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Geometry & Topology 24 (2020) 2127–2148

New differential operator and noncollapsed RCD spaces

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We show characterizations of noncollapsed compact $\mathrm{RCD}(K,N)$ spaces, which in particular confirm a conjecture of De Philippis and Gigli on the implication from the weakly noncollapsed condition to the noncollapsed one in the compact case. The key idea is to give the explicit formula of the Laplacian associated to the pullback Riemannian metric by embedding in L^2 via the heat kernel. This seems to be the first application of geometric flow to the study of RCD spaces.

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Dedicated to Professor Kenji Fukaya on his 60th birthday

1 Introduction

1.1 Main results

De Philippis and Gigli introduced in [16] two special classes of $\mathrm{RCD}(K,N)$ spaces. One of them is the notion of *weakly noncollapsed spaces* and the other one is that of *noncollapsed spaces*. Our main result states that these are essentially same in the compact case.

After the fundamental works of Lott and Villani [40] and Sturm [46; 47], Ambrosio, Gigli and Savaré [3] (when $N=\infty$), Gigli [20] and Erbar, Kuwada and Sturm [18] (when $N<\infty$) introduced the notion of $\mathrm{RCD}(K,N)$ *spaces* for metric measure spaces $(X,\mathsf{d},\mathfrak{m})$, which means a synthetic notion of "$\mathrm{Ric}\geq K$ and $\dim\leq N$ with Riemannian structure". Typical examples are measured Gromov–Hausdorff limit spaces of Riemannian manifolds with Ricci bounds from below and dimension bounds from above, so-called *Ricci limit spaces*. The RCD theory gives a striking framework to treat Ricci limit spaces in a synthetic way.

Cheeger and Colding established the fundamental structure theory of Ricci limit spaces [12; 13; 14]. Thanks to recent quick developments on the study of $\mathrm{RCD}(K,N)$ spaces, most of the theory of Ricci limit spaces, including Colding and Naber's

Published: 10 November 2020 DOI: 10.2140/gt.2020.24.2127

result [15], is covered by the RCD theory (see for instance Bruè and Semola [10]). In particular, whenever $N < \infty$, the essential dimension, denoted by $\dim_{\mathsf{d},\mathfrak{m}}(X)$, of any $\mathrm{RCD}(K, N)$ space $(X, \mathsf{d}, \mathfrak{m})$ makes sense (see Theorem 2.4).

On the other hand, in a special class of Ricci limit spaces, so-called *noncollapsed Ricci limit spaces*, finer properties are obtained by Cheeger and Colding. For instance, the Bishop inequality with the rigidity and the almost Reifenberg flatness are justified in this setting. They are not covered by general Ricci limits/RCD theories.

The properties of noncollapsed $\mathrm{RCD}(K, N)$ spaces introduced in [16] cover finer results on noncollapsed Ricci limit spaces, as explained above. It is worth pointing out that any convex body is not a noncollapsed Ricci limit space, but it is a noncollapsed $\mathrm{RCD}(K, N)$ space.

Let us give the definitions: an $\mathrm{RCD}(K, N)$ space $(X, \mathsf{d}, \mathfrak{m})$ is

- *noncollapsed* if $\mathfrak{m} = \mathcal{H}^N$, where $\mathcal{H}^N$ denotes the N–dimensional Hausdorff measure;
- *weakly noncollapsed* if $\mathfrak{m} \ll \mathcal{H}^N$.

The second definition is equivalent to $\dim_{\mathsf{d},\mathfrak{m}}(X) = N$; this is proved in [16]. Note that some structure results on weakly noncollapsed $\mathrm{RCD}(K, N)$ spaces are obtained in [16] and that Kitabeppu [37] provides a similar notion (which is a priori stronger than the weakly noncollapsed condition, but is a priori weaker than the noncollapsed one) and prove similar structure results.

De Philippis and Gigli conjectured that these notions are essentially same. More precisely:

Conjecture 1.1 *If $(X, \mathsf{d}, \mathfrak{m})$ is a weakly noncollapsed $\mathrm{RCD}(K, N)$ space, then $\mathfrak{m} = a\mathcal{H}^N$ for some $a \in (0, \infty)$.*

For the conjecture the only known development is due to Kapovitch and Ketterer [34] and Han [29]. Kapovitch and Ketterer proved that Conjecture 1.1 is true under assuming bounded sectional curvature from above in the sense of Alexandrov (that is, the metric structure is CAT). Han proved that this conjecture is true for smooth Riemannian manifolds with (not necessary smooth) weighted measures.

We are now in a position to introduce a main result of the paper:

Theorem 1.2 (characterization of noncollapsed RCD spaces) *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a compact* $\mathrm{RCD}(K, N)$ *space with* $n := \dim_{\mathsf{d},\mathfrak{m}}(X)$. *Then the following two conditions* (1) *and* (2) *are equivalent*:

(1) *The following two conditions hold*:

(a) *For every eigenfunction* f *on* X *of* $-\Delta$ *we have*

$$\Delta f = \mathrm{tr}(\mathrm{Hess}_f) \quad \text{in } L^2(X, \mathfrak{m}). \tag{1-1}$$

(b) *There exists* $C > 0$ *such that*

$$\mathfrak{m}(B_r(x)) \geq C r^n \quad \text{for all } x \in X \text{ and } r \in (0, 1). \tag{1-2}$$

(2) $(X, \mathsf{d}, \mathfrak{m})$ *is an* $\mathrm{RCD}(K, n)$ *space with*

$$\mathfrak{m} = \frac{\mathfrak{m}(X)}{\mathcal{H}^n(X)} \mathcal{H}^n. \tag{1-3}$$

It is easy to understand that this theorem gives a contribution to Conjecture 1.1. More precisely, combining a result of Han [27] (see Theorem 2.8) with the Bishop–Gromov inequality yields that all compact weakly noncollapsed $\mathrm{RCD}(K, N)$ spaces satisfy (1) in the theorem, as $n = N$. Therefore:

Corollary 1.3 *Conjecture 1.1 is true in the compact case.*

We will also establish other characterization of noncollapsed RCD spaces. See Section 4.2. Next let us explain how to achieve these results. Roughly speaking, it is to take *canonical deformations* g_t of the Riemannian metric g via the heat kernel.

1.2 Key idea: deformation of Riemannian metric via the heat kernel

In order to prove our main results the key idea is to use the pullback Riemannian metrics $g_t := \Phi_t^* g_{L^2}$ by embeddings $\Phi_t \colon X \to L^2(X, \mathfrak{m})$ via the heat kernel p instead of using the original Riemannian metric g of $(X, \mathsf{d}, \mathfrak{m})$. The definition of Φ_t is

$$\Phi_t(x)(y) := p(x, y, t). \tag{1-4}$$

This map is introduced and studied by Bérard, Besson and Gallot [9] for closed manifolds. They proved that for closed manifolds (M^n, g), as $t \to 0^+$,

$$\omega_n t^{(n+2)/2} g_t = c_n g - \tfrac{2}{3} c_n \bigl(\mathrm{Ric}_g - \tfrac{1}{2} \mathrm{Scal}_g g\bigr) t + O(t^2), \tag{1-5}$$

where Ric_g and Scal_g denote the Ricci and the scalar curvatures, respectively, and

$$\omega_n := \mathcal{L}^n(B_1(0_n)), \quad c_n := \frac{\omega_n}{(4\pi)^n}\int_{\mathbb{R}^n} |\partial_{x_1}(e^{-|x|^2/4})|^2 \, \mathrm{d}\mathcal{L}^n(x). \tag{1-6}$$

Recently, the map Φ_t has also been studied for compact $\mathrm{RCD}(K,N)$ spaces by Ambrosio, Portegies, Tewodrose and the author [5]. In particular, g_t is also well defined in this setting (see Theorem 2.9).

