order, $N_r(\gamma)$ is cyclic. Below is the sentence χ_r, where $\boldsymbol{h}_i$ stands for $(h_{i,1}, \ldots, h_{i,r})$ and $x_i := h_{i,1}^{K!} \cdots h_{i,r}^{K!}$:

$$\forall \gamma \; \forall \boldsymbol{h}_1 \; \forall \boldsymbol{h}_2 \; \forall \boldsymbol{h}_3 \; \exists \boldsymbol{h}_4 \; \Bigl(\bigl(\textstyle\bigwedge_{i=1}^{3} \bigwedge_{j=1}^{r} \psi(h_{i,j}, \gamma) \wedge (\gamma^{K!} \neq 1)\bigr) \Rightarrow \bigl(\textstyle\bigwedge_{j=1}^{r} \psi(h_{4,j}, \gamma) \wedge \theta(x_1, x_2, x_3, x_4)\bigr)\Bigr).$$

Since $\mathrm{Th}_{\forall\exists}(\Gamma) \subset \mathrm{Th}_{\forall\exists}(G)$, the sentence χ_r is true in G as well. It follows that $N_r(g)$ is cyclic. This concludes the proof. □

Corollary 2.18 *Let Ω be a hyperbolic group, let Γ be a subgroup of Ω, and let G be a finitely generated group. Suppose that $\mathrm{Th}_{\forall\exists}(\Gamma) \subset \mathrm{Th}_{\forall\exists}(G)$. Then every abelian subgroup of G is virtually cyclic.*

Proof First, note that $\mathrm{Th}_{\exists}(G) \subset \mathrm{Th}_{\exists}(\Omega)$. By Proposition 2.15, the group G is K–CSA for some K. Let H be an infinite abelian subgroup of G. By Proposition 2.12, H is contained in a unique maximal virtually abelian subgroup of G, denoted by $M(H)$, and $M(H) = M(h)$ for every element h of H of infinite order (note that such an element exists since H is abelian, finitely generated (according to Theorem 2.9) and infinite). By Proposition 2.17, $M(h)$ is virtually cyclic, so H is virtually cyclic. □

2.5 Generalized Baumslag's lemma

If g is an element of infinite order of a hyperbolic group G, we denote by g^+ and g^- the attracting and repelling fixed points of g on the boundary $\partial_\infty G$ of G. The following proposition generalizes a criterion proved by Baumslag in the case of free groups (see [3, Proposition 1] or [2, Lemma 7]). This result is very useful to show that some sequences of homomorphisms taking values in a hyperbolic group are discriminating. A proof in the hyperbolic case can be found in [17, Lemma 2.4].

Proposition 2.19 *Let $a_0, a_1, \dots, a_m, c_1, \dots, c_m$ be elements of a hyperbolic group. Let $w(p_1, \dots, p_m) = a_0 c_1^{p_1} a_1 c_2^{p_2} a_2 \cdots a_{m-1} c_m^{p_m} a_m$ with $p_1, \dots, p_m \geq 0$. Suppose*

(1) *every c_i has infinite order;*

(2) *for every $i \in [\![1, m-1]\!]$, $c_i^- \neq a_i \cdot c_{i+1}^+$.*

Then there exists a constant C such that $w(p_1, \dots, p_m) \neq 1$ for every $p_1, \dots, p_m \geq C$.

If G is a hyperbolic group, each element g of infinite order is contained in a unique maximal virtually abelian subgroup of G, denoted by $M(g)$, namely the stabilizer of the pair of points $P = \{g^+, g^-\}$. An element $h \in G$ belongs to $M(g)$ if and only if $h(P) \cap P \neq \varnothing$. The following straightforward consequence of Proposition 2.19 is easier to use in practice.

Corollary 2.20 *Let $a_0, a_1, \dots, a_m$ and c be elements of a hyperbolic group. Let (ε_i) in $\{-1, +1\}^m$. Let $w(p) = a_0 c^{\varepsilon_1 p} a_1 c^{\varepsilon_2 p} a_2 \cdots a_{m-1} c^{\varepsilon_m p} a_m$ with $p \geq 0$. Suppose*

(1) *c has infinite order;*

(2) *for every $i \in [\![1, m-1]\!]$, $a_i \notin M(c)$.*

Then there exists a constant C such that $w(p) \neq 1$ for $p \geq C$.

Here is an interesting application of Baumslag's lemma.

Proposition 2.21 *Let G be a nonelementary torsion-free hyperbolic group. We have*

$$\mathrm{Th}_{\forall}(G*\mathbb{Z}) = \mathrm{Th}_{\forall}(G).$$

Proof Fix a finite generating set of $G*\langle t\rangle$, and let B_n denote the ball of radius n in $G*\langle t\rangle$ for this generating set. We shall use Baumslag's lemma in order to find a homomorphism ϕ_n from $G*\langle t\rangle$ to G that does not kill any nontrivial element of B_n. As a consequence, the sequence of homomorphisms $(\phi_n)_{n\in\mathbb{N}}$ will be discriminating, and hence $\mathrm{Th}_{\forall}(G*\mathbb{Z}) \subset \mathrm{Th}_{\forall}(G)$. Note that the reverse inclusion $\mathrm{Th}_{\forall}(G*\mathbb{Z}) \supset \mathrm{Th}_{\forall}(G)$ is obvious since $G \subset G*\mathbb{Z}$.

First, note that every nontrivial element x of B_n can be written as a reduced word $x = g_{1,x}t^{n_{1,x}}g_{2,x}t^{n_{2,x}}\cdots t^{n_{k_x,x}}g_{k_x+1,x}$, with $n_{i,x}\neq 0$ and $g_{i,x}\in G\setminus\{1\}$, except $g_{1,x}$ and $g_{k_x+1,x}$, which may be trivial. Let $A=\bigcup_{x\in B_n}\{g_{1,x},\dots,g_{k_x,x}\}$. Since the group G is torsion-free hyperbolic, two elements commute if and only if they fix a common point on the boundary of G. By hypothesis, G is nonelementary, so $\{g^+ \mid g\in G\setminus\{1\}\}$ is infinite. Moreover, every nontrivial element of G fixes exactly two distinct points on the boundary, and A is a finite set, so one can find an element $g\in G$ that does not fix any point on the boundary that is fixed by a nontrivial element of A. Hence, g does not commute with any nontrivial element of A. It follows from Corollary 2.20 that the homomorphism ϕ_n from $G*\langle t\rangle$ to G whose restriction to G is the identity and that sends t to a sufficiently big power of g does not kill any nontrivial element of B_n. Hence, the sequence $(\phi_n)_{n\in\mathbb{N}}$ is discriminating. □

Note that Proposition 2.21 is false if G is not assumed to be torsion-free, as illustrated by Example 1.9 in the introduction. Section 4 is devoted to this problem.

2.6 JSJ decompositions of finitely generated groups

JSJ decompositions for torsion-free hyperbolic groups were introduced by Sela in [24] (see also [5]). Constructions of JSJ decompositions were then given in more general settings by many authors. In [7], Guirardel and Levitt give a simple general definition of JSJ decompositions and explain how to construct JSJ decompositions in a wide range of contexts. We refer the reader to [7] for details.

2.6.1 Existence of JSJ decompositions In this paper, we are mainly concerned with the following JSJ decompositions:

(1) JSJ decompositions of a finitely generated group G over finite subgroups of G. The existence of these decompositions is guaranteed as soon as there exists a constant K such that every finite subgroup of G has order less than K (see [11]). However, such a decomposition is not unique in general. See Section 2.6.6 for more details.

(2) JSJ decompositions of a one-ended finitely generated group G over virtually $\mathbb{Z}$ subgroups of G. Such decompositions exist for one-ended hyperbolic groups, and more generally for one-ended finitely generated K–CSA groups that do not contain $\mathbb{Z}^2$. For a proof of the existence, we refer the reader to the first point of Theorem 9.14 of [7], which is an immediate consequence of Corollary 9.1.

(3) JSJ decompositions of a one-ended finitely generated group G over the family $\mathcal{Z}$ of virtually $\mathbb{Z}$ subgroups of G with infinite center. Such decompositions exist for one-ended hyperbolic groups, and more generally for one-ended finitely generated K–CSA groups that do not contain $\mathbb{Z}^2$. Again, the proof of the existence is an immediate consequence of Corollary 9.1 of [7].

(4) JSJ decompositions of a one-ended finitely generated K–CSA group over virtually abelian groups (see [7], first point of Theorem 9.14).

2.6.2 Uniqueness Let G be a one-ended finitely generated group as in cases (2)–(4). In [7], the authors associate to each JSJ decomposition T of G over a family of subgroups $\mathcal{A}$ a tree called the *tree of cylinders* of T, denoted by T_c (see [7, Definition 7.2]). They prove that T_c is a JSJ decomposition over $\mathcal{A}$ and that T_c does not depend on the initial JSJ decomposition T: if T' is another JSJ decomposition over $\mathcal{A}$, then $T_c = T'_c$ (meaning that there exists a G–equivariant isomorphism between them). This tree T_c is called the *canonical* JSJ decomposition of G over $\mathcal{A}$ (see the third point of [7, Lemma 7.3] for details). If we are in the third case above, we refer to the canonical decomposition as the $\mathcal{Z}$–JSJ splitting of G.

2.6.3 Description of the vertex groups Once we know that JSJ decompositions exist, the essential feature of the JSJ theory is the description of their flexible vertices, as defined below.

Definition 2.22 Let G be a finitely generated group. Let T be a JSJ decomposition of G over a family of subgroups $\mathcal{A}$. A vertex group G_v of T is said to be *rigid* if it is elliptic in every splitting of G over $\mathcal{A}$. If G_v fails to be elliptic in some splitting of G over $\mathcal{A}$, it is said to be *flexible*.

Before giving a description of flexible vertex groups of the $\mathcal{Z}$–JSJ splitting, we shall recall some basic facts about hyperbolic 2–dimensional orbifolds.

A compact connected 2–dimensional orbifold with boundary $\mathcal{O}$ is said to be *hyperbolic* if it is equipped with a hyperbolic metric with totally geodesic boundary. It is the quotient of a closed convex subset $C \subset \mathbb{H}^2$ by a proper discontinuous group of isometries $G_{\mathcal{O}} \subset \mathrm{Isom}(\mathbb{H}^2)$. We denote by $p\colon C \to \mathcal{O}$ the quotient map. By definition, the orbifold fundamental group $\pi_1(\mathcal{O})$ of $\mathcal{O}$ is $G_{\mathcal{O}}$. We may also view $\mathcal{O}$ as the quotient of a compact orientable hyperbolic surface with geodesic boundary by a finite group of isometries. A point of $\mathcal{O}$ is *singular* if its preimages in C have nontrivial stabilizer. A *mirror* is the image by p of a component of the fixed point set of an orientation-reversing element of $G_{\mathcal{O}}$ in C. Singular points not contained in mirrors are *conical points*; the stabilizer of the preimage in $\mathbb{H}^2$ of a conical point is a finite cyclic group consisting of orientation-preserving maps (rotations). The orbifold $\mathcal{O}$ is said to be *conical* if it has no mirror.

Definition 2.23 A group G is called a *finite-by-orbifold group* if it is an extension

$$1 \to F \to G \to \pi_1(\mathcal{O}) \to 1,$$

where $\mathcal{O}$ is a compact connected hyperbolic conical 2–orbifold, possibly with totally geodesic boundary, and F is an arbitrary finite group called the *fiber*. An *extended boundary subgroup* of G is the preimage in G of a boundary subgroup of the orbifold fundamental group $\pi_1(\mathcal{O})$ (for an indifferent choice of regular base point). We define in the same way *extended conical subgroups*.

Definition 2.24 A vertex v of a graph of groups is called *quadratically hanging* (QH) if its stabilizer G_v is a finite-by-orbifold group $1 \to F \to G \to \pi_1(\mathcal{O}) \to 1$ such that $\mathcal{O}$ has nonempty boundary and such that any incident edge group is finite or contained in an extended boundary subgroup of G. We also say that G_v is QH.

The following proposition is crucial (see [7, Section 6, Theorem 6.5 and the paragraph below Remark 9.29]).

Proposition 2.25 *Let G be a one-ended finitely generated K–CSA group that does not contain $\mathbb{Z}^2$. The flexible vertex groups of its $\mathcal{Z}$–JSJ splitting are QH.*

2.6.4 Properties of the $\mathcal{Z}$–JSJ decomposition Proposition 2.26 summarizes the properties of the $\mathcal{Z}$–JSJ decomposition that will be useful in the sequel.

Proposition 2.26 *Let G be a one-ended finitely generated K–CSA group that does not contain $\mathbb{Z}^2$ as a subgroup. Let T be its $\mathcal{Z}$–JSJ decomposition.*

- *The tree T is bipartite: every edge joins a vertex carrying a maximal virtually cyclic group to a vertex carrying a group that is not virtually cyclic.*
- *The action of G on T is acylindrical in the following strong sense: if an element $g \in G$ of infinite order fixes a segment of length ≥ 2 in T, then this segment has length exactly 2 and its midpoint has virtually cyclic stabilizer.*
- *Let v be a vertex of T, and let e and e' be two distinct edges incident to v. If G_v is not virtually cyclic, then the group $\langle G_e, G_{e'} \rangle$ is not virtually cyclic.*
- *There are two kinds of vertices of T carrying a group that is not virtually cyclic: rigid ones and QH ones. If v is a QH vertex of T, every edge group G_e of an edge e incident to v coincides with an extended boundary subgroup of G_v. Moreover, given any extended boundary subgroup B of G_v, there exists a unique incident edge e such that $G_e = B$.*

The following result will be constantly used in the sequel.

Proposition 2.27 *Let Ω be a hyperbolic group and Γ a subgroup of Ω. Let G be a finitely generated group and H a one-ended finitely generated subgroup of G. If $\mathrm{Th}_{\forall\exists}(\Gamma)$ is contained in $\mathrm{Th}_{\forall\exists}(G)$, then H is K–CSA for some constant $K > 0$ and does not contain $\mathbb{Z}^2$ as a subgroup. Consequently, H has a $\mathcal{Z}$–JSJ splitting.*

Proof By Proposition 2.15, the group G is K–CSA for some constant K. Moreover, by Corollary 2.18, every abelian subgroup of G is virtually cyclic. Since H is a subgroup of G, it is K–CSA and every abelian subgroup of H is virtually cyclic. Hence, H has a $\mathcal{Z}$–JSJ splitting. □

2.6.5 The modular group

Definition 2.28 Let G be a one-ended finitely generated K–CSA group that does not contain $\mathbb{Z}^2$ as a subgroup. The modular group $\mathrm{Mod}(G)$ of G is the subgroup of $\mathrm{Aut}(G)$ composed of all automorphisms that act by conjugation on non-QH vertex groups of the $\mathcal{Z}$–JSJ splitting, and on finite subgroups of G, and that act trivially on the underlying graph of the $\mathcal{Z}$–JSJ splitting.

2.6.6 JSJ splittings over finite groups Let G be a finitely generated group. Under the hypothesis that there exists a constant K such that every finite subgroup of G has order less than K, Linnell proved in [11] that G splits as a finite graph of groups with finite edge groups and all of whose vertex groups are finite or one-ended. We call such a splitting a $\mathcal{F}$–JSJ splitting of G, where $\mathcal{F}$ stands for "finite" (in [7], such a splitting is called a Stallings–Dunwoody splitting of G).

Contrary to the $\mathcal{Z}$–JSJ splitting, there is no canonical $\mathcal{F}$–JSJ splitting, but the conjugacy classes of one-ended vertex groups do not depend on the splitting. Moreover, the conjugacy classes of finite vertex groups are the same in all reduced $\mathcal{F}$–JSJ splittings of G. A one-ended subgroup of G that appears as a vertex group of a $\mathcal{F}$–JSJ splitting is called a *one-ended factor* of G. Note that if G is virtually free, then there is no one-ended factors; all vertex stabilizers are finite.

If G is a Γ–limit group, where Γ is hyperbolic, one can prove that there exists a uniform bound on the order of finite subgroups of G (see [20, Lemma 1.18]). As a consequence, G has a $\mathcal{F}$–JSJ splitting.

Definition 2.29 Let G be a finitely generated group whose finite subgroups have bounded order. Let G' be a group. A homomorphism $\phi\colon G \to G'$ is said to be factor-injective if ϕ is injective in restriction to every one-ended factor of G that is not finite-by-orbifold.

2.7 Preliminaries on orbifolds

In the sequel, all orbifolds are compact, connected, 2–dimensional, hyperbolic and conical, ie without mirrors.

2.7.1 Cutting an orbifold into elliptic components

Definition 2.30 A set $\mathcal{C}$ of simple closed curves on a conical orbifold is said to be essential if its elements are non-null-homotopic, two-sided, non-boundary-parallel, pairwise nonparallel, and represent elements of infinite order (in other words, no curve of $\mathcal{C}$ bounds a disk with a single singularity (or cone point)).

Proposition 2.31 *Let $\mathcal{O}$ be a hyperbolic orbifold. Suppose that $S = \pi_1(\mathcal{O})$ acts minimally on a simplicial tree T in such a way that its boundary elements are elliptic. Then there exists an essential set $\mathcal{C}$ of curves on $\mathcal{O}$, and a surjective S–equivariant map $f\colon T_{\mathcal{C}} \to T$, where $T_{\mathcal{C}}$ stands for the Bass–Serre tree associated with the splitting of S dual to $\mathcal{C}$. In other words:*

- *Every element of S corresponding to a loop of $\mathcal{C}$ fixes an edge of T.*
- *Every fundamental group of a connected component of $\mathcal{O} \setminus \mathcal{C}$ is elliptic.*

The proposition above is proved in [16] for surfaces (see Theorem 3.2.6), and the generalization to compact hyperbolic 2–orbifolds of conical type is straightforward since conical subgroups are elliptic in T. Then, Proposition 2.31 extends to finite-by-orbifold groups through the following observation: if G is a finite extension $F \hookrightarrow G \twoheadrightarrow \pi_1(\mathcal{O})$ acting minimally on a tree T, then the action factors through $G/F \simeq \pi_1(\mathcal{O})$, because F acts as the identity on T. Indeed, since F is finite, it fixes a point x of T. Since F is normal, the nonempty subtree of T pointwise-fixed by F is invariant under the action of G. Since this action is minimal, F fixes T pointwise.

2.7.2 Nonpinching homomorphisms In this section, we establish conditions ensuring that a homomorphism between two finite-by-orbifold groups is an isomorphism.

Definition 2.32 Let G and G' be conical finite-by-orbifold groups (see Definition 2.23). A homomorphism from G to G' is called a morphism of finite-by-orbifold groups if it sends each extended boundary subgroup injectively into an extended boundary subgroup and if it is injective on finite subgroups.

Definition 2.33 Let G be a conical finite-by-orbifold group $F \hookrightarrow G \xrightarrow{q} \pi_1(\mathcal{O})$. Let ϕ be a homomorphism from G to a group G'. Let α be an essential curve on $\mathcal{O}$, let $\boldsymbol{\alpha}$ be a corresponding element of the fundamental orbifold group of $\mathcal{O}$ (for a given choice of base point) and let $C_{\boldsymbol{\alpha}} = q^{-1}(\langle \boldsymbol{\alpha} \rangle) \simeq F \rtimes \mathbb{Z}$. The curve α is said to be pinched by ϕ if $\phi(C_{\boldsymbol{\alpha}})$ is finite. The homomorphism ϕ is said to be nonpinching if it does not pinch any two-sided simple loop. Otherwise, ϕ is said to be pinching.

Lemma 2.34 *Let $\mathcal{O}$ and $\mathcal{O}'$ be conical hyperbolic orbifolds. Let G and G' be their orbifold fundamental groups. Let $\phi\colon G \to G'$ be a nonpinching morphism of orbifold groups. Suppose that $\phi(G)$ is not contained in a conical or boundary subgroup of G'. Then $\phi(G)$ has infinite index in G'.*

Proof Suppose towards a contradiction that $\phi(G)$ has infinite index in G'. Then $\phi(G)$ is the fundamental group of an infinite-degree orbifold covering $\overline{\mathcal{O}'}$ of $\mathcal{O}'$. This covering is a geometrically finite hyperbolic conical orbifold of infinite volume. Let K be its convex core. The inclusion $K \subset \overline{\mathcal{O}'}$ induces an isomorphism between the orbifold fundamental groups of K and $\overline{\mathcal{O}'}$. Since $\overline{\mathcal{O}'}$ is of infinite volume and has no cusp end, it has at least one funnel end, so K has at least one boundary component that is not a preimage in $\overline{\mathcal{O}'}$ of a boundary component of $\mathcal{O}'$. Therefore, $\phi(G)$ can be written as a free product

$$\phi(G) = \langle c_1 \rangle * \cdots * \langle c_p \rangle * \langle b_1 \rangle * \cdots * \langle b_q \rangle * F_{r-1},$$

where $c_1, \dots, c_p$ are conical elements of $\phi(G)$, $b_1, \dots, b_q$ denote boundary elements of $\phi(G)$ conjugate to boundary elements of G', and $r \geq 1$ is the number of funnel ends of $\overline{\mathcal{O}'}$. Note that this splitting is nontrivial because, by assumption, $\phi(G)$ is not contained in a boundary subgroup or in a conical subgroup of G'. Let T be the Bass–Serre tree of this splitting. The group G acts on T via ϕ. Moreover, by definition of a morphism of orbifold groups, every boundary element of G is elliptic in T for this action. Then it follows from Proposition 2.31 that there exists a simple loop on $\mathcal{O}$ pinched by ϕ. This is a contradiction. □

An Euler characteristic on a class of groups $\mathcal{G}$ closed under taking subgroups of finite index is a function $\chi\colon \mathcal{G} \to \mathbb{R}$ satisfying the following condition: if $G \in \mathcal{G}$ and H is a subgroup of finite index in G, then $\chi(H) = [G : H]\chi(G)$. Euler characteristics have been defined on various classes of groups, by many authors (see for instance [28]).

In the sequel, we consider the class $\mathcal{G}$ composed of groups G of the form

$$1 \to F \to G \to A_1 * \cdots * A_m * F_n \to 1$$

where F and $A_1, \dots, A_m$ are finite and F_n is the free group of rank $n \geq 0$. Note that finite extensions of fundamental groups of conical hyperbolic orbifolds with nonempty boundary belong to $\mathcal{G}$. In [28, Corollary 3.6], Sykiotis proved that the function χ defined below (Definition 2.35) is an Euler characteristic on $\mathcal{G}$.

