The quantum tropical vertex

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Gross, Pandharipande and Siebert have shown that the 2–dimensional Kontsevich–Soibelman scattering diagrams compute certain genus-zero log Gromov–Witten invariants of log Calabi–Yau surfaces. We show that the q–refined 2–dimensional Kontsevich–Soibelman scattering diagrams compute, after the change of variables $q = e^{i\hbar}$, generating series of certain higher-genus log Gromov–Witten invariants of log Calabi–Yau surfaces.

This result provides a mathematically rigorous realization of the physical derivation of the refined wall-crossing formula from topological string theory proposed by Cecotti and Vafa and, in particular, can be viewed as a nontrivial mathematical check of the connection suggested by Witten between higher-genus open A–model and Chern–Simons theory.

We also prove some new BPS integrality results and propose some other BPS integrality conjectures.

Introduction

Statements of results

We start by giving slightly imprecise versions of the main results of the present paper. For us, a log Calabi–Yau surface is a pair (Y, D), where Y is a smooth complex projective surface and D is a reduced effective normal crossing anticanonical divisor on Y. A log Calabi–Yau surface (Y, D) has maximal boundary if D is singular.

Theorem 0.1 *The functions attached to the rays of the q–deformed 2–dimensional Kontsevich–Soibelman scattering diagrams are, after the change of variables $q = e^{i\hbar}$, generating series of higher-genus log Gromov–Witten invariants — with maximal tangency condition and insertion of the top lambda class — of log Calabi–Yau surfaces with maximal boundary.*

A precise version of Theorem 0.1 is given by Theorems 3.1 and 3.2 in Section 3.

Theorem 0.2 *Higher genus log Gromov–Witten invariants — with maximal tangency condition and insertion of the top lambda class — of log Calabi–Yau surfaces with maximal boundary satisfy an Ooguri–Vafa/open BPS integrality property.*

A precise version of Theorem 0.2 is given by Theorem 8.5 in Section 8.

We also formulate a new conjecture.

Conjecture 0.3 *Higher genus relative Gromov–Witten invariants — with maximal tangency condition and insertion of the top lambda class — of a del Pezzo surface S relative to a smooth anticanonical divisor are related to refined counts of dimension-one stable sheaves on the local Calabi–Yau 3–fold $\mathrm{Tot}\, K_S$, the total space of the canonical line bundle of S.*

A precise version of Conjecture 0.3 is given by Conjecture 8.16 in Section 8.6.

Context and motivations

SYZ The Strominger–Yau–Zaslow [68] picture of mirror symmetry suggests a two-step construction of the mirror of a Calabi–Yau variety admitting a Lagrangian torus fibration: first, construct the "semiflat" mirror by dualizing the nonsingular torus fibers; second, correct the complex structure of the "semiflat" mirror such that it extends across the locus of singular fibers. It is expected — see [68] and Fukaya [22] — that the corrections involved in the second step are determined by some counts of holomorphic discs in the original variety with boundary on torus fibers.

KS In dimension two and with at most nodal singular fibers in the torus fibration, Kontsevich and Soibelman [42] had the insight that algebraic self-consistency constraints on the corrections were strong enough to determine these corrections uniquely. More precisely, they reduced the problem to an algebraic computation of commutators in a group of formal families of symplectomorphisms of the 2–dimensional algebraic torus.

This algebraic formalism, graphically encoded under the form of scattering diagrams, was generalized and extended to higher dimensions by Gross and Siebert [32] and plays an essential role in the Gross–Siebert algebraic approach to mirror symmetry.

GPS In [31], Gross, Pandharipande and Siebert made some progress in connecting the original enumerative expectation and the algebraic recipe of scattering diagrams. They showed that the 2–dimensional Kontsevich–Soibelman scattering diagrams indeed have an enumerative meaning: they compute some genus 0 log Gromov–Witten invariants of some log Calabi–Yau surfaces with maximal boundary, ie complements of a singular normal crossing anticanonical divisor in a smooth projective surface.

This agrees with the original expectation because these geometries admit Lagrangian torus fibrations and these genus 0 log Gromov–Witten invariants should be thought of as algebraic definitions of some counts of holomorphic discs with boundary on Lagrangian torus fibers. For some symplectic approaches, relating counts of holomorphic discs in hyperkähler manifolds of real dimension 4 and the Konstevich–Soibelman wall-crossing formula, we refer to the works of Lin [49] and Iacovino [34].

The combination of 2–dimensional scattering diagrams with their enumerative interpretation given by Gross, Pandharipande and Siebert [31] is the main tool in the Gross–Hacking–Keel [28] construction of mirrors for log Calabi–Yau surfaces with maximal boundary.

Higher genus GPS = refined KS In [42, Section 11.8] (see also [67]), Kontsevich and Soibelman remarked that the 2–dimensional scattering diagram formalism has a natural q–deformation, with the group of formal families of symplectomorphisms of the 2–dimensional algebraic torus replaced by a group a formal families of automorphisms of the 2–dimensional quantum torus, a natural noncommutative deformation of the 2–dimensional algebraic torus. The enumerative meaning of this q–deformed scattering diagram was a priori unclear.

In Section 5.8 of [31], Gross, Pandharipande and Siebert remarked that the genus 0 log Gromov–Witten invariants they consider have a natural extension to higher genus, by integration of the top lambda class, and they asked if there is an interpretation of these higher-genus invariants in terms of scattering diagrams.

The main result of the present paper, Theorem 0.1, is that the two previous questions, the enumerative meaning of the algebraic q–deformation and the algebraic meaning of the higher-genus deformation, are answers to each other.

OV The higher-genus log Gromov–Witten invariants of log Calabi–Yau surfaces that we are considering — with insertion of the top lambda class — should be thought of as an algebrogeometric definition of some counts of higher-genus Riemann surfaces with boundary on a Lagrangian torus fiber in a Calabi–Yau 3–fold geometry, essentially the product of the log Calabi–Yau surface by a third trivial direction; see Section 2.4. For such counts of higher-genus open curves in a Calabi–Yau 3–fold geometry, Ooguri and Vafa [61] have conjectured an open BPS integrality structure. Theorem 0.2, which is a consequence of Theorem 0.1 and of nontrivial algebraic properties of q–deformed scattering diagrams, can be viewed as a check of this BPS integrality structure.

DT The nontrivial integrality properties of q–deformed scattering diagrams are well known to be related to integrality properties of refined Donaldson–Thomas (DT) invariants; see Kontsevich and Soibelman [43]. Indeed, q–deformed scattering diagrams control the wall-crossing behavior of refined DT invariants.

The fact that the integrality structure of DT invariants coincides with the Ooguri–Vafa integrality structure of higher-genus open Gromov–Witten invariants of Calabi–Yau 3–folds, essentially involving the quantum dilogarithm in both cases, can be viewed as an early indication that something like Theorem 0.1 should be true.

As consequence of Theorem 0.1, we obtain explicit relations between refined DT invariants of some quivers and higher-genus log Gromov–Witten invariants of log Calabi–Yau surfaces — see Section 8.5 — generalizing the unrefined/genus 0 relation of Gross and Pandharipande [30] and Reineke and Weist [66].

CV In fact, Cecotti and Vafa [15] have given a physical derivation of the wall-crossing formula in DT theory going through the higher-genus open Gromov–Witten theory of some Calabi–Yau 3–fold. We will explain in Section 9 that Theorems 0.1 and 0.2 are indeed fully compatible with the Cecotti–Vafa argument. In particular, Theorem 0.1 can be viewed as a highly nontrivial mathematical check of the connection predicted by Witten [70] between higher-genus open A–model and quantum Chern–Simons theory.

del Pezzo Theorems 0.1 and 0.2 are about log Calabi–Yau surfaces with maximal boundary, ie with a singular normal crossing anticanonical divisor. Similar questions can be asked for log Calabi–Yau surfaces with respect to a smooth anticanonical divisor. Conjecture 0.3 gives a nontrivial correspondence in such a case, suggested by the similarities between refined DT theory and open higher-genus Gromov–Witten invariants discussed above.

Applications

In a way parallel to how Gross, Hacking and Keel [28] used [31] to construct Poisson varieties as mirrors of log Calabi–Yau surfaces with maximal boundary, we will use Theorem 0.1 in a coming work [11] to construct deformation quantizations of these Poisson varieties.

Comments on the proof of Theorem 0.1

The curve-counting invariants appearing in Theorem 0.1 are log Gromov–Witten invariants, as defined by Gross and Siebert [33] and Abramovich and Chen [16; 1]. The proof of Theorem 0.1 relies on recently developed general properties of log Gromov–Witten invariants, such as the decomposition formula of Abramovich, Chen, Gross and Siebert [2].

The main tool of [31] is a reduction to a tropical setting using the correspondence theorem of Mikhalkin [57] and Nishinou and Siebert [60] between counts of curves in complex toric surfaces and counts of tropical curves in $\mathbb{R}^2$. Similarly, the main tool of the present paper is a reduction to a tropical setting using a correspondence theorem previously proved by the author [10] between generating series of higher-genus curves in toric surfaces and Block–Göttsche q–deformed tropical invariants.

Given that the relation between q–deformed tropical invariants and q–deformed scattering diagrams has already been worked out by Filippini and Stoppa [21], Theorem 0.1 should really be viewed as a combination of [10] and [21]. The new results required for the proof of Theorem 0.1 are the check that the degeneration step used in [31] to go from a log Calabi–Yau setting to a toric setting extends to the higher-genus case and the check that the correspondence proved in [10] has exactly the correct form to be used as input in [21]. To make this paper more self-contained, we will in fact review the content of [21].

The most technical part is the higher-genus version of the degeneration step. As the general version of the degeneration formula in log Gromov–Witten theory is not yet

known, we combine the general decomposition formula of [2] with some situation-specific vanishing statements, which, as in [10], reduce the gluing operations to some torically transverse locus where they are under control, thanks to, for example, Kim, Lho and Ruddat [40].

Comments on the proof of Theorem 0.2

The proof of Theorem 0.2 is a combination of Theorem 0.1 and of the nontrivial integrality results about q–deformed scattering diagrams proved by Kontsevich and Soibelman in Section 6 of [44]. In fact, to get the most general form of Theorem 0.2, the results contained in [44] do not seem to be enough. We use an induction argument on scattering diagrams, parallel to the one used by Gross, Hacking, Keel and Kontsevich in [29, Appendix C3], to reduce the most general case to a case which can be treated by [44].

A small technical point is to keep track of signs, because of the difference between quantum tori and twisted quantum tori; see Section 8.3 on the quadratic refinement for details.

Plan of the paper

In Section 1, we review the notion of 2–dimensional scattering diagrams, both classical and quantum, with an emphasis on the symplectic/Hamiltonian aspects. In Section 2, we introduce a class of log Calabi–Yau surfaces and their log Gromov–Witten invariants.

In Section 3, we state our main result, Theorem 3.1, a precise version of Theorem 0.1, relating 2–dimensional quantum scattering diagrams and generating series of higher-genus log Gromov–Witten invariants of log Calabi–Yau surfaces. We also state a generalization of Theorem 3.1, Theorem 3.2, phrased in terms of orbifold log Gromov–Witten invariants.

Sections 4, 5, 6 and 7 are dedicated to the proof of Theorems 3.1 and 3.2. The general structure of the proof is parallel to [31]. In Section 4, we introduce higher-genus log Gromov–Witten invariants of toric surfaces. In Section 5, the most technical part of this paper, we prove a degeneration formula relating log Gromov–Witten invariants of log Calabi–Yau surfaces defined in Section 2 and appearing in Theorem 3.1, with log Gromov–Witten invariants of toric surfaces defined in Section 4.2. In Section 6, following Filippini and Stoppa [21], we review the connection between quantum scattering diagrams and refined counts of tropical curves. We finish the proof of Theorem 3.1 in Section 7, combining the results of Sections 5 and 6 with the correspondence theorem proved in [10] between refined counts of tropical curves and log Gromov–Witten

invariants of toric surfaces. The orbifold Gromov–Witten computation needed to finish the proof of Theorem 3.2 is done in Section 7.2.

In Section 8.1, we formulate a BPS integrality conjecture for higher-genus log Gromov–Witten invariants of log Calabi–Yau surfaces. In Section 8.2, we state Theorem 8.5, a precise form of Theorem 0.2. The proof of Theorem 8.5 takes Sections 8.3 and 8.4. In Section 8.5, Theorem 8.13 gives an explicit connection with refined DT invariants of quivers. Finally, in Section 8.6, we state Conjecture 8.16, a precise version of Conjecture 0.3.

In Section 9, we explain how Theorem 0.1 can be viewed as a mathematical check of the physics work of Cecotti and Vafa [15] and how Theorem 0.2 is compatible with the Ooguri–Vafa integrality conjecture [61].

1 Scattering

In this section, we first fix our notation for the basic objects considered in this paper: tori, quantum tori and automorphisms of formal families of them. We then introduce scattering diagrams, both classical and quantum, following [42; 32; 31; 21].

1.1 Torus

We fix T, a 2–dimensional complex algebraic torus. Let $M := \mathrm{Hom}(T, \mathbb{C}^*)$ be the 2–dimensional lattice of characters of T. Characters form a linear basis of the algebra of functions on T,

$$\Gamma(\mathcal{O}_T) = \bigoplus_{m \in M} \mathbb{C} z^m,$$

with the product given by $z^m \cdot z^{m'} = z^{m+m'}$. In other words, the algebra of functions on T is the algebra of the lattice M: $\Gamma(\mathcal{O}_T) = \mathbb{C}[M]$.

We fix

$$\langle -, - \rangle \colon \wedge^2 M \xrightarrow{\sim} \mathbb{Z}$$

an orientation of M, ie an integral unimodular skew-symmetric bilinear form on M. This defines a Poisson bracket on $\Gamma(\mathcal{O}_T)$, given by

$$\{z^m, z^{m'}\} = \langle m, m' \rangle z^{m+m'},$$

and a corresponding algebraic symplectic form Ω on T.

If we choose a basis (m_1, m_2) of M such that $\langle m_1, m_2 \rangle = 1$, then, writing $z_1 := z^{m_1}$ and $z_2 := z^{m_2}$, we have identifications $T = (\mathbb{C}^*)^2$, $M = \mathbb{Z}^2$, $\Gamma(\mathcal{O}_T) = \mathbb{C}[z_1^\pm, z_2^\pm]$ and

$$\Omega = \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2}.$$

1.2 Quantum torus

Given the symplectic torus (T, Ω), or equivalently the Poisson algebra $(\Gamma(\mathcal{O}_T), \{-, -\})$, it is natural to look for a "quantization". The quantum torus $\widehat{T}^q$ is the noncommutative "space" whose algebra of functions is the noncommutative $\mathbb{C}[q^{\pm\frac{1}{2}}]$–algebra $\Gamma(\mathcal{O}_{\widehat{T}^q})$, with linear basis indexed by the lattice M,

$$\Gamma(\mathcal{O}_{\widehat{T}^q}) = \bigoplus_{m \in M} \mathbb{C}[q^{\pm\frac{1}{2}}] \hat{z}^m,$$

and with product defined by

$$\hat{z}^m \cdot \hat{z}^{m'} = q^{\frac{1}{2}\langle m, m' \rangle} \hat{z}^{m+m'}.$$

The quantum torus $\widehat{T}^q$ is a quantization of the torus T in the sense that writing $q = e^{i\hbar}$ and taking the limit $\hbar \to 0$ and $q \to 1$, the linear term in $\hbar$ of the commutator $[\hat{z}^m, \hat{z}^{m'}]$ is determined by the Poisson bracket $\{z^m, z^{m'}\}$. Indeed, we have

$$[\hat{z}^m, \hat{z}^{m'}] = (q^{\frac{1}{2}\langle m, m' \rangle} - q^{-\frac{1}{2}\langle m, m' \rangle}) \hat{z}^{m+m'},$$

and so

$$\lim_{\hbar \to 0} \frac{1}{i\hbar} [\hat{z}^m, \hat{z}^{m'}] = \langle m, m' \rangle \hat{z}^{m+m'}.$$

We denote by $\widehat{T}^\hbar$ the noncommutative "space" whose algebra of functions is the $\mathbb{C}((\hbar))$–algebra $\Gamma(\mathcal{O}_{\widehat{T}^\hbar}) := \Gamma(\mathcal{O}_{\widehat{T}^\hbar}) \otimes_{\mathbb{C}[q^{\pm\frac{1}{2}}]} \mathbb{C}((\hbar))$.

1.3 Automorphisms of formal families of tori

Let R be a complete local $\mathbb{C}$–algebra and let $\mathfrak{m}_R$ be the maximal ideal of R. By definition of completeness, we have

$$R = \varprojlim_{\ell} R/\mathfrak{m}_R^{\ell}.$$

We denote by $S := \mathrm{Spf}\, R$ the corresponding formal scheme and s_0 the closed point of S defined by $\mathfrak{m}_R$. Let T_S be the trivial family of 2–dimensional complex algebraic tori parametrized by S, ie $T_S := S \times T$. The corresponding algebra of functions is given by

$$\Gamma(\mathcal{O}_{T_S}) = \varprojlim_{\ell}(R/\mathfrak{m}_R^{\ell} \otimes \Gamma(\mathcal{O}_T)) = \varprojlim_{\ell}(R/\mathfrak{m}_R^{\ell} \otimes \mathbb{C}[M]).$$

Let $\widehat{T}_S^{\hbar}$ be the trivial family of noncommutative 2–dimensional tori parametrized by S, ie $\widehat{T}_S^{\hbar} := S \times \widehat{T}^{\hbar}$. The corresponding algebra of functions is simply given by

$$\Gamma(\mathcal{O}_{\widehat{T}_S^{\hbar}}) = \varprojlim_{\ell}(R/\mathfrak{m}_R^{\ell} \otimes \Gamma(\mathcal{O}_{\widehat{T}^{\hbar}})).$$

The family T_S of tori has a natural Poisson structure, whose symplectic leaves are the torus fibers, and whose Poisson center is R. Explicitly, we have

$$\{H_m z^m, H_{m'} z^{m'}\} = H_m H_{m'}\{z^m, z^{m'}\}$$

for every $H_m, H_{m'} \in R$ and $m, m' \in M$. The family $\widehat{T}_S^{\hbar}$ of noncommutative tori is a quantization of the Poisson variety T_S.

Let

$$H = \sum_{m \in M} H_m z^m$$

be a function on T_S whose restriction to the fiber over the closed point $s_0 \in S$ vanishes, ie such that $H = 0 \mod \mathfrak{m}_R$. Then $\{H, -\}$ defines a derivation of the algebra of functions on T_S and so a vector field on T_S, the Hamiltonian vector field defined by H, whose restriction to the fiber over the closed point $s_0 \in S$ vanishes.

The time-one flow of this vector field defines an automorphism

$$\Phi_H := \exp(\{H, -\})$$

of T_S, whose restriction to the fiber over the closed point $s_0 \in S$ is the identity. Note that Φ_H is well defined because of the assumptions that $H = 0 \mod \mathfrak{m}_R$ and that R is a complete local algebra, ie exp makes sense formally.

Let $\mathbb{V}_R$ be the subset of automorphisms of T_S which are of the form Φ_H for H as above. By the Baker–Campbell–Hausdorff formula, $\mathbb{V}_R$ is a subgroup of the group of automorphisms of T_S. In [31], $\mathbb{V}_R$ is called the tropical vertex group.

Let

$$\hat{H} = \sum_{m \in M} \hat{H}_m \hat{z}^m$$

be a function on $\hat{T}_S^\hbar$ whose restriction to the fiber over the closed point $s_0 \in S$ vanishes, ie such that $\hat{H} = 0$ mod $\mathfrak{m}_R$. Conjugation by $\exp(\hat{H})$ defines an automorphism

$$\hat{\Phi}_{\hat{H}} := \mathrm{Ad}_{\exp(\hat{H})} = \exp(\hat{H})(-)\exp(-\hat{H})$$

of $\hat{T}_S^\hbar$ whose restriction to the fiber over the closed point $s_0 \in S$ is the identity. Note that $\hat{\Phi}_{\hat{H}}$ is well defined because of the assumptions that $\hat{H} = 0$ mod $\mathfrak{m}_R$ and that R is a compete local algebra, ie everything makes sense formally. Let $\hat{\mathbb{V}}_R^\hbar$ be the subset of automorphisms of $\hat{T}_S^\hbar$ which are of the form $\hat{\Phi}_{\hat{H}}$ for $\hat{H}$ as above. By the Baker–Campbell–Hausdorff formula, $\hat{\mathbb{V}}_R^\hbar$ is a subgroup from the group of automorphisms of $\hat{T}_S^\hbar$. We call $\hat{\mathbb{V}}_R^\hbar$ the quantum tropical vertex group. This group is much bigger that the "quantum tropical vertex group" of [44]. We will meet the group of [44] in Section 8, under the name "BPS quantum tropical vertex group".

If the limit

$$H := \lim_{\hbar \to 0}(i\hbar\hat{H})$$

exists, then, replacing $\hat{z}^m$ by z^m, H can be naturally viewed as a function on T_S and is the classical limit of $\hat{H}$. It is easy to check that Φ_H is the classical limit of $\hat{\Phi}_{\hat{H}}$.

1.4 Scattering diagrams

In this section, we work in the 2–dimensional real plane $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$. A *ray* is a half-line $\mathfrak{d}$ in $M_{\mathbb{R}}$ of rational slope with initial point the origin $0 \in M_{\mathbb{R}}$, and we denote by $m_{\mathfrak{d}} \in M - \{0\}$ its primitive integral direction, pointing away from the origin.

Definition 1.1 A *scattering diagram* $\mathfrak{D}$ over R is a set of rays $\mathfrak{d}$ in $M_{\mathbb{R}}$, equipped with functions $H_{\mathfrak{d}}$ such that either

$$H_{\mathfrak{d}} \in \varprojlim_{\ell}(R/\mathfrak{m}_R^\ell \otimes \mathbb{C}[z^{m_{\mathfrak{d}}}])$$

or

$$H_{\mathfrak{d}} \in \varprojlim_{\ell}(R/\mathfrak{m}_R^\ell \otimes \mathbb{C}[z^{-m_{\mathfrak{d}}}])$$

and such that $H_{\mathfrak{d}} = 0 \bmod \mathfrak{m}_R$ and, for every $\ell \geqslant 1$, only finitely many rays $\mathfrak{d}$ have $H_{\mathfrak{d}} \neq 0 \bmod \mathfrak{m}_R^\ell$.

A ray $(\mathfrak{d}, H_{\mathfrak{d}})$ such that

$$H_{\mathfrak{d}} \in \varprojlim_{\ell} (R/\mathfrak{m}_R^\ell \otimes \mathbb{C}[z^{m_{\mathfrak{d}}}])$$

is called *outgoing* and a ray $(\mathfrak{d}, H_{\mathfrak{d}})$ such that

$$H_{\mathfrak{d}} \in \varprojlim_{\ell} (R/\mathfrak{m}_R^\ell \otimes \mathbb{C}[z^{-m_{\mathfrak{d}}}])$$

is called *ingoing*.

Given a ray $(\mathfrak{d}, H_{\mathfrak{d}})$, we define $m(H_{\mathfrak{d}}) := m_{\mathfrak{d}}$ if $(\mathfrak{d}, H_{\mathfrak{d}})$ is outgoing, and $m(H_{\mathfrak{d}}) := -m_{\mathfrak{d}}$ if $(\mathfrak{d}, H_{\mathfrak{d}})$ is ingoing. In both cases, we have

$$H_{\mathfrak{d}} \in \varprojlim_{\ell} (R/\mathfrak{m}_R^\ell \otimes \mathbb{C}[z^{m(H_{\mathfrak{d}})}]),$$

We will always consider scattering diagrams up to the following simplifying operations:

- A ray $(\mathfrak{d}, H_{\mathfrak{d}})$ with $H_{\mathfrak{d}} = 0$ is considered to be trivial and can be safely removed from the scattering diagram.

- If two rays $(\mathfrak{d}_1, H_{\mathfrak{d}_1})$ and $(\mathfrak{d}_2, H_{\mathfrak{d}_2})$ are such that $\mathfrak{d}_1 = \mathfrak{d}_2$ and are both ingoing or outgoing, then they can be replaced by a single ray $(\mathfrak{d}, H_{\mathfrak{d}})$, where $\mathfrak{d} = \mathfrak{d}_1 = \mathfrak{d}_2$ and $H_{\mathfrak{d}} = H_{\mathfrak{d}_1} + H_{\mathfrak{d}_2}$. Note that, because $\{H_{\mathfrak{d}_1}, H_{\mathfrak{d}_2}\} = 0$, we have that $\Phi_{H_{\mathfrak{d}}} = \Phi_{H_{\mathfrak{d}_1}} \Phi_{H_{\mathfrak{d}_2}} = \Phi_{H_{\mathfrak{d}_2}} \Phi_{H_{\mathfrak{d}_1}}$.

Let $\mathfrak{D}$ be a scattering diagram. The *singular locus* of $\mathfrak{D}$ is the union of the set of initial points of rays and the set of nontrivial intersection points of rays. Let $\gamma \colon [0,1] \to M_{\mathbb{R}}$ be a smooth path. We say that γ is admissible if γ does not intersect the singular locus of $\mathfrak{D}$, if the endpoints of γ are not on rays of $\mathfrak{D}$, and if γ intersects transversely all the rays of $\mathfrak{D}$.

Let γ be an admissible smooth path in $M_{\mathbb{R}}$. Let $\ell \geqslant 1$ be a positive integer. By definition, $\mathfrak{D}$ contains only finitely many rays $(\mathfrak{d}, H_{\mathfrak{d}})$ with $H_{\mathfrak{d}} \neq 0 \bmod \mathfrak{m}_R^\ell$. We denote by $0 < t_1 \leqslant \cdots \leqslant t_s < 1$ the times of intersection of γ with rays $(\mathfrak{d}_1, H_{\mathfrak{d}_1}), \ldots, (\mathfrak{d}_s, H_{\mathfrak{d}_s})$ of $\mathfrak{D}$ such that $H_{\mathfrak{d}_r} \neq 0 \bmod \mathfrak{m}_R^\ell$. For every $1 \leqslant r \leqslant s$, we define $\epsilon_r \in \{\pm 1\}$ to be the sign of $\langle m(H_{\mathfrak{d}_i}), \gamma'(t_r) \rangle$. We then define

$$\theta_{\gamma, \mathfrak{D}, \ell} := \Phi_{H_{\mathfrak{d}_s}}^{\epsilon_s} \cdots \Phi_{H_{\mathfrak{d}_1}}^{\epsilon_1}.$$

Taking the limit $\ell \to +\infty$, we define

$$\theta_{\gamma,\mathfrak{D}} := \lim_{\ell \to +\infty} \theta_{\gamma,\mathfrak{D},\ell}.$$

Definition 1.2 A scattering diagram $\mathfrak{D}$ over R is *consistent* if, for every closed admissible smooth path $\gamma \colon [0,1] \to M_{\mathbb{R}}$, we have $\theta_{\gamma,\mathfrak{D}} = \mathrm{id}$.

The following result is due to Kontsevich and Soibelman [42, Theorem 6] (see also [31, Theorem 1.4]).

Proposition 1.3 *Any scattering diagram $\mathfrak{D}$ can be canonically completed by adding only outgoing rays to form a consistent scattering diagram $S(\mathfrak{D})$.*

Proof It is enough to show that for every nonnegative integer ℓ, it is possible to add outgoing rays to $\mathfrak{D}$ to get a scattering diagram $\mathfrak{D}_\ell$ consistent at the order ℓ, ie such that $\theta_{\gamma,\mathfrak{D}_\ell} = \mathrm{id} \bmod \mathfrak{m}_R^{\ell+1}$. The construction is done by induction on ℓ, starting with $\mathfrak{D}_0 = \mathfrak{D}$. Let us assume we have constructed $\mathfrak{D}_{\ell-1}$, consistent at the order $\ell-1$. Let p be a point in the singular locus of $\mathfrak{D}_{\ell-1}$ and let γ be a small anticlockwise closed loop around p. As $\mathfrak{D}_{\ell-1}$ is consistent at the order $\ell-1$, we can write $\theta_{\gamma,\mathfrak{D}_{\ell-1}} = \Phi_H$ for some H with $H = 0 \bmod \mathfrak{m}_R^\ell$. There are finitely many primitive $m_j \in M - \{0\}$ such that we can write

$$H = \sum_j H_j \quad \bmod \mathfrak{m}_R^{\ell+1}$$

with $H_j \in \mathfrak{m}_R^\ell R \otimes \mathbb{C}[z_j^m]$. We construct $\mathfrak{D}_\ell$ by adding to $\mathfrak{D}_{\ell-1}$ the outgoing rays $(p + \mathbb{R}_{\geqslant 0}\, m_j, \Phi_{-H_j})$. $\qquad\square$

Adding hats everywhere, we obtain the definition of a quantum scattering diagram $\widehat{\mathfrak{D}}$, with functions

$$\widehat{H}_{\mathfrak{d}} \in \varprojlim_\ell (R/\mathfrak{m}_R^\ell \otimes \mathbb{C}((\hbar))[\hat{z}^{m_{\mathfrak{d}}}])$$

for outgoing rays and

$$\widehat{H}_{\mathfrak{d}} \in \varprojlim_\ell (R/\mathfrak{m}_R^\ell \otimes \mathbb{C}((\hbar))[\hat{z}^{-m_{\mathfrak{d}}}])$$

for ingoing rays, the notion of consistent quantum scattering diagram, and the fact that every quantum scattering diagram $\widehat{\mathfrak{D}}$ can be canonically completed by adding only outgoing rays to form a consistent quantum scattering diagram $S(\widehat{\mathfrak{D}})$. We will often call $\widehat{H}_{\mathfrak{d}}$ the Hamiltonian attached to the ray $\mathfrak{d}$.

A general notion of scattering diagram, as in Section 2 of [45], takes as input a lattice M and an M–graded Lie algebra $\mathfrak{g}$. What we call a (classical) scattering diagram is the special case where M is the lattice of characters of a 2–dimensional symplectic torus T and where $\mathfrak{g} = (\Gamma(\mathcal{O}_{T_S}), \{-, -\})$. What we call a quantum scattering diagram is the special case where M is the lattice of characters of a 2–dimensional symplectic torus T and where $\mathfrak{g} = (\Gamma(\mathcal{O}_{\widehat{T}_S^\hbar}), [-, -])$.

In our definition of a scattering diagram, we attach to each ray $\mathfrak{d}$ a function

$$H_{\mathfrak{d}} = \sum_{\ell \geqslant 0} H_\ell z^{\ell m(H_{\mathfrak{d}})}$$

such that $H_{\mathfrak{d}} = 0 \bmod \mathfrak{m}_R$, which can be interpreted as a Hamiltonian generating an automorphism

$$\Phi_{H_{\mathfrak{d}}} = \exp(\{H_{\mathfrak{d}}, -\}).$$

In [31; 32; 21], the terminology is slightly different. To a ray $\mathfrak{d}$, they attach a function

$$f_{\mathfrak{d}} = \sum_{\ell \geqslant 0} c_\ell z^{\ell m(H_{\mathfrak{d}})}$$

such that $f_{\mathfrak{d}} = 1 \bmod \mathfrak{m}_R$ and, to a path $\gamma(t)$ intersecting transversely $\mathfrak{d}$ at time t_0, an automorphism

$$\theta_{f_{\mathfrak{d}}, \gamma} \colon z^m \mapsto z^m f_{\mathfrak{d}}^{\langle n_{\mathfrak{d}}, m \rangle},$$

where $n_{\mathfrak{d}}$ is the primitive generator of $\mathfrak{d}$ such that $\langle n_{\mathfrak{d}}, \gamma'(t_0) \rangle > 0$. These two choices are equivalent. Indeed, if ϵ is the sign of $\langle m(H_{\mathfrak{d}}), \gamma'(t_0) \rangle$, we have

$$\Phi_{H_{\mathfrak{d}}}^{\epsilon} = \theta_{f_{\mathfrak{d}}, \gamma}$$

if $H_{\mathfrak{d}}$ and $f_{\mathfrak{d}}$ are related by

$$\log f_{\mathfrak{d}} = \sum_{\ell \geqslant 0} \ell H_\ell z^{\ell m(H_{\mathfrak{d}})}.$$

The formalism of [32] is more general because it treats the Calabi–Yau case and not just a holomorphic symplectic case. In the present paper, focused on a holomorphic symplectic situation, using the Hamiltonians $H_{\mathfrak{d}}$ rather than the functions $f_{\mathfrak{d}}$ makes the quantization step transparent. The quantum version of the functions $f_{\mathfrak{d}}$ will be studied and used in [11].