Let us introduce the new differential operator

$$\Delta^t f := \langle \mathrm{Hess}_f, g_t\rangle + \tfrac{1}{4}\langle \nabla_x \Delta_x p(x,x,2t), \nabla f\rangle. \tag{1-7}$$

This plays the role of the Laplacian associated to g_t; in fact, we will prove

$$\int_X \langle g_t, \mathrm{d}\psi \otimes \mathrm{d}f\rangle \, \mathrm{d}\mathfrak{m} = -\int_X \psi \Delta^t f \, \mathrm{d}\mathfrak{m}, \tag{1-8}$$

which is new even for closed manifolds. See Theorem 3.4 for the precise statement. Then, assuming (1-2), after normalization, taking the limit $t \to 0^+$ in (1-8) with convergence results given in [5] yields the *metric* integration-by-parts formula

$$\int_X \langle \nabla\psi, \nabla f\rangle \, \mathrm{d}\mathcal{H}^n = -\int_X \psi \, \mathrm{tr}(\mathrm{Hess}_f) \, \mathrm{d}\mathcal{H}^n. \tag{1-9}$$

This allows us to prove (1-3) by letting $\psi \equiv 1$ if in addition (1-1) holds because we see that $\mathrm{d}\mathcal{H}^n/\mathrm{d}\mathfrak{m}$ is $L^2(X,\mathfrak{m})$–orthogonal to $\mathrm{tr}(\mathrm{Hess}_f)$ for all f; in particular, $\mathrm{d}\mathcal{H}^n/\mathrm{d}\mathfrak{m}$ is $L^2(X,\mathfrak{m})$–orthogonal to any nontrivial eigenfunction. This implies that $\mathrm{d}\mathcal{H}^n/\mathrm{d}\mathfrak{m}$ must be a constant function.

Finally, let us give a few comments. It is well known that in the smooth setting, there are many geometric flows (eg Ricci flow) which are useful to understand the original space. However, for singular spaces there are not so many (see eg Bamler and Kleiner [8], Gigli and Mantegazza [23] and Kleiner and Lott [38; 39]). In general, RCD spaces have very wild singularities (eg the singular set may be dense). This paper shows us that such flow approaches are also useful even in the RCD setting. Geometric applications of the main results can be found in Honda and Mondello [31] and Kapovitch and Mondino [35]. Moreover, although we discuss only the compact case, the author believes that the techniques provided will be available even in the noncompact case.

The paper is organized as follows:

In Section 2 we give a quick introduction on RCD spaces and prove technical results. In Section 3 we establish (1-8). In the final section, Section 4, we prove the main results stated in Section 1.1 and related results. It is worth pointing out that Section 4.2 is written from the point of view of metric geometry.

Acknowledgements The author is grateful to the referees for careful readings and valuable suggestions. He acknowledges support of the Grant-in-Aid for Young Scientists (B) 16K17585 and Grant-in-Aid for Scientific Research (B) 18H01118.

2 RCD(K, N) spaces

A triple $(X, \mathsf{d}, \mathfrak{m})$ is a *metric measure space* if (X, d) is a complete separable metric space and $\mathfrak{m}$ is a Borel measure on X with $\operatorname{supp}\mathfrak{m} = X$. We consider only the case when X is not a single point below.

2.1 Definitions

Throughout this paper the parameters $K \in \mathbb{R}$ (lower bound on Ricci curvature) and $N \in [1, \infty)$ (upper bound on dimension) will be kept fixed. Instead of giving the original definition of RCD(K, N) spaces, we introduce an equivalent shorter version. See [18; 7; 11; 2] for the proof of the equivalence and the detail.

Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space. The Cheeger energy Ch: $L^2(X, \mathfrak{m}) \to [0, +\infty]$ is a convex and $L^2(X, \mathfrak{m})$–lower semicontinuous functional defined as

$$\mathsf{Ch}(f) := \inf\left\{\liminf_{n\to\infty}\frac{1}{2}\int_X (\operatorname{Lip} f_n)^2\, d\mathfrak{m} \;\middle|\; f_n \in \operatorname{Lip}_b(X, \mathsf{d}) \cap L^2(X, \mathfrak{m}),\ \|f_n - f\|_{L^2} \to 0\right\}, \tag{2-1}$$

where $\operatorname{Lip}_b(X, \mathsf{d})$ denotes the space of all bounded Lipschitz functions and $\operatorname{Lip} f$ is the slope.

The Sobolev space $H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ then coincides with $\{f \in L^2(X, \mathfrak{m}) : \mathsf{Ch}(f) < +\infty\}$. When endowed with the norm $\|f\|_{H^{1,2}} := (\|f\|_{L^2}^2 + 2\,\mathsf{Ch}(f))^{1/2}$, this space is Banach, reflexive if (X, d) is doubling (see [1, Corollary 7.5]), and separable Hilbert if Ch is a quadratic form (see [3, Proposition 4.10]). According to the terminology introduced in [21], we say that $(X, \mathsf{d}, \mathfrak{m})$ is infinitesimally Hilbertian if Ch is a quadratic form.

We assume that $(X, \mathsf{d}, \mathfrak{m})$ is infinitesimally Hilbertian. Then, for all $f_i \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$,

$$\langle \nabla f_1, \nabla f_2\rangle := \lim_{\epsilon\to 0}\frac{|\nabla(f_1 + \epsilon f_2)|^2 - |\nabla f_1|^2}{2\epsilon} \in L^1(X, \mathfrak{m}) \tag{2-2}$$

is well defined, where $|\nabla f| \in L^2(X, \mathfrak{m})$ denotes the minimal relaxed slope of $f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ (see [21, Sections 3 and 4]).

We can now define a densely defined operator $\Delta\colon D(\Delta) \to L^2(X, \mathfrak{m})$ whose domain consists of all functions $f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ satisfying

$$\int_X \psi\varphi \, \mathrm{d}\mathfrak{m} = -\int_X \langle \nabla f, \nabla\varphi \rangle \, \mathrm{d}\mathfrak{m} \quad \text{for all } \varphi \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$$

for some $\psi \in L^2(X, \mathfrak{m})$. The unique ψ with this property is then denoted by Δf.

We are now in a position to introduce the RCD space:

Definition 2.1 (RCD spaces) Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space, let $K \in \mathbb{R}$ and let $\hat{N} \in [1, \infty]$. We say that $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K, \hat{N})$ *space* if the following hold:

(1) **Volume growth** There exist $x \in X$ and $C > 1$ such that $\mathfrak{m}(B_r(x)) \le Ce^{Cr^2}$ for all $r \in (0, \infty)$.

(2) **Bochner's inequality** For all $f \in D(\Delta)$ with $\Delta f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$,

$$\frac{1}{2}\int_X |\nabla f|^2 \Delta\varphi \, \mathrm{d}\mathfrak{m} \ge \int_X \varphi \left(\frac{(\Delta f)^2}{\hat{N}} + \langle \nabla f, \nabla \Delta f \rangle + K|\nabla f|^2 \right) \mathrm{d}\mathfrak{m} \tag{2-3}$$

for all $\varphi \in D(\Delta) \cap L^\infty(X, \mathfrak{m})$ with $\varphi \ge 0$ and $\Delta\varphi \in L^\infty(X, \mathfrak{m})$.

(3) **Sobolev-to-Lipschitz property** Any $f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ with $|\nabla f| \le 1$ $\mathfrak{m}$–a.e. in X has a 1–Lipschitz representative.

