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Isotopies of surfaces in 4–manifolds via banded unlink diagrams

MARK C HUGHES
SEUNGWON KIM
MAGGIE MILLER

We study surfaces embedded in 4–manifolds. We give a complete set of moves relating banded unlink diagrams of isotopic surfaces in an arbitrary 4–manifold. This extends work of Swenton and Kearton–Kurlin in S^4. As an application, we show that bridge trisections of isotopic surfaces in a trisected 4–manifold are related by a sequence of perturbations and deperturbations, affirmatively proving a conjecture of Meier and Zupan. We also exhibit several isotopies of unit surfaces in $\mathbb{C}P^2$ (ie spheres in the generating homology class), proving that many explicit unit surfaces are isotopic to the standard $\mathbb{C}P^1$. This strengthens some previously known results about the Gluck twist in S^4, related to Kirby problem 4.23.

57K45; 57K40

1 Introduction

Knotted surfaces in 4–manifolds play an important role in smooth 4–dimensional topology, analogous to the part played by classical knots in 3–dimensional topology. Much like in the 3–dimensional case, there are a number of surgery operations and

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invariants for a smooth 4–manifold X^4 which are defined in terms of embedded surfaces inside X^4.

Because of their importance in 4–manifold topology, it is useful to have concrete ways of describing these embedded surfaces and their isotopies. When $X^4 = S^4$, there are several ways to describe embedded surfaces and isotopies between them. These include broken surface diagrams with Roseman moves [26]; motion picture presentations with movie moves — see Carter and Saito [1; 2] — and braid charts with chart moves — see Kamada [12; 13].

In this paper we consider two additional methods for describing surfaces in a 4–manifold. When the underlying 4–manifold is S^4, a complete set of moves to describe isotopies of these surfaces has already been established. We focus here on establishing complete sets of moves to describe surface isotopies in an arbitrary 4–manifold.

The first method we consider to describe a surface Σ in a 4–manifold X^4 is via *banded unlink diagrams*. When $X^4 = S^4$, this construction involves putting Σ into Morse position with respect to a standard height function h on S^4, and then encoding the index-0 and index-1 critical points of $h|_\Sigma$ as a classical unlink in S^3 with a collection of embedded bands attached (see Section 3 for a more detailed description). In [33], Yoshikawa presents a set of moves on banded unlink diagrams for surfaces in S^4 which are realizable by isotopies of the underlying surface, and asks if these moves are sufficient to relate banded unlink diagrams of any pair of isotopic surfaces. This question was affirmatively answered by Swenton [29], with an alternative proof being given by Kearton and Kurlin [16].

In this paper we study a generalization of banded unlink diagrams to embedded surfaces in an arbitrary 4–manifold X^4 equipped with a Morse function, where we encode the Morse function by a Kirby diagram $\mathcal{K}$. We describe a set of moves on banded unlinks, called *band moves*, which can be realized by isotopies of the underlying surface Σ. These consist of Yoshikawa's original moves, as well as additional moves which describe the interaction of the surface Σ with the handle structure on X^4. The main theorem we prove is the following:

Theorem 4.3 *Let X^4 be a smooth 4–manifold with Kirby diagram $\mathcal{K}$, and suppose that Σ and Σ' are embedded surfaces in X^4. Let $(\mathcal{K}, L, v)$ and $(\mathcal{K}, L', v')$ be banded unlink diagrams for Σ and Σ', respectively. Then Σ and Σ' are isotopic if and only if $(\mathcal{K}, L, v)$ can be transformed into $(\mathcal{K}, L', v')$ by a finite sequence of band moves.*

The second method we consider to represent an embedded surface Σ is by using a *bridge trisection* of Σ, which allows one to present Σ in terms of intersections with a given trisection of the ambient manifold X^4. Bridge trisections for surfaces in S^4 were introduced by Meier and Zupan in [21], where they provide a stabilization/destabilization move which they prove is sufficient to relate any two bridge trisections of isotopic surfaces. In [22] the same authors generalize this notion of bridge trisections to surfaces in an arbitrary 4–manifold X^4, and prove that every surface $\Sigma \subset X^4$ can be put into bridge trisected position with respect to any given trisection on X^4. They similarly define a stabilization/destabilization move and conjecture that these moves are sufficient to relate any two bridge trisections of isotopic surfaces in X^4. Using Theorem 4.3, we affirmatively answer this conjecture. We give the relevant definitions and some exposition on trisections and bridge trisections in Section 5.

Theorem 5.8 *Let S and S' be surfaces in bridge position with respect to a trisection $\mathcal{T}$ of a closed 4–manifold X^4. Suppose that S is isotopic to S'. Then S can be taken to S' by a sequence of perturbations and deperturbations, followed by a $\mathcal{T}$–regular isotopy.*

As a separate application of Theorem 4.3, we focus on the case of unit surfaces in $\mathbb{C}P^2$. By Melvin [23], the study of unit surfaces is relevant to understanding the Gluck twist surgery of [7]. Melvin showed that the Gluck twist on a sphere $S \subset S^4$ yields S^4 again if and only if there is a diffeomorphism from the pair $(\mathbb{C}P^2, S \# \mathbb{C}P^1)$ to the pair $(\mathbb{C}P^2, \mathbb{C}P^1)$. See Section 6, where we give the relevant definitions and exposition, for more detail.

Theorem 6.25 *Let $F = S \# \mathbb{C}P^1 \subset \mathbb{C}P^2$ be a genus-g unit surface knot, where $S \subset S^4$ is an orientable surface that is 0–concordant to a band-sum of twist-spun knots and unknotted surfaces. Then F is isotopic to $\mathbb{C}P^1 \# gT$, where $\mathbb{C}P^1 \# gT$ indicates the standard $\mathbb{C}P^1$ trivially stabilized g times.*

Outline In Section 2, we define horizontal–vertical position and some nice families of isotopies for surfaces in 4–manifolds. In Section 3, we define banded unlink diagrams. In Section 4, we show that any surface in a 4–manifold is described by a banded unlink diagram which is well defined up to a certain set of moves. In Section 5, we show that a bridge trisection of a surface in a trisected 4–manifold is unique up to perturbation. In Section 6, we consider many examples of surfaces in the generating homology class of $\mathbb{C}P^2$, and show explicitly that these examples are isotopic to $\mathbb{C}P^1$ (perhaps with trivial tubes attached if the original surface is of positive genus).

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2 Level sets and isotopies

2.1 Banded links

Our primary technique for studying embedded surfaces in a 4–manifold will be to arrange them so that their intersections with the level sets of a given Morse function are composed of disjoint unions of embedded disks and *banded links*.

More precisely, let M^3 be an oriented 3–manifold, and let $L \subset M$ be a link. A *band* b for the link L is the image of an embedding $\phi\colon I \times I \hookrightarrow M$, where $I = [-1, 1]$, and $b \cap L = \phi(\{-1, 1\} \times I)$. We call $\phi(I \times \{0\})$ the *core* of the band b. Let L_b be the link defined by

$$L_b = (L \setminus \phi(\{-1, 1\} \times I)) \cup \phi(I \times \{-1, 1\}).$$

Then we say that L_b is the result of performing *band surgery* to L along b. If v is a finite family of pairwise disjoint bands for L, then we will let L_v denote the link we obtain by performing band surgery along each of the bands in v. We say that L_v is the result of *resolving* the bands in v. The union of a link L and a family of disjoint bands for L is called a *banded link*. If L is an unlink, we call the union of L and a family of disjoint bands a *banded unlink*.

2.2 Horizontal and vertical sets

Now let X^4 denote a closed, oriented 4–manifold equipped with a self-indexing Morse function $h\colon X^4 \to [0, 4]$, where h has exactly one index-0 critical point and one index-4 critical point. We will write $\mathcal{K}$ to denote the Kirby diagram of X^4 induced by h (we explain this more precisely in Section 3.2).

In order to study $\Sigma \subset X^4$ via the level sets of h, it will be convenient to have a way of identifying subsets of distinct level sets $h^{-1}(t_1)$ and $h^{-1}(t_2)$. Suppose then that $t_1 \le t_2$, and let $x_1, \dots, x_p$ denote the critical points of h which satisfy $t_1 \le h(x_j) \le t_2$. Let X_{t_1,t_2} denote the complement in X^4 of the ascending and descending manifolds of the critical points $x_1, \dots, x_p$. Then the gradient flow of h defines a diffeomorphism $\rho_{t_1,t_2}\colon h^{-1}(t_1) \cap X_{t_1,t_2} \to h^{-1}(t_2) \cap X_{t_1,t_2}$.

Definition 2.1 We call ρ_{t_1,t_2} the *projection of* $h^{-1}(t_1)$ *to* $h^{-1}(t_2)$. Similarly, we call $\rho^{-1}_{t_1,t_2}$ the *projection of* $h^{-1}(t_2)$ *to* $h^{-1}(t_1)$, which we likewise denote by ρ_{t_2,t_1}.

Note that despite calling ρ_{t_1,t_2} the projection from $h^{-1}(t_1)$ to $h^{-1}(t_2)$, it is only defined on the complement of the ascending and descending manifolds of the critical points that sit between t_1 and t_2. These projection maps allow us to define local product structures away from the ascending and descending manifolds of the critical points of h.

Definition 2.2 Let W be a subset of X^4, and let J either be the closed interval $[t_1, t_2]$ or the open interval (t_1, t_2). Then we say that W is *vertical on the interval* J if $W \subset X_{t_1,t_2}$ and if $\rho_{t,t'}(h^{-1}(t) \cap W) = h^{-1}(t') \cap W$ for all $t, t' \in J$.

In Sections 3 and 4, we will construct isotopies of surfaces in X^4. In this paper, every isotopy of a surface will always extend to ambient isotopy. We generally write "$f\colon F \times I \to X^4 \times I$ is an isotopy" with the understanding that there is in fact a smooth family of diffeomorphisms $g_s\colon X^4 \to X^4$ with $g_0 = \mathrm{id}$ such that $g_s \circ \mathrm{pr}_{X^4} \circ f(F \times 0) = f(F \times s)$ for $s \in [0,1]$. Here we use $\mathrm{pr}_X\colon X^4 \times I \to X^4$ to denote projection to the first factor.

We consider a few special types of isotopy which behave well with respect to h.

Definition 2.3 Let $\Sigma \subset X^4$ be a smoothly embedded surface. Let $f\colon F \times I \to X^4 \times I$ be a smooth isotopy of Σ, so that $\Sigma = f(F \times \{0\})$. If the image of $\mathrm{pr}_X \circ f$ is disjoint from the critical points of h, then we say that f is *h–disjoint*.

We say that f is *horizontal* with respect to h if $h\big(\mathrm{pr}_X(f(x,s))\big)$ is independent of s for all $x \in F$. We say that f is *vertical* with respect to h if for each $x \in F$ the image of $\{x\} \times I$ under $\mathrm{pr}_X \circ f$ is contained in a single orbit of the flow of ∇h. Finally, we say that f is *h–regular* if for each $s \in I$, $h|_{\mathrm{pr}_X(F \times \{s\})}$ is Morse. (See Figure 1 for schematics of horizontal and vertical isotopies.)

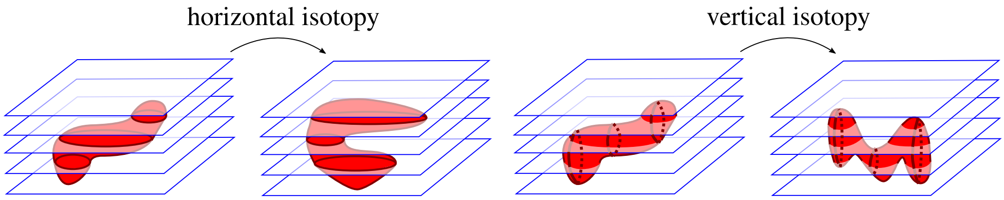

Figure 1: The horizontal planes represent $h^{-1}(t)$ for various values of t. Left: a horizontal isotopy of Σ preserves $h|_\Sigma$ pointwise. Right: a vertical isotopy of Σ moves each $x \in \Sigma$ within a single orbit flow of ∇h.

Intuitively, one should think of h as a height function whose level sets are horizontal. A horizontal isotopy of Σ moves $\Sigma \cap h^{-1}(t)$ within $h^{-1}(t)$, preserving $h|_\Sigma$. A vertical isotopy of Σ changes $h|_\Sigma$, but preserves the projection of Σ onto each level set $h^{-1}(t)$. We will usually just say that f is horizontal or vertical, omitting "with respect to h".

Note that a surface isotopy is generically h–disjoint. We will often explicitly require isotopies to be h–disjoint because we will be interested in how isotopy affects the projection of surfaces to $h^{-1}\left(\frac{3}{2}\right)$, which can be complicated when the isotopy takes the surface through a critical point. The definitions of horizontal and vertical isotopy naturally motivate a "nice" position of a surface embedded in X^4 if we are willing to allow the surface embedding to have corners.

Definition 2.4 Let $\Sigma \subset X^4$ be a PL embedded surface. We say that Σ is in *horizontal–vertical position* with respect to h if there exists a set $T = \{t_1, \dots, t_n\}$ disjoint from $\{0, 1, 2, 3, 4\}$, with $t_1 < t_2 < \cdots < t_n$, such that the following are true:

- For each $1 \le i \le n-1$ the surface Σ is vertical on the interval (t_i, t_{i+1}).
- For each $1 \le j \le n$ the intersection $\Sigma \cap h^{-1}(t_j)$ consists of the disjoint union of a (possibly empty) banded link and a (possibly empty) union of disjoint embedded disks.

In other words, Σ is vertical away from a finite number of nonsingular level sets $h^{-1}(t_i)$, while it intersects the $h^{-1}(t_i)$ in a collection of horizontal disks and a banded link. When h is clear, we may simply say that Σ is *horizontal–vertical*.

Note that from the definition of vertical, a horizontal–vertical surface must be disjoint from critical points of h. Note furthermore that, by an arbitrarily small perturbation in a neighborhood of the level sets $h^{-1}(t_1), \dots, h^{-1}(t_n)$, a horizontal–vertical surface Σ may be isotoped to a surface Σ' with $h|_{\Sigma'}$ Morse. This perturbation can be chosen so that each horizontal band of Σ gives rise to a nondegenerate saddle point in Σ', and each horizontal disk in Σ gives rise to a nondegenerate maximum or minimum point in Σ'. We will thus work largely with nonsmooth surfaces that are horizontal–vertical when constructing isotopies, with the understanding that they may be isotoped into smooth surfaces in Morse position as described above.

The following classical theorem states that arbitrary surfaces can be put into horizontal–vertical position. This is critical to the study of surfaces embedded in 4–manifolds. A proof for orientable surfaces is essentially contained in Section 2 of [15]; the nonorientable case is covered in [11].

Theorem 2.5 [11; 15] *Let $\Sigma \subset X^4$ be a smoothly embedded surface such that $h|_\Sigma$ is Morse and Σ is disjoint from the critical points of h. Then there is an h–disjoint and h–regular isotopy $f\colon F \times I \to X^4 \times I$ with $f(F \times \{0\}) = \Sigma$ such that f is a concatenation of horizontal and vertical isotopies, $f(F \times \{1\})$ is horizontal–vertical and $h|_{f(F\times\{s\})}$ is Morse for all $s \in [0,1)$.*

Both [11] and [15] consider only surfaces embedded in S^4 with the standard height function, but by applying the argument locally the theorem can be extended to surfaces in an arbitrary 4–manifold X^4 with self-indexing Morse function h. We will not cite this theorem directly, but will implicitly prove this result in Lemma 4.9.

3 Banded unlink diagrams for surfaces in 4–manifolds

3.1 Banded unlink position

For ease of notation we let $M_t = h^{-1}(t)$ denote the (possibly singular) 3–dimensional level set at height t for each $t \in \mathbb{R}$. By the projection maps ρ_{t_1,t_2} we may identify subsets of distinct level sets M_t, provided they avoid the appropriate ascending and descending manifolds. We will often make these identifications implicitly, so for example, we may think of a link L as living in both M_{t_1} and M_{t_2} when there is no risk of confusion.

Definition 3.1 We say that an embedded surface $\Sigma \subset X^4$ is in *banded unlink position* if

- $h(\Sigma) = \left[\frac{1}{2}, \frac{5}{2}\right]$,

- Σ is vertical on the intervals $\left(\frac{1}{2}, \frac{3}{2}\right)$ and $\left(\frac{3}{2}, \frac{5}{2}\right)$,
- $\Sigma \cap M_{3/2}$ is a banded unlink disjoint from the descending manifolds of index-2 critical points of h, and
- $\Sigma \cap M_{1/2}$ and $\Sigma \cap M_{5/2}$ are finite collections of disjoint embedded disks.

Letting t denote the height coordinate on X^4 induced by h, we can describe a surface in banded unlink position by a movie in t as follows. Starting at $t = 0$ and increasing, we first encounter a collection of minimal disks of Σ at height $t = \frac{1}{2}$. For $t \in \left(\frac{1}{2}, \frac{3}{2}\right)$, the intersection $\Sigma \cap M_t$ is an unlink in M_t, which we denote by L. The next feature we encounter are the index-1 critical points of X^4 at height $t = 1$, which completes the 1–skeleton of X^4. As we continue upwards, at height $t = \frac{3}{2}$ a family v of bands appear, attached to the link L. Passing $t = \frac{3}{2}$, the resulting level set of Σ becomes L_v, which is obtained from L by resolving the bands v. We then pass the index-2 critical points of X at height $t = 2$, before finally capping off the components of L_v with maximal disks at height $t = \frac{5}{2}$.

Note that the link L is necessarily an unlink in M_t for $t \in \left(\frac{1}{2}, \frac{3}{2}\right)$ (ie it bounds a collection of disjoint embedded disks), and L_v will be an unlink in M_t for $t \in \left(2, \frac{5}{2}\right)$.