2　Gromov–Witten theory of log Calabi–Yau surfaces

Our main result, Theorem 3.1, is an enumerative interpretation of a class of quantum scattering diagrams, as introduced in Section 1, in terms of higher-genus log Gromov–Witten invariants of a class of log Calabi–Yau surfaces. In Section 2.1 we review the definition of these log Calabi–Yau surfaces, following [31]. We define the relevant higher-genus log Gromov–Witten invariants in Sections 2.2 and 2.3. We give a 3–dimensional interpretation of these invariants in Section 2.4. Finally, we give a generalization of these invariants to a certain orbifold context in Section 2.5.

2.1　Log Calabi–Yau surfaces

We fix $\mathfrak{m} = (m_1, \ldots, m_n)$, an n–tuple of primitive nonzero vectors of $M = \mathbb{Z}^2$. The fan in $\mathbb{R}^2$ with rays $-\mathbb{R}_{\geqslant 0} m_1, \ldots, -\mathbb{R}_{\geqslant 0} m_n$ defines a toric surface $\overline{Y}_{\mathfrak{m}}$. Let $D_{m_1}, \ldots, D_{m_n}$ be the corresponding toric divisors. If $m_1, \ldots, m_n$ do not span M, ie if $\overline{Y}_{\mathfrak{m}}$ is noncompact, we add some extra rays to the fan to make it span M and we still denote by $\overline{Y}_{\mathfrak{m}}$ the corresponding compact toric surfaces. The choice of the added rays will be irrelevant for us because of the log birational invariance result in logarithmic Gromov–Witten theory proved in [6]. We denote by $\partial \overline{Y}_{\mathfrak{m}}$ the toric boundary divisor of $\overline{Y}_{\mathfrak{m}}$.

For every $1 \leqslant j \leqslant n$, we blow up $Y_{\mathfrak{m}}$ at a point x_j on the toric divisor D_{m_j} and not on any other toric divisor. We also assume that all the points x_j are distinct. By deformation invariance of log Gromov–Witten invariants, the precise choice of x_j will be irrelevant for us. Note that it is possible to have $\mathbb{R}_{\geqslant 0} m_j = \mathbb{R}_{\geqslant 0} m_{j'}$, and so $D_{m_j} = D_{m_{j'}}$, for $j \neq j'$, and that in this case we blow up several distinct points on the same toric divisor. We denote by $Y_{\mathfrak{m}}$ the resulting projective surface and $\nu \colon Y_{\mathfrak{m}} \to \overline{Y}_{\mathfrak{m}}$ the blow-up morphism. Let $E_j := \nu^{-1}(x_j)$ be the exceptional divisor over x_j. We denote by $\partial Y_{\mathfrak{m}}$ the strict transform of the toric boundary divisor. The divisor $\partial Y_{\mathfrak{m}}$ is an anticanonical cycle of rational curves and so the pair $(Y_{\mathfrak{m}}, \partial Y_{\mathfrak{m}})$ is an example of a log Calabi–Yau surface with maximal boundary.

2.2　Curve classes

We want to consider curves in $Y_{\mathfrak{m}}$ meeting $\partial Y_{\mathfrak{m}}$ in a unique point. We first explain how to parametrize the relevant curve classes in terms of their intersection numbers p_j with the exceptional divisors E_j.

Let $p := (p_1, \ldots, p_n) \in P := \mathbb{N}^n$. We assume that $\sum_{j=1}^{n} p_j m_j \neq 0$ and so we can uniquely write

$$\sum_{j=1}^{n} p_j m_j = \ell_p m_p,$$

for some $m_p \in M$ primitive and some $\ell_p \in \mathbb{N}$.

We explain now how to define a curve class $\beta_p \in H_2(Y_{\mathrm{m}}, \mathbb{Z})$. In short, β_p is the class of a curve in Y_{m} having for every $1 \leqslant j \leqslant n$ intersection number p_j with the exceptional divisor E_j, and exactly one intersection point with the anticanonical cycle ∂Y_{m}.

More precisely, the vector $m_p \in M$ belongs to some cone of the fan of $\overline{Y}_{\mathrm{m}}$ and we write the corresponding decomposition

$$m_p = a_p^L m_p^L + a_p^R m_p^R,$$

where $m_p^L, m_p^R \in M$ are primitive generators of rays of the fan of $\overline{Y}_{\mathrm{m}}$ and where $a_p^L, a_p^R \in \mathbb{N}$. Note that there is only one term in this decomposition if the ray $\mathbb{R}_{\geqslant 0} m_p$ coincides with one of the rays of the fan of $\overline{Y}_{\mathrm{m}}$. Let D_p^L and D_p^R be the toric divisors corresponding to the rays $\mathbb{R}_{\geqslant 0} m_p^L$ and $\mathbb{R}_{\geqslant 0} m_p^R$. Let $\beta \in H_2(\overline{Y}_{\mathrm{m}}, \mathbb{Z})$ be determined by the following intersection numbers with the toric divisors:

- The intersection numbers with those D_{m_j} for $1 \leqslant j \leqslant n$ that are distinct from D_p^L and D_p^R:

$$\beta \cdot D_{m_j} = \sum_{j', D_{m_{j'}} = D_{m_j}} p_{j'}.$$

- The intersection number with D_p^L:

$$\beta \cdot D_p^L = \ell_p a_p^L + \sum_{j, D_{m_j} = D_p^L} p_j.$$

- The intersection number with D_p^R:

$$\beta \cdot D_p^R = \ell_p a_p^R + \sum_{j, D_{m_j} = D_p^R} p_j.$$

- The intersection number with every toric divisor D different from D_{m_j} for every $1 \leqslant j \leqslant n$ and from D_p^L and D_p^R:

$$\beta \cdot D = 0.$$

Such a class $\beta \in H_2(\overline{Y}_{\mathrm{m}}, \mathbb{Z})$ exists and is unique by standard toric geometry because of the relation $\sum_{j=1}^{n} p_j m_j = \ell_p m_p$. Finally, we define

$$\beta_p := \nu^* \beta - \sum_{j=1}^{n} p_j E_j \in H_2(Y_{\mathrm{m}}, \mathbb{Z}).$$

By construction, we have

$$\beta_p \cdot E_j = p_j$$

for every $1 \leqslant j \leqslant n$,

$$\beta_p \cdot D_p^L = \ell_p a_p^L \quad \text{and} \quad \beta_p \cdot D_p^R = \ell_p a_p^R,$$

and

$$\beta_p \cdot D = 0$$

for every component D of ∂Y_{m} distinct from D_p^L and D_p^R.

2.3 Log Gromov–Witten invariants

For every $p = (p_1, \ldots, p_n) \in P = \mathbb{N}^n$, we defined in Section 2.2 positive integers ℓ_p, a_p^L and a_p^R, some components D_p^L and D_p^R of the divisor ∂Y_{m} and a curve class $\beta_p \in H_2(Y_{\mathrm{m}}, \mathbb{Z})$. We would like to consider genus g stable maps $f \colon C \to Y_{\mathrm{m}}$ of class β_p that meet ∂Y_m in a unique point. At this point, such a map necessarily has an intersection number $\ell_p a_p^L$ with D_p^L and $\ell_p a_p^R$ with D_p^R.

The space of such stable maps is not proper in general: a limit of stable maps intersecting ∂Y_{m} in a unique point does not necessarily intersect ∂Y_{m} in a unique point. For example, a component of the limit curve could map entirely inside ∂Y_{m}. A nice compactification of this space is obtained by considering stable log maps. The idea is to allow maps with components possibly mapping entirely inside ∂Y_{m}, but to also remember some additional information under the form of log structures, which give a way to make sense of tangency conditions for points on such components mapping entirely inside ∂Y_{m}. The theory of stable log maps has been developed by Gross and Siebert [33] and Abramovich and Chen [16; 1]. By stable log maps, we always mean basic stable log maps in the sense of [33]. We refer to Kato [38] for elementary notions of log geometry. We consider the divisorial log structure on Y_{m} defined by the divisor ∂Y_{m} and use it to view Y_{m} as a smooth log scheme.

Let $\overline{M}_{g,p}(Y_{\mathrm{m}}/\partial Y_{\mathrm{m}})$ be the moduli space of genus g stable log maps to Y_{m}, of class β_p, with contact order along ∂Y_{m} given by $\ell_p m_p$. According to the main results of [33], this

is a proper Deligne–Mumford stack coming with a g–dimensional virtual fundamental class

$$[\overline{M}_{g,p}(Y_{\mathrm{m}}/\partial Y_{\mathrm{m}})]^{\mathrm{virt}}.$$

If $\pi\colon \mathcal{C} \to \overline{M}_{g,p}(Y_{\mathrm{m}}/\partial Y_{\mathrm{m}})$ is the universal curve, of relative dualizing sheaf ω_{π}, then the Hodge bundle

$$\mathbb{E} := \pi_* \omega_{\pi}$$

is a rank g vector bundle over $\overline{M}_{g,p}(Y_{\mathrm{m}}/\partial Y_{\mathrm{m}})$. Its Chern classes are classically called the lambda classes [58] and denoted by $\lambda_j := c_j(\mathbb{E})$ for $0 \leqslant j \leqslant g$. Finally, we define the genus g log Gromov–Witten invariants $N_{g,p}^{Y_{\mathrm{m}}} \in \mathbb{Q}$ of Y_{m} which will be of interest for us by

$$N_{g,p}^{Y_{\mathrm{m}}} := \int_{[\overline{M}_{g,p}(Y_{\mathrm{m}}/\partial Y_{\mathrm{m}})]^{\mathrm{virt}}} (-1)^g \lambda_g.$$

Note that the top lambda class λ_g has exactly the right degree to cut down the virtual dimension from g to 0, so that $N_{g,p}^{Y_{\mathrm{m}}}$ is not obviously zero.

The fact that the top lambda class should be the natural insertion to consider for some higher-genus version of [31] was already suggested in Section 5.8 of [31]. From our point of view, higher-genus invariants with the top lambda class inserted are the correct objects because it is to them that the correspondence tropical theorem of [10] applies. In Section 9, we will explain how our main result Theorem 3.1 fits into an expected story for higher-genus open holomorphic curves in Calabi–Yau 3–folds. This is probably the most conceptual understanding of the role of the invariants $N_{g,\beta}^{Y_{\mathrm{m}}}$: they are really higher-genus invariants of the log Calabi–Yau 3–fold $Y_{\mathrm{m}} \times \mathbb{P}^1$, and the top lambda class is simply a measure of the difference between surface and 3–fold obstruction theories. This will be made precise in the following Section 2.4, whose analogue for K3 surfaces is well known; see Lemma 7 of [55].

2.4 3–dimensional interpretation of the invariants $N_{g,p}^{Y_{\mathrm{m}}}$

In this section, we rewrite the log Gromov–Witten invariants $N_{g,p}^{Y_{\mathrm{m}}}$ of the log Calabi–Yau surface Y_{m} in terms of 3–dimensional geometries, first $Y_{\mathrm{m}} \times \mathbb{C}$ and then $Y_{\mathrm{m}} \times \mathbb{P}^1$.

We endow the 3–fold $Y_{\mathrm{m}} \times \mathbb{C}$ with the smooth log structure given by the divisorial log structure along the divisor $\partial Y_{\mathrm{m}} \times \mathbb{C}$. Let

$$\overline{M}_{g,p}(Y_{\mathrm{m}} \times \mathbb{C}/\partial Y_{\mathrm{m}} \times \mathbb{C})$$

be the moduli space of genus g stable log maps to Y_{m}, of class β_p, with contact order

along $\partial Y_{\mathrm{m}} \times \mathbb{C}$ given by $\ell_p m_p$. This is a Deligne–Mumford stack coming with a 1–dimensional virtual fundamental class

$$[\overline{M}_{g,p}(Y_{\mathrm{m}}/\partial Y_{\mathrm{m}})]^{\mathrm{virt}}.$$

Because $\mathbb{C}$ is not compact, $\overline{M}_{g,p}(Y_{\mathrm{m}} \times \mathbb{C}/\partial Y_{\mathrm{m}} \times \mathbb{C})$ is not compact and so one cannot simply integrate over the virtual class. Using the standard action of $\mathbb{C}^*$ on $\mathbb{C}$, fixing $0 \in \mathbb{C}$, we obtain an action of $\mathbb{C}^*$ on $\overline{M}_{g,p}(Y_{\mathrm{m}} \times \mathbb{C}/\partial Y_{\mathrm{m}} \times \mathbb{C})$, with its perfect obstruction theory, whose fixed-point locus is the space of stable log maps mapping to $Y_{\mathrm{m}} \times \{0\}$, ie $\overline{M}_{g,p}(Y_{\mathrm{m}}/\partial Y_{\mathrm{m}})$, with its natural perfect obstruction theory. Given the virtual localization formula [27], it is natural to define equivariant log Gromov–Witten invariants

$$N_{g,p}^{Y_{\mathrm{m}} \times \mathbb{C}} := \int_{[\overline{M}_{g,p}(Y_{\mathrm{m}}/\partial Y_{\mathrm{m}})]^{\mathrm{virt}}} \frac{1}{e(\mathrm{Nor}^{\mathrm{virt}})} \in \mathbb{Q}(t),$$

where $\mathrm{Nor}^{\mathrm{virt}}$ is the equivariant virtual normal bundle in $\overline{M}_{g,p}(Y_{\mathrm{m}} \times \mathbb{C}/\partial Y_{\mathrm{m}} \times \mathbb{C})$ of $\overline{M}_{g,p}(Y_{\mathrm{m}}/\partial Y_{\mathrm{m}})$, $e(\mathrm{Nor}^{\mathrm{virt}})$ is its equivariant Euler class, and t is the generator of the $\mathbb{C}^*$–equivariant cohomology of a point.

Lemma 2.1 $$N_{g,p}^{Y_{\mathrm{m}} \times \mathbb{C}} = \frac{1}{t} N_{g,p}^{Y_{\mathrm{m}}}.$$

Proof Because the 3–dimensional geometry $Y_{\mathrm{m}} \times \mathbb{C}$, including the log/tangency conditions, is obtained from the 2–dimensional geometry Y_{m} by a trivial product with a trivial factor $\mathbb{C}$, with $\mathbb{C}^*$ scaling this trivial factor, the virtual normal at a stable log map $f: C \to Y_{\mathrm{m}}$ is $H^0(C, f^*\mathcal{O}) - H^1(C, f^*\mathcal{O}) = t - \mathbb{E}^\vee \otimes t$, so

$$\frac{1}{e(\mathrm{Nor}^{\mathrm{virt}})} = \frac{1}{t}\left(\sum_{i=0}^{g} (-1)^i \lambda_i t^{g-i} \right),$$

and

$$N_{g,p}^{Y_{\mathrm{m}} \times \mathbb{C}} = \int_{[\overline{M}_{g,p}(Y_{\mathrm{m}},\partial Y_{\mathrm{m}})]^{\mathrm{virt}}} \frac{(-1)^g \lambda_g}{t} = \frac{1}{t} N_{g,p}^{Y_{\mathrm{m}}}. \qquad \square$$

The proof of Lemma 2.1 is identical to the proof of Lemma 7 in [55] up to a small point: in [55], counts of expected dimensions work because of the use of a reduced Gromov–Witten theory of K3 surfaces, whereas for us, counts of expected dimensions work because of the use of log Gromov–Witten theory.

We consider now the 3–fold $Z_{\mathrm{m}} := Y_{\mathrm{m}} \times \mathbb{P}^1$ with the smooth log structure given by the divisorial log structure along the divisor

$$\partial Z_{\mathrm{m}} := (\partial Y_{\mathrm{m}} \times \mathbb{P}^1) \cup (Y_{\mathrm{m}} \times \{0\}) \cup (Y_{\mathrm{m}} \times \{\infty\}).$$

The divisor ∂Z_m is anticanonical, containing 0–dimensional strata, and so the pair $(Z_\mathrm{m}, \partial Z_\mathrm{m})$ is an example of log Calabi–Yau 3–fold with maximal boundary.

Let

$$\overline{M}_{g,p}(Z_\mathrm{m}/\partial Z_\mathrm{m})$$

be the moduli space of genus g stable log maps to Z_m, of class β_p, with contact order along ∂Z_m given by $\ell_p m_p$. This is a proper Deligne–Mumford stack coming with a 1–dimensional virtual fundamental class

$$[\overline{M}_{g,p}(Z_\mathrm{m}/\partial Z_\mathrm{m})]^{\mathrm{virt}}.$$

Composing the evaluation map at the contact point with ∂Z_m and the projection $\partial Z_\mathrm{m} \to \mathbb{P}^1$, we obtain a map $\rho\colon \overline{M}_{g,p}(Z_\mathrm{m}/\partial Z_\mathrm{m}) \to \mathbb{P}^1$ and we define log Gromov–Witten invariants

$$N^{Z_\mathrm{m}}_{g,p} := \int_{[\overline{M}_{g,p}(Z_\mathrm{m}/\partial Z_\mathrm{m})]^{\mathrm{virt}}} \rho^*(\mathrm{pt}),$$

where $\mathrm{pt} \in A^1(\mathbb{P}^1)$ is the class of a point.

Lemma 2.2 $$N^{Z_\mathrm{m}}_{g,p} = N^{Y_\mathrm{m}}_{g,p}.$$

Proof We use virtual localization [27] with respect to the action of $\mathbb{C}^*$ on the $\mathbb{P}^1$–factor with weight t at 0 and weight $-t$ at ∞. We lift the class pt of a point to the equivariant class pt_0 of the torus fixed point 0. Because of the insertion of $\mathrm{pt}_0 = t$, only the fixed point $0 \in \mathbb{P}^1$, and not $\infty \in \mathbb{P}^1$, contributes to the localization formula, and so we obtain

$$N^{Z_\mathrm{m}}_{g,p} = t N^{Y_\mathrm{m} \times \mathbb{C}}_{g,p},$$

and hence the result by Lemma 2.1. □

2.5 Orbifold Gromov–Witten theory

We give an orbifold generalization of Sections 2.1, 2.2 and 2.3, which will be necessary to state Theorem 3.2 in Section 7.2.

As in Section 2.1, we fix $\mathrm{m} = (m_1, \ldots, m_n)$, an n–tuple of primitive nonzero vectors of $M = \mathbb{Z}^2$ and this defines a toric surface $\overline{Y}_\mathrm{m}$, with toric divisors D_{m_j} for $1 \leqslant j \leqslant n$. For every $\mathfrak{r} = (r_1, \ldots, r_n)$, an n–tuple of positive integers, we define a projective surface $Y_{\mathrm{m},\mathfrak{r}}$ by blowing up a subscheme of length r_j in general position on the toric divisor D_{m_j} for every $1 \leqslant j \leqslant n$. For $\mathfrak{r} = (1, \ldots, 1)$, we simply have $Y_{\mathrm{m},\mathfrak{r}} = Y_\mathrm{m}$, which was defined in Section 2.1.

Let $\nu \colon Y_{\mathfrak{m},\mathfrak{r}} \to \overline{Y}_{\mathfrak{m}}$ be the blow-up morphism. If $r_j \geqslant 2$, then $Y_{\mathfrak{m},\mathfrak{r}}$ has an A_{r_j-1} singularity on the exceptional divisor $E_j := \nu^{-1}(x_j)$. We will consider $Y_{\mathfrak{m},\mathfrak{r}}$ as a Deligne–Mumford stack by taking the natural structure of smooth Deligne–Mumford stack on an A_{r_j-1} singularity. The exceptional divisor E_j is then a stacky projective line $\mathbb{P}^1[r_j, 1]$, with a single $\mathbb{Z}/r_j$ stacky point $0 \in \mathbb{P}^1[r_j, 1]$. The normal bundle to E_j in $Y_{\mathfrak{m},\mathfrak{r}}$ is the orbifold line bundle $\mathcal{O}_{\mathbb{P}^1[r_j,1]}(-[0]/(\mathbb{Z}/r_j))$ of degree $-1/r_j$, and in particular, we have $E_j^2 = -1/r_j$.

Denote by $P_{\mathfrak{r}}$ the set of $(p_1, \ldots, p_n) \in P = \mathbb{N}^n$ such that r_j divides p_j for every $1 \leqslant j \leqslant n$. Exactly as in Section 2.2, we define for every $p \in P_{\mathfrak{r}}$ a curve class $\beta_p \in H_2(Y_{\mathfrak{m},\mathfrak{r}}, \mathbb{Z})$. The only difference is that now we have

$$\beta_p \cdot E_j = \frac{p_j}{r_j}.$$

We denote by $\partial Y_{\mathfrak{m},\mathfrak{r}}$ the strict transform of the toric boundary divisor $\partial \overline{Y}_{\mathfrak{m}}$ of $\overline{Y}_{\mathfrak{m}}$, and we endow $Y_{\mathfrak{m}}$ with the divisorial log structure define by $\partial Y_{\mathfrak{m}}$. So we view $Y_{\mathfrak{m},\mathfrak{r}}$ as a smooth Deligne–Mumford log stack. Because the nontrivial stacky structure is disjoint from the divisor $\partial Y_{\mathfrak{m},\mathfrak{r}}$ supporting the nontrivial log structure, there is no difficulty in combining orbifold Gromov–Witten theory [4; 17] with log Gromov–Witten theory [33; 16; 1] to get a moduli space $\overline{M}_{g,p}(Y_{\mathfrak{m},\mathfrak{r}}/\partial Y_{\mathfrak{m},\mathfrak{r}})$ of genus g stable log maps to $Y_{\mathfrak{m},\mathfrak{r}}$, of class β_p, with contact order along $\partial Y_{\mathfrak{m},\mathfrak{r}}$ given by $\ell_p m_p$. It is a proper Deligne–Mumford stack coming with a g–dimensional virtual fundamental class

$$[\overline{M}_{g,p}(Y_{\mathfrak{m},\mathfrak{r}}/\partial Y_{\mathfrak{m},\mathfrak{r}})]^{\mathrm{virt}}.$$

We finally define genus g orbifold log Gromov–Witten invariants $N_{g,p}^{Y_{\mathfrak{m},\mathfrak{r}}} \in \mathbb{Q}$ of $Y_{\mathfrak{m},\mathfrak{r}}$ by

$$N_{g,p}^{Y_{\mathfrak{m},\mathfrak{r}}} := \int_{[\overline{M}_{g,p}(Y_{\mathfrak{m},\mathfrak{r}}/\partial Y_{\mathfrak{m},\mathfrak{r}})]^{\mathrm{virt}}} (-1)^g \lambda_g.$$

3 Main results

In Section 3.1, we state the main result of the present paper, Theorem 3.1, a precise form of Theorem 0.1 mentioned in the introduction. In Section 3.2, we give elementary examples illustrating Theorem 3.1. In Section 3.3, we state Theorem 3.2, a generalization of Theorem 3.1 including orbifold geometries. Finally, we give in Section 3.4 some brief comments about the level of generality of Theorems 3.1 and 3.2.

3.1 Statement

Using the notation of Section 1, we define a family of consistent quantum scattering diagrams. Our main result, Theorem 3.1, is that the Hamiltonians attached to the rays of these quantum scattering diagrams are generating series of the higher-genus log Gromov–Witten invariants defined in Section 2.

We fix $\mathfrak{m} = (m_1, \ldots, m_n)$, an n–tuple of primitive nonzero vectors of M. We define $P := \mathbb{N}^n$ and we consider the complete local $\mathbb{C}$–algebra $R := \mathbb{C}[\![P]\!] = \mathbb{C}[\![t_1, \ldots, t_n]\!]$. Let $\widehat{\mathfrak{D}}_\mathfrak{m}$ be the quantum scattering diagram over R consisting of incoming rays $(\mathfrak{d}_j, \hat{H}_{\mathfrak{d}_j})$ for $1 \leqslant j \leqslant n$, where

$$\mathfrak{d}_j = -\mathbb{R}_{\geqslant 0}\, m_j,$$

and

$$\hat{H}_{\mathfrak{d}_j} = -i \sum_{\ell \geqslant 1} \frac{1}{\ell} \frac{(-1)^{\ell-1}}{2\sin\left(\frac{1}{2}\ell\hbar\right)} t_j^\ell \hat{z}^{\ell m_j} = \sum_{\ell \geqslant 1} \frac{1}{\ell} \frac{(-1)^{\ell-1}}{q^{\frac{1}{2}\ell} - q^{-\frac{1}{2}\ell}} t_j^\ell \hat{z}^{\ell m_j},$$

where $q = e^{i\hbar}$.

Let $S(\widehat{\mathfrak{D}}_\mathfrak{m})$ be the corresponding consistent quantum scattering diagram given by Proposition 1.3, obtained by adding outgoing rays to $\widehat{\mathfrak{D}}_\mathfrak{m}$. We can assume that, for every $m \in M - \{0\}$ primitive, $S(\widehat{\mathfrak{D}}_\mathfrak{m})$ contains a unique outgoing ray of support $\mathbb{R}_{\geqslant 0}\, m$.

For every $m \in M - \{0\}$ primitive, let P_m be the subset of $p = (p_1, \ldots, p_n) \in P = \mathbb{N}^n$ such that $\sum_{j=1}^n p_j m_j$ is positively collinear with m:

$$\sum_{j=1}^n p_j m_j = \ell_p m$$

for some $\ell_p \in \mathbb{N}$.

Recall that in Section 2, for every $\mathfrak{m} = (m_1, \ldots, m_n)$, we introduced a log Calabi–Yau surface $Y_\mathfrak{m}$ and for every $p = (p_1, \ldots, p_n) \in P = \mathbb{N}^n$, we defined a certain genus g log Gromov–Witten $N_{g,p}^{Y_\mathfrak{m}}$ of $Y_\mathfrak{m}$.

Theorem 3.1 *For every* $\mathfrak{m} = (m_1, \ldots, m_n)$, *an* n–*tuple of primitive nonzero vectors in* M, *and every* $m \in M - \{0\}$ *primitive, the Hamiltonian* $\hat{H}_m$ *attached to the outgoing ray* $\mathbb{R}_{\geqslant 0}\, m$ *in the consistent quantum scattering diagram* $S(\widehat{\mathfrak{D}}_\mathfrak{m})$ *is given by*

$$\hat{H}_m = \left(-\frac{i}{\hbar}\right) \sum_{p \in P_m} \left(\sum_{g \geqslant 0} N_{g,p}^{Y_\mathfrak{m}} \hbar^{2g}\right) \left(\prod_{j=1}^n t_j^{p_j}\right) \hat{z}^{\ell_p m}.$$

In the classical limit $\hbar \to 0$, Theorem 3.1 reduces to the main result (Theorem 5.4) of [31], expressing the classical scattering diagram $S(\mathfrak{D}_{\mathrm{m}})$ in terms of the genus 0 log Gromov–Witten invariants $N_{0,p}^{Y_{\mathrm{m}}}$. In [31], the genus 0 invariants are defined as relative Gromov–Witten invariants of some open geometry. The fact that they coincide with genus 0 log Gromov–Witten invariants follows from arguments of the type of those used in Section 5 of [10].

The proof of Theorem 3.1 takes Sections 4, 5, 6 and 7. In Section 2.3, we define higher-genus log Gromov–Witten invariants $N_{g,w}^{\overline{Y}_{\mathrm{m}}}$ of toric surfaces $\overline{Y}_{\mathrm{m}}$. In Section 5, we prove a degeneration formula expressing the log Gromov–Witten invariants $N_{g,p}^{Y_{\mathrm{m}}}$ of the log Calabi–Yau surface Y_{m} in terms of log Gromov–Witten invariants $N_{g,w}^{\overline{Y}_{\mathrm{m}}}$ of the toric surface $\overline{Y}_{\mathrm{m}}$. In Section 6, we review, following [21], the relation between quantum scattering diagrams and Block–Göttsche q–deformed tropical curve count. In Section 7, we conclude the proof by using the main result of [10] relating q–deformed tropical curve count and higher-genus log Gromov–Witten invariants of toric surfaces.

The consistency of the quantum scattering diagram $S(\widehat{\mathfrak{D}}_{\mathrm{m}})$ translates into the fact that the product, ordered according to the phase of the rays, of the elements $\widehat{\Phi}_{\widehat{H}_j}$ and $\widehat{\Phi}_{\widehat{H}_m}$ of the quantum tropical vertex group $\widehat{\mathbb{V}}_R^{\hbar}$ is equal to the identity. Therefore, one can paraphrase Theorem 3.1 by saying that the log Gromov–Witten invariants $N_{g,p}^{Y_{\mathrm{m}}}$ produce relations in the quantum tropical vertex group $\widehat{\mathbb{V}}_R^{\hbar}$ or conversely that relations in $\widehat{\mathbb{V}}_R^{\hbar}$ give constraints on the log Gromov–Witten invariants $N_{g,p}^{Y_{\mathrm{m}}}$.

The automorphism $\widehat{\Phi}_{\widehat{H}_j}$ attached to the incoming rays $\mathfrak{d}_j$ of the quantum scattering diagram $S(\widehat{\mathfrak{D}}_{\mathrm{m}})$ are conjugation by $e^{\widehat{H}_{\mathfrak{d}_j}}$, ie by

$$\exp\left(\sum_{\ell \geq 1} \frac{1}{\ell} \frac{(-1)^{\ell-1}}{q^{\frac{1}{2}\ell} - q^{-\frac{1}{2}\ell}} t_j^\ell \hat{z}^{\ell m_j} \right),$$

which can be written as $\Psi_q(-t_j \hat{z}^{m_j})$, where

$$\Psi_q(x) := \exp\left(-\sum_{\ell \geq 1} \frac{1}{\ell} \frac{x^\ell}{q^{\frac{1}{2}\ell} - q^{-\frac{1}{2}\ell}} \right) = \prod_{k \geq 0} \frac{1}{1 - q^{k+\frac{1}{2}} x}$$

is the quantum dilogarithm. We warn that various conventions are used for the quantum dilogarithm throughout the literature. We refer for example to [71] for a nice review of the many aspects of the dilogarithm, including its quantum version.

As the incoming rays of $S(\widehat{\mathfrak{D}}_{\mathrm{m}})$ are expressed in terms of quantum dilogarithms, it is natural to ask if the outgoing rays, which by Theorem 3.1 are generating series of higher-genus log Gromov–Witten invariants, can be naturally expressed in terms of quantum

dilogarithms. This question is related to the multicover/BPS structure of higher-genus log Gromov–Witten theory and is fully answered by Theorem 0.2 in Section 8.

3.2 Examples

In this section, we give some elementary examples illustrating Theorem 3.1.