It is known that a smooth weighted complete Riemannian manifold $(M^n, g, e^{-\varphi} \mathrm{vol}_g)$ with $\varphi \in C^\infty(M^n)$ is an $\mathrm{RCD}(K, N)$ space for $K \in \mathbb{R}$ and $\hat{N} \in (n, \infty]$ if and only if

$$\mathrm{Ric}_g + \mathrm{Hess}^g_\varphi - \frac{\mathrm{d}\varphi \otimes \mathrm{d}\varphi}{\hat{N} - n} \ge Kg. \tag{2-4}$$

See [18, Proposition 4.21]. In particular, if it is an $\mathrm{RCD}(K, n)$ space, then φ must be a constant function because it is also an $\mathrm{RCD}(K, \hat{N})$ space for all $\hat{N} \in (n, \infty]$, which implies by (2-4) that $|\mathrm{d}\varphi| \equiv 0$. Let us denote the heat flow associated to the Cheeger energy by h_t. It holds (without curvature assumption) that

$$\|h_t f\|_{L^2} \le \|f\|_{L^2}, \quad \||\nabla f|\|_{L^2}^2 \le \frac{\|f\|_{L^2}^2}{2t^2}, \quad \|\Delta h_t f\|_{L^2} \le \frac{\|f\|_{L^2}}{t}. \tag{2-5}$$

Then one of the crucial properties of the heat flow on $\mathrm{RCD}(K, \infty)$ spaces is

$$h_t f \in \mathrm{TestF}(X, \mathsf{d}, \mathfrak{m}) \quad \text{for all } f \in L^2(X, \mathfrak{m}) \cap L^\infty(X, \mathfrak{m}) \text{ and } t \in (0, \infty), \tag{2-6}$$

where

(2-7) $\mathrm{TestF}(X,\mathsf{d},\mathfrak{m}) := \{f \in \mathrm{Lip}_b(X,\mathsf{d}) \cap H^{1,2}(X,\mathsf{d},\mathfrak{m}) : \Delta f \in H^{1,2}(X,\mathsf{d},\mathfrak{m})\}.$

See for instance [22] for the crucial role of test functions in the study of RCD spaces. Finally, we end this subsection by giving the following elementary lemma:

Lemma 2.2 *Let* $(X,\mathsf{d},\mathfrak{m})$ *be an* $\mathrm{RCD}(K,\infty)$ *space and let* $f \in D(\Delta)$. *Then there exists a sequence* $f_i \in \mathrm{TestF}(X,\mathsf{d},\mathfrak{m})$ *such that* $\|f_i - f\|_{H^{1,2}} + \|\Delta f_i - \Delta f\|_{L^2} \to 0$.

Proof Let $F_L := (-L) \vee f \wedge L$. Note that $h_t F_L \in \mathrm{TestF}(X,\mathsf{d},\mathfrak{m})$, that $h_t F_L \to h_t f$ in $H^{1,2}(X,\mathsf{d},\mathfrak{m})$ as $L \to \infty$ for all $t > 0$, and that $\Delta h_t F_L$ L^2–weakly converges to $\Delta h_t f$ as $L \to \infty$ for all $t > 0$, where we used (2-5) (see [4, Corollary 10.4]). Since $\Delta h_t f \to \Delta f$ in $L^2(X,\mathfrak{m})$ as $t \to 0^+$, there exist $L_i \to \infty$ and $t_i \to 0^+$ such that $h_{t_i} F_{L_i} \to f$ in $H^{1,2}(X,\mathsf{d},\mathfrak{m})$ and that $\Delta h_{t_i} F_{L_i}$ L^2–weakly converges to Δf. Then, applying Mazur's lemma for the sequence $\{\Delta h_{t_i} F_{L_i}\}_i$ yields that for all $m \geq 1$ there exist $N_m \in \mathbb{N}_{\geq m}$ and $\{t_{m,i}\}_{m \leq i \leq N_m} \subset [0,1]$ such that $\sum_{i=m}^{N_m} t_{m,i} = 1$ and that $\sum_{i=m}^{N_m} t_{m,i} \Delta h_{t_i} F_{L_i} \to \Delta f$ in $L^2(X,\mathfrak{m})$. It is easy to check that $f_m := \sum_{i=m}^{N_m} t_{m,i} h_{t_i} F_{L_i}$ satisfies the desired claim. □

2.2 Heat kernel

It is well known that the Bishop–Gromov theorem holds for any $\mathrm{RCD}(K,N)$ space $(X,\mathsf{d},\mathfrak{m})$ (or more generally for $\mathrm{CD}^*(K,N)$ spaces) and that the local Poincaré inequality holds for $\mathrm{RCD}(K,\infty)$ spaces (or more generally for $\mathrm{CD}(K,\infty)$ spaces). See [48, Theorem 30.11; 43; 42, Theorem 1]. Furthermore, it follows from the Sobolev-to-Lipschitz property that, on any $\mathrm{RCD}(K,N)$ space $(X,\mathsf{d},\mathfrak{m})$, the intrinsic distance

$$\mathsf{d}_{\mathsf{Ch}}(x,y) := \sup\{|f(x) - f(y)| : f \in H^{1,2}(X,\mathsf{d},\mathfrak{m}) \cap C_b(X,\mathsf{d}),\ |\nabla f| \leq 1\}$$

associated to the Cheeger energy Ch coincides with the original distance d. Consequently, applying [44, Proposition 2.3; 45, Corollary 3.3] on the general theory of Dirichlet forms provides the existence of a locally Hölder continuous representative p on $X \times X \times (0,\infty)$ for the heat kernel of $(X,\mathsf{d},\mathfrak{m})$. Let us recall that, by definition,

(2-8) $$h_t f(x) = \int_X p(x,y,t) f(y) \,\mathrm{d}\mathfrak{m}(y) \quad \text{for all } t > 0,\ x \in X \text{ and } f \in L^2(X,\mathfrak{m})$$

and

(2-9) $$p(x,y,t+s) = \int_X p(x,z,t) p(z,y,s) \,\mathrm{d}\mathfrak{m}(z) \quad \text{for all } t > 0,\ s > 0,\ x,y \in X.$$

The sharp Gaussian estimates on this heat kernel have been proved later on in the RCD context [33, Theorem 1.2]: for any $\epsilon > 0$, there exist $C_i := C_i(\epsilon, K, N) > 1$ for $i = 1, 2$, depending only on K, N and ϵ, such that

$$\text{(2-10)} \quad \frac{C_1^{-1}}{\mathfrak{m}(B_{\sqrt{t}}(x))} \exp\left(-\frac{\mathsf{d}^2(x,y)}{(4-\epsilon)t} - C_2 t\right) \le p(x,y,t) \le \frac{C_1}{\mathfrak{m}(B_{\sqrt{t}}(x))} \exp\left(-\frac{\mathsf{d}^2(x,y)}{(4+\epsilon)t} + C_2 t\right)$$

for all $x, y \in X$ and any $t > 0$, where from now on we state our inequalities with the Hölder continuous representative. Combining (2-10) with the Li–Yau inequality [19, Corollary 1.5; 32, Theorem 1.2], we have a gradient estimate [33, Corollary 1.2]:

$$\text{(2-11)} \quad |\nabla_x p(x,y,t)| \le \frac{C_3}{\sqrt{t}\,\mathfrak{m}(B_{\sqrt{t}}(x))} \exp\left(-\frac{\mathsf{d}^2(x,y)}{(4+\epsilon)t} + C_4 t\right) \quad \text{for } \mathfrak{m}\text{–a.e. } x \in X$$

for any $t > 0$ and $y \in X$, where $C_i := C_i(\epsilon, K, N) > 1$ for $i = 3, 4$. Note that in this paper, we will always work with (2-10) and (2-11) in the case $\epsilon = 1$.

Let us assume that $\mathrm{diam}(X, \mathsf{d}) < \infty$, thus (X, d) is compact (because in general (X, d) is proper). Then the doubling condition and a local Poincaré inequality on $(X, \mathsf{d}, \mathfrak{m})$ yields that the canonical embedding map $H^{1,2}(X, \mathsf{d}, \mathfrak{m}) \hookrightarrow L^2(X, \mathfrak{m})$ is a compact operator [26, Theorem 8.1]. In particular, the (minus) Laplacian $-\Delta$ admits a discrete positive spectrum $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \to +\infty$. We denote the corresponding eigenfunctions by $\varphi_0, \varphi_1, \ldots$ with $\|\varphi_i\|_{L^2} = 1$. This provides the following expansions for the heat kernel p:

$$\text{(2-12)} \quad p(x,y,t) = \sum_{i \ge 0} e^{-\lambda_i t} \varphi_i(x)\varphi_i(y) \quad \text{in } C(X \times X)$$

for any $t > 0$ and

$$\text{(2-13)} \quad p(\cdot, y, t) = \sum_{i \ge 0} e^{-\lambda_i t} \varphi_i(y)\varphi_i \quad \text{in } H^{1,2}(X, \mathsf{d}, \mathfrak{m})$$

for any $y \in X$ and $t > 0$ with the Hölder representative of all eigenfunctions. Combining (2-12) and (2-13) with (2-11), we know that φ_i is Lipschitz; in fact,

$$\text{(2-14)} \quad \|\varphi_i\|_{L^\infty} \le C_5 \lambda_i^{N/4}, \quad \|\nabla \varphi_i\|_{L^\infty} \le C_5 \lambda_i^{(N+2)/4}, \quad \lambda_i \ge C_5^{-1} i^{2/N},$$

where $C_5 := C_5(\mathrm{diam}(X, \mathsf{d}), K, N) > 0$. See for instance the appendices in [5; 30] for the proofs.