3.2 Banded unlink diagrams

Surfaces in banded unlink position can be represented in terms of the associated Kirby diagram of X^4 via *banded unlink diagrams*. Suppose that the handle decomposition induced by h on X^4 is represented by the Kirby diagram $\mathcal{K} \subset S^3$. More precisely, $\mathcal{K}$ is a link $L_1 \sqcup L_2 \subset S^3$, where L_1 is an unlink, and each component of L_2 is labeled with an integer framing. The components of L_1 are each decorated with a dot to distinguish them from the components of L_2, and each indicates a 1–handle attached to the 0–handle B^4 along $\partial B^4 = S^3$ as usual (the meridians of L_1 are cores of the 1–handles in the handle decomposition of X^4). The labeled components of L_2 each represent the framed attaching circle of a 2–handle attached to X_1.

Given such a Kirby diagram $\mathcal{K} \subset S^3$ for X^4, the sphere S^3 can be identified with $M_{1/2}$, while the 3–manifold obtained by performing 0–surgery to S^3 along L_1 can be identified with the level set $M_{3/2}$. After performing this surgery, L_2 can again be thought of as a framed link in $M_{3/2}$, and we identify the result of performing Dehn surgery to $M_{3/2}$ along the components of L_2 (where the surgery coefficient of each

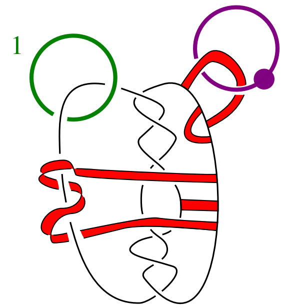

Figure 2: A banded unlink in Kirby diagram $\mathcal{K}$ describing a torus Σ smoothly embedded in $\mathbb{C}P^2 \# (S^1 \times S^3)$. The 2–component unlink in $E(\mathcal{K})$ bounds two minima of Σ. Resolving the unlink along the four bands yields the boundary of two maxima of Σ. Then $\chi(\Sigma) = 2 - 4 + 2 = 0$. One can check also that Σ is orientable.

component is specified by its framing) with $M_{5/2}$. Let $E(\mathcal{K})$ denote the complement $S^3 \setminus (\nu(L_1) \cup \nu(L_2))$ of a small tubular neighborhood of $\mathcal{K} = L_1 \sqcup L_2$ in S^3. Then, given a link $L \subset E(\mathcal{K})$, we may think of L as describing links in $M_{1/2}$, $M_{3/2}$ and $M_{5/2}$ in the obvious way.

A *banded unlink diagram* in the Kirby diagram $\mathcal{K}$ is a triple $(\mathcal{K}, L, v)$, where $L \subset E(\mathcal{K})$ is a link and v is a finite family of *disjoint* bands for L in $E(\mathcal{K})$, such that L bounds a family of pairwise disjoint embedded disks in $M_{1/2}$, and L_v bounds a family of pairwise disjoint embedded disks in $M_{5/2}$. See Figure 2 for an example of a banded unlink diagram.

A banded unlink diagram describes an embedded surface Σ in banded unlink position as follows. We first note that we can identify $E(\mathcal{K})$ with a subset of $M_{3/2}$ in a natural way. When fixing this identification, note that the intersection of $M_{3/2}$ with the descending manifolds (cores) of the 2–handles of X^4 can be thought of as the attaching circles of the 2–handles. Hence, as our banded link $L \cup v$ sits in the complement of a tubular neighborhood of the attaching circles $L_2 \subset S^3$, it can be identified with a subset of $M_{3/2}$, which we denote by $L' \cup v'$, that misses the descending manifolds of the 2–handles of X^4.

Now, as L' is an unlink, we can apply a horizontal isotopy in $M_{3/2}$ if necessary so that L' also avoids the ascending manifolds of the 1–handles of X^4. We can thus extend L' vertically downwards from $M_{3/2}$ to $M_{1/2}$, where it can be capped off by a family of disjoint embedded disks in $M_{1/2}$. Similarly, we can extend the surgered link $L'_{v'}$

vertically upwards from $M_{3/2}$ to $M_{5/2}$, where it can be capped off. As these families of disks are unique up to isotopy rel boundary, the surface we obtain in this way from the banded unlink diagram $(\mathcal{K}, L, v)$ is well defined up to isotopy. We denote this surface by $\Sigma(\mathcal{K}, L, v)$. We say that $(\mathcal{K}, L, v)$ *describes* $\Sigma(\mathcal{K}, L, v)$, or that $(\mathcal{K}, L, v)$ is a *banded unlink diagram for* $\Sigma(\mathcal{K}, L, v)$.

3.3 Band moves

We now proceed to describe a collection of moves which will allow us in Section 4 to define banded unlink diagrams of arbitrary surfaces in X^4, and relate the banded unlink diagrams of any isotopic surfaces. These moves are described in Figures 3 and 4. They consist of *cup* and *cap* moves (Figure 3, top), *band slides* (Figure 3, middle), *band swims* (Figure 3, bottom), 2–*handle-band slides* (Figure 4, top), *dotted circle slides* (Figure 4, middle two rows) and 2–*handle-band swims* (Figure 4, bottom). These operations, together with isotopy in $E(\mathcal{K})$, form a collection of moves which we refer to as *band moves*. (Note that the dotted circle slide may actually move L rather than a band, but we still refer to this as a band move for convenience.) Band moves may transform a banded unlink diagram $(\mathcal{K}, L, v)$ into a banded unlink diagram $(\mathcal{K}, L', v')$, though it is not difficult to verify that the surfaces $\Sigma(\mathcal{K}, L, v)$ and $\Sigma(\mathcal{K}, L', v')$ are isotopic.

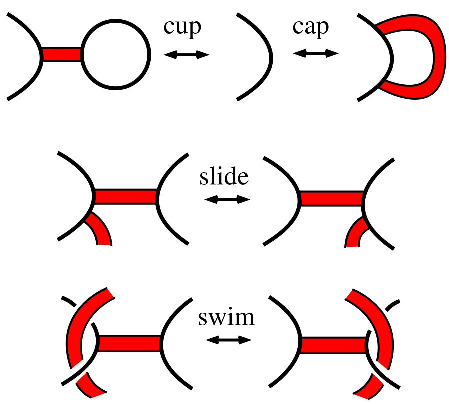

Figure 3: The cup/cap, slide and swim band moves. These band moves do not involve the 2–handle attaching circles of $\mathcal{K}$. The cup/cap moves correspond to 0– and 1– or 2– and 3– stabilization/destabilization of a surface Σ with respect to h. The slide move passes an end of one band along the length of a distinct band. The swim move passes a band lengthwise through the interior of a distinct band.

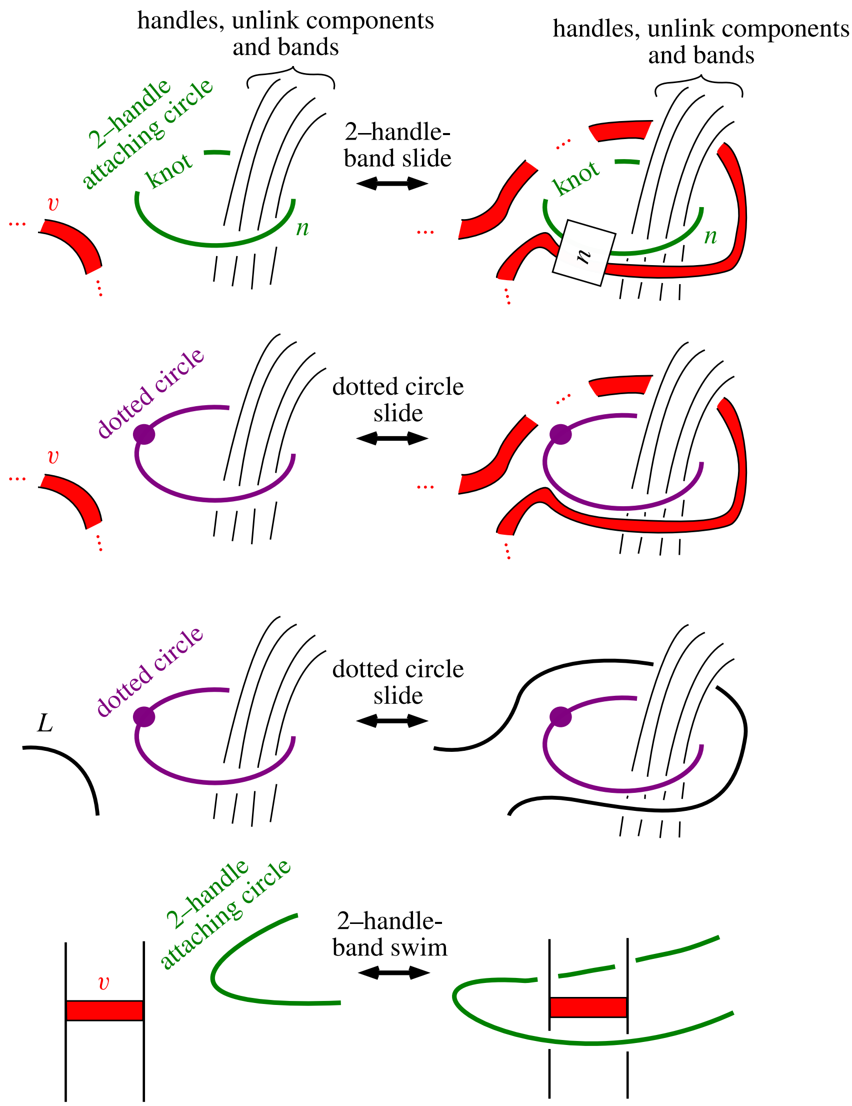

Figure 4: The 2–handle-band slide and 2–handle-band swim moves. These band moves involve the 2–handles attaching circles of $\mathcal{K}$. Top: the 2–handle-band slide move slides a band over a 2–handle, following the usual rules of Kirby calculus. This schematic is meant to indicate that the 2–handle attaching circle may be knotted and link arbitrarily with other circles in $\mathcal{K}$ or unlink or band components (including the band that slides). Second row: a dotted circle slide may pass a band over a dotted circle, following the usual rules of Kirby calculus. Third row: a dotted circle slide may pass the unlink L over a dotted circle, following the usual rules of Kirby calculus. Bottom: the 2–handle-band swim move passes a 2–handle attaching circle lengthwise through the interior of a band.

Notation We refer to the slide, swim, 2–handle-band slide, dotted circle slide, 2–handle-band swim moves and isotopy in $E(\mathcal{K})$ as *Morse-preserving band moves*. The cup and cap moves are not Morse-preserving.

Lemma 3.2 *Let $\Sigma \subset X^4$ be a surface in banded unlink position. There is a procedure to obtain a banded unlink diagram $(\mathcal{K}, L_\Sigma, v_\Sigma)$ such that L_Σ and v_Σ are completely determined by the embedding $\Sigma \hookrightarrow X^4$. Moreover, Σ is isotopic to $\Sigma' := \Sigma(\mathcal{K}, L_\Sigma, v_\Sigma)$.*

Proof Let $L_\Sigma = \Sigma \cap M_{3/2-\epsilon}$ and let v_Σ be the bands of $\overline{(\Sigma \cap M_{3/2}) \setminus L_\Sigma}$. By definition of banded unlink position, L_Σ bounds a system of disks in $M_{1/2}$ (eg $\Sigma \cap M_{1/2}$) and $L_{\Sigma v_\Sigma}$ bounds a system of disks in $M_{5/2}$ (eg $\Sigma \cap M_{5/2}$). Since Σ is in banded unlink position, $L_\Sigma \cup v_\Sigma$ is contained in $E(\mathcal{K})$. Therefore, $(\mathcal{K}, L_\Sigma, v_\Sigma)$ is a banded unlink diagram.

Let $\Sigma' := \Sigma(\mathcal{K}, L_\Sigma, v_\Sigma)$. Then Σ' is also in banded unlink position, with $\Sigma' \cap \left(\frac{1}{2}, \frac{5}{2}\right) = \Sigma \cap \left(\frac{1}{2}, \frac{5}{2}\right)$. Both $\Sigma \cap h^{-1}\left(\left[0, \frac{1}{2}\right]\right)$ and $\Sigma' \cap h^{-1}\left(\left[0, \frac{1}{2}\right]\right)$ are boundary-parallel disk systems with equal boundary in $h^{-1}\left(\left[0, \frac{1}{2}\right]\right) \cong B^4$, so are isotopic rel boundary. Similarly, $\Sigma \cap h^{-1}\left(\left[\frac{5}{2}, 4\right]\right)$ and $\Sigma' \cap h^{-1}\left(\left[\frac{5}{2}, 4\right]\right)$ are isotopic rel boundary in $h^{-1}\left(\left[\frac{5}{2}, 4\right]\right) \cong \natural(S^1 \times B^3)$. Therefore, Σ is isotopic to Σ'. □

Remark 3.3 In Lemma 3.2, we showed that if Σ and Σ' are surfaces in banded unlink position in X^4, then the banded unlink diagrams $\mathcal{D} := (\mathcal{K}, L_\Sigma, v_\Sigma)$ and $\mathcal{D}' := (\mathcal{K}, L_{\Sigma'}, v_{\Sigma'})$ are well defined. However, even if Σ and Σ' are isotopic, we have not yet shown that $\mathcal{D}$ and $\mathcal{D}'$ are related in any way.

4 A calculus of moves on banded unlink diagrams

4.1 Overview

In what follows, let $\mathcal{K}_0$ denote the standard (empty) Kirby diagram induced by the standard height function on S^4. (The handle decomposition described by $\mathcal{K}_0$ has one 0–handle, one 4–handle and no other handles.) When $\mathcal{K} = \mathcal{K}_0$, Swenton [29] and Kearton and Kurlin [16] show that the cup, cap, band slide and band swim moves relate any two banded unlink diagrams of isotopic surfaces.

Theorem 4.1 [29; 16] *Let Σ and $\Sigma' \subset S^4$ be isotopic surfaces described by banded unlink diagrams $\mathcal{D} := (\mathcal{K}_0, L, v)$ and $\mathcal{D}' := (\mathcal{K}_0, L', v')$, respectively. Then $\mathcal{D}'$ can be obtained from $\mathcal{D}$ by a finite sequence of cap/cup, band slides, band swims and isotopies in S^3.*

Note that we have not defined what it means for an arbitrary surface in S^4 to be described by a banded unlink diagram (even with h the standard height function). Part of the content of Theorem 4.1 is that such a diagram is well defined.

Definition 4.2 Let Σ be a surface in S^4. Let Σ' be a surface in banded unlink position (with respect to the standard height function) which is isotopic to Σ. We say that $(\mathcal{K}_0, L_{\Sigma'}, v_{\Sigma'})$ is a banded unlink diagram for Σ. This diagram is well defined up to cap/cup, band slides, band swims and isotopy in S^3 [29; 16].

By including 2–handle-band swims, 2–handle-band slides and dotted circle slides along with the moves in Theorem 4.1, we can generalize Theorem 4.1 to surfaces in arbitrary closed 4–manifolds. We state the theorem now, even though we have not defined what it means for an arbitrary surface in X^4 to be described by a banded unlink diagram.

Theorem 4.3 *Let Σ and Σ' be surfaces X^4, with banded unlink diagrams $\mathcal{D} := (\mathcal{K}, L, v)$ and $\mathcal{D}' := (\mathcal{K}, L', v')$, respectively. Then Σ and Σ' are isotopic if and only if $\mathcal{D}$ can be transformed into $\mathcal{D}'$ by a finite sequence of band moves.*

Note that when $X^4 = S^4$ and $\mathcal{K} = \mathcal{K}_0$, Theorem 4.3 reduces to the statement of Theorem 4.1. Loosely, to prove Theorem 4.3, we will analyze how banded unlink diagrams for Σ change under isotopy of Σ. Here is a brief outline of our strategy for proving Theorem 4.3:

(1) We show that surfaces in banded unlink position admit banded unlink diagrams well defined up to band moves (Section 3.3).

(2) We show that surfaces in horizontal–vertical position (Definition 2.4) admit banded unlink diagrams well defined up to band moves (Section 4.2).

(3) We show that certain isotopies of a horizontal–vertical surface preserve the associated banded unlink diagram up to band moves (Section 4.2).

(4) We show that surfaces in the more general *generic* position (Definition 4.8) admit banded unlink diagrams well defined up to band moves (Section 4.4).

(5) We show that certain isotopies between generic surfaces preserve the associated banded unlink diagrams up to band moves (Section 4.5).

(6) We show that any isotopy of surfaces can be perturbed to an isotopy as in step (5). We define the banded unlink diagram of a surface Σ to be the banded unlink diagram of any generic surface isotopic to Σ (Section 4.5).

4.2 Banded unlink diagrams for horizontal–vertical surfaces

We now extend Lemma 3.2 to horizontal–vertical surfaces, rather than only surfaces in banded unlink position.

Proposition 4.4 *Let $\Sigma \subset X^4$ be a surface in horizontal–vertical position such that all minima of $h|_\Sigma$ are below all saddles of $h|_\Sigma$, which are below all maxima of $h|_\Sigma$. Then we may obtain a banded unlink diagram $\mathcal{D} = (\mathcal{K}, L_\Sigma, v_\Sigma)$ such that L_Σ and v_Σ are determined up to Morse-preserving band moves by the embedding of Σ into X^4. Moreover, Σ is isotopic to $\Sigma(\mathcal{K}, L_\Sigma, v_\Sigma)$.*

Proof We will isotope Σ into banded unlink position and apply Lemma 3.2. If necessary, apply a small horizontal isotopy to the horizontal parts of $\Sigma \cap h^{-1}(3-\epsilon, 4]$ to avoid intersections with the ascending manifolds of index-3 critical points of h. Then isotope $\Sigma \cap h^{-1}\big(\frac{5}{2}-\epsilon, 4\big]$ vertically into $h^{-1}\big(\frac{5}{2}-\epsilon, \frac{5}{2}\big)$.