3.2.1 Trivial scattering: propagation of a ray

We take $n = 1$ and $\mathfrak{m} = (m_1)$ with $m_1 = (1, 0) \in M = \mathbb{Z}^2$. In this case, $R = \mathbb{C}[\![t_1]\!]$, and the quantum scattering diagram $\widehat{\mathfrak{D}}_{\mathfrak{m}}$ contains a unique incoming ray: $\mathfrak{d}_1 = -\mathbb{R}_{\geqslant 0} (1, 0) = \mathbb{R}_{\geqslant 0} (-1, 0)$ equipped with

$$\widehat{H}_{\mathfrak{d}_1} = -i \sum_{\ell \geqslant 1} \frac{1}{\ell} \frac{(-1)^{\ell-1}}{2 \sin\left(\frac{1}{2}\ell\hbar\right)} t_1^\ell \hat{z}^{(\ell,0)}.$$

Then the consistent scattering diagram $S(\widehat{\mathfrak{D}}_{\mathfrak{m}})$ is obtained by simply propagating the incoming ray, ie by adding the outgoing ray $\mathbb{R}_{\geqslant 0} (1, 0)$ equipped with

$$\widehat{H}_{(1,0)} = -i \sum_{\ell \geqslant 1} \frac{1}{\ell} \frac{(-1)^{\ell-1}}{2 \sin\left(\frac{1}{2}\ell\hbar\right)} t_1^\ell \hat{z}^{(\ell,0)}.$$

We start with a fan consisting of the ray $\mathbb{R}_{\geqslant 0} (-1, 0)$. To get a proper toric surface, we add to the fan the rays $\mathbb{R}_{\geqslant 0} (1, 0)$, $\mathbb{R}_{\geqslant 0} (0, 1)$ and $\mathbb{R}_{\geqslant 0} (0, -1)$. The corresponding toric surface $\overline{Y}_{\mathfrak{m}}$ is simply $\mathbb{P}^1 \times \mathbb{P}^1$. We obtain $Y_{\mathfrak{m}}$ by blowing up a point on $\{0\} \times \mathbb{C}^*$, eg $\{0\} \times \{1\}$. Denote by E the exceptional divisor and F the strict transform of $\mathbb{P}^1 \times \{1\}$. We have $E^2 = F^2 = -1$ and $E \cdot F = 1$. For $\ell \in P = \mathbb{N}$, we have $\beta_\ell = \ell[F]$. So, according to Theorem 3.1, one should have, for every $\ell \geqslant 1$,

$$\sum_{g \geqslant 0} N^{Y_{\mathfrak{m}}}_{g,\ell} \hbar^{2g-1} = \frac{1}{\ell} \frac{(-1)^{\ell-1}}{2 \sin\left(\frac{1}{2}\ell\hbar\right)}.$$

As F is rigid, contributions to $N_{g,\ell}$ only come from ℓ to 1 multicoverings of F and the computation of $N_{g,\ell}$ can be reduced to a computation in relative Gromov–Witten theory of $\mathbb{P}^1$. Using Theorem 5.1 of [13], one can check that the above formula is indeed correct. We refer for more details to Lemma 5.9, which plays a crucial role in the proof of Theorem 3.1.

3.2.2 Simple scattering of two rays

We take $n = 2$ and $\mathfrak{m} = (m_1, m_2)$ with $m_1 = (1, 0) \in M = \mathbb{Z}^2$ and $m_2 = (0, 1) \in M = \mathbb{Z}^2$. In this case, $R = \mathbb{C}[\![t_1, t_2]\!]$, and the

quantum scattering diagram $\widehat{\mathfrak{D}}_{\mathrm{m}}$ contains two incoming rays $\mathfrak{d}_1 = \mathbb{R}_{\geqslant 0}(-1, 0)$ and $\mathfrak{d}_2 = \mathbb{R}_{\geqslant 0}(0, -1)$, respectively equipped with

$$\widehat{H}_{\mathfrak{d}_1} = -i \sum_{\ell \geqslant 1} \frac{1}{\ell} \frac{(-1)^{\ell-1}}{2\sin\left(\frac{1}{2}\ell\hbar\right)} t_1^{\ell} \hat{z}^{(\ell, 0)} \quad \text{and} \quad \widehat{H}_{\mathfrak{d}_2} = -i \sum_{\ell \geqslant 1} \frac{1}{\ell} \frac{(-1)^{\ell-1}}{2\sin\left(\frac{1}{2}\ell\hbar\right)} t_2^{\ell} \hat{z}^{(0, \ell)}.$$

Then, because of the Faddeev–Kashaev [20] pentagon identity

$$\Psi_q(z^{(1,0)}) \Psi_q(z^{(0,1)}) = \Psi_q(z^{(0,1)}) \Psi_q(z^{(1,1)}) \Psi_q(z^{(1,0)})$$

satisfied by the quantum dilogarithm Ψ_q, the consistent scattering diagram $S(\widehat{\mathfrak{D}}_{\mathrm{m}})$ is obtained by propagation of the two incoming rays in outgoing rays, as in Section 3.2.1, and by addition of a third outgoing ray $\mathbb{R}_{\geqslant 0}(1, 1)$ equipped with

$$\widehat{H}_{(1,1)} = -i \sum_{\ell \geqslant 1} \frac{1}{\ell} \frac{(-1)^{\ell-1}}{2\sin\left(\frac{1}{2}\ell\hbar\right)} t_1^{\ell} t_2^{\ell} \hat{z}^{(\ell, \ell)}.$$

We start with the fan consisting of the rays $\mathbb{R}_{\geqslant 0}(-1, 0)$ and $\mathbb{R}_{\geqslant 0}(0, -1)$. To get a proper toric surface, we can for example add to the fan the ray $\mathbb{R}_{\geqslant 0}(1, 1)$. The corresponding toric surface $\overline{Y}_{\mathrm{m}}$ is simply $\mathbb{P}^2$, with its toric divisors D_1, D_2 and D_3. We obtain Y_{m} by blowing a point p_1 on D_1 and a point p_2 on D_2, both away from the torus fixed points. We denote by E_1 and E_2 the corresponding exceptional divisors and F the strict transform of the unique line in $\mathbb{P}^2$ passing through p_1 and p_2. We have $E_1^2 = E_2^2 = F^2 = -1$ and $E_1 \cdot F = E_2 \cdot F = 1$. For $\ell \in \mathbb{N}$ and $(\ell, \ell) \in P = \mathbb{N}^2$, we have $\beta_{(\ell, \ell)} = \ell[F]$. So according to Theorem 3.1, one should have, for every $\ell \geqslant 1$,

$$\sum_{g \geqslant 0} N^{Y_{\mathrm{m}}}_{g, (\ell, \ell)} \hbar^{2g-1} = \frac{1}{\ell} \frac{(-1)^{\ell-1}}{2\sin\left(\frac{1}{2}\ell\hbar\right)}.$$

As F is rigid, contributions to $N_{g, (\ell, \ell)}$ only come from ℓ-to-1 multicoverings of F and the computation of $N_{g, (\ell, \ell)}$ reduces to a computation identical to the one used for $N_{g, \ell}$ in Section 3.2.1.

3.2.3 More complicated scatterings Already at the classical level of [31], general scattering diagrams can be very complicated. A fortiori, general quantum scattering diagrams are extremely complicated. Direct computation of the higher-genus log Gromov–Witten invariants $N^{Y_{\mathrm{m}}}_{g, p}$ is a difficult problem in general. In particular, unlike what happens in Sections 3.2.1 and 3.2.2, linear systems defined by β_p and the tangency condition contain in general curves of positive genus, and so genus $g > 0$ stable log maps appearing in the moduli space defining $N^{Y_{\mathrm{m}}}_{g, p}$ do not factor through genus 0 curves in general. As consistent scattering diagrams can be algorithmically computed,

one can view Theorem 3.1 as an answer to the problem of effectively computing the higher-genus log Gromov–Witten invariants $N_{g,p}^{Y_m}$.

3.3 Orbifold generalization

As in Sections 5.5 and 5.6 of [31] for the classical case, we can give an enumerative interpretation of quantum scattering diagrams more general than those considered in Theorem 3.1 if we allow ourselves to work with orbifold Gromov–Witten invariants.

We fix $\mathfrak{m} = (m_1, \ldots, m_n)$, an n–tuple of primitive nonzero vectors of $M = \mathbb{Z}^2$, and $\mathfrak{r} = (r_1, \ldots, r_n)$, an n–tuple of positive integers. We define $P := \mathbb{N}^n$ and we consider the complete local $\mathbb{C}$–algebra $R := \mathbb{C}[\![P]\!] := \mathbb{C}[\![t_1, \ldots, t_n]\!]$. Let $P_\mathfrak{r}$ be the set of $p = (p_1, \ldots, p_n) \in P$ such that r_j divides p_j for every $1 \leqslant j \leqslant n$. Let $\widehat{\mathfrak{D}}_{\mathfrak{m},\mathfrak{r}}$ be the quantum scattering diagram over R consisting of incoming rays $(\mathfrak{d}_j, \hat{H}_{\mathfrak{d}_j})$ for $1 \leqslant j \leqslant n$, where

$$\mathfrak{d}_j = -\mathbb{R}_{\geqslant 0}\, m_j$$

and

$$\hat{H}_{\mathfrak{d}_j} = -i \sum_{\ell \geqslant 1} \frac{1}{\ell} \frac{(-1)^{\ell-1}}{2\sin(\frac{1}{2} r_j \ell \hbar)} t_j^{r_j \ell} \hat{z}^{r_j \ell m_j} = \sum_{\ell \geqslant 1} \frac{1}{\ell} \frac{(-1)^{\ell-1}}{q^{\frac{1}{2} r_j \ell} - q^{-\frac{1}{2} r_j \ell}} t_j^{r_j \ell} \hat{z}^{r_j \ell m_j},$$

where $q = e^{i\hbar}$. Let $S(\widehat{\mathfrak{D}}_{\mathfrak{m},\mathfrak{r}})$ be the corresponding consistent quantum scattering diagram given by Proposition 1.3, obtained by adding outgoing rays to $\widehat{\mathfrak{D}}_{\mathfrak{m},\mathfrak{r}}$. For every $m \in M - \{0\}$, let $P_{\mathfrak{r},m}$ be the subset of $p = (p_1, \ldots, p_n) \in P_\mathfrak{r}$ such that $\sum_{j=1}^n p_j m_j$ is positively collinear with m:

$$\sum_{j=1}^n p_j m_j = \ell_p m$$

for some $\ell_p \in \mathbb{N}$.

Recall that in Section 2.5, for every $\mathfrak{m} = (m_1, \ldots, m_n)$ and $\mathfrak{r} = (r_1, \ldots, r_n)$, we introduced an orbifold log Calabi–Yau surface $Y_{\mathfrak{m},\mathfrak{r}}$ and for every $p = (p_1, \ldots, p_n) \in P_\mathfrak{r}$, we defined a certain genus g orbifold log Gromov–Witten $N_{g,p}^{Y_{\mathfrak{m},\mathfrak{r}}}$ of $Y_{\mathfrak{m},\mathfrak{r}}$.

Theorem 3.2 *For every $\mathfrak{m} = (m_1, \ldots, m_n)$, an n–tuple of primitive nonzero vectors in M, every $\mathfrak{r} = (r_1, \ldots, r_n)$, an n–tuple of positive integers, and every $m \in M - \{0\}$ primitive, the Hamiltonian $\hat{H}_m$ attached to the outgoing ray $\mathbb{R}_{\geqslant 0}\, m$ in the consistent quantum scattering diagram $S(\widehat{\mathfrak{D}}_{\mathfrak{m},\mathfrak{r}})$ is given by*

$$\hat{H}_m = \left(-\frac{i}{\hbar}\right) \sum_{p \in P_{\mathfrak{r},m}} \left(\sum_{g \geqslant 0} N_{g,p}^{Y_{\mathfrak{m},\mathfrak{r}}}\, \hbar^{2g} \right) \left(\prod_{j=1}^n t_j^{p_j} \right) \hat{z}^{\ell_p m}.$$

For $\mathfrak{r} = (1, \ldots, 1)$, Theorem 3.2 reduces to Theorem 3.1. In the classical limit $\hbar \to 0$, Theorem 3.1 reduces to Theorem 5.6 of [31]. The proof of Theorem 3.2 is entirely parallel to the proof of its special case Theorem 3.1. The key point is that orbifold and logarithmic questions never interact in a nontrivial way. The only major needed modification is an orbifold version of the multicovering formula of Lemma 5.9. This is done in Lemma 7.2 in Section 7.2.

3.4 More general quantum scattering diagrams

We still fix $\mathfrak{m} = (m_1, \ldots, m_n)$, an n–tuple of primitive vectors of $M = \mathbb{Z}^2$, and we continue to write $P = \mathbb{N}^n$, so that $R = \mathbb{C}[\![P]\!] = \mathbb{C}[\![t_1, \ldots, t_n]\!]$. One could try to further generalize Theorem 3.2 by starting with a quantum scattering diagram over R consisting of incoming rays $(\mathfrak{d}_j, \hat{H}_{\mathfrak{d}_j})$ for $1 \leqslant j \leqslant n$, where $\mathfrak{d}_j = -\mathbb{R}_{\geqslant 0} \, m_j$ and

$$\hat{H}_{\mathfrak{d}_j} = \sum_{\ell \geqslant 1} \hat{H}_{\mathfrak{d}_j, \ell} t_j^\ell \hat{z}^{\ell m_j}$$

for arbitrary

$$\hat{H}_{\mathfrak{d}_j, \ell} \in \mathbb{C}[\![\hbar]\!].$$

In the classical limit $\hbar \to 0$, Theorem 5.6 of [31], classical limit of our Theorem 3.2, is enough to give an enumerative interpretation of the resulting consistent scattering diagram in such generality. Indeed, the genus 0 orbifold Gromov–Witten story takes as input classical Hamiltonians

$$H_r = \sum_{\ell \geqslant 1} \frac{(-1)^{\ell-1}}{r\ell^2} t^{r\ell} z^{r\ell} = \frac{1}{r}(tz)^r + \mathcal{O}((tz)^{r+1})$$

for all $r \geqslant 0$, which form a basis of $\mathbb{C}[(tz)]$. In particular, at every finite order in $\mathfrak{m}_R$, every classical scattering diagram consisting of n incoming rays meeting at $0 \in \mathbb{R}^2$ coincides with a classical scattering diagram whose consistent completion has an enumerative interpretation in terms of genus 0 orbifold Gromov–Witten invariants. In the quantum case, things are more complicated due to the extra dependence in $\hbar$. More precisely, Theorem 3.2 only covers a particular class of Hamiltonians $\hat{H}_{\mathfrak{d}_j}$ whose form is dictated by the multicovering structure of higher-genus orbifold Gromov–Witten theory.

4 Gromov–Witten theory of toric surfaces

For every $\mathfrak{m} = (m_1, \ldots, m_n)$, an n–tuple of primitive nonzero vectors in $M = \mathbb{Z}^2$, we defined in Section 2.1 a log Calabi–Yau surface $Y_\mathfrak{m}$ obtained as the blow-up of some

toric surface $\overline{Y}_{\mathrm{m}}$, and we introduced in Section 2.3 a collection of log Gromov–Witten invariants $N_{g,p}^{Y_{\mathrm{m}}}$ of Y_{m}. In the present section, we define log Gromov–Witten invariants $N_{g,w}^{\overline{Y}_{\mathrm{m}}}$ of the toric surface $\overline{Y}_{\mathrm{m}}$. In Section 5, we will compare the invariants $N_{g,p}^{Y_{\mathrm{m}}}$ of Y_{m} and $N_{g,w}^{\overline{Y}_{\mathrm{m}}}$ of $\overline{Y}_{\mathrm{m}}$.

4.1 Curve classes on toric surfaces

Recall from Section 2.1 that $\overline{Y}_{\mathrm{m}}$ is a proper toric surface whose fan contains the rays $-\mathbb{R}_{\geqslant 0}\, m_j$ for every $1 \leqslant j \leqslant n$. We denote by $\partial \overline{Y}_{\mathrm{m}}$ the toric boundary divisor of $\overline{Y}_{\mathrm{m}}$. We will consider curves in $\overline{Y}_{\mathrm{m}}$ meeting $\partial \overline{Y}_{\mathrm{m}}$ in a number of prescribed points with prescribed tangency conditions and at one unprescribed point with prescribed tangency condition. In this section, we explain how to parametrize the relevant curve classes in terms of the prescribed tangency conditions w_j at the prescribed points.

Let s be a positive integer and let $w = (w_1, \ldots, w_s)$ be an s–tuple of nonzero vectors in M such that for every $1 \leqslant r \leqslant s$, there exists $1 \leqslant j \leqslant n$ such that $-\mathbb{R}_{\geqslant 0}\, w_r = -\mathbb{R}_{\geqslant 0}\, m_j$. In particular, the ray $-\mathbb{R}_{\geqslant 0}\, w_r$ belongs to the fan of $\overline{Y}_{\mathrm{m}}$ and we denote by D_{w_r} the corresponding toric divisor of $\overline{Y}_{\mathrm{m}}$. Note that we can have $D_{w_r} = D_{w_{r'}}$ even if $r \neq r'$. We denote by $|w_r| \in \mathbb{N}$ the divisibility of $w_r \in M = \mathbb{Z}^2$, ie the largest positive integer k such that one can write $w_r = kv$ with $v \in M$. One should think about w_r as defining a toric divisor D_{w_r} and an intersection number $|w_r|$ with D_{w_r} for a curve in $\overline{Y}_{\mathrm{m}}$.

We assume that $\sum_{r=1}^{s} w_r \neq 0$ and so we can uniquely write

$$\sum_{r=1}^{s} w_r = \ell_w m_w,$$

with $m_w \in M$ primitive and $\ell_w \in \mathbb{N}$.

We explain now how to define a curve class $\beta_w \in H_2(\overline{Y}_{\mathrm{m}}, \mathbb{Z})$. In short, β_w is the class of a curve in $\overline{Y}_{\mathrm{m}}$ having, for every $1 \leqslant r \leqslant s$, an intersection point of intersection number $|w_r|$ with D_{w_r} and exactly one other intersection point with the toric boundary $\partial \overline{Y}_{\mathrm{m}}$.

More precisely, the vector $m_w \in M$ belongs to some cone of the fan of $\overline{Y}_{\mathrm{m}}$ and we write the corresponding decomposition

$$m_w = a_w^L m_w^L + a_w^R m_w^R,$$

where $m_w^L, m_w^R \in M$ are primitive generators of rays of the fan of $\overline{Y}_{\mathrm{m}}$ and where $a_w^L, a_w^R \in \mathbb{N}$. Note that there is only one term in this decomposition if the ray $\mathbb{R}_{\geqslant 0}\, m_w$

coincides with one of the rays of the fan of $\overline{Y}_{\mathrm{m}}$. Let D_w^L and D_w^R be the toric divisors of $\overline{Y}_{\mathrm{m}}$ corresponding to the rays $\mathbb{R}_{\geqslant 0} \, m_w^L$ and $\mathbb{R}_{\geqslant 0} \, m_w^R$. Let $\beta_w \in H_2(\overline{Y}_{\mathrm{m}}, \mathbb{Z})$ be the curve class uniquely determined by the following intersection numbers with the toric divisors:

- The intersection numbers with those D_{w_r} for $1 \leqslant r \leqslant s$ that are distinct from D_w^L and D_w^R:
$$\beta_w \cdot D_{w_r} = \sum_{r', D_{w_{r'}} = D_{w_r}} |w_{r'}|.$$

- The intersection number with D_w^L:
$$\beta_w \cdot D_w^L = \ell_w a_w^L + \sum_{r, D_{w_r} = D_w^L} |w_r|.$$

- The intersection number with D_w^R:
$$\beta_w \cdot D_w^R = \ell_w a_w^R + \sum_{r, D_{w_r} = D_w^R} |w_r|.$$

- The intersection number with every toric divisor D different from D_{w_r} for every $1 \leqslant r \leqslant s$ and from D_w^L and D_w^R:
$$\beta_w \cdot D = 0.$$

Such a class $\beta_w \in H_2(\overline{Y}_{\mathrm{m}}, \mathbb{Z})$ exists by standard toric geometry because of the relation $\sum_{r=1}^s w_r = \ell_w m_w$ and is unique.

4.2 Log Gromov–Witten invariant of toric surfaces

In the previous section, given $w = (w_1, \dots, w_s)$, an s–tuple of nonzero vectors in M, we defined certain positive integers ℓ_w, a_w^L and a_w^R, certain toric divisors D_w^L and D_w^R of $\overline{Y}_{\mathrm{m}}$, and a curve class $\beta_w \in H_2(\overline{Y}_{\mathrm{m}}, \mathbb{Z})$.

We would like to consider genus g stable maps $f \colon C \to \overline{Y}_{\mathrm{m}}$ of class β_w intersecting ∂Y_{m} in $s+1$ points, s of them being on the divisors D_{w_r} with contact order $|w_r|$ for $1 \leqslant r \leqslant s$ and the last one having contact number $\ell_w a_w^L$ with the divisor D_w^L and contact order $\ell_w a_w^R$ with the divisor D_w^R. We also would like to fix the position of the s intersection numbers with the divisors D_{w_r}. It is easy to check that the expected dimension of this enumerative problem is g. As in Section 2.3, we will cut down the virtual dimension from g to zero by integration of the top lambda class.

As in Section 2.3, to get proper moduli spaces, we work with stable log maps. We consider the divisorial log structure on $\overline{Y}_{\mathrm{m}}$ defined by the toric divisor $\partial \overline{Y}_{\mathrm{m}}$ and use it to view $\overline{Y}_{\mathrm{m}}$ as a smooth log scheme. Let $\overline{M}_{g,w}(\overline{Y}_{\mathrm{m}}, \partial \overline{Y}_{\mathrm{m}})$ be the moduli space of genus g stable log maps to $\overline{Y}_{\mathrm{m}}$, of class β_w, with $s+1$ tangency conditions along $\partial \overline{Y}_{\mathrm{m}}$ defined by the $s+1$ vectors $-w_1, \ldots, -w_s, \ell_w m_w$ in M. This is a proper Deligne–Mumford stack coming with a $(g+s)$–dimensional virtual fundamental class

$$[\overline{M}_{g,w}(\overline{Y}_{\mathrm{m}}, \partial \overline{Y}_{\mathrm{m}})]^{\mathrm{virt}}.$$

For every $1 \leqslant r \leqslant s$, we have an evaluation map

$$\mathrm{ev}_r \colon \overline{M}_{g,w}(\overline{Y}_{\mathrm{m}}, \partial \overline{Y}_{\mathrm{m}}) \to D_{w_r}.$$

If $\pi \colon \mathcal{C} \to \overline{M}_{g,w}(\overline{Y}_{\mathrm{m}}, \partial \overline{Y}_{\mathrm{m}})$ is the universal curve, of relative dualizing sheaf ω_π, then the Hodge bundle $\mathbb{E} := \pi_* \omega_\pi$ is a rank g vector bundle over $\overline{M}_{g,w}(\overline{Y}_{\mathrm{m}}, \partial \overline{Y}_{\mathrm{m}})$, of top Chern class $\lambda_g := c_g(\mathbb{E})$.

We define log Gromov–Witten invariants $N_{g,w}^{\overline{Y}_{\mathrm{m}}} \in \mathbb{Q}$ by

$$N_{g,w}^{\overline{Y}_{\mathrm{m}}} := \int_{[\overline{M}_{g,w}(\overline{Y}_{\mathrm{m}}, \partial \overline{Y}_{\mathrm{m}})]^{\mathrm{virt}}} (-1)^g \lambda_g \prod_{r=1}^{s} \mathrm{ev}_r^*(\mathrm{pt}_r),$$

where $\mathrm{pt}_r \in A^1(D_{w_r})$ is the class of a point on D_{w_r}. This is a rigorous definition of the enumerative problem sketched at the beginning of this section.

5 Degeneration from log Calabi–Yau to toric

5.1 Degeneration formula: statement

We fix $\mathfrak{m} = (m_1, \ldots, m_n)$, an n–tuple of primitive nonzero vectors in $M = \mathbb{Z}^2$. In Section 2.1, we defined a log Calabi–Yau surface Y_{m} obtained as blow-up of some toric surface $\overline{Y}_{\mathrm{m}}$. In Section 2.3, we introduced a collection of log Gromov–Witten invariants $N_{g,p}^{Y_{\mathrm{m}}}$ of Y_{m} indexed by n–tuples $p = (p_1, \ldots, p_n) \in P = \mathbb{N}^n$. In Section 4.2, we defined log Gromov–Witten invariants $N_{g,w}^{\overline{Y}_{\mathrm{m}}}$ of the toric surface $\overline{Y}_{\mathrm{m}}$ indexed by s–tuples $w = (w_1, \ldots, w_s) \in M^s$. The main result of this section, Proposition 5.1, is an explicit formula expressing the invariants $N_{g,p}^{Y_{\mathrm{m}}}$ in terms of the invariants $N_{g,w}^{\overline{Y}_{\mathrm{m}}}$.

We first need to introduce some notation to relate the indices $p = (p_1, \ldots, p_n)$ in the invariants $N_{g,p}^{Y_{\mathrm{m}}}$ and the indices $w = (w_1, \ldots, w_s)$ in the invariants $N_{g,w}^{\overline{Y}_{\mathrm{m}}}$. The way it goes is imposed by the degeneration formula in Gromov–Witten theory and hopefully will become conceptually clear in Section 5.6.

We fix $p = (p_1, \ldots, p_n) \in P = \mathbb{N}^n$. We call k a partition of p, and we write $k \vdash p$, if k is an n–tuple $(k_1, \ldots, k_n)$ with k_j a partition of p_j for every $1 \leqslant j \leqslant n$. We encode a partition k_j of p_j as a sequence $k_j = (k_{\ell,j})_{\ell \geqslant 1}$ of nonnegative integers, all zero except finitely many of them, such that

$$\sum_{\ell \geqslant 1} \ell k_{\ell,j} = p_j.$$

Given a partition k of p, we define

$$s(k) := \sum_{j=1}^{n} \sum_{\ell \geqslant 1} k_{\ell,j}.$$

We now define, given a partition k of p, an $s(k)$–tuple

$$w(k) = (w_1(k), \ldots, w_{s(k)}(k))$$

of nonzero vectors in $M = \mathbb{Z}^2$, by the following formula:

$$w_r(k) := \ell m_j \quad \text{if } 1 + \sum_{j'=1}^{j} \sum_{\ell'=1}^{\ell-1} k_{\ell',j'} \leqslant r \leqslant k_{\ell,j} + \sum_{j'=1}^{j} \sum_{\ell'=1}^{\ell-1} k_{\ell',j'}.$$

In particular, for every $1 \leqslant j \leqslant n$ and $\ell \geqslant 1$, the $s(k)$–tuple $w(k)$ contains $k_{\ell,j}$ copies of the vector $\ell m_j \in M$. Note that because m_j is primitive in M, we have $\ell = |w_r(k)|$, where $|w_r(k)|$ is the divisibility of w_r in M. Note also that

$$\sum_{r=1}^{s(k)} w_r(k) = \sum_{j=1}^{n} \sum_{\ell \geqslant 1} k_{\ell,j} \ell m_j = \sum_{j=1}^{n} p_j m_j = \ell_p m_p,$$

and so, comparing the notation of Sections 2.2 and 4.1, $\ell_{w(k)} = \ell_p$ and $m_{w(k)} = m_p$.

Using the above notation, we can now state Proposition 5.1.

Proposition 5.1 For every $\mathfrak{m} = (m_1, \ldots, m_n)$, an n–tuple of primitive nonzero vectors in $M = \mathbb{Z}^2$, and every $p = (p_1, \ldots, p_n) \in P = \mathbb{N}^n$, the log Gromov–Witten invariants $N_{g,p}^{Y_\mathfrak{m}}$ of the log Calabi–Yau surface $Y_\mathfrak{m}$ are expressed in terms of the log Gromov–Witten invariants $N_{g,w}^{\overline{Y}_\mathfrak{m}}$ of the toric surface $\overline{Y}_\mathfrak{m}$ by the following formula:

$$\sum_{g \geqslant 0} N_{g,p}^{Y_\mathfrak{m}} \hbar^{2g-1}$$
$$= \sum_{k \vdash p} \left(\sum_{g \geqslant 0} N_{g,w(k)}^{\overline{Y}_\mathfrak{m}} \hbar^{2g-1+s(k)} \right) \prod_{j=1}^{n} \prod_{\ell \geqslant 1} \frac{1}{k_{\ell,j}!} \ell^{k_{\ell,j}} \left(\frac{(-1)^{\ell-1}}{\ell} \frac{1}{2 \sin\left(\frac{1}{2}\ell\hbar\right)} \right)^{k_{\ell,j}},$$

where the first sum is over all partitions k of p.

The proof of Proposition 5.1 takes the remainder of Section 5. We consider the degeneration from Y_{m} to $\overline{Y}_{\mathrm{m}}$ introduced in Section 5.3 of [31] and we apply a higher-genus version of the argument of [31]. Because the general degeneration formula in log Gromov–Witten theory is not yet available, we give a proof of the needed degeneration formula following the general strategy used in [10], which uses specific vanishing properties of the top lambda class. We assume for simplicity that m_p is distinct from all $-m_j$. It is easy to adapt the argument in this special case.

5.2 Degeneration set-up

We first review the construction of the degeneration considered in Section 5.3 of [31].

We fix $\mathrm{m} = (m_1, \ldots, m_n)$, an n–tuple of primitive nonzero vectors in $M = \mathbb{Z}^2$. Recall from Section 2.1 that $\overline{Y}_{\mathrm{m}}$ is a proper toric surface whose fan contains the rays $-\mathbb{R}_{\geqslant 0} m_j$ for $1 \leqslant j \leqslant n$ and that we denote by D_{m_j} the corresponding toric divisors. For every $1 \leqslant j \leqslant n$, we also choose a point x_j in general position on the toric divisor D_{m_j}. Let $\overline{Y}_{\mathrm{m}} \times \mathbb{C} \to \mathbb{C}$ be the trivial family over $\mathbb{C}$ and let $\{x_j\} \times \mathbb{C}$ be the sections determined by the points x_j. Up to doing some toric blow-ups, which do not change the log Gromov–Witten invariants that we are considering by [6], we can assume that the divisors D_{m_j} are disjoint.

The degeneration of $\overline{Y}_{\mathrm{m}}$ to the normal cone of $D_{m_1} \cup \cdots \cup D_{m_n}$,

$$\epsilon_{\overline{\mathcal{Y}}_{\mathrm{m}}} \colon \overline{\mathcal{Y}}_{\mathrm{m}} \to \mathbb{C},$$

is obtained by blowing up the loci $D_{m_1}, \ldots, D_{m_n}$ over $0 \in \mathbb{C}$ in $\overline{Y}_{\mathrm{m}} \times \mathbb{C}$. The special fiber is given by

$$\epsilon_{\overline{\mathcal{Y}}_{\mathrm{m}}}^{-1}(0) = \overline{Y}_{\mathrm{m}} \cup \bigcup_{j=1}^{n} \mathbb{P}_j,$$

where $\mathbb{P}_j$ is the projective bundle over D_{m_j} obtained by projectivization of the rank-two vector bundle $\mathcal{O}_{D_{m_j}} \oplus N_{D_{m_j}|\overline{Y}_{\mathrm{m}}}$ over D_{m_j}, where $N_{D_{m_j}|\overline{Y}_{\mathrm{m}}}$ denotes the normal line bundle to D_{m_j} in $\overline{Y}_{\mathrm{m}}$. The embeddings

$$\mathcal{O}_{D_{m_j}} \hookrightarrow \mathcal{O}_{D_{m_j}} \oplus N_{D_{m_j}|\overline{Y}_{\mathrm{m}}} \quad \text{and} \quad N_{D_{m_j}|\overline{Y}_{\mathrm{m}}} \hookrightarrow \mathcal{O}_{D_{m_j}} \oplus N_{D_{m_j}|\overline{Y}_{\mathrm{m}}}$$

induce two sections of $\mathbb{P}_j \to D_{m_j}$ that we denote respectively by $D_{m_j,\infty}$ and $D_{m_j,0}$. In $\epsilon_{\overline{\mathcal{Y}}_{\mathrm{m}}}^{-1}(0)$, the divisor D_{m_j} in $\overline{Y}_{\mathrm{m}}$ is glued to the divisor $D_{m_j,0}$ in $\mathbb{P}_j$. The strict transform of the section $\{x_j\} \times \mathbb{C}$ of $\overline{Y}_{\mathrm{m}} \times \mathbb{C}$ is a section S_j of $\epsilon_{\overline{\mathcal{Y}}_{\mathrm{m}}}$, whose intersection with $\epsilon_{\overline{\mathcal{Y}}_{\mathrm{m}}}^{-1}(0)$ is a point $x_{j,\infty} \in D_{m_j,\infty}$.

