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Graph manifolds as ends of negatively curved Riemannian manifolds

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Let M be a graph manifold such that each piece of its JSJ decomposition has the $\mathbb{H}^2\times\mathbb{R}$ geometry. Assume that the pieces are glued by isometries. Then there exists a complete Riemannian metric on $\mathbb{R}\times M$ which is an "eventually warped cusp metric" with the sectional curvature K satisfying $-1\le K<0$.

A theorem by Ontaneda then implies that M appears as an end of a 4–dimensional, complete, noncompact Riemannian manifold of finite volume with sectional curvature K satisfying $-1\le K<0$.

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Dedicated to Professor Kenji Fukaya on his 60th birthday

1 Introduction and main theorem

1.1 Ends of manifolds of negative curvature

If a noncompact manifold N is the interior of some compact manifold with boundary, then each boundary component is called an *end* of N. Let N be a complete, noncompact Riemannian manifold of finite volume such that the sectional curvature K satisfies $-1\le K<0$. It is known by Ballmann, Gromov and Schroeder [4] that N is diffeomorphic to the interior of a compact manifold, $\overline{N}$, with boundary, $\partial\overline{N}$, that has finitely many components, each of which is an end of N.

It is a wide open question to decide which manifolds M can appear as ends of such Riemannian manifolds. An end is a closed manifold and one general obstruction by Gromov [17, 0.5] is that the simplicial volume of $\partial\overline{N}$, and hence of each end, is zero. Also, the ℓ^2–Betti number and the Euler characteristic vanish (see Belegradek [7, Corollary 15.7]). It is a theorem by Avramidi and Nguyen-Phan [3, Corollary 5] that if the dimension of an end we consider is at most 4, then it is aspherical. In this paper we address the question: which aspherical manifolds can appear as such ends?

For example, an n–dimensional torus appears as an end of an $(n+1)$–dimensional hyperbolic manifold of finite volume. Other examples of ends are circle bundles

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over some hyperbolic manifolds of various dimension; see Fujiwara [16] (see also Belegradek [5] and Minemyer [22] for the complex hyperbolic versions). In contrast to tori, such bundles will not be ends of any complete, noncompact Riemannian manifold of finite volume such that $-1 \le K \le -c < 0$ for some $c > 0$, since under this curvature assumption, the fundamental group of an end has to be virtually nilpotent.

In dimension 2, if a closed, aspherical manifold has zero simplicial volume, then it is a torus or a Klein bottle, and it appears as an end, for example in the figure-eight knot complement and in the Gieseking manifold. In dimension 3, any closed aspherical manifold M is irreducible, has an infinite fundamental group and its universal cover is $\mathbb{R}^3$; see Lück [21]. If the simplicial volume of such M is 0 then it is a graph manifold.

A *graph manifold* is an aspherical, closed 3–manifold whose JSJ decomposition along embedded incompressible tori/Klein bottles contains only Seifert fibered spaces. Abresch and Schroeder [1] proved certain graph manifolds appear as ends. Our theorem will provide a large class of graph manifolds that appear as ends, and their examples are contained in our class (but for such a manifold M, their manifold N that has M as an end is different from ours). Also, every 3–dimensional sol-manifold appears as an end; see Nguyen-Phan [23].

Other known examples are infranilmanifolds; see Ontaneda [24] and Belegradek and Kapovitch [8], and Belegradek [6] has an axiomatic construction from known examples.

1.2 Eventually warped product cusp metric

In this paper we show that a family of (3–dimensional) graph manifolds occurs as ends of complete, noncompact, Riemannian manifolds of finite volume whose sectional curvature K satisfies $-1 \le K < 0$.

To explain our strategy, recall the following groundbreaking theorem by Ontaneda [24]. If a (not necessarily connected) manifold B is diffeomorphic to the boundary of a connected, smooth, compact manifold N, then we say that B *bounds* N. We sometimes say B *bounds* without specifying N.

Theorem 1.1 (Ontaneda) *Let B be a closed manifold such that either* $\dim B \le 4$ *or the Whitehead group* $\mathrm{Wh}(B)$ *vanishes.*

Assume $\mathbb{R} \times B$ admits a complete Riemannian metric g such that

(1) *there exists a constant $C < 0$ with the sectional curvature of g satisfying $C \le K < 0$,*

(2) $(-\infty, 0] \times B$ *has finite* g*–volume,*

(3) *there is* $D > 0$ *such that on* $[D, \infty) \times B$*, the metric* g *is of the form* $g = dr^2 + e^{2r} g_B$ *for some Riemannian metric* g_B *on* B*.*

Then $B \sqcup B$ *bounds a manifold whose interior admits a complete Riemannian metric of finite volume and with sectional curvature in* $[-1, 0)$*.*

A metric on $\mathbb{R} \times B$ that satisfies the condition (2) is called a *cusp* metric and an *eventually warped* (*cusp*) *metric* if it satisfies (3). This theorem is stated only implicitly in [24] (see [6], where the result is quoted in this form), since it is an intermediate claim, but a detailed argument is given. The actual value of C is not important and we can take $C = -1$ by rescaling g.

We will show that for a manifold B in certain families there exists a Riemannian metric on $\mathbb{R} \times B$ that is an eventually warped cusp metric with $C \leq K < 0$ for some $C < 0$. Then Theorem 1.1 implies that B is an end of a manifold of negative curvature.

This argument appears in [24] for the infranil manifolds (the existence of a desired metric is known by Buyalo and Kobelsky [10]) then also is used in [6; 23] to construct other examples of ends.

1.3 Graph manifolds and flip manifolds

To illustrate the first family we handle, let W be a 3–dimensional manifold which is diffeomorphic to $\Sigma \times S^1$, where Σ is a compact surface with nonempty boundary and S^1 is a circle. Each boundary component of W is a torus, $S^1 \times S^1$, where the first factor is a boundary component of Σ. We put an orientation to each factor. We call such W a *piece*, and Σ the *base* surface of W. We construct a *closed*, connected, 3–dimensional manifold M, which is a graph manifold, from a finite collection of pieces by gluing a pair of boundary tori by a diffeomorphism, a *gluing map*. There are two special maps for gluing: the trivial map mapping the first factor to the first factor and the second one to the second; and the *flip map* interchanging the first and second factors. We preserve the orientation of the factor. We say M is a *flip manifold* — see Kapovich and Leeb [18] — if each gluing map is either the trivial map or the flip map.

Some remarks are in order. There are eight ways to glue a pair of boundary tori: two ways to put an orientation on each of the two S^1, then a trivial map or a flip map. If Σ is a closed surface, then $\Sigma \times S^1$ is considered as a flip manifold made from one piece of two boundary components, where the gluing map is trivial.

More generally, maybe the S^1–fibers are nonorientable, and/or a piece is a Seifert fibered space, S; see Scott [25, Section 3]. Let $s_1, \dots, s_n$ be the singular fibers of S where the twist at s_i is by the q_i/p_i of a full twist. The pair (q_i, p_i) is called the *orbit invariant* of s_i, which is a pair of coprime integers with $0 < q_i < p_i$. One can say that a Seifert fibered space is an S^1–bundle over a base orbifold, where the singular fibers occur at the orbifold points, while $\Sigma \times S^1$ is a (trivial) S^1–bundle over the surface Σ.

A *generalized flip manifold* is a generalization of a flip manifold where we allow Seifert fibered spaces in addition to products $\Sigma \times S^1$ as pieces in the definition. Of course we only consider gluing maps that are diffeomorphisms.

We call a base surface/orbifold Σ *hyperbolic* if we can put a complete hyperbolic (orbifold) metric of finite area to the interior of Σ. We denote by Σ^o the interior of Σ. An S^1–bundle over Σ^o has a Riemannian metric that is locally a product of the hyperbolic metric and S^1 (see [25]). In other words, it has the *geometry* of $\mathbb{H}^2 \times \mathbb{R}$, or the metric is of *type* $\mathbb{H}^2 \times \mathbb{R}$. We only consider pieces of this kind in this paper.

We truncate a small neighborhood of each cusp of Σ^o such that the each boundary component of the universal cover of the truncated Σ^o with respect to the hyperbolic metric is a horoline in $\mathbb{H}^2$. Since Σ is diffeomorphic to the truncated Σ^o, we obtain a Riemannian metric on the S^1–bundle over Σ such that each boundary torus/Klein bottle is flat. We also say this metric is of *type* $\mathbb{H}^2 \times \mathbb{R}$.

We say a graph manifold M has a *geometrization* if one can put a Riemannian metric of type $\mathbb{H}^2 \times \mathbb{R}$ on all pieces such that every gluing map along the boundary tori/Klein bottle is an *isometry*. Some remarks are in order. We do not assume that each gluing map is a trivial map or a flip map; see Example 2.9. A metric of type $\mathbb{H}^2 \times \mathbb{R}$ on a piece is not unique. Without loss of generality, we may assume that there is a small $c > 0$ such that the length of the fibers of the pieces and the length of the boundary components of the pieces are c. The resulting metric on M after gluing the pieces is only C^1. In [10], they use the term *isometric geometrization* instead of geometrization (see Remark 2.10).

We prove:

Theorem 1.2 *Let M be a graph manifold such that each piece has the geometry of $\mathbb{H}^2 \times \mathbb{R}$. Assume M has a geometrization (ie the gluing is isometric). Then there is a complete Riemannian metric g on $\mathbb{R} \times M$ that is an eventually warped cusp metric with the sectional curvature K satisfying $C \le K < 0$ for some constant $C < 0$.*

Remark 1.3 The metric in the above theorem is taken to be C^∞. This is always the case for other results in this paper too.

By rescaling the metric g we may always take $C = -1$. Theorem 1.2 has a generalization to high-dimensional manifolds; see Theorem 1.6.

Since dim $M = 3$, combining Theorems 1.2 and 1.1, we immediately obtain:

Corollary 1.4 (graph manifolds with a geometrization) *Let M be a graph manifold such that each piece has the geometry of $\mathbb{H}^2 \times \mathbb{R}$. Assume M has a geometrization.*

Then there exists a 4–dimensional, complete, noncompact Riemannian manifold N of finite volume, with sectional curvature K satisfying $-1 \le K < 0$, and with M appearing as an end. More precisely, there is a compact subset C in N such that $N \setminus C$ has two connected components, and that each component is diffeomorphic to $M \times (0, \infty)$.

It is known that among graph manifolds M whose pieces have the geometry of $\mathbb{H}^2 \times \mathbb{R}$, M has a geometrization if and only if it has a Riemannian metric of nonpositive sectional curvature by Leeb [19] and Leeb and Scott [20]. Hence we can rephrase our results as follows:

Corollary 1.5 (graph manifolds of nonpositive curvature) *Let M be a closed graph manifold such that each piece has the geometry of $\mathbb{H}^2 \times \mathbb{R}$. Assume M has a Riemannian metric of nonpositive curvature. Then the conclusions of Theorem 1.2 and Corollary 1.4 hold.*

1.4 High-dimensional graph manifolds

There are several notions of high-dimensional graph manifolds (see Frigerio, Lafont and Sisto [15]) and one can prove a high-dimensional version of Theorem 1.2. The main part of the proof of the theorem is by constructing a suitable Riemannian metric, which is same for the high-dimensional case.

Fix $n \ge 2$ and $m \ge 1$. Let X be an n–dimensional complete, noncompact, hyperbolic manifold of finite volume such that the cross-section of each cusp is an $(n-1)$–dimensional torus. Let $\Sigma \subset X$ be a compact manifold with boundary obtained by truncating a sufficiently small neighborhood of each cusp from X so that each boundary component is a flat torus. The interior of Σ is diffeomorphic to X. Take a

Riemannian product $W = \Sigma \times T^m$, where T^m is an m–dimensional flat torus. Each boundary component of W is an $(n+m-1)$–dimensional torus. We call W a piece, and Σ the base.

Suppose a closed $(n+m)$–dimensional manifold M is obtained from pieces with various bases by gluing pairs of boundary components of the pieces by diffeomorphisms; then we call M a *high-dimensional graph manifold*. We say M has *geometrization* if all gluing maps are isometric with respect to the product metric on the pieces.

Theorem 1.6 (high-dimensional graph manifolds) *Let M be an $(n+m)$–dimensional high-dimensional graph manifold. Assume M has a geometrization. Then M carries a metric of nonpositive curvature, so that $\mathrm{Wh}(M)$ vanishes. Also, there is a complete Riemannian metric g on $\mathbb{R} \times M$ that is an eventually warped cusp metric with the sectional curvature K satisfying $C \le K < 0$ for some constant $C < 0$.*

Remark 1.7 We only consider a product metric on each piece, but we can formulate the result for locally product metrics as in Theorem 1.2.