Next, apply a small horizontal isotopy to the horizontal parts of $\Sigma \cap h^{-1}\big[0, \frac{3}{2}+\epsilon\big)$ to avoid intersections with the descending manifolds of index-1 critical points of h. Then isotope $\Sigma \cap h^{-1}\big[0, \frac{3}{2}+\epsilon\big)$ vertically into $h^{-1}\big(\frac{3}{2}, \frac{3}{2}+\epsilon\big)$.

Now isotope horizontal neighborhoods of the minima and maxima of $h|_\Sigma$ horizontally to avoid the ascending and descending manifolds of index-2 critical points of h. Isotope the minima vertically to $h^{-1}\big(\frac{1}{2}\big)$ and the maxima vertically to $h^{-1}\big(\frac{5}{2}\big)$ (apply further horizontal isotopies to the minima and maxima as necessary to ensure no self-intersections are introduced to Σ).

Let b be the horizontal neighborhood of an index-1 critical point of $h|_\Sigma$ (so b is a band). If $h(b) > 2$ and b intersects the ascending manifold of some index-2 critical point of h, then isotope b slightly horizontally to either side of the ascending manifold. (These two choices eventually give rise to banded unlink diagrams which differ by a

2–handle-band slide; see Figure 5, top left.) Do this for each such intersection, and then isotope b vertically to $h^{-1}\left(\frac{3}{2}, 2\right)$ (vertically isotope other bands downward in $h^{-1}\left(\frac{3}{2}, 2\right)$ as necessary to avoid self-intersections).

Repeat for every other index-1 critical point of $h|_\Sigma$. Take the bands to lie in distinct heights, by vertical isotopy. Say the bands are $b_1, \dots, b_n$, with $\frac{3}{2} < h(b_1) < \cdots < h(b_n) < 2$. Set $L_\Sigma := \Sigma \cap M_{3/2}$, $v_i := \rho_{h(b_i),3/2}$ and $v_\Sigma = \bigcup_i v_i$.

Note $(\mathcal{K}, L_\Sigma, v_\Sigma)$ is not yet a banded unlink diagram, due to the following disallowed situations that may occur when projecting the bands b_i to $M_{3/2}$:

- It might be that an end of some v_i is attached to another band v_j. This implies $j < i$. If so, slide the end of v_i off of v_j and onto either L_Σ or v_k, with $k < j$. Repeat until both ends of v_i are on L_Σ. There are two choices to make at each step (that is, there is a choice of which direction to slide). The two obtainable diagrams differ by a sequence of band slides. (See the left of the second row of Figure 5 for the simplest case when v_j has both ends on L_Σ.)
- It might be that a band v_i intersects the interior of another band v_j. This implies $j < i$. If so, swim v_i out the length of v_j. If v_j intersects the interior of another band v_k (necessarily $k < j$), this introduces new intersections between v_i and v_k. Repeat on each intersection of v_i with another band until v_i does not intersect any other bands. There are two choices at each step (that is, there is a choice of which direction to swim). The two obtainable diagrams differ by a sequence of band swims. (See the left of the third row of Figure 5 for the simplest case when v_j does not intersect the interior of any other band.)
- It might be that a segment of a band v_i passes through the descending manifold of some index-2–critical point of h. In $\mathcal{K}$, this means that v_i intersects a 2–handle attaching circle C. Swim C through v to remove the intersection. There are two choices of directions in which to swim. The two obtainable diagrams differ by a 2–handle-band swim. (See the right of the top row of Figure 5 for the simplest case when v intersects exactly one attaching circle, exactly once.)
- It might be that L_Σ or a band v_i still do not lie in $E(\mathcal{K})$ because they intersect the ascending manifold of an index-1 critical point. Then push L_Σ or v_i horizontally off the ascending manifold. For each such intersection, there are two choices of which direction to push. The two obtainable diagrams differ by a dotted circle slide. (See the right of the second and third rows of Figure 5.)

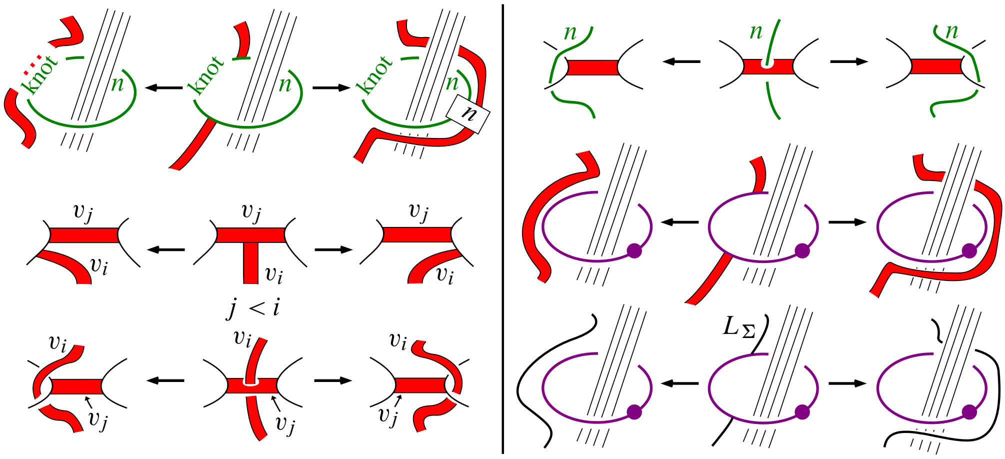

Figure 5: Left, top row: If a band v_i intersects the ascending manifold of an index-2 critical point of h, we must choose how to slide the band off of the ascending manifold before projecting to $h^{-1}\left(\frac{3}{2}\right)$. In $\mathcal{K}$, these two choices yield diagrams that differ by a 2–handle-band slide of v_i over the corresponding 2–handle attaching circle. Left, second row: If the end of band v_i lies on v_j, then we must choose which way to slide v_i off of v_j. (Here we draw only the simple case that both ends of v_j lie on L_Σ. We do not care about interior intersections of bands or intersections with 2–handle attaching circles.) Left, third row: If a band v_i intersects the interior of band v_j, then we must choose which way to swim v_i through and out of v_j. (Here we draw only the simple case that v_j does not intersect any other bands. We do not care about intersections with 2–handle attaching circles.) Right, top row: If a band v_i intersects the descending manifold of an index-2 critical point, we must choose how to swim the corresponding attaching circle in $\mathcal{K}$ out of the band v_i. (Here, we draw only the simple case that only one 2–handle attaching circle intersects v_i, in one point.) Right, second and third rows: If a band v_i or L (respectively) intersect the ascending manifold of an index-1 critical point, then we push horizontally off. The resulting diagrams differ by a dotted circle slide.

Changing any choices made during this operation changes the diagram by Morse-preserving band moves. See Figure 5 for a summarizing schematic. The point of ordering the bands and proceeding in order from lowest to highest is to ensure that this procedure terminates, and eventually after a finite number of choices all bands will be simultaneously projected (disjointly) to $E(\mathcal{K})$.

Each move on the projections v_i can be induced by a horizontal isotopy supported in a neighborhood of b_i. After this procedure and a vertical isotopy of the bands to $M_{3/2}$, we find an isotopy from Σ to a surface in banded unlink position, whose banded unlink diagram we denote by $\mathcal{D} = (\mathcal{K}, L_\Sigma, v_\Sigma)$. Via the isotopy constructed above we see that Σ is indeed isotopic $\Sigma(\mathcal{K}, L_\Sigma, v_\Sigma)$.

We must check that the choices of h–regular and h–disjoint horizontal and vertical isotopies used to position Σ (that is, the choice of Σ') do not affect the resulting diagram up to Morse-preserving band moves (ie that L_Σ and v_Σ are well defined up to Morse-preserving band moves). It is sufficient to prove the following proposition:

Proposition 4.5 *Let S and S' be horizontal–vertical surfaces in X^4, with all minima below all bands, which are in turn below all maxima. Assume all bands of S and S' can be projected to $M_{3/2}$ (ie assume that the bands of S and S' do not intersect the ascending manifold of any index-3 critical point of h or the descending manifold of any index-1 critical point of h). Suppose there is an h–disjoint and h–regular horizontal or vertical isotopy f taking S to S'. Let $\mathcal{D}_S = (\mathcal{K}, L_S, v_S)$ be a banded unlink diagram obtained by setting $L_S = S \cap M_{t_0+\epsilon}$ for t_0 the height of the highest minima of S, and v_S the bands obtained by projecting the bands of S to $M_{3/2}$ (viewed as containing a copy of L_S, projected vertically) and (as in Proposition 4.4) choosing slides, swims, 2–handle-band slides/swims and dotted circle slides as necessary to make the projected bands disjointly lie in $E(\mathcal{K})$. Similarly, choose a banded unlink diagram $\mathcal{D}_{S'} = (\mathcal{K}, L_{S'}, v_{S'})$ using S'. Then $\mathcal{D}_S$ and $\mathcal{D}_{S'}$ are related by Morse-preserving band moves.*

Proof We have two cases:

Case 1 (the isotopy is horizontal) Since the isotopy is horizontal, L_S is isotopic to $L_{S'}$ in $h^{-1}\left(\frac{3}{2}\right)$. Therefore, L_S is isotopic to $L_{S'}$ in $E(S)$ up to dotted circle slides.

Let v_i be a band in S. The isotopy f takes the banded link $\widetilde{L} \cup v_1 \cup \cdots \cup v_k = S \cap M_{h(v_i)}$ to the banded link $\widetilde{L}' \cup v_1' \cup \cdots \cup v_k' = S' \cap M_{h(v_i)}$, up to relabeling of bands (for some isotopic links $\widetilde{L}$ and $\widetilde{L}'$). Since $\widetilde{L}$ and $\widetilde{L}'$ are isotopic, there is a natural identification between $\widetilde{L}$ and $\widetilde{L}'$. Say that v_i goes to band v_i', with their endpoints on $\widetilde{L}$ identified.

Suppose $h(v_i) > 2$. As per the above argument in Proposition 4.4, if the isotopy passes v_i through the ascending manifold of an index-2 critical point of h, then this effects a 2–handle-band slide in $\mathcal{D}_S$.

For any value of $h(v_i)$, if f takes the ends of $\rho_{h(v_i),3/2}(v_i)$ over any other projection $\rho_{h(v_j),3/2}(v_j)$ with $h(v_j) < h(v_i)$, then as in Proposition 4.4 this effects a band slide in $\mathcal{D}$. If f takes the interior of $\rho_{h(v_i),3/2}(v_i)$ through any other projection $\rho_{h(v_j),3/2}(v_j)$ with $h(v_j) < h(v_i)$, then as in Proposition 4.4 this effects a band swim in $\mathcal{D}_S$. If f takes v_i through the descending manifold of an index-2 critical point of h, then as in Proposition 4.4 this effects a 2–handle-band swim in $\mathcal{D}_S$.

Finally, if f takes $\rho_{h(v_i),3/2}(v_i)$ through the ascending manifold of an index-1 critical point, then as in Proposition 4.4 this effects a dotted circle slide in $\mathcal{D}$.

If none of the above happen to v_i during f, then the replacement $v_i \mapsto v_i'$ just isotopes the projection of v_i in $h^{-1}\left(\frac{3}{2}\right)$, ie changes the projection of v_i by isotopy in $E(K)$ and dotted circle slides.

Case 2 (the isotopy is vertical) By assumption, the vertical isotopy does not introduce new critical points of $h|_S$ and preserves the projections of S pointwise to each M_t. Then $L_{S'}$ differs by L_S by isotopy in $h^{-1}\left(\frac{3}{2}\right)$, ie isotopy in $E(K)$ and dotted circle slides. Moreover, the vertical isotopy does not affect the projections of the bands of S (after identifying L_S and $L_{S'}$), so these projections agree with those of S'. Then $\mathcal{D}_S$ and $\mathcal{D}_{S'}$ agree up to Morse-preserving band moves which arise from varying choices of how to separate the projections of bands to $M_{3/2}$ (as seen above in Proposition 4.4). □

Thus, $\mathcal{D}$ is well defined from Σ up to Morse-preserving band moves. Moreover, Σ is isotopic to Σ', so, by Lemma 3.2, Σ is isotopic to $\Sigma(\mathcal{K}, L_\Sigma, v_\Sigma)$. This completes the proof of Proposition 4.4. □

Given Σ as in Proposition 4.4, we let $(\mathcal{K}, L_\Sigma, v_\Sigma)$ denote the banded unlink diagram resulting from the proof of Proposition 4.4.

We now show that we need not restrict the ordering of critical points of a horizontal–vertical surface in order to obtain a banded unlink diagram.

Lemma 4.6 *Let Σ be a surface in horizontal–vertical position. Then we may obtain a banded unlink diagram $(\mathcal{K}, L_\Sigma, v_\Sigma)$ such that L_Σ and v_Σ are determined up to Morse-preserving band moves by the embedding of Σ into X^4. Moreover, Σ is isotopic to $\Sigma(\mathcal{K}, L_\Sigma, v_\Sigma)$.*

Proof Choose an ordering $x_1, \dots, x_n$ of the critical points of $h|_\Sigma$, with all index-0 critical points coming before all index-1 critical points, which come before all index-2 critical points. Perform h–regular and h–disjoint horizontal and vertical isotopy to Σ to reorder the horizontal regions according to this ordering, to obtain surface Σ' (the vertical isotopies move horizontal regions to the appropriate height; we apply horizontal isotopy as necessary to ensure horizontal regions never intersect). Set $L_\Sigma = L_{\Sigma'}$ and $v_\Sigma = v_{\Sigma'}$.

Suppose $y_1, \dots, y_n$ is another ordering of the critical points of $h|_\Sigma$ with all index-0 critical points coming before all index-1 critical points, which come before all index-2 critical points. Let Σ'' be the surface obtained by isotoping Σ to reorder the horizontal regions of Σ according to this ordering. Then Σ'' can be transformed into Σ' by a sequence of h–regular and h–disjoint horizontal and vertical isotopies (and the surface is in horizontal–vertical position after each isotopy, with minima below bands below maxima — essentially, we reorder minima, keeping them below all bands; then we reorder bands, keeping them between the minima and maxima; then we reorder the maxima, keeping them above the bands). By Proposition 4.5, the diagrams $(\mathcal{K}, L_{\Sigma'}, v_{\Sigma'})$ and $(\mathcal{K}, L_{\Sigma''}, v_{\Sigma''})$ agree up to Morse-preserving band moves. Therefore, $(\mathcal{K}, L_\Sigma, v_\Sigma)$ does not depend on the choice of Σ'. By Proposition 4.4, $(\mathcal{K}, L_\Sigma, v_\Sigma)$ is well defined up to Morse-preserving band moves. Moreover, $\Sigma(\mathcal{K}, L_\Sigma, v_\Sigma)$ is isotopic to Σ', so is isotopic to Σ. □

Given Σ as in Lemma 4.6, we let $(\mathcal{K}, L_\Sigma, v_\Sigma)$ denote the banded unlink diagram resulting from the proof of Lemma 4.6.

Remark 4.7 We can immediately extend Proposition 4.5 to say that if Σ and Σ' are horizontal–vertical surfaces which are isotopic through an h–regular and h–disjoint horizontal or vertical isotopy, then $(\mathcal{K}, L_\Sigma, v_\Sigma)$ is related to $(\mathcal{K}, L_{\Sigma'}, v_{\Sigma'})$ by Morse-preserving band moves.

4.3 Interlude: surface singularities

The following definitions generalize the classes of surface singularities identified and studied in [16].

Definition 4.8 [16, Definition 2.5] For a fixed surface F, let CS be the space of all smoothly embedded surfaces $\Sigma \subset X^4$ with $\Sigma \cong F$. CS inherits the induced topology from the Whitney topology on the space of smooth maps $F \to X$. From now on, when we write "CS", the topology of the embedded surface is implicitly understood to be fixed. We say that $\Sigma \in \mathrm{CS}$ is

- *generic* if $h|_\Sigma$ is Morse and all critical points of $h|_\Sigma$ have distinct height values under h;

- an $A_1^+A_1^+$*–singularity* if $h|_\Sigma$ fails to be generic because of two nondegenerate extrema of h that have the same h value;
- an $A_1^+A_1^-$*–singularity* if $h|_\Sigma$ fails to be generic because a nondegenerate saddle and extremum of h that have the same h value;
- an $A_1^-A_1^-$*–singularity* if $h|_\Sigma$ fails to be generic because of two nondegenerate saddles of h that have the same h value;
- an A_2*–singularity* if $h|_\Sigma$ fails to be generic because of a singularity of $h|_\Sigma\colon \Sigma\to\mathbb{R}$ having the form $h(x,y)=x^2-y^3$ in local coordinates (x,y) on Σ.

Let $\mathcal{S}_h\subset \mathrm{CS}$ be the subspace of all surfaces which are $A_1^+A_1^+$–, $A_1^+A_1^-$–, $A_1^-A_1^-$– or A_2–singularities. We call $\mathcal{S}_h$ the *singular subspace of X^4 with respect to h.* When $\mathcal{K}=\mathcal{K}_0$, then $\mathcal{S}_h$ agrees with the singular subspace defined by Kearton and Kurlin [16].

4.4 Banded unlink diagrams for generic surfaces

The following lemma is analogous to [16, Proposition 2.6(i)].

Lemma 4.9 *Let $\Sigma\subset X^4$ be a generic surface which is disjoint from critical points of h. Then we may obtain a banded unlink diagram $(\mathcal{K},L_\Sigma,v_\Sigma)$ determined by $\Sigma\hookrightarrow X^4$ up to Morse-preserving band moves. Moreover, Σ is isotopic to $\Sigma(\mathcal{K},L_\Sigma,v_\Sigma)$.*

Proof We will isotope Σ into horizontal–vertical position and apply Lemma 4.6. See Figure 6 for a schematic of the isotopy.

