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Asymmetric L–space knots

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We construct the first examples of asymmetric L–space knots in S^3. More specifically, we exhibit a construction of hyperbolic knots in S^3 with both (i) a surgery that may be realized as a surgery on a strongly invertible link such that the result of the surgery is the double branched cover of an alternating link and (ii) trivial isometry group. In particular, this produces L–space knots in S^3 which are not strongly invertible. The construction also immediately extends to produce asymmetric L–space knots in any lens space, including $S^1 \times S^2$.

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1 Introduction

A knot K in S^3 is an *L–space knot* if it admits a nontrivial Dehn surgery to a manifold Y such that $\operatorname{rk} \widehat{\mathrm{HF}}(Y) = |H_1(Y,\mathbb{Z})|$; see Ozsváth and Szabó [34]. The knot K is *strongly invertible* if there is an orientation-preserving involution of S^3 whose fixed point set is a circle that intersects the knot in two points and takes the knot K to itself. Watson makes a passing comment that "many L–space knots are strongly invertible" preceding [44, Theorem 1.2]. In the absence of evidence of L–space knots that are not strongly invertible, this has been promoted to a question (eg Motegi [30] and Hom [23]) and eventually a conjecture; see eg Lidman and Moore [26] and Watson [45].

Conjecture 1.1 (eg [26, Example 20] and [45, Conjecture 30]) *An L–space knot in S^3 is strongly invertible.*

We define the *symmetry group* of a 3–manifold N to be $\pi_0(\mathrm{Diff}(N))$, the group of isotopy classes of diffeomorphisms of N. Let K be a knot in a 3–manifold Y whose complement admits a complete hyperbolic metric of finite volume. We refer to such a knot as a *hyperbolic knot* in Y. We say that K is *asymmetric* if the symmetry group of its complement is trivial, that is, if any diffeomorphism of the knot complement is isotopic to the identity.

The main result of this paper is:

Theorem 1.2 *There exist asymmetric hyperbolic L–space knots K in S^3.*

The inclusion of the isometry group of a hyperbolic knot complement into the symmetry group of the knot complement is an isomorphism; see for example the paragraph preceding Theorem 6.2 in Bonahon [8]. On the other hand, the orbifold theorem shows that the isometry group of a strongly invertible hyperbolic knot is nontrivial. The symmetry group of a strongly invertible hyperbolic knot then is nontrivial. Thus Theorem 1.2 shows that Conjecture 1.1 is false, by giving examples of L–space knots in S^3 which cannot be strongly invertible.

Remark 1.3 Let the finite group G act on a 3–manifold N which admits a finite volume hyperbolic structure with totally geodesic boundary. If its fixed set has dimension at least one, then the action is conjugate to an isometric action by a diffeomorphism of N which is isotopic to the identity. See [8, Theorem 6.3].

Dunfield, Hoffman, and Licata [18] found examples of asymmetric hyperbolic L–space knots in lens spaces with nontrivial surgeries to other lens spaces, the double branched covers of 2–bridge links. This then allowed them to demonstrate the existence of asymmetric hyperbolic L–spaces [18, Theorem 1.2]. Similarly, Theorem 1.2 along with [18, Lemma 2.2] (see also Hodgson and Weeks [21]) and [34, Proposition 2.1] shows that such manifolds can be obtained by Dehn surgery on knots in S^3.

Corollary 1.4 *There exist asymmetric hyperbolic L–spaces that are obtained as Dehn surgery on knots in* S^3. □

The double branched cover of a nonsplit alternating link is an L–space; see Ozsváth and Szabó [35]. When a nontrivial Dehn surgery on a knot yields the double branched cover of a nonsplit alternating link, we refer to this Dehn surgery as an *alternating surgery* on the knot. Thus any knot in S^3 admitting an alternating surgery is an L–space knot.

Along the lines of Conjecture 1.1, McCoy proposed:

Conjecture 1.5 [27, Conjecture 1] *If a knot in* S^3 *has an alternating surgery, then the knot corresponds to the dealternation crossing in an almost-alternating diagram of the unknot.*

The knot corresponding to a dealternation crossing is strongly invertible. Our proof of Theorem 1.2 is a consequence of the following theorem which shows McCoy's conjecture is also false.