Definition 2.35 Let $G \in \mathcal{G}$, and let Δ be a $\mathcal{F}$–JSJ splitting of G. Denote by $V(\Delta)$ the set of vertices of Δ, and by $E(\Delta)$ its set of edges. We define $\chi(\Delta)$ by

$$\chi(\Delta) = \sum_{e \in E(\Delta)} \frac{1}{|G_e|} - \sum_{v \in V(\Delta)} \frac{1}{|G_v|}.$$

This number does not depend on the choice of the $\mathcal{F}$–JSJ splitting Δ of G. The Euler characteristic of G is defined as $\chi(G) = \chi(\Delta)$, for any $\mathcal{F}$–JSJ splitting Δ of G.

Lemma 2.36 *Let $G, G' \in \mathcal{G}$. Let Δ and Δ' be $\mathcal{F}$–JSJ splittings of G and G' respectively. Suppose that the edge groups of Δ and Δ' are trivial. Let $\phi\colon G \twoheadrightarrow G'$ be an epimorphism that is injective on finite subgroups of G. Then $\chi(G) \geq \chi(G')$, with equality if and only if ϕ is injective.*

Proof If Δ is reduced to a point, then G is finite, so ϕ is injective by assumption. From now on, we will suppose that Δ has at least two vertices. Let T and T' be Bass–Serre trees of Δ and Δ' respectively. We build a ϕ–equivariant map $f\colon T \to T'$

in the following way: for every vertex v of T, since $\phi(G_v)$ is finite (because G_v is finite), it fixes a vertex v' of T'. Moreover, v' is unique since ϕ is injective on finite subgroups, and edge groups of Δ' are trivial. We let $f(v) = v'$. Next, if e is an edge of T, with endpoints v and w, there exists a unique path e' from $f(v)$ to $f(w)$ in T'. We let $f(e) = e'$. Let us denote by d' the natural distance function on T'. Up to subdividing the edges of T, we can assume that $d'(f(v), f(w)) \in \{0,1\}$ for every pair of adjacent vertices $v, w \in T$. We will prove that ϕ can be written as a composition $i \circ \pi_n \circ \cdots \circ \pi_0$, with $n \geq 0$, $\pi_0 = \mathrm{id}$ and i injective, such that $\chi(\pi_\ell \circ \cdots \circ \pi_0(G)) \geq \chi(\pi_{\ell+1} \circ \cdots \circ \pi_0(G))$ for every $0 \leq \ell < n$ (if $n > 0$), with equality if and only if $\pi_{\ell+1} = \mathrm{id}$.

Step 1 Assume there are two adjacent vertices v and w such that $d'(f(v), f(w)) = 0$. Let e be the edge between v and w. We collapse e in T, as well as all its translates under the action of G. Collapsing e gives rise to a new G–tree S with a new vertex x labeled by $G_x = \langle G_v, G_w \rangle$ if v and w are not in the same orbit, or $G_x = \langle G_v, g \rangle$ if $w = g \cdot v$. Let N be the kernel of the restriction of ϕ to G_x and let $\langle\langle N \rangle\rangle$ denote the subgroup of G normally generated by N. Let $G_1 = G/\langle\langle N \rangle\rangle$, let us denote by $\pi_1 \colon G \twoheadrightarrow G_1$ the associated epimorphism, and let $\phi_1 \colon G_1 \to G$ be the unique morphism such that $\phi = \phi_1 \circ \pi_1$. The group G_1 splits as a graph of groups Δ_1 obtained from the splitting S/G of G by replacing the vertex group G_x by $G_x/N \simeq \phi(G_x)$. Note that G_x/N is finite since $\phi(G_x)$ fixes the vertex $f(v) = f(w)$ of T'. Hence, Δ_1 is a $\mathcal{F}$–JSJ splitting of G_1. Let us compare $\chi(G_1)$ with $\chi(G)$. It is not hard to see that

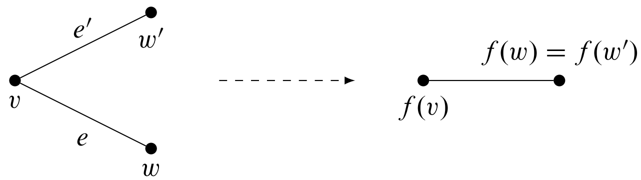
$$\chi(G) - \chi(G_1) = \begin{cases} 1 - \dfrac{1}{|G_v|} - \dfrac{1}{|G_w|} + \dfrac{1}{|G_x/N|} & \text{if } v \text{ and } w \text{ are not in the same orbit,} \\ 1 - \dfrac{1}{|G_v|} + \dfrac{1}{|G_x/N|} & \text{if } v \text{ and } w \text{ are in the same orbit.} \end{cases}$$

Hence, if v and w are in the same orbit, it is clear that $\chi(G) > \chi(G_1)$. If v and w are not in the same orbit, there are four distinct cases: if $|G_v| \geq 2$ and $|G_w| \geq 2$, it is clear that $\chi(G) > \chi(G_1)$; if $|G_v| = 1$ and $|G_w| \geq 2$ (respectively $|G_w| = 1$ and $|G_v| \geq 2$), then $G_x = G_w$ (respectively $G_x = G_v$) and $\chi(G) \geq \chi(G_1)$ with equality if and only if $N = \{1\}$, ie $\pi_1 = \mathrm{id}$; if $|G_v| = |G_w| = 1$, then $G_x = N = \{1\}$, ie $\pi_1 = \mathrm{id}$.

Let T_1 be the Bass–Serre tree of the splitting Δ_1 of G_1. We build a ϕ_1–equivariant map $f_1 \colon T_1 \to T'$ as above. If f_1 collapses some edge, we repeat the previous operation. Since T has only finitely many orbits of edges under the action of G, the procedure terminates after finitely many steps.

Up to replacing G by the last group of the sequence of quotients of G built above, we can assume without loss of generality that f sends adjacent vertices to adjacent vertices.

Step 2 Assume that f folds some pair of edges, as pictured below:



Let us fold e and e' together in T, as well as all their translates under the action of G. Note that e and e' are not in the same G_v–orbit since T' has trivial edge stabilizers and ϕ is injective on G_v. Folding e and e' together gives rise to a new G–tree S with a new vertex x labeled by $G_x = \langle G_w, G'_w \rangle$ if w and w' are not in the same orbit, or $G_x = \langle G_w, g \rangle$ if $w' = g \cdot w$.

Let N be the kernel of the restriction of ϕ to G_x and let $\langle\langle N \rangle\rangle$ denote the subgroup of G normally generated by N. Let $G_1 = G/\langle\langle N \rangle\rangle$, let us denote by $\pi_1\colon G \twoheadrightarrow G_1$ the associated epimorphism, and let $\phi_1\colon G_1 \to G$ be the unique morphism such that $\phi = \phi_1 \circ \pi_1$. The group G_1 splits as a graph of groups Δ_1 obtained from the splitting S/G of G by replacing the vertex group G_x by $G_x/N \simeq \phi(G_x)$. Note that G_x/N is finite since $\phi(G_x)$ fixes a vertex of T'. Hence, Δ_1 is a $\mathcal{F}$–JSJ splitting of G_1. Let us compare $\chi(G_1)$ with $\chi(G)$. As in the first step, we can see that $\chi(G) > \chi(G_1)$. Again, we can repeat this operation only finitely many times since T has only finitely many orbits of edges under the action of G. Let T_1 denote the Bass–Serre tree of Δ_1. At the end of the process, with obvious notation, we get a ϕ_n–equivariant map $f_n\colon T_n \to T'$ that is locally injective, thus injective. It remains to prove that ϕ_n is injective: if $\phi_n(g) = 1$, then for every vertex v of T_n, $f_n(gv) = f_n(v)$, so $gv = v$. Since G_n acts on T_n with trivial edge stabilizers, we get $g = 1$. □

Proposition 2.37 follows easily from Lemmas 2.34 and 2.36.

Proposition 2.37 *Let $\mathcal{O}$ and $\mathcal{O}'$ be conical hyperbolic orbifolds, with nonempty boundary. Denote by G and G' their orbifold fundamental groups. If $\phi\colon G \to G'$ is a nonpinching morphism of orbifold groups such that $\phi(G)$ is not contained in a conical or boundary subgroup of G', then $\chi(G) \geq \chi(G')$, with equality if and only if ϕ is an isomorphism.*

Proof By Lemma 2.34, $d := [G' : \phi(G)]$ is finite. Since χ is an Euler characteristic (see [28]), $\chi(\phi(G)) = d\chi(G')$. By Lemma 2.36, $\chi(G) \geq \chi(\phi(G))$, with equality if and only if ϕ is injective. It follows that $\chi(G) \geq \chi(G')$, with equality if and only if ϕ is an isomorphism. $\square$

We conclude this section by generalizing Proposition 2.37 to finite extensions of conical hyperbolic orbifold groups. First, we need the following lemma.

Lemma 2.38 *Let $\mathcal{O}$ and $\mathcal{O}'$ be conical hyperbolic orbifolds. Let G and G' be two finite extensions*

$$F \hookrightarrow G \twoheadrightarrow \pi_1(\mathcal{O}) \quad \text{and} \quad F' \hookrightarrow G' \twoheadrightarrow \pi_1(\mathcal{O}').$$

If $\phi\colon G \to G'$ is a homomorphism whose restriction to F is injective and whose image is infinite, then $\phi(F) \subset F'$. As a consequence, ϕ induces a homomorphism σ from $\pi_1(\mathcal{O})$ to $\pi_1(\mathcal{O}')$:

$$\begin{array}{ccccc} F & \hookrightarrow & S & \xrightarrow{\ \pi\ } & \pi_1(\mathcal{O}) \\ \downarrow & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \sigma} \\ F' & \hookrightarrow & S' & \xrightarrow{\ \pi'\ } & \pi_1(\mathcal{O}') \end{array}$$

Proof First, we make the following observation: if A is a finite subgroup of G' which is not contained in F', then the normalizer $N_{G'}(A)$ of A in G' is finite. Indeed, if A is not contained in F', then $\pi(A)$ is a nontrivial finite subgroup of $\mathcal{O}'$, so its normalizer is a finite cyclic group (since we can see $\pi(A)$ as an elliptic subgroup of $\mathrm{PSL}(2,\mathbb{R})$ acting on $\mathbb{H}^2$). Now, if $\phi(F)$ is not contained in F', then $\phi(G) \subset N_{G'}(\phi(F))$, which is finite. This is a contradiction. $\square$

Proposition 2.39 *Let $\mathcal{O}$ and $\mathcal{O}'$ be conical hyperbolic orbifolds, with nonempty boundary. Let G and G' be two finite extensions*

$$F \hookrightarrow G \twoheadrightarrow \pi_1(\mathcal{O}) \quad \text{and} \quad F' \hookrightarrow G' \twoheadrightarrow \pi_1(\mathcal{O}').$$

Let $\phi\colon G \to G'$ be a nonpinching morphism of finite-by-orbifold groups such that $\phi(G)$ is not contained in an extended conical or boundary subgroup of G'. Then $\chi(G) \geq \chi(G')$, with equality if and only if ϕ is an isomorphism.

Proof Note that $\chi(\pi_1(\mathcal{O})) = |F|\chi(G)$ and $\chi(\pi_1(\mathcal{O}')) = |F'|\chi(G')$. By Lemma 2.38, ϕ induces a nonpinching homomorphism of orbifolds $\sigma\colon \pi_1(\mathcal{O}) \to \pi_1(\mathcal{O}')$ whose image is not contained in a boundary or conical subgroup of $\pi_1(\mathcal{O}')$. According to Proposition 2.37, $\chi(\pi_1(\mathcal{O})) \geq \chi(\pi_1(\mathcal{O}'))$. But $\phi(F) \subset F'$, and ϕ is injective in restriction to F, so $|F'| \geq |F|$. Thus, $\chi(G) \geq \chi(G')$. Moreover, if $\chi(G) = \chi(G')$, then $\chi(\pi_1(\mathcal{O})) = \chi(\pi_1(\mathcal{O}'))$, so $\sigma\colon \pi_1(\mathcal{O}) \to \pi_1(\mathcal{O}')$ is an isomorphism by Proposition 2.37, and $|F| = |F'|$. Thus, $\phi|_F\colon F \to F'$ is an isomorphism, so ϕ is an isomorphism. □

3 How to extract information from the JSJ using first-order logic

3.1 A particular case

In the introduction, we proved a particular case of Theorems 1.2 and 1.3 under the hypotheses that G and Γ are one-ended and have finitely many outer automorphisms, namely Claim 1.7. We shall now consider another particular case of these theorems, which is little more general, and much more instructive. First of all, note that we cannot express the full statement of the shortening argument Corollary 2.4 in first-order logic, since precomposition by a modular automorphism is not expressible by a first-order formula in general. To deal with this problem, let us consider the following corollary of Corollary 2.4, which follows immediately from the definition of the modular group.

Corollary 3.1 *Let Γ be a group that embeds into a hyperbolic group, and let G be a one-ended finitely generated group. There exists a finite set $F \subset G \setminus \{1\}$ with the following property: for every noninjective homomorphism $\phi\colon G \to \Gamma$, there exists a homomorphism $\phi'\colon G \to \Gamma$ such that:*

- $\ker(\phi') \cap F \neq \varnothing$.
- *For every non-QH vertex group G_v of the $\mathcal{Z}$–JSJ splitting of G, there exists an element $\gamma \in \Gamma$ such that $\phi' = \mathrm{ad}(\gamma) \circ \phi$ in restriction to G_v.*
- *For every finite group F of G, there exists an element $\gamma \in \Gamma$ such that $\phi' = \mathrm{ad}(\gamma) \circ \phi$ in restriction to F.*

We say that ϕ and ϕ' are JSJ–related (see Definition 3.4).

Here is a particular case of Theorems 1.2 and 1.3.

Proposition 3.2 *Let Γ be a hyperbolic group, and let G be a finitely generated group such that $\mathrm{Th}_{\forall\exists}(\Gamma) = \mathrm{Th}_{\forall\exists}(G)$. Suppose that G is one-ended, and that there is no QH vertex in the $\mathcal{Z}$–JSJ decomposition of G. Then G embeds into Γ. Suppose moreover that Γ is one-ended and that there is no QH vertex in the $\mathcal{Z}$–JSJ decomposition of Γ. Then Γ and G are isomorphic.*

Remark 3.3 This result generalizes Claim 1.7, because the existence of a QH vertex group in the $\mathcal{Z}$–JSJ decomposition of G (or Γ) gives rise to infinitely many outer automorphisms.

Proof First, we prove that G embeds into Γ. Argue by contradiction and suppose that every homomorphism from G to Γ is noninjective. By Corollary 3.1, there exists a finite set $F \subset G \setminus \{1\}$ such that for every homomorphism $\phi\colon G \to \Gamma$, there exists $\phi'\colon G \to \Gamma$ that kills an element of F and that coincides with ϕ up to conjugacy on every vertex group of the canonical $\mathcal{Z}$–JSJ splitting Δ of G. One easily sees that this fact can be expressed by a $\forall\exists$–sentence satisfied by Γ (see Lemma 3.7 for details). Since $\mathrm{Th}_{\forall\exists}(\Gamma) \subset \mathrm{Th}_{\forall\exists}(G)$, this sentence is also satisfied by G, and its interpretation in G yields the following: for every endomorphism $\phi\colon G \to G$, there exists an endomorphism $\phi'\colon G \to G$ that kills an element of F and that coincides with ϕ up to conjugacy on every vertex group of the canonical $\mathcal{Z}$–JSJ splitting Δ of G. Taking $\phi = \mathrm{id}_G$, we get an endomorphism of G that kills an element of F and that is inner in restriction to every vertex group. But we will prove later that such an endomorphism is necessarily injective (see Proposition 7.1). This contradicts the fact that ϕ' kills an element of F. Hence, we have shown that G embeds into Γ. Now, using Corollary 3.1, we prove in the same way that Γ embeds into G. As a one-ended hyperbolic group, Γ is co-Hopfian, so $G \simeq \Gamma$. □

New difficulties arise when Γ and G are not supposed to be one-ended, or when the $\mathcal{Z}$–JSJ splittings of Γ and G contain QH vertices. However, the example above highlights the crucial role played by endomorphisms of G that coincide up to conjugacy with the identity of G on non-QH vertex groups. This example also brings out the key idea to obtain these special homomorphisms, due to Sela–Perin, that consists in expressing a consequence of the shortening argument Theorem 2.2 by a $\forall\exists$–sentence that Γ satisfies (assuming G is one-ended). Since Γ and G have the same $\forall\exists$–theory, G satisfies this sentence as well, and its interpretation in G endows us with a special endomorphism of G. This example leads us to the definition of related homomorphisms.

3.2 Related homomorphisms and preretractions

The next definition is similar (but slightly different) to [19, Definitions 5.9 and 5.15].

Definition 3.4 (related homomorphisms) Let G be a one-ended finitely generated K–CSA group that does not contain $\mathbb{Z}^2$, and let G' be a group. Let Δ be the $\mathcal{Z}$–JSJ splitting of G. Let ϕ and ϕ' be two homomorphisms from G to G'. We say ϕ and ϕ' are *JSJ–related* or *Δ–related* if the following two conditions hold:

- For every non-QH vertex v of Δ, there exists an element $g_v \in G'$ such that
$$\phi'|_{G_v} = \mathrm{ad}(g_v) \circ \phi|_{G_v}.$$
- For every finite subgroup F of G, there exists an element $g \in G'$ such that
$$\phi'|_F = \mathrm{ad}(g) \circ \phi|_F.$$

Note that being JSJ–related is an equivalence relation on $\mathrm{Hom}(G, G')$.

Remark 3.5 The second condition above can be reformulated as follows: for every QH vertex v, and for every finite subgroup F of G_v, there exists an element $g \in G'$ such that $\phi|_F = \mathrm{ad}(g) \circ \phi'|_F$. Indeed, every finite group has Serre's property (FA), so every finite subgroup F of G is contained in a conjugate of some vertex group G_w of Δ. Furthermore, if w is a non-QH vertex, it follows from the first condition that there exists an element $g \in G'$ such that $\phi|_F = \mathrm{ad}(g) \circ \phi'|_F$.

Definition 3.6 (preretraction) Let G be a finitely generated K–CSA group that does not contain $\mathbb{Z}^2$, and let H be a one-ended subgroup of G. Let Δ be the $\mathcal{Z}$–JSJ decomposition of H. A *Δ–preretraction* or *JSJ–preretraction* from H to G is a homomorphism $H \to G$ that is JSJ–related to the inclusion of H into G in the sense of Definition 3.4. A JSJ–preretraction is said to be nondegenerate if it sends each QH subgroup of H isomorphically to a conjugate of itself.

The following lemma shows that being JSJ–related can be expressed in first-order logic.

Lemma 3.7 (compare with [19, Lemma 5.18]) *Let G be a finitely generated one-ended K–CSA group that does not contain $\mathbb{Z}^2$. Let $\{g_1, \dots, g_n\}$ be a generating set of G. Let G' be a group. There exists an existential formula $\psi(x_1, \dots, x_{2n})$ with $2n$ free variables such that for every $\phi, \phi' \in \mathrm{Hom}(G, G')$, ϕ and ϕ' are JSJ–related if and only if G' satisfies $\psi(\phi(g_1), \dots, \phi(g_n), \phi'(g_1), \dots, \phi'(g_n))$.*

Proof Let Δ be the $\mathcal{Z}$–JSJ splitting of G. First, note that there exist finitely many (say $p \geq 1$) conjugacy classes of finite subgroups of QH vertex groups of Δ (indeed, a QH vertex group possesses finitely many conjugacy classes of finite subgroups, and Δ has finitely many vertices). Denote by $F_1, \dots, F_p$ a system of representatives of those conjugacy classes. Denote by $R_1, \dots, R_m$ the non-QH vertex groups of Δ.

Note that these groups are finitely generated since G and the edge groups of Δ are finitely generated. Denote by $\{A_i\}_{1\leq i\leq p+m}$ the union of $\{F_i\}_{1\leq i\leq p}$ and $\{R_i\}_{1\leq i\leq m}$. For every $i\in[\![1,m+p]\!]$, let $\{a_{i,1},\dots,a_{i,k_i}\}$ be a finite generating set of A_i. For every $i\in[\![1,m+p]\!]$ and $j\in[\![1,k_i]\!]$, there exists a word $w_{i,j}$ in n letters such that $a_{i,j}=w_{i,j}(g_1,\dots,g_n)$. Let $\psi(x_1,\dots,x_{2n})$ be the sentence

$$\exists u_1\dots\exists u_{m+p}\ \bigwedge_{i=1}^{m+p}\bigwedge_{j=1}^{k_i} w_{i,j}(x_1,\dots,x_n)=u_i w_{i,j}(x_{n+1},\dots,x_{2n})u_i^{-1}.$$

Since $\phi(a_{i,j})=w_{i,j}(\phi(g_1),\dots,\phi(g_n))$ and $\phi'(a_{i,j})=w_{i,j}(\phi'(g_1),\dots,\phi'(g_n))$ for every $i\in[\![1,m+p]\!]$ and $j\in[\![1,k_i]\!]$, the homomorphisms ϕ and ϕ' are JSJ–related if and only if the sentence $\psi(\phi(g_1),\dots,\phi(g_n),\phi'(g_1),\dots,\phi'(g_n))$ is satisfied by G'. □

3.3 Centered graph of groups

We need to define relatedness in a more general context. To deal with groups that are not assumed to be one-ended, we define the notion of a centered graph of groups. We denote by $\overline{\mathcal{Z}}$ the class of groups that are either finite or virtually cyclic with infinite center.

Definition 3.8 (centered graph of groups) A graph of groups over $\overline{\mathcal{Z}}$, with at least two vertices, is said to be *centered* if the following conditions hold:

- The underlying graph is bipartite, with a QH vertex v such that every vertex different from v is adjacent to v; moreover, the finite-by-orbifold group G_v is conical.
- For every edge e incident to v, the edge group G_e coincides with an extended boundary subgroup or with an extended conical subgroup of G_v (see Definition 2.23).
- Given any extended boundary subgroup B, there exists a unique edge e incident to v such that G_e is conjugate to B in G_v.
- If an element of infinite order fixes a segment of length ≥ 2 in the Bass–Serre tree of the splitting, then this segment has length exactly 2 and its endpoints are translates of v.

The vertex v is called the central vertex. See Figure 1.

Recall that a subgroup H of a group G is called almost malnormal if $H\cap gHg^{-1}$ is finite for every g in $G\setminus H$. In particular, every finite subgroup of G is malnormal.