Flatten Σ in a small neighborhood of each local extremum. Say the smallest value of t for which $M_t\cap\Sigma$ is nonempty is t_0; so $\Sigma\cap M_{t_0+\epsilon}$ is an unlink for very small

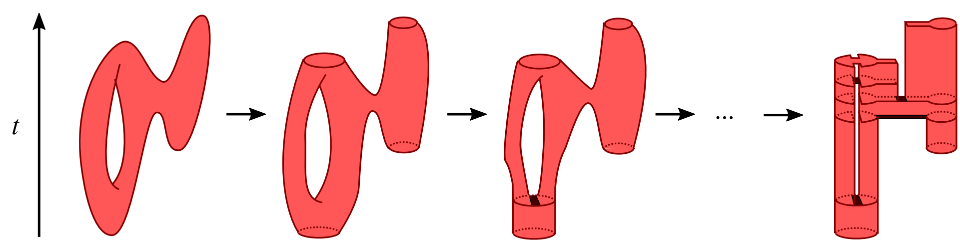

Figure 6: To isotope a generic surface $\Sigma\subset X^4$ into horizontal–vertical position with respect to h, we first flatten a neighborhood of each extremum of $h|_\Sigma$. Then we isotope Σ to be vertical below each critical point of $h|_\Sigma$, starting from the lowest critical point and working upward. We give more detail in the proof of Lemma 4.9.

$\epsilon > 0$. Take the values of critical points of $h|_\Sigma$ to be $t_0 < t_1 < \cdots < t_n$. Perturb Σ vertically if necessary so that $\{t_0, \ldots, t_n\} \cap \{0, 1, 2, 3, 4\} = \varnothing$ (for small perturbations, the choice of perturbation does not affect the construction of Lemma 4.6). Since Σ is generic, there is exactly one critical point of $h|_\Sigma$ in each M_{t_i}.

Fixing $\Sigma \cap M_{t_0}$, horizontally isotope $\Sigma \cap h^{-1}(t_0, 4]$ so that Σ is vertical in $h^{-1}[0, t_1 - \epsilon)$. Then the projection of $\Sigma \cap h^{-1}[t_1 - \epsilon, t_1 + \epsilon]$ to M_{t_1} is a bounded (perhaps disconnected) surface S. Specifically, if x is the critical point of $h|_\Sigma$ in M_{t_1}, then this S is

$$\begin{cases} \text{the disjoint union of an annulus and a disk} & \text{if } x \text{ is a minimum,} \\ \text{a planar surface with three boundary components} & \text{if } x \text{ is a saddle,} \\ \text{a disk} & \text{if } x \text{ is a maximum.} \end{cases}$$

If x is an extremum, perform a horizontal isotopy of Σ in $h^{-1}[t_1 - \epsilon, t_1 + \epsilon]$ so that $\Sigma \cap h^{-1}(t_0, t_1)$ is vertical. The cross-section $\Sigma \cap h^{-1}(t_1)$ includes one horizontal disk.

Say that x is a saddle. Then, up to isotopy in M_{t_1}, S can be uniquely contracted to a 1–complex $\Sigma \cap M_{t_1 - \epsilon} \cup \{\text{edge } E\}$. Vertically isotope Σ in $h^{-1}[t_1 - \epsilon, t_1 + \epsilon]$ so that Σ is vertical in $h^{-1}(t_0, t_1)$ and $\Sigma \cap M_{t_1} = (\Sigma \cap M_{t_1 - \epsilon}) \cup (\text{band along } E)$. The framing of the band along the arc E agrees with the surface framing S induces on E.

Now, for $i = 1, \ldots, n-1$ (in order), repeat this procedure. That is, horizontally isotope $\Sigma \cap h^{-1}(t_i, 4]$ so that Σ is vertical in $h^{-1}(t_i, t_{i+1} - \epsilon)$. If the critical point of $h|_\Sigma$ in $M_{t_{i+1}}$ is an extremum, then horizontally isotope Σ near $M_{t_{i+1}}$ so that Σ is vertical in $h^{-1}(t_i, t_{i+1})$ and $\Sigma \cap M_{t_{i+1}}$ is a disjoint union of a (possibly empty) link and a disk. If the critical point of $h|_\Sigma$ in $M_{t_{i+1}}$ is a saddle, then again (up to isotopy in $M_{t_{i+1}}$) there is a unique band b that can be attached to $\Sigma \cap M_{t_{i+1} - \epsilon}$ so that resolving b yields $\Sigma \cap M_{t_{i+1} + \epsilon}$. Vertically isotope $\Sigma \cap h^{-1}[t_{i+1} - \epsilon, t_{i+1} + \epsilon]$ so that Σ is vertical in $h^{-1}(t_i, t_{i+1})$ and $\Sigma \cap h^{-1}(t_{i+1})$ is the banded link $\Sigma \cap M_{t_{i+1} - \epsilon} \cup b$.

Call the resulting surface Σ', so the original Σ is isotopic through h–regular and h–disjoint horizontal and vertical isotopies to Σ', where Σ' is in horizontal–vertical position. Suppose Σ'' is another horizontal–vertical surface isotopic to Σ through h–disjoint and h–regular horizontal and vertical isotopies. Then Σ' and Σ'' are isotopic through h–disjoint and h–regular horizontal and vertical isotopies. By Proposition 4.5 and Remark 4.7, $(\mathcal{K}, L_{\Sigma'}, v_{\Sigma'})$ is related to $(\mathcal{K}, L_{\Sigma''}, v_{\Sigma''})$ by Morse-preserving band moves.

Set $(\mathcal{K}, L_\Sigma, v_\Sigma) := (\mathcal{K}, L_{\Sigma'}, v_{\Sigma'})$. By Lemma 4.6, Σ determines a banded unlink diagram $(\mathcal{K}, L_\Sigma, v_\Sigma)$ well defined up to Morse-preserving band moves. Moreover, Σ is isotopic to $\Sigma(\mathcal{K}, L_\Sigma, v_\Sigma)$. □

Given Σ as in Lemma 4.9, we let $(\mathcal{K}, L_\Sigma, v_\Sigma)$ denote the banded unlink diagram resulting from the proof of Lemma 4.9. We say that $(\mathcal{K}, L_\Sigma, v_\Sigma)$ *describes* or *is a banded unlink diagram for* Σ.

Remark 4.10 When Σ is a generic surface in banded unlink position, the definitions of $(\mathcal{K}, L_\Sigma, v_\Sigma)$ in Lemmas 3.2 and 4.9 agree.

4.5 Banded unlink diagrams for arbitrary surfaces

The following lemma is analogous to [16, Proposition 2.6(ii)–(iv)]:

Lemma 4.11 *Let Σ and Σ' be generic surfaces in X^4. Let $\mathcal{D} = (\mathcal{K}, L, v)$ and $\mathcal{D}' = (\mathcal{K}, L', v')$ be banded unlink diagrams associated to Σ and Σ', respectively, as in Lemma 4.9.*

(i) *If Σ and Σ' are h–disjoint isotopic through generic surfaces, then $\mathcal{D}$ and $\mathcal{D}'$ are related by Morse-preserving band moves.*

(ii) *If Σ and Σ' are h–disjoint isotopic through generic surfaces and one $A_1^{\pm}A_1^{\pm}$–singularity, then $\mathcal{D}$ and $\mathcal{D}'$ are related by Morse-preserving band moves.*

(iii) *If Σ and Σ' are h–disjoint isotopic through generic surfaces and one A_2–singularity, then $\mathcal{D}$ and $\mathcal{D}'$ are related by band moves.*

Proof (i) An isotopy through generic surfaces preserves the level sets $\Sigma \cap h^{-1}(t)$ up to isotopy and reparametrization of h. Therefore, this isotopy does not affect the construction of Lemma 4.9. The unlink L is taken to L' by isotopy in $h^{-1}\left(\frac{3}{2}\right)$, so L' can be obtained from L by isotopy in $E(\mathcal{K})$ and dotted circle slides. Each band projection is then isotoped in $E(K)$ unless the corresponding band meets an

- ascending manifold of an index-2 critical point of h, inducing a 2–handle-band slide;
- descending manifold of an index-2 critical point of h, inducing a 2–handle-band swim;
- ascending manifold of an index-1 critical point of h, inducing a dotted circle slide.

(ii) Perturb the isotopy so that near the singularity, the isotopy is a vertical exchange of heights between two critical points. Say the isotopy goes from Σ to Σ_0, then vertically to Σ_1 and then to Σ', where the $A_1^{\pm}A_1^{\pm}$ singularity appears during the isotopy from Σ_0 to Σ_1. We consider each of the following possibilities:

- If the exchanged critical points are both extrema, then this does not affect the construction of Lemma 4.9.
- If the exchanged critical points are an extremum and an index-1 critical point, then this does not affect the construction of Lemma 4.9. (These critical points correspond to a minimum or maximum disk and a band which does not intersect the interior of that disk.)
- If the exchanged critical points are both index-1, then they correspond to bands whose projections to $M_{3/2}$ must be disjoint (since at some point during this vertical isotopy, they live in a common M_t). Therefore, this does not affect the diagram resulting from Lemma 4.9.

Therefore, Σ_0 and Σ_1 have banded unlink diagrams equivalent up to Morse-preserving band moves. The claim follows from part (i).

(iii) When passing an A_2–singularity (away from critical points of h) a nondegenerate saddle and extremum appear or disappear [16, Claim 4.3(iv)]. In the case when the extremum created is a minimum, this corresponds to performing a cup move, while in the case of a maximum the banded unlinks are related by a cap move. Away from the A_2–singularity, this is an isotopy through generic surfaces, so the claim follows from part (i). □

The following analysis appears in [16]. Although they state this lemma in S^4 rather than an arbitrary 4–manifold X^4, their techniques hold generally.

Lemma 4.12 [16, Claim 4.3] (i) *The subspace $\bar{S}_h$ is codimension-*1 *in* CS.

(ii) *Every element of* CS $\setminus \bar{S}_h$ *is generic.*

(iii) *Any h–disjoint isotopy of a surface in X^4 can be deformed to an h–disjoint isotopy so that all intermediate surfaces are generic except for finitely many singularities as in Definition 4.8.*

Lemma 4.13 *Let Σ and Σ' be surfaces which are disjoint from the critical points of h. Let f be an isotopy with $f(F \times I) = \Sigma$ and $f(F \times 1) = \Sigma'$. Then f can be deformed to an h–disjoint isotopy, fixing $f|_{F\times 0}$ and $f|_{F\times 1}$.*

Proof Let $H \subset X^4$ be the critical points of h. Then $F \times I$ is a smooth codimension-2 submanifold of $X^4 \times I$, while $H \times I$ is a dimension-1 submanifold of $X^4 \times I$. We may generically perturb $F \times I$ (and hence f) rel boundary to be disjoint from $H \times I$, to obtain an h–disjoint isotopy. □

Lemma 4.14 *Let Σ and Σ' be isotopic generic surfaces embedded in X^4 which are both disjoint from critical points of h. Say that Σ and Σ' have banded unlink diagrams $\mathcal{D}$ and $\mathcal{D}'$, respectively. Then $\mathcal{D}$ can be transformed into $\mathcal{D}'$ by band moves.*

Proof Let $f\colon F \times I \to X^4 \times I$ be an isotopy from $\Sigma = f(F \times 0)$ to $\Sigma' = f(F \times 1)$ By Lemma 4.13, f can be taken to be h–disjoint. By Lemma 4.12, f can be perturbed slightly so that $f(F \times s)$ is generic except for finitely many values of s (at which time $f(F \times s)$ is a singularity as in Definition 4.8), and f is still h–disjoint. Fix $0 < s_1 < \cdots < s_n < 1$ such that $f(F \times s)$ is generic if $t \notin \{s_i\}$, and $f(F \times s_i)$ is a singularity as in Definition 4.8.

For $i = 1, \ldots, n-1$, let $\Sigma_i := f(F \times (s_i + \epsilon))$. Let $\Sigma_0 := \Sigma$ and $\Sigma_n := \Sigma'$. Let $\mathcal{D}_i = (\mathcal{K}, L_{\Sigma_i}, v_{\Sigma_i})$. By Lemma 4.11, $\mathcal{D}_i$ is obtained from $\mathcal{D}_{i-1}$ by band moves. Thus, $\mathcal{D}_n = \mathcal{D}'$ is obtained from $\mathcal{D}_0 = \mathcal{D}$ by band moves. □

Any surface in X^4 can be perturbed to be generic and away from critical points of h. Therefore, Lemma 4.14 allows us to make the following definition, completing the proof of Theorem 4.3:

Definition 4.15 Let Σ be a surface embedded in X^4. Let Σ' be a generic surface which is disjoint from critical points of h such that Σ is isotopic to Σ'. We say that $(\mathcal{K}, L_{\Sigma'}, v_{\Sigma'})$ *describes* or *is a banded unlink diagram for* Σ. By Lemma 4.14, this diagram is well defined up to band moves.

5 Uniqueness of bridge trisections

First, we recall the definition of trisection of a closed 4–manifold.

Definition 5.1 [6] Let X^4 be a closed 4–manifold. A (g,k)*–trisection* of X^4 is a triple (X_1, X_2, X_3) where

- $X_1 \cup X_2 \cup X_3 = X^4$,
- $X_i \cong \natural_{k_i} S^1 \times B^3$,
- $X_i \cap X_j = \partial X_i \cap \partial X_j \cong \natural_g S^1 \times B^2$
- $X_1 \cap X_2 \cap X_3 \cong \Sigma_g$,

where Σ_g is the closed orientable surface of genus g. Here, g is an integer while $k = (k_1, k_2, k_3)$ is a triple of integers. If $k_1 = k_2 = k_3$, then the trisection is said to be *balanced*.

Briefly, a trisection is a decomposition of a 4–manifold into three elementary pieces, analogous to a Heegaard splitting of a 3–manifold into two elementary pieces. Intuitively, one should think that the need for an "extra" piece of this decomposition when the dimension increases corresponds to an "extra" type of handle. That is, given a Heegaard splitting $M^3 = H_1 \cup H_2$, one can view H_1 as containing the 0– and 1–handles of M^3 while H_2 contains the 2– and 3–handles of M^3. Similar is true for a trisection (X_1, X_2, X_3) of X^4; one can view X_1 as containing the 0– and 1–handles of X^4 and X_3 as containing the 3– and 4–handles of X^4 while X_2 contains the 2–handles of X^4. See [20] for a clear description of a trisection from this point of view.

Note that from the definition, $(\Sigma_g, X_i \cap X_j, X_i \cap X_k)$ gives a Heegaard splitting of ∂X_i. By Laudenbach and Poénaru [18], X^4 is specified by its *spine*, $\Sigma_g \cup_{i,j} (X_i \cap X_j)$. Therefore, we usually describe a trisection (X_1, X_2, X_3) by a *trisection diagram* $(\Sigma_g, \alpha, \beta, \gamma)$ where each of α, β and γ consist of g independent curves bounding disks in the handlebodies $X_1 \cap X_2$, $X_2 \cap X_3$ and $X_1 \cap X_3$, respectively.

We do not require much knowledge about trisections for this paper. For more exposition of trisections, refer to [6].

Definition 5.2 By the *standard trisection of* S^4 we mean the unique $(0,0)$–trisection (X_1, X_2, X_3). View $S^4 = \mathbb{R}^4 \cup \infty$, with coordinates (x, y, r, θ) on $\mathbb{R}^4$, where (x, y) are Cartesian planar coordinates of a plane and (r, θ) are polar planar coordinates. Up to isotopy, $X_i = \{\theta \in [\frac{2\pi}{3} \cdot i, \frac{2\pi}{3} \cdot (i+1)]\} \cup \infty$. Then $X_i \cong B^4$, $X_i \cap X_{i+1} = \{\theta = \frac{2\pi}{3} \cdot (i+1)\} \cup \infty \cong B^3$ and $X_1 \cap X_j \cap X_k = \{r = 0\} \cup \infty \cong S^2$.

In [21], Meier and Zupan introduce bridge trisections of surfaces in S^4. In [22], they extend this notion to surfaces in an arbitrary closed 4–manifold.

Definition 5.3 [21; 22] Let S be a surface embedded in X^4. Let $\mathcal{T} = (X_1, X_2, X_3)$ be a trisection of a closed 4–manifold X^4. We say that S is in (c, b)–bridge position with respect to $\mathcal{T}$ if

- $S \cap X_i$ is a disjoint union of c boundary-parallel disks;
- $S \cap X_i \cap X_j$ is a trivial tangle of b arcs.

Here, b is an integer and $c = (c_i, c_j, c_k)$ is a triple of integers. Note $\chi(S) = \sum c_i - b$.

See Figure 7 for an example of a surface in bridge position.

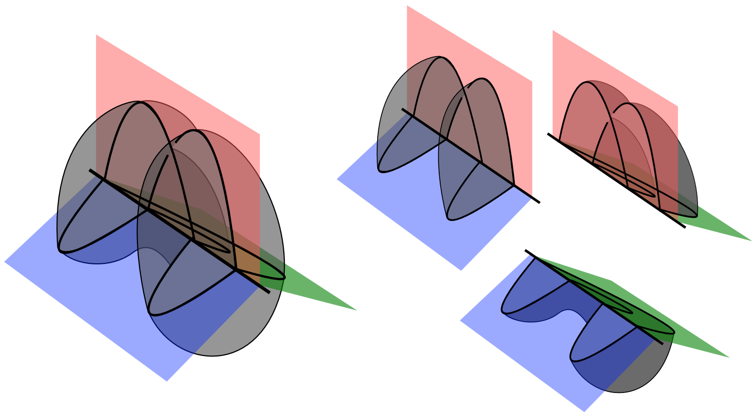

Figure 7: An $\mathbb{R}P^2$ in S^4 in $(1,1,1;2)$–bridge position with respect to the standard trisection. Left: we draw the whole surface (projected to $\mathbb{R}^3$). The three planes indicate the three 3–balls $\{X_i \cap X_j\}$. Right: we show each disk system (each have one component) individually.