As before, a corollary follows from Theorem 1.1:

Corollary 1.8 *Let M be an $(n+m)$–dimensional high-dimensional graph manifold. Assume it has a geometrization. Then there exists an $(n+m+1)$–dimensional, complete, noncompact Riemannian manifold N of finite volume, with sectional curvature K satisfying $-1 \le K < 0$, and with M appearing as an end.*

1.5 The other construction

We discuss the other family of examples of ends. This family contains manifolds of various dimension, and in dimension 3, it contains all flip manifolds without a piece whose base surface has genus at most 1. Although it is not necessary, we only treat the orientable case to make the account simple and clear.

A manifold in this family is also obtained by gluing pieces along their boundary, and each boundary component is a circle bundle over a circle bundle over a hyperbolic manifold N. If $\dim N = 0$, the boundary is a torus and we obtain graph manifolds.

Here is a precise description. Let M_i for $i = 1, 2$ be n–dimensional closed hyperbolic manifolds, and N_i totally geodesic, closed submanifolds of codimension two in M_i, respectively, such that $b\colon N_1 \to N_2$ is an isometry. For a sufficiently small $\epsilon > 0$, let P_i be S^1–bundles over $V_i = M_i \setminus N_\epsilon(N_i)$, respectively, with Riemannian metrics which are locally product of the hyperbolic metric on the base manifolds and the circle.

The $\partial P_i = P_i \mid \partial V_i$ are flat torus-bundles over N_i, respectively. We glue P_1 and P_2 along their boundaries by a bundle map whose base map is the isometry $b\colon N_1 \to N_2$ and on the fiber it is a diffeomorphism, for example a trivial map or a flip map, as in the graph manifold case. This gives an $(n+1)$–dimensional manifold, W. If the bundle map is an isometry, then we say it satisfies *the gluing condition* and W has a *geometrization*.

Then:

Theorem 3.1 *Assume W has a geometrization. Then W carries a metric of nonpositive curvature, so that $\mathrm{Wh}(W)$ vanishes. Also, $\mathbb{R} \times W$ carries a complete Riemannian metric that is an eventually warped cusp metric with $C \le K < 0$ for some constant $C < 0$.*

Combining Theorems 3.1 and 1.1 we obtain:

Corollary 1.9 (piecewise S^1–bundles) *Assume W has a geometrization. Then W appears as an end of an $(n+2)$–dimensional Riemannian manifold Z that is complete, noncompact, of finite volume, with the sectional curvature K satisfying $-1 \le K < 0$.*

1.6 Gluing condition

We examine the gluing condition for a geometrization in the case $n = 3$ in some detail, where n is the dimension of M_i for $i = 1, 2$. Submanifolds N_i for $i = 1, 2$ are simple closed geodesics, and we denote them by γ_i, respectively. By our assumption they have same length. Let m_i be the meridian curve for γ_i in M_i. Each X_i is an S^1–bundle over $M_i \setminus N_\epsilon(\gamma_i)$. We denote by σ_i the fiber circle of X_i. With respect to the Riemannian metric, we can measure the monodromy (ie rotation) along the curve γ_i for σ_i and m_i, respectively. We denote them by $0 \le \theta(\sigma_i), \theta(m_i) < 2\pi$.

We now consider a bundle map $\phi\colon \partial X_1 \to \partial X_2$ such that the base map is the isometry b and that it is a flip map on the torus fiber:

$$\phi(m_1) = \sigma_2, \quad \phi(\sigma_1) = m_2.$$

Then we can arrange ϕ to be an isometry, ie the gluing condition is satisfied, if and only if

$$\theta(m_1) = \theta(\sigma_2), \quad \theta(\sigma_1) = \theta(m_2) \tag{1-1}$$

We conclude the introduction with examples $\{(M_i, N_i)\}$ that admit circle bundles X_i satisfying (1-1), which give W of $\dim W = 4$ by Theorem 3.1.

Example 1.10 (S^1–bundle with a given monodromy) Take a closed, oriented, hyperbolic 3–manifold M with a simple closed (oriented) geodesic γ which is nontrivial in $H_1(M, \mathbb{R})$. Let m be a *meridian curve* around γ in M. Namely, take an ϵ–neighborhood of γ in M for a small $\epsilon > 0$. Its boundary is a torus, T. Take a small hyperbolic disc D in M perpendicular to γ, then set $m = D \cap T$. Let $\theta(m)$ be the monodromy of m along γ. Another way to define $\theta(m)$ is using the universal cover of M. Lift γ to an (oriented) infinite geodesic $\widetilde{\gamma}$, then take the element $g \in \pi_1(M)$ such that $\widetilde{\gamma}/g = \gamma$, where g shifts to the positive direction. Then g rotates $\widetilde{\gamma}$ by $\theta(m)$.

Set $V = M \setminus N_\epsilon(\gamma)$ with a small $\epsilon > 0$. We will construct an S^1–bundle X over V, and glue X and a copy of X along their boundary and obtain W. Let σ denote the fiber circle. For our construction, we need to arrange $\theta(m) = \theta(\sigma)$.

For example, let M be such that all of its closed geodesics are simple (such examples exist; see Chinburg and Reid [13]). Taking a finite cover if necessary, we may assume that $H_1(M, \mathbb{R})$ is nontrivial; see Agol [2]. Let $p\colon \pi_1(M) \to H_1(M, \mathbb{R})$ be the homomorphism obtained from abelianization. Take any closed (simple) geodesic γ with $p([\gamma]) \neq 0$. Then $H_1(\gamma, \mathbb{R})$ injects to $H_1(M, \mathbb{R})$.

Set $\theta_0 = \theta(m)$. Then there is a homomorphism $h\colon \pi_1(M) \to S^1$ such that $h([\gamma]) = \theta_0$. Indeed, we take a homomorphism $f\colon \mathbb{Z} \to S^1$ such that $f(p([\gamma])) = \theta_0$ then set $h = f \circ p$.

Now take an S^1–bundle X over M, which is locally a Riemannian product whose monodromy representation of $\pi_1(M)$ to S^1 is h. Then $\theta(\sigma) = \theta_0$.

Now take (M, γ) and its copy; then this pair satisfies the condition (1-1), so that Theorem 3.1 applies.

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2 Proof of Theorems 1.2 and 1.6

We will prove Theorem 1.2. We first treat the case where every piece in a graph manifold is the product of a circle and a surface, then discuss the general case.

We then prove Theorem 1.6. The main part of the proof overlaps with the proof of Theorem 1.2, which is Proposition 2.5.

2.1 Geometric idea

Before we start the metric construction, we explain the difficulty and outline our method to construct a desired metric on $\mathbb{R} \times M$, where each piece of M is a trivial bundle over a surface.

As the first step, we put a Riemannian metric of nonpositive curvature on M. This part is straightforward. We review the metric construction; see [18]. By assumption the interior of the base surface Σ_i of each piece P_i has a hyperbolic metric g_0 of finite volume.

Choose a small constant $c > 0$. Truncate the interior of Σ_i with the metric g_0 at each cusp so that each boundary circle has constant geodesic curvature and has length c. We identify this truncated surface with g_0 and Σ_i.

To express the idea clearly, we first assume M has a geometrization with respect to a product metric on each piece, $P_i = \Sigma_i \times S^1$, namely it is a Riemannian product $\Sigma_i \times S^1(c)$, where $S^1(c)$ is a circle of length c. The curvature satisfies $-1 \le K \le 0$. Each boundary component of P_i is $S^1(c) \times S^1(c)$, so that we can glue the P_i along their torus boundaries by the prescribed gluing homeomorphisms (up to homotopy), which are isometries by our assumption, and obtain a metric of nonpositive curvature on M. This metric has singularities along the tori where pieces are glued, but we can smooth it out keeping the curvature condition $C \le K \le 0$ for some $C < 0$.

In the next step we want to put a desired metric on $\mathbb{R} \times M$. As a first attempt, we consider a warped product

$$\mathbb{R} \times_{e^r} M$$

but it does not work in general for the following reason. Since M is compact, the volume of $(-\infty, 0] \times M$ is finite. Also, since M satisfies $C < K \le 0$, the curvature on $\mathbb{R} \times_{e^r} M$ satisfies $K < 0$, but as $r \to -\infty$, the diameter of M tends to 0 and the curvature K tends to $-\infty$, while K tends to -1 as r tends to ∞. So, this construction does not give a desired curvature bound from below for $r \to -\infty$.

As a second attempt, we next replace the warping function e^r by a smooth convex function, $h(r)$, such that $h(r) = e^r$ for $r > 1$ and $h(r) = \frac{1}{2}(1 + e^r)$ for $r < -1$. Then the curvature K is negative and bounded, but the volume of $(-\infty, 0] \times M$ is infinity, which is not good.

To suppress the volume at $r \to \infty$, we try to change the metric we have on M depending the value of $r \in (-\infty, 0]$, namely we rescale the S^1–fiber of each P_i so as to shrink its length as $r \to -\infty$. Remember that the metric on M was obtained by gluing the pieces along their torus boundaries by isometries. The metric of each boundary component used to be $S^1(c) \times S^1(c)$, but after shrinking the S^1–fiber, one factor of $S^1(c) \times S^1(c)$ is rescaled. But the gluing homeomorphisms are, typically, flipping the fiber circles and the boundary circles (of the base surfaces) of a pair of neighboring pieces so that they are no longer isometric in general. In other words, on each $\mathbb{R} \times P_i$, the metric we consider here is a "doubly warped product" in the form of

$$\mathbb{R} \times_{h(r)} (\Sigma_i \times S^1(c) \times_{g(r)} S^1(c))$$

with $g(r)$ some suitable function, but they do not match up along the gluing tori, so that a doubly warped product metric on $\mathbb{R} \times M$ will not do.

Therefore, as a remedy, we also modify the metric on the base surface of each piece depending on r as well for $r < 0$ so that the gluing maps are isometric. This modification is not by rescaling the metric on the surface by a constant, but making its cusp part longer as $r \to -\infty$, so that its boundary circle becomes smaller. Take $a > 0$ such that $e^r \le c$ if $r < -a$. For each $r \in (-\infty, -a]$, truncate the initial complete hyperbolic metric g_0 on the interior of Σ_i so that the boundary circle has length e^r, which we denote by $\Sigma_i(e^r)$. Take a Riemannian product $\Sigma_i(e^r) \times S^1(e^r)$, which is the metric structure on P_i at r. Each boundary component is $S^1(e^r) \times S^1(e^r)$. Now glue them by the given isometries and obtain the metric on M at r, which we denote by M_r. As before we smooth out near the gluing tori. In this way we obtain a metric on $(-\infty, -a] \times M$, which we write as $(-\infty, -a] \times M_r$. Notice that the volume of M_r is (more or less) proportional to e^r, so that one expects the volume of $(-\infty, -a] \times M_r$ is finite. Also, we arrange that the curvature satisfies $C \le K < 0$ (see [16]).

In the final step, we interpolate $(-\infty, -a] \times M_r$ and $[a, \infty) \times_{e^r} M_0$ between $r \in [-a, a]$, where M_0 is M with a Riemannian metric, say, as constructed in the previous paragraph (maybe we rescale it by a constant). Note that the metric on M_0 is fixed for $r \in [a, \infty)$ while M_r keeps changing for $r \in (-\infty, a]$. Also, notice that the diameter of M_r tends to ∞ as $r \to -\infty$.

Lastly, we address the issue that a metric on a piece is in fact only a locally Riemannian product. It turns out that this is not a serious problem. The piece is topologically a trivial S^1–bundle, and it has a locally product metric such that the fiber circles have the same length. The difference from the Riemannian product case is encoded in the monodromy representation of the fundamental group of the base Σ into S^1, viewed as a group, which acts on the fiber circles by rotations. By assumption, our manifold M admits a geometrization, ie pieces are glued by isometries along the tori. In conclusion, the method we explained above will work in this generality without any change because we only use the property that the gluing maps are isometric.

2.2 Metric construction

We denote the group of isometries of $\mathbb{R}^n$ by $\mathrm{Isom}(\mathbb{R}^n)$. In the following we consider a product

$$Y = \mathbb{R} \times \mathbb{R} \times \mathbb{R}^l \times \mathbb{R}^m$$

and, for example, an element of $\mathrm{Isom}(\mathbb{R}^l)$ naturally acts on Y by an isometry that is trivial except on $\mathbb{R}^l$. The Euclidean metrics on $\mathbb{R}^l$ and $\mathbb{R}^m$ are denoted by $d\rho^2$ and $d\tau^2$, respectively. We denote a flat torus of dimension n by T^n. For $n = 1$, we may also write it as S^1.