Finally, let us remark that it follows from this observation with (2-23) that

$$\Delta_x p(x,x,t) = 2\sum_{i\geq 0} e^{-\lambda_i t}\big(-\lambda_i(\varphi_i(x))^2 + |\nabla\varphi_i|^2(x)\big) \quad \text{in } H^{1,2}(X,\mathsf{d},\mathfrak{m}) \tag{2-15}$$

holds because (2-23) implies that

$$\sup_k \left\| \Delta\Big(\sum_{i\geq 0}^{k} e^{-\lambda_i t}\varphi_i^2\Big)\right\|_{H^{1,2}} = \sup_k \left\| 2\sum_{i\geq 0}^{k} e^{-\lambda_i t}(-\lambda_i\varphi_i^2 + |\nabla\varphi_i|^2)\right\|_{H^{1,2}} < \infty. \tag{2-16}$$

In particular, thanks to (2-12), we see that $p(x,x,t) \in D(\Delta)$ satisfies the equality in (2-15) in $L^2(X,\mathfrak{m})$. Then, applying Mazur's lemma for the sequence

$$\left\{\Delta\Big(\sum_{i\geq 0}^{k} e^{-\lambda_i t}\varphi_i^2\Big)\right\}_k$$

with (2-16) allows us to prove that the equality in (2-15) holds in $H^{1,2}(X,\mathsf{d},\mathfrak{m})$.

2.3 Infinitesimal structure

Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ space.

Definition 2.3 (regular set $\mathcal{R}_k$) For any $k \geq 1$, we denote by $\mathcal{R}_k$ the k*–dimensional regular set* of $(X,\mathsf{d},\mathfrak{m})$, namely the set of points $x \in X$ such that

$$\big(X, r^{-1}\mathsf{d}, \mathfrak{m}(B_r(x))^{-1}\mathfrak{m}, x\big)$$

pointed measured Gromov–Hausdorff converge to $(\mathbb{R}^k, \mathsf{d}_{\mathbb{R}^k}, \omega_k^{-1}\mathcal{L}^k, 0_k)$ as $r \to 0^+$.

We are now in a position to introduce the latest structural result for $\mathrm{RCD}(K,N)$ spaces.

Theorem 2.4 (essential dimension of $\mathrm{RCD}(K,N)$ spaces) *Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ space. Then, there exists a unique integer $n \in [1,N]$, denoted by $\dim_{\mathsf{d},\mathfrak{m}}(X)$, such that*

$$\mathfrak{m}(X \setminus \mathcal{R}_n) = 0. \tag{2-17}$$

In addition, the set $\mathcal{R}_n$ is $(\mathfrak{m},n)$–rectifiable and $\mathfrak{m}$ is representable as $\theta\mathcal{H}^n \llcorner \mathcal{R}_n$ for some nonnegative-valued function $\theta \in L^1_{\mathrm{loc}}(X,\mathcal{H}^n)$.

Note that the rectifiability of all sets $\mathcal{R}_k$ was inspired by [12; 13; 14] and proved in [41, Theorem 1.1], together with the concentration property $\mathfrak{m}\big(X \setminus \bigcup_k \mathcal{R}_k\big) = 0$, with the crucial uses of [24; 20]; the absolute continuity of $\mathfrak{m}$ on regular sets with respect to the corresponding Hausdorff measure was proved afterwards and is a consequence of [36, Theorem 1.2; 17, Theorem 1.1; 25, Theorem 3.5]. Finally, in the very recent work [10, Theorem 0.1] it is proved that only one set $\mathcal{R}_n$ has positive $\mathfrak{m}$–measure, leading to (2-17) and to the representation $\mathfrak{m} = \theta\mathcal{H}^n \llcorner \mathcal{R}_n$. Recall that our main target here is θ.

By slightly refining the definition of n–dimensinoal regular set, passing to a reduced set $\mathcal{R}_n^*$, general results of measure differentiation provide also the converse absolute continuity property $\mathcal{H}^n \ll \mathfrak{m}$ on $\mathcal{R}_n^*$. We summarize here the results obtained in this direction in [6, Theorem 4.1]:

Theorem 2.5 (weak Ahlfors regularity) *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *space,* $n := \dim_{\mathsf{d},\mathfrak{m}}(X)$, $\mathfrak{m} = \theta\mathcal{H}^n \llcorner \mathcal{R}_n$ *and set*

$$\mathcal{R}_n^* := \left\{ x \in \mathcal{R}_n \;\middle|\; \lim_{r\to 0+} \frac{\mathfrak{m}(B_r(x))}{\omega_n r^n} \in (0,\infty) \text{ exists} \right\}. \tag{2-18}$$

Then $\mathfrak{m}(\mathcal{R}_n \setminus \mathcal{R}_n^*) = 0$, $\mathfrak{m}\llcorner\mathcal{R}_n^*$ *and* $\mathcal{H}^n\llcorner\mathcal{R}_n^*$ *are mutually absolute continuous, and*

$$\lim_{r\to 0+} \frac{\mathfrak{m}(B_r(x))}{\omega_n r^n} = \theta(x) \qquad \textit{for } \mathfrak{m}\textit{–a.e. } x \in \mathcal{R}_n^*, \tag{2-19}$$

$$\lim_{r\to 0+} \frac{\omega_n r^n}{\mathfrak{m}(B_r(x))} = 1_{\mathcal{R}_n^*}(x)\frac{1}{\theta(x)} \qquad \textit{for } \mathfrak{m}\textit{–a.e. } x \in X. \tag{2-20}$$

Moreover, $\mathcal{H}^n(\mathcal{R}_n \setminus \mathcal{R}_n^*) = 0$ *if* $n = N$.

2.4 Second-order differential structure and Riemannian metric

Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, \infty)$ space.

Inspired by [49], the theory of the second-order differential structure on $(X, \mathsf{d}, \mathfrak{m})$ based on L^2–normed modules is established in [22]. To keep our presentation short, we omit several notions, for instance the spaces of L^2–vector fields, denoted by $L^2(T(X, \mathsf{d}, \mathfrak{m}))$, and of L^2–tensor fields of type $(0,2)$, denoted by $L^2((T^*)^{\otimes 2}(X, \mathsf{d}, \mathfrak{m}))$. See [22] for details. We denote the pointwise Hilbert–Schmidt norm and the pointwise scalar product by $|T|_{\mathrm{HS}}$ and $\langle T, S\rangle$, respectively (see also [22, Section 3.2; 4, Section 10]).

One of the important results in [22] we will use later is that for all $f \in D(\Delta)$, the Hessian $\mathrm{Hess}_f \in L^2((T^*)^{\otimes 2}(X, \mathsf{d}, \mathfrak{m}))$ is well defined and satisfies

$$\begin{aligned} &\langle \mathrm{Hess}_f, \mathsf{d}f_1 \otimes \mathsf{d}f_2\rangle \\ &\quad = \tfrac{1}{2}\big(\langle \nabla f_1, \nabla\langle\nabla f, \nabla f_2\rangle\rangle + \langle \nabla f_2, \nabla\langle\nabla f, \nabla f_1\rangle\rangle - \langle \nabla f, \nabla\langle\nabla f_1, \nabla f_2\rangle\rangle\big) \\ &\qquad\qquad \text{for } \mathfrak{m}\text{–a.e. } x \in X \text{ for all } f_i \in \mathrm{TestF}(X, \mathsf{d}, \mathfrak{m}) \end{aligned} \tag{2-21}$$

and the Bochner inequality with the Hessian term

$$\frac{1}{2}\Delta|\nabla f|^2 \geq |\mathrm{Hess}_f|_{\mathrm{HS}}^2 + \langle \nabla\Delta f, \nabla f\rangle + K|\nabla f|^2 \tag{2-22}$$

in the weak sense (see [22, Section 4]). In particular,

$$\int_X |\mathrm{Hess}_f|_{\mathrm{HS}}^2 \,\mathrm{d}\mathfrak{m} \leq \int_X ((\Delta f)^2 - K|\nabla f|^2)\,\mathrm{d}\mathfrak{m}. \tag{2-23}$$

Let us introduce the notion of Riemannian metrics on $(X, \mathsf{d}, \mathfrak{m})$. In order to simplify our argument we assume that $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K, N)$ space with $n = \dim_{\mathsf{d},\mathfrak{m}}(X)$ and $\mathrm{diam}(X, \mathsf{d}) < \infty$ below. Although we defined the notion as a bilinear form on $L^2(T(X, \mathsf{d}, \mathfrak{m}))$ in [5], we adopt an equivalent formulation by using tensor fields in this paper. Moreover, we consider only L^2 ones, which is enough for our purposes.