Theorem 5.4 [21; 22] *Let S be a surface embedded in X^4 with a trisection (X_1, X_2, X_3). Then, for some c and b, S can be isotoped into (c,b)–bridge position with respect to $\mathcal{T}$. We may take $c_1 = c_2 = c_3$.*

Because a collection of boundary-parallel disks in $\natural(S^1 \times B^3)$ is uniquely determined by its boundary (up to isotopy rel boundary), a surface S in bridge position is determined up to isotopy by $S \cap \left(\bigcup_{i \neq j} X_i \cap X_j\right)$.

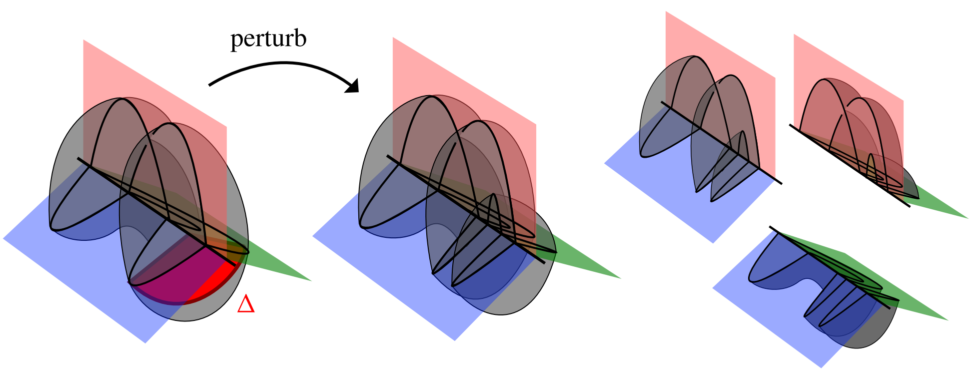

Figure 8: Left: $\Sigma \cong \mathbb{R}P^2$ in $(1,1,1;2)$–bridge position in S^4 (with respect to the standard trisection). We indicate a disk Δ along which we perturb. Right: after perturbing, Σ is in $(1,2,1;3)$–bridge position (up to permuting X_1, X_2 and X_3).

Definition 5.5 Let X^4 be a 4–manifold with trisection $\mathcal{T}(X_1, X_2, X_3)$. We say that an isotopy f of X^4 is *$\mathcal{T}$–regular* if $f_t(X_i) = X_i$ for each $i = 1, 2, 3$ for all t.

There is a natural perturbation of a surface in bridge position, analogous to perturbation of a knot in bridge position within a 3–manifold.

Definition 5.6 [21; 22] Let $S \subset X^4$ be a surface in (c, b)–bridge position with respect to $\mathcal{T} = (X_1, X_2, X_3)$. Let $\Delta \subset X_i \setminus \nu(S)$ be a properly embedded disk such that $\partial\Delta$ consists of one arc in $\partial\nu(S)$, one arc in $X_i \cap X_{i+1}$ and one arc in $X_i \cap X_{i-1}$. Obtain S' by compressing S along Δ. Note S' is isotopic to S and is in $(c', b{+}1)$–bridge position, where $c'_i = c_i + 1, c'_{i+1} = c_{i+1}, c'_{i-1} = c_{i-1}$. We say that S' is obtained from S by elementary perturbation, while S is obtained from S' by elementary deperturbation (see Figure 8).

Theorem 5.4 shows existence of bridge trisections. The following theorem of [21] gives uniqueness of bridge trisections with respect to the standard trisection of S^4.

Theorem 5.7 [21] *Let S and S' be surfaces in bridge position with respect to the standard trisection $\mathcal{T}_0$ of S^4. Suppose S is isotopic to S'. Then S can be taken to S' by a sequence of perturbations and deperturbations, followed by a $\mathcal{T}_0$–regular isotopy.*

Theorem 5.7 relies on Theorem 4.1, which is specific to S^4. In [21], Meier and Zupan give an equivalence between bridge trisections and banded unlink diagrams, and then show how to translate moves on banded unlink diagrams into sequences of perturbations and deperturbations. These moves do not occur in order; in particular, they do not show that all the deperturbations can come after the perturbations.

In [22], Meier and Zupan state the following theorem as a conjecture, and comment that they believe it would follow from a generalized version of Theorem 4.1 following a proof similar to that of Theorem 5.7. We will prove the following theorem using Theorem 4.3.

Theorem 5.8 *Let S and S' be surfaces in bridge position with respect to a trisection $\mathcal{T}$ of a closed 4–manifold X^4. Suppose S is isotopic to S'. Then S can be taken to S' by a sequence of perturbations and deperturbations, followed by a $\mathcal{T}$–regular isotopy.*

Before proving Theorem 5.8, we state several necessary definitions and lemmas from [22].

Definition 5.9 Let T be a tangle of properly embedded arcs in a solid handlebody H. We say that T is *trivial* if T is boundary-parallel, ie cobounding disjoint disks D with arcs $T' \subset \partial H$. We call T' a *shadow* of T.

Definition 5.10 [21; 22] Let T be a trivial tangle in a handlebody H with shadow T'. Let v be a set of bands attached to T, with core arcs η disjoint from a core of H. Project η to ∂H. We call η a *shadow* of v. We say that η is *dual* to T' if $\mathring{\eta} \cap T' = \varnothing$ and each component of $\eta \cup T'$ is simply connected.

Notation Given a Kirby diagram $\mathcal{K}$, let $L_1 \subset S^3 \supset \mathcal{K}$ be the unlink of dotted circles (defining 1–handles) in $\mathcal{K}$. Recall that $M_{3/2} = S^3$ surgered along L_1 with 0–framing. Let $L_2 \subset S^3$ be the link of 2–handle attaching circles in $\mathcal{K}$.

Recall $E(\mathcal{K}) = S^3 \setminus \nu(L_1 \cup L_2)$.

Definition 5.11 [22] Let $\mathcal{K}$ be a Kirby diagram. Let $H \cup_F H'$ be a Heegaard splitting of $M_{3/2}$ such that a core of H contains L_2 and a core of H' contains L_1.

Let $(\mathcal{K}, L, v)$ be a banded unlink. We say that $(\mathcal{K}, L, v)$ is in *bridge position* with respect to the Heegaard splitting $H \cup_F H'$ if the following are true:

- $L \cap H$ and $L \cap H'$ are each trivial tangles with no closed components.
- The bands v are all contained in H and there is a shadow η of $v \subset H$ such that the surface framing ∂H induces on η agrees with the framing v induces on η.
- There is a shadow L' of $L \cap H$ such that η and L' are dual.

See Figure 9 for an example of a banded unlink in bridge position. Meier and Zupan [22] show that every banded unlink can be put into bridge position with respect to a given Heegaard splitting of $M_{3/2}$ (the proof is similar to the fact that every knot in a Heegaard-split 3–manifold can be isotoped into bridge position).

Lemma 5.12 [22, Lemma 4.4] *Let $(\mathcal{K}, L, v)$ be a banded unlink diagram. Let $H \cup_F H'$ be a Heegaard splitting of M_1. After isotopy of $L \cup v$ in $E(\mathcal{K})$, we may assume $(\mathcal{K}, L, v)$ is in bridge position with respect to $H \cup_F H'$.*

Lemma 5.13 [22, Lemma 4.5] *Let $\mathcal{T} = (X_1, X_2, X_3)$ be a trisection of a 4–manifold X^4. Fix a self-indexing Morse function $h\colon X^4 \to I$, so that X_1 contains all the index-0 and index-1 critical points of h, X_2 contains all the index-2 critical points, and X_3 contains all the index-3 and index-4 critical points* (see [6, Lemma 14]).

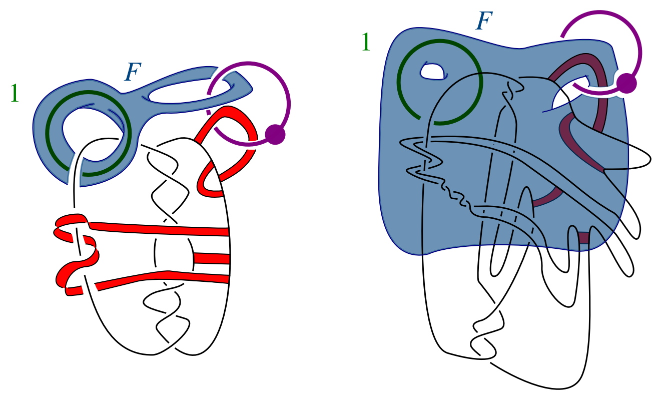

Figure 9: Left: a banded unlink for a torus in $\mathbb{C}P^2\#(S^1\times S^3)$. The surface F induces a genus-2 Heegaard splitting $H\cup_F H'$ on $M_1=S^1\times S^2$. Right: we isotope the banded unlink into bridge position with respect to the Heegaard splitting $H\cup_F H'$, as in Definition 5.11.

Let $\mathcal{K}$ be the Kirby diagram of X^4 induced by h, so $M_1\cong\partial X_1$ comes with a Heegaard splitting $M_{3/2}=H\cup_F H'$, where $H=X_1\cap X_2$, $H'=X_1\cap X_3$ and $F=X_1\cap X_2\cap X_3$. (We will say that $\mathcal{K}$ is a Kirby diagram of X^4 induced by $\mathcal{T}$.)

Let $(\mathcal{K},L,v)$ be a banded unlink diagram describing a surface $S\subset X^4$. If $(\mathcal{K},L,v)$ is in bridge position with respect to $H\cup_F H'$, then the Heegaard splitting $H\cup_F H'$ induces a trisection $\mathcal{T}$ on X^4 such that S is naturally in bridge position with respect to $\mathcal{T}$. Similarly, if S' is a surface in bridge position with respect to $\mathcal{T}$, then we may obtain a banded unlink $(\mathcal{K},L',v')$ for S' which is in bridge position with respect to the Heegaard splitting $M_{3/2}=H\cup_F H'$.

Now we are ready to prove Theorem 5.8, mirroring the proof of Theorem 5.7 in [21].

Proof Let S and S' be isotopic surfaces in X^4 which are both in bridge position with respect to a trisection $\mathcal{T}=(X_1,X_2,X_3)$. Let $\mathcal{K}$ be a Kirby diagram for X^4 as in Lemma 5.13. Again let $H=X_1\cap X_2$ and $H'=X_1\cap X_3$ be such that $H\cup_F H'$ is a Heegaard splitting of $M_{3/2}$.

By Lemma 5.13, $\mathcal{T}$ induces a banded unlink diagram $\mathcal{D}:=(\mathcal{K},L,v)$ for S, where $\mathcal{D}$ is in bridge position with respect to the Heegaard splitting $H\cup_F H'$. Similarly, $\mathcal{T}$ induces a banded unlink diagram $\mathcal{D}'=(\mathcal{K},L',v')$ for S' which is in bridge position with respect to $H\cup_F H'$.

By Theorem 4.3, $\mathcal{D}$ can be taken to be isotopic to $\mathcal{D}'$ after performing cap/cup, band slide, band swim, 2–handle-band slide, dotted circle slide and 2–handle-band swim moves. Meier and Zupan show explicitly how to achieve the cap/cup, band slide and band swim moves by perturbing and deperturbing S with respect to $\mathcal{T}$ in [21, Theorem 1.6].

Suppose $\mathcal{D}'$ is obtained from $\mathcal{D}$ by a single 2–handle-band slide or swim. Let z be the framed arc from a band $\nu \in v_1$ to a 2–handle attaching circle $C \subset L_2 \subset M_{3/2}$ along which the slide or swim takes place. Project C onto F so that the surface framing on the projection C' is the handle framing of C. Take C' to be disjoint from a shadow of the trivial tangle $H \cap L$. As in [21] (the proof that band swims can be realized by perturbation), perturb (L, v) so that C' is also disjoint from shadows of the bands b and then further perturb (L, v) until the projection of z to F is embedded and disjoint in its interior from the shadows of $L \cap H$ and b, and so the surface framing in this projection of z agrees with the framing of z. Each of these perturbations induces perturbation of the bridge trisection $\mathcal{T}$.

Now performing the 2–handle-band slide or swim induces isotopy on S which fixes $S \cap (X_1 \cap X_3)$ and isotopes $S \cap (X_1 \cap X_2)$ and $S \cap (X_2 \cap X_3)$ within $X_1 \cap X_2$ and $X_2 \cap X_3$, respectively. This can therefore be taken to be a $\mathcal{T}$–regular isotopy. (Diagrammatically, the 2–handle-band slide or swim induces disk-slides on a bridge trisection diagram of S. We do not consider the diagrammatic point of view on bridge trisections in this paper; see [22].)

Claim 5.14 *After a sequence of perturbations and deperturbations of S, we may take (L, v) to be isotopic to (L', v') in $E(\mathcal{K})$ up to dotted circle slides.*

Proof The claim almost follows from Theorem 4.3, except that we did not show we could take the dotted circle slides to happen after the other band moves (except for isotopy in $E(\mathcal{K})$, which we did implicitly show could be taken to happen at the end of the equivalence from $(\mathcal{K}, L, v)$ to $(\mathcal{K}, L', v')$). Two diagrams that differ by dotted slides agree up to isotopy in $S(\mathcal{K}) := (S^3 \setminus \nu(L_2))_0(L_1)$, ie the 3–manifold with boundary obtained from $\mathcal{K}$ by deleting the 2–handle attaching circles and surgering the dotted circles. Therefore, if there is a sequence of (nonisotopy) band moves $m_1 m_2 \cdots m_n$ which takes $(\mathcal{K}, L, v)$ to $(\mathcal{K}, L', v')$ up to isotopy in $E(\mathcal{K})$, then we may delete all of the dotted circle slides to find a sequence of (nonisotopy and nondotted circle slide) band moves $m'_1 m'_2 \cdots m'_{n'}$ from $(\mathcal{K}, L, v)$ to $(\mathcal{K}, L', v')$ up to isotopy in $S(\mathcal{K})$. Then there exist dotted circle slides $s_1, \ldots, s_k$ such that $m'_1 m'_2 \cdots m'_{n'} s_1 \cdots s_k$ takes $(\mathcal{K}, L, v)$ to $(\mathcal{K}, L', v')$ up to isotopy in $E(\mathcal{K})$. □

Thus, after the above perturbations and deperturbations of S, we may take (L, v) to be isotopic to (L', v') in $S(\mathcal{K})$. The isotopy may not respect the bridge splitting with respect to $H \cup_F H'$. By the same argument as [21, Theorem 1.6] (this cites [35], which is stated for bridge splittings of a tangle in a punctured 3–sphere but works just as well for a punctured handlebody), we may perturb and deperturb $\mathcal{D}_1$ (and isotope a neighborhood of F, taking F to F setwise) so that $L \cap F = L' \cap F$, $(L \cup v) \cap H$ is isotopic to $(L' \cup v') \cap H$ rel boundary in $H \cap E(K)$, and $L \cap H'$ is isotopic to $L \cap H'$ rel boundary in H'. Thus, after the listed sequence of perturbations and deperturbations, we find that S is $\mathcal{T}$–regular isotopic to S'. □

6 Examples in $\mathbb{C}P^2$

In this section, we construct isotopies of surfaces embedded in $\mathbb{C}P^2$. In particular, we study unit surfaces.

Definition 6.1 Let Σ be a surface in $\mathbb{C}P^2$. If Σ intersects the standard $\mathbb{C}P^1 \subset \mathbb{C}P^2$ in exactly one point, then we say that Σ is a *unit surface*.

Note that by Freedman [4], an oriented unit sphere is topologically isotopic to $\mathbb{C}P^1$. Similarly, by Sunukjian [27] a genus-g orientable unit surface is topologically isotopic to the connected sum of $\mathbb{C}P^1$ with an unknotted surface of genus-g contained in a 4–ball.

One motivation for studying unit surfaces in the (potentially more interesting) smooth category is to understand the Gluck twist operation [7]. This is a surgery operation on a 2–sphere $\Sigma \subset X^4$ as long as Σ has trivial normal bundle. In particular, the Gluck twist on S^4 about any embedded 2–sphere yields a homotopy 4–sphere. The homotopy 4–sphere resulting from a Gluck twist along $\Sigma \subset S^4$ is known to be diffeomorphic to S^4 for many families of Σ, including ribbon knots [7; 31], spun knots [7], twist-spun knots [8], band-sums of ribbon and twist-spun knots [9] and knots 0–concordant to any of the others in this list [23]. We will define each of these families later in this section.

If Σ is a sphere in S^4, then we can take the connected sum of the pairs (S^4, Σ) and $(\mathbb{C}P^2, \mathbb{C}P^1)$ to obtain the 4–manifold $\mathbb{C}P^2 = S^4 \# \mathbb{C}P^2$ and an embedded surface which we denote by $\Sigma \# \mathbb{C}P^1$. Melvin [23] showed that the Gluck twist along $\Sigma \subset S^4$ is diffeomorphic to S^4 if and only if there is a pairwise diffeomorphism from $(\mathbb{C}P^2, \Sigma \# \mathbb{C}P^1)$ to $(\mathbb{C}P^2, \mathbb{C}P^1)$. This in part motivates the following questions of Melvin and Gabai:

Question 6.2 Let $F \subset \mathbb{C}P^2$ be a sphere in the generating homology class $[\mathbb{C}P^1] \in H_2(\mathbb{C}P^2; \mathbb{Z})$ with $F \cap \mathbb{C}P^1 = \{\text{pt}\}$.

(i) [23] Is $(\mathbb{C}P^2, F)$ diffeomorphic as a pair to $(\mathbb{C}P^2, \mathbb{C}P^1)$?

(ii) [5, Question 10.17.i] Is F isotopic to the standard $\mathbb{C}P^1$ in $\mathbb{C}P^2$?

Note that by the previously stated work of [23], Question 6.2(i) is equivalent to Kirby problem 4.23 [17] ("Is the Gluck twist of S^4 about an arbitrary 2–sphere diffeomorphic to S^4?").