Lemma 3.9 *Edge groups in a centered graph of groups are almost malnormal in the central vertex group.*

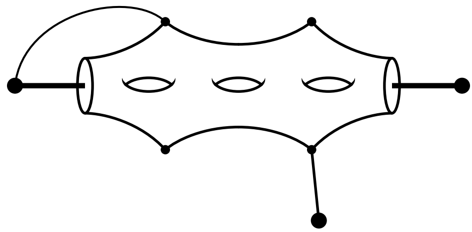


Figure 1: A centered graph of groups. Edges with infinite stabilizer are thickened.

Proof Let G be a group with a centered splitting Δ_G. Let v be the central vertex and let e be an edge of Δ_G incident to v. By the second condition of Definition 3.8, G_e is an extended conical subgroup or an extended boundary subgroup of the finite-by-orbifold group G_v. In the first case, G_e is finite, so it is clearly malnormal. In the second case, G_e is virtually cyclic infinite maximal in G_v, so its fixed-point set $P \subset \partial_\infty G_v$ is a pair of points, and we have $\mathrm{Stab}_{G_v}(P) = G_e$. Hence, if $g \in G_v$ is such that $G_e \cap gG_eg^{-1}$ is infinite, then gG_eg^{-1} fixes P. It follows that g fixes P, ie that g belongs to G_e. □

We will need the following two classical results.

Proposition 3.10 [4, pages 99 and 100] *Let $G = A *_C B$ be an amalgamated product such that A and B are hyperbolic and C is virtually cyclic* (possibly finite) *and almost malnormal in A or B. Then G is hyperbolic.*

Let $G = A*_C$ be an HNN extension such that A is hyperbolic, C is virtually cyclic (possibly finite)*, and the two copies C_1 and C_2 of C form an almost malnormal family of subgroups, ie at least one of them is almost malnormal in A, and $C_1 \cap gC_2g^{-1}$ is finite for every $g \in G$. Then G is hyperbolic.*

Proposition 3.11 [5, Proposition 1.2] *If a hyperbolic group splits over quasiconvex subgroups, then every vertex group is quasiconvex* (hence hyperbolic).

The following result is an immediate consequence of the previous two propositions.

Proposition 3.12 *Let G be a group that splits as a centered graph of groups, with central vertex v. Then the following assertions are equivalent:*

(1) *G is hyperbolic.*

(2) *For every vertex $w \neq v$, G_w is hyperbolic.*

Proof The implication (2) $\Rightarrow$ (1) is an immediate consequence of the combination theorem of Bestvina and Feighn (see Proposition 3.10 above) together with Lemma 3.9. The implication (1) $\Rightarrow$ (2) follows from Proposition 3.11. $\square$

We define relatedness for centered splittings (compare JSJ–relatedness, Definition 3.4), and preretractions (compare JSJ–preretractions, Definition 3.6).

Definition 3.13 (related homomorphisms) Let G and G' be two groups. Let Δ be a centered splitting of G, with central vertex v. Let ϕ and ϕ' be two homomorphisms from G to G'. We say ϕ and ϕ' are Δ*–related* if the following two conditions hold:

- For every vertex $w \neq v$, there exists an element $g_w \in G'$ such that
$$\phi'|_{G_w} = \mathrm{ad}(g_w) \circ \phi|_{G_w}.$$
- For every finite subgroup F of G, there exists an element $g \in G'$ such that
$$\phi'|_F = \mathrm{ad}(g) \circ \phi|_F.$$

Note that being Δ–related is an equivalence relation on $\mathrm{Hom}(G, G')$.

Definition 3.14 Let G be a group, and let Δ be a centered splitting of G. Let v be the central vertex of Δ. An endomorphism ϕ of G is called a Δ*–preretraction* if it is Δ–related to the identity of G in the sense of Definition 3.13. A preretraction is said to be *nondegenerate* if it sends G_v isomorphically to a conjugate of itself.

4 Torsion-saturated groups

To deal with certain pathologies arising from torsion (see Example 1.9), we prove in the current section that every hyperbolic group G embeds into a hyperbolic group $\overline{G}$ such that the class of $\overline{G}$–limit groups is closed under HNN extensions and amalgamated free products over finite groups.

Definition 4.1 We say that a hyperbolic group G is torsion-saturated if the following two conditions hold:

(1) For every isomorphism $\alpha\colon F_1 \to F_2$ between finite subgroups F_1 and F_2 of G, there exists an element $g \in G$ such that $gxg^{-1} = \alpha(x)$ for every $x \in F_1$.

(2) For every finite subgroup F of G, there exists an infinite subset $\{g_1, g_2, \dots\}$ of G such that g_n has infinite order, $M(g_n) = \langle g_n \rangle \times F$ for every n, and the intersection $M(g_n) \cap M(g_m)$ is equal to F whenever $n \neq m$.

Lemma 4.2 *Let G be a hyperbolic group, and let F be a finite subgroup of G. Suppose that there exist two elements $a, b \in G$ generating a free group of rank two and such that $M(a) = \langle a \rangle \times F$ and $M(b) = \langle b \rangle \times F$. Then there exists an infinite subset $\{g_1, g_2, \dots\}$ of G such that g_n has infinite order, $M(g_n) = \langle g_n \rangle \times F$ for every n, and $M(g_n) \cap M(g_m) = F$ whenever $n \neq m$.*

Proof Note $H = N_G(F)$ is nonelementary since it contains the free group $\langle a, b \rangle \simeq F_2$. By [17, Proposition 1], there is a unique maximal finite subgroup of G normalized by H, denoted by $E_G(H)$. By [17, Lemma 3.8], there is an infinite subset $\{g_1, g_2, \dots\}$ of H such that g_n has infinite order, $M(g_n) = \langle g_n \rangle \times E_G(H)$ for every n, and $M(g_n) \cap M(g_m) = E_G(H)$ whenever $n \neq m$. To conclude, we just have to show $E_G(H) = F$. First, note that F is obviously contained in $E_G(H)$. Then, note $E_G(H)$ is normalized by a. Because $E_G(H)$ is finite, a^r centralizes $E_G(H)$ for some integer $r \geq 1$. Then $E_G(H)$ preserves the pair of points $\{a^+, a^-\}$ fixed by a in the boundary of G. But the stabilizer of this pair of points is equal to $M(a)$ (by maximality of $M(a)$), which equals $\langle a \rangle \times F$ by hypothesis. Thus, $E_G(H) \subset F$, so $E_G(H) = F$. □

We shall see that every hyperbolic group G embeds into a torsion-saturated hyperbolic group $\overline{G}$ obtained from G by performing finitely many HNN extensions over finite groups (see Theorem 4.8). The main interest of torsion-saturated groups resides in the following result.

Theorem 4.3 *Let G be a torsion-saturated hyperbolic group. Then the class of G–limit groups is closed under HNN extensions and amalgamated free products over finite groups.*

Before proving Theorem 4.3, we need some preliminary results.

Lemma 4.4 *Let G be a hyperbolic group and H a G–limit group. Let $\alpha\colon F_1 \to F_2$ be an isomorphism between finite subgroups F_1 and F_2 of H. Then there exists an isomorphism $\beta\colon F_1' \to F_2'$ between finite subgroups F_1' and F_2' of G such that $H*_\alpha$ is a $G*_\beta$–limit group.*

Lemma 4.5 *Let G be a hyperbolic group. Suppose the first condition of Definition 4.1 holds. Let A and B be G–limit groups. Let F be a finite group that embeds into A and B. Then there exists a finite subgroup F' of G and an HNN extension*

$$G' = \langle G, t \mid [t, x] = 1 \text{ for all } x \in F \rangle$$

of G such that $A *_F B$ is a G'–limit group.

Lemma 4.6 *Let G be a hyperbolic group, and let $\alpha\colon F_1 \to F_2$ be an isomorphism between finite subgroups F_1 and F_2 of G. Suppose that the following conditions hold:*

(1) *There exists an element $g \in G$ such that $gxg^{-1} = \alpha(x)$ for every $x \in F_1$.*

(2) *There exists an infinite subset $E = \{g_1, g_2, \dots\} \subset G$ such that g_n has infinite order, $M(g_n) = \langle g_n \rangle \times F_1$ for every n, and $M(g_n) \neq M(g_m)$ whenever $n \neq m$.*

Then $G*_\alpha$ is a G–limit group.

Proof of Lemma 4.4 Let $(\phi_n)_{n\in\mathbb{N}}$ be a discriminating sequence of homomorphisms from H to G. Without loss of generality, we can suppose that ϕ_n is injective in restriction to F_1 and F_2. We will construct a discriminating sequence $(\rho_n)_{n\in\mathbb{N}}$ of homomorphisms from $H*_\alpha$ to $G*_\beta$, for some β that will be defined below. Since G has finitely many conjugacy classes of finite subgroups, there exist two finite subgroups F_1' and F_2' of G such that, up to extracting a subsequence, $\phi_n(F_1)$ is conjugate to F_1' and $\phi_n(F_2)$ is conjugate to F_2' for every n. Up to composing ϕ_n with an inner automorphism of G, one can assume that $\phi_n(F_1) = F_1'$ and $\mathrm{ad}(g_n)\circ\phi_n(F_2) = F_2'$ for some $g_n \in G$. Denote by β_n the isomorphism from F_1' to F_2' making the following diagram commute:

$$\begin{array}{ccc} F_1 & \xrightarrow{\ \phi_n|_{F_1}\ } & F_1' \\ \Big\downarrow{\scriptstyle\alpha} & & \Big\downarrow{\scriptstyle\beta_n} \\ F_2 & \xrightarrow{\ \mathrm{ad}(g_n)\circ\phi_n|_{F_2}\ } & F_2' \end{array}$$

Since $\mathrm{Isom}(F_1', F_2')$ is finite, there exists an isomorphism β between F_1' and F_2' such that (up to extracting a subsequence) $\beta_n = \beta$ for every n. Let t and u be the stable letters of $H*_\alpha$ and $G*_\beta$; that is, $txt^{-1} = \alpha(x)$ for all $x \in F_1$ and $uyu^{-1} = \beta(y)$ for all $y \in F_1'$. For every n, we define a map ρ_n from $H*_\alpha$ to $G*_\beta$ by

$$\rho_n(x) = \begin{cases} \phi_n(x) & \text{if } x \in H, \\ g_n^{-1}u & \text{if } x = t. \end{cases}$$

The map ρ_n clearly extends to a homomorphism since the diagram commutes, and we claim that the sequence $(\rho_n)_{n\in\mathbb{N}}$ is discriminating. Let x be a nontrivial element of $H*_\alpha$. If x lies in H, it is obvious that $\rho_n(x)$ is nontrivial for every n large enough since $(\phi_n)_{n\in\mathbb{N}}$ is discriminating. Assume now that $x \notin H$. Then x can be written in reduced form as $x = h_0 t^{\varepsilon_1} h_1 t^{\varepsilon_2} h_2 \cdots t^{\varepsilon_p} h_{p+1}$ with $p > 0$, $h_i \in H$, $\varepsilon_i = \pm 1$, $h_i \notin F_1$ if $\varepsilon_i = -\varepsilon_{i+1} = 1$ and $h_i \notin F_2$ if $\varepsilon_i = -\varepsilon_{i+1} = -1$. One has

$$\rho_n(x) = \phi_n(h_0)(g_n u)^{\varepsilon_1}\phi_n(h_1)(g_n u)^{\varepsilon_2}\phi_n(h_2)\cdots(g_n u)^{\varepsilon_p}\phi_n(h_{p+1}).$$

One has to prove that $\rho_n(x) \neq 1$ for every n large enough. To apply Britton's lemma, one verifies that, for every subword of $\rho_n(x)$ (written as above) of the form uvu^{-1} with v not involving u, the element v does not lie in F_1', and that for every subword of $\rho_n(x)$ of the form $u^{-1}vu$ with v not involving u, the element v does not lie in F_2':

- If uvu^{-1} is a subword of w with v not involving u, then v is of the form $\phi_n(h_i)$ with $h_i \notin F_1$, and $\phi_n(h_i) \notin F_1'$ for every n large enough because $(\phi_n)_{n\in\mathbb{N}}$ is discriminating.
- Similarly, if $u^{-1}vu$ is a subword of w with v not involving u, then v is of the form $g_n\phi_n(h_i)g_n^{-1}$ with $h_i \notin F_2$, and $g_n\phi_n(h_i)g_n^{-1} \notin F_2' = g_n\phi_n(F_2)g_n^{-1}$ for every n large enough because the sequence $(\mathrm{ad}(g_n)\circ\phi_n)_{n\in\mathbb{N}}$ is discriminating.

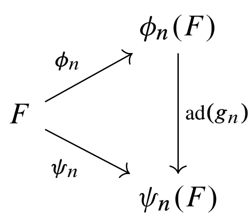
Hence, it follows from Britton's lemma that $\rho_n(x) \neq 1$ for every n large enough. □

Remark 4.7 In the previous proof, the hypothesis that G is hyperbolic is only used in order to ensure that G has finitely many classes of finite subgroups.

Proof of Lemma 4.5 Let $(\phi_n)_{n\in\mathbb{N}}$ be a discriminating sequence from A to G, and let $(\psi_n)_{n\in\mathbb{N}}$ be a discriminating sequence from B to G. Up to extracting a subsequence, we can assume that ϕ_n and ψ_n are injective on F. Hence, $\phi_n(F)$ and $\psi_n(F)$ are isomorphic to F, for every n.

Since G has only finitely many conjugacy classes of finite subgroups, we can assume there exists a finite subgroup F' of G such that $\phi_n(F) = F'$ for every n, up to extracting a subsequence of $(\phi_n)_{n\in\mathbb{N}}$ and precomposing ϕ_n with an inner automorphism.

By hypothesis, there exists an element $g_n \in G$ making the following diagram commute:



Let $\chi_n = \mathrm{ad}(g_n^{-1}) \circ \psi_n$. Then χ_n and ϕ_n coincide on F.

Let $G*_{F'} = \langle G, t \mid [t,x]=1 \text{ for all } x \in F'\rangle$ be the HNN extension of G over the identity of F'. For every n, since ϕ_n and χ_n coincide on F, we define a homomorphism ρ_n from $A *_F B$ to $G*_{F'}$ by

$$\rho_n(x) = \begin{cases} \phi_n(x) & \text{if } x \in A, \\ \mathrm{ad}(t)\circ\chi_n(x) & \text{if } x \in B. \end{cases}$$

We claim that the sequence $(\rho_n)_{n\in\mathbb{N}}$ is discriminating, ie that for every nontrivial element $x \in A *_F B$, we have $\rho_n(x) \neq 1$ for every n large enough. If $x \in F \setminus \{1\}$, then $\rho_n(x) = \phi_n(x)$, so $\rho_n(x)$ is nontrivial for every n large enough since $(\phi_n)_{n\in\mathbb{N}}$ is discriminating. Assume now that $x \notin F$. Then x can be written in a reduced form $a_1 b_1 a_2 b_2 \cdots a_k b_k$ with $a_i \in A \setminus F$ and $b_i \in B \setminus F$ (except maybe a_1 and b_k). Then

$$\rho_n(x) = \phi_n(a_1) t \chi_n(b_1) t^{-1} \phi_n(a_2) t \chi_n(b_2) t^{-1} \cdots \phi_n(a_k) t \chi_n(b_k) t^{-1}.$$

Since $(\phi_n)_{n\in\mathbb{N}}$ and $(\chi_n)_{n\in\mathbb{N}}$ are discriminating, $\phi_n(a_i) \notin F'$ and $\chi_n(b_i) \notin F'$ for every n large enough (except maybe $\phi_n(a_1)$ and $\chi_n(b_k)$). Hence, it follows from Britton's lemma that $\rho_n(x) \neq 1$ for every n large enough. □

Proof of Lemma 4.6 Let $G = \langle S \mid R \rangle$ be a presentation of G, and let

$$G*_\alpha = \langle S, t \mid R,\ txt^{-1} = \alpha(x) \text{ for all } x \in F_1 \rangle$$

be a presentation of $G*_\alpha$. By hypothesis, there exists an element $g \in G$ such that $gxg^{-1} = \alpha(x)$ for every $x \in F_1$. Up to replacing t by $g^{-1}t$, we can assume that $F_1 = F_2$ and that α is the identity of F_1. The presentation of $G*_\alpha$ becomes

$$\langle S, t \mid R,\ [t, x] = 1 \text{ for all } x \in F_1 \rangle.$$

For every $n \in \mathbb{N}$, and for every integer p, we define a map $\phi_{p,n}$ from $G*_\alpha$ to G by

$$\phi_{p,n}\colon \begin{cases} z \mapsto z & \text{if } z \in G, \\ t \mapsto g_n^p. \end{cases}$$

The map $\phi_{p,n}$ clearly extends to a homomorphism since $\mathrm{ad}(t)$ and $\mathrm{ad}(g_n^p)$ coincide on F_1 for every p, because $M(g_n) = \langle g_n \rangle \times F_1$ by hypothesis.

Denote by B_m the ball of radius m in $G*_\alpha$ (for a given generating set). We shall prove the existence of two sequences $(n_m)_{n\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ and $(p_{n_m})_{n\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ such that $\phi_{p_{n_m}, n_m}(x) \neq 1$ for every $x \in B_m \setminus \{1\}$, for every m. Let $x \in B_m \setminus \{1\}$. If x lies in G, $\phi_{p,n}(x) = x \neq 1$ for all p and n. Assume now that x does not belong to G. Then x can be written in a reduced form as

$$x = y_0 t^{\varepsilon_1} y_1 t^{\varepsilon_2} \cdots t^{\varepsilon_k} y_k$$

with $k > 0$, $\varepsilon_i = \pm 1$ and $y_i \notin F_1$ if $\varepsilon_i = -\varepsilon_{i+1}$. We claim that for p and n sufficiently large, the homomorphism $\phi_{p,n}$ satisfies $\phi_{p,n}(x) \neq 1$. In order to prove this, we will use Baumslag's lemma (see Corollary 2.20) with $c = g_n$. We have

$$\phi_{p,n}(x) = y_0 g_n^{\varepsilon_1 p} y_1 g_n^{\varepsilon_2 p} \cdots g_n^{\varepsilon_k p} y_k.$$

First, let us rewrite $\phi_{p,n}(x)$ under a more convenient form. Let $i \in [\![1, k-1]\!]$, and suppose that $\varepsilon_i = \varepsilon_{i+1} = 1$, and that y_i lies in F_1. Then, since $[g_n, F_1] = 1$, we replace the subword $g_n^p y_i g_n^p y_{i+1}$ by $g_n^{2p} y_i y_{i+1}$. In the case where $\varepsilon_{i+2} = -1$, note that $y_i y_{i+1}$ does not belong to F_1, since $y_i \in F_1$ and $y_{i+1} \notin F_1$. In the case where $\varepsilon_{i+2} = 1$ and y_{i+1} belongs to F_1, we repeat the previous operation, and so on. Similarly, if $\varepsilon_i = \varepsilon_{i+1} = -1$ and y_i lies in F_1, we replace the subword $g_n^{-p} y_i g_n^{-p} y_{i+1}$ of $\phi_{p,n}$ by $g_n^{-2p} y_i y_{i+1}$, and so on. At the end of this process, we have

$$\phi_{p,n}(x) = z_0 g_n^{\varepsilon_1 n_1 p} z_1 g_n^{\varepsilon_2 n_2 p} \cdots g_n^{\varepsilon_\ell n_\ell p} z_\ell,$$

with $n_1, \dots, n_\ell \in \mathbb{N}^*$, and $z_i \notin F_1$ for every $i \in [\![1, \ell-1]\!]$.

We can now use Baumslag's lemma (see Corollary 2.20) with $c = g_n$. We claim that there exists $n(x)$ such that z_i does not belong to $M(g_n)$ for every $n \geq n(x)$ and for every $i \in [\![1, \ell-1]\!]$. Indeed, suppose that z_i lies in $M(g_n) = \langle g_n \rangle \times F_1$, for some n. Since z_i does not belong to F_1, it has infinite order, so $M(z_i)$ is well defined and $M(z_i) = M(g_n)$. Then the claim follows from the facts that $M(g_m) \cap M(g_n) = F_1$ if $n \neq m$, and $z_i \notin F_1$.

Let $n_m := \max\{n(x) \mid x \in B_m\}$. By Corollary 2.20, there exists an integer p_{n_m} such that $\ker(\phi_{p_{n_m}, n_m}) \cap B_m = \{1\}$. This concludes the proof. □

Proof of Theorem 4.3 Let H be a G–limit group. Let $\alpha\colon F_1 \to F_2$ be an isomorphism between two finite subgroups F_1 and F_2 of H. We shall prove that the HNN extension $H*_\alpha$ is a G–limit group. By Lemma 4.4, there exists an isomorphism $\beta\colon F_1' \to F_2'$ between finite subgroups F_1' and F_2' of G such that $H*_\alpha$ is a $G*_\beta$–limit group. But $G*_\beta$ is a G–limit group thanks to Lemma 4.6, so $H*_\alpha$ is a G–limit group. It remains to consider the case of an amalgamated free product. Let A and B be G–limit groups. Let F be a finite group that embeds into A and B. According to Lemma 4.5, there exists a finite subgroup F' of G such that $A *_F B$ is a $G*_{F'}$–limit group, and $G*_{F'}$ is a G–limit group by Lemma 4.6, so $A *_F B$ is a G–limit group as well. □

Theorem 4.8 *Every hyperbolic group embeds into a torsion-saturated hyperbolic group.*

Proof Let G be a hyperbolic group. Denote by $F_1, \dots, F_m$ a system of representatives of the conjugacy classes of finite subgroups of G. Define $\theta\colon [\![1, m]\!]^2 \to \{0, 1\}$ by $\theta(i, j) = 1$ if and only if F_i and F_j are isomorphic. For every $(i, j) \in \theta^{-1}(1)$, let $n_{i,j} = |\mathrm{Isom}(F_i, F_j)|$ and $\mathrm{Isom}(F_i, F_j) = \{\alpha_{i,j,1}, \dots, \alpha_{i,j,n_{i,j}}\}$. If $i = j$, one can assume that $\alpha_{i,i,1}$ is the identity of F_i. Let us define $\overline{G}$ by adding new generators and relations to G, as follows:

$$\overline{G} = \big\langle G, \{s_{i,j,k}, t_{i,j,k}\}_{(i,j)\in\theta^{-1}(1)}^{1\le k\le n_{i,j}} \;\big|\; \mathrm{ad}(s_{i,j,k})|_{F_i} = \mathrm{ad}(t_{i,j,k})|_{F_i} = \alpha_{i,j,k}\big\rangle.$$

This group is hyperbolic by the combination theorem of Bestvina and Feighn (see [4, pages 99 and 100] or Proposition 3.10 above); indeed, $\overline{G}$ is obtained iteratively from the hyperbolic group G by performing HNN extensions over finite groups, and two finite subgroups always form an almost malnormal family. Let us prove that $\overline{G}$ is torsion-saturated. Let F_1 and F_2 be two finite subgroups of $\overline{G}$. Since finite groups have property (FA), there exist two elements g_1 and g_2 of $\overline{G}$ such that $g_1F_1g_1^{-1}$ and $g_2F_2g_2^{-1}$ are contained in G. Without loss of generality, we now assume that $F_1, F_2 \subset G$. By definition of $\overline{G}$, for every isomorphism $\alpha\colon F_1 \to F_2$ between finite subgroups of $\overline{G}$, there exists an element g of $\overline{G}$ such that $gxg^{-1} = \alpha(x)$ for all $x \in F_1$. Hence the first condition of Definition 4.1 is satisfied by $\overline{G}$. It remains to verify that the second condition holds. Let F be a finite subgroup of $\overline{G}$. We can assume that $F = F_i$ for some $1 \le i \le m$. Let a and b be the two generators of $\overline{G}$ corresponding to the identity of F_i, ie $a := s_{i,i,1}$ and $b := t_{i,i,1}$. The group $\langle a, b\rangle$ is free, $M(a) = \langle a\rangle \times F$, and $M(b) = \langle b\rangle \times F$. This concludes the proof, thanks to Lemma 4.2. □

5 Quasifloors and quasitowers

5.1 Definitions

Hyperbolic towers were introduced by Sela [25] under the name of nonelementary hyperbolic ω–residually free towers (see the paragraph before Proposition 6 in [26], and [25, Definition 6.1]). We also refer the reader to the NTQ groups of Kharlampovich and Myasnikov [10].