Theorem 1.6 *There exist asymmetric hyperbolic knots in* S^3 *with alternating surgeries.*

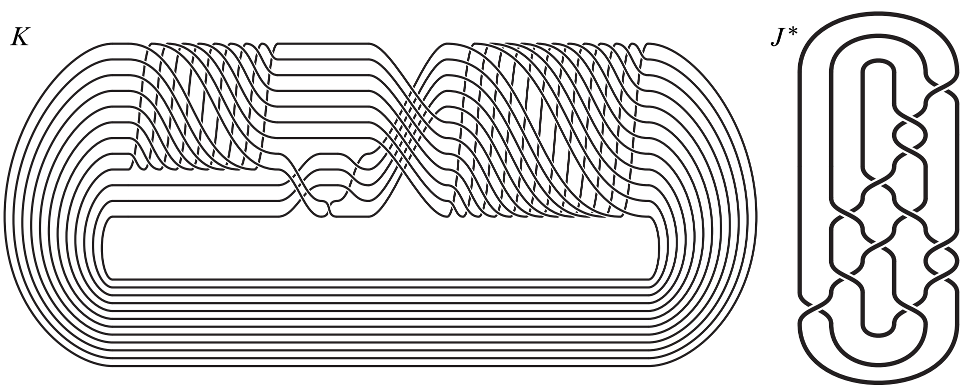

Figure 1: The knot K (left) is the closure of a 12–strand positive braid of length 249. It is an asymmetric hyperbolic L–space knot. The result of 272–surgery on K produces the double branched cover of the 12–crossing alternating link J^* (right).

In Conjecture 11.2 we propose a modification of Conjecture 1.5.

Example 1.7 Our simplest example is the genus 119 knot K that is the closure of the positive 12–strand braid of length 249 shown in Figure 1, left. The result of 272–surgery on K produces the double branched cover of the 12–crossing alternating link J^* shown in Figure 1, right, which is also known as the link $L12a955$.

SnapPy [16] and Sage [37] confirm the complement of K is hyperbolic and asymmetric and compute that its complement has volume ~ 10.20098 with cusp shape $\sim 0.41433 + 1.19820i$ and canonical triangulation comprised of 13 tetrahedra.[1]

Remark 1.8 Boileau, González-Acuña, and Montesinos [7] and Callahan [12] have created knots in S^3 with no symmetry of finite order that yet admit a nontrivial Dehn surgery to double branched covers of S^3. However it is not clear whether L–space knots that are not strongly invertible may arise from their constructions.

Our construction giving Theorem 1.6 easily adapts with S^3 replaced by any of the lens spaces, including $S^1 \times S^2$. This adaptation is detailed in Theorem 9.2. Following Rasmussen and Rasmussen [36], the exterior of a knot in $S^1 \times S^2$ with a nontrivial L–space surgery is a *generalized solid torus*; every nontrivial surgery on such a knot is an L–space. Thus the adaptation gives the following theorem.

Theorem 1.9 *There exist asymmetric hyperbolic generalized solid tori.* □

1.1 Heegaard genus and tunnel number

As suggested by Baker and Moore [5], an upper bound on the tunnel number of hyperbolic L–space knots in S^3 is not expected, though the familiar examples all have tunnel number 1. Motegi [30] has demonstrated strongly invertible hyperbolic L–space knots with tunnel number 2.

Since knots with tunnel number 1 are all strongly invertible, the tunnel number of an asymmetric hyperbolic L–space knot must be at least 2. The families of knots we create in this article have bounded tunnel number. Indeed, as shown in Proposition 10.1, the lashing construction (Section 2) applied to a genus 2 Heegaard splitting produces knots

[1] Through a search of the SnapPy census, Nathan Dunfield found nine asymmetric L–space knots in S^3 (personal communication, May 2019). Their complements are the census manifolds `t12533`, `t12681`, `o9_38928`, `o9_39162`, `o9_40363`, `o9_40487`, `o9_40504`, `o9_40582`, and `o9_42675`. These knots have smaller genus (from genus 12 to genus 33) than those discussed here. Do they nevertheless arise from a variation of the construction presented in this article?

with tunnel number at most 3. Moreover, Proposition 10.2 shows that the asymmetric L–space knots we construct have tunnel number 2, and Proposition 10.3 shows that the Heegaard genus of the double branched cover of the alternating link obtained by surgery is at most 3. As evidenced by the fact that the link J^* of Example 1.7 is a 3–bridge link, our asymmetric hyperbolic L–space knots sometimes admit surgeries to Heegaard genus 2 L–spaces.