For every $1 \leqslant j \leqslant n$, we blow up the section S_j in $\overline{\mathcal{Y}}_{\mathfrak{m}}$ and we obtain a family

$$\epsilon_{\mathcal{Y}_{\mathfrak{m}}} : \mathcal{Y}_{\mathfrak{m}} \to \mathbb{C},$$

whose fibers away from zero are isomorphic to the surface $Y_{\mathfrak{m}}$, and whose special fiber is given by

$$\mathcal{Y}_{m,0} := \epsilon_{\mathcal{Y}_{\mathfrak{m}}}^{-1}(0) = \overline{Y}_{\mathfrak{m}} \cup \bigcup_{j=1}^{n} \widetilde{\mathbb{P}}_j,$$

where $\widetilde{\mathbb{P}}_j$ is the blow-up of $\mathbb{P}_j$ at all the points $x_{j',\infty}$ such that $\mathbb{P}_{j'} = \mathbb{P}_j$. We denote by $E_{j'}$ the corresponding exceptional divisor in $\widetilde{\mathbb{P}}_j$ and $C_{j'}$ the strict transform in $\widetilde{\mathbb{P}}_j$ of the unique $\mathbb{P}^1$–fiber of $\mathbb{P}_j \to D_{m_j}$ containing $x_{j',\infty}$. We have $E_{j'} \cdot C_{j'} = 1$ in $\widetilde{\mathbb{P}}_j$. We still denote by $D_{m_j,0}$ and $D_{m_j,\infty}$ the strict transforms of $D_{m_j,0}$ and $D_{m_j,\infty}$. We denote by $\partial \widetilde{\mathbb{P}}_j$ the "boundary" of $\widetilde{\mathbb{P}}_j$, which is the union of $D_{m_j,0}$ and $D_{m_j,\infty}$ and of the strict transforms of the two $\mathbb{P}^1$–fibers of $\mathbb{P}_j \to D_{m_j}$ over the two intersection points of D_{m_j} with the remaining part of $\partial \overline{Y}_{\mathfrak{m}}$.

We would like to obtain Proposition 5.1 by application of a degeneration formula in log Gromov–Witten theory to the family

$$\epsilon_{\mathcal{Y}_{\mathfrak{m}}} : \mathcal{Y}_{\mathfrak{m}} \to \mathbb{C},$$

to relate the invariants $N_{g,p}^{Y_{\mathfrak{m}}}$ of the general fiber $Y_{\mathfrak{m}}$ to the invariants $N_{g,w}^{\overline{Y}_{\mathfrak{m}}}$ of $\overline{Y}_{\mathfrak{m}}$ which appears as component of the special fiber $\mathcal{Y}_{\mathfrak{m},0}$. In [31], Gross, Pandharipande and Siebert work with an ad hoc definition of the genus 0 invariants as relative Gromov–Witten invariants of some open geometry and they only need to apply the usual degeneration formula in relative Gromov–Witten theory. In our present setting, with log Gromov–Witten invariants in arbitrary genus, we cannot follow exactly the same path.

Because the general degeneration formula in log Gromov–Witten theory is not yet available, we follow the strategy used in [10]. We first apply the decomposition formula of Abramovich, Chen, Gross and Siebert [2]. We then use the vanishing property of the top lambda class to restrict the terms appearing in the decomposition formula and to prove a gluing formula by working only with torically transverse stable log maps. We review the decomposition formula of [2] in Section 5.3. In Section 5.4, we derive constraints on the terms contributing to the decomposition formula. In Section 5.5, we prove a gluing formula computing each of these terms. We end the classification of the terms contributing to the decomposition formula in Section 5.6. We conclude the proof of Proposition 5.1 in Section 5.7.

5.3 Statement of the decomposition formula

We consider $\mathcal{Y}_{\mathrm{m}}$ as a smooth log scheme for the divisorial log structure defined by the union of the "vertical" divisor $\mathcal{Y}_{\mathrm{m},0}$ with the strict transform of the "horizontal" divisor $\partial \overline{Y}_{\mathrm{m}} \times \mathbb{C}$. Viewing $\mathbb{C}$ as a smooth log scheme for the divisorial log structure defined by the divisor $\{0\} \subset \mathbb{C}$, we obtain that $\epsilon_{\mathcal{Y}_{\mathrm{m}}}$ is a log smooth morphism. Restricting to the special fiber gives a structure of log scheme on $\mathcal{Y}_{\mathrm{m},0}$ and a log smooth morphism

$$\epsilon_{\mathcal{Y}_{\mathrm{m},0}} : \mathcal{Y}_{\mathrm{m},0} \to \mathrm{pt}_{\mathbb{N}},$$

where the standard log point $\mathrm{pt}_{\mathbb{N}}$ is obtained by restriction to $\{0\} \subset \mathbb{C}$ of the divisorial log structure on $\mathbb{C}$.

Let $\overline{M}_{g,p}(\mathcal{Y}_{\mathrm{m},0})$ be the moduli space of genus g stable log maps to $\epsilon_{\mathcal{Y}_{\mathrm{m},0}} : \mathcal{Y}_{\mathrm{m},0} \to \mathrm{pt}_{\mathbb{N}}$, of class β_p, with a marked point of contact order $\ell_p m_p$. This is a proper Deligne–Mumford stack coming with a g–dimensional virtual fundamental class

$$[\overline{M}_{g,p}(\mathcal{Y}_{\mathrm{m},0})]^{\mathrm{virt}}.$$

By deformation invariance of the virtual fundamental class on moduli spaces of stable log maps in log smooth families, we have

$$N_{g,p}^{Y_{\mathrm{m}}} = \int_{[\overline{M}_{g,p}(\mathcal{Y}_{\mathrm{m},0})]^{\mathrm{virt}}} (-1)^g \lambda_g.$$

The decomposition formula of [2] gives a decomposition of $[\overline{M}_{g,p}(\mathcal{Y}_{\mathrm{m},0})]^{\mathrm{virt}}$ indexed by tropical curves mapping to the tropicalization of $\mathcal{Y}_{\mathrm{m},0}$. These tropical curves encode the intersection patterns of irreducible components of stable log maps mapping to the special fiber of the degeneration. We refer to Appendix B of [33] and Section 2 of [2] for the general notion of tropicalization of a log scheme. We denote by $\Sigma(X)$ the tropicalization of a log scheme X. The tropicalization of a log scheme is a cone complex, ie an abstract gluing of cones.

We start by describing the tropicalization $\Sigma(\mathcal{Y}_{\mathrm{m},0})$ of $\mathcal{Y}_{\mathrm{m},0}$. Tropicalizing the log morphism $\epsilon_{\mathcal{Y}_{\mathrm{m},0}} : \mathcal{Y}_{\mathrm{m},0} \to \mathrm{pt}_{\mathbb{N}}$, we obtain a morphism of cone complexes

$$\Sigma(\epsilon_{\mathcal{Y}_{\mathrm{m},0}}) : \Sigma(\mathcal{Y}_{\mathrm{m},0}) \to \Sigma(\mathrm{pt}_{\mathbb{N}}).$$

We have $\Sigma(\mathrm{pt}_{\mathbb{N}}) = \mathbb{R}_{\geqslant 0}$ and $\Sigma(\mathcal{Y}_{\mathrm{m},0})$ is naturally identified with the cone over the fiber $\Sigma(\epsilon_{\mathcal{Y}_{\mathrm{m},0}})^{-1}(1)$ at $1 \in \mathbb{R}_{\geqslant 0}$. It is thus enough to describe the polyhedral complex

$$\mathcal{Y}_{\mathrm{m},0}^{\mathrm{trop}} := \Sigma(\epsilon_{\mathcal{Y}_{\mathrm{m},0}})^{-1}(1).$$

The polyhedral complex $\mathcal{Y}_{\mathrm{m},0}^{\mathrm{trop}}$ has one vertex v_0 dual to $\overline{Y}_{\mathrm{m}}$ and vertices v_j dual to $\widetilde{\mathbb{P}}_j$ for all $1 \leqslant j \leqslant n$. For every $1 \leqslant j \leqslant n$, there is an edge $e_{j,0}$ of integral length 1,

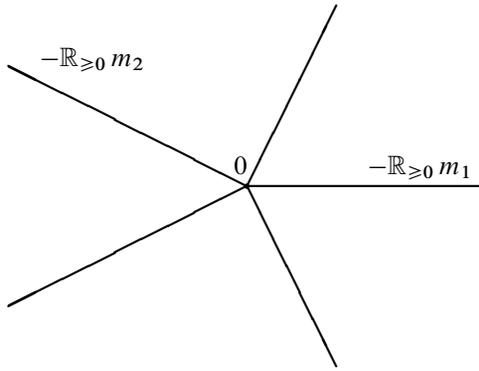Figure 1: Tropicalization of $\overline{Y}_{\mathrm{m}}$.

connecting v_0 and v_j, dual to $D_{m_j,0}$, and an unbounded edge $e_{j,\infty}$ attached to v_j, dual to $D_{m_j,\infty}$.

The best way to understand $\mathcal{Y}_{\mathrm{m},0}^{\mathrm{trop}}$ probably is to think about it as a modification of the tropicalization of $\overline{Y}_{\mathrm{m}}$. As $\overline{Y}_{\mathrm{m}}$ is simply a toric surface, its tropicalization $\Sigma(\overline{Y}_{\mathrm{m}})$ can be naturally identified with $\mathbb{R}^2$ endowed with the fan decomposition. In particular, $\Sigma(\overline{Y}_{\mathrm{m}})$ has one vertex $v_0 = 0 \in \mathbb{R}^2$ and unbounded edges $-\mathbb{R}_{\geqslant 0}\, m_j$, attached to v_0 and dual to the toric boundary divisors D_{m_j}. To go from $\Sigma(\overline{Y}_{\mathrm{m}})$ to $\mathcal{Y}_{\mathrm{m},0}^{\mathrm{trop}}$, one adds a vertex v_j on each primitive integral point of $-\mathbb{R}_{\geqslant 0}\, m_j$, which has the effect of splitting $-\mathbb{R}_{\geqslant 0}\, v_j$ into a bounded edge $e_{j,0}$ and an unbounded edge $e_{j,\infty}$. One still has to cut along $e_{j,\infty}$ and to insert there two 2–dimensional cones dual to the two "corners" of $\partial\widetilde{\mathbb{P}}_j$ which
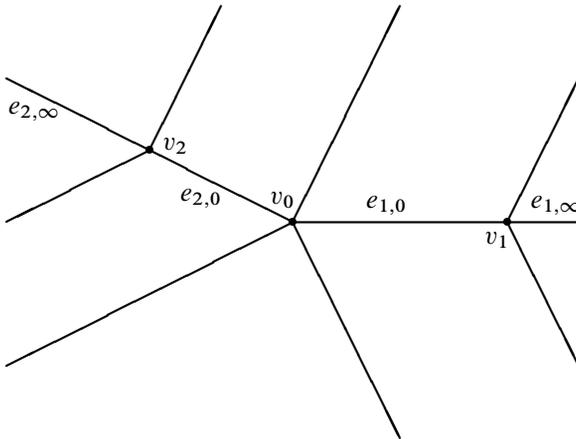

Figure 2: $\mathcal{Y}_{\mathrm{m},0}^{\mathrm{trop}}$.

are on $D_{m_j,\infty}$. In particular, for every $1 \leqslant j \leqslant n$, the vertex v_j is 4–valent and looks locally like the fan of the Hirzebruch surface $\mathbb{P}_j$. In general, there is no global linear embedding of $\mathcal{Y}_{\mathrm{m},0}^{\mathrm{trop}}$ in $\mathbb{R}^2$.

We refer to Definition 2.5.3 of [2] for the general definition of parametrized tropical curve $h \colon \Gamma \to \mathcal{Y}_{\mathrm{m},0}^{\mathrm{trop}}$. It is a natural generalization of the notion of parametrized tropical curve in $\mathbb{R}^2$ that we will use and review in Section 6.1. In particular, Γ is a finite graph, with bounded and unbounded edges mapped by h to $\mathcal{Y}_{\mathrm{m},0}^{\mathrm{trop}}$ in an affine linear way and vertices V of Σ are decorated by some genus $g(V)$. The total genus g of the parametrized tropical curve is defined by $g_\Gamma + \sum_V g(V)$, where g_Γ is the genus of the graph Γ and where the sum is taken over the vertices of Γ.

Some distinction between $\mathcal{Y}_{\mathrm{m},0}^{\mathrm{trop}}$ and $\mathbb{R}^2$, related to the fact that the components $\widetilde{\mathbb{P}}_j$ of $\mathcal{Y}_{\mathrm{m},0}$ are nontoric, is that the usual form of the balancing condition for a tropical curve in $\mathbb{R}^2$ is not necessarily valid at vertices of Γ mapping to a vertex v_j of $\mathcal{Y}_{\mathrm{m},0}^{\mathrm{trop}}$ for some $1 \leqslant j \leqslant n$. For vertices of Γ mapping away from the vertices $v_1, \ldots, v_n$ of $\mathcal{Y}_{\mathrm{m},0}^{\mathrm{trop}}$, the usual balancing condition applies.

Following [2, Definition 4.2.1], a decorated parametrized tropical curve is a parametrized tropical curve $h \colon \Gamma \to \mathcal{Y}_{\mathrm{m},0}^{\mathrm{trop}}$ where each vertex V has a further decoration by a curve class $\beta(V)$ in the stratum of $\mathcal{Y}_{\mathrm{m},0}$ dual to the stratum of $\mathcal{Y}_{\mathrm{m},0}^{\mathrm{trop}}$ where this vertex is mapped. The curve class $\beta(V)$ is assumed to be numerically compatible with the tangency conditions imposed by the edges incident to V. In short, a decorated parametrized tropical curve to $\mathcal{Y}_{\mathrm{m},0}^{\mathrm{trop}}$ encodes all the necessary combinatorial information to be a fiber of the tropicalization of a stable log maps to $\mathcal{Y}_{\mathrm{m},0}$.

The decomposition formula of [2] involves decorated parametrized tropical curves which are rigid in their combinatorial type. This is easy to understand intuitively: the decomposition formula is supposed to describe how the moduli space of stable log maps breaks into pieces under degeneration. If the moduli space of tropical curves were the tropicalization, and so the dual intersection complex, of the moduli space of stable log maps, components of the moduli space of stable log maps should be in bijection with the 0–dimensional strata of the moduli space of tropical curves, ie with rigid tropical curves. According to the decomposition formula of [2], this intuitive picture is correct at the virtual level.

The tropical curves relevant to the study of $\overline{M}_{g,p}(\mathcal{Y}_{\mathrm{m},0})$ are genus g decorated parametrized tropical curves $\Gamma \to \mathcal{Y}_{\mathrm{m},0}^{\mathrm{trop}}$ of type p, ie with only one unbounded edge, of weight ℓ_p and of direction m_p, and with total curve class β_p. According

to [2, Section 4.4], for every such $h\colon \Gamma \to \mathcal{Y}_{\mathrm{m},0}^{\mathrm{trop}}$, we have the notion of stable log map marked by h, and a moduli space $\overline{M}_{g,p}^{h}(\mathcal{Y}_{\mathrm{m},0})$ of stable log maps marked by h, which is a proper Deligne–Mumford stack equipped with a virtual fundamental class $[\overline{M}_{g,p}^{h}(\mathcal{Y}_{\mathrm{m},0})]^{\mathrm{virt}}$. Forgetting the marking by h gives a morphism

$$i_h\colon \overline{M}_{g,p}^{h}(\mathcal{Y}_{\mathrm{m},0}) \to \overline{M}_{g,p}(\mathcal{Y}_{\mathrm{m},0}).$$

We can finally state the decomposition formula, Theorem 4.8.1 of [2]:

$$[\overline{M}_{g,p}(\mathcal{Y}_{\mathrm{m},0})]^{\mathrm{virt}} = \sum_{h} \frac{n_h}{|\mathrm{Aut}(h)|} (i_h)_* [\overline{M}_{g,p}^{h}(\mathcal{Y}_{\mathrm{m},0})]^{\mathrm{virt}},$$

where the sum is over rigid genus g decorated parametrized tropical curves $h\colon \Gamma \to \mathcal{Y}_{\mathrm{m},0}^{\mathrm{trop}}$ of type p, n_h is the smallest positive integer such that, after scaling by n_h, h gets integral vertices and integral lengths, and $|\mathrm{Aut}(h)|$ is the order of the automorphism group of h.

5.4 Constraints on rigid tropical curves

In order to extract some explicit information from the decomposition formula, the first step is to identify the rigid decorated parametrized tropical curves $h\colon \Gamma \to \mathcal{Y}_{\mathrm{m},0}^{\mathrm{trop}}$ of type p. This is in general a difficult question. But because we are only interested in numerical invariants obtained by integration of the top lambda class λ_g, and not in the full virtual class, the situation is much simpler by the following vanishing result.

Lemma 5.2 *Let* $h\colon \Gamma \to \mathcal{Y}_{\mathrm{m},0}^{\mathrm{trop}}$ *be a rigid genus g decorated parametrized tropical curve of type p with Γ of positive genus. Then*

$$\int_{[\overline{M}_{g,p}^{h}(\mathcal{Y}_{\mathrm{m},0})]^{\mathrm{virt}}} (-1)^g \lambda_g = 0.$$

Proof If $f\colon C \to \mathcal{Y}_{\mathrm{m},0}^{\mathrm{trop}}$ is a stable log map in $\overline{M}_{g,p}^{h}(\mathcal{Y}_{\mathrm{m},0})$, then, by definition of the marking by h, the dual intersection complex of C retracts onto Γ and, in particular, has genus bigger than the genus of Γ, which is positive by hypothesis. It follows that C contains a cycle of irreducible components. It is then a general property of λ_g that it vanishes on families of curves containing cycles of irreducible components (eg see Lemma 8 of [10]). □

For every $h\colon \Gamma \to \mathcal{Y}_{\mathrm{m},0}^{\mathrm{trop}}$, a rigid genus g decorated parametrized tropical curve of type p, we define

$$N_{g,p}^{h} := \int_{[\overline{M}_{g,p}^{h}(\mathcal{Y}_{\mathrm{m},0})]^{\mathrm{virt}}} (-1)^g \lambda_g.$$

Proposition 5.3 *We have*

$$N_{g,p}^{Y_m} = \sum_h \frac{n_h}{|\mathrm{Aut}(h)|} N_{g,p}^h,$$

where the sum is over rigid genus g decorated parametrized tropical curves $h\colon \Gamma \to \mathcal{Y}_{m,0}^{\mathrm{trop}}$ of type p with Γ of genus 0.

Proof This follows from integrating $(-1)^g \lambda_g$ over the decomposition formula of [2], reviewed at the end of Section 5.3. By Lemma 5.2, rigid tropical curves $h\colon \Gamma \to \mathcal{Y}_{m,0}^{\mathrm{trop}}$ with Γ of positive genus do not contribute. $\qquad\square$

Proposition 5.4 *Let $h\colon \Gamma \to \mathcal{Y}_{m,0}^{\mathrm{trop}}$ be a rigid genus g decorated parametrized tropical curve of type p, with Γ of genus 0 and $N_{g,p}^h \neq 0$. Then:*

- *For every vertex V of Γ, we have $h(V) = v_j$ for some $0 \leqslant j \leqslant n$.*

- *Every bounded edge E of Γ connects a vertex V_0 with $h(V_0) = v_0$ and a vertex V_+ with $h(V_+) = v_j$ for some $1 \leqslant j \leqslant n$. In particular, the image $h(E)$ is the bounded edge $e_{j,0}$ of $\mathcal{Y}_{m,0}^{\mathrm{trop}}$.*

- *For every vertex V of Γ such that $h(V) = v_j$ for some $1 \leqslant j \leqslant n$, there exists a unique index $j(V)$ such that $1 \leqslant j(V) \leqslant n$ and the curve class $\beta(V)$ is a positive multiple of the curve class $[C_{j(V)}]$. In particular, we have $h(V) = v_{j(V)}$.*

The proof of Proposition 5.4 takes the remainder of Section 5.4. The argument is similar to the one used in the proof of Proposition 11 of [10], itself a tropical version of the properness argument given in Proposition 4.2 of [31]. By iterative application of the balancing condition, we will argue that the source Γ of a rigid decorated parametrized tropical curve $h\colon \Gamma \to \mathcal{Y}_{m,0}^{\mathrm{trop}}$ violating the conclusions of Proposition 5.4 necessarily contains a cycle and so has positive genus. We refer to Proposition 1.15 of [33] for the general form of the balancing condition in log Gromov–Witten theory.

Let $h\colon \Gamma \to \mathcal{Y}_{m,0}^{\mathrm{trop}}$ be a rigid genus g parametrized tropical curve of type p. As h is rigid, there is no edge of Γ contracted by h. The fact that h has type p implies that h has only one unbounded edge and this unbounded edge has weight ℓ_p and direction m_p.

Lemma 5.5 *If there exists a vertex V of Γ such that $h(V) \notin \{v_0, v_1, \ldots, v_n\}$, then Γ has positive genus.*

Proof We first assume $h(V)$ is contained in the interior of one of the 2–dimensional cones $\mathcal{C}$ of $\mathcal{Y}_{m,0}^{\mathrm{trop}}$. Because $h(V)$ is away from the vertices v_j, the situation is locally toric and the balancing condition has to be satisfied at $h(V)$. If $h(V) \notin \mathbb{R}_{\geq 0} \, m_p$, there is no unbounded edge of Γ incident to V, and so by balancing, not all edges attached to $h(V)$ can point towards the vertex of $\mathcal{C}$, ie at least one of those edges points towards a boundary ray of $\mathcal{C}$. If $h(V) \in \mathbb{R}_{\geq 0} \, m_p$, we can get the same conclusion: if all edges passing through $h(V)$ were parallel to $\mathbb{R}_{\geq 0} \, m_p$, this would contradict the rigidity of h because one could move $h(V)$ along $\mathbb{R}_{\geq 0} \, m_p$.

Next, we follow the proof of Proposition 11 of [10]. Fixing a cyclic orientation on the collection of cones and rays of $\mathcal{Y}_{m,0}^{\mathrm{trop}}$, we can assume that this edge points towards the left (from the point of view of the vertex of the cone, looking inside the cone) ray of $\mathcal{C}$. If this edge ends on some vertex still contained in the interior of $\mathcal{C}$, then the balancing condition still applies and so there is still an edge attached to this vertex pointing towards the left ray of $\mathcal{C}$. Because Γ has finitely many vertices, iterating this construction finitely many times, we construct a path starting from $h(V)$ and ending at some vertex $h(V')$ on the left boundary ray of $\mathcal{C}$.

Let $\mathcal{C}'$ be the 2–dimensional cone of $\mathcal{Y}_{m,0}^{\mathrm{trop}}$ incident to $\mathcal{C}$ near $h(V')$. Then we claim that by the balancing condition, there exists an edge attached to $h(V')$ pointing towards the left ray of $\mathcal{C}'$. Indeed, the only case for which the balancing condition is not a priori satisfied is if $h(V') = v_j$ for some j. But at v_j, the nontoric nature of $\widetilde{\mathbb{P}}_j$ only modifies the balancing condition in the direction parallel to $e_{j,0}$ and $e_{j,\infty}$: if there is an incoming edge with nonzero transversal direction, then there is still an outgoing edge with nonzero transversal direction. Indeed, $\widetilde{\mathbb{P}}_j$ is obtained from the Hirzebruch surface $\mathbb{P}_j \to D_{m_j}$ by blowing up points on the divisors $D_{m_j,\infty}$: this does not affect the fact that the general fibers of $\widetilde{\mathbb{P}}_j \to D_{m_j}$ are still linearly equivalent.

Iterating this construction, we obtain a path in Γ whose image by h in $\mathcal{Y}_{m,0}^{\mathrm{trop}}$ is a path which intersects successive rays in the anticlockwise order. Because Γ has finitely many edges, this path has to close eventually and so Γ contains a nontrivial cycle, ie Γ has positive genus.

It remains to treat the case where $h(V)$ is in the interior of a 1–dimensional ray of $\mathcal{Y}_{m,0}^{\mathrm{trop}}$. If all the edges attached to $h(V)$ were parallel to the ray, this would contradict the rigidity of h because one could move $h(V)$ along the ray. So at least one of the edges attached to $h(V)$ is not parallel to the ray and by balancing, we can assume that there is an edge attached to $h(V)$ pointing towards the 2–dimensional cone of $\mathcal{Y}_{m,0}^{\mathrm{trop}}$ left to the ray. We can then apply the iterative argument described above. $\square$

We continue the proof of Proposition 5.4. We are assuming that Γ has genus 0. By Lemma 5.5, every vertex V of Γ maps to one of the vertices $v_0, v_1, \ldots, v_n$ of Γ. If there were an edge connecting a vertex mapped to v_j with a vertex mapped to $v_{j'}$, $1 \leqslant j, j' \leqslant n$ with $j \neq j'$, then we could apply the iterative argument used in the proof of Lemma 5.5, and this would contradict the assumption that Γ has genus 0. It follows that every bounded edge in Γ incident to some vertex mapped to v_j for some $1 \leqslant j \leqslant n$ is also incident to some vertex mapped to v_0.

Let V be a vertex of Γ such that $h(V) = v_j$ for some $1 \leqslant j \leqslant n$. As we are assuming that $m_p \neq -m_j$ for every j, the unique unbounded edge of Γ is not incident to V, and so all edges incident to V are also incident to a vertex mapped to v_0. The only curve classes numerically compatible with these tangency conditions are positive multiples of the classes $[C_j]$. Therefore, there exists a unique $1 \leqslant j(V) \leqslant n$ such that $\beta(V)$ is a positive multiple of $[C_{j(V)}]$. This concludes the proof of Proposition 5.4.

5.5 Gluing formula

According to Proposition 5.3, the computation of the log Gromov–Witten invariants $N_{g,p}^{Y_{\mathrm{m}}}$ is reduced to the computation of the invariants

$$N_{g,p}^h := \int_{[\overline{M}_{g,p}^h(\mathcal{Y}_{\mathrm{m},0})]^{\mathrm{virt}}} (-1)^g \lambda_g,$$

where $h \colon \Gamma \to \mathcal{Y}_{\mathrm{m},0}^{\mathrm{trop}}$ is a rigid genus g decorated parametrized tropical curve of type p with Γ of genus 0, and where $\overline{M}_{g,p}^h(\mathcal{Y}_{\mathrm{m},0})$ is a moduli space of stable log maps to $\mathcal{Y}_{\mathrm{m},0}$ marked by h, ie whose tropicalization is equipped with a retraction to h.

According to Proposition 5.4, the image by h of the bounded edges of Γ is contained in the 1–dimensional part of the polyhedral decomposition of $\mathcal{Y}_{\mathrm{m},0}^{\mathrm{trop}}$. Therefore, cutting the bounded edges, we obtained well-defined numerical data to define moduli spaces M_V of stable log maps attached to the vertices V of Γ. More precisely, we define the moduli spaces M_V as follows.

For V a vertex such that $h(V) = v_0$, M_V is the moduli space of genus $g(V)$ and class $\beta(V)$ stable log maps to $\overline{Y}_{\mathrm{m}}$ with tangency conditions along $\partial \overline{Y}_{\mathrm{m}}$ defined by the weighted edges of Γ attached to V. As $\overline{Y}_{\mathrm{m}}$ is toric, the class $\beta(V)$ is uniquely determined by the numerical compatibility with the tangency conditions along $\partial \overline{Y}_{\mathrm{m}}$.

Let V be a vertex such that $h(V) \neq v_0$. By Proposition 5.4, there exists a unique $1 \leqslant j(V) \leqslant n$ such that $h(V) = v_{j(V)}$ and $\beta(V)$ is a positive multiple of $[C_{j(V)}]$. We

endow $\widetilde{\mathbb{P}}_{j(V)}$ with the divisorial log structure defined by $\partial\widetilde{\mathbb{P}}_{j(V)}$ Then M_V is the moduli space of genus $g(V)$ class $\beta(V)$ stable log maps to $\widetilde{\mathbb{P}}_{j(V)}$ with contact orders along $\partial\widetilde{\mathbb{P}}_j$ defined by the weighted edges of Γ attached to V. As such an edge always connects V with a vertex mapped to v_0, these contact orders are only nontrivial along the divisor $D_{m_{j(V)},0}$. The positive integer $\ell(V)$ such that $\beta(V) = \ell(V)[C_{j(V)}]$ is necessarily equal to the sum of weights of edges incident to V by numerical compatibility of $\beta(V)$ with the tangency conditions.

For every bounded edge E of Γ, we define $j(E) := j(V)$, where V is the unique vertex of Γ incident to E such that $h(V) \neq v_0$, and we set $D_E := D_{m_{j(E)}}$. We denote by $w(E)$ the weight of E.

By general log Gromov–Witten theory [33], each moduli space M_V comes with a virtual fundamental class $[M_V]^{\mathrm{virt}}$. For V a vertex with $h(V) = v_0$, we define

$$N_V := \int_{[M_V]^{\mathrm{virt}}} (-1)^{g(V)} \lambda_{g(V)} \prod_{V \in E} \mathrm{ev}_E^*(\mathrm{pt}_E),$$

where the product is over the bounded edges E incident to V, ev_E is the evaluation map at the corresponding contact point with the divisor D_E, and $\mathrm{pt}_E \in A^1(D_E)$ is the class of a point on D_E. For a vertex V with $h(V) \neq v_0$, we define

$$N_V := \int_{[M_V]^{\mathrm{virt}}} (-1)^{g(V)} \lambda_{g(V)}.$$

Proposition 5.6 *For every $h\colon \Gamma \to \mathcal{Y}_{\mathrm{m},0}^{\mathrm{trop}}$, a rigid genus g decorated parametrized tropical curve of type p with Γ of genus 0, and $N_{g,p}^h \neq 0$, we have*

$$N_{g,p}^h = \frac{\prod_E w(E)}{\mathrm{lcm}\{w(E)\}_E} \prod_V N_V,$$

where the index E runs over bounded edges of Γ and the index V over vertices of Γ.

The proof of Proposition 5.6 takes the remainder of Section 5.5. The moduli space $\overline{M}_{g,p}^h(\mathcal{Y}_{\mathrm{m},0})$ parametrizes stable log maps marked by h, and so in general contains stable log maps whose tropicalization is not h but only retracts onto h. Such stable log maps might interact in a complicated way with the log structure of $\mathcal{Y}_{\mathrm{m},0}$ and their general cutting and gluing properties have not been worked out yet.

We go around this issue by following the strategy used in Section 6 of [10]. On an open locus of torically transverse stable maps, the above-mentioned problems do not arise and the difficulty of the gluing problem is of the same level as the usual

degeneration formula in relative Gromov–Witten theory. The log version of this gluing problem has been recently treated in full detail by Kim, Lho and Ruddat [40]. On the complement of the nice locus of torically transverse stable log maps, a combinatorial argument described in Proposition 11 of [10] implies that one of the relevant curves will always contain a nontrivial cycle of components. By standard vanishing properties of the lambda class, it follows that we can ignore this bad locus if we only care about numerical invariants obtained by integration of a top lambda class, which is our case.

We give now an outline of the proof, referring to [40] and Section 6 of [10] for some of the steps. We have an evaluation morphism

$$\mathrm{ev}\colon \prod_V M_V \to \prod_E D_E^2,$$

where the left product is over the vertices of Γ and where the right product is over the bounded edges of Γ. Let

$$\delta\colon \prod_E D_E \to \prod_E D_E^2$$

be the diagonal morphism. Using the morphisms ev and δ, we define the fiber product

$$\mathcal{M} := \left(\prod_V M_V \right) \times_{\prod_E D_E^2} \left(\prod_E D_E \right).$$

We define a cycle class $[\mathcal{M}]^{\mathrm{virt}}$ on $\mathcal{M}$ by

$$[\mathcal{M}]^{\mathrm{virt}} := \delta^! \left(\prod_V [M_V]^{\mathrm{virt}} \right),$$

where $\delta^!$ is the refined Gysin morphism (see Section 6.2 of [23]) defined by δ.

The following lemma will play for us the same role played by Lemma 16 in Section 6 of [10].