Sela used hyperbolic towers in [26] to solve Tarski's problem about the elementary equivalence of free groups, and in [27] to classify all finitely generated groups with the same first-order theory as a given torsion-free hyperbolic group. A hyperbolic tower is a group obtained by successive addition of hyperbolic floors. We refer the reader to [19]. Here is a slightly different.

Definition 5.1 Let G be a torsion-free finitely generated group, Δ_G a centered splitting of G with exactly one vertex w other than the central vertex v, and r a retraction from G onto $H = G_w$. We say that (G, H, Δ_G, r) is a *weak hyperbolic floor* or, more simply, that the group G is a weak hyperbolic floor over H. We say that G is a *hyperbolic floor* over H if, in addition, $r(G_v)$ is nonabelian and the underlying surface of G_v has topological Euler characteristic at most -2 or is a punctured torus (ie is not a punctured Klein bottle, a twice punctured projective plane or a pair of pants).

This definition is suitable for dealing with hyperbolic groups without torsion. In order to handle torsion, we need new definitions.

Definition 5.2 Let G and H be two finitely generated groups, Δ_G a centered splitting of G, Δ_H a splitting of H with finite edge groups, r a homomorphism from G to H and j a homomorphism from H to G. Let V_G be the set of vertices of Δ_G, and v the central vertex. Let V_H be the set of vertices of Δ_H. Let $V_H = V_H^1 \sqcup V_H^2$ be a partition of V_H and let $s\colon V_G \setminus \{v\} \to V_H^1$ be a bijection such that the following conditions hold:

- $j \circ r$ is a Δ_G–preretraction, ie $j \circ r$ coincides up to conjugacy with id_G on every vertex group G_w with $w \neq v$, and on every finite subgroup of G.
- For every w in $V_G \setminus \{v\}$, the homomorphism r maps G_w isomorphically to $h_w H_{s(w)} h_w^{-1}$ for some $h_w \in H$.
- For every $u \in V_H^2$, the vertex group H_u is finite and j is injective on H_u.

We say that $(G, H, \Delta_G, \Delta_H, r, j)$ is a *quasifloor*. If, moreover, there exists a one-ended subgroup A of G such that $A \cap \ker(r) \neq \{1\}$, the quasifloor is said to be *strict*.

For reasons of brevity, one sometimes wants to avoid the notation $(G, H, \Delta_G, \Delta_H, r, j)$. More simply, one writes (G, H, r, j), and one says that G is a (strict) quasifloor over H (with respect to r and j).

Lemma 5.3 (notation as in Definition 5.2) (1) *The homomorphism $r \circ j$ is inner on every vertex group H_u with $u \in V_H^1$ (in particular on every one-ended subgroup of H).*

(2) *The homomorphism j is injective on one-ended subgroups of H.*

Proof Let $u \in V_H^1$. By definition, there exists a vertex $w = s^{-1}(u)$ of Δ_G such that $r(G_w) = xH_u x^{-1}$ for some $x \in H$. Moreover, there exists an element $g \in G$ such that $j \circ r = \mathrm{ad}(g)$ on G_w. Let h be an element of H_u, and set $h' = xhx^{-1}$. Note that h' belongs to $r(G_w)$. Since r induces an isomorphism from G_w to $r(G_w)$, we can write $(r \circ j)(h') = (r \circ j)\big(r(r^{-1}(h'))\big)$. But $(j \circ r)(r^{-1}(h')) = gr^{-1}(h')g^{-1}$, and therefore $(r \circ j)(h') = r(g)h'r(g)^{-1}$. It follows that $(r \circ j)(h) = yhy^{-1}$ with $y = ((r \circ j)(x))^{-1} r(g) x$. Thus $(r \circ j) = \mathrm{ad}(y)$ on H_u. This proves the first assertion.

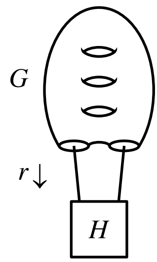


Figure 2: The group G is a (weak) hyperbolic floor over H.

Let T be the Bass–Serre tree of the splitting Δ_H of H, and let A be a one-ended subgroup of H. Since edge groups of T are finite, the subgroup A fixes a vertex of T. Hence, A is contained in a conjugate of H_u for some $u \in V_H$. According to the previous paragraph, the homomorphism $r \circ j$ coincides with an inner automorphism on H_u. In particular, it is injective on A. Hence, j is injective on A. This proves the second point. □

Remark 5.4 In Definition 5.2, no assumption is made about the image of the QH subgroup G_v, nor about the orbifold Euler characteristic of the underlying conical orbifold of G_v, whereas in the definition of a hyperbolic floor, Definition 5.1, $r(G_v)$ is assumed to be nonabelian and the underlying topological surface of G_v has Euler characteristic at most -2, or is a punctured torus. In fact, these technical hypotheses are not necessary in order to prove that hyperbolicity is preserved under elementary equivalence.

Sometimes it is convenient to think of G and H as subgroups of a bigger group G' obtained from G by performing amalgamated products and HNN extensions over finite groups, with the property that there exists an epimorphism $\rho\colon G' \twoheadrightarrow H$ satisfying $\rho|_G = r$ and $\rho|_H = \mathrm{id}_H$. This group G' can be built as follows. Let $w_1, \dots, w_d$

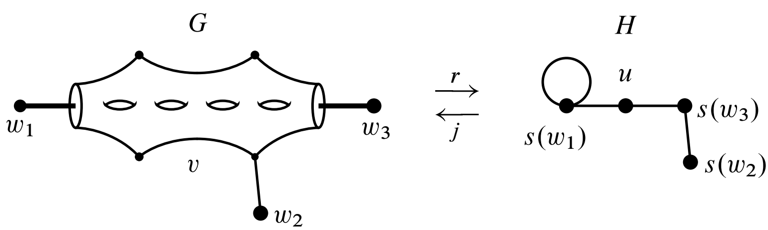


Figure 3: The group G is a quasifloor over H. Here $V_H^2 = \{u\}$ and $V_H^1 = \{s(w_1), s(w_2), s(w_3)\}$. The vertex u is labeled by a finite group. Edges with infinite stabilizer are thickened.

denote the vertices of Δ_G different from the central vertex v. First, let

$$\widehat{G} = \langle G, H \mid g = r(g) \text{ for all } g \in G_{w_1} \rangle,$$

which is an amalgamated product $G *_{G_{w_1}} H$, and let $\hat{r}\colon \widehat{G} \twoheadrightarrow H$ be the retraction defined by $\hat{r}|_G = r$ and $\hat{r}|_H = \mathrm{id}_H$. Then, let us define an overgroup G' of $\widehat{G}$ and a retraction $\rho\colon G' \twoheadrightarrow H$ as follows:

$$G' = \langle \widehat{G}, t_2, \dots, t_d \mid t_k g t_k^{-1} = r(g) \text{ for all } g \in G_{w_k} \text{ and all } k \in [\![1, d]\!] \rangle,$$

$\rho|_{\widehat{G}} = \hat{r}$ and $\rho(t_k) = 1$ for every $2 \leq k \leq d$. The group G' can be viewed in an equivalent way as the fundamental group of the graph of groups Λ obtained from Δ_G and Δ_H by identifying, for every vertex $w \in V_G \setminus \{v\}$, the vertex group G_w with $H_{s(w)}$ via the isomorphism $\mathrm{ad}(h_w^{-1}) \circ r|_{G_w}\colon G_w \to H_{s(w)}$, with the same notation as in Definition 5.2. Figure 4 illustrates this construction (to be compared with Figure 3).

In the case where G is torsion-free, the group G' defined above is a free product of G with a free group of rank n depending on $|V_G \setminus \{v\}|$ and $\mathrm{rk}(\pi_1(\mathcal{G}))$, where $\mathcal{G}$ stands for the underlying graph of Δ_H. It follows from this observation that, in the torsion-free case, the definition of a weak hyperbolic floor, Definition 5.1, and the definition of a quasifloor, Definition 5.2, are equivalent, up to replacing G by $G * F_n$. This equivalence is explained in detail in the following statement.

Proposition 5.5 *Let G be a torsion-free group. If (G, H, Δ_G, r) is a weak hyperbolic floor, then $(G, H, \Delta_G, \Delta_H, r, i)$ is a quasifloor, where i stands for the inclusion of H into G, and Δ_H stands for the splitting of H reduced to a point. Conversely, if $(G, H, \Delta_G, \Delta_H, r, j)$ is a quasifloor, then there exist a free group F_n and a retraction $\rho\colon G' \twoheadrightarrow H$, where $G' = G * F_n$, such that $\rho|_G = r$ and $(G', H, \Delta_{G'}, \rho)$ is a weak hyperbolic floor, where $\Delta_{G'}$ is the splitting of G' obtained from the graph Λ defined above by collapsing to a point the graph Δ_H, viewed as a subgraph of Λ.*

Remark 5.6 Let Γ be a torsion-free hyperbolic group and let G be a Γ–limit group. It is worth noting that $G * F_n$ is a Γ–limit group as well, since Γ and $\Gamma * F_n$ have the same existential theory by Proposition 2.21, and even the same theory, by [26]. Hence, replacing G by $G * F_n$ does not affect the first-order theory, in the torsion-free case.

It is important to emphasize that, maybe, our construction of a quasifloor in Section 7 might be modified to ensure that r is an epimorphism, and that j is a monomorphism. However, it seemed to us that this could give rise to a number of new technical difficulties.

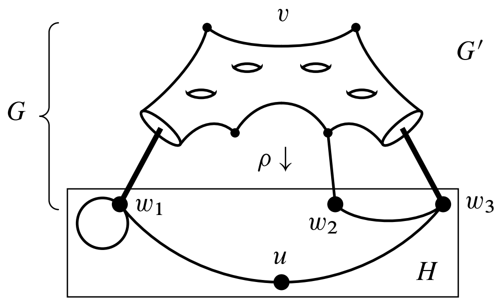


Figure 4: We construct a group G' by identifying w with $s(w)$ for every $w \in V_G \setminus \{v\}$. Edges with infinite stabilizer are thickened.

These complications will be avoided by using Theorem 4.8 stating that every hyperbolic group embeds into a torsion-saturated hyperbolic group (see Definition 4.1).

Example 5.7 shows a quasifloor in the presence of torsion.

Example 5.7 Let $A = \big(\langle x\rangle \times F(x_1,x_2)\big) * \big(\langle y\rangle \times F(x_3,x_4)\big)$, with x of order 6 and with y of order 10, where $F(x_i,x_j)$ stands for the free group on two generators x_i and x_j. The group A admits the presentation

$$A = \langle x, y, x_1, x_2, x_3, x_4 \mid [x,x_1] = [x,x_2] = [y,x_3] = [y,x_4] = x^6 = y^{10} = 1\rangle.$$

Let Σ be the orientable surface of genus two with two boundary components, and let $S = \pi_1(\Sigma)$. Call its two boundary subgroups (up to conjugation) $\langle b_1\rangle$ and $\langle b_2\rangle$. Let $B = \langle z\rangle \times S$ with z of order 2. The group B admits the presentation

$$B = \langle z, s_1, s_2, s_3, s_4, b_1, b_2 \mid [s_3,s_4][s_1,s_2]b_1b_2 = [z,b_i] = [z,s_j] = z^2 = 1\rangle.$$

Let us define a graph of groups Δ_G with two vertices labeled by A and B, and two edges linking these vertices, identifying the extended boundary subgroup $\langle z\rangle \times \langle b_1\rangle$ of B with the subgroup $\langle x^3\rangle \times \langle [x_1,x_2]\rangle$ of A by $z \mapsto x^3$ and $b_1 \mapsto [x_2,x_1]$ and the extended boundary subgroup $\langle z\rangle \times \langle b_2\rangle$ of B with the subgroup $\langle y^5\rangle \times \langle [x_3,x_4]\rangle$ of A by $z \mapsto y^5$ and $b_2 \mapsto [x_4,x_3]$. Denote by G the fundamental group of this graph of groups Δ_G.

First, note that G cannot be a quasifloor over A. To see this, note that each element of G of order 2 commutes with an element of order 3 and with an element of order 5, whereas there are two conjugacy classes of elements of order 2 in A: those commuting with an element of order 3, and those commuting with an element of order 5. Hence, there cannot exist any homomorphism $G \to A$ that is injective on finite subgroups.

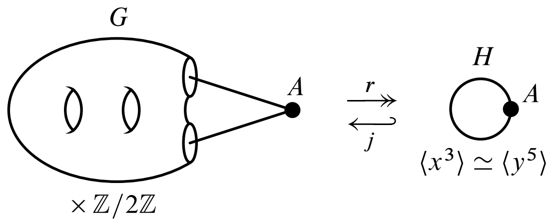


Figure 5: G is a quasifloor over H, but not over A.

Let $H=\langle A,t \mid tx^3t^{-1}=y^5\rangle$, let $j\colon H\to G$ denote the monomorphism which is the identity on A and on t, and let Δ_H be the splitting of H with exactly one vertex v such that $H_v=A$ and exactly one edge identifying $\langle x^3\rangle$ with $\langle y^5\rangle$.

Here are two finite presentations of G and H:

$$G=\langle A,B,t \mid z=x^3,\ b_1=[x_2,x_1],\ tb_2t^{-1}=[x_4,x_3],\ tzt^{-1}=y^5\rangle$$

$$=\left\langle x,y,x_1,x_2,x_3,x_4,s_1,s_2,s_3,s_4,t \;\middle|\; \begin{array}{c} tx^3t^{-1}=y^5,\ x^6=y^{10}=1,\\ [s_3,s_4][s_1,s_2][x_2,x_1]t^{-1}[x_4,x_3]t=1,\\ [x,x_1]=[x,x_2]=[y,x_3]=[y,x_4]=1 \end{array}\right\rangle,$$

$$H=\left\langle x,y,x_1,x_2,x_3,x_4,t \;\middle|\; \begin{array}{c} tx^3t^{-1}=y^5,\ x^6=y^{10}=1,\\ [x,x_1]=[x,x_2]=[y,x_3]=[y,x_4]=1 \end{array}\right\rangle$$

$$=\langle A,t \mid tx^3t^{-1}=y^5\rangle.$$

The group G retracts onto H via the epimorphism $r\colon G\to H$ defined by

$$r\colon \begin{cases} a\mapsto a & \text{if } a\in\{x,y,x_1,x_2,x_3,x_4,t\},\\ s_i\mapsto x_i & \text{if } 1\le i\le 2,\\ s_i\mapsto t^{-1}x_it^1 & \text{if } 3\le i\le 4. \end{cases}$$

Hence, $(G,H,\Delta_G,\Delta_H,r,j)$ is a quasifloor. Indeed, $j\circ r$ coincides with the identity on A, and r maps $A\subset G$ isomorphically to $A\subset H$.

See Figure 5.

We now define quasitowers, which are obtained by successive addition of quasifloors.

Definition 5.8 A *quasitower* is a finite sequence of quasifloors

$$(G_k,H_k,\Delta_{G_k},\Delta_{H_k},r_k,j_k)_{1\le k\le n}$$

such that $G_{k+1}=H_k$ for every $1\le k\le n-1$. Note that $\Delta_{G_{k+1}}$ and Δ_{H_k} are two distinct splittings of $G_{k+1}=H_k$.

If, moreover, every quasifloor is strict, then the quasitower is said to be strict. The integer n is called the *height* of the quasitower. The homomorphisms $r_k \circ \cdots \circ r_1$ and $j_1 \circ \cdots \circ j_k$ are denoted by $\overline{r}_k$ and $\overline{j}_k$ respectively.

Let $G := G_1$, $H := H_n$, $r := \overline{r}_n$ and $j := \overline{j}_n$. For reasons of brevity, one sometimes writes (G, H, r, j) instead of $(G_k, H_k, \Delta_{G_k}, \Delta_{H_k}, r_k, j_k)_{1 \leq k \leq n}$.

We end this subsection with two definitions that are comparable to Sela's elementary prototype [27, Definition 7.3] and Sela's elementary core [27, Definition 7.5].

Definition 5.9 A *quasiprototype* is a finitely generated group G that is not a strict quasifloor over any group H.

Example 5.10 A one-ended hyperbolic group whose $\mathcal{Z}$–JSJ splitting does not contain any QH vertex is a quasiprototype.

Definition 5.11 Let G be a finitely generated group. If G is not a quasiprototype, a *quasicore* of G is a group C satisfying the following two conditions:

- G is a strict quasitower over C.
- C is a quasiprototype.

If G is a quasiprototype, we define G as the only quasicore of G.

Remark 5.12 If a quasicore exists, it is not unique a priori.

5.2 Inheritance of hyperbolicity

Here is an easy but crucial proposition.

Proposition 5.13 *Let G and H be two groups. Suppose that G is a quasitower over H. The following hold*:

- *G is hyperbolic if and only if H is hyperbolic.*
- *G embeds into a hyperbolic group if and only if H embeds into a hyperbolic group.*
- *G is hyperbolic and cubulable if and only if H is hyperbolic and cubulable.*

The proof of the third claim is postponed to Section 6.4 (see Proposition 6.19).

Proof We shall prove the proposition in the case where G is a quasifloor over H, and the general case follows immediately by induction. Let Δ_G and Δ_H be the splittings

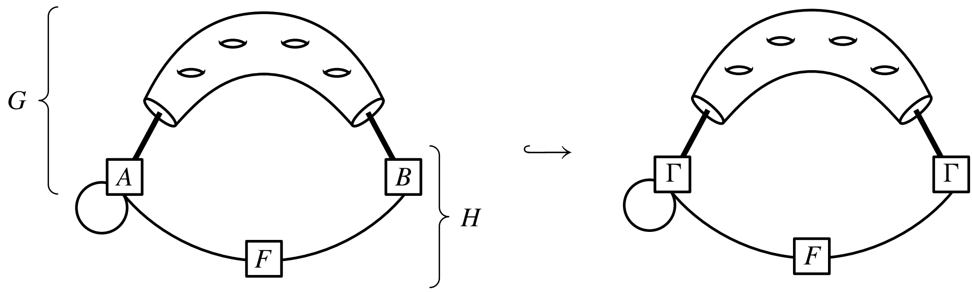


Figure 6: Second point of Proposition 5.13: The group G is a quasifloor over H; if H embeds into a hyperbolic group Γ, then G embeds into the hyperbolic group obtained by replacing A and B by Γ. Edges with infinite stabilizer are thickened.

of G and H associated with the quasifloor structure. By definition, Δ_G is a centered graph of groups. Let V_G be its set of vertices, and let v be the central vertex. Let $V_H = V_H^1 \sqcup V_H^2$ be a partition of the set of vertices V_H of Δ_H as in Definition 5.2. By definition, there exists a bijection $s\colon V_G \setminus \{v\} \to V_H^1$ such that $G_w \simeq H_{s(w)}$ for every $w \in V_G \setminus \{v\}$.

We prove the first claim. Suppose that H is hyperbolic. Then, by Proposition 3.11, H_u is hyperbolic for every vertex u of Δ_H. As a consequence, the vertex groups of Δ_G are hyperbolic. It follows from Proposition 3.12 that G is hyperbolic. Conversely, if G is hyperbolic, then the vertex groups of Δ_G are hyperbolic by Proposition 3.12. Thus, the vertex groups of Δ_H are hyperbolic. Since Δ_H has finite edge groups, the combination theorem of Bestvina and Feighn (see Proposition 3.10) applies and shows that H is hyperbolic.

Now, let us prove the second claim (see Figure 6). Suppose that H embeds into a hyperbolic group Γ. In particular, each H_u embeds into Γ. As a consequence, each G_w embeds into Γ, for $w \in V_G \setminus \{v\}$. We construct a graph of groups Δ_G^Γ from Δ_G by replacing each vertex group G_w by Γ. Denote by Ω the fundamental group of Δ_G^Γ, and let us observe that Δ_G^Γ is a centered splitting of Ω. Since Γ is hyperbolic, Proposition 3.12 tells us that Ω is hyperbolic. In addition, it is clear that G embeds into Ω. Conversely, if G embeds into a hyperbolic group, we prove in the same way that H embeds into a hyperbolic group. □

5.3 Every Γ–limit group has a quasicore

This subsection is devoted to a proof of the following result.

Proposition 5.14 *Let Γ be a hyperbolic group and G a Γ–limit group. Then G has a quasicore C. Moreover, G is hyperbolic if and only if C is hyperbolic.*

The second part is an immediate consequence of Proposition 5.13. It remains to prove the first part, which is the following proposition.

Proposition 5.15 *If Γ is a hyperbolic group, then every Γ–limit group has a quasicore.*

In other words, the previous proposition claims that a Γ–limit group is either a quasiprototype or a strict quasitower over a quasiprototype. Proposition 5.15 is an easy consequence of the following lemma, which will be proved below.