1.2 Overview of construction of asymmetric L–space knots

Our construction begins with a family of *lashings* of a pair of pants P embedded in a 3–manifold Y. Let H_P be a genus 2 handlebody properly containing P that is identified with the closed product neighborhood $P \times [-1, 1]$ so that P is identified with $P \times \{0\}$. Described more explicitly in Section 2, a p/q*–lashing* of P is a particular framed knot in H_P, relative to a labeling of ∂P as $\mathcal{C} = C_\nu \cup C_\mu \cup C_\lambda$. Lemma 2.2 shows that this p/q–lashing of P has the property that framed surgery on the lashing is equivalent to $(+1, -q/(p+q), -p/(p+q))$–surgery on $\mathcal{C} = C_\nu \cup C_\mu \cup C_\lambda$ where the surgery coefficients are taken with respect to the framing by P. Figure 5 illustrates the surgery calculus that proves this lemma.

While every framed knot in Y can be expressed as a lashing of P for some pair of pants P, we find utility in lashings when ∂P enjoys properties that are not apparent in the lashings of P. Suppose Y admits a genus 2 Heegaard surface $\Sigma = P \cup_{\mathcal{C}} Q$ expressed as the union of two pairs of pants P and Q such that ∂P and ∂Q are both identified as the triple of curves $\mathcal{C}$. Since each component of $\mathcal{C}$ is nonseparating in Σ, the hyperelliptic involution on Σ may be isotoped to exchange P and Q while fixing $\mathcal{C}$ componentwise. This involution on Σ extends to an involution ι on Y in which $\mathcal{C}$ is a strongly invertible link, and yet, because of the exchanging of P and Q by ι, the lashings of P seem unlikely to be isotopic to a knot invariant under ι — at least in general.

We use this setup to construct knots proving Theorem 1.2 as follows:

(1) We find a family of 3–bridge presentations of the unknot parametrized by an alternating 3–braid α and integer m so that the triple of $(+1, -q/(p+q), -p/(p+q))$–rational tangle replacements along a triple of arcs $c = c^{\alpha,m}$ in the bridge sphere produces an alternating link for coprime integers p and q with $p/q > 0$. Using the Montesinos trick, this lifts to $(+1, -q/(p+q), -p/(p+q))$–surgery on a 3–component link $\mathcal{C} = \mathcal{C}^{\alpha,m}$ in S^3. Being the double branched cover of a nonsplit alternating link, the resulting manifold is an L–space; see Ozsváth and Szabó [35]. The bridge sphere

lifts to a genus 2 Heegaard surface Σ divided by $\mathcal{C}$ into pairs of pants $P = P^{\alpha,m}$ and $Q = Q^{\alpha,m}$. As in Lemma 2.2, the surgery on $\mathcal{C}$ may be obtained by framed surgery on a p/q–lashing of P. Hence the p/q–lashing is an L–space knot.

(2) The manifold $M = Y - \mathcal{N}(P)$ is the union of two genus 2 handlebodies along the pair of pants Q. The exterior of the p/q–lashing of P in Y may be identified as $M[\gamma]$, the result of attaching a 2–handle to M along a particular nonseparating curve γ in the genus 2 boundary of M. For "suitable" choices of P and Q in Y, based upon the disk-busting and annulus-busting nature of $\mathcal{C}$ in the two handlebodies $M \setminus Q$, we show that M is a simple 3–manifold (Lemma 5.3) and that Q is the unique separating, incompressible, boundary incompressible pair of pants in M (Lemmas 5.4 and 5.2). Then, for "sufficiently complicated" lashings of P, Theorem 4.10 shows that the complement of the lashing is a hyperbolic manifold, and any diffeomorphism of its exterior is isotopic to the identity. The latter is done by first ensuring any such diffeomorphism may be isotoped to restrict to a diffeomorphism of M, then to one that is the identity on Q, and eventually to one that is the identity on M and γ.