Definition 2.6 (L^2–Riemannian metric) We say that $T \in L^2((T^*)^{\otimes 2}(X, \mathsf{d}, \mathfrak{m}))$ is a *Riemannian metric* if for all $\eta_i \in L^\infty(T^*(X, \mathsf{d}, \mathfrak{m}))$ (which means that $\eta_i \in L^2(T^*(X, \mathsf{d}, \mathfrak{m}))$ with $|\eta_i|_{\mathrm{HS}} \in L^\infty(X, \mathfrak{m})$), it holds that

$$\langle T, \eta_1 \otimes \eta_2\rangle = \langle T, \eta_2 \otimes \eta_1\rangle, \quad \langle T, \eta_1 \otimes \eta_1\rangle \geq 0 \quad \text{for } \mathfrak{m}\text{–a.e. } x \in X \tag{2-24}$$

and that if $\langle T, \eta_1 \otimes \eta_1\rangle = 0$ for $\mathfrak{m}$–a.e. $x \in X$, then $\eta_1 = 0$ in $L^2(T^*(X, \mathsf{d}, \mathfrak{m}))$.

Proposition 2.7 (the canonical metric g) *There exists a unique Riemannian metric $g \in L^2((T^*)^{\otimes 2}(X, \mathsf{d}, \mathfrak{m}))$ such that*

$$\langle g, \mathrm{d}f_1 \otimes \mathrm{d}f_2\rangle = \langle \nabla f_1, \nabla f_2\rangle \quad \textit{for } \mathfrak{m}\textit{–a.e. on } X$$

for all Lipschitz functions f_i on X. Then

$$|g|_{\mathrm{HS}} = \sqrt{n} \quad \textit{for } \mathfrak{m}\textit{–a.e. } x \in X. \tag{2-25}$$

Note that for $T \in L^2((T^*)^{\otimes 2}(X, \mathsf{d}, \mathfrak{m}))$, *the trace* $\mathrm{tr}(T) \in L^2(X, \mathfrak{m})$ is $\mathrm{tr}(T) := \langle T, g\rangle$. The following result, proved in [27, Proposition 3.2], will play a crucial role in the proof of Theorem 1.2:

Theorem 2.8 (Laplacian is trace of Hessian under maximal dimension) *Assume that N is an integer with $\dim_{\mathsf{d},\mathfrak{m}}(X) = N$. Then, for all $f \in D(\Delta)$, we see that*

$$\Delta f = \mathrm{tr}(\mathrm{Hess}_f) \quad \textit{for } \mathfrak{m}\textit{–a.e. } x \in X. \tag{2-26}$$

Let us introduce the pullback Riemannian metrics by embeddings via the heat kernel (see [5, Proposition 4.7]).

Theorem 2.9 (the pullback metrics) *For every $t > 0$ there exists a unique Riemannian metric $g_t \in L^2((T^*)^{\otimes 2}(X, \mathsf{d}, \mathfrak{m}))$ such that*

$$\int_X \langle g_t, \eta_1 \otimes \eta_2 \rangle \, \mathrm{d}\mathfrak{m} = \int_X \int_X \langle \mathrm{d}_x p(x,y,t), \eta_1(x) \rangle \langle \mathrm{d}_x p(x,y,t), \eta_2(x) \rangle \, \mathrm{d}\mathfrak{m}(x) \, \mathrm{d}\mathfrak{m}(y) \quad \text{for all } \eta_i \in L^\infty(T^*(X, \mathsf{d}, \mathfrak{m})). \tag{2-27}$$

Moreover, it is representable as the HS-convergent series

$$g_t = \sum_{i=1}^{\infty} e^{-2\lambda_i t} \mathrm{d}\varphi_i \otimes \mathrm{d}\varphi_i \quad \text{in } L^2((T^*)^{\otimes 2}(X, \mathsf{d}, \mathfrak{m})). \tag{2-28}$$

Finally, the rescaled metric $t\mathfrak{m}(B_{\sqrt{t}}(\cdot))g_t$ satisfies

$$t\mathfrak{m}(B_{\sqrt{t}}(\cdot))g_t \le C(K,N)g \quad \text{for all } t \in (0,1), \tag{2-29}$$

which means that, for all $\eta \in L^\infty(T^*(X, \mathsf{d}, \mathfrak{m}))$,

$$t\mathfrak{m}(B_{\sqrt{t}}(x)) \langle g_t, \eta \otimes \eta \rangle (x) \le C(K,N) |\eta|^2_{\mathrm{HS}}(x) \quad \text{for } \mathfrak{m}\text{–a.e. } x \in X.$$

Note that since

$$\mathrm{Test}(T^*)^{\otimes 2}(X, \mathsf{d}, \mathfrak{m}) := \left\{ \sum_{i \ge 1}^{k} f_{1,i} \, \mathrm{d} f_{2,i} \otimes \mathrm{d} f_{3,i} \;\middle|\; k \in \mathbb{N},\ f_{j,i} \in \mathrm{TestF}(X, \mathsf{d}, \mathfrak{m}) \right\}$$

is dense in $L^2((T^*)^{\otimes 2}(X, \mathsf{d}, \mathfrak{m}))$ [22, (3.2.7)], it is easily checked that

$$\int_X \langle g_t, T \rangle \, \mathrm{d}\mathfrak{m} = \int_X \int_X \langle \mathrm{d}_x p \otimes \mathrm{d}_x p, T \rangle \, \mathrm{d}\mathfrak{m}(x) \, \mathrm{d}\mathfrak{m}(y) \quad \text{for all } T \in L^2((T^*)^{\otimes 2}(X, \mathsf{d}, \mathfrak{m})). \tag{2-30}$$

A main convergence result proved in [5] is the following:

Theorem 2.10 (L^p–convergence to the original metric) *We have*

$$|t\mathfrak{m}(B_{\sqrt{t}}(\cdot))g_t - c_n g|_{\mathrm{HS}} \to 0 \quad \text{in } L^p(X, \mathfrak{m}) \tag{2-31}$$

for all $p \in [1, \infty)$, where we recall (1-6) *for the definition of c_n.*

See [5, Theorem 5.10] for proofs of the results above. It is worth pointing out that in general we can not improve this L^p–convergence to the L^∞ one (see [5, Remark 5.11]).