Lemma 5.16 *There does not exist any infinite sequence $(G_n)_{n\in\mathbb{N}}$ of finitely generated groups such that G_0 is a Γ–limit group and for every integer n, the group G_n is a strict quasifloor over G_{n+1}.*

Before proving this lemma, let us make some observations. First, note that if Γ is a torsion-free hyperbolic group, it follows from the descending chain condition (see Theorem 2.8) that there does not exist any infinite sequence $(G_n)_{n\in\mathbb{N}}$ such that G_0 is a Γ–limit group and each G_n is a hyperbolic floor over G_{n+1} (in the sense of Sela).

In the presence of torsion, however, Definition 5.2 has two drawbacks that seem to be obstacles to the use of the descending chain condition: if G is a Γ–limit group, and if G is a quasifloor over H, then, in general,

(1) H is not a Γ–limit group,

(2) H is not a quotient of G.

We remedy the first issue (see Propositions 5.17 and 5.18) by using the fact that every hyperbolic group embeds into a torsion-saturated hyperbolic group (see Theorem 4.8).

Proposition 5.17 *Let Γ be a hyperbolic group, and G a Γ–limit group. Let $\overline{\Gamma}$ be a torsion-saturated hyperbolic group containing Γ. Suppose that G is a quasifloor over a finitely generated group H. Then H is a $\overline{\Gamma}$–limit group.*

Proof Let Δ_H be the splitting of H over finite groups associated to the structure of a quasifloor. It follows from the definition of a quasifloor that every vertex group of Δ_H embeds into G, so is a Γ–limit group. By Theorem 4.3, the class of $\overline{\Gamma}$–limit groups is closed under HNN extensions and amalgamated free products over finite groups, so H is a $\overline{\Gamma}$–limit group. □

The following corollary is immediate.

Corollary 5.18 *Let Γ be a hyperbolic group. Let $\overline{\Gamma}$ be a torsion-saturated hyperbolic group containing Γ. Let $(G_n)_{n\geq 0}$ be a sequence of finitely generated groups such that G_0 is a Γ–limit group and G_n is a quasifloor over G_{n+1} for every n. Then every G_n is a $\overline{\Gamma}$–limit group.*

Next, the following proposition remedies the lack of surjectivity in the definition of a quasifloor (see the second issue listed above), under the assumption that each quasifloor is a Γ–limit group.

Proposition 5.19 *Let Γ be a hyperbolic group. There does not exist any infinite sequence $(G_n)_{n\in\mathbb{N}}$ of Γ–limit groups such that for every integer n, the group G_n is a strict quasifloor over G_{n+1}.*

Proof Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of Γ–limit groups. Suppose that G_n is a quasifloor over G_{n+1}, and let $r_n\colon G_n\to G_{n+1}$ and $j_n\colon G_{n+1}\to G_n$ be the associated homomorphisms, for every integer n. We shall prove that there exists an integer n_0 such that for every $n\geq n_0$, the quasifloor (G_n,G_{n+1},r_n,j_n) is not strict.

Since the homomorphisms r_n are not assumed to be surjective, one cannot apply the descending chain condition (see Theorem 2.8) to the sequence $(r_n\colon G_n\to G_{n+1})_{n\in\mathbb{N}}$. Let $G'_0=G_0$ and $G'_{n+1}=r_n(G'_n)$ for every integer n. The descending chain condition, applied to the sequence of epimorphisms $(r_n|_{G'_n}\colon G'_n\twoheadrightarrow G'_{n+1})_{n\in\mathbb{N}}$, ensures that $r_n|_{G'_n}$ is injective for every n large enough.

In order to conclude the proof, we shall verify that the restriction of r_n to G'_n is not injective if G_n is a strict quasifloor over G_{n+1}.

If G_n is a strict quasifloor over G_{n+1}, there is by definition a one-ended subgroup A_n of G_n such that $A_n\cap\ker(r_n)\neq\{1\}$. Set $A'_n=(r_{n-1}\circ\cdots\circ r_0\circ j_0\circ\cdots\circ j_n)(A_n)$. This is a subgroup of $G'_n=(r_{n-1}\circ\cdots\circ r_0)(G_0)$. In order to prove that r_n is noninjective in restriction to G'_n, it is enough to prove that r_n is noninjective in restriction to A'_n. Indeed, let us prove that $A'_n=gA_ng^{-1}$ for some element $g\in G_n$. This is enough to conclude because we know that there is a nontrivial element x in $A_n\cap\ker(r_n)$, and hence gxg^{-1} is a nontrivial element of $A'_n\cap\ker(r_n)$.

Let us prove that A'_n and A_n are conjugate in G_n. Set $B_n=(j_0\circ\cdots\circ j_{n-1})(A_n)$. Since $(j_1\circ\cdots\circ j_{n-1})(A_n)$ is a one-ended subgroup of G_1, it follows from Lemma 5.3

that $(r_0 \circ j_0)((j_1 \circ \cdots \circ j_{n-1})(A_n)) = r_0(B_n)$ is conjugate to $(j_1 \circ \cdots \circ j_{n-1})(A_n)$. Therefore, $(r_1 \circ r_0)(B_n)$ is conjugate to $(r_1 \circ j_1 \cdots \circ j_{n-1})(A_n)$, which is conjugate to $(j_2 \circ \cdots \circ j_{n-1})(A_n)$ by Lemma 5.3, since $(j_2 \circ \cdots \circ j_{n-1})(A_n)$ is a one-ended subgroup of G_2. An immediate induction shows that $(r_{n-1} \circ \cdots \circ r_0)(B_n) = gA_n g^{-1}$ for some $g \in G_n$. Thus, $A'_n = gA_n g^{-1}$. □

We can now prove Lemma 5.16, which implies Proposition 5.15.

Proof Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of finitely generated groups such that G_0 is a Γ–limit group and for every integer n, the group G_n is a strict quasifloor over G_{n+1}. We aim to prove that this sequence is finite. By Corollary 5.18, every G_n is a $\overline{\Gamma}$–limit group, where $\overline{\Gamma}$ stands for a torsion-saturated hyperbolic group containing Γ. Then it follows from Proposition 5.19 that the sequence $(G_n)_{n\in\mathbb{N}}$ is finite. □

5.4 Quasitowers and relatedness

In this section, we collect some lemmas that will be useful in the proofs of our main theorems.

Lemma 5.20 *Let $(G, H, \Delta_G, \Delta_H, r, j)$ be a quasifloor. Let A be a one-ended finitely generated K–CSA group that does not contain $\mathbb{Z}^2$, and let Δ_A be its $\mathcal{Z}$–JSJ splitting. Then*:

1. *For every monomorphism $\phi\colon A \hookrightarrow G$, $j \circ r \circ \phi$ and ϕ are Δ_A–related.*
2. *For every monomorphism $\phi\colon A \hookrightarrow H$, $r \circ j \circ \phi$ and ϕ are Δ_A–related.*

Proof We shall prove the first assertion. We have to prove that ϕ and $j \circ r \circ \phi$ coincide, up to conjugacy by an element of G, on every non-QH vertex group of Δ_A, as well as on every finite subgroup of A. The condition concerning finite subgroups is obvious, since $j \circ r$ is inner on each finite subgroup of G. Now, let R be an infinite non-QH vertex group of Δ_A. We shall prove that $\phi(R)$ fixes a non-QH vertex of the centered splitting Δ_G. Since R is non-QH, it is rigid (see Proposition 2.25). Hence, R acts elliptically on every tree on which A acts with finite edge stabilizers or with edge stabilizers in $\mathcal{Z}$. Therefore, $\phi(R)$ is elliptic in the Bass–Serre tree T of Δ_G (indeed, the preimage under ϕ of every $\mathcal{Z}$–subgroup (resp. finite subgroup) of G is a $\mathcal{Z}$–subgroup (resp. finite subgroup) of A, because ϕ is injective). We make the following observation: let S be a finite-by-orbifold group, and let B be a subgroup of S. If B is elliptic in every splitting of S over $\mathcal{Z}$, then B is finite or lies in an extended boundary subgroup (see [7, Corollary 5.24(5)]). As a consequence, if R is rigid, $\phi(R)$ lies in a conjugate of a noncentral vertex group of Δ_G. In the case where R is virtually cyclic, then $\phi(R)$ may possibly lie in a boundary subgroup of the central QH vertex group of Δ_G. In any case, $\phi(R)$ lies in a conjugate of a noncentral vertex group of Δ_G. Hence, since $j \circ r$ is Δ_G–related to the identity of G, there exists an element g in G such that

$$j \circ r \circ \phi|_R = \mathrm{ad}(g) \circ \phi|_R.$$

This finishes the proof of the first assertion.

We now prove the second assertion. By Lemma 5.3, $r \circ j$ is inner on every one-ended vertex group of Δ_H. Since $\phi(A) \simeq A$ is one-ended, it is contained in a one-ended vertex group of Δ_H. Thus, $r \circ j$ is inner on $\phi(A)$. □

Lemma 5.21 follows from Lemma 5.20 by induction.

Lemma 5.21 *Let $(G_k, H_k, \Delta_{G_k}, \Delta_{H_k}, r_k, j_k)_{1\leq k\leq n}$ be a quasitower of height $n \geq 1$. Let $G = G_1$ and $H = G_n$. Let A be a one-ended finitely generated K–CSA group that does not contain $\mathbb{Z}^2$, and let Δ_A be its $\mathcal{Z}$–JSJ splitting. Then:*

1. *For every monomorphism $\phi\colon A \hookrightarrow G$, if $\overline{r}_{n-1} \circ \phi$ is injective, then $j \circ r \circ \phi$ and ϕ are Δ_A–related.*
2. *For every monomorphism $\phi\colon A \hookrightarrow H$, $r \circ j \circ \phi$ and ϕ are Δ_A–related.*

Proof Set $\overline{r}_0 = \mathrm{id}_G$. Let us prove the first point by induction on the integer n. By Lemma 5.20, the claim is true if $n = 1$. Now, suppose that the claim is true for a given integer n, and let us prove that it is true for $n+1$. Let $\phi\colon A \hookrightarrow G$ be a monomorphism such that $\overline{r}_n \circ \phi$ is injective, where $\overline{r}_n = r_n \circ \cdots \circ r_1$. We claim that $j \circ r \circ \phi$ and ϕ are Δ_A–related, where $j = \overline{j}_{n+1} = j_1 \circ \cdots \circ j_{n+1}$ and $r = \overline{r}_{n+1} = r_{n+1} \circ \overline{r}_n$. By the induction hypothesis, the morphisms $j_1 \circ \cdots \circ j_n \circ r_n \circ \cdots \circ r_1 \circ \phi$ and ϕ are Δ_A–related. Let R be a rigid vertex group of Δ_A. Since $\overline{r}_n \circ \phi = r_n \circ \cdots \circ r_1 \circ \phi$ is injective, the group $(\overline{r}_n \circ \phi)(R)$ is a rigid subgroup of G_n. As a consequence, it is contained in a noncentral vertex group of the centered splitting Δ_{G_n} of G_n. By definition of a quasifloor, the restriction of the morphism $j_{n+1} \circ r_{n+1}$ to $(\overline{r}_n \circ \phi)(R)$ is a conjugacy. Hence, $(j_{n+1} \circ r_{n+1} \circ \overline{r}_n \circ \phi)(R)$ is conjugate to $(\overline{r}_n \circ \phi)(R)$. It follows that $(\overline{j}_n \circ j_{n+1} \circ r_{n+1} \circ \overline{r}_n \circ \phi)(R) = (j \circ r \circ \phi)(R)$ is conjugate to $(\overline{j}_n \circ \overline{r}_n \circ \phi)(R)$, which is conjugate to $\phi(R)$ by induction hypothesis. This proves that $j \circ r \circ \phi$ and ϕ are Δ_A–related and concludes the proof of the first point. The proof of the second point is similar and is left to the reader. □

By combining Lemma 5.21 with the shortening argument Corollary 3.1, we get the following result.

Lemma 5.22 *Let (Γ, Γ', r, j) be a quasitower. Suppose that Γ embeds into a hyperbolic group. Let H be a one-ended finitely generated K–CSA group that does not contain $\mathbb{Z}^2$. Let Δ_H be its $\mathcal{Z}$–JSJ splitting. Assume that H is not a finite-by-orbifold group. There exists a finite set $F \subset H \setminus \{1\}$ such that for every homomorphism $\phi\colon H \to \Gamma$, either*

- *$r \circ \phi\colon H \to \Gamma'$ is injective, or*
- *ϕ is Δ_H–related to a homomorphism $\phi'\colon H \to \Gamma$ that kills an element of F.*

Proof Let $F \subset H \setminus \{1\}$ be the finite set given by Corollary 3.1 applied to $\mathrm{Hom}(H, \Gamma)$. Let ϕ be a homomorphism from H to Γ. Suppose the homomorphism $r \circ \phi\colon H \to \Gamma'$ is noninjective, and let $n \geq 1$ be the smallest integer such that $\overline{r}_n \circ \phi$ is noninjective. Since $\overline{r}_{n-1} \circ \phi$ is injective, $\overline{j}_n \circ \overline{r}_n \circ \phi$ and ϕ are Δ_H–related, by the first assertion of Lemma 5.21. Thanks to Corollary 3.1, there exists a homomorphism ϕ' that kills an element of F and that is Δ–related to $\overline{j}_n \circ \overline{r}_n \circ \phi$, which is a noninjective homomorphism from H to Γ. By transitivity of the Δ–relatedness, the morphism ϕ' is Δ–related to ϕ. □

We need a variant of Lemma 5.22 in the case where H is not one-ended. Recall that a homomorphism is said to be factor-injective if it is injective in restriction to every one-ended factor that is not finite-by-orbifold (see Definition 2.29). The proof of the following result is similar to that of Lemma 5.22 above.

Lemma 5.23 *Let (Γ, Γ', r, j) be a quasitower. Suppose that Γ embeds into a hyperbolic group. Let H be a finitely generated K–CSA group that does not contain $\mathbb{Z}^2$. Let $H_1, \dots, H_n$ be its one-ended factors (for a fixed $\mathcal{F}$–JSJ splitting of H) that are not finite-by-orbifold. Let Δ_k be the $\mathcal{Z}$–JSJ splitting of H_k for $1 \leq k \leq n$. There exists a finite set $F \subset H \setminus \{1\}$ such that for every homomorphism $\phi\colon H \to \Gamma$, either*

- *$r \circ \phi\colon H \to \Gamma'$ is factor-injective, or*
- *there exists an integer $1 \leq k \leq n$ such that $\phi|_{H_k}$ is Δ_k–related to a homomorphism $\phi'\colon H_k \to \Gamma$ that kills an element of F.*

6 Proofs of Theorems 1.2–1.5

In the current section, we prove our main theorems by admitting a technical result, namely Proposition 6.3, whose proof is postponed to Section 7 for the sake of clarity.

6.1 How to build a quasifloor using first-order logic

Proposition 6.1 *Let (Γ, Γ', r, j) and (G, G', ρ, η) be two quasitowers. Suppose that Γ embeds into a hyperbolic group and that $\mathrm{Th}_{\forall\exists}(\Gamma) \subset \mathrm{Th}_{\forall\exists}(G)$. Let H be a one-ended factor of G'. Suppose that H is not a finite-by-orbifold group. Then either*

- *there exists a morphism $\phi\colon H \to \Gamma$ such that $r \circ \phi$ is injective, or*
- *there exists a noninjective preretraction $H \to G'$.*

Before proving this result, let us justify why H and its canonical $\mathcal{Z}$–JSJ splitting are well defined. Let Ω denote a hyperbolic group in which Γ embeds. There exists a constant K such that all finite subgroups of Ω have order less than K. Since $\Gamma \subset \Omega$ and $\mathrm{Th}_{\exists}(G) \subset \mathrm{Th}_{\exists}(\Gamma)$, and since the morphism $\eta\colon G' \to G$ is injective on finite subgroups of G', every finite subgroup of G' has order less than K. As a consequence, G' has a $\mathcal{F}$–JSJ decomposition (see [11] and Section 2.6.6). Hence, one-ended factors of G' and in particular H are well defined. Then, by the second assertion of Lemma 5.3, the morphism η is injective on H, because H is one-ended. Proposition 2.27 therefore guarantees that the canonical $\mathcal{Z}$–JSJ splitting of H exists. We now prove Proposition 6.1.

Proof Let Δ_H denote the $\mathcal{Z}$–JSJ splitting of H. Suppose that for every homomorphism $\phi\colon H \to \Gamma$, the homomorphism $r \circ \phi\colon H \to \Gamma'$ is not injective in restriction to H. We will prove that there exists a noninjective Δ_H–preretraction from H to G'.

Step 1 By Lemma 5.22, there exists a finite set $F \subset H \setminus \{1\}$ such that every homomorphism $\phi\colon H \to \Gamma$ is Δ_H–related to a homomorphism $\phi'\colon H \to \Gamma$ that kills an element of F. Set $F = \{h_1, \dots, h_\ell\}$.

Step 2 We claim that there exists a noninjective homomorphism $p\colon H \to G$ that is Δ_H–related to $\eta|_H$. The proof of this claim consists in expressing by a $\forall\exists$–sentence the statement of Step 1, and by interpreting this sentence in G. Note that H is finitely generated, because G' is finitely generated by definition of a quasitower. Let $H = \langle s_1, \dots, s_n \mid w_1, w_2, \dots \rangle$ be a (possibly infinite) presentation of H. Let $W_i = \{w_1, \dots, w_i\}$ for every $i \geq 0$. Denote by H_i the finitely presented group $\langle s_1, \dots, s_n \mid W_i \rangle$. By hypothesis, Γ embeds into a hyperbolic group Ω. As a hyperbolic group, Ω is equationally noetherian (see [20, Corollary 6.13]). Note that Γ and G are Ω–limit groups, because $\mathrm{Th}_\exists(G) \subset \mathrm{Th}_\exists(\Gamma) \subset \mathrm{Th}_\exists(\Omega)$. This implies that Γ and G are equationally noetherian (see [18, Corollary 2.10]). As a consequence, the sets $\mathrm{Hom}(H, \Gamma)$ and $\mathrm{Hom}(H, G)$ are respectively in bijection with $\mathrm{Hom}(H_i, \Gamma)$ and $\mathrm{Hom}(H_i, G)$ for i large enough (see [20, Lemma 5.2]). Hence, there exists an integer i such that $\mathrm{Hom}(H, \Gamma)$ (resp. $\mathrm{Hom}(H, G)$) is in bijection with the n–tuples in Γ^n (resp. G^n) that are solutions of the following system of equations, denoted by $\Sigma_i(x_1, \dots, x_n)$:

$$\begin{cases} w_1(x_1, \dots, x_n) = 1, \\ \quad \vdots \\ w_i(x_1, \dots, x_n) = 1. \end{cases}$$

Let ϕ and ϕ' be homomorphisms from H to Γ. Recall that there exists an existential formula $\psi(x_1, \dots, x_{2n})$ with $2n$ free variables such that

$$\Gamma \models \phi(\phi(s_1), \dots, \phi(s_n), \phi'(s_1), \dots, \phi'(s_n))$$

if and only if ϕ and ϕ' are Δ_H–related (see Lemma 3.7).

We can write a $\forall\exists$–first-order sentence μ, satisfied by Γ, whose interpretation in Γ is the statement of Step 1: for every homomorphism $\phi\colon H \to \Gamma$, there exists a homomorphism $\phi'\colon H \to \Gamma$ that is Δ_H–related to ϕ and that kills some h_ℓ. The sentence μ is the following:

$$\forall x_1 \dots \forall x_n \ \exists y_1 \dots \exists y_n \ \big(\Sigma_i(x_1, \dots, x_n) = 1 \\ \Rightarrow \big(\Sigma_i(y_1, \dots, y_n) = 1 \wedge \psi(x_1, \dots, x_n, y_1, \dots, y_n) \wedge \textstyle\bigvee_{\ell \in [\![1,k]\!]} h_\ell(y_1, \dots, y_n) = 1\big)\big).$$

In this form, it is not totally clear that μ is a $\forall\exists$–sentence. Recall that $\psi(\boldsymbol{x}, \boldsymbol{y})$ is of the form $\exists \boldsymbol{u}\ \theta(\boldsymbol{x}, \boldsymbol{y})$, where θ is a quantifier-free formula, and $\boldsymbol{x}$ and $\boldsymbol{y}$ are two n–tuples of variables. Hence, μ can be rewritten in the following manner:

$$\forall \boldsymbol{x} \ \exists \boldsymbol{y} \ \exists \boldsymbol{u} \ \Sigma_i(\boldsymbol{x}) \neq 1 \vee \Sigma_i(\boldsymbol{y}) = 1 \wedge \theta(\boldsymbol{x}, \boldsymbol{y}) \wedge \textstyle\bigvee_{\ell \in [\![1,k]\!]} h_\ell(\boldsymbol{y}) = 1.$$

Since $\mu \in \mathrm{Th}_{\forall\exists}(\Gamma) \subset \mathrm{Th}_{\forall\exists}(G)$, the sentence μ is true in G as well. The interpretation of μ in G is the following: for every homomorphism $\phi\colon H \to G$, there exists a homomorphism $\phi'\colon H \to G$ that is Δ_H–related to ϕ and that kills some h_ℓ. By taking $\phi = \eta|_H$, we get a noninjective homomorphism $p\colon H \to G$ that is Δ_H–related to $\eta|_H$.

Conclusion Because the homomorphism $p\colon H \to G$ is Δ_H–related to $\eta|_H$, the homomorphism $\rho \circ p\colon H \to G'$ is Δ_H–related to $\rho \circ \eta|_H$. Indeed, the fact that p and $\eta|_H$ coincide up to conjugacy on finite subgroups of H and on vertex groups of Δ_H remains true for $\rho \circ p$ and $\rho \circ \eta|_H$. Moreover, $\rho \circ \eta|_H$ is Δ_H–related to the inclusion of H into G' by the second assertion of Lemma 5.21. This concludes the proof. □

In the same way, one can prove the following result, which generalizes Proposition 6.1.

Proposition 6.2 *Let (Γ, Γ', r, j) and (G, G', ρ, η) be two quasitowers. Suppose Γ embeds into a hyperbolic group and $\mathrm{Th}_{\forall\exists}(\Gamma) \subset \mathrm{Th}_{\forall\exists}(G)$. Let $G'_1, \dots, G'_n$ be the one-ended factors of G' that are not finite-by-orbifold groups. Then either*

- *there exists a morphism $\phi\colon G' \to \Gamma$ such that $r \circ \phi$ is factor-injective* (ie injective on each G'_i)*, or*
- *there exists an integer $1 \le k \le n$ and a noninjective preretraction $G'_k \to G'$.*

In Section 7, we shall prove the following proposition, whose proof is quite technical.