(3) Finally, we verify that the parameters α and m may be chosen so that $P = P^{\alpha,m}$ and $Q = Q^{\alpha,m}$ are "suitable" and that there are coprime integers p and q with $p/q > 0$ for which the p/q–lashing of P is "sufficiently complicated".

1.3 Basic notation

Throughout, $A \setminus B$ means the closure of $A - B$ in the path metric, and $\mathcal{N}(B)$ denotes a regular tubular neighborhood of B. If A and B are properly embedded 1–submanifolds of a surface C, then $\Delta_C(A, B)$ denotes the geometric intersection number, the minimal number of intersections of A and B up to proper isotopy.

If $L = L_1 \cup \cdots \cup L_n$ is an n–component link in a 3–manifold Y, then $Y_L(r_1, \ldots, r_n)$ denotes the result of r_i–Dehn surgery on the component L_i for each $i = 1, \ldots, n$. If $X = Y \setminus L$, then this may also be denoted $X(r_1, \ldots, r_n)$.

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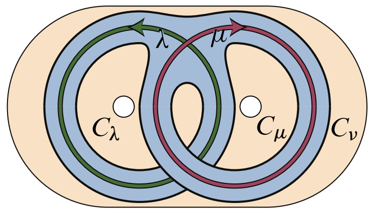
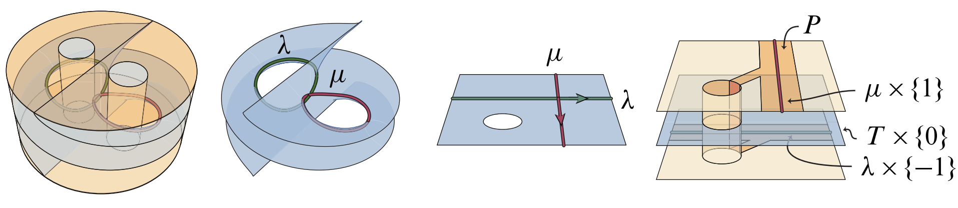


Figure 2: A projection to P of the once-punctured torus $T \subset P \times I$ with its basis of curves μ and λ along with the three boundary curves C_ν, C_μ, and C_λ.

2 Lashings

Consider an oriented pair of pants P embedded in a 3–manifold Y with an oriented once-punctured torus T embedded in a closed product neighborhood $P \times [-1,1] = P \times I \subset Y$ so that T has a projection to P as shown in Figure 2. We identify P with $P \times \{0\}$.

A basis of curves μ and λ in T are also shown, oriented so that $\mu \cdot \lambda = +1$. An unoriented, nontrivial, simple closed curve τ in T has *slope* p/q if it is homologous to $p[\mu] + q[\lambda]$ for some orientation. A curve in T of slope p/q with respect to this basis is a knot $K = K(p/q)$ in $P \times I \subset Y$. We say the knot K with framing given by T is a *p/q–lashing* of P.



$P \times I \cong T \times I$ $\quad$ $T \times \{0\}$ $\quad$ T $\quad$ $T \times I$

Figure 3: The handlebody $P \times I\,(\cong T \times I)$ containing the properly embedded once-punctured torus $T = T \times \{0\}$, along with this configuration of $T \times \{0\}$ isolated and the curves λ and μ labeled (leftmost two). The once-punctured torus T and its curves λ and μ in a more standard form where opposite sides of the rectangle are identified (second from right). On the far right is a view of the handlebody $T \times I\,(\cong P \times I)$ with the pair of pants $P = P \times \{1\}$ marked on its boundary. (This figure is essentially Figures 1 and 2 from [2] except that μ and λ are swapped. Equivalently, one may change the orientation of T in [2, Figure 2].)

Note that the three boundary components of P are each isotopic in $P \times I$ to one of the lashings $\nu = K\left(\frac{-1}{1}\right)$, $\mu = K\left(\frac{1}{0}\right)$, and $\lambda = K\left(\frac{0}{1}\right)$. We call them C_ν, C_μ, and C_λ accordingly and collectively $\mathcal{C} = \partial P (= \partial P \times \{0\})$.