In this section, we will show that many of these unit surfaces (including all the examples listed above) are in fact isotopic to the standard $\mathbb{C}P^1$ using the moves of Theorem 4.3.

First, we give an alternative definition of $\Sigma \# \mathbb{C}P^1$.

Definition 6.3 Let Σ be a surface in S^4. Let x be a point in S^4 far from Σ, so that $\Sigma \subset S^4 \setminus \nu(x) \cong B^4$. We can view Σ as living in $\mathbb{C}P^2$, inside the 4–ball $\mathbb{C}P^2 \setminus \nu(\mathbb{C}P^1)$. Let h be the radial Morse function on B^4 and isotope Σ so that $h|_\Sigma$ has a unique global maximum at $y \in \Sigma$. Let γ be an arc from y extending radially outward in B^4 until reaching $\mathbb{C}P^2$.

Let U_Σ be a copy of Σ tubed to $\mathbb{C}P^1$ along an arc γ so that $\gamma \cap (\mathbb{C}P^2 \setminus \nu(\mathbb{C}P^1))$ and $\gamma \cap \nu(\mathbb{C}P^1)$ are each single intervals. We call U_Σ the *unit surface associated to* Σ. We write $U_\Sigma = \Sigma \# \mathbb{C}P^1$.

Equivalently, $(S^4, \Sigma) \# (\mathbb{C}P^2, \mathbb{C}P^1) = (\mathbb{C}P^2, U_\Sigma)$.

Remark 6.4 In Definition 6.3, the identification of $S^4 \setminus \nu(x)$ with $\mathbb{C}P^2 \setminus \nu(\mathbb{C}P^1)$ does not affect the embedding $\Sigma \subset \mathbb{C}P^2$ up to ambient isotopy. The framing of γ does not matter so long as the orientation on U_Σ agrees with the orientations on Σ and $\mathbb{C}P^1$.

Remark 6.5 We may obtain a banded unlink diagram for U_Σ as follows. Let $(\mathcal{K}_0, L, v)$ be a banded unlink diagram for $\Sigma \subset S^4$. Add a 1–framed 2–handle to $\mathcal{K}_0$ as a small meridian of L, far away from v, to obtain a Kirby diagram $\mathcal{K}_1$ for $\mathbb{C}P^2$. Then $(\mathcal{K}_1, L, v)$ is a banded unlink diagram for $U_\Sigma \subset \mathbb{C}P^2$.

6.1 Ribbon and 0–concordant surfaces

We now proceed to define ribbon surfaces, as well as a diagrammatic framework for describing these surfaces and their isotopies, called *chord diagrams*. We will use chord diagrams to show that all ribbon unit surfaces are unknotted in $\mathbb{C}P^2$.

Definition 6.6 Let S_1 and S_2 be surfaces in S^4, and let $\pi_I\colon S^4 \times I \to I$ be the projection to the unit interval. We say that S_1 is *ribbon-concordant* to S_2 if there exists an embedding $f\colon \Sigma \times I \to S^4 \times I$ such that $f(\Sigma \times 0) = S_1 \times 0$ and $f(\Sigma \times 1) = S_2 \times 1$ and such that $\pi_I \circ f$ has finitely many critical points with distinct critical values, all of which are of index 0 or 1. We say that a surface S is *ribbon* if the unknotted sphere is ribbon-concordant to S.

Kawauchi [14] gives an equivalent definition of ribbonness without the cobordism perspective, by using semiunknotted punctured handlebodies.

Definition 6.7 (eg [14]) A genus-g surface $R \subset X^4$ is *ribbon* if R bounds a punctured 3–dimensional handlebody V embedded in X^4 so that $\partial V = R \cup O$, where O is an trivial unlink of unknotted spheres.

From this definition comes a diagrammatic description of ribbon surfaces. We recall the definition of a chord diagram of an oriented ribbon surface knot $R \subset S^4$.

Definition 6.8 [14] A *chord graph* for a ribbon-surface in S^4 consists of an oriented unlink o of circles in S^3 and arcs α in S^3 with endpoints on o and interiors disjoint from o. This graph indicates the same ribbon-surface as the banded unlink $(\mathcal{K}_0, o, v)$, where v consists of pairs of dual bands attached along the α curves, as in Figure 10. (Twisting this pair does not affect the resulting surface, as long as they describe an orientable surface; see Remark 6.9.) A *chord diagram* is a planar diagram of a chord graph. (See [14] for details.)

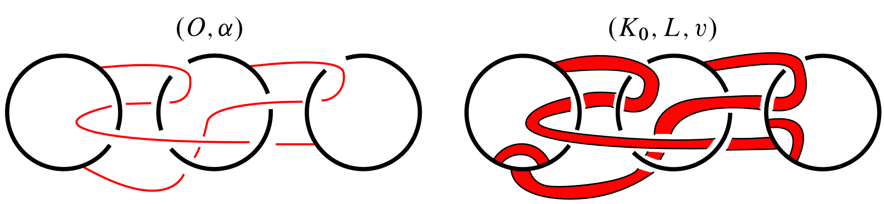

Figure 10: Left: a chord diagram for a ribbon surface R in S^4. Each unlink component is oriented counterclockwise. Right: a banded unlink diagram for the same ribbon surface R in S^4.

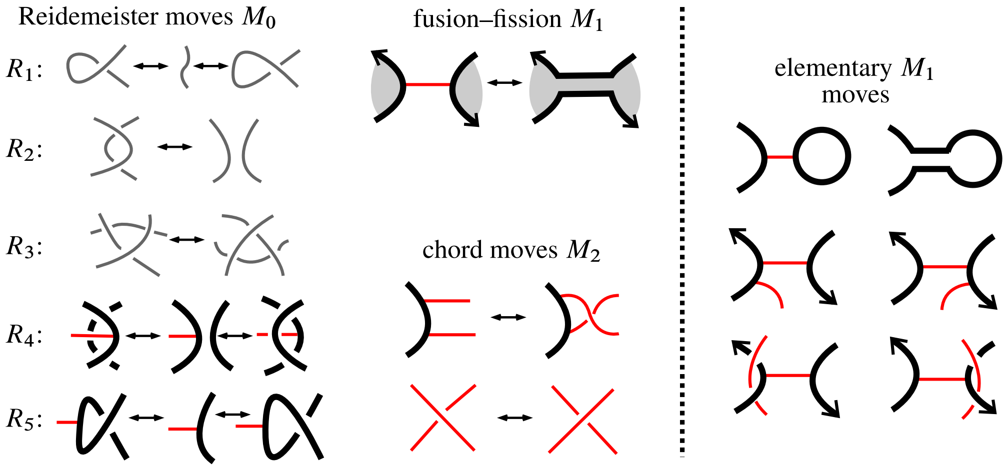

Figure 11: Kawauchi's M_0, M_1 and M_2 moves for chord diagrams in S^4. In moves R_4, R_5, M_1 and M_2 the red (narrow) curves represent α arcs, while black (bold) curves represent components of o. The (gray) curves of moves R_1, R_2 and R_3 represent arcs from either α or o.

Kawauchi [14] gives a list of diagrammatic moves on chord diagrams that represent isotopies of a ribbon surface S^4. (This list is incomplete; see eg the supplement to [14].) These moves are illustrated in Figure 11.

Remark 6.9 We do not specify the framings on the chords α in a chord diagram (O,α) for R. In fact, if two framings of α give descriptions of orientable surfaces, then the surfaces are isotopic. Similarly, we may add whole twists to the framing of v in the associated banded unlink $(\mathcal{K}_0, L, v)$ for R without changing the isotopy type of the described surface.

Exercise 6.10 Let (O,α) be a chord diagram and $(\mathcal{K}_0, L, v)$ the associated banded unlink.

- An M_0 move on (O,α) induces isotopy on $(\mathcal{K}_0, L, v)$.
- Each elementary M_1 move on (O,α) can be achieved by performing a sequence of fusion–fission M_1 moves and isotopies on (O,α). Conversely, every fusion–fission M_1 move can be achieved by a sequence of elementary M_1 moves and isotopies.
- An elementary M_1 move on (O,α) induces isotopy and a sequence of band moves on $(\mathcal{K}_0, L, v)$.

- An M_2 move on (O,α) induces isotopy and a sequence of band slides and swims on $(\mathcal{K}_0, L, v)$. (Very informally, the M_2 move most uses the property that (O,α) describes a ribbon surface. In a general banded unlink diagram, we cannot hope to pass bands through one another.)

Note that there is not a converse to Exercise 6.10. In general, performing band moves on the banded unlink $(\mathcal{K}_0, L, v)$ will destroy the symmetry arising from the association between $(\mathcal{K}_0, L, v)$ and (O,α). Chord diagrams exist only for ribbon surfaces and naturally describe these surfaces with a clear symmetry; banded unlink diagrams describe arbitrary surfaces and need not respect any symmetry of the underlying surface. In particular, we do *not* claim to prove that M_0, M_1 and M_2 moves relate any two chord diagrams of isotopic surfaces.

When R is ribbon, we can define a useful isotopy of U_R using the moves of Theorem 4.3. We call this the M_3 move; see Figure 12. We slide a band over the 2–handle, and then swim the 2–handle through the dual band (or perform the same moves in the opposite order). This move will allow us to change the linking number of L and v in $E(\mathcal{K}_1)$.

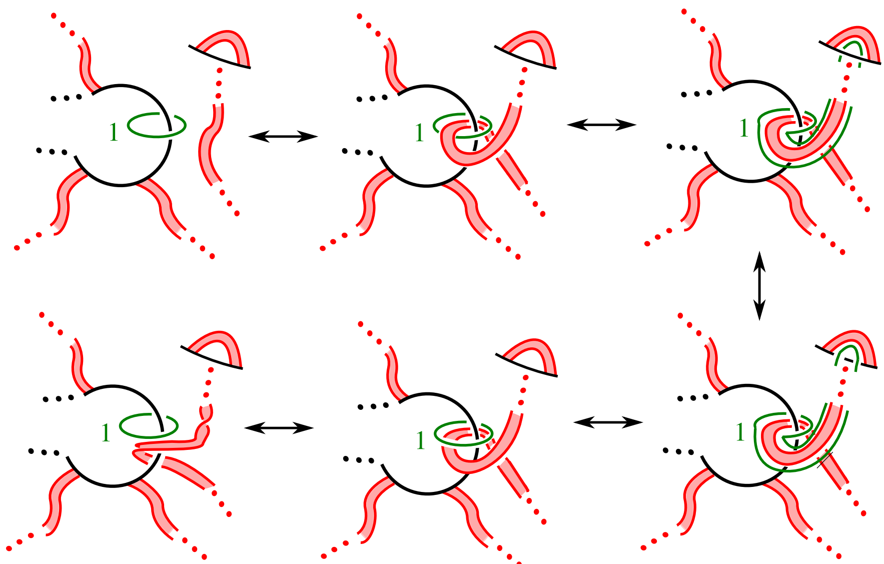

Figure 12: The M_3 move consists of one 2–handle-band slide and one 2–handle-band swim. We slide a band over the 1–framed 2–handle, then swim the 2–handle through the dual band (or perform the same moves in the opposite order). With this move, we can add or remove linking between L and v in $E(\mathcal{K}_1)$.

Lemma 6.11 *Let $R \subset S^4$ be a ribbon surface of genus g. Then $U_R \subset \mathbb{C}P^2$ is isotopic to $\mathbb{C}P^1 \# gT$, where T is an unknotted torus in B^4.*

Proof Let $(\mathcal{K}_0, L, v)$ be a banded unlink diagram for R, as in Definition 6.8. Then $(\mathcal{K}_1, L, v)$ is a banded unlink diagram for U_R. Perform the M_3 move on $L \cup v$ in $\mathcal{K}_1$ finitely many times until v does not link L. (That is, until the bands v do not meet disks bounded by L in $S^3 \supset \mathcal{K}_1$.) To achieve this unlinking, we use 2–handle-band swims to take the 2–handle attaching circle of $\mathcal{K}_1$ to be a meridian of any desired component of L. We then perform M_3 moves until v no longer links that component. We then use 2–handle-band swims to move the 2–handle attaching circle to be a meridian of the next component of L, and repeat until eventually v is unlinked with every component of L.

We then do moves M_0 and M_2 finitely many times to trivialize L and v so that, in the projection $\pi\colon E(K_1) \to \mathbb{R}^2$, $\pi(L \cup v)$ is embedded (recall we may choose the framing of v to be the framing induced by the plane). Finally, we perform the fusion M_1 move finitely many times to get a banded unlink diagram with one circle and g pairs of dual bands. This is a banded unlink diagram for $\mathbb{C}P^1$ trivially stabilized g times, ie $\mathbb{C}P^1 \# gT$. □

Using the above argument, we actually prove a stronger fact about a more general class of knots. We recall a form of concordance introduced by Melvin [23] for spheres, and extended to positive-genus surfaces by Sunukjian [27].

Definition 6.12 [23] Let S_1 and S_2 be genus-g surfaces in X^4. We say that S_1 is 0–concordant to S_2 if there exists an embedding $f\colon \Sigma_g \times I \to X^4 \times I$ such that $f(\Sigma_g \times 0) = S_1 \times 0$, $f(\Sigma_g \times 1) = S_2 \times 1$, $\pi_I \circ f$ has finitely many critical points mapped to distinct t's, and, away from critical values of t, $f^{-1}(X^4 \times t)$ is a disjoint union of a genus-g surface and some number of spheres.

Sunukjian [28] showed that there exist infinitely many pairwise non-0–concordant 2–knots in S^4. Under the equivalence relation of 0–concordance (and operation of connected sum), the set of 2–knots in S^4 becomes a monoid $\mathcal{M}$ somewhat analogous to the concordance group $\mathcal{C}$ of classical knots in S^3. Dai and the third author [3] showed that $\mathcal{M}$ is not finitely generated (ie contains a copy of $\mathbb{N}^\infty$). Joseph [10] showed that $\mathcal{M}$ is not a group: there exist 2–knots $K \subset S^4$ such that $K \# J$ is not 0–concordant to the unknot for any 2–knot J.

Sunukjian [27] noted the following relation between 0–concordance and ribbon-concordance. This lemma is a key fact in [28; 3].

Lemma 6.13 [27, Lemma 8.1] *Let S_1 and $S_2 \subset X^4$ be genus-g surfaces such that S_1 is 0–concordant to S_2. Then there exists a genus-g surface S such that S_1 and S_2 are both ribbon-concordant to S.*

The proof of the above lemma is to show that one can view a 0–concordance as a concordance consisting of 0–handles, then 1–handles each attached between distinct surface components, then 2–handles which each create a new sphere component, and finally 3–handles.

Theorem 6.14 *Let $S, S' \subset S^4$ be genus-g surfaces such that S is 0–concordant to S'. Then $S \# \mathbb{C}P^1$ is isotopic to $S' \# \mathbb{C}P^1$ in $\mathbb{C}P^2$. In particular, if S is 0–concordant to the unknot, then $S \# \mathbb{C}P^1$ is isotopic to $\mathbb{C}P^1 \# gT$.*

Proof By Lemma 6.13, there exists a surface S'' such that S and S' are each ribbon-concordant to S''. It is sufficient to prove the following statement:

Proposition 6.15 *Suppose S is ribbon-concordant to S' via a ribbon-concordance consisting of k index-0 critical points and k index-1 critical points. Then $S \# \mathbb{C}P^1$ is isotopic to $S' \# \mathbb{C}P^1$ in $\mathbb{C}P^2$.*

Proof The above setup is equivalent to saying that S' is given by tubing S to an unlink $\bigsqcup_k O$ of k unknotted spheres along k narrow tubes around arcs b. (There is an extra restriction on the endpoints of b, as this tubing must yield a connected surface.) Via move M_3 (Figure 12), in $\mathbb{C}P^2$ we may remove all intersections of b with the balls bounded by $\bigsqcup_k O$. That is, $S'\#\mathbb{C}P^1$ is isotopic to $(S\#O\#\cdots\#O)\#\mathbb{C}P^1 = S\#\mathbb{C}P^1$. □

This completes the proof of Theorem 6.14. □

In fact, essentially the proof of Theorem 6.14 can be used to prove the following more general statement:

Theorem 6.16 *Let X^4 be a geometrically simply connected 4–manifold (ie X admits a handle decomposition with no 1–handles). Let F and F' be 0–concordant genus-g surfaces in X. Assume that $X \setminus F$ and $X \setminus F'$ are simply connected. Then F and F' are isotopic.*

Sunukjian [27, Theorem 8.2] has previously shown that there exists a diffeomorphism $(X, F) \cong (X, F')$.

Proof Note that the condition that $X \setminus F$ is simply connected is equivalent to the existence of a 2–sphere G immersed in X that intersects F transversely exactly once.

Again, by Lemma 6.13 (whose proof carries out in a general 4–manifold), there exists a surface F'' such that F and F' are each ribbon-concordant to F''. The surface F'' is obtained by tubing F to many unlinked, unknotted 2–spheres in X. These unlinked 2–spheres may be moved far from G, and by dimensionality the tubes may also be taken to be disjoint from G. Therefore, F'' also intersects G transversely exactly once, so it is sufficient to prove the following statement:

Proposition 6.17 *Suppose F and F' are as in Theorem 6.14 and also that F is ribbon-concordant to F' via a ribbon-concordance consisting of k index-0 critical points and k index-1 critical points. Then F and F' are isotopic.*

Proof Since X^4 is geometrically simply connected, there exists a Kirby diagram $\mathcal{K}$ of X^4 with no dotted circles.