The goal of the following few subsections is Proposition 2.5, which shows that a certain Riemannian metric g that is invariant by $\mathrm{Isom}(\mathbb{R}^l)$ and $\mathrm{Isom}(\mathbb{R}^m)$ exists on Y. The proof of Proposition 2.5 is by concretely constructing a metric g. If tori T^l and T^m are given as quotients of $\mathbb{R}^l$ and $\mathbb{R}^m$ by isometric actions, then the metric g descends to

$$X = \mathbb{R} \times \mathbb{R} \times T^l \times T^m,$$

which will be used later to prove theorems.

To define the metric g, we prepare several functions. Pick a C^∞ function, R, on $\mathbb{R}$ such that

$$R(r) = \begin{cases} r & \text{if } r \le 1, \\ 3 & \text{if } r \ge 5, \end{cases} \qquad R' > 0 \quad \text{on } (1,5), \qquad -\tfrac{1}{2} \le R'' \le 0.$$

We take a nonnegative C^∞ function, λ, supported in $[-1,1]$ and with $\int_{-1}^{1}\lambda(x)\,dx=1$, then define the *convolution product* of λ and a locally Lebesgue integrable function φ on $\mathbb{R}$ by

$$\lambda*\varphi(x):=\int_{-\infty}^{\infty}\varphi(t)\lambda(x-t)\,dt.$$

Note that $\lambda*\varphi$ is also defined in the case where φ is a finite Borel measure on $\mathbb{R}$, and $\lambda*\varphi$ is a C^∞ function.

Since $\lambda*e^t$ satisfies $(\lambda*e^t)''=\lambda*e^t$, we have $\lambda*e^t=ce^t$, where $c:=(\lambda*e^t)(0)$. We put

$$\bar{f}(s):=\begin{cases}1 & \text{if } s\le 0,\\ e^s & \text{if } s>0,\end{cases}$$

then define a C^∞ function by

$$f:=\lambda*\bar{f}.$$

By the definition,

$$f(s)=\begin{cases}1 & \text{if } s\le -1,\\ ce^s & \text{if } s\ge 1.\end{cases}\tag{2-1}$$

We observe

$$\bar{f}'(s)=\begin{cases}0 & \text{if } s<0,\\ e^s & \text{if } s>0,\end{cases}\tag{2-2}$$

$$\bar{f}''(s)=\begin{cases}0 & \text{if } s<0,\\ \delta_0 & \text{if } s=0,\\ e^s & \text{if } s>0,\end{cases}\tag{2-3}$$

where δ_0 denotes Dirac's delta measure at 0 and we consider the distributional derivative for $\bar{f}''$. Note that $f'=\lambda*\bar{f}'$ and $f''=\lambda*\bar{f}''$. It holds that $f\ge 1$ and $f',f''\ge 0$.

We pick a C^∞ function, h, on $\mathbb{R}$ such that

$$h(r)=\begin{cases}1+e^r & \text{if } r\le -1,\\ 2e^r & \text{if } r\ge 1,\end{cases}\qquad h\ge 1,\qquad h',h''>0\quad\text{on } \mathbb{R}.$$

Let b be a positive constant and F the C^∞ function on $\mathbb{R}^2$ defined by

$$F(r,t):=be^{R(r)}f(t-R(r)).$$

(We may take $b:=1$ in this section; b is needed in the later sections.) Note that $F_t,F_{tt}\ge 0$, where F_t and F_{tt} are the partial derivatives of F. Note also that $F=bce^t$ for all $t\ge 4$.

On Y, we consider the metric

$$g = dr^2 + h(r)^2(dt^2 + b^2 e^{2R(r)}\, d\rho^2 + F(r,t)^2\, d\tau^2), \tag{2-4}$$

where $d\rho^2 = \sum_{\alpha=1}^{l} d\rho_\alpha^2$ is the l–dimensional Euclidean metric and $d\tau^2 = \sum_{\beta=1}^{m} d\tau_\beta^2$ the m–dimensional Euclidean metric. Let us set

$$g = \sum_{i=1}^{n} g_i\, dx_i^2,$$

ie $x_1 := r$, $x_2 := t$, $x_{2+\alpha} := \rho_\alpha$, $x_{2+l+\beta} := \tau_\beta$, $g_1 := 1$, $g_2 := h(r)^2$, $g_{2+\alpha} := H(r)^2$, $H(r) := be^{R(r)}h(r)$ and $g_{2+l+\beta} := h(r)^2 F(r,t)^2$ for $\alpha = 1, 2, \ldots, l$ and $\beta = , 2, \ldots, m$.

Note that, for $t \geq 4$,

$$dt^2 + F(r,t)^2\, d\tau^2 = dt^2 + bce^{2t}\, d\tau^2$$

is a hyperbolic metric.

We calculate the Christoffel symbols:

$$\begin{aligned}
\Gamma_{12}^{2} &= \frac{h'}{h}, & \Gamma_{2,2+l+\beta}^{2+l+\beta} &= \frac{F_t}{F},\\
\Gamma_{22}^{1} &= -hh', & \Gamma_{2+\alpha,2+\alpha}^{1} &= -HH',\\
\Gamma_{1,2+\alpha}^{2+\alpha} &= \frac{H'}{H}, & \Gamma_{2+l+\beta,2+l+\beta}^{1} &= -h^2 FF_r - hh'F^2,\\
\Gamma_{1,2+l+\beta}^{2+l+\beta} &= \frac{F_r}{F} + \frac{h'}{h}, & \Gamma_{2+l+\beta,2+l+\beta}^{2} &= -FF_t
\end{aligned}$$

for $\alpha = 1, 2, \ldots, l$ and $\beta = 1, 2, \ldots, m$. We have the symmetry $\Gamma_{ij}^k = \Gamma_{ji}^k$. Except for this, the rest of the Γ_{ij}^k are zero.

We then calculate the curvature tensor

$$R_{ijk}{}^{l} = \partial_i \Gamma_{jk}^{l} - \partial_j \Gamma_{ik}^{l} + \sum_{m=1}^{n} (\Gamma_{im}^{l}\Gamma_{jk}^{m} - \Gamma_{jm}^{l}\Gamma_{ik}^{m}), \quad R_{ijkl} = R_{ijk}^{l} g_l$$

in the following:

$$\begin{aligned}
R_{1221} = R_{1,2+\alpha,2+\alpha,1} &= -hh'',\\
R_{1,2+l+\beta,2+l+\beta,1} &= -hh''F^2 - h^2 FF_{rr} - 2hh'FF_r,\\
R_{1,2+l+\beta,2+l+\beta,2} &= -h^2 FF_{rt},
\end{aligned}$$

$$
\begin{aligned}
R_{2,2+\alpha,2+\alpha,2} &= -hh'HH', \\
R_{2,2+l+\beta,2+l+\beta,2} &= -h^3h'FF_r - h^2(h')^2F^2 - h^2FF_{tt}, \\
R_{2+\alpha,2+\alpha',2+\alpha',2+\alpha} &= -H^2(H')^2, \\
R_{2+\alpha,2+l+\beta,2+l+\beta,2+\alpha} &= -h^2FF_rHH' - hh'F^2HH', \\
R_{2+l+\beta,2+l+\beta',2+l+\beta',2+l+\beta} &= -h^4F^2F_r^2 - 2h^3h'F^3F_r - h^2(h')^2F^4 - h^2F^2F_t^2
\end{aligned}
$$

for $\alpha, \alpha' = 1, 2, \dots, l$ with $\alpha < \alpha'$ and $\beta, \beta' = 1, 2, \dots, m$ with $\beta < \beta'$. We have the (skew-)symmetry for R_{ijkl}. Except for this, the rest of the R_{ijkl} are zero. Note that the nonzero R_{ijkl} are only of the form R_{ijji} and R_{1jj2} up to the (skew-)symmetry.

The sectional curvatures for the plane spanned by $\{\partial/\partial x_i, \partial/\partial x_j\}$ are

$$
\begin{aligned}
K_{12} &= -\frac{h''}{h}, \\
K_{1,2+\alpha} &= -\frac{H''}{H}, \\
K_{1,2+l+\beta} &= -\frac{F_{rr}}{F} - \frac{2h'F_r}{hF} - \frac{h''}{h}, \\
K_{2,2+\alpha} &= -\frac{h'H'}{hH}, \\
K_{2,2+l+\beta} &= -\frac{F_{tt}}{h^2F} - \frac{h'F_r}{hF} - \frac{(h')^2}{h^2}, \\
K_{2+\alpha,2+\alpha'} &= -\frac{(H')^2}{H^2}, \\
K_{2+\alpha,2+l+\beta} &= -\frac{F_rH'}{FH} - \frac{h'H'}{hH}, \\
K_{2+l+\beta,2+l+\beta'} &= -\frac{F_r^2}{F^2} - \frac{2h'F_r}{hF} - \frac{F_t^2}{h^2F^2} - \frac{(h')^2}{h^2}
\end{aligned}
$$

for $\alpha, \alpha' = 1, 2, \dots, l$ and $\beta, \beta' = 1, 2, \dots, m$ with $\alpha < \alpha'$ and $\beta < \beta'$.

To estimate the sectional curvatures, we prepare a few lemmas.

Lemma 2.1 (1) *$f' \le f$.*

(2) *$f' \le f''$.*

Proof We see $\bar{f}' \le \bar{f}$ and $\bar{f}' \le \bar{f}''$ from (2-2) and (2-3). Taking the convolution product of them with λ yields the lemma. □

Lemma 2.2 (1) *$h \ge 1$ and $h', h'' > 0$.*

(2) $H, H', H'' > 0$.

(3) $F > 0$ and $F_t, F_{tt}, F_r \geq 0$.

(4) *The following functions are all uniformly bounded*:

$$\frac{f'}{f}, \quad \frac{f''}{f}, \quad \frac{h'}{h}, \quad \frac{h''}{h}, \quad \frac{H'}{H}, \quad \frac{H''}{H}, \quad \frac{F_r}{F}, \quad \frac{F_{rr}}{F}, \quad \frac{F_t}{F}, \quad \frac{F_{tt}}{F}.$$

Proof (1) is obvious.

We prove (2). The derivative of $H = bhe^R$ is

$$H' = bhe^R R' + bh'e^R,$$

which is positive. Differentiating it again, we see

$$H'' = be^R(hR'' + h(R')^2 + 2h'R' + h'') \geq be^R(hR'' + h'').$$

If $r \leq 1$, then $hR'' + h'' = h'' > 0$. If $r > 1$, then $hR'' + h'' > -\frac{1}{2}h + h'' = e^r > 0$. Therefore H'' is positive everywhere.

We prove (3). It is clear that $F > 0$. It follows from $f', f'' \geq 0$ that $F_t, F_{tt} \geq 0$. We see

$$F_r = bR'e^R(f(t-R) - f'(t-R)),$$

which is nonnegative by Lemma 2.1 and $R' \geq 0$.

(4) is clear.

We prove (5). The boundedness of f'/f and f''/f follows from (2-1). The boundedness of h'/h, h''/h, T'/T and T''/T are derived from their definitions. We see

$$\begin{aligned}
\frac{H'}{H} &= \frac{hR' + h'}{h}, & \frac{H''}{H} &= \frac{hR'' + h(R')^2 + 2h'R' + h''}{h}, \\
\frac{F_r}{F} &= \frac{R'(f - f')}{f}, & \frac{F_{rr}}{F} &= \frac{(R')^2(f - 2f' + f'') + R''(f - f')}{f}, \\
\frac{F_t}{F} &= \frac{f'}{f}, & \frac{F_{tt}}{F} &= \frac{f''}{f},
\end{aligned}$$

which are all bounded. This completes the proof of the lemma. □

Lemma 2.3 *There is a constant $C < 0$ such that $C \leq K_{ij} < 0$ for all $i \neq j$.*

Proof The lemma is readily seen from Lemma 2.2 except the negativity of $K_{1,2+l+\beta}$. We remark that $F_{rr} \geq 0$ does not hold. We have

$$K_{1,2+l+\beta} = -\frac{\varphi}{fh},$$

where

$$\varphi := h(R')^2(f - 2f' + f'') + hR''(f - f') + 2R'h'(f - f') + h'' f.$$

If $r \leq 1$, then $R = r$, which together with Lemma 2.1 implies $\varphi > 0$. If $r \geq 1$, then $h = 2e^r$ and so

$$\frac{\varphi}{h} = (R')^2(f - 2f' + f'') + R''(f - f') + 2R'(f - f') + f.$$

By $R' \geq 0$, $-\frac{1}{2} \leq R'' \leq 0$, and by Lemma 2.1, we obtain

$$\frac{\varphi}{h} \geq R''(f - f') + f \geq R'' f + f \geq \tfrac{1}{2} f > 0.$$

Therefore, $K_{1,2+l+\beta}$ is negative. □

Let σ be any 2–plane (ie two-dimensional linear subspace) in the tangent space at any point of Y, and take an orthogonal basis, $\{u, v\}$, of σ. Since $\|u \times v\| = \|u\| \cdot \|v\|$, the sectional curvature for σ is

$$K_\sigma = \frac{\langle R(u,v)v, u\rangle}{\|u \times v\|^2} = \frac{\sum_{i,j,k,l} u^i v^j v^k u^l R_{ijkl}}{\sum_{a,b} (u^a)^2 (v^b)^2 g_a g_b},$$

where $u = \sum_i u^i \partial/\partial x_i$, $v = \sum_j v^j \partial/\partial x_j$.