It will be helpful to have another way to view a lashing inside of $P \times I$. Observe that T may be isotoped in $P \times I$ to be properly embedded so that $P \times I \cong T \times I$ where the boundary of $T \times \{0\}$ crosses $\partial P \times \{0\}$ in only two points of C_ν. This is shown in Figure 3. Figure 3 then also represents $P \times \{1\}$ in $T \times I$. In particular, in $T \times I$, the curves C_μ and C_λ are respectively isotopic in $T \times \{1\}$ and $T \times \{-1\}$ to $\mu \times \{1\}$ and $\lambda \times \{-1\}$. As shown in Figure 11, the curve C_ν however is isotopic in $\partial(T \times I)$ to a union of the arcs $-\mu' \times \{1\}$ and $\lambda' \times \{-1\}$ and two arcs in $\partial T \times I$. (For an essential simple closed curve α in T, we let α' denote the essential properly embedded arc in T that is disjoint from α.)

Remark 2.1 Lashings, though without this name, were previously used in [9] and [2] to create knots in handlebodies with cosmetic surgeries and bridge number greater than 1. Note that we have reversed the roles of μ and λ from the convention in [2]. In particular, our $K(p/q)$ would be $K(-q/p)$ in [2].

Lemma 2.2 *Let K be the p/q–lashing of the pair of pants P. Then T–framed surgery on K is equivalent to $(+1, -q/(p+q), -p/(p+q))$–surgery on the link $C_\nu \cup C_\mu \cup C_\lambda = \partial P$ with respect to the framing by P.*

Before proving the lemma, we first establish continued fraction conventions and recall surgery realizations of Dehn twists that facilitate a surgery description of a lashing. From this surgery description, Figure 5 illustrates the relevant surgery calculus from which the lemma follows.

We use the *continued fraction* convention

$$[r_n, r_{n-1}, \dots, r_2, r_1] = -\cfrac{1}{r_n - \cfrac{1}{r_{n-1} - \cfrac{1}{\cdots - \cfrac{1}{r_2 - \cfrac{1}{r_1}}}}}$$

used in [25]. This corresponds to the continued fraction $[0, r_n, \dots, r_1]$ used in [38] and to the continued fraction $-[r_n, \dots, r_1] = [-r_n, \dots, -r_1]$ used in [1].

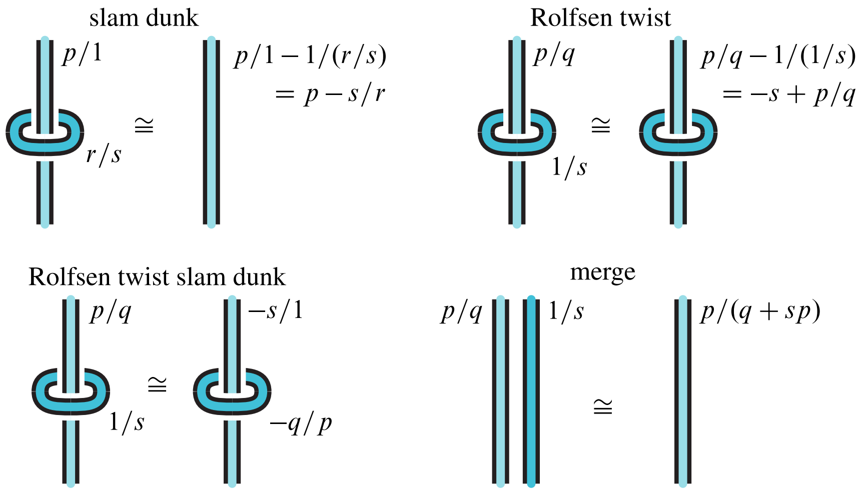


Figure 4: A few basic moves on surgery diagrams. The merge move amalgamates two surgery curves cobounding an annulus along their 0–framings.

If L is a simple closed curve in an oriented surface S, then ϕ_L is a *positive Dehn twist* of S along L. If S is embedded in an oriented 3–manifold, let L_+ and L_- denote the push-offs of L to the positive and negative side of S, so that they cobound an annulus $\widehat{R}$ intersecting S in the curve L. If K is a simple closed curve in S, then $\phi_L^n(K)$ may be obtained as the image of K after $(-1/n, 1/n)$–Dehn surgery on the link $L_+ \cup L_-$, as framed by $\widehat{R}$. (Specifically, the homeomorphism on the exterior of $L_+ \cup L_-$ gotten by twisting n–times along (the restriction of) $\widehat{R}$ extends to a homeomorphism from the $(-1/n, 1/n)$–Dehn surgery on $L_+ \cup L_-$ to the $\left(\frac{-1}{0}, \frac{1}{0}\right)$–Dehn surgery. Under this homeomorphism K goes to $\phi^n(K)$. See also [4, Definition 1.2].)