Lemma 5.7 *Let*

$$
\begin{array}{ccc}
C & \xrightarrow{\ f\ } & \mathcal{Y}_{\mathrm{m},0} \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \epsilon_{\mathcal{Y}_{\mathrm{m},0}}} \\
W & \xrightarrow{\ g\ } & \mathrm{pt}_{\mathbb{N}}
\end{array}
$$

be a point of $\overline{M}_{g,p}^h(\mathcal{Y}_{\mathrm{m},0})$. Let

$$
\begin{array}{ccc}
\Sigma(C) & \xrightarrow{\Sigma(f)} & \Sigma(\mathcal{Y}_{\mathrm{m},0}) \\
\downarrow{\scriptstyle \Sigma(\pi)} & & \downarrow{\scriptstyle \Sigma(\epsilon_{\mathcal{Y}_{\mathrm{m},0}})} \\
\Sigma(W) & \xrightarrow{\Sigma(g)} & \Sigma(\mathrm{pt}_{\mathbb{N}})
\end{array}
$$

be its tropicalization. For every $b \in \Sigma(g)^{-1}(1)$, let

$$\Sigma(f)_b \colon \Sigma(C)_b \to \Sigma(\epsilon_{\mathcal{Y}_{\mathrm{m},0}})^{-1}(1) = \mathcal{Y}_{\mathrm{m},0}^{\mathrm{trop}}$$

be the fiber of $\Sigma(f)$ over b. For every bounded edge E of Γ, let $E_{\Sigma(f)_b}$ be the edge of $\Sigma(f)_b$ marked by E. Then

$$h(E_{\Sigma(f)_b}) \subset h(E) = e_{j(E),0}.$$

Proof This follows from the fact that C_j is the unique curve in $\widetilde{\mathbb{P}}_j$ of class $[C_j]$. $\square$

Given a stable log map $f \colon C \to \mathcal{Y}_{\mathrm{m},0}$ marked by h, we have nodes of C in correspondence with the bounded edges of Γ. Cutting C along these nodes, we obtain a morphism

$$\mathrm{cut} \colon \overline{M}_{g,p}^h(Y_{\mathrm{m}}/\partial Y_{\mathrm{m}}) \to \mathcal{M}.$$

By Lemma 5.7, each cut is locally identical to the corresponding cut in a degeneration along a smooth divisor and so we can refer to Section 6 of [10] or Section 5 of [40] for a precise definition of the cut morphism dealing with log structures.

We say that a stable log map $f \colon C \to \overline{Y}_{\mathrm{m}}$ is torically transverse if its image does not contain any of the torus fixed points of $\overline{Y}_{\mathrm{m}}$, ie if its image does not pass through the "corners" of the toric boundary divisor $\partial \overline{Y}_{\mathrm{m}}$, ie if its tropicalization has no vertex mapping in the interior of one of the 2–dimensional cones of the fan of $\overline{Y}_{\mathrm{m}}$.

For every vertex V of Γ such that $h(V) = v_0$, let M_V^0 be the open substack of M_V consisting of torically transverse stable log maps. We define

$$\mathcal{M}^0 := \left(\prod_{\{V : h(V) = v_0\}} M_V^0 \times \prod_{\{V : h(V) \neq v_0\}} M_V \right) \times_{\prod_E D_E^2} \left(\prod_E D_E \right),$$

$$\overline{M}_{g,p}^{h,0}(Y_{\mathrm{m}}/\partial Y_{\mathrm{m}}) := \mathrm{cut}^{-1}(\mathcal{M}^0),$$

and we denote by

$$\mathrm{cut}^0 \colon \overline{M}_{g,p}^{h,0}(Y_{\mathrm{m}}/\partial Y_{\mathrm{m}}) \to \mathcal{M}^0$$

the corresponding restriction of the cut morphism.

Lemma 5.8 *The morphism*

$$\mathrm{cut}^0 \colon \overline{M}_{g,p}^{h,0}(Y_{\mathrm{m}}/\partial Y_{\mathrm{m}}) \to \mathcal{M}^0$$

is étale of degree

$$\frac{\prod_E w(E)}{\mathrm{lcm}\{w(E)\}_E},$$

where the index E runs over bounded edges of Γ and $w(E)$ is the weight of E.

Proof Because of the restriction to the torically transverse locus, the gluing question is locally isomorphic to the corresponding gluing question in a degeneration along a smooth divisor, and so the result follows from formula (6.13) and Lemma 9.2 of [40]. In the corresponding argument in Section 6 of [10], the denominator of the formula did not appear because the relevant tropical curves had edges all of integral length. □

Restricted to the torically transverse locus, the comparison of obstruction theories on $\overline{M}^h_{g,p}(Y_{\mathrm{m}}/\partial Y_{\mathrm{m}})$ and $\mathcal{M}$ reduces to the same question studied in Section 9 of [40] for a degeneration along a smooth divisor. In particular, combining Lemma 5.8 with [40, (9.14)], we obtain that the cycle classes

$$(\mathrm{cut})_* ([\overline{M}^h_{g,p}(Y_{\mathrm{m}}/\partial Y_{\mathrm{m}})]^{\mathrm{virt}}) \quad \text{and} \quad \frac{\prod_E w(E)}{\mathrm{lcm}\{w(E)\}_E} [\mathcal{M}]^{\mathrm{virt}}$$

have the same restriction to the open substack $\mathcal{M}^0$ of $\mathcal{M}$. By [23, Proposition 1.8], it follows that their difference is rationally equivalent to a cycle supported on the closed substack

$$Z := \mathcal{M} - \mathcal{M}^0.$$

At a point of Z, the corresponding stable log map $f\colon C \to \overline{Y}_{\mathrm{m}}$ to $\overline{Y}_{\mathrm{m}}$ is not torically transverse. Using Lemma 5.7, we can apply Proposition 11 of [10] to get that C contains a nontrivial cycle of components. As $\lambda_g = 0$ for a family of curves containing a nontrivial cycle of components (see eg Lemma 8 of [10]), we deduce as in Section 6 of [10] that

$$N^h_{g,p} = \frac{\prod_E w(E)}{\mathrm{lcm}\{w(E)\}_E} \int_{[\mathcal{M}]^{\mathrm{virt}}} (-1)^g \lambda_g,$$

and using the gluing properties of lambda classes (see eg Lemma 9 of [10]) that

$$\int_{[\mathcal{M}]^{\mathrm{virt}}} (-1)^g \lambda_g = \prod_V N_V.$$

For this last step, in order to compute $[\mathcal{M}]^{\mathrm{virt}}$, we had to insert the class $1 \times \mathrm{pt}_E + \mathrm{pt}_E \times 1$ of the diagonal $D_E \hookrightarrow D^2_E$. Each bounded edge E in Γ connects a vertex V_0 with $h(V_0) = v_0$ and a vertex V_+ with $h(V_+) \neq v_0$. A stable log map in M_{V_+} is a cover of the curve $C_{j(V_+)}$, which intersects the divisor D_E in a specific point. Therefore, the only term in the class of the diagonal leading to a possibly nonvanishing contribution is the one with the insertion of pt_E on M_{V_0} and the insertion of 1 on M_{V_+}, which is exactly the way we defined the invariants N_V. This concludes the proof of Proposition 5.6.

We remark that the most general form of the gluing formula in log Gromov–Witten theory, work in progress of Abramovich, Chen, Gross and Siebert, requires the use of punctured Gromov–Witten invariants; see [3]. We do not see punctured invariants in our gluing formula because the only contributing rigid tropical curves are contained in the 1–dimensional part of the polyhedral decomposition of $\mathcal{Y}_{m,0}^{\mathrm{trop}}$.

5.6 Classification of contributing rigid tropical curves

Lemma 5.9 *Let $h\colon \Gamma \to \mathcal{Y}_{m,0}^{\mathrm{trop}}$ be a rigid genus g decorated parametrized tropical curve of type p with Γ of genus 0 and $N_{g,p}^{h} \neq 0$. Let V be a vertex of Γ such that $h(V) \neq v_0$, so that $\beta(V) = \ell(V)[C_{j(V)}]$. Let n_V be the number of bounded edges of Γ incident to V. If $n_V > 1$, then we have $N_V = 0$. If $n_V = 1$, then N_V is the coefficient of $\hbar^{2g(V)-1}$ in*

$$\frac{(-1)^{\ell(V)-1}}{\ell(V)} \frac{1}{2\sin\left(\frac{1}{2}\ell(V)\hbar\right)}.$$

Proof As the curve $C_{j(V)} \simeq \mathbb{P}^1$ is rigid in $\widetilde{\mathbb{P}}_{j(V)}$, with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$, every stable log map defining a point of M_V factors through $C_{j(V)}$. Therefore, the moduli space M_V coincides with the moduli space $M_V(\mathbb{P}^1/\infty)$ of genus $g(V)$ degree $\ell(V)$ stable log maps to $\mathbb{P}^1$, relative to $\infty \in \mathbb{P}^1$ with n_V contact points above ∞ and contact orders given by the weights of the n_V bounded edges incident to V. However, the surface obstruction theory on M_V defining the class $[M_V]^{\mathrm{virt}}$ differs from the curve obstruction theory on $M_V(\mathbb{P}^1/\infty)$ defining the class $[M_V(\mathbb{P}^1/\infty)]^{\mathrm{virt}}$. Denoting by $\pi\colon \mathcal{C} \to M_V(\mathbb{P}^1/\infty)$ the universal source log curve and by $f\colon \mathcal{C} \to \mathbb{P}^1$ the universal log map, the two obstruction theories differ by

$$R^1\pi_* f^* N_{C_{j(V)}|\widetilde{\mathbb{P}}_{j(V)}} = R^1\pi_* f^* \mathcal{O}_{\mathbb{P}^1}(-1),$$

and so

$$N_V = \int_{[M_V(\mathbb{P}^1/\infty)]^{\mathrm{virt}}} e\big(R^1\pi_* f^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))\big),$$

where $e(-)$ is the Euler class. The virtual dimension of $[M_V(\mathbb{P}^1/\infty)]^{\mathrm{virt}}$ is

$$2g(V) - 2 + \ell(V) + n_V,$$

whereas the complex degree of the integrand is

$$2g(V) - 1 + \ell(V).$$

Therefore, we have $N_V = 0$ for $n_V \neq 1$. To compute N_V for $n_V = 1$, we remark that log Gromov–Witten invariants of $(\mathbb{P}^1, \infty)$ coincide with relative Gromov–Witten

invariants of $(\mathbb{P}^1, \infty)$ by the general log/relative comparison theorem of [5] for a smooth divisor. The corresponding relative Gromov–Witten invariants of $(\mathbb{P}^1, \infty)$ have been computed by Bryan and Pandharipande in [13]; see the proof of Theorem 5.1 in [13]. □

Let $h\colon \Gamma \to \mathcal{Y}_{\mathrm{m},0}^{\mathrm{trop}}$ be a rigid genus g decorated parametrized tropical curve of type p with Γ of genus 0 and $N_{g,p}^h \neq 0$. By Proposition 5.4, every vertex V of Γ is mapped by h to some v_j for some $0 \leqslant j \leqslant n$, and every bounded edge E of Γ connects a vertex V_0 with $h(V_0) = v_0$ and a vertex V_+ with $h(V_+) = v_j$ for some $1 \leqslant j \leqslant n$. Furthermore, the combination of Proposition 5.6 and Lemma 5.9 shows that every vertex V of Γ with $h(V) \neq v_0$ is incident to a single bounded edge. As Γ is connected, this implies that there is a unique vertex V_0 in Γ with $h(V_0) = v_0$. In other words, $h\colon \Gamma \to \mathcal{Y}_{\mathrm{m},0}^{\mathrm{trop}}$ is one of the rigid genus g decorated parametrized tropical curves $h_{k,\vec{g}}\colon \Gamma_{k,\vec{g}} \to \mathcal{Y}_{\mathrm{m},0}^{\mathrm{trop}}$ defined as follows.

Recall that we defined in Section 5.1 what a partition of p is and that we associated to such a partition k of p a positive integer $s(k)$ and an $s(k)$–tuple $(w_1(k), \ldots, w_{s(k)}(k))$ of nonzero vectors in $M = \mathbb{Z}^2$. In particular, each $w_r(k)$ can be naturally written $w_r(k) = \ell m_j$ for some $\ell \geqslant 0$ and some $1 \leqslant j \leqslant n$. For every partition k of p and every $\vec{g} = (g_0, g_1, \ldots, g_{s(k)})$, an $(s(k){+}1)$–tuple of nonnegative integers such that $|\vec{g}| := g_0 + \sum_{r=1}^{s(k)} g_r = g$, we define a rigid genus g decorated parametrized tropical curve $h_{k,\vec{g}}\colon \Gamma_{k,\vec{g}} \to \mathcal{Y}_{\mathrm{m},0}^{\mathrm{trop}}$.

Let $\Gamma_{k,\vec{g}}$ be the genus 0 graph consisting of vertices $V_0, V_1, \ldots, V_{s(k)}$, for every $1 \leqslant r \leqslant s(k)$ bounded edges E_r connecting V_0 to V_r, and an unbounded edge E_p attached to V_0. We define a structure of tropical curve on $\Gamma_{k,\vec{g}}$ by assigning:

- **Genera to the vertices** We assign g_0 to V_0 and g_r to V_r for all $1 \leqslant r \leqslant s(k)$.

- **Lengths to the bounded edges** We assign the length

$$\ell(E_r) := \frac{1}{|w_r(k)|} = \frac{1}{\ell}$$

  to the bounded edge E_r, for all $1 \leqslant r \leqslant s(k)$.

Finally, we define a decorated parametrized tropical curve

$$h_{k,\vec{g}}\colon \Gamma_{k,\vec{g}} \to \mathcal{Y}_{\mathrm{m},0}^{\mathrm{trop}}$$

by

$$h_{k,\vec{g}}(V_0) := v_0 \quad \text{and} \quad h_{k,\vec{g}}(V_r) := v_j \quad \text{for } 1 \leqslant r \leqslant s(k),$$

where we write $w_r(k) = \ell m_j$, and the following data:
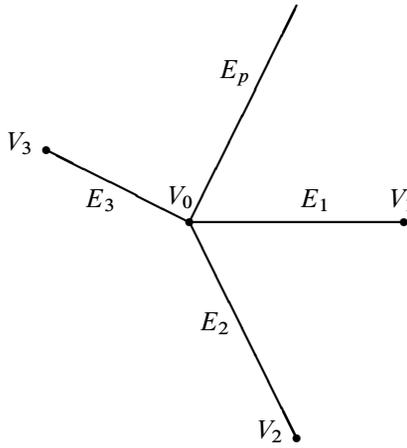
Figure 3: $\Gamma_{k,\vec{g}}$.

- **Edge markings of bounded edges** We define $v_{V_0,E_r} := w_r$ for all $1 \leqslant r \leqslant s(k)$. In particular, the bounded edge E_r has weight $|w_r(k)| = \ell$. This is a valid choice because

$$h(V_r) - h(V_0) = m_j = \frac{1}{\ell}\ell m_j = \ell(E_r)v_{V_0,E_r}.$$

  This uniquely specifies an affine linear map $h_{k,\vec{g}}|_{E_r}$.

- **Edge marking of the unbounded edge** We define $v_{V_0,E_p} := \ell_p m_p$. In particular, the unbounded edge E_p has weight ℓ_p. This uniquely specifies an affine linear map $h_{k,\vec{g}}|_{E_p}$.

- **Decoration of vertices by curve classes** We decorate V_0 with the curve class $\beta_{w(k)} \in H_2(\overline{Y}_{\mathrm{m}}, \mathbb{Z})$. Writing $w_r(k) = \ell m_j$, we decorate the vertex V_r with the curve class $\ell[C_j] \in H_2(\widetilde{\mathbb{P}}_j, \mathbb{Z})$.

We summarize our classification of contributing rigid tropical curves by the following proposition.

Proposition 5.10 *Every rigid genus g decorated parametrized tropical curve*

$$h \colon \Gamma \to \mathcal{Y}_{\mathrm{m},0}^{\mathrm{trop}}$$

of type p with Γ of genus 0 and $N_{g,p}^h \neq 0$ is of the form $h_{k,\vec{g}}$ for some partition k of p and some $(s(k)+1)$–tuple $\vec{g} = (g_0, g_1, \ldots, g_{s(k)})$ of nonnegative integers such that $|\vec{g}| = g$.

In the remaining part of Section 5.6, we compute for $h = h_{k,\vec{g}}$ the numerical factors n_h and $|\mathrm{Aut}(h)|$ entering in the decomposition formula of Proposition 5.3. We fix a partition k of p and $\vec{g}$ such that $|\vec{g}| = g$, and we consider the decorated parametrized tropical curve $h\colon \Gamma_{k,\vec{g}} \to \mathcal{Y}_{\mathrm{m},0}^{\mathrm{trop}}$.

Lemma 5.11 $\qquad\qquad n_{h_{k,\vec{g}}} = \mathrm{lcm}\{|w_r(k)| : 1 \leqslant r \leqslant s(k)\}.$

Proof Recall that $n_{h_{k,\vec{g}}}$ is the smallest positive integer such that after scaling by $n_{k,\vec{g}}$, $h_{k,\vec{g}}$ gets integral vertices and integral lengths. By definition of $h_{k,\vec{g}}$, vertices of $h_{k,\vec{g}}$ are already mapped to integral points of $\mathcal{Y}_{\mathrm{m},0}^{\mathrm{trop}}$. On the other hand, bounded edges E_r of $\Gamma_{k,\vec{g}}$ have fractional lengths $1/|w_r(k)|$. It follows that $n_{k,\vec{g}}$ is the least common multiple of the positive integers $|w_r(k)|$ such that $1 \leqslant r \leqslant s(k)$. $\qquad\square$

For $1 \leqslant j \leqslant n$, $\ell \geqslant 1$ and $a \geqslant 0$, denote by $k_{\ell,j,a}$ the number of vertices of $\Gamma_{k,\vec{g}}$ having genus a among the $k_{\ell,j}$ ones having curve class decoration $\ell[C_j]$. Note that

$$k_{\ell,j} = \sum_{a \geqslant 0} k_{\ell,j,a} \quad \text{and} \quad \sum_{r=1}^{s(k)} g_r = \sum_{j=1}^{n} \sum_{\ell \geqslant 1} \sum_{a \geqslant 0} a k_{\ell,j,a}.$$

Lemma 5.12 *The order of the automorphism group of the decorated parametrized tropical curve* $h_{k,\vec{g}}\colon \Gamma_{k,\vec{g}} \to \mathcal{Y}_{\mathrm{m},0}^{\mathrm{trop}}$ *is given by*

$$|\mathrm{Aut}(h_{k,\vec{g}})| = \prod_{j=1}^{n} \prod_{\ell \geqslant 1} \prod_{a \geqslant 0} k_{\ell,j,a}!.$$

Proof For every $1 \leqslant j \leqslant n$, $\ell \geqslant 1$ and $a \geqslant 0$, there are $k_{\ell,j,a}$ of the vertices V_r having the same curve class decoration $\ell[C_j]$, the same genus a and attached edges of the same weight ℓm_j, so permutations of these $k_{\ell,j,a}$ vertices define automorphisms of the decorated tropical curve $h_{k,\vec{g}}$. Any other permutation of the vertices of Σ_k permutes vertices having different curve class decorations and/or different genus, and so cannot be an automorphism of the decorated tropical curve. $\qquad\square$

5.7 End of the proof of the degeneration formula

In this section, we end the proof of Proposition 5.1. By Proposition 5.3, we have

$$N_{g,p}^{Y_{\mathrm{m}}} = \sum_{h} \frac{n_h}{|\mathrm{Aut}(h)|} N_{g,p}^{h},$$

where the sum is over rigid genus g decorated parametrized tropical curves $h\colon \Gamma \to \mathcal{Y}_{\mathrm{m},0}^{\mathrm{trop}}$ of type p with Γ of genus 0. By Proposition 5.10, such tropical curves h with $N_{g,p}^{h}$

are necessarily of the form $h_{k,\vec{g}}$ for some partition k of p and some $(s(k)+1)$–tuple $\vec{g} = (g_0, g_1, \ldots, g_{s(k)})$ of nonnegative integers such that $|\vec{g}| = g$. For $h = h_{k,\vec{g}}$, the factors n_h and $|\mathrm{Aut}(h)|$ are computed by Lemmas 5.11 and 5.12. On the other hand, $N_{g,p}^{h_{k,\vec{g}}}$ can be computed using the gluing formula of Proposition 5.6 and Lemma 5.9. After simplification, we obtain the formula of Proposition 5.1.

6   Scattering and tropical curves

In this section, we review the connection established in [21] between quantum scattering diagrams and refined tropical curve counting.

6.1   Refined tropical curve counting

In this section, we review the definition of the refined tropical curve counts used in [21]. The relevant tropical curves are identical to those considered in [31]. The only difference is that they are counted with the Block–Göttsche refined multiplicity [8], q–deformation of the usual Mikhalkin multiplicity [57].

We first recall the definition of a parametrized tropical curve to $\mathbb{R}^2$ by simply repeating the presentation we gave in [10]. For us, a graph Γ has a finite set $V(\Gamma)$ of vertices, a finite set $E_f(\Gamma)$ of bounded edges connecting pairs of vertices and a finite set $E_\infty(\Gamma)$ of legs attached to vertices that we view as unbounded edges. By edge, we refer to a bounded or unbounded edge. We will always consider connected graphs.

A parametrized tropical curve $h\colon \Gamma \to \mathbb{R}^2$ is the following data:

- A nonnegative integer $g(V)$ for each vertex V, called the genus of V.

- A labeling of the elements of the set $E_\infty(\Gamma)$.

- A vector $v_{V,E} \in \mathbb{Z}^2$ for every vertex V and E an edge incident to V. If $v_{V,E}$ is not zero, the divisibility $|v_{V,E}|$ of $v_{V,E}$ in $\mathbb{Z}^2$ is called the weight of E and is denoted by $w(E)$. We require that $v_{V,E} \neq 0$ if E is unbounded and that for every vertex V, the following balancing condition is satisfied:
$$\sum_E v_{V,E} = 0,$$
where the sum is over the edges E incident to V. If E is an unbounded edge, we write v_E for $v_{V,E}$, where V is the unique vertex to which E is attached.

- A nonnegative real number $\ell(E)$ for every bounded edge of E, called the length of E.

- A proper map $h\colon \Gamma \to \mathbb{R}^2$ such that:
  - If E is a bounded edge connecting the vertices V_1 and V_2, then h maps E affine linearly on the line segment connecting $h(V_1)$ and $h(V_2)$, and
  $$h(V_2) - h(V_1) = \ell(E)v_{V_1,E}.$$
  - If E is an unbounded edge of vertex V, then h maps E affine linearly to the ray $h(V) + \mathbb{R}_{\geqslant 0}\, v_{V,E}$.

The genus g_h of a parametrized tropical curve $h\colon \Gamma \to \mathbb{R}^2$ is defined by
$$g_h := g_\Gamma + \sum_{V \in V(\Gamma)} g(V),$$
where g_Γ is the genus of the graph Γ.

Let $w = (w_1, \ldots, w_s)$ be an s–tuple of nonzero vectors in M. We fix $x = (x_1, \ldots, x_s)$, an element of $(\mathbb{R}^2)^s$. We say that a parametrized tropical curve $h\colon \Gamma \to \mathbb{R}^2$ is of type (w, x) if Γ has exactly $s+1$ unbounded edges, labeled $E_0, E_1, \ldots, E_s$, such that

- $v_{E_0} = \sum_{r=1}^s w_r$,
- $v_{E_r} = -w_r$ for every $1 \leqslant r \leqslant s$,
- E_r asymptotically coincides with the half-line $-\mathbb{R}_{\geqslant 0}\, w_r + x_r$ for every $1 \leqslant r \leqslant s$.

Let $T_{w,x}$ be the set of genus 0 parametrized tropical curves $h\colon \Gamma \to \mathbb{R}^2$ of type (w, x) without contracted edges. If $x \in (\mathbb{R}^2)^s$ is general enough (in some appropriate open dense subset), then it follows from [57] or [60] that $T_{w,x}$ is a finite set, and that if $(h\colon \Gamma \to \mathbb{R}^2) \in T_{w,x}$, then Γ is trivalent and h is an immersion (distinct vertices have distinct images and two distinct edges have at most one point in common in their images).

For $h\colon \Gamma \to \mathbb{R}^2$ a parametrized tropical curve in $\mathbb{R}^2$ and V a trivalent vertex with incident edges E_1, E_2 and E_3, the multiplicity of V is the integer defined by
$$m(V) := |\det(v_{V,E_1}, v_{V,E_2})|.$$
Thanks to the balancing condition
$$v_{V,E_1} + v_{V,E_2} + v_{V,E_3} = 0,$$
this definition is symmetric in E_1, E_2 and E_3. The Block–Göttsche [8] multiplicity of V is a Laurent polynomial in a formal variable $q^{\frac{1}{2}}$:
$$[m_V]_q := \frac{q^{\frac{1}{2}m(V)} - q^{-\frac{1}{2}m(V)}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} = q^{-\frac{1}{2}(m(V)-1)}(1 + q + \cdots + q^{\frac{1}{2}(m(V)-1)}) \in \mathbb{N}[q^{\pm\frac{1}{2}}].$$

For $h\colon \Gamma \to \mathbb{R}^2$ a parametrized tropical curve with Γ trivalent, the refined multiplicity of h is defined by

$$m_h(q^{\frac{1}{2}}) := \prod_{V \in V(\Gamma)} [m(V)]_q,$$

where the product is over the vertices of Γ.

If $x \in (\mathbb{R}^2)^s$ is in general position, we count the elements of $T_{w,x}$ with refined multiplicities and we obtain a refined count of tropical curves:

$$N_{w,x}^{\mathrm{trop}}(q^{\frac{1}{2}}) := \sum_{(h\colon \Gamma \to \mathbb{R}^2) \in T_{w,x}} m_h(q^{\frac{1}{2}}) \in \mathbb{N}[q^{\pm \frac{1}{2}}].$$

According to Itenberg and Mikhalkin [35], $N_{w,x}^{\mathrm{trop}}(q^{\frac{1}{2}})$ does not depend on x if x is general. This also follows from the correspondence theorem of [10]. Therefore, we simply denote by $N_w^{\mathrm{trop}}(q^{\frac{1}{2}})$ the corresponding invariant.

6.2 Elementary quantum scattering

Let m_1 and m_2 be two nonzero vectors in $M = \mathbb{Z}^2$. Let $\widehat{\mathfrak{D}}$ be the quantum scattering diagram over an Artinian ring R consisting of two incoming rays $-\mathbb{R}_{\geqslant 0}\, m_1$ and $-\mathbb{R}_{\geqslant 0}\, m_2$ equipped with the Hamiltonians

$$\widehat{H}_1 = \frac{f_1}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \hat{z}^{m_1} \quad \text{and} \quad \widehat{H}_2 = \frac{f_2}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \hat{z}^{m_2},$$

where $f_1, f_2 \in R$ satisfy $f_1^2 = f_2^2 = 0$. Let $S(\widehat{\mathfrak{D}})$ be the resulting consistent quantum scattering diagram given by Proposition 1.3. The following result is Lemma 4.3 of [21].

Lemma 6.1 *The consistent quantum scattering diagram $S(\widehat{\mathfrak{D}})$ is obtained from $\widehat{\mathfrak{D}}$ by adding three outgoing rays:*

- $(\mathbb{R}_{\geqslant 0}\, m_1, \widehat{H}_1)$,
- $(\mathbb{R}_{\geqslant 0}\, m_2, \widehat{H}_2)$,
- $(\mathbb{R}_{\geqslant 0}\,(m_1 + m_2), \widehat{H}_{12})$, *where*

$$\widehat{H}_{12} := [\langle m_1, m_2 \rangle]_q \frac{f_1 f_2}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \hat{z}^{m_1 + m_2},$$

$$[\langle m_1, m_2 \rangle]_q := \frac{q^{\frac{1}{2}\langle m_1, m_2 \rangle} - q^{-\frac{1}{2}\langle m_1, m_2 \rangle}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}.$$

Proof Using

$$[\hat{z}^{m_1}, \hat{z}^{m_2}] = (q^{\frac{1}{2}\langle m_1, m_2 \rangle} - q^{-\frac{1}{2}\langle m_1, m_2 \rangle}) \hat{z}^{m_1 + m_2},$$

we compute that

$$[\hat{H}_1, \hat{H}_2] = [\langle m_1, m_2 \rangle]_q \frac{f_1 f_2}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \hat{z}^{m_1 + m_2}.$$

As $f_1^2 = f_2^2 = 0$, it follows that $\hat{H}_1$ and $\hat{H}_2$ commute with $[\hat{H}_1, \hat{H}_2]$. Using an easy case of the Baker–Campbell–Hausdorff formula, according to which $e^a e^b = e^{a+b+\frac{1}{2}[a,b]}$ if a and b commute with $[a, b]$, we obtain

$$e^{\hat{H}_2} e^{-\hat{H}_1} e^{-\hat{H}_2} e^{\hat{H}_1} = e^{[\hat{H}_1, \hat{H}_2]},$$

and so

$$\hat{\Phi}_{\hat{H}_2}^{-1} \hat{\Phi}_{\hat{H}_1} \hat{\Phi}_{\hat{H}_2} \hat{\Phi}_{\hat{H}_1}^{-1} = \hat{\Phi}_{[\hat{H}_1, \hat{H}_2]},$$

and hence the result. $\qquad\square$

6.3 Quantum scattering from refined tropical curve counting

In this section, we review the result of Filippini and Stoppa [21] expressing the Hamiltonians attached to the rays of the consistent quantum scattering diagram $S(\hat{\mathfrak{D}}_{\mathrm{m}})$, defined in Section 3.1, in tropical terms. We use the notation introduced at the beginning of Section 5.1.

Proposition 6.2 *For every* $\mathrm{m} = (m_1, \ldots, m_n)$, *an* n–*tuple of primitive nonzero vectors in* M, *and every* $m \in M - \{0\}$, *the Hamiltonian* $\hat{H}_m$ *attached to the outgoing ray* $\mathbb{R}_{\geqslant 0} m$ *in the consistent quantum scattering diagram* $S(\hat{\mathfrak{D}}_{\mathrm{m}})$ *is given by*

$$\hat{H}_m = \sum_{p \in P_m} \sum_{k \vdash p} N_{w(k)}^{\mathrm{trop}}(q^{\frac{1}{2}}) \left(\prod_{j=1}^{n} \prod_{\ell \geqslant 1} \frac{1}{k_{\ell,j}!} \left(\frac{(-1)^{\ell-1}}{\ell} \frac{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}{q^{\frac{1}{2}\ell} - q^{-\frac{1}{2}\ell}} \right)^{k_{\ell,j}} \right)$$
$$\times \left(\prod_{j=1}^{n} t_j^{p_j} \right) \frac{\hat{z}^{\ell_p m}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}},$$

where $q = e^{i\hbar}$, *and the innermost sum is over all partitions* k *of* p.

Proof This follows from the main result, Corollary 4.9, of [21], which is a q–deformed version of the proof of Theorem 2.8 of [31]. A higher-dimensional generalization of this argument has been given by Mandel in [50]. For completeness and because we organize the combinatorics in a slightly different way, we provide a proof.