Lemma 2.4 *There is a constant $C < 0$ such that $C \leq K_\sigma < 0$ for all σ.*

Proof As we pointed out, the nonzero R_{ijkl} are only of the form R_{ijji} and R_{1jj2} up to the (skew-)symmetry for our metric g. Also, $R_{ijji} < 0$ by Lemma 2.3. Therefore, for the proof of the negativity of K_σ, it suffices to prove that

$$(u^1)^2(v^j)^2 R_{1jj1} + 2u^1(v^j)^2 u^2 R_{1jj2} + (u^2)^2(v^j)^2 R_{2jj2} < 0$$

for all $u^1, u^2, v^j \neq 0$, where $j = 2 + l + \beta$. This is equivalent to

$$(R_{1jj2})^2 < R_{1jj1} R_{2jj2}. \tag{2-5}$$

We see $(R_{1jj2})^2 = h^4 F^2 F_{rt}^2$ and

$$R_{1jj1} R_{2jj2} = (hh'' F^2 + h^2 F F_{rr} + 2hh' F F_r)(h^3 h' F F_r + h^2 (h')^2 F^2 + h^2 F F_{tt}).$$

We first assume $r \leq 1$. Note that $R = r$ in this case. For (2-5), it is sufficient to prove $F_{rt}^2 \leq F_{rr} F_{tt}$. Since $F_{rt} = be^R(f' - f'')$, $F_{rr} = be^R(f - 2f' + f'')$ and $F_{tt} = be^R f''$, the inequality $F_{rt}^2 \leq F_{rr} F_{tt}$ follows from Lemma 2.1 and $f \geq 1$.

We next assume $r > 1$. In this case, we see $h = 2e^r$, so that (2-5) boils down to

$$(R')^2(f''-f')^2 < [f+(R''+(R')^2)(f-f')+(R')^2(f''-f')+2R'(f-f')] \times [4e^{2r}(f - f') + 4e^{2r} f + f'']. \tag{2-6}$$

We have $f+(R''+(R')^2)(f-f') > 0$ by $R'' \geq -\frac{1}{2}$. We also have $(R')^2(f''-f')^2 \leq (R')^2(f'' - f') \times f''$. Therefore, (2-6) is obtained. The negativity of K_σ has been proved.

We prove the boundedness of K_σ. It suffices to prove the boundedness of each

$$A_{ijkl} := \frac{|u^i v^j v^k u^l R_{ijkl}|}{\sum_{a,b} (u^a)^2 (v^b)^2 g_a g_b}.$$

We have, for all $i < j$,

$$A_{ijji} \leq \frac{|R_{ijji}|}{g_i g_j} = |K_{ij}|,$$

which is bounded by Lemma 2.3. Let $j := 2 + l + \beta$. If $u^1 u^2 = 0$, then $A_{1jj2} = 0$. For $u^1 u^2 \neq 0$, setting $s := |u^1/u^2|$, we have

$$A_{1jj2} \leq \frac{|u^1 u^2 R_{1jj2}|}{\sum_a (u^a)^2 g_a g_j} \leq \frac{|R_{1jj2}|}{(sg_1 + (1/s)g_2)g_j} = \frac{h^2 R'(f' - f'')}{(s + h^2/s)f} \leq \frac{hR'(f' - f'')}{2f},$$

which is bounded since $f' - f''$ has compact support. This completes the proof of Lemma 2.4. □

2.3 Properties of g

Let b and c be the constants that previously appeared.

Proposition 2.5 *For $l, m > 0$, there is a Riemannian metric g on $Y = \mathbb{R} \times \mathbb{R} \times \mathbb{R}^l \times \mathbb{R}^m$ that is invariant by* $\mathrm{Isom}(\mathbb{R}^l)$ *and* $\mathrm{Isom}(\mathbb{R}^m)$ *satisfying the following* (1)–(7):

(1) *There is a constant $C < 0$ such that the sectional curvature K satisfies $C \leq K < 0$ on Y.*

(2) *Let T^l and T^m be flat tori obtained as quotients of $\mathbb{R}^l$ and $\mathbb{R}^m$ by isometries. Then g defines a metric on $\mathbb{R}\times\mathbb{R}\times T^l\times T^m$ such that the volume of the following subset is finite*:

$$\{(r,t,\rho,\tau)\mid r\in(-\infty,-1],\ t\in[r-1,2],\ \rho\in T^l,\ \tau\in T^m\}.$$

(3) *For $r\le 0$ and $t\le r-1$,*

$$g=dr^2+h(r)^2(dt^2+b^2e^{2r}\,d\rho^2+b^2e^{2r}\,d\tau^2).$$

(4) *For $r\ge 0$ and $t\le -1$,*

$$g=dr^2+h(r)^2(dt^2+b^2e^{2R(r)}\,d\rho^2+b^2e^{2R(r)}\,d\tau^2).$$

(5) *For $r\in\mathbb{R}$ and $t\ge 4$,*

$$g=dr^2+h(r)^2(dt^2+b^2e^{2R(r)}\,d\rho^2+b^2c^2e^{2t}\,d\tau^2).$$

(6) *For $r\ge 5$, g is a warped metric of the form*

$$g=dr^2+4e^{2r}\hat{g},$$

where $\hat{g}$ is the metric on $\mathbb{R}\times\mathbb{R}^l\times\mathbb{R}^m$ defined by

$$\hat{g}:=dt^2+b^2e^6\,d\rho^2+b^2e^6f(t-3)^2\,d\tau^2.$$

(7) *The metric $\hat{g}$ in* (6) *has nonpositive curvature.*

Remark 2.6 (i) C does not depend on l or m.

(ii) By (5), for all r and for $t\ge 4$ the metric is

$$g=dr^2+h(r)^2(d_{\mathrm{hyp}}+b^2e^{2R(r)}\,d\rho^2),$$

where $d_{\mathrm{hyp}}:=dt^2+b^2e^{2t}\,d\tau^2$ is a hyperbolic metric with $K=-1$.

(iii) In the proof of Theorem 1.2, setting $l=m=1$, g will be used to put a Riemannian metric on a neighborhood of a boundary component of $\mathbb{R}\times P$, where $P=\Sigma\times S^1$ is a piece of the flip manifold M. Outside of the neighborhood, we use a metric from a hyperbolic metric on Σ, which coincides with the metric g at $t=4$ as in (ii). For Theorem 1.6, the general form of g is used.

Proof Let g be the metric given by (2-4).

(1) By Lemma 2.4.

(2) Without loss of generality we may assume that $\mathrm{vol}(T^l)=1$ and $\mathrm{vol}(T^m)=1$ with respect to $d\rho^2$ and $d\tau^2$, respectively, since the volume of the concerned set is proportional to the product $\mathrm{vol}(T^l)\,\mathrm{vol}(T^m)$ because of the form of g.

For $r\le -1$, we have $h(r)=1+e^r$ and $R(r)=r$. We divide the subset into two according to t:

(i) (the part for $t\in[r+1,2]$) Since $t-R(r)=t-r\ge 1$, we have $f(t-R(r))=ce^{t-R(r)}$, hence

$$g=dr^2+(1+e^r)^2(dt^2+b^2e^{2r}\,d\rho^2+b^2c^2e^{2t}\,d\tau^2).$$

Fix r. The metric $dt^2+b^2c^2e^{2t}\,d\tau^2$ is hyperbolic, and its volume for the part $t\in[r+1,2]$, $\tau\in T^m$ is at most $b^mc^m\int_{-\infty}^2 e^{mt}\,dt=b^mc^me^{2m}$. Hence, the volume of the part $t\in[r+1,2]$, $\rho\in T^l$, $\tau\in T^m$ for the metric $dt^2+b^2e^{2r}\,d\rho^2+b^2c^2e^{2t}\,d\tau^2$ is at most $b^{l+m}c^me^{2m}e^{lr}$. Now the g–volume for the part $r\le -1$, $t\in[r+1,2]$, $\rho\in T^l$ and $\tau\in T^m$ is, since $1+e^r\le 2$, at most $2^{l+m+1}b^{l+m}c^me^{2m}\int_{-\infty}^{-1}e^{lr}\,dr=2^{l+m+1}b^{l+m}c^me^{2m-l}/l$.

(ii) (the part for $t\in[r-1,r+1]$) In this part, we have $t-R(r)=t-r\in[-1,1]$, so $f(t-R(r))\le ce$. The metric is

$$g=dr^2+(1+e^r)^2\bigl(dt^2+b^2e^{2r}\,d\rho^2+b^2e^{2r}f(t-R(r))^2\,d\tau^2\bigr).$$

Since the volume of $be^rf(t-R(r))T^m$ is at most $b^me^{mr}c^me^m$, the volume for (t,τ) with $t\in[r-1,r+1]$ and $\tau\in T^m$ is at most $2b^mc^me^me^{mr}$, so that the volume of $dt^2+b^2e^{2r}\,d\rho^2+b^2e^{2r}f(t-R(r))^2\,d\tau^2$ is at most $2b^{l+m}c^me^me^{(l+m)r}$. Finally, the volume of this part is, since $1+e^r\le 2$, at most

$$2^{l+m+2}b^{l+m}c^me^m\int_{-\infty}^{-1}e^{(l+m)r}\,dr=\frac{2^{l+m+2}b^{l+m}c^me^{-l}}{l+m}.$$

Combining (i) and (ii), the volume of the subset is at most

$$2^{l+m+1}b^{l+m}c^me^{-l}\Bigl(\frac{e^{2m}}{l}+\frac{2}{l+m}\Bigr).$$

(3) We fix $r\le 0$. Then $R(r)=r$. For $t\le r-1$, we have $t-R(r)=t-r\le -1$, so that $f(t-R(r))=1$. Thus, $g=dr^2+h(r)^2(dt^2+b^2e^{2r}\,d\rho^2+b^2e^{2r}\,d\tau^2)$.

(4) Fix $r\ge 0$. Note that then $0\le R(r)\le 3$. So, if $t\le -1$ then $t-R(r)\le -1$, so that $f(t-R(r))=1$. Substitute them into the definition of g.

(5) $R(r) \le 3$. Since $t \ge 4$, we have $t - R(r) \ge 1$, so that $f(t - R(r)) = ce^{t-R(r)}$. Substitute this into the definition of g.

(6) If $r \ge 5$, then $R(r) = 3$, $h(r) = 2e^r$ and $f(t - R(r)) = f(t-3)$, so that $g = dr^2 + 4e^{2r}(dt^2 + b^2e^6\,d\rho^2 + b^2e^6 f(t-3)^2\,d\tau^2)$, which is a desired warped metric.

(7) This follows from $f'' \ge 0$.

This completes the proof. □

2.4 Proof of Theorem 1.2 where the pieces are products

Proof By assumption the graph manifold M has a geometrization, ie each piece has a locally product Riemannian metric of type $\mathbb{H}^2 \times \mathbb{R}$, and the gluing maps are isometries. In the following, we first give an argument assuming that M has a geometrization with respect to a product metric on each piece. Then we will explain that in fact our argument applies to the locally product case as well.

Step 1 Let P_i be the pieces of M. Suppose $P_i = \Sigma_i \times S^1$. We will put a Riemannian metric on each $\mathbb{R} \times P_i$ so that they match up for gluing along the boundary, which defines a Riemannian metric on $\mathbb{R} \times M$. First, put a complete, hyperbolic metric of finite volume in the interior of each Σ_i. Let $\mathrm{vol}_{\mathrm{hyp}}(\Sigma_i)$ denote its volume. There is a constant $L > 0$ such that the interior of each Σ_i contains a compact subset K_i homeomorphic to Σ_i such that each connected component of $\Sigma_i \setminus K_i$ is isometric to an annulus $(-\infty, 0) \times S^1(bce^2L)$ with the metric $dt^2 + e^{2t}\,d\tau^2$, ie the warped product $(-\infty, 0) \times_{e^t} S^1(bce^2L)$, where $S^1(a) := \mathbb{R}/a\mathbb{Z}$ is a circle of length $a > 0$.