Lemma 2.3 (eg [1, Lemma 2.1]) *If $p/q = [r_n, \dots, r_1]$ with n odd, then*

$$K(p/q) = \phi_\lambda^{r_n} \circ \cdots \circ \phi_\mu^{r_2} \circ \phi_\lambda^{r_1}(\mu),$$

where $K(p/q)$ is the essential simple closed curve in T homologous to $p\mu + q\lambda$ for some orientation.

Proof With changes of notation and continued fraction convention, this lemma is [1, Lemma 2.1]. However, to be explicit about our parametrizations, we sketch the proof here.

With respect to the oriented basis $\langle \mu, \lambda \rangle$ for $H_1(T)$, we have that $\phi_\mu = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ and $\phi_\lambda = \left[\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right]$ Then $\phi_\lambda^{r_1}(\mu) = \mu - r_1\lambda$ and its slope is $-1/r_1 = [r_1]$. Now say $a/b = [r_{n-2}, \dots, r_1]$ with n odd. Then

$$\phi_\lambda^{r_n} \circ \phi_\mu^{r_{n-1}}(a\mu + b\lambda) = (a + r_{n-1}b)\mu + (-r_n(a + r_{n-1}b) + b)\lambda$$

has slope

$$\frac{a + r_{n-1}b}{-r_n(a + r_{n-1}b) + b} = -\cfrac{1}{r_n - \cfrac{1}{r_{n-1} + \cfrac{a}{b}}} = -\cfrac{1}{r_n - \cfrac{1}{r_{n-1} + -\cfrac{1}{r_{n-2} - \cfrac{1}{\dots - \cfrac{1}{r_2 - \cfrac{1}{r_1}}}}}}$$
$$= [r_n, r_{n-1}, r_{n-2}, \dots, r_2, r_1].$$

Hence induction gives the desired result. □

Proof of Lemma 2.2 Take a continued fraction expansion $p/q = [r_n, \dots, r_1]$ with n odd (as may always be done since $[r_n, \dots, r_1] = [r_n, \dots, r_1 + 1, 1]$). Since the p/q–lashing $K = K(p/q)$ is an essential simple closed curve in T with slope p/q, it may be expressed as the sequence of Dehn twists $\phi_\lambda^{r_n} \circ \dots \circ \phi_\mu^{r_2} \circ \phi_\lambda^{r_1}(\mu)$ of μ in T along λ and μ alternately by Lemma 2.3. From this sequence of Dehn twists taking μ to K, we construct a link in $P \times I$ formed around μ from alternating, nested pairs of push-offs of λ and μ. Using the surgery realization of Dehn twists, the sequence of surgeries

$$-\frac{1}{r_n},\quad -\frac{1}{r_{n-1}},\quad \dots,\quad -\frac{1}{r_2},\quad -\frac{1}{r_1},\quad *,\quad \frac{1}{r_1},\quad \frac{1}{r_2},\quad \dots,\quad \frac{1}{r_{n-1}},\quad \frac{1}{r_n}$$

on this link in order, from above T down to below, takes the central μ to K (see [1, Section 3.1]). These surgeries are all framed with respect to T, which coincides with the blackboard framing of λ and μ in Figure 2 (because T is "flat" in this projection). The single $*$ indicates that no surgery is being done on the central μ. This gives a surgery presentation of K. For the T–framed surgery on K, we then replace the $*$ with 0 in the surgery sequence. For $n = 5$, a diagram in P of the link with its surgery coefficients is shown in the lower left of Figure 5. Note that in Figure 5 the central μ is yellow.