Proposition 6.3 *Let G be a finitely generated K–CSA group that does not contain $\mathbb{Z}^2$. Let A be a one-ended factor of G. Suppose that A is not a finite-by-orbifold group. If there exists a noninjective preretraction $p\colon A \to G$, then there exist a finitely generated group H and two morphisms $r\colon G \to H$ and $j\colon H \to G$ such that (G, H, r, j) is a strict quasifloor.*

Together, Propositions 6.1 and 6.3 endow us with a machinery to build quasifloors.

Proposition 6.4 *Let (Γ, Γ', r, j) be a quasitower. Suppose that Γ embeds into a hyperbolic group. Let G be a finitely generated group. Suppose $\mathrm{Th}_{\forall\exists}(\Gamma) \subset \mathrm{Th}_{\forall\exists}(G)$. In particular, G has a quasicore G'. If H is a one-ended factor of G', then H embeds into Γ'.*

Similarly, by putting together Propositions 6.2 and 6.3, one gets the following more general statement.

Proposition 6.5 *Let (Γ, Γ', r, j) be a quasitower. Suppose that Γ embeds into a hyperbolic group. Let G be a finitely generated group. Suppose $\mathrm{Th}_{\forall\exists}(\Gamma) \subset \mathrm{Th}_{\forall\exists}(G)$. In particular, G has a quasicore G'. Then there exists a factor-injective homomorphism from G' to Γ'.*

6.2 Proofs of Theorems 1.3 and 1.4

Theorem 6.6 *Let Γ be a group that embeds into a hyperbolic group Ω, and let G be a finitely generated group. If $\mathrm{Th}_{\forall\exists}(\Gamma) \subset \mathrm{Th}_{\forall\exists}(G)$, then G embeds into a hyperbolic group.*

Proof Since G is a Ω–limit group, it possesses a quasicore G' (by Proposition 5.15). There exists an integer K such that all finite subgroups of Ω have order $\leq K$. As a consequence, all finite subgroups of G' have order $\leq K$. Therefore, by [11], G' has a $\mathcal{F}$–JSJ splitting. We claim that all one-ended factors of G' are hyperbolic. Let H be such a one-ended factor. If H is a finite-by-orbifold group, then it is hyperbolic by definition. If H is not a finite-by-orbifold group, it follows from Proposition 6.4 that H embeds into Γ, and hence into Ω. Now, let $H_1, \dots, H_p$ be the one-ended factors of G' (well-defined up to conjugation) that are not finite-by-orbifold groups, and let Λ be the graph of groups obtained by replacing each H_k in a $\mathcal{F}$–JSJ splitting of G' by Ω. Let Ω' be the fundamental group of Λ. It is clear that G' embeds into Ω'. In addition, vertex groups of Λ being hyperbolic and edge groups of Λ being finite, the group Ω' is hyperbolic thanks to the combination theorem of Bestvina and Feighn (see Proposition 3.10). Thus, by Proposition 5.13, G embeds into a hyperbolic group Ω''. □

If Ω is locally hyperbolic then, with the same notation as in the previous proof, the groups Ω' and Ω'' are locally hyperbolic as well. As a consequence, the following theorem holds.

Theorem 6.7 *Let Ω be a locally hyperbolic group, let Γ be a subgroup of Ω, and let G be a finitely generated group. If $\mathrm{Th}_{\forall\exists}(\Gamma) \subset \mathrm{Th}_{\forall\exists}(G)$, then G is a locally hyperbolic group.*

Remark 6.8 We stress that the theorem above can be derived more directly from Sela's shortening argument, without using quasitowers. Sela [25, Corollary 4.4] proved that a limit group is hyperbolic if and only if it does not contain $\mathbb{Z}^2$. Kharlampovich and Myasnikov [9, Corollary 4] also proved this result. It follows that a finitely generated group G satisfying $\mathrm{Th}_{\forall\exists}(F_2) \subset \mathrm{Th}_{\forall\exists}(G)$ is hyperbolic, thanks to Corollary 2.18. In fact, Sela's proof shows that G is locally hyperbolic, and his proof remains valid if we replace F_2 by any locally hyperbolic group.

6.3 Proof of Theorem 1.2

We will prove Theorem 1.2 using the following result, whose proof is postponed.

Proposition 6.9 *Let* Γ *be a hyperbolic group and* G *a finitely generated group such that* $\mathrm{Th}_{\forall\exists}(\Gamma)=\mathrm{Th}_{\forall\exists}(G)$*. Let* G' *be a quasicore of* G *and* Γ' *a quasicore of* Γ*. Then every one-ended factor of* G' *that is not a finite-by-orbifold group is isomorphic to a one-ended factor of* Γ'*. Therefore,* G' *is hyperbolic.*

More precisely, we prove the following result, which immediately implies Theorem 1.2.

Theorem 6.10 *Let* Γ *be a hyperbolic group and* G *a finitely generated group such that* $\mathrm{Th}_{\forall\exists}(\Gamma)=\mathrm{Th}_{\forall\exists}(G)$*. Then* G *is hyperbolic.*

Proof Let Γ' be a quasicore of Γ, and let G' be a quasicore of G. By Proposition 5.13 about inheritance of hyperbolicity, the group Γ' is hyperbolic. Thus, by Proposition 3.11, all one-ended factors of Γ' are hyperbolic. Then Proposition 6.9 shows that all one-ended factors of G' are hyperbolic, which implies that G is hyperbolic according to the combination theorem of Bestvina and Feighn (see Proposition 3.10). Finally, Proposition 5.13 proves that G is hyperbolic, as a quasitower over a hyperbolic group. □

The rest of this subsection is devoted to a proof of Proposition 6.9. Let $G'_1,\dots,G'_n$ be the one-ended factors (well-defined up to conjugacy) of G' that are not finite-by-orbifold groups. Let $\Gamma'_1,\dots,\Gamma'_m$ be the one-ended factors of Γ' that are not finite-by-orbifold groups. By Proposition 6.5, there exist a factor-injective homomorphism ϕ from G' to Γ', and a factor-injective homomorphism ψ from Γ' to G'. For every integer $1\le i\le n$, the group $\phi(G'_i)$ is contained in a conjugate of Γ'_j for some $1\le j\le m$. Therefore, one can associate to ϕ a map $\tau(\phi)\colon [\![1,n]\!]\to[\![1,m]\!]$ such that for every $1\le i\le n$, the group $\phi(G'_i)$ is contained in a conjugate of $\Gamma'_{\tau(\phi)(i)}$. Likewise, one can associate to ψ a map $\sigma(\psi)\colon [\![1,m]\!]\to[\![1,n]\!]$ such that for every $1\le j\le m$, the group $\psi(\Gamma'_j)$ is contained in a conjugate of $G'_{\sigma(\psi)(j)}$. In order to prove Proposition 6.9, it suffices to show that we can choose $\tau(\phi)$ and $\sigma(\psi)$ so that $\sigma(\psi)\circ\tau(\phi)$ is a permutation of $[\![1,n]\!]$, since one-ended hyperbolic groups are co-Hopfian (see [24] for torsion-free hyperbolic groups and [15], based on [6], for hyperbolic groups with torsion).

Lemma 6.11 *Let* Γ *be a hyperbolic group and* G *a finitely generated group such that* $\mathrm{Th}_{\forall\exists}(\Gamma)=\mathrm{Th}_{\forall\exists}(G)$*. Let* G' *be a quasicore of* G *and* Γ' *a quasicore of* Γ*. We keep the same notation as above. There exists a factor-injective homomorphism* ϕ *from* G' *to* Γ' *such that* $\tau(\phi)$ *satisfies the following condition: for every* $i\in[\![1,n]\!]$*, if* $\phi(G'_i)$ *is equal to* $\Gamma'_{\tau(\phi)(i)}$ *up to conjugacy by an element of* Γ' *and if* $\Gamma'_{\tau(\phi)(i)}$ *is hyperbolic, then* i *is the unique preimage of* $\tau(\phi)(i)$ *by* $\tau(\phi)$*.*

Proof Assume towards a contradiction that the lemma is false.

Step 1 We claim that there exist some finite subsets $X_1 \subset G'_1, \dots, X_n \subset G'_n$ containing only elements of infinite order, and a finite set $F \subset G' \setminus \{1\}$ such that for every homomorphism ϕ from G' to Γ, either

- there exists an integer $k \in [\![1,n]\!]$ such that $\phi|_{G'_k}$ is Δ_k–related to a homomorphism $\phi' \colon G'_k \to \Gamma$ that kills an element of F, where Δ_k stands for the $\mathcal{Z}$–JSJ splitting of G'_k, or
- there exist an element $\gamma \in \Gamma$ and two elements $x_k \in X_k$ and $x_\ell \in X_\ell$ (with $k \neq \ell$) such that $\phi(x_k) = \gamma\phi(x_\ell)\gamma^{-1}$.

Proof of Step 1 Let F be the set given by Lemma 5.23. We shall define the sets $X_1, \dots, X_n$. Let I be the subset of $[\![1,n]\!]$ consisting of the integers i such that G'_i is hyperbolic. In the case where I is empty, let $X_1 = \dots = X_n = \varnothing$. Now, assume that I is nonempty. For each $k \in [\![1,n]\!]$, since G'_k is one-ended and is not a finite-by-orbifold group, there exists at least one infinite non-QH vertex group A_k in the $\mathcal{Z}$–JSJ splitting Δ_k of G'_k. Fix an element of infinite order $x_k \in A_k$. For each $\ell \in [\![1,n]\!] \setminus \{k\}$, if G'_ℓ is not hyperbolic, let $X_{k,\ell} := \varnothing$; if G'_ℓ is hyperbolic, let $Y_{k,\ell} := \{f(x_k) \mid f \in \mathrm{Mono}(G'_k, G'_\ell)\}$ and let $X_{k,\ell}$ be a set of representatives for the orbits of $Y_{k,\ell}$ under the action of G'_ℓ by conjugation. We claim that the set $X_{k,\ell}$ is finite (possibly empty). Indeed, thanks to Sela's shortening argument, Theorem 2.5, there exist finitely many monomorphisms $f_1, \dots, f_r \colon G'_k \hookrightarrow G'_\ell$ such that for every monomorphism $f \colon G'_k \hookrightarrow G'_\ell$, there exist an integer $1 \le p \le r$ and a modular automorphism σ of G'_k such that f and $f_p \circ \sigma$ coincide up to conjugacy. Since σ coincides with a conjugacy on each rigid subgroup of G'_k, the elements $f(x_k)$ and $f_p(x_k)$ are in the same orbit under the action of G'_ℓ by conjugation, which proves that $X_{k,\ell}$ is finite. Finally, we define

$$X_k := \{x_k\} \cup \bigcup_{1 \le \ell \neq k \le n} X_{k,\ell}.$$

Now, we will prove that these sets have the expected property. Since Γ' is a quasicore of Γ, there exists a quasitower (Γ, Γ', r, j). By Lemma 5.23, for every homomorphism $\phi \colon G' \to \Gamma$, either

- there exists an integer $1 \le k \le n$ such that the restriction $\phi|_{G'_k}$ is Δ_k–related to a homomorphism $\phi' \colon G'_k \to \Gamma$ that kills an element of F, or
- $\theta := r \circ \phi$ is factor-injective.

As explained in the paragraph before Lemma 6.11, one can associate to θ a map $\tau(\theta)\colon [\![1,n]\!] \to [\![1,m]\!]$ such that for every $1 \le i \le n$, the group $\theta(G_i')$ is contained in a conjugate of $\Gamma'_{\tau(\theta)(i)}$. In the event that the second alternative above holds, since we have assumed that Lemma 6.11 is false (towards a contradiction), there exist two integers $k \neq \ell$ such that $\tau(\theta)(k) = \tau(\theta)(\ell)$ and $\theta(G_\ell') = \Gamma'_{\tau(\theta)(\ell)}$ up to conjugacy and such that $\Gamma'_{\tau(\theta)(\ell)}$ is hyperbolic. Thus, $\theta(G_k') \subset \mathrm{ad}(\gamma)(\theta(G_\ell')) = \mathrm{ad}(\gamma')(\Gamma'_{\tau(\theta)(\ell)})$ for some $\gamma, \gamma' \in \Gamma'$, and G_ℓ' is hyperbolic. We have

$$(\theta|_{G_\ell'})^{-1} \circ \mathrm{ad}(\gamma^{-1}) \circ (\theta|_{G_k'}) \in \mathrm{Mono}(G_k', G_\ell').$$

Therefore, by definition of X_k and X_ℓ, there exist $x_k \in X_k$ and $x_\ell \in X_\ell$ such that $\rho(x_k) = \mathrm{ad}(\gamma)(\theta(x_\ell))$. So $(j \circ \theta)(x_k) = \mathrm{ad}(\gamma'')((j \circ \theta)(x_\ell))$ for some $\gamma'' \in \Gamma$. But $j \circ \theta = j \circ r \circ \phi$ is Δ_k–related to ϕ thanks to Lemma 5.20. Therefore, $\phi(x_k) = \mathrm{ad}(\gamma''') \circ \phi(x_\ell)$ for some $\gamma''' \in \Gamma$. This concludes the proof of the first step.

Step 2 The statement of Step 1 is expressible by a $\forall\exists$–sentence, denoted by μ, which is true in Γ; more precisely, the first point of Step 1 is expressible as in the second step of the proof of Proposition 6.1 (see Step 2), and the second point is easily expressible using the fact that X_k and X_ℓ are finite. Since $\mathrm{Th}_{\forall\exists}(\Gamma) \subset \mathrm{Th}_{\forall\exists}(G)$, μ is true in G as well. Thus, for every $\phi \in \mathrm{Hom}(G', G)$, either

- there exist a vertex group G_k' together with a homomorphism $\phi'\colon G_k' \to G$ which is Δ_k–related to $\phi|_{G_k'}$ and kills an element of F, or
- there exist an element $g \in G$ and two elements $x_k \in X_k$ and $x_\ell \in X_\ell$ (with $k \neq \ell$) such that $\phi(x_k) = g\phi(x_\ell)g^{-1}$, with x_k of infinite order.

By definition, G is a quasitower over G'. Let $\rho\colon G \to G'$ and $\eta\colon G' \to G$ be the two homomorphisms associated with this structure of a quasitower. Taking $\phi := \eta$, the second claim above is false. Otherwise, $\eta(G_k') \cap g\eta(G_\ell')g^{-1}$ is infinite, and thus $(\rho \circ \eta)(G_k') \cap \rho(g)(\rho \circ \eta)(G_\ell')\rho(g)^{-1}$ is infinite, since ρ is injective in restriction to $\eta(G_k')$. But $\rho \circ \eta$ is inner on G_k' and on G_ℓ'. Hence, there exists an element $h \in G'$ such that $G_k' \cap hG_\ell'h^{-1}$ is infinite. This is a contradiction, since G_k' and G_ℓ' are two different vertex groups of a $\mathcal{F}$–JSJ splitting of G'.

As a consequence, the first claim is necessarily true (for $\phi := \eta$). There exist a one-ended factor G_k' together with a homomorphism $\phi'\colon G_k' \to G$ that is Δ_k–related to $\eta|_{G_k'}$ and kills an element of F. Thus $\rho \circ \phi'$ is Δ_k–related to $\rho \circ \eta|_{G_k'}$ and kills an element of F. Let i denote the inclusion of G_k' into G'. By Lemma 5.20, $\rho \circ \eta|_{G_k'} = \rho \circ \eta|_{G_k'} \circ i$ is Δ_k–related to i. Hence, $\rho \circ \phi'\colon G_k' \to G'$ is Δ_k–related to i and kills an element of F.

In other words, $\rho \circ \phi'$ is a noninjective preretraction. It follows from Proposition 6.3 that G' is a strict quasifloor. This is a contradiction, since G' is a quasicore by hypothesis. $\square$

Before proving Proposition 6.9, we need the following definition.

Definition 6.12 Let X be a set and let $f\colon X \to X$ be a map. An element $x \in X$ is said to be periodic if there exists an integer $k \geq 1$ such that $f^k(x) = x$.

Proof of Proposition 6.9 We keep the same notation as above. Since Γ is hyperbolic, Γ' is hyperbolic by Proposition 5.13, so $\Gamma'_1, \dots, \Gamma'_m$ are hyperbolic by Proposition 3.11. Let $\phi\colon G' \to \Gamma'$ and $\psi\colon \Gamma' \to G'$ be the factor-injective homomorphisms given by Lemma 6.11. If $i \in [\![1,n]\!]$ is periodic under $\sigma(\psi) \circ \tau(\phi)$, then i is the unique preimage of $\tau(\phi)(i)$ by the map $\tau(\phi)$. Indeed, if i is periodic, then ϕ induces an isomorphism between G'_i and $\phi(G'_i) = \Gamma'_{\tau(\phi)(i)}$, because one-ended hyperbolic groups are co-Hopfian. It follows from Lemma 6.11 that i is the unique preimage of $\tau(\phi)(i)$ by the map $\tau(\phi)$. Similarly, if $j \in [\![1,m]\!]$ is periodic under $\tau(\phi) \circ \sigma(\psi)$, then j is the unique preimage of $\sigma(\psi)(j)$ by the map $\sigma(\psi)$. The conclusion follows immediately from Lemma 6.13. $\square$

Lemma 6.13 *Suppose that $n, m \geq 1$ are two integers. Let $\tau\colon [\![1,n]\!] \to [\![1,m]\!]$ and $\sigma\colon [\![1,m]\!] \to [\![1,n]\!]$ be two maps. Suppose that the following two conditions hold*:

(1) *For every $i \in [\![1,n]\!]$, if i is periodic under $\sigma \circ \tau$, then i is the unique preimage of $\tau(i)$ by τ.*

(2) *For every $j \in [\![1,m]\!]$, if j is periodic under $\tau \circ \sigma$, then j is the unique preimage of $\sigma(j)$ by σ.*

Then $n = m$, and τ and σ are bijective.

Proof We claim that every $i \in [\![1,n]\!]$ is periodic under $\sigma \circ \tau$. Assume towards a contradiction that i is not periodic under $\sigma \circ \tau$. Let $k \geq 1$ be the smallest integer such that $(\sigma \circ \tau)^k(i)$ is periodic under $\sigma \circ \tau$. Let $j = \tau \circ (\sigma \circ \tau)^{k-1}(i)$. If j is periodic under $\tau \circ \sigma$, there exists an integer $p \in [\![1,n]\!]$ periodic under $\sigma \circ \tau$ such that $\tau(p) = j$. But $q = (\sigma \circ \tau)^{k-1}(i)$ is not periodic under $\sigma \circ \tau$ by definition of k, so $q \neq p$ (since p is periodic and q is not periodic), and $\tau(q) = j = \tau(p)$. This contradicts the second condition. If j is not periodic under $\tau \circ \sigma$, we find a contradiction with the first condition, in the same way. Hence, every element $i \in [\![1,n]\!]$ is periodic. It follows from condition (1) that τ is injective. Likewise, σ is injective. Therefore, $n = m$ and τ and σ are bijective. $\square$

6.4 Proof of Theorem 1.5

In this subsection, we prove the following results.

Theorem 6.14 *Let Γ be a hyperbolic group and let G be a finitely generated group. Suppose that $\mathrm{Th}_{\forall\exists}(\Gamma) = \mathrm{Th}_{\forall\exists}(G)$. Then Γ is cubulable if and only if G is cubulable.*

Corollary 6.15 *Let Γ and G be finitely generated groups. Suppose that $\mathrm{Th}_{\forall\exists}(\Gamma) = \mathrm{Th}_{\forall\exists}(G)$. Then Γ is hyperbolic and cubulable if and only if G is hyperbolic and cubulable.*

Note that Corollary 6.15 follows immediately from Theorem 6.14 and from the fact that hyperbolicity is preserved under $\forall\exists$–equivalence among finitely generated groups (see Proposition 6.9).

We refer the reader to [22] for a definition of $\mathrm{CAT}(0)$ and special cube complexes. A group G is said to be *cubulable* (resp. *special*) if it acts properly and cocompactly by isometries on a locally finite $\mathrm{CAT}(0)$ (resp. special) cube complex, and *virtually special* if it has a finite-index subgroup which is special.

Let us recall Wise's quasiconvex hierarchy theorem for hyperbolic cubulable groups.

Theorem 6.16 [30, Theorems 13.1 and 13.3] *Let G be a hyperbolic group. Suppose that G splits as a graph of groups all of whose vertex groups are virtually special cubulable and quasiconvex in G. Then G is virtually special cubulable.*

Using results of Haglund and Wise, Agol proved the following celebrated theorem.

Theorem 6.17 [1, Theorem 1.1] *Every hyperbolic and cubulable group is virtually special.*

We will also need the following proposition, claiming that cubulability is inherited by quasiconvex subgroups.

Proposition 6.18 [8, Corollary 2.29] *Let G be a hyperbolic and cubulable group. If H is a quasiconvex subgroup of G, then H is cubulable.*

Before proving Theorem 6.14, we establish the following preliminary result.

Proposition 6.19 *Let $(G, H, \Delta_G, \Delta_H)$ be a quasitower. Suppose that G is hyperbolic. Then G is cubulable if and only if H is cubulable.*

Proof We prove the proposition in the case where G is a quasifloor over H, and the general case follows immediately by induction on the height of the quasitower.

Let V_G and V_H denote the sets of vertices of Δ_G and Δ_H. Let v be the central vertex of Δ_G. By definition of a quasifloor, there exists a subset $V_H^1 \subset V_H$ and a bijection $s\colon V_G \setminus \{v\} \to V_H^1$ such that $G_w \simeq H_{s(w)}$ for every $w \in V_G \setminus \{v\}$.

Suppose that G is cubulable. Then, by Propositions 3.11 and 6.18, all vertex groups of Δ_G are hyperbolic and cubulable. By the theorem of Agol, Theorem 6.17, all vertex groups of Δ_G are therefore virtually special. As a consequence, every vertex group H_u with $u \in V_H^1$ is virtually special cubulable (see the bijection s above). Moreover, recall that $V_H = V_H^1 \sqcup V_H^2$, with H_u finite for every $u \in V_H^2$. Hence, all vertex groups of Δ_H are virtually special cubulable. The group H being hyperbolic thanks to Proposition 5.13 (since G is hyperbolic), one can now apply Wise's quasiconvex hierarchy Theorem 6.16 to conclude that H is virtually special cubulable, using the fact that vertex groups of Δ_H are quasiconvex in H, by Proposition 3.11.