Step 2 For each $r \in \mathbb{R}$, we consider a Riemannian product

$$K_i \times S^1(be^{R(r)-2}L),$$

then further take a "generalized" warped product with $\mathbb{R}$ as follows:

$$J_i = \mathbb{R} \times_{h(r)} (K_i \times S^1(be^{R(r)-2}L)),$$

where at each r, the metric of the fiber $K_i \times S^1(be^{R(r)-2}L)$ is rescaled by $h(r)$. We say this is a generalized warped product since the metric on the fiber at r depends on r. Then:

Lemma 2.7 (1) *The subset of J_i for the part $r < 0$ has finite volume, which is bounded above by $8beL\,\mathrm{vol}_{\mathrm{hyp}}(\Sigma_i)$.*

(2) *For the part* $r > 5$, J_i *is a warped product*,

$$(5, +\infty) \times_{2e^r} (K_i \times S^1(beL)).$$

(3) *The sectional curvature of* J_i *is bounded*:

$$C \le K < 0,$$

where $C < 0$ *is the constant from Proposition 2.5.*

(4) *Each boundary component of* J_i *is isometric to*

$$\mathbb{R} \times_{h(r)} (S^1(bce^2 L) \times S^1(be^{R(r)-2} L)).$$

Proof (1) At each $r < 0$, $R(r) = r$, hence the volume of $K_i \times S^1(be^{R(r)-2} L)$ is $\le \mathrm{vol}_{\mathrm{hyp}}(\Sigma_i) \cdot be^{r-2} L$. Since $h(r) \le h(1) \le 2e$ for $r \le 0$, the volume of J_i for the part $r \le 0$ is

$$\le (2e)^3 \,\mathrm{vol}_{\mathrm{hyp}}(\Sigma_i) L b \int_{-\infty}^{0} e^{r-2}\, dr = 8ebL \,\mathrm{vol}_{\mathrm{hyp}}(\Sigma_i).$$

(2) Suppose $r > 5$. Then $R(r) = 3$ and $h(r) = 2e^r$. Substitute them into the definition of the metric on J_i.

(3) The metric of J_i is written as

$$g = dr^2 + h(r)^2 (d_{\mathrm{hyp}} + e^{2R(r)}\, d\rho^2),$$

where ρ is for $S^1(bLe^{-2})$. Now this metric and the metric that appears in Proposition 2.5(5) are locally isometric to each other (see Remark 2.6(ii)), but that metric satisfies $C \le K < 0$ for the constant C in the proposition.

(4) This is because each boundary of K_i is isometric to $S^1(bce^2 L)$. □

Step 3 We set $l = m = 1$ in Proposition 2.5. We prepare a manifold with boundary

$$A = \{(r, t, \rho, \tau) \mid r \in \mathbb{R},\ t \in [R(r) - 2, 4],\ \rho \in S^1(Le^{-2}),\ \tau \in S^1(Le^{-2})\}$$

with the metric g given in (2-4):

$$g = dr^2 + h(r)^2 (dt^2 + b^2 e^{2R(r)}\, d\rho^2 + b^2 e^{2R(r)} f(t - R(r))^2\, d\tau^2).$$

The manifold A has two boundary components, $\partial_0 A$ and $\partial_1 A$, where $\partial_1 A$ is the component at $t = 4$ and $\partial_0 A$ at $t = R(r) - 2$. For $t = 4$, we have $f(t - R(r)) = f(4 - R(r)) = ce^{4-R(r)}$, so that $\partial_1 A$ is isometric to

$$\mathbb{R} \times_{h(r)} (S^1(be^{R(r)-2} L) \times S^1(bce^2 L)). \tag{2-7}$$

Hence $\partial_1 A$ is isometric to each boundary component of every J_i by Lemma 2.7(4), so that we are able to glue A to the boundary component of J_i along $\partial_1 A$. By Proposition 2.5(5) (see also Remark 2.6(ii)), no singularity of the metric occurs by this gluing. In this way we obtain a Riemannian manifold diffeomorphic to P_i (or a Riemannian metric on P_i) such that

- P_i is diffeomorphic to $\mathbb{R} \times (\Sigma_i \times S^1)$, where the first parameter is r;
- every connected component of the boundary of P_i is isometric to

$$\partial_0 A = \mathbb{R} \times_{h(r)} (S^1(be^{R(r)-2}L) \times S^1(be^{R(r)-2}L)), \tag{2-8}$$

  and moreover the 1–neighborhood of $\partial_0 A$ is isometric to the direct Riemannian product $\partial_0 A \times [0,1]$ since $f(t - R(r)) = 1$ for $t \in [R(r)-2, R(r)-1]$;
- the volume of the subset P_i for the part $r \le -1$ is finite (by Proposition 2.5 for the part isometric to A and for J_i it is by Lemma 2.7(1));
- $C \le K < 0$ on P_i (for A by Proposition 2.5, and for J_i by Lemma 2.7(3));
- the metric on P_i is a warped product w.r.t. the function $2e^r$ for $r > 5$ (for A by Proposition 2.5(6) and for J_i by Lemma 2.7(2)).

Step 4 Now our metric on $\mathbb{R} \times P_i$ will give a Riemannian metric on $\mathbb{R} \times M$. Indeed, by the second bullet in the above, the two boundary circles have the same length at each r, so that we can glue the $\mathbb{R} \times P_i$ by the given gluing maps at each r.

We finish the proof by checking this metric satisfies all the properties in Theorem 1.1. By the third bullet, the volume of the part $(-\infty, -1] \times M$ is finite since there are only finitely many pieces for M, which implies that the volume for $(-\infty, 0] \times M$ is finite since M is compact. The sectional curvature K satisfies $C \le K < 0$ on $\mathbb{R} \times M$ by the fourth bullet. The metric is a warped product for $r \ge 5$ w.r.t. the function $2e^r$ and some metric g_M on M by the last bullet and Proposition 2.5(6). Now we rescale the metric g_M to $\frac{1}{4} g_M$, which we still denote by g_M, then the warping function becomes e^r. Then we have $g = dr^2 + e^{2r} g_M$ for $r \ge 5$. Set $D = 5$. Finally, since $\dim M = 3$, we are done.

The proof of Theorem 1.2 is complete in the case without Seifert fibered spaces, provided that M has a geometrization with respect to a product metric on every piece.

Locally product case Now, suppose some pieces are only locally Riemannian product. We handle this case by following the product case, and we only explain the changes we need to make. Let $P_i = \Sigma_i \times S^1$ (the trivial bundle) be a piece which is a locally Riemannian product with respect to which M has a geometrization. Let

$$\theta_i\colon \pi_1(\Sigma_i) \to S^1$$

be the monodromy representation defined by the Riemannian metric on P_i.

No change is necessary in Step 1. In Step 2, instead of the Riemannian product $K_i \times S^1(be^{R(r)-2}L)$, we take the locally Riemannian product with respect to θ_i, which we denote by

$$K_i \times_{\theta_i} S^1(be^{R(r)-2}L).$$

Accordingly we also use $K_i \times_{\theta_i} S^1(be^{R(r)-2}L)$ in the statement of Lemma 2.7, but the proof is nearly same: for example in the proof of (3), $g = dr^2 + h(r)^2(d_{\mathrm{hyp}} + e^{2R(r)}\,d\rho^2)$ does not hold any more, but g is only locally isometric to the right-hand side. But this is enough since the sectional curvature depends only locally on g.

In Step 3, when we define the manifold A, we use the same definition, but the metric on A is a locally product metric with respect to the monodromy θ_i on the fiber circle for ρ. We call this circle ρ–circle in the following. Accordingly, in the description (2-7), $\partial_1 A$ becomes only a locally Riemannian product with respect to θ_i on the ρ–circle (which is the first S^1 acted by the second S^1 via ρ). This also happens in the metric description of $\partial_0 A$ in (2-8).

Finally, in Step 4, the two circles in (2-8) have the same length in this case, and we keep using the same monodromy θ_i on each piece P_i; therefore, the given gluing maps are all isometric. This finishes the proof in this case, and the proof of Theorem 1.2 for flip manifolds without Seifert space pieces is complete. □

2.5 Proof of Theorem 1.2 for the general case

We now handle a graph manifolds such that possibly some pieces are Seifert fibered spaces or fibers are nonorientable (from now on we consider a Seifert fibered space contains the latter case). The argument is identical to the previous case with nontrivial monodromy representation θ_i of $\pi_1(\Sigma_i)$, where Σ_i is the base surface of a piece P_i. The only difference is that Σ_i is maybe an orbifold and $\pi_1(\Sigma_i)$ is the orbifold fundamental group. In the following we only explain that part. A good reference for the geometry of Seifert fibered spaces is [25].

Proof Let P be a piece in M. Suppose P is a Seifert fibered space; otherwise we do not have to change anything. We remember that when P is a trivial circle bundle over a surface, we can choose the length of the fiber circle when we put a locally product Riemannian metric.

Let Σ be the base orbifold of P. Let $x_1, \dots, x_n$ be the singular points of Σ such that the twist parameter at x_i is q_i/p_i. Since P has nonempty boundary, P admits the geometry of $\mathbb{H}^2 \times \mathbb{R}$ [25, Theorem 5.3(ii)]. We explain this part in some detail (see [25, Proof of Theorem 5.3(ii); 19, Lemma 2.5]). We put on Σ a complete hyperbolic orbifold metric of finite volume, then view P as an S^1–bundle over the orbifold Σ with a Riemannian metric that is locally isometric to $\mathbb{H}^2 \times \mathbb{R}$. The global geometry is described by the monodromy representation of $\pi_1(\Sigma)$ into the group S^1 if the fibers are oriented, otherwise into $S^1 \rtimes \mathbb{Z}_2$, the isometry group of a circle. Here the fundamental group is in the orbifold sense, and $\mathbb{Z}_2$ means $\mathbb{Z}/2\mathbb{Z}$.

First, assume Σ is orientable. Let X_i denote a loop around the singular point x_i, and $b_1, \dots, b_n$ the curves around the punctures (boundary components) of Σ. Let g be the genus of Σ then take loops $\alpha_1, \beta_1; \dots; \alpha_g, \beta_g$ associated to the genus such that X_i, b_i, α_i and β_i generate the fundamental group of Σ satisfying a well-known relation (after choosing orientations of the loops suitably),

$$\prod_i [\alpha_i, \beta_i] \prod_i X_i \prod_i b_i = 1.$$

Let $\theta(\alpha_i)$, $\theta(\beta_i)$, $\theta(X_i)$ and $\theta(b_i)$ denote the monodromy along those loops for the S^1–fiber. We set, for each i,

$$\theta(X_i) = 2\pi q_i/p_i.$$

We choose $\theta(b_i)$, $\theta(\alpha_i)$ and $\theta(\beta_i)$ for each i such that, in $S^1 \rtimes \mathbb{Z}_2$,

$$\theta\Bigl(\prod_i [\alpha_i, \beta_i] \prod_i X_i \prod_i b_i\Bigr) = 1.$$

Then there is a locally product Riemannian metric on P whose monodromy representation is θ. Note that if $\theta(\alpha_i), \theta(\beta_i) \in S^1$ then, since S^1 is abelian, we always have $\theta\bigl(\prod_i [\alpha_i, \beta_i]\bigr) = 0$ in S^1.

Conversely, the monodromy representation induced by a locally product Riemannian metric is obtained in the above way.

If Σ is not oriented, the relation in the fundamental group is slightly different, but the rest is the same and we omit repeating it.

Note that when we put a Riemannian metric on P, as before we can choose the length of the S^1–fiber (at a regular point) as we want. Also each boundary component of P is a flat torus/Klein bottle.

We take a compact subset K homeomorphic to Σ such that all singular points are contained in K, and that each connected component of $\Sigma \setminus K$ satisfies the same metric property as the nongeneralized case described in Step 2 in the previous section. We do not need to alter the argument since we modify the metric only outside of K, then that Σ is an orbifold does not cause any difference.

Now we proceed in the same way as the previous case, and complete the proof of Theorem 1.2 in general. □

We give an example of a flip manifold with a geometrization made from a Seifert fibered space.