We end this subsection by giving the following technical lemma:

Lemma 2.11 *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a compact* $\mathrm{RCD}(K, N)$ *space with* $n := \dim_{\mathsf{d},\mathfrak{m}}(X)$. *Assume that there exists* $C > 0$ *such that*

$$\mathfrak{m}(B_r(x)) \ge C r^n \quad \text{for all } x \in X \text{ and } r \in (0,1). \tag{2-32}$$

Then, as $t \to 0^+$, *we see that*

$$t^{(n+2)/2} p(x,x,t) \to 0 \quad \text{in } H^{1,2}(X, \mathsf{d}, \mathfrak{m}) \tag{2-33}$$

and that

$$\left| \omega_n t^{(n+2)/2} g_t - c_n \frac{\mathrm{d}\mathcal{H}^n}{\mathrm{d}\mathfrak{m}} g \right|_{\mathrm{HS}} \to 0 \quad \text{in } L^p(X, \mathfrak{m}) \quad \text{for all } p \in [1, \infty). \tag{2-34}$$

Proof By (2-10), we see that, for all $x \in X$ and all $t \in (0,1)$,

$$t^{(n+2)/2} p(x,x,t) \le \frac{t}{C} \mathfrak{m}(B_{\sqrt{t}}(x)) p(x,x,t) \le \frac{C(K,N)}{C} t. \tag{2-35}$$

In particular, $t^{(n+2)/2} p(x,x,t) \to 0$ in $C(X)$. On the other hand, (2-15) and (2-28) yield

$$\begin{aligned} \int_X |\Delta_x p(x,x,t)| \,\mathrm{d}\mathfrak{m}(x) &\le \sum_i 4\lambda_i e^{-\lambda_i t} = 4 \int_X \langle g, g_{t/2} \rangle \,\mathrm{d}\mathfrak{m} \\ &\le 4\sqrt{n} \int_X |g_{t/2}|_{\mathrm{HS}} \,\mathrm{d}\mathfrak{m}. \end{aligned} \tag{2-36}$$

In particular, (2-35) and (2-36) yield

$$\begin{aligned} \int_X |\nabla_x (t^{(n+2)/2} p(x,x,t))|^2 \,\mathrm{d}\mathfrak{m}(x) &= -\int_X t^{n+2} p(x,x,t) \Delta_x p(x,x,t) \,\mathrm{d}\mathfrak{m}(x) \\ &\le \frac{4C(K,N)\sqrt{n}}{C} t \int_X |t^{(n+2)/2} g_{t/2}|_{\mathrm{HS}} \,\mathrm{d}\mathfrak{m}(x) \\ &\le \frac{4C(K,N)\sqrt{n}}{C^2} 2^{(n+2)/2} t \int_X \left| \tfrac{1}{2} t \mathfrak{m}(B_{\sqrt{t/2}}(x)) g_{t/2} \right|_{\mathrm{HS}} \mathrm{d}\mathfrak{m}(x) \to 0 \end{aligned} \tag{2-37}$$

as $t \to 0^+$, where we used (2-29). Thus we get (2-33).

Next let us prove (2-34). First let us remark that (2-32) yields $\mathcal{H}^n \ll \mathfrak{m}$. Combining this with Theorem 2.5 shows that $\mathrm{d}\mathcal{H}^n/\mathrm{d}\mathfrak{m} \in L^\infty(X, \mathfrak{m})$ and that, as $r \to 0^+$,

$$\frac{\omega_n r^n}{\mathfrak{m}(B_r(x))} \to \frac{\mathrm{d}\mathcal{H}^n}{\mathrm{d}\mathfrak{m}}(x) \quad \text{for } \mathfrak{m}\text{–a.e. } x \in X. \tag{2-38}$$

Then since, as $t \to 0$,

$$\int_X \left| \omega_n t^{(n+2)/2} g_t - t\mathfrak{m}(B_{\sqrt{t}}(x)) \frac{\mathrm{d}\mathcal{H}^n}{\mathrm{d}\mathfrak{m}}(x) g_t \right|_{\mathrm{HS}}^p \mathrm{d}\mathfrak{m} = \int_X \left| \frac{\omega_n \sqrt{t}^n}{\mathfrak{m}(B_{\sqrt{t}}(x))} - \frac{\mathrm{d}\mathcal{H}^n}{\mathrm{d}\mathfrak{m}}(x) \right|^p |t\mathfrak{m}(B_{\sqrt{t}}(x)) g_t|_{\mathrm{HS}}^p \mathrm{d}\mathfrak{m} \le C(K,N)^p \int_X \left| \frac{\omega_n \sqrt{t}^n}{\mathfrak{m}(B_{\sqrt{t}}(x))} - \frac{\mathrm{d}\mathcal{H}^n}{\mathrm{d}\mathfrak{m}}(x) \right|^p \mathrm{d}\mathfrak{m} \to 0, \tag{2-39}$$

we conclude because of Theorem 2.10, where we used the dominated convergence theorem. □

3 Laplacian on $(X, g_t, \mathfrak{m})$

Let $(X, \mathsf{d}, \mathfrak{m})$ be a compact $\mathrm{RCD}(K,N)$ space. We rewrite our new differential operators:

Definition 3.1 (Laplacian on $(X, g_t, \mathfrak{m})$) For all $f \in D(\Delta)$ we define *the Laplacian* $\Delta^t f$ *associated to* g_t by

$$\Delta^t f := \langle \mathrm{Hess}_f, g_t \rangle + \tfrac{1}{4} \langle \nabla_x \Delta_x p(x,x,2t), \nabla f \rangle \in L^1(X, \mathfrak{m}). \tag{3-1}$$

Let us start our calculation:

Lemma 3.2 *For all* $f \in D(\Delta)$ *and* $\psi \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$, *we have*

$$\int_X \int_X \psi(x) \langle \nabla_x p, \nabla_x \langle \nabla_x p, \nabla f \rangle \rangle \,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y) = -\int_X \langle g_t, \mathrm{d}f \otimes \mathrm{d}\psi \rangle \,\mathrm{d}\mathfrak{m} + \frac{1}{4} \int_X \mathrm{div}(\psi \nabla f) \frac{\mathrm{d}}{\mathrm{d}t} p(x,x,2t)\,\mathrm{d}\mathfrak{m}. \tag{3-2}$$

Proof Note that

$$\begin{aligned} \int_X \int_X \psi(x) \langle \nabla_x p, \nabla_x \langle \nabla_x p, \nabla f \rangle \rangle \,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y) &= -\int_X \int_X \mathrm{div}_x(\psi \nabla_x p) \langle \nabla_x p, \nabla f \rangle \,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y) \\ &= -\int_X \int_X (\langle \nabla \psi, \nabla_x p \rangle + \psi(x) \Delta_x p) \langle \nabla_x p, \nabla f \rangle \,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y) \\ &= -\int_X \langle g_t, \mathrm{d}f \otimes \mathrm{d}\psi \rangle \,\mathrm{d}\mathfrak{m} - \int_X \int_X \psi(x) \Delta_x p \langle \nabla_x p, \nabla f \rangle \,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y) \end{aligned} \tag{3-3}$$

and that

$$\begin{aligned}
(3\text{-}4)\quad &-\int_X\int_X \psi(x)\Delta_x p\langle\nabla_x p, \nabla f\rangle \,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y)\\
&= -\int_X\int_X \psi(x)\Big(-\sum_{i\ge 0}\lambda_i e^{-\lambda_i t}\varphi_i(x)\varphi_i(y)\Big)\\
&\qquad\qquad\times\Big(\sum_{i\ge 0} e^{-\lambda_i t}\varphi_i(y)\langle\nabla\varphi_i, \nabla f\rangle(x)\Big)\,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y)\\
&= \sum_{i\ge 0}\lambda_i e^{-2\lambda_i t}\int_X \psi(x)\varphi_i(x)\langle\nabla f, \nabla\varphi_i\rangle(x)\,\mathrm{d}\mathfrak{m}(x)\\
&= \frac{1}{2}\sum_{i\ge 0}\lambda_i e^{-2\lambda_i t}\int_X \langle\psi\nabla f, \nabla\varphi_i^2\rangle\,\mathrm{d}\mathfrak{m}\\
&= -\frac{1}{2}\int_X \mathrm{div}(\psi\nabla f)\Big(\sum_{i\ge 0}\lambda_i e^{-2\lambda_i t}\varphi_i^2\Big)\,\mathrm{d}\mathfrak{m}.
\end{aligned}$$

On the other hand, since

$$(3\text{-}5)\qquad \frac{\mathrm{d}}{\mathrm{d}t}p(x,x,2t) = -2\sum_{i\ge 0}\lambda_i e^{-2\lambda_i t}\varphi_i(x)^2,$$

we have

$$(3\text{-}6)\quad -\frac{1}{2}\int_X \mathrm{div}(\psi\nabla f)\Big(\sum_i \lambda_i e^{-2\lambda_i t}\varphi_i^2\Big)\,\mathrm{d}\mathfrak{m} = \frac{1}{4}\int_X \mathrm{div}(\psi\nabla f)\frac{\mathrm{d}}{\mathrm{d}t}p(x,x,2t)\,\mathrm{d}\mathfrak{m},$$

which completes the proof because of (3-3) and (3-4). □

Lemma 3.3 *For all* $f\in D(\Delta)$ *and* $\psi\in H^{1,2}(X,\mathsf{d},\mathfrak{m})$, *we have*

$$\begin{aligned}
(3\text{-}7)\quad &-\frac{1}{2}\int_X\int_X \psi(x)\langle\nabla f, \nabla_x|\nabla_x p|^2\rangle\,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y)\\
&= -\frac{1}{4}\int_X \mathrm{div}(\psi\nabla f)\frac{\mathrm{d}}{\mathrm{d}t}p(x,x,2t)\,\mathrm{d}\mathfrak{m}(x)\\
&\qquad\qquad + \frac{1}{4}\int_X \mathrm{div}(\psi\nabla f)\Delta_x p(x,x,2t)\,\mathrm{d}\mathfrak{m}(x).
\end{aligned}$$

Proof By Lemma 2.2 it is enough to prove (3-7) under assuming $f\in \mathrm{TestF}(X,\mathsf{d},\mathfrak{m})$.