By definition, $S(\hat{\mathfrak{D}}_{\mathrm{m}})$ is the consistent quantum scattering diagram obtained from the quantum scattering diagram $\hat{\mathfrak{D}}_{\mathrm{m}}$ consisting of incoming rays $(\mathfrak{d}_j, \hat{H}_{\mathfrak{d}_j})$, for $1 \leqslant j \leqslant n$,

where

$$\mathfrak{d}_j = -\mathbb{R}_{\geq 0}\, m_j \quad \text{and} \quad \hat{H}_{\mathfrak{d}_j} = \sum_{\ell \geq 1} \frac{1}{\ell} \frac{(-1)^{\ell-1}}{q^{\frac{1}{2}\ell} - q^{-\frac{1}{2}\ell}} t_j^l \hat{z}^{\ell m_j}.$$

Let us work over the ring $\mathbb{C}[t_1, \ldots, t_n]/(t_1^{N+1}, \ldots, t_n^{N+1})$. We embed this ring into

$$\mathbb{C}[\{u_{ja} \mid 1 \leq j \leq n \text{ and } 1 \leq a \leq N\}]/\langle\{u_{ja}^2 \mid 1 \leq j \leq n \text{ and } 1 \leq a \leq N\}\rangle$$

by

$$t_j = \sum_{a=1}^N u_{ja}$$

for all $1 \leq j \leq n$. We then have

$$t_j^\ell = \sum_{\substack{A \subset \{1,\ldots,N\} \\ |A|=\ell}} \ell! \prod_{a \in A} u_{ja},$$

and so

$$\hat{H}_{\mathfrak{d}_j} = \sum_{\ell=1}^N \sum_{\substack{A \subset \{1,\ldots,N\} \\ |A|=\ell}} \left(\frac{1}{\ell} \frac{(-1)^{\ell-1}}{q^{\frac{1}{2}\ell} - q^{-\frac{1}{2}\ell}} \right) \ell! \left(\prod_{a \in A} u_{ja} \right) \hat{z}^{\ell m_j}.$$

This suggests we consider the quantum scattering diagram $\hat{\mathfrak{D}}_{\mathfrak{m}}^{\text{split}}$ consisting of incoming rays $(\mathfrak{d}_{j\ell A}, \hat{H}_{\mathfrak{d}_{j\ell A}})$ for $1 \leq \ell \leq N$, $A \subset \{1, \ldots, N\}$ and $|A| = \ell$, where

$$\mathfrak{d}_{j\ell A} = -\mathbb{R}_{\geq 0}\, m_j + c_{j\ell A}$$

for $c_{j\ell A} \in \mathbb{R}^2$ in general position, and

$$\hat{H}_{\mathfrak{d}_{j\ell A}} = \left(\frac{1}{\ell} \frac{(-1)^{\ell-1}}{q^{\frac{1}{2}\ell} - q^{-\frac{1}{2}\ell}} \right) \ell! \left(\prod_{a \in A} u_{ja} \right) \hat{z}^{\ell m_j}.$$

If we had taken all $c_{j\ell A}$ to be 0, then $\hat{\mathfrak{D}}_{\mathfrak{m}}^{\text{split}}$ would have been equivalent to $\hat{\mathfrak{D}}_{\mathfrak{m}}$. But for $c_{j\ell A} \in \mathbb{R}^2$ in general position, $\hat{\mathfrak{D}}_{\mathfrak{m}}^{\text{split}}$ is a perturbation of $\hat{\mathfrak{D}}_{\mathfrak{m}}$: each ray $(\mathfrak{d}_j, \hat{H}_{\mathfrak{d}_j})$ of $\mathfrak{D}_{\mathfrak{m}}$ splits into various rays $(\mathfrak{d}_{j\ell A}, \hat{H}_{\mathfrak{d}_{j\ell A}})$ of $\hat{\mathfrak{D}}_{\mathfrak{m}}^{\text{split}}$.

The key simplifying fact is that the consistent scattering diagram $S(\hat{\mathfrak{D}}_{\mathfrak{m}}^{\text{split}})$ can be obtained from $\hat{\mathfrak{D}}_{\mathfrak{m}}^{\text{split}}$ by a recursive procedure involving only elementary scatterings in the sense of Lemma 6.1. When two rays of $\hat{\mathfrak{D}}_{\mathfrak{m}}^{\text{split}}$ intersect, we are in the situation of Lemma 6.1 because $u_{ja}^2 = 0$. The local consistency at this intersection is then guaranteed by emitting a third ray according to Lemma 6.1. Further intersections of the old and newly created rays can similarly be treated by application of Lemma 6.1. Indeed, the assumption of general position of the $c_{j\ell A}$ guarantees that only double intersections occur.

The asymptotic scattering diagram of $S(\widehat{\mathfrak{D}}_{\mathfrak{m}}^{\text{split}})$ is the scattering diagram obtained by taking all the rays of $S(\widehat{\mathfrak{D}})$ and placing their origin at $0 \in \mathbb{R}^2$. By uniqueness of the consistent completion, the asymptotic scattering diagram is precisely $S(\widehat{\mathfrak{D}}_{\mathfrak{m}})$. To get the Hamiltonian $\widehat{H}_m$ attached to an outgoing ray $\mathbb{R}_{\geqslant 0}\, m$ in $S(\widehat{\mathfrak{D}}_{\mathfrak{m}})$, it is then enough to collect the various contributions to the corresponding asymptotic ray of $S(\widehat{\mathfrak{D}}_{\mathfrak{m}}^{\text{split}})$ coming from the recursive construction of $S(\widehat{\mathfrak{D}}_{\mathfrak{m}}^{\text{split}})$.

Let us study how the recursive construction of $S(\widehat{\mathfrak{D}}_{\mathfrak{m}}^{\text{split}})$ can produce a ray $\mathfrak{d}$ asymptotic to $\mathbb{R}_{\geqslant 0}\, m$ and equipped with a function $\widehat{H}_{\mathfrak{d}}$ proportional to $\hat{z}^{\ell_{\mathfrak{d}} m}$ for some $\ell_{\mathfrak{d}} \geqslant 1$. Such a ray is obtained by successive applications of Lemma 6.1 starting from a subset of the initial incoming rays of $\widehat{\mathfrak{D}}_{\mathfrak{m}}^{\text{split}}$.

We focus on one particular sequence of such elementary scatterings. Such a sequence naturally defines a graph $\overline{\Gamma}$ in $\mathbb{R}^2$. This graph starts with unbounded edges given by the initial rays taking part in the sequence of scatterings. When two of these rays meet, they scatter and produce a third ray given by Lemma 6.1. If this third ray does not contribute to further scatterings ultimately contributing to $\widehat{H}_{\mathfrak{d}}$, we do not include it in $\overline{\Gamma}$ and we continue $\overline{\Gamma}$ by propagating the two initial rays. In particular, $\overline{\Gamma}$ contains a 4–valent vertex given by the two initial rays crossing without nontrivial interaction.

If the third ray does contribute to further scatterings ultimately contributing to $\widehat{H}_{\mathfrak{d}}$, we include it in $\overline{\Gamma}$ and we do not propagate the two initial rays. In particular, $\overline{\Gamma}$ gets a trivalent vertex given by the two initial rays meeting and producing the third ray. Iterating this construction, we obtain one trivalent vertex for each elementary scattering ultimately giving a contribution to $\widehat{H}_{\mathfrak{d}}$. At the end of this process, the last elementary scattering produces the ray $\mathfrak{d}$ which becomes an unbounded edge of the graph.

The graph $\overline{\Gamma}$ has two kinds of vertices: trivalent vertices where a nontrivial elementary scattering happens and 4–valent vertices where two rays cross without nontrivial interaction. For each 4–valent vertex, we can separate the two rays crossing, and we obtain a trivalent graph Γ and a map $h\colon \Gamma \to \overline{\Gamma} \subset \mathbb{R}^2$ which is one-to-one except over the 4–valent vertices of $\overline{\Gamma}$ where it is two-to-one. It follows from the iterative construction that the trivalent graph Γ is a tree, ie a graph of genus 0.

The function attached to initial ray of $\widehat{\mathfrak{D}}_{\mathfrak{m}}^{\text{split}}$ is a monomial in $\hat{z}$, whose power is proportional to the direction of the ray. By Lemma 6.1, this property is preserved under elementary scattering. Each edge E of our Γ is thus equipped with a function proportional to $\hat{z}^{m_E}$ for some $m_E \in M = \mathbb{Z}^2$ proportional to the direction of E. Furthermore, in an elementary scattering of two edges E_1 and E_2 equipped with m_{E_1} and m_{E_2},

the produced edge E_3 is equipped with $m_{E_1} + m_{E_2}$ by Lemma 6.1. In other words, the balancing condition is satisfied at each vertex and so we can view $h \colon \Gamma \to \mathbb{R}^2$ as a parametrized tropical curve to $\mathbb{R}^2$ in the sense of Section 6.1.

For every $1 \leqslant j \leqslant n$ and $\ell \geqslant 1$, there is a number $k_{\ell,j}$ of subsets A of $\{1, \ldots, n\}$, of size ℓ, such that $\mathfrak{d}_{j\ell A}$ is one of the initial rays appearing in Γ. Denote by $\mathcal{A}_{j\ell}^{\Gamma}$ this set of subsets of $\{1, \ldots, n\}$. Writing $p_j := \sum_{\ell \geqslant 1} \ell k_{\ell,j}$, we have, by the balancing condition,

$$\sum_{j=1}^{n} p_j = \ell_{\mathfrak{d}} m$$

and, in particular, $\ell_{\mathfrak{d}} = \ell_p$.

It follows from an iterative application of Lemma 6.1 that the contribution of Γ to $\widehat{H}_{\mathfrak{d}}$ is given by

$$m_{\Gamma}(q^{\frac{1}{2}}) \left(\prod_{j=1}^{n} \prod_{\ell \geqslant 1} \left(\frac{(-1)^{\ell-1}}{\ell} \frac{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}{q^{\frac{1}{2}\ell} - q^{-\frac{1}{2}\ell}} \right)^{k_{\ell,j}} (\ell!)^{k_{\ell,j}} \left(\prod_{A \in \mathcal{A}_{j\ell}^{\Gamma}} \prod_{a \in A} u_{ja} \right) \right) \frac{\hat{z}^{\ell_p m}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}},$$

where $m_{\Gamma}(q^{\frac{1}{2}})$ is the refined multiplicity of the tropical curve Γ.

To get the complete expression for $\widehat{H}_{\mathfrak{d}}$, we have to sum over the possible Γ.

If we fix $p = (p_1, \ldots, p_n) \in P = \mathbb{N}^n$, a partition k of p and, for every $1 \leqslant j \leqslant n$ and $\ell \geqslant 1$, a set $\mathcal{A}_{j\ell}$ of $k_{\ell,j}$ disjoint subsets of $\{1, \ldots, N\}$ of size ℓ, we can consider the set $T_{j\ell \mathcal{A}_{j\ell}}$ of genus 0 tropical curves Γ having one unbounded edge of asymptotic direction $\mathbb{R}_{\geqslant 0} m$ and weight $\ell_p m$ and, for every $1 \leqslant j \leqslant n$, $\ell \geqslant 1$ and $A \in \mathcal{A}_{j\ell}$, an unbounded edge of weight ℓm_j asymptotically coinciding with $\mathfrak{d}_{j\ell A}$. By Section 6.1, this set is finite.

We already saw how a sequence of elementary scatterings contributing to $\widehat{H}_{\mathfrak{d}}$ produces an element $\Gamma \in T_{j\ell \mathcal{A}_{j\ell}}$. Conversely, any $\Gamma \in T_{j\ell \mathcal{A}_{j\ell}}$ will define a sequence of elementary scatterings appearing in the construction of $S(\widehat{\mathfrak{D}}_{\mathfrak{m}}^{\mathrm{split}})$ and contributing to $\widehat{H}_{\mathfrak{d}}$.

It follows that, for every $m \in M - \{0\}$, we have

$$\widehat{H}_m = \sum_{p \in P_m} \sum_{k \vdash p} \sum_{\mathcal{A}_{j\ell}} \left(\sum_{\Gamma \in T_{j\ell \mathcal{A}_{j\ell}}} m_{\Gamma}(q^{\frac{1}{2}}) \right) \left(\prod_{j=1}^{n} \prod_{\ell \geqslant 1} \left(\frac{(-1)^{\ell-1}}{\ell} \frac{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}{q^{\frac{1}{2}\ell} - q^{-\frac{1}{2}\ell}} \right)^{k_{\ell,j}} (\ell!)^{k_{\ell,j}} \right.$$

$$\left. \times \left(\prod_{A \in \mathcal{A}_{j\ell}} \prod_{a \in A} u_{ja} \right) \right) \frac{\hat{z}^{\ell_p m}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}},$$

But by [Section 6.1](), we have

$$\sum_{\Gamma \in T_{j\ell_{\mathcal{A}_{j\ell}}}} m_\Gamma(q^{\frac{1}{2}}) = N_{w(k)}^{\text{trop}}(q^{\frac{1}{2}}),$$

which is in particular independent of $\mathcal{A}_{j\ell}$. So we can do the sum over $\mathcal{A}_{j\ell}$. Given an $\mathcal{A}_{j\ell}$, we can form

$$B := \bigcup_{A \in \mathcal{A}_{j\ell}} A,$$

a subset of $\{1, \ldots, N\}$ of size $\sum_{\ell \geq 1} \ell k_{\ell,j} = p_j$. Conversely, the number of ways to write a set B of $p_j = \sum_{\ell \geq 1} \ell k_{\ell,j}$ elements as a disjoint union of subsets, $k_{\ell,j}$ of them being of size ℓ, is equal to

$$\frac{p_j!}{\prod_{\ell \geq 1} k_{\ell,j}!(\ell!)^{k_{\ell,j}}}.$$

Replacing the sum over $\mathcal{A}_{j\ell}$ by a sum over B, we obtain

$$\hat{H}_m = \sum_{p \in P_m} \sum_{k \vdash p} N_{w(k)}^{\text{trop}}(q^{\frac{1}{2}}) \left(\prod_{j=1}^{n} \prod_{\ell \geq 1} \frac{1}{k_{\ell,j}!} \left(\frac{(-1)^{\ell-1}}{\ell} \frac{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}{q^{\frac{1}{2}\ell} - q^{-\frac{1}{2}\ell}} \right)^{k_{\ell,j}} \right)$$
$$\times \left(\prod_{j=1}^{n} \sum_{\substack{B \subset \{1,\ldots,N\} \\ |B|=p_j}} p_j! \prod_{b \in B} u_{jb} \right) \frac{\hat{z}^{\ell_p m}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}.$$

Finally, using that

$$t_j^{p_j} = \sum_{\substack{B \subset \{1,\ldots,N\} \\ |B|=p_j}} p_j! \prod_{b \in B} u_{jb},$$

we obtain the desired formula for $\hat{H}_m$. $\qquad \square$

Corollary 6.3 *We have*

$$\hat{H}_m = \sum_{p \in P_m} \sum_{k \vdash p} N_{w(k)}^{\text{trop}}(q^{\frac{1}{2}}) \left(\prod_{j=1}^{n} \prod_{\ell \geq 1} \frac{1}{k_{\ell,j}!} \left(\frac{(-1)^{\ell-1}}{\ell} \frac{1}{q^{\frac{1}{2}\ell} - q^{-\frac{1}{2}\ell}} \right)^{k_{\ell,j}} \right)$$
$$\times (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{s(k)-1} \hat{z}^{\ell_p m}.$$

Proof We simply rearrange the factors $(q^{\frac{1}{2}} - q^{-\frac{1}{2}})$ in [Proposition 6.2]() and use that

$$s(k) = \sum_{j=1}^{n} \sum_{\ell \geq 1} k_{\ell,j}. \qquad \square$$

7 End of the proof of Theorems 3.1 and 3.2

7.1 End of the proof of Theorem 3.1

In this section, we finish the proof of Theorem 3.1. We have to express the Hamiltonians attached to the rays of the consistent quantum scattering diagram $S(\widehat{\mathfrak{D}}_{\mathrm{m}})$ in terms of the log Gromov–Witten invariants $N_{g,p}^{Y_{\mathrm{m}}}$ of the log Calabi–Yau surface Y_{m}. We already know

- Corollary 6.3, expressing the Hamiltonians attached to the rays of $S(\widehat{\mathfrak{D}}_{\mathrm{m}})$ in terms of the refined counts $N_w^{\mathrm{trop}}(q^{\frac{1}{2}})$ of tropical curves in $\mathbb{R}^2$;

- Proposition 5.1, relating the log Gromov–Witten invariants $N_{g,p}^{Y_{\mathrm{m}}}$ of the log Calabi–Yau surface Y_{m} to the log Gromov–Witten invariants $N_{g,w}^{\overline{Y}_{\mathrm{m}}}$ of the toric surface $\overline{Y}_{\mathrm{m}}$.

It remains to connect the refined tropical counts $N_w^{\mathrm{trop}}(q^{\frac{1}{2}})$ to the log Gromov–Witten invariants $N_{g,w}^{\overline{Y}_{\mathrm{m}}}$ of the toric surface $\overline{Y}_{\mathrm{m}}$. This is given by Proposition 7.1, which is a special case of the main result, Theorem 6, of [10].

Proposition 7.1 *For every* $\mathfrak{m} = (m_1, \ldots, m_n)$, *an* n–*tuple of nonzero primitive vectors in* $M = \mathbb{Z}^2$, *every* $p = (p_1, \ldots, p_n) \in P = \mathbb{N}^n$ *and every partition* k *of* p, *we have*

$$\sum_{g \geqslant 0} N_{g,w(k)}^{\overline{Y}_{\mathrm{m}}} \hbar^{2g-1+s(k)} = N_{w(k)}^{\mathrm{trop}}(q^{\frac{1}{2}}) \left(\prod_{r=1}^{s(k)} \frac{1}{|w_r|} \right) \left(2\sin(\tfrac{1}{2}\hbar) \right)^{s(k)-1}$$

$$= N_{w(k)}^{\mathrm{trop}}(q^{\frac{1}{2}}) \left(\prod_{j=1}^{n} \prod_{\ell \geqslant 1} \frac{1}{\ell^{k_{\ell,j}}} \right) \left(2\sin(\tfrac{1}{2}\hbar) \right)^{s(k)-1}.$$

Proof We simply explain the change in notation needed to translate from Theorem 6 of [10].

In [10], we fix Δ, a balanced collection of vectors in $\mathbb{Z}^2$, specifying a toric surface X_Δ and tangency conditions for a curve along the toric divisors. We also fix a subset Δ^F of Δ for which the corresponding tangency conditions happen at prescribed positions on the toric divisors. Finally, we fix a nonnegative integer n. Theorem 6 of [10] is a correspondence theorem between log Gromov–Witten invariants of X_Δ, counting curves in X_Δ satisfying the tangency constraints imposed by Δ and Δ^F, and passing through n points in general position, and refined counts of tropical curves in $\mathbb{R}^2$ satisfying the tropical analogue of these constraints.

To get Proposition 7.1, we take

$$\Delta = (w_1(k), \ldots, w_{s(k)}(k), k_w m_w), \quad \Delta^F = (w_1(k), \ldots, w_{s(k)}(k))$$

and $n = 0$. We then have $X_\Delta = \overline{Y}_m$ up to some toric blow-ups, which do not change the relevant log Gromov–Witten invariants by [6]. Using the notation of [10], we have $|\Delta| = s(k) + 1$, $|\Delta^F| = s(k)$ and $g^{\Delta^F}_{\Delta,n} = 0$. As the variable u keeping track of the genus in [10] is denoted by $\hbar$ in the present paper, we see that Theorem 6 of [10] reduces to Proposition 7.1. □

By comparison of the explicit formulas of Corollary 6.3 and Propositions 7.1 and 5.1, and using the relation

$$s(k) = \sum_{j=1}^{n} \sum_{\ell \geqslant 1} k_{\ell,j}$$

to collect the powers of i, we find exactly the formula given in Theorem 3.1 for the Hamiltonians of the quantum scattering diagram $S(\widehat{\mathfrak{D}}_m)$ in terms of the log Gromov–Witten invariants $N^{Y_m}_{g,p}$ of the log Calabi–Yau surface Y_m. This ends the proof of Theorem 3.1.

7.2 End of the proof of Theorem 3.2

The proof of Theorem 3.2 follows the one of Theorem 3.1, up to minor notational changes. The only needed serious modification is an orbifold version of the multicovering formula of Lemma 5.9. This is provided by Lemma 7.2 below.

We fix positive integers r and ℓ. Let $\mathbb{P}^1[r, 1]$ be the stacky projective line with a single orbifold point of isotropy group $\mathbb{Z}/r$ at 0. Let $\overline{M}_{g,\ell}(\mathbb{P}^1[r, 1]/\infty)$ be the moduli space of genus g orbifold stable maps to $\mathbb{P}^1[r, 1]$, relative to $\infty \in \mathbb{P}^1[r, \infty]$, of degree $r\ell$, with maximal tangency order $r\ell$ along ∞. This is a proper Deligne–Mumford stack of virtual dimension $2g - 1 + \ell$, admitting a virtual fundamental class

$$[\overline{M}_{g,\ell}(\mathbb{P}^1[r, 1]/\infty)]^{\mathrm{virt}} \in A_{2g-1+\ell}\big(\overline{M}_{g,\ell}(\mathbb{P}^1[r, 1]/\infty), \mathbb{Q}\big).$$

Let $\mathcal{O}_{\mathbb{P}^1[r,1]}(-[0]/(\mathbb{Z}/r))$ be the orbifold line bundle on $\mathbb{P}^1[r, 1]$ of degree $-1/r$. Denoting by $\pi \colon \mathcal{C} \to \overline{M}_{g,\ell}(\mathbb{P}^1[r, 1]/\infty)$ the universal source curve and $f \colon \mathcal{C} \to \mathbb{P}^1[r, 1]$ the universal map, we define

$$N^\ell_{g,r} := \int_{[\overline{M}_{g,\ell}(\mathbb{P}^1[r,1]/\infty)]^{\mathrm{virt}}} (-1)^g \lambda_g \, e\big(R^1\pi_* \, f^*\big(\mathcal{O}_{\mathbb{P}^1[r,1]}(-[0]/(\mathbb{Z}/r))\big)\big),$$

where $e(-)$ is the Euler class.

Lemma 7.2 *For all positive integers r and ℓ, we have*

$$\sum_{g \geq 0} N_{g,r}^{\ell} \hbar^{2g-1} = \frac{(-1)^{\ell-1}}{\ell} \frac{1}{2 \sin(\frac{1}{2} r \ell \hbar)}.$$

Proof This is a higher-genus version of Proposition 5.7 of [31] and an orbifold version of Theorem 5.1 of [13]. Very similar localization computations of higher-genus orbifold Gromov–Witten invariants can be found in [36]. The main thing we need to explain is the replacement in the orbifold case for the Mumford relation $c(\mathbb{E})c(\mathbb{E}^{\vee}) = 1$ playing a key role in the proof of Theorem 5.1 of [13]. We will simply have to twist the usual Hodge theoretic argument of [58] by a local system.

We consider the action of $\mathbb{C}^*$ on $\mathbb{P}^1[r, 1]$ with tangent weights $[1/r, -1]$ at the fixed points $[0, \infty]$. We choose the equivariant lifts of

$$\mathcal{O}_{\mathbb{P}^1[r,1]}(-[0]/(\mathbb{Z}/r))$$

and $\mathcal{O}_{\mathbb{P}^1[r,1]}$ having fibers over the fixed points $[0, \infty]$ of weights $[-1/r, 0]$ and $[0, 0]$ respectively. For such choices, the argument given in the proof of Theorem 5.1 of [13] shows that only one graph Γ contributes to the $\mathbb{C}^*$–localization formula computing $N_{g,r}^l$. The graph Γ consists of a genus g vertex over 0, a unique edge of degree $r\ell$ and a degenerate genus 0 vertex over ∞.

The contribution of Γ is computed using the virtual localization formula of [27]. We assume that $g > 0$. The case $g = 0$ is simpler and treated in Proposition 5.7 of [31]. The corresponding $\mathbb{C}^*$–fixed locus is the fiber product

$$\overline{M}_{g,1}(B\mathbb{Z}/r) \times_{\overline{I}B\mathbb{Z}/r} B\mathbb{Z}/(rd),$$

where $\overline{M}_{g,1}(B\mathbb{Z}/r)$ is the moduli stack of 1–pointed (with a trivial stacky structure at the marked point) genus g orbifold stable maps to the classifying stack $B\mathbb{Z}/r$, $\overline{I}B\mathbb{Z}/r$ is the rigidified inertia stack of $B\mathbb{Z}/r$, and the classifying stack $B\mathbb{Z}/(rd)$ appears as moduli space of $\mathbb{C}^*$–invariant Galois covers $\mathbb{P}^1 \to \mathbb{P}^1[r, 1]$ of degree $r\ell$. This fibered product is a cover of $\overline{M}_{g,1}(B\mathbb{Z}/r)$ of degree $r/(r\ell)$.

Let $\pi_0: \mathcal{C}_0 \to \overline{M}_{g,1}(B\mathbb{Z}/r)$ be the universal source curve over $\overline{M}_{g,1}(B\mathbb{Z}/r)$. The data of an orbifold stable map $f_0: \mathcal{C}_0 \to B\mathbb{Z}/r$ is equivalent to the data of an (orbifold) $\mathbb{Z}/r$–local system L on $\mathcal{C}_0$. We denote by t the generator of the $\mathbb{C}^*$–equivariant cohomology of a point.

The computation of the inverse of the equivariant Euler class of the equivariant virtual bundle is done in Section 2.2 [36] and gives

$$e\left(R^1(\pi_0)_*\left(\mathcal{O}_{C_0} \otimes L \otimes \frac{t}{r}\right)\right) \frac{(r\ell)^\ell}{t^\ell \ell!} \frac{1}{t/(r\ell) - \psi} \left(\frac{r}{t}\right)^{\delta_{L,0}} \frac{t}{r},$$

where $\delta_{L,0} = 1$ if L is the trivial $\mathbb{Z}/r$–local system and 0 otherwise. The vector bundle

$$R^1(\pi_0)_*\left(\mathcal{O}_{C_0} \otimes L \otimes \frac{t}{r}\right)$$

over $\overline{M}_{g,1}(B\mathbb{Z}/r)$ comes from the equivariant orbifold line bundle

$$T_{\mathbb{P}^1[r,1]}(-\infty)|_{[0]/(\mathbb{Z}/r)}$$

over $B\mathbb{Z}/r$, which is the restriction over $[0]/(\mathbb{Z}/r)$ of the degree $1/r$ orbifold line bundle $T_{\mathbb{P}^1[r,1]}(-\infty)$ over $\mathbb{P}^1[r,1]$.

The contribution of the integrand in the definition of $N_{g,r}^\ell$ is

$$(-1)^g \lambda_g e\left(R^1(\pi_0)_*\left(\mathcal{O}_{C_0} \otimes \left(L \otimes \frac{t}{r}\right)^\vee\right)\right) \left(-\frac{t}{r}\right)^{1-\delta_{R,0}} (-1)^{\ell-1} \frac{(\ell-1)!}{(r\ell)^{\ell-1}} t^{\ell-1}.$$

The vector bundle $R^1(\pi_0)_*(\mathcal{O}_{C_0} \otimes (L \otimes t/r)^\vee)$ over $\overline{M}_{g,1}(B\mathbb{Z}/r)$ comes from the equivariant orbifold line bundle $\mathcal{O}_{\mathbb{P}^1[r,1]}(-[0]/(\mathbb{Z}/r))|_{[0]/(\mathbb{Z}/r)}$ over $B\mathbb{Z}/r$, which is the restriction over $[0]/(\mathbb{Z}/r)$ of the degree $-1/r$ orbifold line bundle $\mathcal{O}_{\mathbb{P}^1[r,1]}(-[0]/(\mathbb{Z}/r))$ over $\mathbb{P}^1[r,1]$.

By Serre duality, we have

$$R^1(\pi_0)_*\left(\mathcal{O}_{C_0} \otimes \left(L \otimes \frac{t}{r}\right)^\vee\right) = \left((\pi_0)_*\left(\omega_{\pi_0} \otimes L \otimes \frac{t}{r}\right)\right)^\vee,$$

and so

$$e\left(R^1(\pi_0)_*\left(\mathcal{O}_{C_0} \otimes \left(L \otimes \frac{t}{r}\right)^\vee\right)\right) = (-1)^{\mathrm{rk}} e\left((\pi_0)_*\left(\omega_{\pi_0} \otimes L \otimes \frac{t}{r}\right)\right)$$

$$= (-1)^{\mathrm{rk}} \left(\frac{t}{r}\right)^{\mathrm{rk}} \sum_{j=0}^{\mathrm{rk}} \left(\frac{r}{t}\right)^j c_j((\pi_0)_*(\omega_{\pi_0} \otimes L))$$

$$= (-1)^{\mathrm{rk}} \left(\frac{t}{r}\right)^{\mathrm{rk}} c_{r/t}((\pi_0)_*(\omega_{\pi_0} \otimes L)),$$

where rk is the rank of $(\pi_0)_*(\omega_{\pi_0} \otimes L)$, a locally constant function on $\overline{M}_{g,1}(B\mathbb{Z}/r)$, equal to g on the component with L trivial and to $g-1$ on the components with L nontrivial, and where

$$c_x(E) := \sum_{j \geqslant 0} x^j c_j(E)$$

is the Chern polynomial of a vector bundle E. Similarly, we have

$$e\left(R^1(\pi_0)_*\left(\mathcal{O}_{\mathcal{C}_0} \otimes L \otimes \frac{t}{r}\right)\right) = \left(\frac{t}{r}\right)^{\mathrm{rk}} \sum_{j=0}^{\mathrm{rk}} \left(\frac{r}{t}\right)^j c_j(R^1(\pi_0)_*(\mathcal{O}_{\mathcal{C}_0} \otimes L))$$

$$= \left(\frac{t}{r}\right)^{\mathrm{rk}} c_{r/t}(R^1(\pi_0)_*(\mathcal{O}_{\mathcal{C}_0} \otimes L)).$$

We twist now the Hodge theoretic argument of [58] (see formulas (5.4) and (5.5)) (see also Proposition 3.2 of [12]) by the local system L. The complex

$$\omega_{\mathcal{C}_0}^\bullet \colon 0 \to \mathcal{O}_{\mathcal{C}_0} \xrightarrow{d} \omega_{\pi_0} \to 0,$$

twisted by L, gives rise to an exact sequence

$$0 \to (\pi_0)_*(\omega_{\pi_0} \otimes L) \to R^1(\pi_0)_*(\omega_{\mathcal{C}_0}^\bullet \otimes L) \to R^1(\pi_0)_*(\mathcal{O}_{\mathcal{C}_0} \otimes L) \to 0.$$

By Hodge theory, we have the Gauss–Manin connection on the restriction of

$$R^1(\pi_0)_*(\omega_{\mathcal{C}_0}^\bullet \otimes L)$$

to the open dense subset of $\overline{M}_{g,1}(B\mathbb{Z}/r)$ given by smooth curves, with regular singularities and nilpotent residue along the divisor of nodal curves. This is enough to imply

$$c_x(R^1(\pi_0)_*(\omega_{\mathcal{C}_0}^\bullet \otimes L)) = 1,$$

and so

$$c_x((\pi_0)_*(\omega_{\pi_0} \otimes L))c_x(R^1(\pi_0)_*(\mathcal{O}_{\mathcal{C}_0} \otimes L)) = 1.$$

Using this relation to simplify the above expressions, we obtain

$$N_{g,r}^\ell = \frac{r}{r\ell} \int_{\overline{M}_{g,1}(B\mathbb{Z}/r)} (-1)^{\ell-1}(-1)^{g+\mathrm{rk}+1-\delta_{L,0}} \left(\frac{t}{r}\right)^{2\,\mathrm{rk}-2\delta_{L,0}+1} \frac{\lambda_g}{t/(r\ell) - \psi}.$$

Using that $\mathrm{rk} = g - 1 + \delta_{R,0}$, this can be rewritten as

$$N_{g,r}^\ell = \int_{\overline{M}_{g,1}(B\mathbb{Z}/r)} \frac{(-1)^{\ell-1}}{\ell} \left(\frac{t}{r}\right)^{2g-1} \frac{\lambda_g}{(t/r\ell) - \psi}.$$

As the dimension of $\overline{M}_{g,1}(B\mathbb{Z}/r)$ is $3g - 2$, we have to extract the term proportional to ψ^{2g-2} and we obtain

$$N_{g,r}^\ell = \int_{\overline{M}_{g,1}(B\mathbb{Z}/r)} \frac{(-1)^{\ell-1}}{\ell} \ell^{2g-1} \lambda_g \psi^{2g-2}.$$

The integrand is now the pullback from the moduli space $\overline{M}_{g,1}$ of 1–pointed genus g stable maps. The forgetful map $\overline{M}_{g,1}(B\mathbb{Z}/r) \to \overline{M}_{g,n}$ has degree r^{2g-1}. Indeed,

there are r^{2g} $\mathbb{Z}/r$–local systems on a smooth genus g curve, each with a $\mathbb{Z}/r$ group of automorphisms. Therefore, we have

$$N_{g,r}^{\ell} = \frac{(-1)^{\ell-1}}{\ell}(r\ell)^{2g-1} \int_{\overline{M}_{g,1}} \lambda_g \psi^{2g-2},$$

and the result then follows, as in the proof of Theorem 5.1 of [13], from the Hodge integrals computations of [19]. □

8 Integrality results and conjectures

In Section 8.1, we state Conjecture 8.3, a log BPS integrality conjecture. In Section 8.2, we state Theorem 8.5, a precise version of Theorem 0.2 of the introduction, establishing the validity of Conjecture 8.3 for $(Y_{\mathfrak{m}}, \partial Y_{\mathfrak{m}})$. The proof of Theorem 0.2 takes Sections 8.3 and 8.4. In Section 8.5, we describe an explicit connection with refined Donaldson–Thomas theory of quivers. Finally, in Section 8.6, we discuss del Pezzo surfaces with a smooth anticanonical divisor and we formulate Conjecture 8.16, a precise form of Conjecture 0.3 of the introduction.