Example 2.8 (Seifert fibered space as a piece) We give an example of a Seifert fibered space that can appear as a piece in a flip manifold with a geometrization. Let Γ be a three-punctured sphere. There is an obvious action of $G = \mathbb{Z}/3\mathbb{Z}$ rotating the three punctures with a generator ρ. Put a complete hyperbolic metric on Γ which is ρ–invariant. Now set $\Sigma = \Gamma/G$, which is a hyperbolic orbifold with two singular points, p_1 and p_2, and with one puncture. Take the product $\Gamma \times S^1$ and let G act on it such that ρ acts on S^1 by the rotation of $\frac{2\pi}{3}$. This is a free action and the quotient $(\Gamma \times S^1)/G$ is a three-dimensional manifold P, which is a Seifert fibered space over Σ such that the twists at p_1 and p_2 are $\frac{1}{3}$ and $\frac{2}{3}$, respectively. $\bigl($One can say that the twist at p_2 is $-\frac{1}{3}$.$\bigr)$ Now P has only one boundary component, which is a Riemannian product of the fiber circle and a loop around the puncture of Σ since the monodromy is trivial. Now, for example, we prepare another copy of this, then glue the two along their boundary by a trivial or flip map, and obtain a flip manifold which admits a geometrization.

We also record an example of a graph manifold M with a geometrization whose gluing map is neither a trivial nor a flip map; see [11, Example 1.5].

Example 2.9 (graph manifold of nonpositive curvature) Consider the parallelogram of side length 1 with the angles of the corners equal to $\frac{\pi}{3}$, $\frac{2\pi}{3}$, $\frac{\pi}{3}$ and $\frac{2\pi}{3}$. Choose a vertex of angle $\frac{\pi}{3}$ and call it O, then call the adjacent vertices A and B. The last vertex is called D. We obtain a flat torus T by gluing the sides OA and BD, and OB and AD. We regard T as a circle bundle over a circle where the base circle is OB and the fiber circle is OA. The monodromy with respect to the flat metric is π.

The torus T has an interesting isometry ϕ that is defined by mapping

$$OA \mapsto BA, \quad OB \mapsto OA.$$

Notice ϕ is not homotopic to the trivial map nor the flip map of T.

We define a graph manifold using ϕ. Let Σ be a compact orientable surface of genus one with two boundary components, $a+$ and $a-$. Orient those two curves using the orientation of Σ. Let P be a trivial circle bundle over Σ and we put a locally product metric of type $\mathbb{H}^2 \times \mathbb{R}$ on P such that the monodromy satisfies $\theta(a+) = \pi$ and $\theta(a-) = \pi$. We arrange that there is a small constant $c > 0$ such that the two boundary tori $T+$ and $T-$ of P at $a+$ and $a-$, respectively, are isometric to T with the metric rescaled by c. Now we glue $T+$ to $T-$ by ϕ, which is an isometry. The map ϕ is not a flip nor trivial. In this way we obtain an oriented graph manifold M that has a Riemannian metric of nonpositive curvature.

Remark 2.10 As we said, the property that a graph manifold M has a geometrization formulated differently in [10]. Although we put a complete, finite-volume hyperbolic metric on the base surface/orbifold of a piece, they put a hyperbolic metric with a geodesic boundary (ie if you lift it to the universal cover, then it is a geodesic in $\mathbb{H}^2$). In both settings we can see the piece as a circle bundle over the base, and it defines a monodromy representation of the fundamental group of the base into S^1, which coincides for the two settings. So, if M admits isometric geometrization, then its monodromy representation can be used to put a locally product Riemannian metric on each piece that gives a geometrization on M in our sense.

2.6 Proof of Theorem 1.6

Proof The proof of Theorem 1.6 is nearly identical to the proof of the version of Theorem 1.2 where each piece is a product of a surface and a circle, which is exactly the case where $l = m = 1$ in Theorem 1.6. The main body of the argument for Theorem 1.2 is Proposition 2.5, which is already shown for general l and m. So we do not repeat the argument, except we make one remark. Suppose $W_1 = \Sigma_1 \times T_1^m$ and $W_2 = \Sigma_2 \times T_2^m$ are pieces such that $S_1^l \times T_1^m$ and $S_2^l \times T_2^m$ are glued by an isometry, where S_i^l is a boundary torus of Σ_i. Also, suppose $\Sigma_i \subset X_i$. By taking Σ_i larger in X_i if necessary, one may assume the metric on S_i is rescaled by any constant $0 < c < 1$. Also one can rescale the fibers T_i^m by the same constant c, which leaves the gluing isometric.

It follows from Proposition 2.5(7) that M carries a metric of nonpositive curvature, so that $\mathrm{Wh}(M)$ vanishes. This completes the proof. $\square$

3 The other family

We discuss the other examples of manifolds that will be ends.

3.1 Construction

Let M_1 and M_2 be n–dimensional closed, orientable hyperbolic manifolds with totally geodesic, orientable submanifold N_1 and N_2, respectively, of codimension two. Assume that N_1 and N_2 are isometric by an isometry $b\colon N_1 \to N_2$.

The unit normal bundle of N_1 in M_1 is an S^1–bundle, (X_1, N_1, S^1), with oriented fibers, which we also denote by

$$X_1 = N_1 \ltimes S_1 \quad \text{or} \quad X_1 = S_1 \rtimes N_1.$$

We will use this notation for bundles in this paper, which does not mean a semidirect product of group structures.

The metric of M_1 induces a Riemannian metric on this bundle which is locally a Riemannian product of the hyperbolic metric on N_1 and S^1. Similarly we have an S^1–bundle over N_2, (X_2, N_2, S^1), which is locally a Riemannian product.

For a sufficiently small constant $\epsilon > 0$, the boundary of $V_1 = M_1 \setminus N_\epsilon(N)$ is canonically identified with (X_1, N_1, S^1). Also, $V_2 = M_2 \setminus N_\epsilon(N)$ is identified with (X_2, N_2, S^1).

Suppose S^1–bundles over M_1 and M_2 with Riemannian metrics which are locally product of M_1 and M_2, respectively, and S^1 are given. We denote them by (Y_1, M_1, S^1) and (Y_2, M_2, S^1), and the restriction of them to N_1 and N_2 by $(Y_1 \mid N_1, N_1, S^1)$ and $(Y_2 \mid N_2, N_2, S^1)$.

We assume (X_1, N_1, S^1) is isometric to $(Y_2 \mid N_2, N_2, S^1)$ by a bundle map (f_1, b), where b is the isometry between N_1 and N_2, and also $(Y_1 \mid N_1, N_1, S^1)$ is isometric to (X_2, N_2, S^1) by a bundle map (f_2, b) in the same manner. It then follows that the fiber product $(X_1 \times Y_1 \mid N_1, N_1, S^1 \times S^1)$ is isometric to the fiber product $(X_2 \times Y_2 \mid N_2, N_2, S^1 \times S^1)$ by the *flip map*

$$\phi\colon (n, (s_1, s_2)) \mapsto \bigl(b(n), (f_2(s_2), f_1(s_1))\bigr), \quad n \in N_1,\ (s_1, s_2) \in S^1 \times S^1,$$

or the trivial map

$$\phi\colon (n, (s_1, s_2)) \mapsto \bigl(b(n), (f_1(s_1), f_2(s_2))\bigr), \quad n \in N_1,\ (s_1, s_2) \in S^1 \times S^1.$$

Note that the metric on the fiber $S^1 \times S^1$ of the two bundles is a product metric since X_i are defined over M_i.

The fiber products $(X_1 \times Y_1 \mid N_1, N_1, S^1 \times S^1)$ and $(X_2 \times Y_2 \mid N_2, N_2, S^1 \times S^1)$ are identified with the boundary of $Y_1 \mid V_1$ and $Y_2 \mid V_2$.

Now we define

$$W = (Y_1 \mid V_1, V_1, S^1) \cup_\phi (Y_2 \mid V_2, V_2, S^1)$$

by identifying their boundaries $(X_1 \times Y_1 \mid N_1, N_1, S^1)$ and $(X_2 \times Y_2 \mid N_2, N_2, S^1)$ using ϕ.

For example, if $n = 2$ then N_1 and N_2 are points and W is a flip manifold.

We recall the theorem from the introduction:

Theorem 3.1 *Assume W has a geometrization (ie the gluing maps are isometric). Then W carries a metric of nonpositive curvature, so that $\mathrm{Wh}(W)$ vanishes. Also, $\mathbb{R} \times W$ carries a complete Riemannian metric that is an eventually warped cusp metric with $C \le K < 0$ for some constant $C < 0$.*

Remark 3.2 As in the construction of 3–dimensional graph manifolds, as a generalization of the theorem, one can use a finite collection of codimension 2 submanifolds $N_1, \dots, N_l$, each of which appears two times in the union of n–dimensional closed hyperbolic manifolds $M_1, \dots, M_k$ as totally geodesic, mutually disjoint submanifolds. For a sufficiently small $\epsilon > 0$ we remove the ϵ–neighborhoods of the N_i, then glue the two boundaries of $N_\epsilon(N_i)$ by either the trivial map or the flip map. In this way we obtain a closed manifold W for which Theorem 3.1 holds.

Note that Theorem 1.2 follows from the generalized version of Theorem 3.1 if all of the base surfaces have genus at least two.

3.2 Gluing condition

We discuss the condition for a flip map to be isometric in the case $n=3$ in some detail. The N_i are simple closed geodesics, and we denote them by γ_i. By our assumption they have same length. Let m_i be the meridian curve for γ_i in M_i. We denote by σ_i the fiber circle of X_i. With respect to the Riemannian metric, we can measure the monodromy (ie rotation) along the curve γ_i for σ_i and m_i, respectively. We denote them by $0 \le \theta(\sigma_i), \theta(m_i) < 2\pi$.

Notice that the flip map ϕ is an isometry if and only if

$$\theta(m_1) = \theta(\sigma_2), \quad \theta(\sigma_1) = \theta(m_2). \tag{3-1}$$

In general, ie if $\dim N \ge 1$, then let $\rho_N(m_1)$ be the monodromy representation of $\pi_1(N)$ to S^1, in terms of the meridian curve m_1. Let $\rho_N(\sigma_1)$ be the monodromy representation in terms of σ_1. Similarly we define $\rho_N(m_2)$ and $\rho_N(\sigma_2)$. We then assume

$$\rho_N(m_1) = b^*\rho_N(\sigma_2), \quad \rho_N(\sigma_1) = b^*\rho_N(m_2). \tag{3-2}$$

It is an interesting question if the bundles X_i satisfying this property exists for given (N_i, M_i). One sufficient condition is that $H_1(N_i, \mathbb{Z})$ injects into $H_1(M_i, \mathbb{Z})$ for both $i = 1, 2$. Indeed, if so then first define a circle bundle over N_2 using $\rho_N(m_1) = b^*\rho_N(\sigma_2)$ (here, we use that S^1 is abelian), then extend it to M_2 (use that H_1 injects), which will be X_2. Similarly we can define X_1.

We realize that it is enough if X_i are defined over V_i for our construction. But in this case we need an additional condition since the metric on the fiber $S^1 \times S^1$ is flat, but not a Riemannian product any more. Hence the monodromies $\theta_{m_i}(\sigma_i)$ and $\theta_{\sigma_i}(m_i)$ are not trivial in general, and we need

$$\theta_{m_1}(\sigma_1) = \theta_{\sigma_2}(m_2), \quad \theta_{\sigma_1}(m_1) = \theta_{m_2}(\sigma_2). \tag{3-3}$$

It turns out that if one is satisfied then the other one follows. We will assume this condition if we consider bundles that are defined only on V_i.

Example 3.3 We discuss the case that $\dim M = 2$ and $\dim N = 1$. If X is defined over M, then the boundary of V is a torus which is a Riemannian product. But if X is defined only on $M \setminus N_\epsilon(N)$, then maybe $\theta_m(\sigma) \ne 0$, and the boundary of V is a flat torus, but not a product. Then we need to arrange that $\theta_m(\sigma)$ coincides for a pair of tori which are identified.

3.3 Outline of proof of Theorem 3.1

The proof of Theorem 3.1 is parallel to Theorem 1.2.

We denote $Y_i \mid V_i$ by P_i and call it a *piece*. The N_i are isometric to each other by the isometry b, so we may write them as N.

We will put metrics on $J_i = \mathbb{R} \times P_i$ so that they match up for gluing by $\mathrm{id} \times \phi$, which gives a desired metric on $\mathbb{R} \times W$ to apply Theorem 1.1. Each J_i has a product metric using the (noncomplete) hyperbolic metric on V_i, but there will be singularity when we glue them. So we deform the original metric near ∂J_i. A small neighborhood of ∂J_i is diffeomorphic to $\mathbb{R} \times [0,\infty) \times ((S^1 \times S^1) \rtimes N)$. In view of that we will construct a complete Riemannian metric g of negative curvature on

$$\mathbb{R} \times \mathbb{R} \times S^1 \times S^1 \times N,$$

which is invariant by a rotation on each S^1. We arrange that there is a constant a such that for every $r \in \mathbb{R}$ the metric on $\{r\} \times [a,\infty) \times S^1 \times S^1 \times N$ is identical to the original product metric on P_i up to scaling by a constant depending on r (see Proposition 3.4(5)). Here, the identification of the metric is canonically done between the fiber bundle $(S^1 \times S^1) \rtimes N$ and $S^1 \times S^1 \times N$ since the metric on the product is invariant by rotations on both the S^1–factors.