First assume $\psi \in \mathrm{TestF}(X, \mathsf{d}, \mathfrak{m})$. Let $\varphi := \mathrm{div}(\psi \nabla f) \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$. Then

$$
\begin{aligned}
(3\text{-}8)\quad & -\frac{1}{2}\int_X \int_X \psi(x)\langle \nabla_x f, \nabla_x |\nabla_x p|^2\rangle \,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y) \\
&= \frac{1}{2}\int_X \int_X \varphi(x)|\nabla_x p|^2 \,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y) \\
&= \frac{1}{2}\int_X \int_X \langle \nabla_x p, \nabla_x(\varphi p)\rangle \,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y) - \frac{1}{2}\int_X \int_X \langle p\nabla_x p, \nabla_x \varphi\rangle \,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y) \\
&= -\frac{1}{2}\int_X \int_X \varphi(x) p \Delta_x p \,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y) - \frac{1}{4}\int_X \int_X \langle \nabla_x p^2, \nabla_x \varphi\rangle \,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y) \\
&= -\frac{1}{2}\int_X \int_X \varphi(x) p \frac{\mathrm{d}}{\mathrm{d}t} p \,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y) + \frac{1}{4}\int_X \int_X \varphi \Delta_x p^2 \,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y) \\
&= -\frac{1}{4}\int_X \varphi(x)\left(\int_X \frac{\mathrm{d}}{\mathrm{d}t} p^2 \,\mathrm{d}\mathfrak{m}(y)\right) \mathrm{d}\mathfrak{m}(x) \\
&\qquad\qquad + \frac{1}{4}\int_X \varphi(x)\left(\int_X \Delta_x p^2 \,\mathrm{d}\mathfrak{m}(y)\right) \mathrm{d}\mathfrak{m}(x) \\
&= -\frac{1}{4}\int_X \varphi(x)\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_X p^2 \,\mathrm{d}\mathfrak{m}(y)\right) \mathrm{d}\mathfrak{m}(x) \\
&\qquad\qquad + \frac{1}{4}\int_X \varphi(x)\Delta_x\left(\int_X p^2 \,\mathrm{d}\mathfrak{m}(y)\right) \mathrm{d}\mathfrak{m}(x) \\
&= -\frac{1}{4}\int_X \varphi(x)\frac{\mathrm{d}}{\mathrm{d}t} p(x,x,2t) \,\mathrm{d}\mathfrak{m}(x) + \frac{1}{4}\int_X \varphi(x)\Delta_x p(x,x,2t)\,\mathrm{d}\mathfrak{m}(x),
\end{aligned}
$$

which proves (3-7), where we used (2-9).

Finally, let us prove (3-7) for general $\psi \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$. Let $\psi_L := (-L) \vee \psi \wedge L \in H^{1,2}(X, \mathsf{d}, \mathfrak{m}) \cap L^\infty(X, \mathfrak{m})$. Since (3-7) holds as $\psi = h_t(\psi_L)$ for all $t > 0$, letting $L \to \infty$ and then letting $t \to 0^+$ shows the desired claim. □

Theorem 3.4 (integration-by-parts on $(X, g_t, \mathfrak{m})$) *For all $f \in D(\Delta)$ and $\psi \in H^{1,2}(X, \mathsf{d}, \mathfrak{m}) \cap L^\infty(X, \mathfrak{m})$, we have*

$$
(3\text{-}9)\qquad \int_X \langle g_t, \mathrm{d}\psi \otimes \mathrm{d} f\rangle \,\mathrm{d}\mathfrak{m} = -\int_X \psi \Delta^t f \,\mathrm{d}\mathfrak{m}.
$$

Proof Lemmas 3.2 and 3.3 yield

$$
\begin{aligned}
&\int_X \psi \langle \mathrm{Hess}_f, g_t\rangle \,\mathrm{d}\mathfrak{m} \\
&\quad = \int_X \int_X \psi(x)\langle \mathrm{Hess}_f, \mathrm{d}_x p \otimes \mathrm{d}_x p\rangle \,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y)
\end{aligned}
$$

$$
\begin{aligned}
&= \int_X \int_X \psi(x) \langle \nabla_x p, \nabla_x \langle \nabla_x p, \nabla f \rangle \rangle \, \mathrm{d}\mathfrak{m}(x) \, \mathrm{d}\mathfrak{m}(y) \\
&\qquad\qquad - \frac{1}{2} \int_X \int_X \psi(x) \langle \nabla f, \nabla_x |\nabla_x p|^2 \rangle \, \mathrm{d}\mathfrak{m}(x) \, \mathrm{d}\mathfrak{m}(y) \\
&= -\int_X \langle g_t, \mathrm{d} f \otimes \mathrm{d}\psi \rangle \, \mathrm{d}\mathfrak{m} + \frac{1}{4} \int_X \mathrm{div}(\psi \nabla f) \Delta_x p(x,x,2t) \, \mathrm{d}\mathfrak{m}(x) \\
&= -\int_X \langle g_t, \mathrm{d} f \otimes \mathrm{d}\psi \rangle \, \mathrm{d}\mathfrak{m} - \frac{1}{4} \int_X \psi \langle \nabla f, \nabla_x \Delta_x p(x,x,2t) \rangle \, \mathrm{d}\mathfrak{m}(x),
\end{aligned}
$$

which proves (3-9). □

4 Characterization of noncollapsed RCD spaces

4.1 Proof of Theorem 1.2

Assume that (2) holds. Then the Bishop–Gromov inequality yields that (1-3) holds. Moreover, it follows from Theorem 2.8 that (1-1) holds. This proves the implication from (2) to (1).

Next we assume that (1) holds. Fix a nonconstant eigenfunction f of $-\Delta$ on $(X, \mathsf{d}, \mathfrak{m})$ with eigenvalue $\lambda > 0$. Applying Theorem 3.4 as $\psi \equiv 1$ shows

$$
\begin{aligned}
\text{(4-1)} \quad 0 = &-\int_X \langle \mathrm{Hess}_f, \omega_n t^{(n+2)/2} g_t \rangle \, \mathrm{d}\mathfrak{m} \\
&\qquad - \frac{1}{4} \int_X \langle \nabla \omega_n t^{(n+2)/2} p(x,x,2t), \nabla \Delta f(x) \rangle \, \mathrm{d}\mathfrak{m}(x).
\end{aligned}
$$

Lemma 2.11 yields that, as $t \to 0^+$, the first term of the right-hand side of (4-1) converges to

$$
-c_n \int_X \mathrm{tr}(\mathrm{Hess}_f) \frac{\mathrm{d}\mathcal{H}^n}{\mathrm{d}\mathfrak{m}} \, \mathrm{d}\mathfrak{m} = c_n \lambda \int_X f \frac{\mathrm{d}\mathcal{H}^n}{\mathrm{d}\mathfrak{m}} \, \mathrm{d}\mathfrak{m}. \tag{4-2}
$$

On the other hand, Lemma 2.11 yields that, as $t \to 0^+$, the second term of the right-hand side of (4-1) converges to 0. Thus (4-2) is equal to 0; in particular, $\mathrm{d}\mathcal{H}^n/\mathrm{d}\mathfrak{m}$ is L^2–orthogonal to f, which shows that $\mathrm{d}\mathcal{H}^n/\mathrm{d}\mathfrak{m}$ must be a constant function.