Conversely, if H is cubulable, we prove in exactly the same way that G is cubulable. □

We now prove Theorem 6.14.

Proof Let Γ be a hyperbolic group and let G be a finitely generated group such that $\mathrm{Th}_{\forall\exists}(\Gamma) = \mathrm{Th}_{\forall\exists}(G)$. Suppose that Γ is cubulable. We will prove that G is cubulable. Let us denote by G' a quasicore of G and by Γ' a quasicore of Γ. According to Proposition 6.19, Γ' is cubulable, so each vertex group of a $\mathcal{F}$–JSJ splitting of Γ' is cubulable, by Proposition 6.18 (and the trivial case of finite groups). Moreover, according to Proposition 6.9, every vertex group of a $\mathcal{F}$–JSJ splitting of G' that is not a finite-by-orbifold group is isomorphic to a vertex group of a $\mathcal{F}$–JSJ splitting of Γ'. Consequently, by Theorem 6.16, the group G' is cubulable. Then, it follows from Proposition 6.19 that G is cubulable. Symmetrically, if G is cubulable, then so is Γ (because G is hyperbolic, by Theorem 6.10). □

7 From a preretraction to a quasifloor: proof of Proposition 6.3

In the previous section, we proved our main theorems by admitting Proposition 6.3, which claims that one can build a strict quasifloor from a noninjective preretraction. This section is devoted to a proof of Proposition 6.3.

7.1 A preliminary proposition

Proposition 7.1 *Let G be a one-ended finitely generated K–CSA group that does not contain $\mathbb{Z}^2$. Let Δ denote the $\mathcal{Z}$–JSJ splitting of G, and let $p\colon G \to G$ be a nondegenerate Δ–preretraction. Then p is injective.*

Recall that a Δ–preretraction p is said to be nondegenerate if it sends each QH subgroup of G isomorphically to a conjugate of itself (see Definition 3.6).

Proof Let T be the Bass–Serre tree of Δ. We denote by V the set of vertices of T. First of all, let us recall some properties of Δ that will be useful in the sequel (see Section 2.6.4):

(1) The graph Δ is bipartite, with every edge joining a vertex carrying a virtually cyclic group to a vertex carrying a group that is not virtually cyclic.

(2) Let v be a vertex of T, and let e and e' be two distinct edges incident to v. If G_v is not virtually cyclic, then the group $\langle G_e, G_{e'}\rangle$ is not virtually cyclic.

(3) The action of G on T is 2–acylindrical: if an element $g \in G$ fixes a segment of length > 2 in T, then g has finite order.

If Δ is reduced to a point, then p is obviously injective. From now on, we will suppose that Δ has at least two vertices.

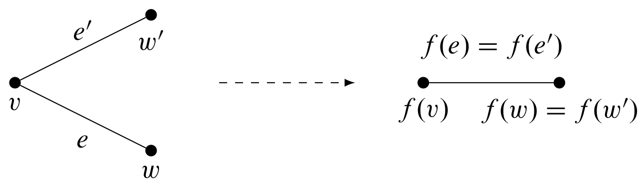
As a first step, we build a p–equivariant map $f\colon T \to T$. Let $v_1, \dots, v_n$ be some representatives of the orbits of vertices. For every $1 \leq k \leq n$, there exists $g_k \in G$ such that $p(G_{v_k}) = g_k G_{v_k} g_k^{-1}$. We let $f(v_k) = g_k v_k$, so that $p(G_{v_k}) = G_{f(v_k)}$. Then we define f on each vertex of T by p–equivariance. Next, we define f on the edges of T in the following way: if e is an edge of T, with endpoints v and w, there exists a unique injective path e' from $f(v)$ to $f(w)$ in T. We let $f(e) = e'$.

Now we will prove that f is injective, which allows to conclude that p is injective. Indeed, if $p(g) = 1$ for some $g \in G$, then for every vertex $v \in V$ one has that $p(g) \cdot f(v) = f(g \cdot v) = f(v)$, and thus $g \cdot v = v$ for every v. Since the action of G on Δ is 2–acylindrical, g has finite order. Moreover the restriction of p to every element of finite order is injective (by definition of Δ–relatedness), so $g = 1$, which proves that p is injective.

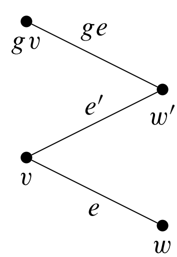
We now prove that f is injective. The proof will proceed in two steps: first, one shows that f sends adjacent vertices to adjacent vertices, then one proves that there are no foldings.

The map f sends adjacent vertices to adjacent vertices Let's consider two adjacent vertices v and w of T. One has $d(f(v), f(w)) \leq 2$, because the action of G on T is 2–acylindrical and p is injective on edge groups, which are virtually cyclic and infinite. Since the tree is bipartite and since f sends a vertex of T to a vertex of the same type, the integers $d(f(v), f(w))$ and $d(v, w) = 1$ have the same parity. Hence $d(f(v), f(w)) = 1$.

There are no foldings Let v be a vertex of T, and let w and w' be two distinct vertices adjacent to v. Denote by e and e' the edges between v and w and between v and w' respectively. Argue by contradiction and suppose that $f(w) = f(w')$. Then $f(e) = f(e')$ since there are no circuits in a tree:



If G_v is not virtually cyclic, then $\langle G_e, G_{e'} \rangle$ is not virtually cyclic (see the second property of Δ recalled above), so $p(\langle G_e, G_{e'} \rangle)$ is not virtually cyclic since $\langle G_e, G_{e'} \rangle$ is contained in G_v, and p is injective on G_v. But $p(\langle G_e, G_{e'} \rangle)$ is contained in the stabilizer of $f(e)$, which is virtually cyclic. This is a contradiction. Hence G_v is virtually cyclic. There exists an element $g \in G$ such that $w' = g \cdot w$. Since f is p–equivariant, $p(g) \cdot f(w) = f(w)$, ie $p(g) \in G_{f(w)} = p(G_w)$. As a consequence, there exists an element $h \in G_w$ such that $p(g) = p(h)$. Up to multiplying g by the inverse of h, one can assume that $p(g) = 1$. Then g does not fix a point of T, because p is injective on vertex groups and $g \neq 1$. It follows that g is hyperbolic, with translation length equal to 2:



The group $\langle G_{e'}, G_{ge} \rangle$ is not virtually cyclic since $G_{w'}$ is not virtually cyclic. It follows that $p(\langle G_{e'}, G_{ge} \rangle)$ is not virtually cyclic (indeed, p is injective on $G_{w'}$).

On the other hand, $p(\langle G_{e'}, G_{ge}\rangle)$ is equal to $p(\langle G_{e'}, G_e\rangle)$ because $p(g)=1$. Thus $p(\langle G_{e'}, G_e\rangle)$, which is contained in the stabilizer of $f(e)$, is not virtually cyclic. This is a contradiction. □

7.2 Building a quasifloor from a noninjective preretraction

Definition 7.2 (maximal pinched set, pinched quotient, pinched decomposition) Let G be a group that splits as a centered splitting Δ_G, with central vertex v. The stabilizer G_v of v is a conical finite-by-orbifold group $F \hookrightarrow G_v \twoheadrightarrow \pi_1(\mathcal{O})$. Denote by q the epimorphism from G_v onto $\pi_1(\mathcal{O})$. Let G' be a group, and let $p\colon G \to G'$ be a homomorphism. Let S be an essential set of curves on $\mathcal{O}$ (see Definition 2.30). Suppose that each element of S is pinched by p — meaning that $p(q^{-1}(\alpha))$ is finite for every $\alpha \in S$, with $q^{-1}(\alpha)$ well-defined up to conjugacy — and that S is maximal for this property. The set S is called a maximal pinched set for p. Note that S may be empty.

For every $\alpha \in S$, the group $q^{-1}(\alpha)$ (well-defined up to conjugacy) is a virtually cyclic subgroup of G_v, isomorphic to $F \rtimes \mathbb{Z}$. Let $N_\alpha = \ker(p|_{q^{-1}(\alpha)})$, and let N be the subgroup of G normally generated by $\{N_\alpha\}_{\alpha\in S}$. The quotient group $Q = G/N$ is called the pinched quotient of G associated with S. Let $\pi\colon G \twoheadrightarrow Q$ be the quotient epimorphism. Since each N_α has finite index in $q^{-1}(\alpha)$, and since p is injective on finite subgroups, killing N gives rise to new QH vertices with new conical points, and new edges whose stabilizers coincide with the new conical groups (see Figure 7). The group Q splits naturally as a graph of groups Δ_Q obtained by replacing the vertex v in Δ_G by the splitting of $\pi(G_v)$ over finite groups obtained by killing N (see Figure 7). Δ_Q is called the pinched decomposition of Q.

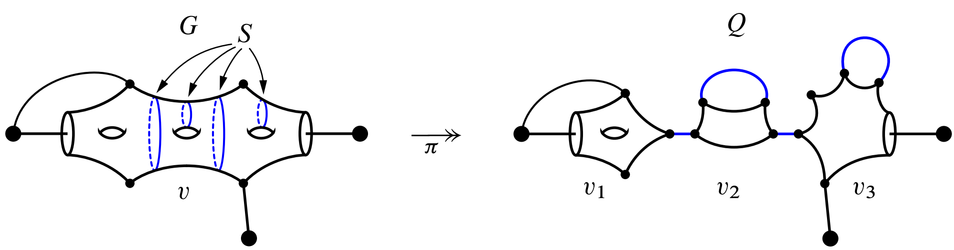


Figure 7: For convenience, in this figure, F is trivial. For each $\alpha \in S$, there exists a smallest integer $n \geq 1$ such that $p(\alpha^n)=1$. Killing $\langle \alpha^n \rangle$ gives rise to a conical point of order n, and a new edge group equal to the conical group. The new QH vertices coming from v are denoted by v_1, v_2 and v_3.

We keep the same notation. Suppose that there exists an endomorphism p of G that is Δ_G–related to the identity of G. Let S be a maximal pinched set for p, let Q be the pinched quotient and let Δ_Q be its pinched decomposition. Denote by $v_1, \dots, v_n$ the new vertices coming from v (see Figure 7). Let $\pi\colon G \twoheadrightarrow Q$ be the quotient epimorphism. There exists a unique homomorphism $\phi\colon Q \to G$ such that $p = \phi \circ \pi$. This homomorphism is nonpinching in restriction to Q_{v_i} for $1 \le i \le n$, since S is assumed to be maximal. We will need a lemma.

Lemma 7.3 *We keep the same notation. Assume that p does not send G_v isomorphically to a conjugate of itself, and denote by $\mathcal{F}$ the set of edges of Δ_Q with finite stabilizer* (note that $\mathcal{F}$ contains all the new edges arising from the pinching). *Let $V = \{v_1, \dots, v_n\}$ be the set of vertices of Δ_Q coming from v, and let W be the set of vertices of Δ_Q that do not belong to V. Let Y be a connected component of $\Delta_Q \setminus \mathcal{F}$, and let Q_Y be the fundamental group of Y. Then*:

- *$\phi(Q_Y)$ is elliptic in the Bass–Serre tree of Δ_G.*
- *Y contains at most one vertex w of W, and if it does, then $\phi(Q_Y) = \phi(Q_w)$.*

Proof **Step 1** First, we will prove that $\phi(Q_{v_k})$ is elliptic in the Bass–Serre tree T of Δ_G, for every $k \in [\![1, n]\!]$.

Denote by O_k the underlying orbifold of Q_{v_k}. We will use Proposition 2.31 in order to split Q_{v_k} as a graph of groups all of whose vertex groups are elliptic in T via ϕ. If C is an extended boundary subgroup of Q_{v_k}, then C is of the form $\pi(C')$, where C' stands for an extended boundary subgroup of G_v, so $\phi(C) = p(C')$ is elliptic in T by definition of Δ_G–relatedness. Consequently, by Proposition 2.31, there exists an essential set $\mathcal{C}_k$ of simple loops on $\mathcal{O}_k$ such that if X is a connected component of $\mathcal{O}_k \setminus \mathcal{C}_k$ and H is the preimage of the orbifold fundamental group $\pi_1(X)$ in Q_{v_k}, then $\phi(H)$ is elliptic in the Bass–Serre tree T.

Let $\Delta_Q(\mathcal{C}_k)$ be the splitting of Q obtained by replacing v_k in Δ_Q by the splitting of Q_{v_k} dual to $\mathcal{C}_k$. First, note that if $\mathcal{C}_k$ is empty, then $\phi(Q_{v_k})$ is obviously elliptic in T. Assume now that $\mathcal{C}_k$ is nonempty and denote by $v_{k,1}, \dots, v_{k,m}$ the new vertices coming from v_k. Let T' be the Bass–Serre tree of $\Delta_Q(\mathcal{C}_k)$. Let $w_{k,i}, w_{k,j} \in T'$ be two representatives of $v_{k,i}, v_{k,j} \in \Delta_Q(\mathcal{C}_k)$ that are adjacent in T' and linked by an edge with infinite stabilizer. By the previous paragraph, there exists a nonempty subset $I \subset T$ pointwise-fixed by $\phi(Q_{w_{k,i}})$, and a nonempty subset $J \subset T$ pointwise-fixed by $\phi(Q_{w_{k,j}})$. Let $x \in I$ and $y \in J$ be such that $d(x, y) = d(I, J)$, where d is the natural metric on T. By definition of a centered splitting, if an element of G of infinite order fixes a segment of length ≥ 2 in T, then this segment has length exactly 2 and its endpoints are translates of the central vertex v. Therefore, since ϕ is nonpinching on Q_{v_k}, the distance $d(x,y)$ between x and y belongs to $\{0,1,2\}$, because otherwise ϕ would pinch a common boundary element of $Q_{w_{k,i}}$ and $Q_{w_{k,j}}$. In addition, if the equality $d(x,y)=2$ holds, then x and y are translates of v.

Claim $d(x,y)=0$.

First, suppose that $d(x,y)=2$. Then we can assume without loss of generality that $x=v$, ie $\phi(Q_{w_{k,i}})\subset G_v$. Since ϕ is nonpinching on $Q_{w_{k,i}}$, the group $\phi(Q_{w_{k,i}})$ is infinite, so it is not contained in an extended conical subgroup of G_v. Let us suppose towards a contradiction that $\phi(Q_{w_{k,i}})$ is not contained in an extended boundary subgroup of G_v. Then, it follows from Proposition 2.39 that $\chi(Q_{w_{k,i}})\geq\chi(G_v)$ with equality if and only if f induces an isomorphism from $Q_{w_{k,i}}$ to G_v. But $\chi(Q_{w_{k,i}})<\chi(G_v)$ since the complexity decreases as soon as we cut along a loop or pinch a loop and since $Q_{w_{k,i}}$ is not isomorphic to G_v. This is a contradiction. Therefore, $\phi(Q_{w_{k,i}})$ is necessarily contained in an extended boundary subgroup of G_v. Then $\phi(Q_{w_{k,i}})$ fixes a point z in T such that $d(x,z)=1$. As a consequence, $d(z,y)=1$ or $d(z,y)=3$. This last case is impossible since an element of G of infinite order fixes a segment of length ≤ 2 in T. Thus $d(z,y)=1$, and this contradicts the definition of x, since x, which is at distance 2 from y, is the closest point to y fixed by $\phi(Q_{w_{k,i}})$.

Now, suppose that $d(x,y)=1$. Since Δ_G is bipartite, one can assume, up to composing ϕ with an inner automorphism and permuting x and y, that $x=v$. If $\phi(Q_{w_{k,i}})$ is not contained in an extended boundary subgroup of G_v, we get a contradiction thanks to Proposition 2.39, as above. Thus $\phi(Q_{w_{k,i}})$ is contained in an extended boundary subgroup of G_v. So $\phi(Q_{w_{k,i}})$ has a fixed point z in T such that $d(x,z)=1$, so $d(z,y)=0$ or $d(z,y)=2$. This last case is impossible since z and y are not translates of v and, by definition of a centered splitting, if an element of G of infinite order fixes a segment of length 2 in T, then its endpoints are translates of the central vertex v. As a consequence, $d(z,y)=0$, and this contradicts the definition of x.

Hence, we have proved that $d(x,y)=0$. This concludes the proof of the claim.

We are now ready to complete the proof of the first step. Thanks to the previous claim, we know that $\phi(Q_{w_{k,i}})$ and $\phi(Q_{w_{k,j}})$ have a common fixed point in the Bass–Serre tree T of Δ_G. We will now deduce that $\phi(Q_{v_k})$ is elliptic in T. Let T'' be the tree obtained from T' by collapsing the adjacent vertices $w_{k,i}$ and $w_{k,j}$ to a point, and let u be the new vertex. The group $\phi(Q_u)$ is elliptic in T. One can repeat the previous proof with the set $S' = \{u\} \cup \{v_{k,\ell} \mid \ell \neq i, j\}$ instead of $S = \{v_{k,1}, \dots, v_{k,m}\}$. Since $|S'| = |S| - 1 < |S|$, a straightforward iteration proves that $\phi(Q_{v_k})$ is elliptic in T. This concludes the proof of the first step.

Step 2 Let Y be a connected component of $\Delta_Q \setminus \mathcal{F}$, and let Q_Y be the fundamental group of Y. Note that Y contains at least one vertex v_k coming from the central vertex v. We distinguish two cases:

(1) Y does not contain a vertex w different from the vertices v_i coming from the central vertex v. Then Y is reduced to v_k, and we have proved above that $\phi(Q_{v_k})$ is elliptic in the Bass–Serre tree T of Δ_G.

(2) Y contains a vertex w different from the vertices v_i. We will prove that w is the only vertex of Y different from the v_i and that $\phi(Q_{v_k})$ is contained in $\phi(Q_w)$. This will prove that $\phi(Q_Y) = \phi(Q_w)$.

Suppose we are in the situation described in case (2). The vertices w and v_k are linked by an edge with infinite stabilizer. Let T'' be the Bass–Serre tree of Δ_Q. For convenience, we still denote by w and v_k two adjacent representatives of w and v_k in T'' linked by an edge with infinite stabilizer. We shall prove that $\phi(Q_{v_k})$ is contained in $\phi(Q_w)$.

We have already proved the existence of a subset $I \subset T$ pointwise-fixed by $\phi(Q_{v_k})$. Since $p|_{G_w}$ is inner, $\phi(Q_w) = p(G_w)$ fixes a vertex $y = gw$ of T, and we have $\phi(Q_w) = G_y = gG_w g^{-1}$. Let x be a point of I such that $d(x, y) = d(I, y)$. Recall that, by definition of a centered splitting, if an element of G of infinite order fixes a segment of length ≥ 2 in T, then this segment has length exactly 2 and its endpoints are translates of the central vertex v. Therefore, since y is not a translate of v, we have $d(x, y) \leq 1$.

Suppose towards a contradiction that $d(x, y) = 1$. Then we can assume without loss of generality that $x = v$. Hence, ϕ induces a nonpinching morphism of finite-by-orbifold groups from Q_{v_k} to G_v. We claim that $\phi(Q_{v_k})$ is contained in an extended boundary subgroup of G_v. First, observe that the group $\phi(Q_{v_k})$ is infinite, since ϕ is nonpinching. Thus $\phi(Q_{v_k})$ is not contained in an extended conical subgroup of G_v. Assume towards a contradiction that $\phi(Q_{v_k})$ is not contained in an extended boundary subgroup of G_v. Then, it follows from Proposition 2.39 that $\chi(Q_{v_k}) \geq \chi(G_v)$, with equality if and only if ϕ is an isomorphism. But the complexity decreases as soon as we cut along a loop or pinch a loop, so $\chi(G_v) = \chi(Q_{v_k})$ and p sends G_v isomorphically to a conjugate of itself. This contradicts the hypothesis of the lemma, and proves that $\phi(Q_{v_k})$ is contained in an extended boundary subgroup of G_v. Thus, $\phi(Q_{v_k})$ fixes a point z in T such that $d(x,z) = 1$. As a consequence, $z = y$ or $d(z,y) = 2$. This last case is impossible since y and z are not translates of v, so $z = y$ and this contradicts the definition of x. Hence, we have proved that $\phi(Q_{v_k})$ fixes the vertex y, whose stabilizer is $\phi(Q_w)$. Therefore, $\phi(Q_{v_k}) \subset \phi(Q_w)$.

In order to prove that $\phi(Q_Y) = \phi(Q_w)$, it is enough to prove that w is the only vertex of Y different from the vertices v_i. Assume towards a contradiction that there is another vertex $w_2 \in Y$ with this property. One can suppose without loss of generality that $d(w,w_2) = 2$ and that w and w_2 are both linked to v_k by an edge with infinite stabilizer (up to replacing v_k by another vertex $v_i \in Y$), because Y is bipartite: the vertices arising from v on the one hand, and the vertices not arising from v on the other hand. We have already shown that $\phi(Q_{v_k}) \subset \phi(Q_{w_1})$ and $\phi(Q_{v_k}) \subset \phi(Q_{w_2})$. Note, in addition, that Q_{v_k} has an extended boundary subgroup C such that $\phi(C)$ is infinite. Hence, $\phi(Q_{w_1}) \cap \phi(Q_{w_2})$ is infinite. By definition of a centered splitting, if an element of G of infinite order fixes a segment of length ≥ 2 in T, then this segment has length exactly 2 and its endpoints are translates of the central vertex v. But w and w_2 are not translates of v. This proves that $w_1 = w_2$ and completes the proof of the lemma. □

Proposition 7.4 *Let G be a finitely generated group. Suppose that G is not a finite-by-orbifold group. Let Δ_G be a centered splitting of G, with central vertex v. Suppose that there exists a noninjective degenerate Δ_G–preretraction $p\colon G \to G$. Suppose moreover that G has a one-ended subgroup A such that $p|_A$ is noninjective. Then there exist a group H, a splitting Δ_H of H and two morphisms $r\colon G \to H$ and $j\colon H \to G$ such that $(G, H, \Delta_G, \Delta_H, r, j)$ is a strict quasifloor.*

Recall that a Δ_G–preretraction p is said to be degenerate if it does not send the central vertex group G_v isomorphically to a conjugate of itself (see Definition 3.14).