Let $(\mathcal{K}, L, v)$ be a banded unlink diagram for F. Let $U_1, \dots, U_k$ be unknots unlinked from $\mathcal{K}$, L and v. That is, take $U_1, \dots, U_k$ to bound disjoint disks $D_1, \dots, D_k$ (respectively) in $S^3 \setminus (\mathcal{K} \cup L \cup v)$. Then $\left(\mathcal{K}, \bigsqcup U_i, \varnothing\right)$ is a banded unlink diagram of an unlink of k spheres $O_1 \sqcup \cdots \sqcup O_k \subset X \setminus (F \cup G)$. Recall F' is obtained from $F \sqcup O_1 \sqcup \cdots \sqcup O_k$ by surgery along k tubes (3D 1–handles) connecting F to each O_i. Then we may obtain a banded unlink diagram $\left(\mathcal{K}, L \sqcup \bigsqcup_i U_i, v'\right)$, where v' is obtained from v by adding k pairs of bands v_i and v_i', where v_i connects L to U_i and v_i' is dual to v_i. (See Figure 13.)

Now we describe some moves on $(\mathcal{K}, L \sqcup U_i, v')$ that induce isotopy of F'.

Note that we may achieve crossing changes of v_i with any $b \in (v \cup_i v_i)$ by swimming b through v_i'. (Here, b may be v_i.) We may similarly achieve crossing changes of v_i with a 2–handle attaching circle C in $\mathcal{K}$ by swimming C through v_i'. These moves are analogous to the M_2 chord moves. See Figure 13, top.

Moreover, we may unlink v_i with $A \subset L \sqcup U_i$ by using G as follows: Fix a point $z \in A$. Isotope G in a neighborhood of F' so that G intersects F' exactly once, in the point z. Let $\Delta \subset G$ be a small embedded disk containing z, perturbed to lie in $S^3 \setminus \mathcal{K}$. Say the tube represented by v_i and v_i' has core arc α_i. Via a homotopy parallel to $G \setminus \Delta$, we see that α_i is homotopic (and hence isotopic) in X^4 to the result of sliding the core of v_i over $\partial\Delta$. We conclude that sliding v_i over $\partial\Delta$ describes an isotopy of F' (the choice of framing is irrelevant). See Figure 13, bottom.

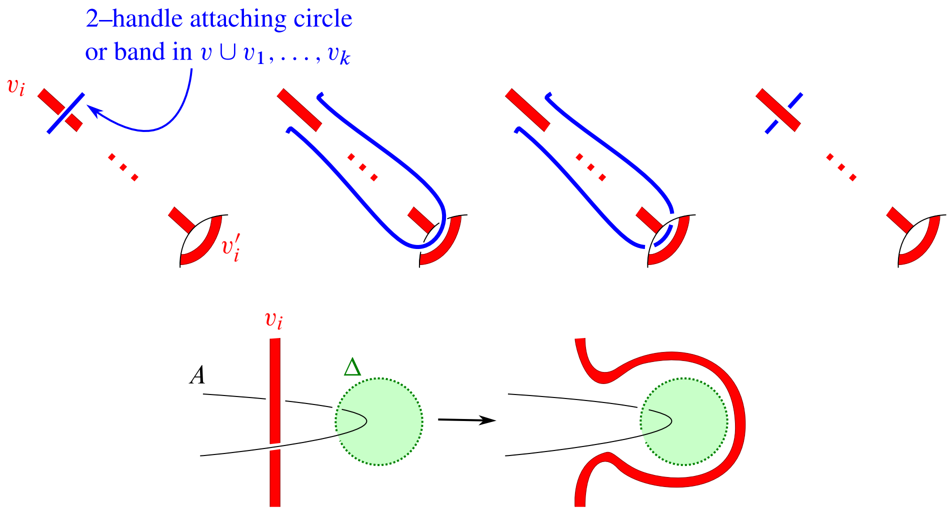

Figure 13: The proof of Theorem 6.16: in a geometrically simply connected 4–manifold, two 0–concordant surfaces with simply connected complements are ambiently isotopic. Top: We may change crossings between $\mathcal{K}$ and v_i or between v_j and v_i by band moves (and hence isotopy of F'). Bottom: We may change crossings of v_i with L by sliding the core of the tube represented by v_i over an immersed 2–sphere dual to F'.

Therefore, via band moves, we may transform the banded unlink diagram of F' until the disk D_i bounded by U_i is disjoint in its interior from v_j for all i and j. Then we may remove each $v_1, v_1', \dots, v_k, v_k'$ and $U_1, \dots, U_k$ via k cap and k cup moves, finally obtaining $(\mathcal{K}, L, v)$. We therefore conclude that F' and F are isotopic. □

This completes the proof of Theorem 6.16. □

The proof of Lemma 6.11 also yields a result about band-sums. Recall the definition of band-summing:

Definition 6.18 Let S_1 and S_2 be oriented surfaces in S^4, contained in disjoint balls. Let γ be an arc from a point on S_1 to a point on S_2. Then the band-sum $S_1 \#_\gamma S_2$ is the surface $((S_1 \sqcup S_2) \setminus \nu(\gamma)) \cup (\gamma \times S^1)$.

The framing on γ to determine the S^1–bundle over γ does not affect the resulting surface up to isotopy, so long as $\nu(\gamma) \cap (S_1 \#_\gamma S_2)$ is oriented consistently with the orientations on S_1 and S_2.

Note that connect-summing is a specific example of band-summing. Now we show that blowing up S^4 trivializes the band-sum.

Theorem 6.19 *Let $S_1 \#_\gamma S_2$ be a band-sum in S^4, where S_1 and S_2 are any smooth surfaces in S^4 contained in disjoint 4–balls and γ is any path between them. Then the unit surfaces $S_1 \#_\gamma S_2 \# \mathbb{C}P^1$ and $S_1 \# S_2 \# \mathbb{C}P^1$ are isotopic in $\mathbb{C}P^2$.*

Proof Fix band diagrams for S_1 and S_2. Then $S_1 \#_\gamma S_2 \# \mathbb{C}P^1$ has a banded unlink diagram $(\mathcal{K}_1, L, v)$ consisting of the union of the banded unlink diagrams for S_1 and S_2 with two bands for the band-sum tube and the $+1$–surgery curve of $\mathbb{C}P^2$. The M_3 move allows one to change any crossing of L with γ. (The M_2 move, or rather the equivalent sequence of band moves of Exercise 6.10, allows one to change any crossing of v with γ and to slide the endpoints of γ along L, through v.) Therefore, in $\mathbb{C}P^2$ we may take γ to be an arbitrary arc, and in particular find $S_1 \#_\gamma S_2 \# \mathbb{C}P^1$ is isotopic to $S_1 \# S_2 \# \mathbb{C}P^1$. □

Remark 6.20 It is not hard to show that there is a ribbon-concordance from $S_1 \#_\gamma S_2$ to $S_1 \# S_2$. Therefore, Theorem 6.14 is actually a corollary of Theorem 6.19. Similarly, Theorem 6.19 follows from Theorem 6.16, but we prefer to state these results separately given the focus of this section on $\mathbb{C}P^2$.

The construction of the ribbon-concordance is the same as in [24], where Miyazaki shows that a connected sum of classical knots K_1 and K_2 is ribbon-concordant to any band-sum of K_1 and K_2. The direction of the word "ribbon" is reversed in the statement for classical knots (if we use consistent definitions; we caution the reader that Miyazaki uses the opposite convention and we have translated his statement to be consistent with our definition of ribbon-concordance). Repeating the construction of Miyazaki a dimension up yields a concordance from $S_1 \# \gamma S_2$ to $S_1 \# S_2$ with only index-2 and index-3 critical points. Turning this concordance upside down, we find that $S_1 \# S_2$ is ribbon-concordant to $S_1 \#_\gamma S_2$.

6.2 Deform-spun knots

We move on from 0–concordant knots to twist-spun knots [34] and deform-spun knots [19].

Definition 6.21 (see [19] generally and [34] for twist-spun knots) Let $K^1 \subset B^3$ be a 1–stranded tangle. Let $f\colon B^3 \to B^3$ be a diffeomorphism fixing ∂B^3 and K^1 pointwise. Then the *f–deform-spun knot* of K^1 is $fK^1 = K^1 \times I/{\sim}$, contained in

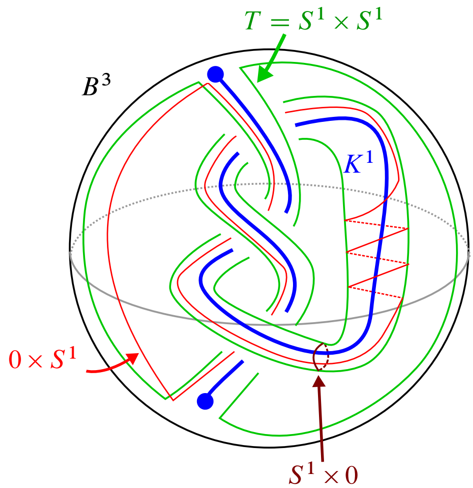

Figure 14: The setup for Litherland's description of twist-roll spun knots [19]. Here, K^1 is a 1–stranded tangle in ball B^3. The torus T is given by $\partial(\nu(K^1 \cup \partial B^3))$. We parametrize $T \cong S^1 \times S^1$ so that its longitude $0 \times S^1$ is nullhomologous in $B^3 \setminus \nu(K^1)$, and its meridian $S^1 \times 0$ bounds a disk in $\nu(K^1)$.

$S^4 = (B^3 \times I)/\big((x,1) \sim (f(x),0),\ y \times S^1 \sim \text{pt for } y \in \partial B^3\big)$. Let $T \subset B^3$ be the torus $\partial(\nu(K^1 \cup \partial B^3))$. Parametrize $T = S^1 \times S^1$ so that $S^1 \times 0$ is a meridian of K^1 and $[0 \times S^1] = 0 \in H_1(B^3 \setminus K^1)$ (ie $0 \times S^1$ is a 0–framed longitude for K^1). Let $T \times I$ be a regular neighborhood of T contained in the interior of $B^3 \setminus K^1$.

Define $\tau, \rho\colon T \times I \to T \times I$ by $\tau(x,y,t) = (x + 2\pi t, y, t)$ and $\rho(x,y,t) = (x, y + 2\pi t, t)$. Extend τ and ρ to the rest of B^3 by the identity map. Then $\tau^n \rho^p K^1$ is called the *n–twist p–roll spun knot* of K^1. (When $n = 0$ or $p = 0$, we may say p–roll spun knot of K^1 or n–twist spun knot of K^1, respectively.) See Figure 14 if this construction is unfamiliar.

Let K be a classical knot in S^3 (that is, a 1–*knot*) and $B \subset S^3$ a small 3–ball meeting K in a trivial arc. If $(B^3, K^1) = (S^3 \setminus B, K \setminus B)$, then we may write fK to indicate fK^1, and refer to the f–deform-spun knot of K^1 and the f–deform-spun knot of K interchangeably.

Theorem 6.22 *Let K be a 1–knot, so that $\tau^n K$ is the n–twist spun knot of K. Then $U_{\tau^n K}$ is isotopic to $\mathbb{C}P^1$.*

Proof In Figure 15, we demonstrate an isotopy in $\mathbb{C}P^2$ taking $U_{\tau^n K}$ to $U_{\tau^{n+1} K}$.

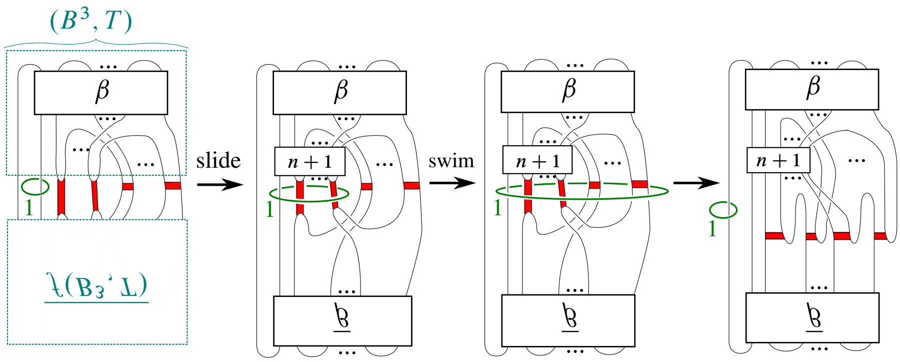

Figure 15: Left to right: a banded unlink diagram for $\tau^n K \# \mathbb{C}P^1$ (see eg [21, Section 5.2]), where K in bridge position is the closure of tangle β. We slide half the bands over the 2–handle of $\mathbb{C}P^2$, and then swim the 2–handle through the remaining bands. We obtain a banded unlink for $\tau^{n+1} K \# \mathbb{C}P^1$.

Inductively, $U_{\tau^n K}$ is isotopic to $U_{\tau K}$. By [34], τK is the unknot, so $U_{\tau^n K}$ is isotopic to $\mathbb{C}P^1$. □

We can say something stronger about the general family of deform-spun knots.

Theorem 6.23 *Let K be a 1–knot. Let fK be a deform-spun knot of K. Then $fK \# \mathbb{C}P^1$ is isotopic to $\tau^n fK \# \mathbb{C}P^1$ for any n.*

Proof Figure 16 shows an explicit isotopy from $fK \# \mathbb{C}P^1$ to $\tau f K \# \mathbb{C}P^1$. □

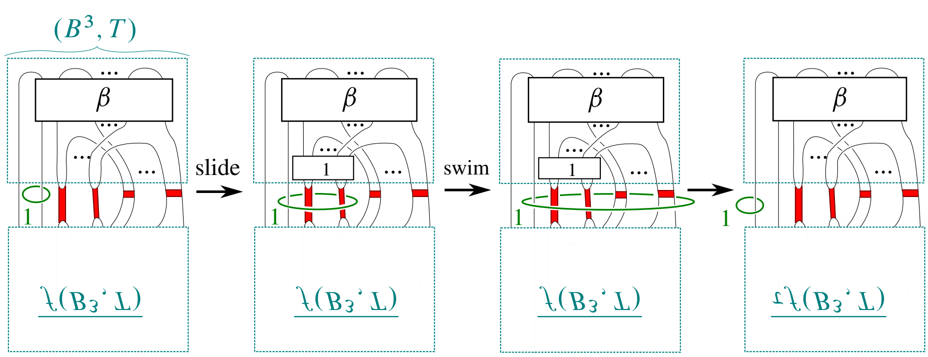

Figure 16: Left to right: a diagram for $fK\#\mathbb{C}P^1$, where K in bridge position is the closure of the tangle β. We slide half the bands over the 2–handle of $\mathbb{C}P^2$, and then swim the 2–handle over the remaining bands. We obtain a diagram for $\tau f K \# \mathbb{C}P^1$.

Corollary 6.24 *Let K be a knot with an integral lens space surgery. Then $\tau^n \rho K \# \mathbb{C}P^1$ is isotopic to the standard $\mathbb{C}P^1$ for any $n \in \mathbb{Z}$.*

Proof Say k–surgery on K yields a lens space $L(k,q)$. Teragaito [30] observed that $\tau^k \rho K$ is the unknot, as follows:

By Litherland [19], $\tau^k \rho K$ is a fibered knot, whose fiber is obtained by 1–Dehn filling the k–fold cyclic cover of $S^3 \setminus \nu(K)$ and then deleting an open 3–ball. That is, the closure of the fiber is a k–fold cover of $L(k,q)$, so the closure of the fiber is S^3. Therefore, $\tau^k \rho K$ bounds a smooth 3–ball.

By Theorem 6.23, $\tau^n \rho K \# \mathbb{C}P^1$ is isotopic to $\tau^k \rho K \# \mathbb{C}P^1 = \mathbb{C}P^1$ for any n. □

6.3 Satellites and miscellaneous examples

Consider the family of 2–knots K_{pq} illustrated in Figure 17, top. Nash and Stipsicz [25] showed via Kirby calculus that the Gluck twist on any of these 2–knots yields S^4. In fact, by translating their handle slides into band moves, we observe that $K_{pq} \# \mathbb{C}P^1$ is isotopic to the standard $\mathbb{C}P^1$ in $\mathbb{C}P^2$.

Most of the results of Sections 6.1 and 6.2 can be consolidated into the single following statement:

Theorem 6.25 *Let $F = S \# \mathbb{C}P^1 \subset \mathbb{C}P^2$ be a genus-g unit surface knot, where $S \subset S^4$ is an orientable surface that is 0–concordant to a band-sum of twist-spun knots and unknotted surfaces. Then F is isotopic to $\mathbb{C}P^1 \# gT$, where $\mathbb{C}P^1 \# gT$ indicates the standard $\mathbb{C}P^1$ trivially stabilized g times.*

The results of Theorems 6.22 and 6.23 extend to satellite knots. This illustrates the strength of this diagrammatic packaging, as in general these knots may not be twist-spins or even fibered (see eg [32]).

Definition 6.26 Let $K_P \subset V$ be a 2–sphere embedded in $V \cong S^2 \times D^2$. Let K_C be a 2–sphere embedded in S^4. Fix a diffeomorphism $\phi\colon V \to \nu(K_C)$. Let $K = f(K_P) \subset S^4$. We call K the *satellite* of *companion* K_C with *pattern* (K_P, V).