For all $\psi \in D(\Delta)$, since

$$
\psi = \sum_{i \geq 0} \Big(\int_X \psi \varphi_i \, \mathrm{d}\mathfrak{m} \Big) \varphi_i \quad \text{in } H^{1,2}(X, \mathsf{d}, \mathfrak{m}) \tag{4-3}
$$

and

$$\Delta\psi = -\sum_{i\geq 0}\lambda_i\left(\int_X \psi\varphi_i \,d\mathfrak{m}\right)\varphi_i \quad \text{in } L^2(X,\mathfrak{m}) \tag{4-4}$$

(see the appendices of [5; 30]), combining (4-3) and (4-4) with (2-23) yields

$$\mathrm{Hess}_\psi = \sum_{i\geq 0}\left(\int_X \psi\varphi_i \,d\mathfrak{m}\right)\mathrm{Hess}_{\varphi_i} \quad \text{in } L^2((T^*)^{\otimes 2}(X,\mathsf{d},\mathfrak{m})). \tag{4-5}$$

In particular,

$$\begin{aligned}\mathrm{tr}(\mathrm{Hess}_\psi) &= \left\langle \sum_{i\geq 0}\left(\int_X \psi\varphi_i \,d\mathfrak{m}\right)\mathrm{Hess}_{\varphi_i}, g\right\rangle = \sum_{i\geq 0}\left(\int_X \psi\varphi_i \,d\mathfrak{m}\right)\langle \mathrm{Hess}_{\varphi_i}, g\rangle \\ &= \sum_{i\geq 0}\left(\int_X \psi\varphi_i \,d\mathfrak{m}\right)\Delta\varphi_i \\ &= -\sum_{i\geq 0}\lambda_i\left(\int_X \psi\varphi_i \,d\mathfrak{m}\right)\varphi_i = \Delta\psi \quad \text{in } L^2(X,\mathfrak{m}).\end{aligned} \tag{4-6}$$

Therefore, if $\Delta\psi \in H^{1,2}(X,\mathsf{d},\mathfrak{m})$, then in the weak sense it holds that

$$\begin{aligned}\tfrac{1}{2}\Delta|\nabla\psi|^2 &\geq |\mathrm{Hess}_\psi|^2 + \langle\nabla\Delta\psi,\nabla\psi\rangle + K|\nabla\psi|^2 \\ &\geq \frac{(\mathrm{tr}(\mathrm{Hess}_\psi))^2}{n} + \langle\nabla\Delta\psi,\nabla\psi\rangle + K|\nabla\psi|^2 \\ &= \frac{(\Delta\psi)^2}{n} + \langle\nabla\Delta\psi,\nabla\psi\rangle + K|\nabla\psi|^2.\end{aligned} \tag{4-7}$$

This shows that $(X,\mathsf{d},\mathfrak{m})$ is an $\mathrm{RCD}(K,n)$ space. Thus we get (2). □

4.2 Witten Laplacian on RCD spaces

Let us recall that for a closed manifold (M^n,g) with a smooth function $\varphi \in C^\infty(M^n)$, the corresponding Laplacian of the weighted space $(M^n, g, e^{-\varphi}\,\mathrm{vol}_g)$ is the *Witten Laplacian* Δ_φ, that is,

$$\int_{M^n}\langle\nabla f_1,\nabla f_2\rangle e^{-\varphi}\,d\mathrm{vol}_g = -\int_{M^n} f_1\Delta_\varphi f_2 e^{-\varphi}\,d\mathrm{vol}_g \quad \text{for all } f_i \in C^\infty(M^n), \tag{4-8}$$

where $\Delta_\varphi f := \mathrm{tr}(\mathrm{Hess}_f) - \langle\nabla\varphi,\nabla f\rangle$. By using the formula (3-9), we can prove an analogous result in the nonsmooth setting. Compare with [28, Proposition 3.5].

Theorem 4.1 (Witten Laplacian on RCD spaces) *Let $n \in [1,\infty)$ and let (X,d) be a compact metric space satisfying that there exists $C > 0$ such that*

$$\mathcal{H}^n(B_r(x)) \geq Cr^n \quad \textit{for all } x \in X \textit{ and } r \in (0,1). \tag{4-9}$$

If $(X, \mathsf{d}, e^{-\varphi}\mathcal{H}^n)$ *is an* $\mathrm{RCD}(K, N)$ *space for some* $N \in [1, \infty)$ *and some Lipschitz function* φ *on* X, *then for all* $f \in D(\Delta)$ *we have*

$$\Delta f = \mathrm{tr}(\mathrm{Hess}_f) - \langle \nabla\varphi, \nabla f\rangle \quad \textit{in } L^2(X, \mathfrak{m}). \tag{4-10}$$

Proof Let $\mathfrak{m} := e^{-\varphi}\mathcal{H}^n$. By Lemma 2.2 it is enough to prove (4-10) under assuming $f \in \mathrm{TestF}(X, \mathsf{d}, \mathfrak{m})$. Note that

$$\mathfrak{m}(B_r(x)) = \int_{B_r(x)} e^{-\varphi}\, \mathrm{d}\mathcal{H}^n \geq Ce^{-\max\varphi} r^n \quad \text{for all } x \in X \text{ and } r \in (0,1). \tag{4-11}$$

Then, by an argument similar to the proof of Theorem 1.2, we see that, for all $\psi \in \mathrm{TestF}(X, \mathsf{d}, \mathfrak{m})$,

$$\int_X \langle \nabla\psi, \nabla f\rangle e^{\varphi}\, \mathrm{d}\mathfrak{m} = -\int_X \psi\, \mathrm{tr}(\mathrm{Hess}_f) e^{\varphi}\, \mathrm{d}\mathfrak{m}. \tag{4-12}$$

Since the left-hand side of (4-12) is equal to

$$\begin{aligned}\int_X \langle \nabla(\psi e^{\varphi}), \nabla f\rangle\, \mathrm{d}\mathfrak{m} - \int_X \langle \nabla\varphi, \nabla f\rangle \psi e^{\varphi}\, \mathrm{d}\mathfrak{m} \\ = -\int_X \psi e^{\varphi} \Delta f\, \mathrm{d}\mathfrak{m} - \int_X \langle \nabla\varphi, \nabla f\rangle \psi e^{\varphi}\, \mathrm{d}\mathfrak{m},\end{aligned} \tag{4-13}$$

we have

$$\int_X (-\Delta f - \langle \nabla\varphi, \nabla f\rangle + \mathrm{tr}(\mathrm{Hess}_f))\psi e^{\varphi}\, \mathrm{d}\mathfrak{m} = 0, \tag{4-14}$$

which completes the proof of (4-10) because ψ is arbitrary. □

We end this paper by giving another characterization of noncollapsed RCD spaces:

Corollary 4.2 *Let* $n, N \in [1, \infty)$ *and let* $(X, \mathsf{d}, \mathcal{H}^n)$ *be a compact* $\mathrm{RCD}(K, N)$ *space. Then the following two conditions are equivalent*:

(1) *There exists* $C > 0$ *such that*

$$\mathcal{H}^n(B_r(x)) \geq Cr^n \quad \textit{for all } x \in X \textit{ and } r \in (0,1). \tag{4-15}$$

(2) $(X, \mathsf{d}, \mathcal{H}^n)$ *is an* $\mathrm{RCD}(K, n)$ *space*, *that is*, *it is a noncollapsed space.*

Proof The implication from (2) to (1) is trivial because of the Bishop–Gromov inequality.

Assume that (1) holds. Then applying Theorem 4.1 as $\varphi \equiv 0$ yields that (1-1) holds. Therefore, Theorem 1.2 shows that (2) holds. □

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Proposed: Tobias H Colding Received: 28 June 2019
Seconded: Gang Tian, Bruce Kleiner Revised: 18 August 2019