Proof **Case A (nonpinching)** Suppose that p is nonpinching on G_v. By hypothesis, p does not send G_v isomorphically to a conjugate of itself, so it follows from Lemma 7.3 that Δ_G has only one vertex w different from v, and that $p(G_v) \subset p(G_w)$. Since p is inner on $H := G_w$, there exists an element $g \in G$ such that $r := \mathrm{ad}(g) \circ p$ is a retraction from G onto H. Let Δ_H be the splitting of H reduced to a point, and let j denote the inclusion of H into G. The tuple $(G, H, \Delta_G, \Delta_H, r, j)$ is a

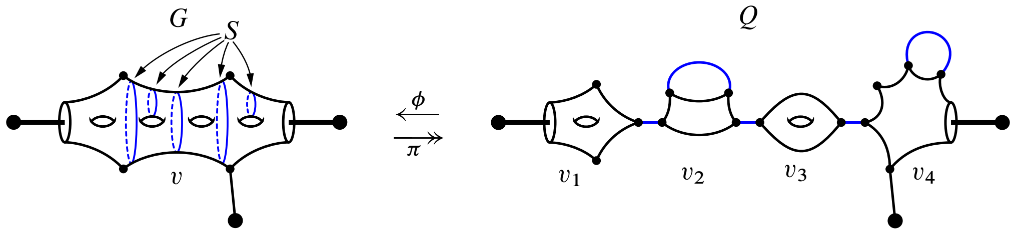


Figure 8: Step 1 of Case B of the proof of Proposition 7.4: Edges with infinite stabilizer are thickened.

quasifloor. Moreover, this quasifloor is strict since, by hypothesis, there exists a one-ended subgroup A of G such that $p|_A$, and therefore $r|_A$, is noninjective.

Case B (pinching) Step 1 (pinching a maximal set of simple loops) Let S be a maximal pinched set for (p, Δ_G), let Q be the pinched quotient and let Δ_Q be its pinched decomposition. Denote by $v_1, \dots, v_n$ the new vertices coming from v (see Figure 8). Let $\pi\colon G \twoheadrightarrow Q$ be the quotient epimorphism. There exists a unique homomorphism $\phi\colon Q \to G$ such that $p = \phi \circ \pi$. This homomorphism is nonpinching since S is assumed to be maximal.

Let $\mathcal{F}$ be the set of edges of Δ_Q with finite stabilizer. For every $k \in [\![1, n]\!]$, we denote by Y_k the connected component of $\Delta_Q \setminus \mathcal{F}$ containing v_k, and we denote by Q_{Y_k} its fundamental group. By Lemma 7.3, $\phi(Q_{Y_k})$ is elliptic in the Bass–Serre tree of the splitting Δ_G (in particular, $\phi(Q_{v_k})$ is elliptic). Hence, there exists a vertex w of Δ_G such that $\phi(Q_{Y_k})$ is contained in a conjugate of G_w. If w is unique, we define $w_k := w$. If w is not unique, then we can assume that $w \neq v$, and we define $w_k := w$.

We will construct the group H together with its splitting Δ_H by eliminating the new vertices $v_1, \dots, v_n$ coming from the central vertex v. We will illustrate each step of the construction in the case of the example shown in Figure 8.

Step 2 (eliminating orbifolds with nonempty boundary) Let v_k be a vertex such that the underlying orbifold of Q_{v_k} has nonempty boundary. Then Y_k contains a vertex w different from the vertices $v_1, \dots, v_n$. By Lemma 7.3, w is the only vertex of Y_k with this property, and we have $\phi(Q_{Y_k}) = \phi(Q_w) = gG_wg^{-1}$ for some $g \in G$. Since w is unique, we have $w = w_k$ (when viewed as a vertex of Δ_G).

Therefore, the quotient $Q' = Q/N$ of Q by the subgroup N normally generated by

$$\{\ker(\phi|_{Q_{Y_k}}) \mid \text{the underlying orbifold of } Q_{v_k} \text{ has nonempty boundary}\}$$

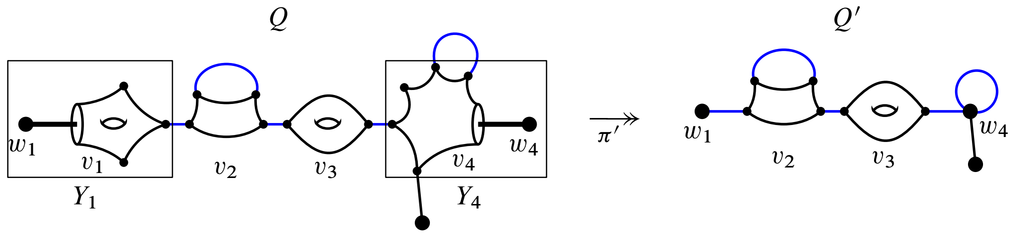


Figure 9: Step 2: Edges with infinite stabilizer are thickened. Note that, by construction, $\Delta_{Q'}$ has finite edge groups.

splits naturally as a graph of groups $\Delta_{Q'}$ obtained by replacing in Δ_Q each subgraph Y_k as above by a new vertex labeled by $\phi(Q_{Y_k}) = \phi(Q_{w_k}) = (\phi\circ\pi)(G_{w_k}) = p(G_{w_k}) = gG_{w_k}g^{-1}$ for some $g \in G$. With a little abuse of notation, this new vertex is still denoted by w_k (see Figure 9). Note that the graph of groups $\Delta_{Q'}$ has finite edge groups. Let $\pi'\colon Q \twoheadrightarrow Q'$ be the quotient epimorphism. There exists a unique homomorphism $\phi'\colon Q' \to G$ such that $\phi = \phi'\circ\pi'$, so $p = \phi'\circ\pi'\circ\pi$.

Step 3 (eliminating vertices v_k such that $w_k = v$) Let $k \in [\![1,n]\!]$ be such that $w_k = v$. Note that, by definition of w_k, $\phi'(Q'_{v_k}) = \phi(Q_{v_k})$ is not contained in an extended boundary subgroup of G_v. Since ϕ' is nonpinching on Q'_{v_k} (by maximality of S), and since the complexity of Q'_{v_k} is strictly less than the complexity of G_v, it follows from Proposition 2.39 that $\phi'(Q'_{v_k})$ is contained in an extended conical subgroup of gG_vg^{-1}; in particular, $\phi'(Q'_{v_k})$ is a finite group. As in the previous step, we replace the vertex v_k by a new vertex labeled by $\phi'(Q'_{v_k})$. This new vertex is called x_k (see Figure 10). We perform the previous operation for each $k \in [\![1,n]\!]$ such that $w_k = v$. Let Λ be the resulting graph of groups. Call its fundamental group Q'', and let $\Delta_{Q''} := \Lambda$. Let $\pi''\colon Q' \twoheadrightarrow Q''$ be the quotient epimorphism. There exists a unique homomorphism $\phi''\colon Q'' \to G$ such that $\phi' = \phi''\circ\pi''$, so $p = \phi''\circ(\pi''\circ\pi'\circ\pi)$.

Step 4 (eliminating the remaining QH vertices) Denote by V the set of vertices of $\Delta_{Q''}$ coming from the initial pinching of G_v and that have not been treated yet. For each $v_k \in V$, recall that w_k stands for a vertex of Δ_G such that $\phi(Q_{v_k})$ is contained in a conjugate of G_{w_k}. Note that w_k is not the central vertex v of Δ_G (see Step 3).

To complete the proof of Proposition 7.4, it is convenient to adopt a topological point of view. Let X_G be a $K(G,1)$ obtained as a graph of spaces using, for each vertex or edge w of Δ_G, a $K(G_w,1)$ denoted by X_G^w (see [23]). Let $X_{Q''}$ be a $K(Q'',1)$ obtained in the same way. There exists a morphism of CW–complexes $f\colon X_{Q''} \to X_G$

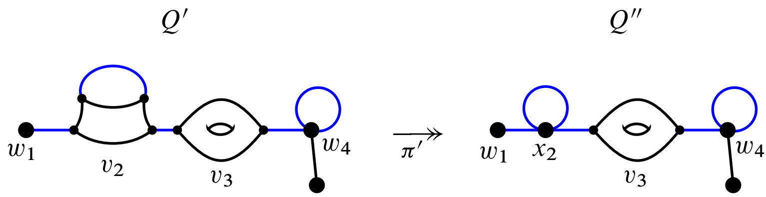


Figure 10: Step 3: In this example, $w_2 = v$. The vertex v_2 is replaced by a vertex x_2 labeled by the finite group $\phi'(Q'_{v_2})$.

inducing $\phi''\colon Q'' \to G$ at the level of fundamental groups such that $f(X^{v_k}_{Q''}) \subset f(X^{w_k}_{Q''})$ for each remaining vertex v_k coming from v, and f induces a homeomorphism between $X^w_{Q''}$ and X^w_G for each vertex w that does not come from v. We define an equivalence relation $\sim$ on $X_{Q''}$ by $x \sim y$ if $x = y$, or if $x \in X^{v_k}_{Q''}$, $y \in X^{w_k}_{Q''}$ and $f(x) = f(y)$. Let $g\colon X_{Q''} \twoheadrightarrow (X_{Q''}/\sim)$ be the quotient map. There exists a unique continuous function $h\colon (X_{Q''}/\sim) \to X_G$ such that $f = h \circ g$. Hence $\phi'' = h_* \circ g_*$. Note that the homomorphism g_* is not surjective in general. Denote by H the fundamental group of $X_{Q''}/\sim$, let $j = h_*$ and $r = g_* \circ \pi'' \circ \pi' \circ \pi$, so that $p = j \circ r$. Note that $X_{Q''}/\sim$ naturally has the structure of a graph of spaces, and denote by Δ_H the corresponding splitting of H. We claim that G is a strict quasifloor over H (see below).

Let us explain the topological construction above from an algebraic point of view in the case of our example. The only remaining vertex coming from the central vertex v is v_3 (see Figure 10). Up to replacing Q'' by $Q''/\langle\langle \ker(\varphi) \rangle\rangle$, where φ stands for the restriction of ϕ'' to the stabilizer Q''_{v_3} of v_3 in Q'', we can assume ϕ'' is injective on Q''_{v_3}. We know $\phi''(Q''_{v_3})$ is contained in $gG_{w_3}g^{-1}$ for some $g \in G$, where w_3 is the vertex associated with v_3 defined in Step 1. Moreover, ϕ'' sends Q''_{w_3} isomorphically onto $G^h_{w_3}$ for some $h \in G$. Therefore, $i := (\phi'')^{-1} \circ \mathrm{ad}(hg^{-1}) \circ \phi''\colon {Q''}_{v_3} \to {Q''}_{w_3}$ is a monomorphism. We add an edge e to the graph of groups $\Delta_{Q''}$ between v_3 and w_3

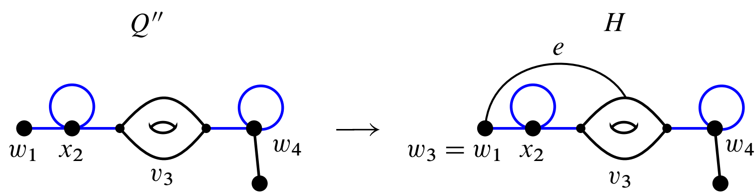


Figure 11: In this example, $w_3 = w_1$. We define H as the fundamental group of the graph of groups obtained by adding an edge e to the graph $\Delta_{Q''}$, identifying Q''_{v_3} with $i(Q''_{v_3}) \subset Q''_{w_3}$. The natural homomorphism from Q'' to H is not surjective in general.

identifying Q''_{v_3} with its image $i(Q''_{v_3})$ in Q''_{w_3} (see Figure 11). Denote by H the fundamental group of this graph of groups. In other words, we obtain H by adding a new generator t to Q'', as well as the relation $\mathrm{ad}(t)(x) = i(x)$ for every $x \in Q''_{v_3}$. Last, we collapse the edge e, and we denote by Δ_H the resulting splitting of H (see Figure 12). We define $r\colon G \to H$ as the composition of $\pi'' \circ \pi' \circ \pi\colon G \to Q''$ with the natural homomorphism from Q'' to H. Note that r is not surjective in general. Then, we define a morphism j from H to G that extends $\phi''\colon Q'' \to G$ by sending t to hg^{-1}. Since $p = \phi'' \circ (\pi'' \circ \pi' \circ \pi)$, we have $p = j \circ r$.

It remains to verify that $(G, H, \Delta_G, \Delta_H, r, j)$ is a strict quasifloor:

- $j \circ r = p$ is Δ_G–related to the identity of G by definition of p.
- Let V_G be the set of vertices of Δ_G, and let V_H be the set of vertices of Δ_H. Recall that v stands for the central vertex of Δ_G. By construction of H and Δ_H, the homomorphism $r\colon G \to H$ induces a bijection s between $V_G \setminus \{v\}$ and a subset $V_1 \subset V_H$ such that $r(G_w) = H_{s(w)}$, for every $w \in V_G \setminus \{v\}$.
- Let $V_2 = V_H \setminus V_1$. By construction (see Step 3 above), for every $w \in V_2$, the vertex group H_w is finite and j is injective on H_w.
- By hypothesis, there exists a one-ended subgroup A of G such that $p|_A$ is noninjective, with $p = j \circ r$. We claim that r is not injective on A. Assume towards a contradiction that r is injective on A. Then $r(A)$ is a one-ended subgroup of H. Because j is injective on one-ended subgroups of H, it is injective on $r(A)$. Therefore, $p = j \circ r$ is injective on A. This is a contradiction. Hence, we have $A \cap \ker(r) \neq \{1\}$, which proves that the quasifloor is strict. □

Before proving Proposition 6.3, we need an easy lemma.

Lemma 7.5 *Let G be a group with a splitting over finite groups. Let T denote the Bass–Serre tree associated with this splitting. Let H be a group with a splitting over infinite groups, and let S be the associated Bass–Serre tree. If $p\colon H \to G$ is a homomorphism injective on edge groups of S and such that $p(H_v)$ is elliptic in T for every vertex v of S, then $p(H)$ is elliptic in T.*

Proof Consider two adjacent vertices v and w in S. Let H_v and H_w be their stabilizers. The group $H_v \cap H_w$ is infinite by hypothesis. Moreover, p is injective on edge groups, so $p(H_v \cap H_w)$ is infinite. Hence $p(H_v) \cap p(H_w)$ is infinite. Since edge groups of T are finite, $p(H_v)$ and $p(H_w)$ fix necessarily the same unique vertex x of T. As a consequence, for each vertex v of S, the group $p(H_v)$ fixes x.

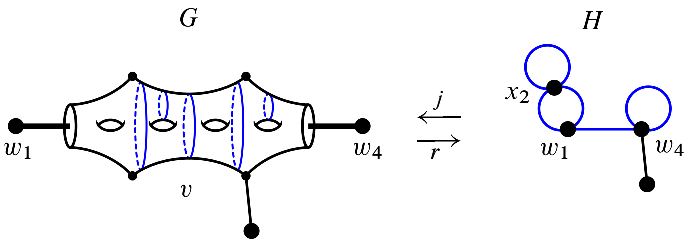


Figure 12: After collapsing the edge e, we get the desired splitting Δ_H of H.

Now, let h be an element of H, and let v be any vertex of S. By the previous paragraph, $p(H_v)$ and $p(H_{hv}) = p(h)p(H_v)p(h)^{-1}$ are both contained in G_x. Thus, $p(H_v)$ is contained in the intersection $G_x \cap p(h)^{-1} G_x p(h)$. Since H_v is infinite and p is injective on H_v, the intersection $G_x \cap p(h)^{-1} G_x p(h)$ is infinite as well. Moreover, edge groups of S are finite. It follows that the vertex groups G_x and $p(h)^{-1} G_x p(h)$ are equal. Thus, $p(h)$ belongs to G_x. Therefore, $p(H)$ is contained in G_x. □

We now prove the main result of this section, Proposition 6.3.

Proposition 7.6 *Let G be a finitely generated K–CSA group that does not contain $\mathbb{Z}^2$. Let A be a one-ended factor of G. Suppose that A is not a finite-by-orbifold group. If there exists a noninjective preretraction $p\colon A \to G$, then there exist a finitely generated group H and two morphisms $r\colon G \to H$ and $j\colon H \to G$ such that (G, H, r, j) is a strict quasifloor.*

Proof Let Δ be the $\mathcal{Z}$–JSJ splitting of A, which exists since A is K–CSA and does not contain $\mathbb{Z}^2$. Let Λ be a $\mathcal{F}$–JSJ splitting of G containing a vertex v_A with stabilizer A. Let T be the Bass–Serre tree of Λ.

Step 1 We shall prove that there exists a QH vertex v of Δ such that A_v is not sent isomorphically to a conjugate of itself by p.

If w is a non-QH vertex of Δ, then $p(A_w)$ is conjugate to A_w by definition of a Δ–preretraction. In particular, $p(A_w)$ is elliptic in T. Suppose for the sake of contradiction that each stabilizer A_v of a QH vertex v of Δ is sent isomorphically to a conjugate of itself by p. Then $p(A_v)$ is elliptic in T, for every QH vertex v. Therefore, it follows from Lemma 7.5 that $p(A)$ is elliptic in T, because p is injective on edge groups of Δ (indeed, each edge group of Δ is contained in a non-QH vertex group of Δ), and T has finite edge groups. Moreover, since p is inner on non-QH vertices of Δ, $p(A)$ is contained in gAg^{-1} for some $g \in G$ (note that there exists at least one non-QH vertex since A is not finite-by-orbifold by hypothesis). Up to composing p with the conjugation by g^{-1}, one can thus assume that p is an endomorphism of A. Now, by Proposition 7.1, p is injective. This is a contradiction. Hence, we have proved that there exists a QH vertex v of Δ such that A_v is not sent isomorphically to a conjugate of itself by p.

Step 2 We shall complete the proof using Proposition 7.4. For this purpose, we shall construct a centered splitting of G, together with an endomorphism of G satisfying the hypotheses of Proposition 7.4.

Note that every stabilizer G_e of an edge e of Λ incident to v_A is elliptic in the splitting Δ of A, as a finite group. First, we refine Λ by replacing the vertex v_A by the splitting Δ of A. The resulting splitting of A is denoted by Λ_2. With a little abuse of notation, we still denote by v the vertex of Λ_2 corresponding to the QH vertex v of Δ defined in the previous step. Then, we collapse to a point every connected component of the complement of $\mathrm{star}(v)$ in Λ_2 (where $\mathrm{star}(v)$ stands for the subgraph of Λ_2 composed of v and all its incident edges); the stabilizer of such a point in the new graph of groups is the fundamental group of the corresponding connected component, viewed as a graph of groups. This new graph of groups, denoted by Λ_3, is nontrivial, since A is not finite-by-orbifold (by hypothesis). Thus, Λ_3 is a centered splitting of G, with central vertex v.

The homomorphism $p\colon A \to G$ is well defined on G_v because $G_v = A_v$ is contained in A. Moreover, p restricts to a conjugation on the group of each edge e of Λ_3 incident to v. Indeed, either e is an edge coming from Δ or G_e is a finite subgroup of A; in both cases, $p|_{G_e}$ is a conjugation since p is Δ–related to the inclusion of A into G.

Now, we claim that there exists an endomorphism $q\colon G \to G$ that coincides with p on $G_v = A_v$ and that coincides with a conjugation on every vertex group of Λ_3 different from G_v. We proceed by induction on the number of edges of the graph of groups Λ_3. It is enough to prove the claim in the case where Λ_3 has only one edge. If $G = A_v *_C B$ with $p|_C = \mathrm{ad}(g)$, one defines $q\colon G \to G$ by $q|_{A_v} = p$ and $q|_B = \mathrm{ad}(g)$. If $G = A_v *_C = \langle A_v, t \mid tct^{-1} = \alpha(c) \text{ for all } c \in C \rangle$ with $p|_C = \mathrm{ad}(g_1)$ and $p|_{\alpha(C)} = \mathrm{ad}(g_2)$, one defines $q\colon G \to G$ by $q|_{A_v} = p$ and $q(t) = g_2^{-1} t g_1$.

The endomorphism q defined above is Λ_3–related to the identity of G (in the sense of Definition 3.13), by construction. Moreover, q does not send G_v isomorphically to a conjugate of itself, by Step 1. In other words, q is a degenerate Λ_3–preretraction of G. In order to apply Proposition 7.4 to the group G with the splitting Λ_3 and the degenerate Λ_3–preretraction q, we will prove that the restriction of q to the one-ended subgroup A of G is noninjective. In the case where q kills an element of $A_v \subset A$, this is obvious. Now, let us suppose that the restriction of q to A_v is injective. Then q is a fortiori nonpinching on A_v, and we claim that $q(A_v)$ is elliptic in T. First, let us observe that the images by q of the extended boundary subgroups of the finite-by-orbifold group A_v are elliptic in T, since q restricts to a conjugacy on these subgroups. Then, by Proposition 2.31 and the paragraph following it, one can cut the underlying orbifold of A_v into connected components that are elliptic in T via q; but edge groups of T are finite, and q is nonpinching on A_v, so $q(A_v)$ is elliptic in T by Lemma 7.5. Again by Lemma 7.5, $q(A)$ is contained in A (up to conjugacy), so q induces an endomorphism of A. Now, we are ready to find a nontrivial element in $\ker(q) \cap A$. Let Δ' be the splitting of A obtained by collapsing every connected component of the complement of $\mathrm{star}(v)$ in the $\mathcal{Z}$–JSJ splitting Δ of A. With an abuse of notation, we still denote by v the vertex of Δ' coming from the vertex v of Δ. The splitting Δ' is centered, with central vertex v, and $q|_A$ is an endomorphism of A that does not send A_v isomorphically to a conjugate of itself, by Step 1 and since $q|_{A_v} = p|_{A_v}$. Moreover, $q|_A$ is Δ'–related to the identity of A (in the sense of Definition 3.13); indeed, if w is a vertex of Δ' different from v, there exists a vertex $\widetilde{w} \in \Lambda_3$ such that A_w is contained in $G_{\widetilde{w}}$, and q restricts to a conjugation on $G_{\widetilde{w}}$ since q is Λ–related to the identity of G, by construction. So it follows from Lemma 7.3 that Δ' has only one vertex w different from v, and that $q(A_v) \subset q(A_w)$. Since q is inner on A_w, there exists an element $a \in A$ such that $\mathrm{ad}(a) \circ q$ is a retraction from A onto A_w. Let x be an element of A_v that does not belong to A_w. Let $y = \mathrm{ad}(a) \circ q(x)$; we have seen that y lies in A_w, so $\mathrm{ad}(a) \circ q(y) = y$. Hence, $\mathrm{ad}(a) \circ q(xy^{-1}) = 1$, with $xy^{-1} \in A \setminus \{1\}$. We have proved that the restriction of q to A is noninjective. Now, it follows from Proposition 7.4 that there exist a finitely generated group H and two morphisms $r\colon G \to H$ and $j\colon H \to G$ such that (G, H, r, j) is a strict quasifloor. This concludes the proof of the proposition. □

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