Let K be the satellite of companion K_C with pattern (K_P, V). To obtain a diagram of K, we view $S^4 = S^3 \times [-1,1]/(S^3 \times 1 \sim \text{pt},\ S^3 \times -1 \sim \text{pt})$, where $K_C \cap S^3 \times 0$

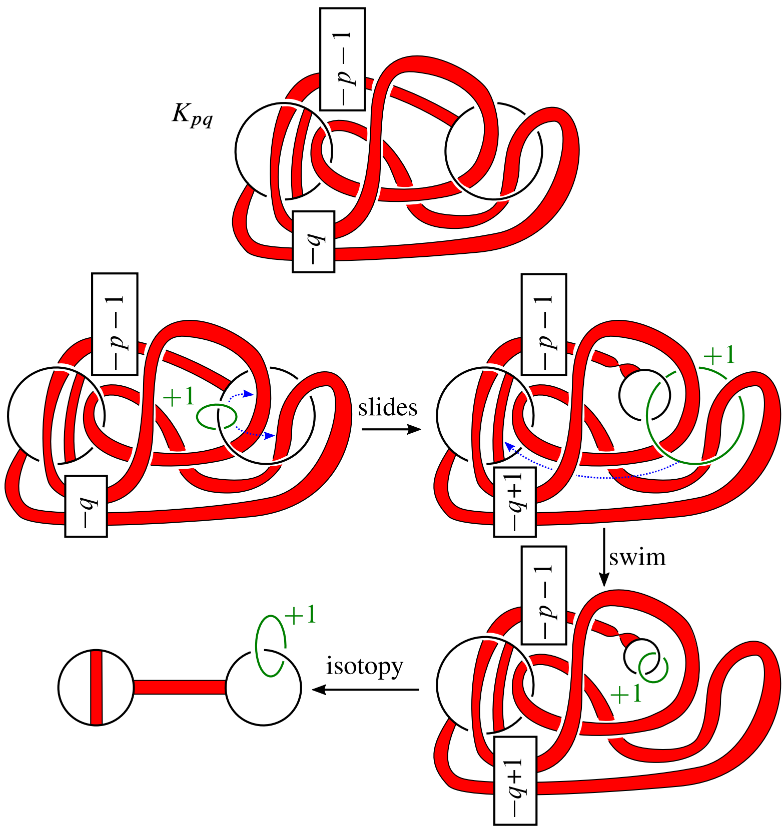

Figure 17: Top: the 2–knot $K_{pq} \subset S^4$. Nash and Stipsicz [25] showed that performing the Gluck twist on K_{pq} yields S^4. Bottom two rows: an isotopy from $K_{pq} \# \mathbb{C}P^1$ to the standard $\mathbb{C}P^1$ in $\mathbb{C}P^2$.

is a connected knot and $K_C \cap (S^3 \times [-1,0])$ and $K_C \cap (S^3 \times [0,1])$ are ribbon disks (this is the normal form of [15]).

We take $V = S^2 \times D^2 \subset S^4$ (a neighborhood of the standard unknotted sphere) so that $W := V \cap (S^3 \times 0) \cong S^1 \times D^2$, and $V \cap (S^3 \times [-1,0]) \cong V \cap (S^3 \times [0,1]) \cong D^2 \times D^2$. See Figure 18, left, for a schematic.

Draw a banded unlink (in $\mathcal{K}_0$) for K_P sitting inside W. (Note the original unlink and the one obtained by resolving all bands are unlinked in $B^3 = W \cup (0\text{–framed 2–handle})$ but may be nontrivial within the solid torus W.) Fix a meridian disk Δ of W disjoint from all bands (note that if Δ intersects bands k times, then we can remove these intersections at the cost of adding $2k$ canceling intersections between Δ and the unlink)

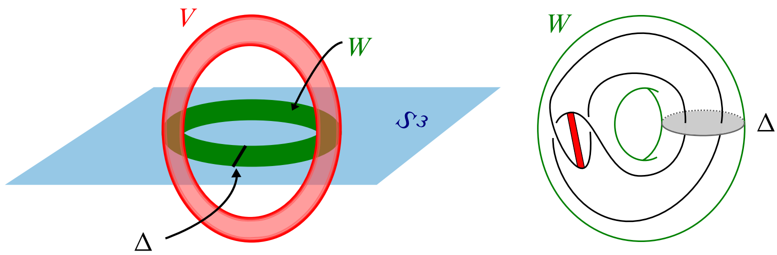

Figure 18: Left: a schematic of V embedded in S^4. The intersection $V \cap S^3 \times 0$ is the solid torus W. Right: a banded unlink of a pattern $K_P \subset V$. With respect to this choice of Δ, the pattern has geometric winding 2. (Note in this diagram K_P is unknotted inside V, so could be isotoped to a pattern of geometric winding 0.)

in the diagram for K_P, and take K_P to be transverse to Δ. We say the *geometric winding* of K_P is the unsigned intersection $K_P \cap \Delta$. (Note this number depends on Δ.)

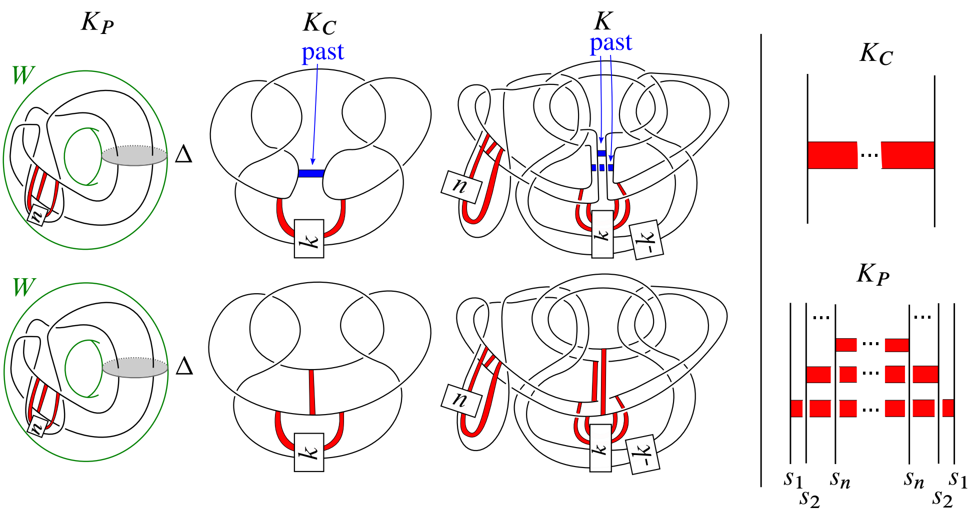

Figure 19: Left, top row: We draw a banded unlink diagram for K_P and a normal form diagram for K_C (a band diagram in which some bands lie below the pictured cross-section, as indicated, so that intersection of K_C with this time slice is a knot). From these two diagrams, we obtain a diagram of K. Left, bottom row: We push all bands above the pictured cross-section, so that each diagram is a banded unlink diagram. Right: We show how to draw the n bands of a banded unlink diagram for K corresponding to a band in the banded unlink diagram for K_C. In this picture, s_i indicates one arc in $\phi(\Delta \times I)$.

Now draw a banded unlink (in $\mathcal{K}_0$) for K_C, isotoped to lie inside

$$W' := \nu(K_C) \cap (S^3 \times 0) \cong S^1 \times D^2$$

(we first draw the diagram in $S^3 = S^3 \times 0$ and perturb to be disjoint from $0 \times S^1 \subset S^3 \setminus W \cong B^3 \times S^1$, and then project to W). Fix a meridian disk Δ' for W' which intersects the banded unlink transversely in one point.

Isotope the diffeomorphism $\phi\colon V \to \nu(K_C)$ so that $\phi(W) = W'$ and $\phi(\Delta \times I) = W' \setminus (\Delta' \times I)$. Choose ϕ so that every saddle of K either corresponds to a saddle of K_P or lies above or below a saddle of K_C. Each saddle of K_C gives rise to $|K_P \cap \Delta|$ saddles of K.

Therefore, the satellite K has $a + b|K_P \cap \Delta|$ critical points of index 1, where a is the number of index-1 critical points of K_P and b the number of index-1 critical points of K_C. Then we obtain a banded unlink diagram (in $\mathcal{K}_0$) for K by taking the standard 0–framed satellite of $K_P \cap (S^3 \times 0) \subset W$ around $K_C \cap (S^3 \times 0)$ (both of these cross-sections are knots), attaching the bands corresponding to K_P, and then

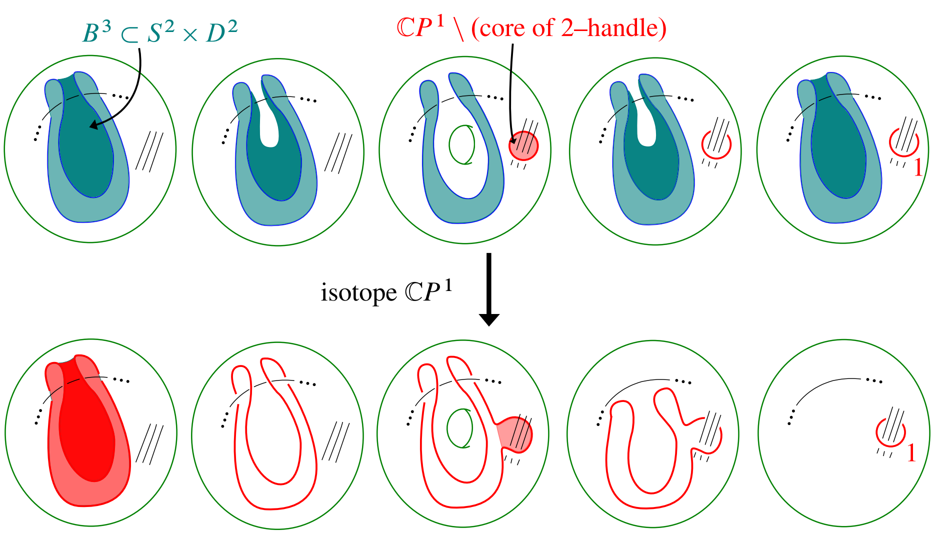

Figure 20: Top: a movie of cross-sections of $S^2 \times D^2 \# \mathbb{C}P^2$. The central cross-section is a solid torus W. In shaded blue, we draw a ball B in $S^2 \times D^2$. In red (to the right), we draw the disk $\mathbb{C}P^1 \cap S^2 \times D^2$. The rest of $\mathbb{C}P^1$ is a core of the 1–framed 2–handle. Black strands are contained in $K_P \subset S^2 \times D^2$. Bottom: we isotope $\mathbb{C}P^1$ through B while fixing the unit surface for K_P.

attaching $|K_P \cap \Delta|$ copies of each band corresponding to K_C (pushing them above $S^3 \times 0$). See Figure 19.

Very roughly, we obtain a banded unlink diagram for K by tubing a diagram for K_P to $|K_P \cap \Delta|$ parallel (up to orientation) diagrams for K_C.

We give a new isotopy move of $\mathbb{C}P^1$ inside $(S^2 \times D^2) \# \mathbb{C}P^2$: the double slide. Refer to Figure 20. The effect of the double slide move is to change the intersection of $\mathbb{C}P^1$ with W by two slides over a longitude of W (with opposite sign). Via this move, we may replace the intersection γ of $\mathbb{C}P^1$ with W with any curve $\gamma' \subset W$ such that γ and γ' are isotopic in $W \cup$ (0–framed 2–handle) and represent the same element of $H_1(W)$.

Theorem 6.27 *Let K be a satellite of companion K_C with pattern (K_P, V). Say $[K_P] = m[S^2 \times \mathrm{pt}]$ in $H_2(V;\mathbb{Z})$. View $V \subset S^4$ as a neighborhood of an unknotted 2–sphere. Assume $K_P \subset V \subset S^4$ is 0–concordant to a band-sum of twist-spun knots.*

- *If $m = 0$, then $K \# \mathbb{C}P^1$ is isotopic to $\mathbb{C}P^1$ in $\mathbb{C}P^2$.*
- *If $m = \pm 1$, then $K \# \mathbb{C}P^1$ is isotopic to $\pm K_C \# \mathbb{C}P^1$ in $\mathbb{C}P^2$.*
- *If $|m| > 1$, then $K \# \mathbb{C}P^1$ is isotopic to $K' \# \mathbb{C}P^1$, where K' is a satellite with companion K_C and pattern (O_m, V), where $O_m \subset V \subset S^4$ is the unknotted sphere K_0 and $[O_m] = m[S^2 \times \mathrm{pt}] \in H_2(V;\mathbb{Z})$.*

We are careful to distinguish between the unknotted sphere K_0 in S^4 and the degree-m unknotted pattern (O_m, V).

Remark 6.28 The pattern $O_m \subset V$ is well defined. We have $V = S^4 \setminus \nu(\gamma)$, where γ is a curve in $S^4 \setminus K_0$ with $[\gamma] = m \subset \mathbb{Z} \cong \pi_1(S^4 \setminus K_0)$. In this dimension, γ is unique up to isotopy, so $O_m \subset V$ is uniquely determined by m. See Figure 21 for a banded unlink diagram (with no bands) for $O_m \subset V$. The satellite knot K' is isotopic to $|m|$ parallel copies of $(m/|m|)K_C$ tubed together.

Proof of Theorem 6.27 Let F be the surface $K_P \# \mathbb{C}P^1 \subset (S^2 \times D^2) \# \mathbb{C}P^2$. The banded unlink diagram for $K_P \# \mathbb{C}P^1 \subset \mathbb{C}P^2$ sits inside the solid torus $W \subset S^3$, with a 1–framed 2–handle attaching circle γ at the site of the blowup. See Figure 22. Apply the isotopy of Theorem 6.25 to unknot the banded unlink diagram. So long as γ stays in W, this isotopy induces isotopy on F in $(S^2 \times D^2) \# \mathbb{C}P^2$. But γ passes outside of W ("through the hole of W") an even number of times — that is, γ appears to slide over a longitude of W an even number of times, an equal number of each direction of

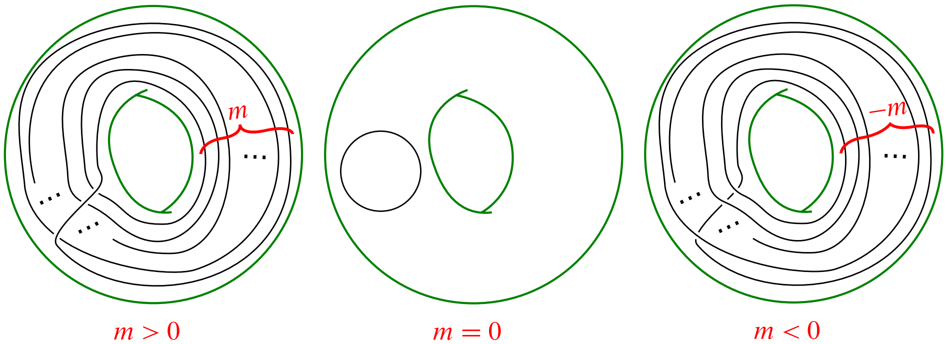

Figure 21: A banded unlink diagram for $O_m \subset V$. In this diagram, O_m has one local minimum, one local maximum and zero saddles (ie zero bands).

slide. Achieve these slides through a sequence of double slide moves. See Figure 22. This isotopy does not fix the standard $\mathbb{C}P^1$, but replaces F with $O_m\#(\text{standard } \mathbb{C}P^1)$.

Let $K' \subset S^4$ be the satellite knot with pattern O_m and companion K_C. Then $K\#\mathbb{C}P^1$ is isotopic to $K'\#\mathbb{C}P^1$ in $\mathbb{C}P^2$. Note that if $m=0$, then K' is the unknot. If $m=\pm 1$, then $K'=\pm K_C$. □

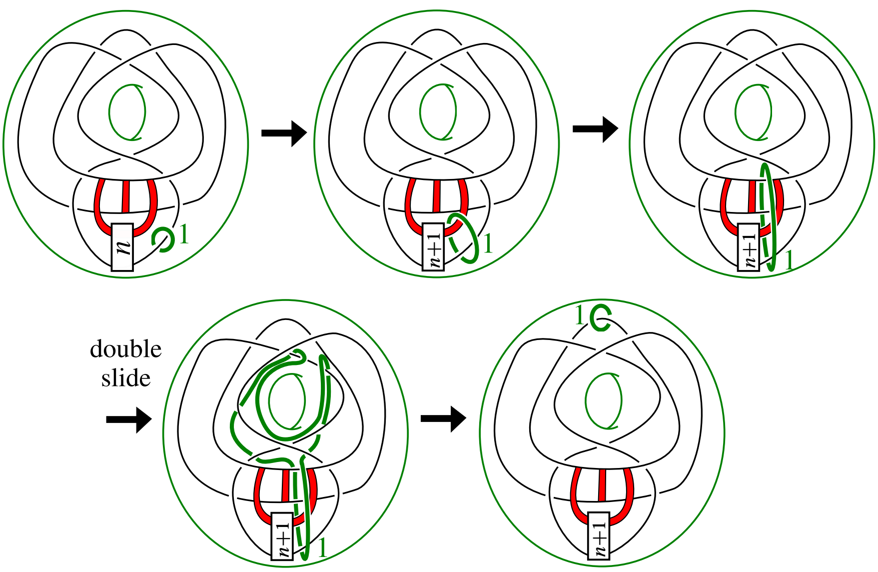

Figure 22: We unknot $K_P\#\mathbb{C}P^1$ in $S^2\times D^2\#\mathbb{C}P^2$ when K_P is 0–concordant to a band-sum of twist-spun knots. We perform the isotopy of Theorem 6.25, performing the double slide move to slide $\mathbb{C}P^1\cap W$ over the longitude of W and back.

The following corollary follows immediately from Theorem 6.27 and the relation of unit surfaces to the Gluck twist from [23]:

Corollary 6.29 *Let K be a satellite with companion K_C and pattern (K_P, V). Say $[K_p] = m[S^2 \times \mathrm{pt}]$ in $H_2(V;\mathbb{Z})$. View $V \subset S^4$ as a neighborhood of an unknotted 2–sphere. Assume $K_P \subset V \subset S^4$ is 0–concordant to a band-sum of twist-spun knots.*

- *If $m = 0$, then the Gluck twist on S^4 about K is diffeomorphic to S^4.*
- *If $m = \pm 1$ and the Gluck twist on S^4 about K_C yields S^4, then so does the Gluck twist on S^4 about K.*

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Department of Mathematics, Brigham Young University
Provo, UT, United States

Center for Geometry and Physics, Institute for Basic Science (IBS)
Pohang, South Korea

Department of Mathematics, Massachusetts Institute of Technology
Cambridge, MA, United States

hughes@mathematics.byu.edu, math751@gmail.com, maggiehm@mit.edu

https://math.byu.edu/~hughes/, https://sites.google.com/view/seungwonkim, https://web.math.princeton.edu/~maggiem

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