8.1 Integrality conjecture

We formulate a higher-genus analogue of the log BPS integrality conjecture, Conjecture 6.2 of [31]. We start by formulating a rationality conjecture, Conjecture 8.1, before stating the integrality conjecture, Conjecture 8.3.

Let Y be a smooth projective surface and let $\partial Y \subset Y$ be a reduced normal crossing effective divisor. We endow Y with the divisorial log structure defined by ∂Y and we obtain a smooth log scheme. Following Section 6.1 of [31], we say that $(Y, \partial Y)$ is log Calabi–Yau with respect to some nonzero class $\beta \in H_2(Y, \mathbb{Z})$ if $\beta \cdot (\partial Y) = \beta \cdot (-K_Y)$.

Some basic examples are:

- For every $\mathfrak{m} = (m_1, \ldots, m_n)$, an n–tuple of primitive nonzero vectors in $M = \mathbb{Z}^2$, the pair $(Y_{\mathfrak{m}}, \partial Y_{\mathfrak{m}})$ defined in Section 2.1 provides an example. Strictly speaking, $Y_{\mathfrak{m}}$ is not smooth, but log smooth. We can either make $Y_{\mathfrak{m}}$ smooth by toric blow-ups or allow log smooth objects in the definition of log Calabi–Yau. Then $(Y_{\mathfrak{m}}, \partial Y_{\mathfrak{m}})$ is log Calabi–Yau with respect to every class $\beta \in H_2(Y_{\mathfrak{m}}, \mathbb{Z})$ and so in particular with respect to the classes $\beta_p \in H_2(Y_{\mathfrak{m}}, \mathbb{Z})$ defined in Section 2.2.

- If Y is a del Pezzo surface and ∂Y a smooth anticanonical divisor, then $(Y, \partial Y)$ is log Calabi–Yau with respect to every class $\beta \in H_2(Y, \mathbb{Z})$.

We fix $(Y, \partial Y)$ log Calabi–Yau with respect to some $\beta \in H_2(Y, \mathbb{Z})$ such that $\beta \cdot (\partial Y) \neq 0$. Let $\overline{M}_{g,\beta}(Y/\partial Y)$ be the moduli space of genus g stable log maps to Y of class β and full tangency of order $\beta \cdot (\partial Y)$ at a single unspecified point of D. This is a proper Deligne–Mumford stack coming with a g–dimensional virtual fundamental class

$$[\overline{M}_{g,\beta}(Y/\partial Y)]^{\text{virt}}.$$

We define

$$N_{g,\beta}^{Y/\partial Y} := \int_{[\overline{M}_{g,\beta}(Y/\partial Y)]^{\text{virt}}} (-1)^g \lambda_g.$$

If $(Y, \partial Y)$ is of the form $(Y_{\mathrm{m}}, \partial Y_{\mathrm{m}})$ and β is of the form β_p — see Section 2.2 — then we have $N_{g,\beta}^{Y/\partial Y} = N_{g,p}^{Y_{\mathrm{m}}}$, where $N_{g,p}^{Y_{\mathrm{m}}}$ are the invariants defined in Section 2.3.

We can now formulate the rationality conjecture.

Conjecture 8.1 *Let $(Y, \partial Y)$ be a log Calabi–Yau pair with respect to some class $\beta \in H_2(Y, \mathbb{Z})$ such that $\beta \cdot (\partial Y) \neq 0$. Then there exists a rational function*

$$\overline{\Omega}_\beta(q^{\frac{1}{2}}) \in \mathbb{Q}(q^{\pm\frac{1}{2}})$$

such that we have the equality of power series in $\hbar$,

$$\overline{\Omega}_\beta(q^{\frac{1}{2}}) = (-1)^{\beta \cdot (\partial Y)+1}\big(2\sin(\tfrac{1}{2}\hbar)\big)\bigg(\sum_{g \geqslant 0} N_{g,\beta}^{Y/\partial Y}\hbar^{2g-1}\bigg),$$

after the change of variables $q = e^{i\hbar}$.

Note that such a rational function $\overline{\Omega}_\beta(q^{\frac{1}{2}})$ is unique if it exists. If the rational function $\overline{\Omega}_\beta(q^{\frac{1}{2}})$ exists, then it is invariant under $q^{\frac{1}{2}} \mapsto q^{-\frac{1}{2}}$, because its power series expansion in $\hbar$ after $q = e^{i\hbar}$ has real coefficients. Given the 3–dimensional interpretation of the invariants $N_{g,\beta}^{Y,\partial Y}$ given in Section 2.4, Conjecture 8.1 should follow from a log version of the MNOP conjectures [53; 54], once an appropriate theory of log Donaldson–Thomas invariants is developed. If ∂Y is smooth, then Conjecture 8.1 indeed follows from the relative MNOP conjectures; see Section 3.3 of [54].

Let $(Y, \partial Y)$ be a log Calabi–Yau pair with respect to some primitive class $\beta \in H_2(Y, \mathbb{Z})$ such that $\beta \cdot (\partial Y) \neq 0$. Let us assume that Conjecture 8.1 is true for all classes that are multiples of β. So, for every $n \geqslant 1$, we have a rational function $\overline{\Omega}_{n\beta}(q^{\frac{1}{2}}) \in \mathbb{Q}(q^{\pm\frac{1}{2}})$. We define a collection of rational functions $\Omega_{n\beta}(q^{\frac{1}{2}}) \in \mathbb{Q}(q^{\pm\frac{1}{2}})$ for $n \geqslant 1$, invariant under $q^{\frac{1}{2}} \mapsto q^{-\frac{1}{2}}$, by the relations

$$\overline{\Omega}_{n\beta}(q^{\frac{1}{2}}) = \sum_{\ell \mid n} \frac{1}{\ell}\frac{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}{q^{\frac{1}{2}\ell}-q^{-\frac{1}{2}\ell}}\Omega_{(n/\ell)\beta}(q^{\frac{1}{2}\ell}).$$

Lemma 8.2 *These relations have a unique solution, given by*

$$\Omega_{n\beta}(q^{\frac{1}{2}}) = \sum_{\ell \mid n} \frac{\mu(\ell)}{\ell} \frac{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}{q^{\frac{1}{2}\ell} - q^{-\frac{1}{2}\ell}} \overline{\Omega}_{(n/\ell)\beta}(q^{\frac{1}{2}\ell}),$$

where μ is the Möbius function.

Proof Indeed, we have

$$\sum_{\ell \mid n} \frac{1}{\ell} \frac{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}{q^{\frac{1}{2}\ell} - q^{-\frac{1}{2}\ell}} \left(\sum_{\ell' \mid (n/\ell)} \frac{\mu(\ell')}{\ell'} \frac{q^{\frac{1}{2}\ell} - q^{-\frac{1}{2}\ell}}{q^{\frac{1}{2}\ell\ell'} - q^{-\frac{1}{2}\ell\ell'}} \overline{\Omega}_{(n/(\ell\ell'))\beta}(q^{\frac{1}{2}\ell\ell'}) \right)$$

$$= \sum_{\ell \mid n} \sum_{\ell' \mid (n/\ell)} \frac{\mu(\ell')}{\ell\ell'} \frac{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}{q^{\frac{1}{2}\ell\ell'} - q^{-\frac{1}{2}\ell\ell'}} \overline{\Omega}_{(n/(\ell\ell'))\beta}(q^{\frac{1}{2}\ell\ell'})$$

$$= \sum_{m \mid n} \frac{1}{m} \frac{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}{q^{\frac{1}{2}m} - q^{-\frac{1}{2}m}} \Omega_{(n/m)\beta}(q^{\frac{1}{2}m}) \left(\sum_{\ell' \mid m} \mu(\ell') \right)$$

$$= \sum_{m \mid n} \frac{1}{m} \frac{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}{q^{\frac{1}{2}m} - q^{-\frac{1}{2}m}} \Omega_{(n/m)\beta}(q^{\frac{1}{2}m}) \delta_{m,1} = \overline{\Omega}_{n\beta}(q^{\frac{1}{2}}),$$

where we used the Möbius inversion formula $\sum_{\ell' \mid m} \mu(\ell') = \delta_{m,1}$. $\square$

We can now formulate the integrality conjecture.

Conjecture 8.3 *Let $(Y, \partial Y)$ be a log Calabi–Yau pair with respect to some class $\beta \in H_2(Y, \mathbb{Z})$ such that $\beta \cdot (\partial Y) \neq 0$ and such that the rationality conjecture Conjecture 8.1 is true for all multiples of β, so that the rational functions $\Omega_{n\beta}(q^{\frac{1}{2}}) \in \mathbb{Q}(q^{\pm\frac{1}{2}})$ are defined. Then, in fact, for every $n \geqslant 1$, $\Omega_{n\beta}(q^{\frac{1}{2}})$ is a Laurent polynomial in $q^{\frac{1}{2}}$ with integer coefficients, ie*

$$\Omega_{n\beta}(q^{\frac{1}{2}}) \in \mathbb{Z}[q^{\pm\frac{1}{2}}],$$

invariant under $q^{\frac{1}{2}} \mapsto q^{-\frac{1}{2}}$.

In Section 9, we explain why this integrality conjecture can be interpreted in some cases as a mathematically well-defined example of the general integrality for open Gromov–Witten invariants in Calabi–Yau 3–folds predicted by Ooguri and Vafa [61]. In particular, the log BPS invariants $\Omega_\beta(q^{\frac{1}{2}})$ should be thought of as examples of Ooguri–Vafa/open BPS invariants.

In the classical limit $\hbar \to 0$, the integrality of $\Omega_{n\beta} := \Omega_{n\beta}(q^{\frac{1}{2}} = 1)$ is equivalent to Conjecture 6.2 of [31]. If $\beta^2 = -1$, $\beta \cdot (\partial Y) = 1$, and the class β only contains a smooth rational curve, then it follows from the proof of Lemma 5.9 that Conjecture 8.3

is true. More precisely, we have

$$\overline{\Omega}_{n\beta}(q^{\frac{1}{2}}) = \frac{1}{n} \frac{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}{q^{\frac{1}{2}n} - q^{-\frac{1}{2}n}}$$

for every $n \geq 1$, and so $\Omega_\beta(q^{\frac{1}{2}}) = 1$ and $\Omega_{n\beta}(q^{\frac{1}{2}}) = 0$ for $n > 1$.

8.2 Integrality result

Lemma 8.4 *For every $\mathfrak{m} = (m_1, \ldots, m_n)$, an n–tuple of primitive nonzero vectors in $M = \mathbb{Z}^2$, and $p \in P = \mathbb{N}^n$, the rationality conjecture Conjecture 8.1 is true for the log Calabi–Yau pair $(Y_\mathfrak{m}, \partial Y_\mathfrak{m})$ with respect to the curve class $\beta_p \in H_2(Y, \mathbb{Z})$.*

Proof This follows from Theorem 3.1, expressing the generating series of invariants $N_{g,p}^{Y_\mathfrak{m}}$ as a Hamiltonian $\hat{H}_m$ attached to some ray of the quantum scattering diagram $S(\hat{\mathfrak{D}}_\mathfrak{m})$, and from Proposition 6.2, giving a formula for $\hat{H}_\mathfrak{m}$ whose coefficients are manifestly in $\mathbb{Q}[q^{\pm\frac{1}{2}}][(1 - q^\ell)^{-1}]_{\ell \geq 1}$.

Alternatively, one could argue that, because the initial quantum scattering diagram $\hat{\mathfrak{D}}_\mathfrak{m}$ is defined over $\mathbb{Q}[q^{\pm\frac{1}{2}}][(1 - q^\ell)^{-1}]_{\ell \geq 1}$, the resulting consistent quantum scattering diagram $S(\hat{\mathfrak{D}}_\mathfrak{m})$ is also defined over $\mathbb{Q}[q^{\pm\frac{1}{2}}][(1 - q^\ell)^{-1}]_{\ell \geq 1}$ and so Lemma 8.4 follows directly from Theorem 3.1.   □

By Lemma 8.4, we have rational functions

$$\overline{\Omega}_p^{Y_\mathfrak{m}}(q^{\frac{1}{2}}) \in \mathbb{Q}(q^{\pm\frac{1}{2}})$$

such that

$$\overline{\Omega}_p^{Y_\mathfrak{m}}(q^{\frac{1}{2}}) = (-1)^{\ell_p + 1} \left(2 \sin\left(\tfrac{1}{2}\hbar\right) \right) \left(\sum_{g \geq 0} N_{g,p}^{Y_\mathfrak{m}} \hbar^{2g-1} \right),$$

as power series in $\hbar$, after the change of variables $q = e^{i\hbar}$. Note that we used the fact that $\beta_p.(\partial Y_\mathfrak{m}) = \ell_p$.

The following result is, after Theorem 3.1, the second main result of this paper. It is a precise form of Theorem 0.2 in the introduction.

Theorem 8.5 *For every $\mathfrak{m} = (m_1, \ldots, m_n)$, an n–tuple of primitive nonzero vectors in $M = \mathbb{Z}^2$, and $p \in P = \mathbb{N}^n$, the integrality conjecture Conjecture 8.3 is true for the log Calabi–Yau pair $(Y_\mathfrak{m}, \partial Y_\mathfrak{m})$ with respect to the class $\beta_p \in H_2(Y_m, \mathbb{Z})$. In other words, there exists $\Omega_p^{Y_\mathfrak{m}}(q^{\frac{1}{2}}) \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$ such that*

$$\overline{\Omega}_p^{Y_\mathfrak{m}}(q^{\frac{1}{2}}) = \sum_{p = \ell p'} \frac{1}{\ell} \frac{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}{q^{\frac{1}{2}\ell} - q^{-\frac{1}{2}\ell}} \Omega_{p'}^{Y_\mathfrak{m}}(q^{\frac{1}{2}\ell}).$$

The proof of Theorem 8.5 takes Sections 8.3 and 8.4.

8.3 Quadratic refinement

According to Theorem 3.1, generating series of the log Gromov–Witten invariants $N_{g,p}^{Y_m}$ are Hamiltonians attached to the rays of some quantum scattering diagram $S(\widehat{\mathfrak{D}}_m)$. Our integrality result, Theorem 8.5, will follow from a general integrality result for scattering diagrams. Our main input, the integrality result of [44], is phrased in terms of twisted quantum scattering diagrams, ie scattering diagrams valued in automorphisms of twisted quantum tori. The comparison with quiver DT invariants, done in Section 8.5, also requires us to consider twisted quantum scattering diagrams.

In this section, we explain how to compare the quantum scattering diagram $S(\widehat{\mathfrak{D}}_m)$ with a twisted quantum scattering diagram $S(\widehat{\mathfrak{D}}_m^{tw})$. This comparison requires the notion of quadratic refinement. A short and to the point discussion by Neitzke can be found in [59]. Some related discussion can be found in Appendix A of [49].

We start with $P = \mathbb{N}^n = \bigoplus_{j=1}^n \mathbb{N} e_j$. For $p = (p_1, \ldots, p_n) \in P = \mathbb{N}^n$, we define $\operatorname{ord}(p) := \sum_{j=1}^n p_j$. An n–tuple $\mathfrak{m} = (m_1, \ldots, m_n)$ of primitive nonzero vectors in $M = \mathbb{Z}^2$ naturally defines an additive map

$$r \colon P \to M, \quad e_j \mapsto m_j.$$

For every $\mathbb{Z}[q^{\pm\frac{1}{2}}]$–algebra A, we denote by $\widehat{T}_{P,tw}^A$ the noncommutative "space" whose algebra of functions is the algebra $\Gamma(\mathcal{O}_{\widehat{T}_{P,tw}^A})$ given by $A[[P]]$, power series in $\hat{x}^p$ for $p \in P$, with coefficients in A, with the product defined by

$$\hat{x}^p \cdot \hat{x}^{p'} = (-1)^{\langle r(p), r(p') \rangle} q^{\frac{1}{2} \langle r(p), r(p') \rangle} \hat{x}^{p+p'}.$$

The main difference with respect to the formalism of Section 1 is the twist by the extra sign $(-1)^{\langle r(p), r(p') \rangle}$.

We will use $A = \mathbb{Z}[[q^{\pm\frac{1}{2}}]]$, $\mathbb{Z}((q^{\frac{1}{2}}))$ and $\mathbb{Q}((q^{\frac{1}{2}}))$. We have obviously the inclusions

$$\Gamma(\mathcal{O}_{\widehat{T}_{P,tw}^{\mathbb{Z}[[q^{\pm 1/2}]]}}) \subset \Gamma(\mathcal{O}_{\widehat{T}_{P,tw}^{\mathbb{Z}((q^{1/2}))}}) \subset \Gamma(\mathcal{O}_{\widehat{T}_{P,tw}^{\mathbb{Q}((q^{1/2}))}}).$$

Every

$$\widehat{H}^{tw} = \sum_{p \in P} \widehat{H}_p^{tw} \hat{x}^p \in \Gamma(\mathcal{O}_{\widehat{T}_{P,tw}^{\mathbb{Q}((q^{1/2}))}})$$

such that $\widehat{H}^{tw} = 0 \mod P$ defines via conjugation by $\exp(\widehat{H}^{tw})$ an automorphism

$$\widehat{\Phi}_{\widehat{H}^{tw}}^{tw} = \operatorname{Ad}_{\exp(\widehat{H}^{tw})} = \exp(\widehat{H}^{tw})(-)\exp(-\widehat{H}^{tw})$$

of $\Gamma(\mathcal{O}_{\widehat{T}_{P,tw}^{\mathbb{Q}((q^{1/2}))}})$.

Definition 8.6 A twisted quantum scattering diagram $\widehat{\mathfrak{D}}^{\text{tw}}$ over $(r\colon P \to M)$ is a set of rays $\mathfrak{d}$ in $M_\mathbb{R}$, equipped with elements

$$\widehat{H}_\mathfrak{d}^{\text{tw}} \in \Gamma(\mathcal{O}_{\widehat{T}_{P,\text{tw}}^{\mathbb{Q}((q^{1/2}))}})$$

such that:

- There exists a primitive $p \in P$ (which is necessarily unique) such that $\widehat{H}_\mathfrak{d}^{\text{tw}} \in \hat{x}^p \mathbb{Q}((q^{\frac{1}{2}}))[\![\hat{x}^p]\!]$ and either $r(p) \in -\mathbb{N}_{\geqslant 1} m_\mathfrak{d}$ or $r(p) \in \mathbb{N}_{\geqslant 1} m_\mathfrak{d}$. We say that the ray $(\mathfrak{d}, \widehat{H}_\mathfrak{d}^{\text{tw}})$ is ingoing if $r(p) \in -\mathbb{N}_{\geqslant 1} m_\mathfrak{d}$ and outgoing if $r(p) \in \mathbb{N}_{\geqslant 1} m_\mathfrak{d}$. We call p the P–direction of the ray $(\mathfrak{d}, \widehat{H}_\mathfrak{d}^{\text{tw}})$.

- For every $\ell \geqslant 0$, there are only finitely many rays $\mathfrak{d}$ of P–direction p satisfying $\mathrm{ord}(p) \leqslant \ell$.

Using the automorphisms $\widehat{\Phi}_{\widehat{H}^{\text{tw}}}$, we define as in Section 1.4 the notion of consistent twisted quantum scattering diagram and one can prove that every twisted quantum scattering diagram $\mathfrak{D}^{\text{tw}}$ can be canonically completed by adding only outgoing rays to form a consistent twisted quantum scattering diagram $S(\mathfrak{D}^{\text{tw}})$.

The following lemma will give us a way to go back and forth between quantum scattering diagrams and twisted quantum scattering diagrams.

Lemma 8.7 The map $\sigma_M\colon M \to \{\pm 1\}$, defined by $\sigma_M(0) = 1$ and $\sigma_M(m) = (-1)^{|m|}$ for $m \in M$ nonzero, where $|m|$ is the divisibility of m in M, is a quadratic refinement of

$$\textstyle\bigwedge^2 M \to \{\pm 1\}, \quad (m_1, m_2) \mapsto (-1)^{\langle m_1, m_2 \rangle},$$

ie we have

$$\sigma_M(m_1 + m_2) = (-1)^{\langle m_1, m_2 \rangle} \sigma_M(m_1) \sigma_M(m_2)$$

for every $m_1, m_2 \in M$. It is the unique quadratic refinement such that $\sigma_M(m) = -1$ for every $m \in M$ primitive.

Proof We fix a basis of M and we denote by $m = (m^x, m^y)$ the coordinates of some $m \in M$ in this basis. We define $\sigma_M'\colon M \to \{\pm 1\}$ by

$$\sigma_M'(m) = (-1)^{m^x m^y + m^x + m^y}.$$

It is easy to check that σ_M' is a quadratic refinement of $(-1)^{\langle -, - \rangle}$: the parity of

$$(m_1^x + m_2^x)(m_1^y + m_2^y) + m_1^x + m_2^x + m_1^y + m_2^y$$

differs from the parity of

$$m_1^x m_1^y + m_1^x + m_1^y + m_2^x m_2^y + m_2^x + m_2^y$$

by $m_1^x m_2^y + m_2^x m_1^y$, which has the parity of $\langle m_1, m_2 \rangle$.

If $m \in M$ is primitive, then (m^x, m^y) is equal to $(1,0)$, $(0,1)$ or $(1,1)$ modulo two, and in all three cases, we obtain $\sigma'_M(m) = -1$. Combined with the fact that σ'_M is a quadratic refinement, this implies that, for every $m \in M$, we have $\sigma'_M(m) = (-1)^{|m|}$, ie $\sigma'_M = \sigma_M$. In particular, σ_M is a quadratic refinement and σ'_M is independent of the choice of basis.

The uniqueness statement follows from the fact that a quadratic refinement is determined by its value on a basis of M. □

Let $\widehat{\mathfrak{D}}^{\mathrm{tw}}_{\mathrm{m}}$ be the twisted quantum scattering diagram consisting of incoming rays $(\mathfrak{d}_j, \widehat{H}^{\mathrm{tw}}_{\mathfrak{d}_j})$ for $1 \leqslant j \leqslant n$, where

$$\mathfrak{d}_j = -\mathbb{R}_{\geqslant 0}\, m_j,$$

and

$$\widehat{H}^{\mathrm{tw}}_{\mathfrak{d}_j} = -\sum_{\ell \geqslant 1} \frac{1}{\ell} \frac{1}{q^{\frac{1}{2}\ell} - q^{-\frac{1}{2}\ell}} \hat{x}^{\ell e_j} \in \Gamma(\mathcal{O}_{\widehat{T}^{\mathbb{Q}((q^{1/2}))}_{P,\mathrm{tw}}}),$$

where we consider

$$\frac{1}{q^{\frac{1}{2}\ell} - q^{-\frac{1}{2}\ell}} = -q^{\frac{1}{2}\ell} \sum_{k \geqslant 0} q^{k\ell} \in \mathbb{Q}((q^{\frac{1}{2}})).$$

Let $S(\widehat{\mathfrak{D}}^{\mathrm{tw}}_{\mathrm{m}})$ be the corresponding consistent twisted quantum scattering diagram obtained by adding only outgoing rays.

Define $\sigma_P \colon P \to \{\pm 1\}$ by $\sigma_P := \sigma_M \circ r$. It follows from [Lemma 8.7](#) that σ_P is a quadratic refinement and so

$$\left(\prod_{j=1}^n t_j^{p_j} \right) \hat{z}^{r(p)} \mapsto \sigma_P(p)\hat{x}^p$$

is an algebra isomorphism between quantum tori and twisted quantum tori. Using this isomorphism, we can construct a twisted quantum scattering diagram $S(\widehat{\mathfrak{D}}_{\mathrm{m}})^{\mathrm{tw}}$ from the quantum scattering diagram $\widehat{\mathfrak{D}}_{\mathrm{m}}$.

The incoming rays of $S(\widehat{\mathfrak{D}}_{\mathrm{m}})^{\mathrm{tw}}$ are $(\mathfrak{d}_j, \widehat{H}^{\mathrm{tw}}_{\mathfrak{d}_j})$ for $1 \leqslant j \leqslant n$, where $\mathfrak{d}_j = -\mathbb{R}_{\geqslant 0}\, m_j$ and

$$\widehat{H}^{\mathrm{tw}}_{\mathfrak{d}_j} = -\sum_{\ell \geqslant 1} \frac{1}{\ell} \frac{1}{q^{\frac{1}{2}\ell} - q^{-\frac{1}{2}\ell}} \hat{x}^{\ell e_j}.$$

The outgoing rays of $S(\widehat{\mathfrak{D}}_{\mathrm{m}})^{\mathrm{tw}}$ are $(\mathbb{R}_{\geqslant 0}\, m, \widehat{H}^{\mathrm{tw}}_m)$, where

$$\widehat{H}^{\mathrm{tw}}_m = -\sum_{p \in P_m} \frac{\overline{\Omega}_p(q^{\frac{1}{2}})}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \hat{x}^p = -\sum_{p \in P_m} \sum_{p = \ell p'} \frac{1}{\ell} \frac{1}{q^{\frac{1}{2}\ell} - q^{-\frac{1}{2}\ell}} \Omega^{Y_m}_{p'}(q^{\frac{1}{2}\ell}) \hat{x}^p.$$

Lemma 8.8 $$S(\widehat{\mathfrak{D}}_{\mathrm{m}}^{\mathrm{tw}}) = S(\widehat{\mathfrak{D}}_{\mathrm{m}})^{\mathrm{tw}}.$$

Proof   As $\left(\prod_{j=1}^{n} t_j^{p_j}\right)\hat{z}^{r(p)} \mapsto \sigma_P(p)\hat{x}^p$ is an algebra isomorphism, the twisted quantum scattering diagram $S(\widehat{\mathfrak{D}}_{\mathrm{m}})^{\mathrm{tw}}$ is consistent and so the result follows from the uniqueness of the consistent completion of twisted quantum scattering diagrams.   □

8.4   Proof of the integrality theorem

We give below the proof of Theorem 8.5. It is a combination of the scattering arguments of Appendix C3 of [29] with the formalism of quantum admissible series of [44]. Because of the structure of the induction argument, we will in fact prove a more general statement than Theorem 8.5. We will prove, as Proposition 8.11, that the consistent completion of any (twisted) quantum scattering with incoming rays equipped with Hamiltonians satisfying some BPS integrality condition has outgoing rays equipped with Hamiltonians satisfying the BPS integrality condition.

We fix $p \in P$ primitive. Consider

$$\widehat{H}^{\mathrm{tw}} = \sum_{\ell \geqslant 1} \widehat{H}_\ell^{\mathrm{tw}}(q^{\frac{1}{2}})\hat{x}^{\ell p} \in \hat{x}^p \mathbb{Q}((q^{\frac{1}{2}}))[\![\hat{x}^p]\!].$$

We define

$$\overline{\Omega}_\ell(q^{\frac{1}{2}}) := -(q^{\frac{1}{2}} - q^{-\frac{1}{2}})\widehat{H}_\ell^{\mathrm{tw}}(q^{\frac{1}{2}}) \in \mathbb{Q}((q^{\frac{1}{2}})),$$

and

$$\Omega_\ell(q^{\frac{1}{2}}) := \sum_{\ell' | \ell} \frac{\mu(\ell')}{\ell'} \frac{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}{q^{\frac{1}{2}\ell} - q^{-\frac{1}{2}\ell}} \overline{\Omega}_{\ell/\ell'}(q^{\frac{1}{2}\ell}) \in \mathbb{Q}((q^{\frac{1}{2}})).$$

It follows from Lemma 8.2 that we have

$$\widehat{H}^{\mathrm{tw}} = -\sum_{n \geqslant 1} \sum_{\ell \geqslant 1} \frac{1}{\ell} \frac{\Omega_n(q^{\frac{1}{2}\ell})}{q^{\frac{1}{2}\ell} - q^{-\frac{1}{2}\ell}} \hat{x}^{\ell n p}.$$

Definition 8.9   We say that $\widehat{H}^{\mathrm{tw}} \in \hat{x}^p \mathbb{Q}((q^{\frac{1}{2}}))[\![\hat{x}^p]\!]$ satisfies the BPS integrality condition if each corresponding $\Omega_\ell(q^{\frac{1}{2}}) \in \mathbb{Q}((q^{\frac{1}{2}}))$ is in fact a Laurent polynomial with integer coefficients, ie $\Omega_\ell(q^{\frac{1}{2}}) \in \mathbb{Z}[q^{\frac{1}{2}}]$.

The function $\widehat{H}^{\mathrm{tw}}$ satisfies the BPS integrality condition if and only if $\exp(\widehat{H}^{\mathrm{tw}})$ is admissible in the sense of Section 6 of [44].

It follows from the product form of the quantum dilogarithm, as recalled in Section 3.1, that if $\widehat{H}^{\mathrm{tw}}$ satisfies the BPS integrality condition, then $\widehat{\Phi}_{\widehat{H}^{\mathrm{tw}}}^{\mathrm{tw}}$ preserves the subring $\Gamma(\mathcal{O}_{\widehat{T}_{P,\mathrm{tw}}^{\mathbb{Z}[\![q^{1/2}]\!]}})$ of $\Gamma(\mathcal{O}_{\widehat{T}_{P,\mathrm{tw}}^{\mathbb{Q}((q^{1/2}))}})$. We refer to the subgroup of automorphisms

of $\Gamma(\mathcal{O}_{\hat{T}_{P,\mathrm{tw}}^{\mathbb{Z}[\![q^{1/2}]\!]}})$ generated by automorphisms of the form $\hat{\Phi}_{\hat{H}^{\mathrm{tw}}}^{\mathrm{tw}}$ with $\hat{H}^{\mathrm{tw}}$ satisfying the BPS integrality condition as the BPS quantum tropical vertex group; this subgroup is called the quantum tropical vertex group in [44].

We fix a choice of twisted quantum scattering diagram in each equivalence class by considering to be distinct rays with different P–directions and by merging rays with coinciding supports and with the same P–direction.

Recall that for $p = (p_1, \ldots, p_n) \in P = \mathbb{N}^n$, we define $\mathrm{ord}(p) := \sum_{j=1}^{n} p_j$. This is simply the total degree of the monomial in several variables $\prod_{j=1}^{n} t_j^{p_j}$.