Moreover, the metric g will be defined on $\mathbb{R} \times \mathbb{R} \times S^1 \times S^1 \times \mathbb{R}^{n-2}$. The factor $\mathbb{R}^{n-2}$ is identified with $\widetilde{N}$ and g is invariant by the action by $\pi_1(N)$ which acts trivially on the other factors. In this way, $(\mathbb{R} \times S^1 \times \mathbb{R}^{n-2})/\pi_1(N)$ is identified with $N_\epsilon(N) \setminus N$. The other S^1 is for the fiber circle in P_i, and we can regard g as a metric on $J_i = \mathbb{R} \times P_i$.

We show the following (see Proposition 2.5). Recall that $S^1(a)$ is a circle of length a.

Proposition 3.4 *Let $c_1, c_2 > 0$ be constants. Then there is a Riemannian metric g on $\mathbb{R} \times \mathbb{R} \times S^1(c_1) \times S^1(c_2) \times N$ that is invariant by rotations on each S^1 satisfying the following* (1)–(7):

(1) *There is an absolute constant $C < 0$, which does not depend on c_1 and c_2, such that $C \le K < 0$ on $\mathbb{R} \times \mathbb{R} \times S^1(c_1) \times S^1(c_2) \times N$.*

(2) *The volume of the following subset is finite*:

$$\{(r,t,\rho,\tau,n) \mid r \in (-\infty,-1],\ t \in [r-1,2],\ \rho \in S^1(c_1),\ \tau \in S^1(c_2),\ n \in N\}.$$

(3) *For $r \le 0$ and $t \le r-1$,*

$$g = dr^2 + h(r)^2\Bigl(dt^2 + b^2e^{2r}\,d\rho^2 + b^2e^{2r}\,d\tau^2 + dw^2 + \sum_{j=1}^{n-3} e^{2w}\,dw_j^2\Bigr).$$

(4) *For $r \ge 0$ and $t \le -1$,*

$$g = dr^2 + h(r)^2\Bigl(dt^2 + b^2e^{2R(r)}\,d\rho^2 + b^2e^{2R(r)}\,d\tau^2 + dw^2 + \sum_{j=1}^{n-3} e^{2w}\,dw_j^2\Bigr).$$

(5) *For $r \in \mathbb{R}$ and $t \ge a$,*

$$g = dr^2 + h(r)^2\Bigl(dt^2 + b^2e^{2R(r)}\,d\rho^2 + \sinh^2(t-5)\,d\tau^2 + \cosh^2(t-5)\Bigl(dw^2 + \sum_{j=1}^{n-3} e^{2w}\,dw_j^2\Bigr)\Bigr).$$

(6) *For $r \ge 5$, the metric g is a warped metric of the form*

$$g = dr^2 + 4e^{2r}\,\hat{g},$$

where $\hat{g}$ is the metric on $\mathbb{R} \times S^1(c_1) \times S^1(c_2) \times N$ defined by

$$\hat{g} := dt^2 + b^2e^6\,d\rho^2 + \widetilde{F}(r,t)^2\,d\tau^2 + T(t)^2\Bigl(dw^2 + \sum_{j=1}^{n-3} e^{2w}\,dw_j^2\Bigr).$$

Here, $\widetilde{F}(r,t)$ (and hence $\hat{g}$ too) is independent of r for $r \ge 5$.

(7) *The metric $\hat{g}$ in* (6) *has nonpositive curvature for $r \ge 5$.*

We postpone proving this proposition and prove Theorem 3.1 using it.

3.4 Proof of Theorem 3.1

Proof First, W carries a metric of nonpositive curvature by Proposition 3.4(7). This implies that $\mathrm{Wh}(W)$ vanishes [14].

We now show the claim for $\mathbb{R} \times W$. We closely follow each step of the argument for Theorem 1.2. But there is one additional issue and we make a remark on that. For Theorem 1.2, each piece P is a trivial bundle $\Sigma \times S^1$ (for the nongeneral case). We glue pieces along boundaries by isometries, and a boundary component of P is $S^1 \times S^1$, where the first S^1 is a boundary component of Σ. On the other hand, for Theorem 3.1, a boundary component of a piece P will be an S^1–bundle over an S^1–bundle over a hyperbolic manifold N, $S^1 \rtimes (S^1 \rtimes N)$. But notice that any metric g on $S^1 \times S^1 \times N$ that is invariant by rotations on both circles gives a metric to the boundary which is locally isometric to g. In view of this, when we construct a metric (see Section 3.5), we consider only rotationally invariant ones on a product space then descend it to a space with circle bundle structures, so that the bundle issue is not an extra problem for us. In the following, we may write $S^1 \rtimes (S^1 \rtimes N)$ simply as $S^1 \rtimes S^1 \rtimes N$.

Step 1 Fix a small constant $\epsilon > 0$. Set $\Sigma_i = M_i - N_\epsilon(N_i)$. The boundary of Σ_i is a circle bundle over N_i. Set $P_i = (M_i - N_\epsilon(N_i)) \ltimes S^1$.

Step 2 We will put a metric on P_i and glue them along the boundary. Set $K_i = M_i - N_{2\epsilon}(N_i)$. Then $\Sigma_i - K_i$ is isometric to $[\epsilon, 2\epsilon) \times (S^1(2\pi) \rtimes N)$ with the metric

$$dt^2 + \sinh(t)\, d\tau^2 + \cosh(t) g_N.$$

Step 3 For each $r \in \mathbb{R}$, we consider an S^1–bundle which is locally a Riemannian product,

$$(M_i - N_\epsilon(N_i)) \ltimes S^1(be^{R(r)-2}L),$$

then further take a "generalized" warped product with $\mathbb{R}$ as follows:

$$J_i = \mathbb{R} \times_{h(r)} \{(M_i - N_\epsilon(N_i)) \ltimes S^1(be^{R(r)-2}L)\},$$

where at each r, the metric of the fiber $K_i \ltimes S^1(be^{R(r)-2}L)$ is rescaled by $h(r)$. We say this is a generalized warped product since the metric on the fiber at r depends on r.

Then we have the following lemma. The argument is similar to Lemma 2.7 and we skip it.

Lemma 3.5 (1) *The subset of J_i for the part $r < 0$ has finite volume, which is bounded above by $2^{n+1} e^{n-1} bL \operatorname{vol}_{\mathrm{hyp}}(M_i)$.*

(2) *For the part $r > 5$, J_i is a warped product*

$$(5, +\infty) \times_{2e^r} (\mathbb{R} \times S^1 \rtimes S^1 \rtimes N).$$

(3) *The sectional curvature of J_i is bounded:*

$$C \le K < 0,$$

where $C < 0$ is the constant from Proposition 3.4.

(4) *Each boundary component of* J_i *is isometric to*

$$\mathbb{R} \times_{h(r)} (S^1(bce^2 L) \rtimes S^1(be^{R(r)-2} L) \rtimes N).$$

Step 4 Similar. We use Proposition 3.4. We skip the details.

Step 5 Similar. We use Proposition 3.4. We skip the details.

Theorem 3.1 is proved. □

3.5 Metric construction

We are left with proving Proposition 3.4. It is done by constructing g. For any constant $a > 5$, we put $\delta := \delta(a) := \frac{1}{2}(a-5)$ and $b := b(a) := c^{-1}e^{-(a-\delta)}\sinh(a-\delta-5)$, where we recall $c = (\lambda * e^t)(0)$. There is a C^∞ function, $\widetilde{F}$, on $\mathbb{R}^2$ such that

(i) for all r and t,
$$\widetilde{F}(r,t) = \begin{cases} F(r,t) = be^{R(r)} f(t - R(r)) & \text{if } t \le 5, \\ \sinh(t-5) & \text{if } t \ge a, \end{cases}$$

(ii) $\widetilde{F}_t, \widetilde{F}_{tt} \ge 0$ everywhere,

(iii) $\widetilde{F}(r,t)$ is independent of r if $t \ge 4$ or $r \ge 5$.

Let us explain why such a function $\widetilde{F}$ exists. Assume $t \ge 4$. Since $R \le 3$, we have $t - R \ge 1$ and so

$$be^R f(t-R) = bce^t \begin{cases} > \sinh(t-5) & \text{if } t < a-\delta, \\ = \sinh(t-5) & \text{if } t = a-\delta, \\ < \sinh(t-5) & \text{if } t > a-\delta. \end{cases}$$

Therefore, there is a C^∞ approximation, $\widetilde{F}$, of the continuous function

$$\begin{cases} be^{R(r)} f(t - R(r)) & \text{if } t \le a-\delta, \\ \sinh(t-5) & \text{if } t > a-\delta, \end{cases}$$

satisfying the required conditions.

Take a C^∞ function, T, on $\mathbb{R}$ such that

$$T(t) = \begin{cases} 1 & \text{if } t \le 4, \\ \cosh(t-5) & \text{if } t \ge a, \end{cases} \qquad T \ge 1, \qquad T', T'' \ge 0.$$

For $n \ge 2$, we consider the metric

$$g = \sum_{i=1}^{n} g_i \, dx_i^2,$$

where $x_1 := r,\ x_2 := t,\ x_3 := \rho,\ x_4 := \tau,\ x_5 := w,\ x_i := w_{i-5}$ for $i \geq 6$, $g_1 := 1,\ g_2 := h(r)^2,\ g_3 := H(r)^2,\ H(r) := be^{R(r)}h(r),\ g_4 := h(r)^2\widetilde{F}(r,t)^2$, $g_5 := h(r)^2T(t)^2$ and $g_i := e^{2w}h(r)^2T(t)^2$ for $i \geq 6$. We see that

$$g = dr^2 + h(r)^2\biggl(dt^2 + b^2e^{2R(r)}\,d\rho^2 + \widetilde{F}(r,t)^2\,d\tau^2 + T(t)^2\biggl(dw^2 + \sum_{j=1}^{n-3} e^{2w}\,dw_j^2\biggr)\biggr),$$

where the term

$$T(t)^2\biggl(dw^2 + \sum_{j=1}^{n-3} e^{2w}\,dw_j^2\biggr)$$

vanishes for $n = 2$. Note that, for $t \geq a$,

$$\begin{aligned} dt^2 + \widetilde{F}(r,t)^2\,d\tau^2 + T(t)^2\biggl(dw^2 + \sum_{j=1}^{n-3} e^{2w}\,dw_j^2\biggr) \\ = dt^2 + \sinh^2(t-5)\,d\tau^2 + \cosh^2(t-5)\biggl(dw^2 + \sum_{j=1}^{n-3} e^{2w}\,dw_j^2\biggr) \end{aligned}$$

is a hyperbolic metric.

We calculate the Christoffel symbols:

$$\begin{aligned}
&\Gamma^i_{1i} = \frac{h'}{h} \quad \text{for } i = 2,\ i \geq 5, && \Gamma^3_{13} = \frac{H'}{H}, && \Gamma^4_{14} = \frac{\widetilde{F}_r}{\widetilde{F}} + \frac{h'}{h},\\
&\Gamma^1_{22} = -hh', && \Gamma^4_{24} = \frac{\widetilde{F}_t}{\widetilde{F}}, && \Gamma^i_{2i} = \frac{T'}{T} \quad \text{for } i \geq 5,\\
&\Gamma^1_{33} = -HH', && \Gamma^1_{44} = -h^2\widetilde{F}\widetilde{F}_r - hh'\widetilde{F}^2, && \Gamma^2_{44} = -\widetilde{F}\widetilde{F}_t,\\
&\Gamma^1_{55} = -hh'T^2, && \Gamma^2_{55} = -TT', && \Gamma^i_{5i} = 1 \quad \text{for } i \geq 6,\\
&\Gamma^1_{ii} = -e^{2w}hh'T^2, && \Gamma^2_{ii} = -e^{2w}TT', && \Gamma^5_{ii} = -e^{2w} \quad \text{for } i \geq 6.
\end{aligned}$$

The curvature tensor is calculated as follows, for $j > i \geq 6$:

$$\begin{aligned}
R_{1221} &= -hh'',\\
R_{1331} &= -HH'',\\
R_{1441} &= -hh''\widetilde{F}^2 - h^2\widetilde{F}\widetilde{F}_{rr} - 2hh'\widetilde{F}\widetilde{F}_r,\\
R_{1442} &= -h^2\widetilde{F}\widetilde{F}_{rt},\\
R_{1551} &= -hh''T^2,\\
R_{1ii1} &= -e^{2w}hh''T^2,
\end{aligned}$$

$$
\begin{aligned}
R_{2332} &= -hh'HH',\\
R_{2442} &= -h^3h'\widetilde{F}\widetilde{F}_r - h^2(h')^2\widetilde{F}^2 - h^2\widetilde{F}\widetilde{F}_{tt},\\
R_{2552} &= -h^2(h')^2T^2 - h^2TT'',\\
R_{2ii2} &= -e^{2w}h^2(h')^2T^2 - e^{2w}h^2TT'',\\
R_{3443} &= -h^2\widetilde{F}\widetilde{F}_rHH' - hh'\widetilde{F}^2HH',\\
R_{3553} &= -hh'HH'T^2,\\
R_{3ii3} &= -e^{2w}hh'HH'T^2,\\
R_{4554} &= -h^3h'\widetilde{F}\widetilde{F}_rT^2 - h^2(h')^2\widetilde{F}^2T^2 - h^2\widetilde{F}\widetilde{F}_tTT',\\
R_{4ii4} &= -e^{2w}h^3h'\widetilde{F}\widetilde{F}_rT^2 - e^{2w}h^2(h')^2\widetilde{F}^2T^2 - e^{2w}h^2\widetilde{F}\widetilde{F}_tTT',\\
R_{5ii5} &= -e^{2w}h^2(h')^2T^4 - e^{2w}h^2T^2(1+(T')^2),\\
R_{ijji} &= -e^{4w}h^2(h')^2T^4 - e^{4w}h^2T^2(1+(T')^2).
\end{aligned}
$$

The sectional curvatures are

$$
K_{1j} = -\frac{h''}{h} \quad \text{for } j=2,\ j\ge 5, \qquad K_{13} = -\frac{H''}{H}, \qquad K_{14} = -\frac{\widetilde{F}_{rr}}{\widetilde{F}} - \frac{2h'\widetilde{F}_r}{h\widetilde{F}} - \frac{h''}{h},
$$

$$
K_{23} = -\frac{h'H'}{hH}, \quad K_{24} = -\frac{\widetilde{F}_{tt}}{h^2\widetilde{F}} - \frac{h'\widetilde{F}_r}{h\widetilde{F}} - \frac{(h')^2}{h^2}, \quad K_{2j} = -\frac{T''}{h^2T} - \frac{(h')^2}{h^2} \quad \text{for } j\ge 5,
$$

$$
K_{34} = -\frac{\widetilde{F}_rH'}{\widetilde{F}H} - \frac{h'H'}{hH}, \qquad K_{3j} = -\frac{h'H'}{hH} \quad \text{for } j\ge 5,
$$

$$
K_{4j} = -\frac{\widetilde{F}_tT'}{h^2\widetilde{F}T} - \frac{h'\widetilde{F}_r}{h\widetilde{F}} - \frac{(h')^2}{h^2} \quad \text{for } j\ge 5,
$$

$$
K_{ij} = -\frac{(T')^2}{h^2T^2} - \frac{1}{h^2T^2} - \frac{(h')^2}{h^2} \quad \text{for } j>i\ge 5.
$$

Lemma 3.6 (1) $\widetilde{F}>0$ *and* $\widetilde{F}_t, \widetilde{F}_{tt}, \widetilde{F}_r \ge 0$.

(2) $T\ge 1$, $T'\ge 0$ *and* $T''\ge 0$.

(3) *The following functions are all uniformly bounded*:

$$
\frac{\widetilde{F}_r}{\widetilde{F}}, \quad \frac{\widetilde{F}_{rr}}{\widetilde{F}}, \quad \frac{\widetilde{F}_t}{\widetilde{F}}, \quad \frac{\widetilde{F}_{tt}}{\widetilde{F}}, \quad \frac{T'}{T}, \quad \frac{T''}{T}.
$$

Proof (1) follows from the definition of $\widetilde{F}$ and Lemma 2.2.

(2) is clear.

We prove (3). The boundedness of T'/T and T''/T are derived from the definition of T. For $t \le 5$, we see that $\widetilde{F} = F$ and the boundedness of $\widetilde{F}_r/\widetilde{F}$, $\widetilde{F}_{rr}/\widetilde{F}$, $\widetilde{F}_t/\widetilde{F}$ and $\widetilde{F}_{tt}/\widetilde{F}$ follow from Lemma 2.2. For $t \ge 5$, we see that $\widetilde{F}$ is independent of r, so that $\widetilde{F}_r = \widetilde{F}_{rr} = 0$ and that $\widetilde{F}_t/\widetilde{F}$ and $\widetilde{F}_{tt}/\widetilde{F}$ are bounded for $t \in [5, a]$. For $t \ge a$, we have $\widetilde{F} = \sinh(t-5)$, for which $\widetilde{F}_t/\widetilde{F}$ and $\widetilde{F}_{tt}/\widetilde{F}$ are bounded because of $a > 5$. We thus obtain (3). This completes the proof of the lemma. □

Lemma 3.7 *There is a constant $C < 0$ such that $C \le K_{ij} < 0$ for all $i \ne j$.*

Proof The negativity and boundedness of K_{ij} is readily seen from Lemmas 2.2 and 3.6 except the negativity of K_{14}. We remark that $\widetilde{F}_{rr} \ge 0$ does not hold.

In the case where $r \ge 5$, we see that $\widetilde{F}$ is independent of r and then

$$K_{14} = -\frac{h''}{h},$$

which is negative and bounded by Lemma 2.2.

In the case where $r \le 5$, we see $\widetilde{F} = F$, in which case the negativity and the boundedness of K_{14} are proved in the same way as in Lemma 2.4. This completes the proof. □

Lemma 3.8 *There is a constant $C < 0$ such that $C \le K_\sigma < 0$ for all 2–planes σ of the tangent spaces at all points.*

Proof We prove the lemma in a similar way to that of Lemma 2.4.

As is already seen in (2-5), for the negativity of K_σ, it suffices to prove

$$(R_{1442})^2 \le R_{1441} R_{2442}. \tag{3-4}$$

If $t > 5$, then $\widetilde{F}$ is independent of r and so $\widetilde{F}_{rt}^2 = \widetilde{F}_{rr}\widetilde{F}_{tt} = 0$, which implies (3-4). If $t \le 5$, then $\widetilde{F} = F$ and the calculation in the proof of Lemma 2.4 yields (3-4). The negativity of K_σ follows.

We prove the boundedness of K_σ for all σ. It suffices to estimate A_{ijji} for $i < j$ and A_{1442}, where A_{ijkl} is as defined in the proof of Lemma 2.4. By $A_{ijji} \le |K_{ij}|$ and by Lemma 3.7, we have the boundedness of A_{ijji}. The same calculation as in the proof of Lemma 2.4 leads us to

$$A_{1442} \le \frac{\widetilde{F}_{rt}}{2h\widetilde{F}} \le \frac{\widetilde{F}_{rt}}{2\widetilde{F}}.$$

If $t > 5$, then $\widetilde{F}$ is independent of r and so $\widetilde{F}_{rt} = 0$. If $t \le 5$, then $\widetilde{F} = F = be^R f(t-R)$ and so

$$\frac{\widetilde{F}_{rt}}{2\widetilde{F}} = \frac{R'(f' - f'')}{f},$$

which is bounded since $f' - f''$ has compact support. This completes the proof. □

We are ready to prove Proposition 3.4.

Proof of Proposition 3.4 First of all, the rotational invariance of g is clear by the form of g.

(1) holds by Lemma 3.8.

Checking (2)–(6) is similar to (2)–(6) of Proposition 2.5. We omit it.

We prove (7). Assume $r \ge 5$. The curvature tensor for the metric $\widehat{g}$ is obtained as, for $j > i \ge 5$,

$$\begin{aligned} R_{1331} &= -\widetilde{F}\widetilde{F}_{tt}, \\ R_{1441} &= -TT'', \\ R_{1ii1} &= -e^{2w}TT'', \end{aligned} \qquad \begin{aligned} R_{3443} &= -\widetilde{F}\widetilde{F}_t TT', \\ R_{3ii3} &= -e^{2w}\widetilde{F}\widetilde{F}_t TT', \\ R_{4ii4} &= -e^{2w}T^2(1 + (T')^2), \\ R_{ijji} &= -e^{4w}T^2(1 + (T')^2), \end{aligned}$$

which are the unique nonzero values of R_{ijkl} under the (skew-)symmetry. This together with Lemma 3.6(1)–(2) implies the nonpositivity of all the sectional curvatures. We have proved Proposition 3.4. □

4 Questions

4.1 More complicated examples

As we explained in Section 1.5, a flip manifold can be obtained as follows: take two surfaces V_1 and V_2, remove a small neighborhood of a point p_i from each of them, then consider an S^1–bundle over each. The boundary of each manifold is an $(S^1 \times S^1)$–bundle over a point (p_1 and p_2), and now we glum them by a flip map.

Regarding the above example, one can view V_1 and V_2 as intersecting in one point. In view of this, a similar construction can be done in dimension $2n$ for $n \ge 1$, with a more complicated intersection pattern. The above case is for $n = 1$, and we describe the case for $n = 2$. Let V_1 be a closed hyperbolic 4–manifold with two, isometric, totally geodesic embedded closed 2–submanifolds V_{12} and V_{13} intersecting at one point V_{123} transversally. Prepare two other copies: V_2 with submanifolds V_{23} and V_{21}, and V_3 with submanifolds V_{31} and V_{32}.

Fix a small $\epsilon > 0$, and consider an S^1–bundle,

$$X_1 = (V_1 \setminus N_\epsilon(V_{12} \cup V_{13})) \ltimes S^1,$$

whose boundary is $\partial N_\epsilon(V_{12} \cup V_{13}) \ltimes S^1$. Note that $\partial N_\epsilon(V_{12} \cup V_{13})$ is a flip manifold embedded in V_1,

$$(V_{12} \setminus N_\epsilon(V_{123}) \ltimes S^1) \cup_{V_{123} \ltimes S^1 \ltimes S^1} (V_{13} \setminus N_\epsilon(V_{123}) \ltimes S^1),$$

where we flip the two S^1–fibers in $V_{123} \ltimes S^1 \ltimes S^1$ when we glue the left piece to the right one. Similarly, consider S^1–bundles X_2 and X_3 for V_2 and V_3, respectively. We put a locally product metric on each X_i.

Now from X_1, X_2 and X_3, we define a 5–manifold

$$M^5 = (X_1 \cup X_2 \cup X_3)/\sim,$$

where $\sim$ means gluing among the boundaries of X_1, X_2 and X_3:

$$\begin{aligned}
\partial X_1 &= (V_{12} \setminus N_\epsilon(V_{123})) \ltimes S^1 \ltimes S^1 \cup_{V_{123} \ltimes S^1 \ltimes S^1 \ltimes S^1} (V_{13} \setminus N_\epsilon(V_{123})) \ltimes S^1 \ltimes S^1,\\
\partial X_2 &= (V_{21} \setminus N_\epsilon(V_{123})) \ltimes S^1 \ltimes S^1 \cup_{V_{123} \ltimes S^1 \ltimes S^1 \ltimes S^1} (V_{23} \setminus N_\epsilon(V_{123})) \ltimes S^1 \ltimes S^1,\\
\partial X_3 &= (V_{32} \setminus N_\epsilon(V_{123})) \ltimes S^1 \ltimes S^1 \cup_{V_{123} \ltimes S^1 \ltimes S^1 \ltimes S^1} (V_{31} \setminus N_\epsilon(V_{123})) \ltimes S^1 \ltimes S^1.
\end{aligned}$$

A gluing map is described as follows for each pair (i, j): use the obvious identification $V_{ij} \setminus N_\epsilon(V_{123}) = V_{ji} \setminus N_\epsilon(V_{123})$ and flip the two S^1–fibers. The common manifold $V_{123} \ltimes S^1 \ltimes S^1 \ltimes S^1$ is shared by all of them in M. We assume that the identification is done by isometries.

It would be interesting to know if M appears as an end (see [1]; see also [9] on the topology of those ends). In view of our strategy, as the first step we want to know if M has a metric of nonpositive curvature, but the curvature estimate becomes more subtle when we look for an eventually warped cusp metric for $\mathbb{R} \times M$.

4.2 Graph manifolds

Among graph manifolds W, we proved that W appears as an end if it has a Riemannian metric of nonpositive curvature (Corollary 1.5). See [12] on the question to decide which graph manifolds carry Riemannian metrics of nonpositive curvature. Leeb [19] gave an example of a graph manifold that does not have a metric of nonpositive curvature. It would be interesting to know if his examples will/will not appear as an end.

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