Lemma 8.10 *Let N, n and n_I be positive integers. Let $r\colon P = \mathbb{N}^n \to M$ be an additive map. Let $(p^1, \ldots, p^{n_I})$ be an n_I–tuple of primitive vectors in P. Let $\hat{\mathfrak{D}}^{\mathrm{tw}}$ be a twisted quantum scattering diagram over $(r\colon P \to M)$, consisting of incoming rays $(\mathfrak{d}_j, \hat{H}_{\mathfrak{d}_j}^{\mathrm{tw}})$ for $1 \leqslant j \leqslant n_I$ with $\mathfrak{d}_j = -\mathbb{R}_{\geqslant 0} \, r(p^j)$ and $\hat{H}_{\mathfrak{d}_j}^{\mathrm{tw}} \in \hat{x}^{p^j} \mathbb{Q}((q^{\frac{1}{2}}))[\![\hat{x}^{p^j}]\!]$ satisfying the BPS integrality condition. Then every outgoing ray $(\mathfrak{d}, \hat{H}_{\mathfrak{d}}^{\mathrm{tw}})$ of the consistent twisted quantum scattering diagram $S(\hat{\mathfrak{D}}^{\mathrm{tw}})$ whose P–direction p satisfies $\mathrm{ord}(p) \leqslant N$ is such that $\hat{H}_{\mathfrak{d}}^{\mathrm{tw}} \in \hat{x}^p \mathbb{Q}((q^{\frac{1}{2}}))[\![\hat{x}^p]\!]$ satisfies the BPS integrality condition.*

Proof We prove the result by induction on N. The result is obviously true for $N = 1$: the only outgoing rays with P–direction p satisfying $\mathrm{ord}(p) = 1$ are obtained by straight propagation of the initial rays and so satisfy the BPS integrality condition if it is the case for the initial rays.

Let $N > 1$ be an integer. We assume by induction that Lemma 8.10 is true for all integers strictly less than N and we want to prove it for N. As in Step III of Appendix C3 of [29], up to applying the perturbation trick, which consists of separating transversally and generically the initial rays with the same support and then looking at the new local scatterings, we can assume that at most two initial rays have order one.

We now use the change of monoid trick, as in Steps I and IV of Appendix C3 of [29]. Write $P' = \bigoplus_{j=1}^{n_I} \mathbb{N} e_j'$ and

$$r'\colon P' \to M, \quad e_j' \mapsto r'(e_j) := r(p^j).$$

Let $\hat{\mathfrak{D}}^{\mathrm{tw}'}$ be the twisted quantum scattering diagram over $(r'\colon P' \to M)$ obtained by replacing $\hat{x}^{p^j}$ by $\hat{x}^{e_j'}$ in $\hat{H}_{\mathfrak{d}_j}^{\mathrm{tw}}$. Write

$$u\colon P' \to P, \quad e_j' \mapsto p^j.$$

Let $(\mathfrak{d}, \hat{H}_{\mathfrak{d}}^{\mathrm{tw}})$ be an outgoing ray of $S(\widehat{\mathfrak{D}}^{\mathrm{tw}})$, whose P–direction p satisfies $\mathrm{ord}(p) = N$. Then $(\mathfrak{d}, \hat{H}_{\mathfrak{d}}^{\mathrm{tw}})$ is the sum of images by u of outgoing rays of $S(\widehat{\mathfrak{D}}^{\mathrm{tw}'})$, of P'–direction mapping to p by u. Let $(\mathfrak{d}', \hat{H}_{\mathfrak{d}'}^{\mathrm{tw}})$ be such an outgoing ray of $S(\widehat{\mathfrak{D}}^{\mathrm{tw}'})$.

Writing $p' = \sum_{j=1}^{n_I} p'_j e'_j$, where $(p'_1, \ldots, p'_n) \in \mathbb{N}^{n_I}$, we have

$$\mathrm{ord}(p') = \mathrm{ord}\left(\sum_{j=1}^{n_I} p'_j e'_j \right) = \sum_{j=1}^{n_I} p'_j,$$

whereas

$$\mathrm{ord}(p) = \mathrm{ord}\left(\sum_{j=1}^{n_I} p'_j p^j \right) = \sum_{j=1}^{n_I} p'_j \, \mathrm{ord}(p^j).$$

If only two p'_j are nonzero, then the ray $(\mathfrak{d}', \hat{H}_{\mathfrak{d}'}^{\mathrm{tw}})$ belongs to a twisted quantum scattering diagram with two incoming rays and so its BPS integrality follows from Proposition 9 of [44]. If more than two of the p'_j are nonzero, then at least one of the p^j with $n_j \neq 0$ satisfies $\mathrm{ord}(p^j) \geq 2$ and so $\mathrm{ord}(p') < \mathrm{ord}(p)$. The BPS integrality of the ray $(\mathfrak{d}', \hat{H}_{\mathfrak{d}'}^{\mathrm{tw}})$ then follows by the induction hypothesis. $\quad\square$

Proposition 8.11 *Let n_I be a positive integer, and let $(p^1, \ldots, p^{n_I})$ be an n_I–tuple of primitive vectors in P. Let $\widehat{\mathfrak{D}}^{\mathrm{tw}}$ be a twisted quantum scattering diagram over $(r \colon P \to M)$, consisting of incoming rays $(\mathfrak{d}_j, \hat{H}_{\mathfrak{d}_j}^{\mathrm{tw}})$ for $1 \leq j \leq n_I$ with $\mathfrak{d}_j = -\mathbb{R}_{\geq 0} \, r(p^j)$ and $\hat{H}_{\mathfrak{d}_j}^{\mathrm{tw}} \in \hat{x}^{p^j} \mathbb{Q}((q^{\frac{1}{2}}))[\![\hat{x}^{p^j}]\!]$ satisfying the BPS integrality condition. Then the consistent twisted quantum scattering diagram $S(\widehat{\mathfrak{D}}^{\mathrm{tw}})$ is such that for every outgoing ray $(\mathfrak{d}, \hat{H}_{\mathfrak{d}}^{\mathrm{tw}})$ of P–direction $p \in P$, we have that $\hat{H}_{\mathfrak{d}}^{\mathrm{tw}} \in \hat{x}^p \mathbb{Q}((q^{\frac{1}{2}}))[\![\hat{x}^p]\!]$ satisfies the BPS integrality condition.*

Proof This follows immediately from Lemma 8.10. $\quad\square$

We can now finish the proof of Theorem 8.5. By Theorem 3.1 and Lemma 8.8, it is enough to show that the outgoing rays of the twisted quantum scattering diagram $S(\widehat{\mathfrak{D}}_{\mathfrak{m}}^{\mathrm{tw}})$ satisfy the BPS integrality condition. As the initial rays of $S(\widehat{\mathfrak{D}}_{\mathfrak{m}}^{\mathrm{tw}})$ satisfy the BPS integrality condition, the result follows from Proposition 8.11.

8.5 Integrality and quiver DT invariants

We refer to [43; 37; 63; 64; 56] for the Donaldson–Thomas (DT) theory of quivers.

For every $\mathfrak{m} = (m_1, \ldots, m_n)$, an n–tuple of primitive nonzero vectors in $M = \mathbb{Z}^2$, we define a quiver $Q_{\mathfrak{m}}$, with set of vertices $\{1, 2, \ldots, n\}$ and, for every $1 \leq j, k \leq n$,

$\langle m_j, m_k \rangle_+ := \max(\langle m_j, m_k \rangle, 0)$ arrows from the vertex j to the vertex k. We identify $P = \bigoplus_{j=1}^n \mathbb{N} e_j$ with the set of dimension vectors for the quiver $Q_{\mathfrak{m}}$.

Lemma 8.12 *The quiver $Q_{\mathfrak{m}}$ is acyclic, ie does not contain any oriented cycle, if and only if the n vectors $m_1, \ldots, m_n$ are all contained in a closed half-plane of $M_{\mathbb{R}} = \mathbb{R}^2$.*

Proof The quiver $Q_{\mathfrak{m}}$ contains an arrow from the vertex i to the vertex j if and only if (m_i, m_j) is an oriented basis of $\mathbb{R}^2$. □

Let us assume that the quiver $Q_{\mathfrak{m}}$ is acyclic. Every $\theta = (\theta_j)_{1 \le j \le n} \in \mathbb{Z}^n$ defines a notion of stability for representations of $Q_{\mathfrak{m}}$. For every $p \in P$, we then have a projective variety $M_p^{\theta-\mathrm{ss}}$, moduli space of θ–semistable representations of $Q_{\mathfrak{m}}$ of dimension p, containing the open smooth locus $M_p^{\theta-\mathrm{st}}$ of θ–stable representations. Let $\iota: M_p^{\theta-\mathrm{st}} \hookrightarrow M_p^{\theta-\mathrm{ss}}$ be the natural inclusion. The main result of [56] is that the Laurent polynomials

$$\Omega_p^{Q_{\mathfrak{m}}, \theta}(q^{\frac{1}{2}}) := (-1)^{\dim M_p^{\theta-\mathrm{ss}}} q^{-\frac{1}{2} \dim M_p^{\theta-\mathrm{ss}}} \sum_{j=0}^{\dim M_p^{\theta-\mathrm{st}}} (\dim H^{2j}(M_p^{\theta-\mathrm{ss}}, \iota_{!*} \mathbb{Q})) q^j$$

$$\in (-1)^{\dim M_p^{\theta-\mathrm{ss}}} q^{-\frac{1}{2} \dim M_p^{\theta-\mathrm{ss}}} \mathbb{N}[q]$$

are the refined DT invariants of $Q_{\mathfrak{m}}$ for the stability θ. In the above formula, $\iota_{!*}$ is the intermediate extension functor defined by ι and so $\iota_{!*} \mathbb{Q}$ is a perverse sheaf on $M_p^{\theta-\mathrm{ss}}$.

As $Q_{\mathfrak{m}}$ is acyclic, we can assume, up to relabeling $m_1, \ldots, m_n$, that $\langle m_j, m_k \rangle \ge 0$ if $j \le k$. If $\theta_1 < \theta_2 < \cdots < \theta_n$, then $\Omega_{e_j}^{Q_{\mathfrak{m}}, \theta}(q^{\frac{1}{2}}) = 1$ for all $1 \le j \le n$, and $\Omega_p^{Q_{\mathfrak{m}}, \theta}(q^{\frac{1}{2}}) = 0$ for $p \in P - \{e_1, \ldots, e_n\}$. We call such θ a trivial stability condition.

If $\theta_1 > \theta_2 > \cdots > \theta_n$, we call θ a maximally nontrivial stability condition. We simply write $\Omega_p^{Q_{\mathfrak{m}}}(q^{\frac{1}{2}})$ for $\Omega_p^{Q_{\mathfrak{m}}, \theta}(q^{\frac{1}{2}})$ when is θ a maximally nontrivial stability condition.

Theorem 8.13 *For every $\mathfrak{m} = (m_1, \ldots, m_n)$ such that the quiver $Q_{\mathfrak{m}}$ is acyclic and every $p \in P = \mathbb{N}^n$, we have the equality*

$$\Omega_p^{Q_{\mathfrak{m}}}(q^{\frac{1}{2}}) = \Omega_p^{Y_{\mathfrak{m}}}(q^{\frac{1}{2}})$$

between the refined DT invariant $\Omega_p^{Q_{\mathfrak{m}}}(q^{\frac{1}{2}})$ of the quiver $Q_{\mathfrak{m}}$ and the log BPS invariant $\Omega_p^{Y_{\mathfrak{m}}}(q^{\frac{1}{2}})$ of the log Calabi–Yau surface $Y_{\mathfrak{m}}$.

Proof According to Theorem 3.1 and Lemma 8.8, the log Gromov–Witten invariants of $Y_{\mathfrak{m}}$ are computed by the twisted quantum scattering diagram $S(\widehat{\mathfrak{D}}_{\mathfrak{m}}^{\mathrm{tw}})$. On the

other hand, the consistency of $S(\widehat{\mathfrak{D}}_{\mathfrak{m}}^{\mathrm{tw}})$ is equivalent to the wall-crossing formula for refined DT invariants of $Q_{\mathfrak{m}}$ between the trivial and the maximally nontrivial stability conditions [63]. $\qquad\square$

In the limit $q^{\frac{1}{2}} \to 1$, if $Q_{\mathfrak{m}}$ is complete bipartite, then Theorem 8.13 reduces to the Gromov–Witten/Kronecker correspondence of [30; 66; 65].

Theorem 8.13 can be viewed as a concrete example of equality between open BPS invariants and DT invariants of quivers. The expectation for this kind of relation goes back at least to [15], as reviewed in Section 9. Related recent stories include [46; 47], where some knot invariants, which via some string theoretic duality should be examples of open BPS invariants, are identified with some quiver DT invariants, and [72], where a precise correspondence between open BPS invariants of a certain class of Lagrangian submanifolds in $\mathbb{C}^3$ and some DT invariants of quivers is conjectured.

Theorem 8.13 gives a different proof of Theorem 8.5 when $Q_{\mathfrak{m}}$ is acyclic. When $Q_{\mathfrak{m}}$ is not acyclic, it is unclear a priori how to relate the log BPS invariants $\Omega_p^{Y_{\mathfrak{m}}}(q^{\frac{1}{2}})$ to some DT quiver theory. In the physics language, one should remove the contributions of nontrivial single-centered (pure Higgs) indices (see [51] and follow-ups). It is still an open question to define mathematically the corresponding operation in DT quiver theory. The fact that the integrality given by Theorem 8.5 holds even if $Q_{\mathfrak{m}}$ is not acyclic is probably additional evidence that it should be possible.

When $Q_{\mathfrak{m}}$ is acyclic, Theorem 8.13 gives a positivity result for the log BPS invariants $\Omega_p^{Y_{\mathfrak{m}}}(q^{\frac{1}{2}})$. It is unclear how to prove a similar positivity result if $Q_{\mathfrak{m}}$ is not acyclic.

We finish this section with a remark about signs. The definition of $\Omega_p^{Y_{\mathfrak{m}}}(q^{\frac{1}{2}})$ given in Section 8.2 includes a global sign $(-1)^{\ell_p - 1} = (-1)^{\beta_p \cdot (\partial Y_{\mathfrak{m}}) - 1}$, whereas the formula given above for $\Omega_p^{Q_{\mathfrak{m}}}(q^{\frac{1}{2}})$ includes a global sign $(-1)^{\dim M_p^{\theta-\mathrm{ss}}}$. Using that $\beta_p \cdot (\partial Y_{\mathfrak{m}})$ and β_p^2 have the same parity by Riemann–Roch on $Y_{\mathfrak{m}}$, the following result gives a direct proof that these two signs are identical.

Lemma 8.14 *For every $p \in P$, we have*
$$\dim M_p^{\theta-\mathrm{ss}} = \beta_p^2 + 1.$$

Proof We write $p = \sum_{j=1}^n p_j e_j \in P$. By standard quiver theory, we have
$$\dim M_p^{\theta-\mathrm{ss}} = \sum_{j=1}^n \sum_{k=1}^n \langle m_j, m_k \rangle + p_j p_k - \sum_{j=1}^n p_j^2 + 1.$$

By definition (Section 2.2) we have

$$\beta_p = \nu^* \beta - \sum_{j=1}^{n} p_j E_j,$$

where $\nu \colon Y_{\mathrm{m}} \to \overline{Y}_{\mathrm{m}}$ is the blow-up morphism and $\beta \in H_2(\overline{Y}_{\mathrm{m}}, \mathbb{Z})$ is defined by certain intersection numbers. It follows that

$$\beta_p^2 = \beta^2 - \sum_{j=1}^{n} p_j^2.$$

From the intersection numbers defining β, we see that the convex polygon dual to β is obtained by successively adding the vectors $p_j m_j$ and $\ell_p m_p$, in the order given by the counterclockwise ordering of the m_j and m_p given by their argument. By standard toric geometry, β^2 is given by twice the area of the dual polygon and so we have

$$\beta^2 = \sum_{j=1}^{n} \sum_{k=1}^{n} \langle m_j, m_k \rangle + p_j \, p_k.$$

It follows that

$$\beta_p^2 = \sum_{j=1}^{n} \sum_{k=1}^{n} \langle m_j, m_k \rangle + p_j \, p_k - \sum_{j=1}^{n} p_j^2 = \dim M_p^{\theta-\mathrm{ss}} - 1. \qquad \square$$

8.6 del Pezzo surfaces

In this section, we study the conjectures of Section 8.1 in the case where Y is a del Pezzo surface S and ∂Y is a smooth anticanonical divisor E of Y. In particular, E is a smooth genus 1 curve. We formulate Conjecture 8.16, a precise form of Conjecture 0.3 of the introduction.

Lemma 8.15 *Let S be a del Pezzo surface, and let E be a smooth anticanonical divisor of S. Then, for every $\beta \in H_2(Y, \mathbb{Z})$, the rationality conjecture Conjecture 8.1 is true for the log Calabi–Yau pair (S, E) with respect to the curve class β.*

Proof As in Section 2.4, the invariants $N_{g,\beta}^{S/E}$ can be written as equivariant Gromov–Witten invariants of the 3–fold $S \times \mathbb{C}$ relative to the divisor $E \times \mathbb{C}$. The rationality result then follows from the Gromov–Witten/stable pairs correspondence for the relative 3–fold geometry $S \times \mathbb{C}/E \times \mathbb{C}$.

This case of the Gromov–Witten/stable pairs correspondence can be proved following Section 5.3 of [55]. This involves considering the degeneration of $S \times \mathbb{C}$ to the

normal cone of $E \times \mathbb{C}$. Let N be the normal bundle to E in S. The degeneration formula expresses equivariant Gromov–Witten/stable pairs theories of $S \times \mathbb{C}$, without insertions, in terms of the relative equivariant Gromov–Witten/stable pairs theories, without insertions, of $S \times \mathbb{C}/E \times \mathbb{C}$ and $\mathbb{P}(N \oplus \mathcal{O}_E) \times \mathbb{C}/E \times \mathbb{C}$.

The 3–fold $S \times \mathbb{C}$ is deformation equivalent to a toric 3–fold. Indeed, a del Pezzo surface is deformation equivalent to a (not necessarily del Pezzo) toric surface: if S is a blow-up of $\mathbb{P}^2$ in n points, then S is deformation equivalent to a surface obtained by n successive toric blow-ups of $\mathbb{P}^2$. Therefore, the Gromov–Witten/stable pairs correspondence for $S \times \mathbb{C}$, without insertions, follows from Section 5.1 of [55].

The equivariant Gromov–Witten/stable pairs theory of $\mathbb{P}(N \oplus \mathcal{O}_E) \times \mathbb{C}/E \times \mathbb{C}$ coincides with the nonequivariant Gromov–Witten theory of $\mathbb{P}(N \oplus \mathcal{O}_E) \times E/E \times E$. The 3–fold $\mathbb{P}(N \oplus \mathcal{O}_E) \times E$ is a $\mathbb{P}^1$–bundle over $E \times E$ and we are considering curves of degree 0 over the second E factor. As $E \times E$ is holomorphic symplectic, the Gromov–Witten/stable pairs theories vanish unless the curve class has also degree 0 over the first E factor. The Gromov–Witten/stable pairs correspondence for

$$\mathbb{P}(N \oplus \mathcal{O}_E) \times E/E \times E,$$

without insertions, thus follows from the Gromov–Witten/stable pairs correspondence, without insertions, for local curves.

It follows from Proposition 6 of [62] that the degeneration formula can be inverted to imply the Gromov–Witten/stable pairs correspondence, without insertions, for $S \times \mathbb{C}/E \times \mathbb{C}$. □

By Lemma 8.15, we have rational functions

$$\overline{\Omega}_\beta^{S/E}(q^{\frac{1}{2}}) \in \mathbb{Q}(q^{\pm\frac{1}{2}})$$

such that

$$\overline{\Omega}_\beta^{S/E}(q^{\frac{1}{2}}) = (-1)^{\beta \cdot E + 1} \big(2 \sin\big(\tfrac{1}{2}\hbar\big)\big)\bigg(\sum_{g \geq 0} N_{g,\beta}^{S/E} \hbar^{2g-1}\bigg),$$

as power series in $\hbar$, after the change of variables $q = e^{i\hbar}$.

We define

$$\Omega_\beta^{S/E}(q^{\frac{1}{2}}) = \sum_{\beta = \ell\beta'} \frac{\mu(\ell)}{\ell} \frac{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}{q^{\frac{1}{2}\ell} - q^{-\frac{1}{2}\ell}} \overline{\Omega}_{\beta'}(q^{\frac{1}{2}\ell}) \in \mathbb{Q}(q^{\frac{1}{2}}).$$

According to Conjecture 8.3, one should have $\Omega_\beta^{S/E}(q^{\frac{1}{2}}) \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$.

Let M_β be the moduli space of dimension-one stable sheaves on S of class $\beta \in H_2(S, \mathbb{Z})$ and Euler characteristic 1. This is a smooth projective variety of dimension $\beta^2 + 1$. Let

$$\chi_q(M_\beta) := q^{-\frac{1}{2}(\beta^2+1)} \sum_{j,k=0}^{\beta^2+1} (-1)^{j+k} h^{j,k}(M_\beta) q^j \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$$

be the normalized Hirzebruch genus of M_β, where $h^{j,k}$ are the Hodge numbers. It follows from Theorem 2 of [52], following [18] and [7], that $h^{j,k}(M_\beta) = 0$ if $j \neq k$. In particular, $\chi_q(M_\beta)$ coincides with the normalized Poincaré polynomial of M_β.

Conjecture 8.16 $\Omega_\beta^{S/E}(q^{\frac{1}{2}}) = (-1)^{\beta^2+1}(\beta \cdot E)\chi_q(M_\beta)$.

Note that we have $\beta^2 = \beta \cdot E \mod 2$ by the Riemann–Roch theorem. In the limit $q^{\frac{1}{2}} \to 1$, Conjecture 8.16 reduces to

$$N_{0,\beta}^{S/E} = (-1)^{\beta \cdot E-1} \sum_{\beta=\ell\beta'} (-1)^{(\beta')^2+1} \frac{(\beta' \cdot E)}{\ell^2} e(M_{\beta'})$$

$$= (-1)^{\beta \cdot E-1}(\beta \cdot E) \sum_{\beta=\ell\beta'} \frac{1}{\ell^3}(-1)^{(\beta')^2+1} e(M_{\beta'}),$$

which is a known result. Indeed, by an application of the degeneration formula originally due to Graber and Hassett and generalized in [24], we have

$$N_{0,\beta}^{S/E} = (-1)^{\beta \cdot E+1}(\beta \cdot E)N_{0,\beta}^X,$$

where X is the local Calabi–Yau 3–fold given by the total space of the canonical line bundle K_S of S, and $N_{0,\beta}^X$ is the genus 0, class β, Gromov–Witten invariant of X. So the previous formula is equivalent to

$$N_{0,\beta}^X = \sum_{\beta=\ell\beta'} \frac{1}{\ell^3}(-1)^{(\beta')^2+1} e(M_{\beta'}),$$

which is exactly the Katz conjecture [39, Conjecture 2.3] for X. As X is deformation equivalent to a toric Calabi–Yau 3–fold, the Katz conjecture for X follows from the combination of the Gromov–Witten/stable pairs correspondence [55, Section 5.1], the integrality result of [41] and Theorem 6.4 of [69].

The right-hand side $(-1)^{\beta^2+1}\chi_q(M_\beta)$ should be thought of as a refined DT invariant of X, counting dimension-one sheaves. From this point of view, Conjecture 8.16 is an equality between a log BPS invariant on one side and a refined DT invariant on the other

side, in a way completely parallel to Theorem 8.13. Further conceptual evidence for Conjecture 8.16 and a further refinement of Conjecture 8.16 will be presented elsewhere.

9 Relation with Cecotti–Vafa

In this last section, we make no claim of mathematical results or mathematical precision. We briefly explain how the main results of this paper are related to some previous expectations in the theoretical physics literature.

In [15], Cecotti and Vafa have given a physical derivation of the fact that the refined BPS indices of a $\mathcal{N} = 2$ 4d quantum field theory admitting a Seiberg–Witten curve satisfy the refined Kontsevich–Soibelman wall-crossing formula. To make a connection with Theorem 3.1, we focus on only one part of the argument, establishing the relation between open Gromov–Witten invariants and the wall-crossing formula via Chern–Simons theory. In particular, we do not discuss the application to the BPS spectrum of $\mathcal{N} = 2$ 4d quantum field theories, which would be related to our Section 8.5 on quiver DT invariants.

9.1 Summary of the Cecotti–Vafa argument

Let U be a noncompact hyperkähler manifold, (I, J, K) be a quaternionic triple of compatible complex structures, $(\omega_I, \omega_J, \omega_K)$ be the corresponding triple of real symplectic forms and $(\Omega_I, \Omega_J, \Omega_K)$ be the corresponding triple of holomorphic symplectic forms. In [15], Cecotti and Vafa consider $U = \mathbb{C}^2$ but the generalization to an arbitrary hyperkähler surface is clear and is considered for example in [14] (in particular Appendix B).

Let $\Sigma \subset U$ be an I–holomorphic Lagrangian subvariety of U, ie a submanifold such that $\Omega_I|_\Sigma = 0$. It is a complex subvariety for the complex structure I and a real Lagrangian for any of the real symplectic forms $(\cos\theta)\omega_J + (\sin\theta)\omega_K$ for $\theta \in \mathbb{R}$. There is in fact a twistor sphere J_ζ, where $\zeta \in \mathbb{P}^1$, of compatible complex structures such that $I = J_0$, $J = J_1$ and $K = J_i$.

Let X be the noncompact Calabi–Yau 3–fold of underlying real manifold $U \times \mathbb{C}^*$, equipped with a complex structure twisted in a twistorial way, ie such that the fiber over $\zeta \in \mathbb{C}^*$ is the complex variety (U, J_ζ). Consider $S^1 \subset \mathbb{C}^*$ and $L := \Sigma \times S^1 \subset X$.

We consider counts of holomorphic maps $(C, \partial C) \to (X, L)$ from an open Riemann surface C to X with boundary ∂C mapping to L. Usually, boundary conditions for

counts of open holomorphic curves are taken be Lagrangian submanifolds. In fact, L is not Lagrangian in X but only totally real. Combined with specific aspects of the twistorial geometry, it is probably enough to have well-defined open Gromov–Witten invariants. As suggested in [15], it would be interesting to clarify this point. We restrict ourselves to open Riemann surfaces with only one boundary component. Given a class $\beta \in H_2(X, L)$, let $N_{g,\beta} \in \mathbb{Q}$ be the "count" of holomorphic maps $\varphi: (C, \partial C) \to (X, L)$ with C a genus g Riemann surface with one boundary component and $[\varphi(C, \partial C)] = \beta$. We write

$$\partial \beta = [\partial C] \in H_1(L)$$

to denote the image of β by the natural boundary map $H_2(X, L) \to H_1(L)$. A holomorphic map $\varphi: (C, \partial C) \to (X, L)$ of class $\beta \in H_2(X, L)$ is a $J_{e^{i\theta}}$–holomorphic map to U, at a constant value $e^{i\theta} \in S^1$, where θ is the argument of $\int_\beta \Omega_I$.

According to Witten [70], one should encode these counts of holomorphic maps as deformations of Chern–Simons theory of gauge group $U(1)$ on L. The field of this theory is a $U(1)$ gauge field A and its action is

$$I_{\mathrm{CS}}(A) := \tfrac{1}{2} \int_L A \wedge dA.$$

According to Section 4.4 of [70], this Chern–Simons action is deformed by additional terms involving the counts of holomorphic maps:

$$I(A) = I_{\mathrm{CS}}(A) + \sum_\beta \sum_{g \geqslant 0} N_{g,\beta} \hbar^{2g} e^{-\int_\beta \omega} e^{\int_{\partial \beta} A}.$$

The partition function of the deformed theory can be written as a correlation function in Chern–Simons theory

$$Z = \int DA\, e^{i(I(A)/\hbar)}$$

$$= \left\langle \exp\left(i \sum_{\beta \in H_2(X,L)} \sum_{g \geqslant 0} N_{g,\beta} \hbar^{2g-1} e^{-\int_\beta \omega} e^{\int_{\partial \beta} A} \right) \right\rangle_{\mathrm{CS}}.$$

As $L = \Sigma \times S^1$, we can adopt a Hamiltonian description where S^1 plays the role of the time direction. The classical phase space of $U(1)$ Chern–Simons theory on $L = \Sigma \times S^1$ is the space of $U(1)$ flat connections on Σ. When Σ is a torus, the classical phase space is the dual torus T. For every $m \in H_1(L)$, the holonomy around m defined a function z^m on T, ie a classical observable,

$$z^m(A) := e^{\int_m A}.$$

The algebra structure is given by $z^m z^{m'} = z^{m+m'}$ and the Poisson structure by $\{z^m, z^{m'}\} = \langle m, m' \rangle z^{m+m'}$. The algebra of quantum observables is given by the noncommutative torus, $\hat{z}^m \hat{z}^{m'} = q^{\frac{1}{2}\langle m, m' \rangle} \hat{z}^{m+m'}$, where $q = e^{i\hbar}$. Writing $t^\beta = e^{-\int_\beta \omega}$, we obtain

$$Z = \mathrm{Tr}_{\mathcal{H}}\left(T \prod_{\beta \in H_2(X,L)} \mathrm{Ad}_{\exp\left(-i \sum_{g \geqslant 0} N_{g,\beta} \hbar^{2g-1} t^\beta \hat{z}^m\right)}\right),$$

where $\mathcal{H}$ is the Hilbert space of quantum Chern–Simons theory and where $T \prod_\beta$ is a time ordered product, with ordering according to the phase of $\int_\beta \Omega_I$.

The key physical input used by Cecotti and Vafa [15] is the continuity of the partition function Z as function of the position of L in X. It follows that the jump of the invariants $N_{g,\beta}$ under variation of L in X is controlled by the refined Kontsevich–Soibelman wall-crossing formula formulated in terms of products of automorphisms of the quantum torus.

9.2 Comparison with Theorem 3.1

Our main result, Theorem 3.1, expresses the log Gromov–Witten theory of a log Calabi–Yau surface $(Y_{\mathrm{m}}, \partial Y_{\mathrm{m}})$ in terms of a 2–dimensional Kontsevich–Soibelman scattering diagram. The complement $U_{\mathrm{m}} := Y_{\mathrm{m}} - \partial Y_{\mathrm{m}}$ is a noncompact holomorphic symplectic surface admitting a SYZ real Lagrangian torus fibration. In some cases, U_{m} admits a hyperkähler metric such that the original complex structure of U_{m} is the compatible complex structure J and such that the SYZ fibration becomes I–holomorphic Lagrangian. Typical examples include 2–dimensional Hitchin moduli spaces; see [9] for a nice review. In such cases, we can apply the Cecotti–Vafa story summarized above to $U := U_{\mathrm{m}}$, with Σ a torus fiber of the SYZ fibration.

The log Gromov–Witten invariants with insertion of a top lambda class $N_{g,\beta}$, introduced in Section 2, should be viewed as a rigorous definition of the open Gromov–Witten invariants in the twistorial geometry X, with boundary on a torus fiber Σ "near infinity". An early reference for the interpretation of some open Gromov–Witten invariants in terms of relative stable maps is [48]. The intuitive picture to have in mind is that an open Riemann surface with a boundary on a torus fiber very close to the divisor at infinity can be capped off by a holomorphic disc meeting the divisor at infinity in one point. This is in part justified by the 3–dimensional interpretation of the invariants $N_{g,\beta}^{Y_{\mathrm{m}}}$ given in Section 2.4 and in particular by Lemma 2.2.

Automorphisms of the quantum torus appearing in Section 9.1 coincide with the automorphisms of the quantum torus appearing in Theorem 3.1. It follows that Theorem 3.1 can be viewed as a mathematically rigorous check of the physical argument given by Cecotti and Vafa [15], based on the continuity of Chern–Simons correlation functions and on the connection predicted by Witten [70] between higher-genus open Gromov–Witten invariants and quantum Chern–Simons theory.

Finally, Ooguri and Vafa [61] have given a physical derivation of an integrality result for open Gromov–Witten invariants of Calabi–Yau 3–folds, parallel to the Gopakumar–Vafa [25; 26] integrality for closed Gromov–Witten invariants of Calabi–Yau 3–folds. Given the heuristic interpretation of the log Gromov–Witten invariants $N_{g,\beta}$ as open Gromov–Witten invariants, this integrality coincides with the integrality of Conjecture 8.3 and Theorem 8.5.

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