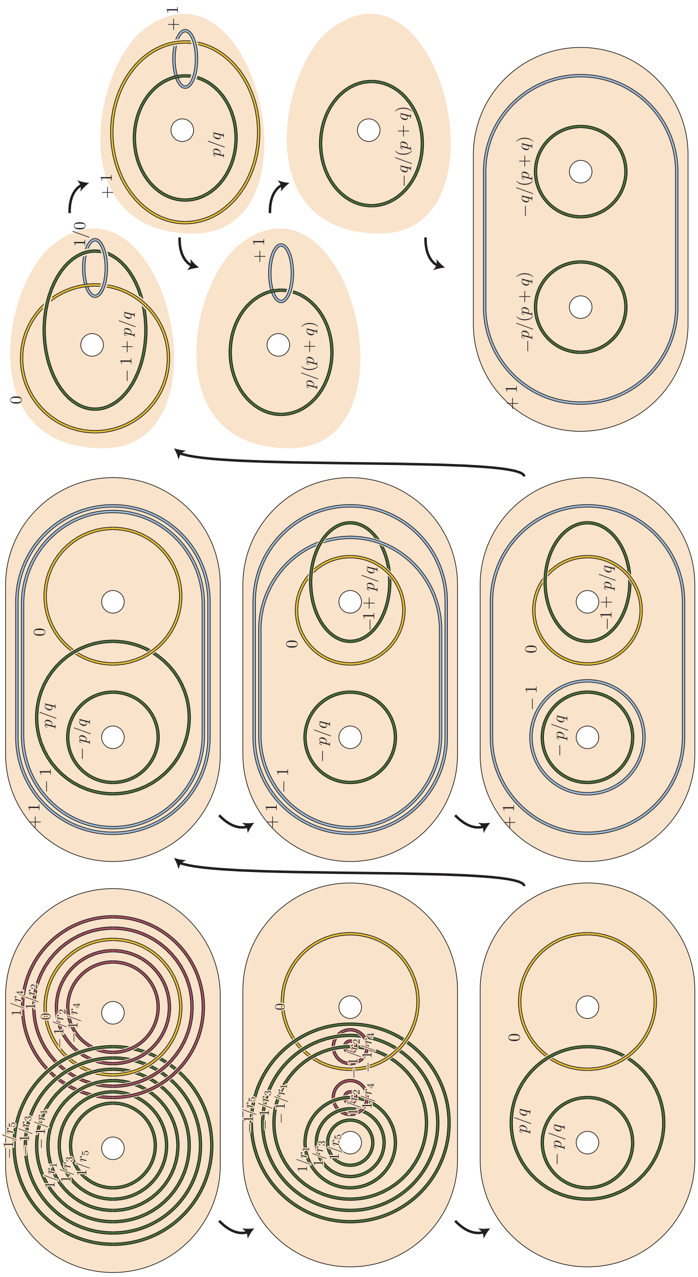


Figure 5: A sequence of equivalences of surgery diagrams. In the bottom-left figure, the surgery coefficients on the two sets of concentric circles from outside to inside are $-1/r_5$, $-1/r_3$, $-1/r_1$, $1/r_1$, $1/r_3$, $1/r_5$ and $1/r_4$, $1/r_2$, $0, -1/r_2$, $-1/r_4$.

Now we perform a sequence of equivalences of Dehn surgery diagrams, transforming the surgery on the large nested link in $P \times I$ to a surgery on a link isotopic in $P \times I$ to ∂P. It begins with sliding the push-offs of μ across the 0–framed central μ followed by slam dunks and mergers (see Figure 4 for terminology) to produce a 3–component link that is close to, but not quite, what we want. Further applications of blow-ups, mergers, and slides produce the desired link. This is explicitly illustrated in Figure 5 for the situation with $n = 5$. The general case follows directly. □

3 A construction of L–space knots via lashings and the Montesinos trick

The center diagram of Figure 7 gives a family of diagrams of the unknot $J = J^{\alpha,m}$ in a 3–bridge position indexed by the 3–braid α and integer m. On the right are the diagrammatic conventions for the braid α and the numbered twist boxes. The diagram on the left more clearly exhibits J as an unknot since the braiding above and below the horizontal line are inverses except for an extra twist at the bottom and the offset capping of the six strands. The diagram in the center is obtained from an isotopy that flips the α^{-1} braid box in the left figure along its vertical axis and rearranges the strands below the horizontal line. Note that if $m \geq 0$ and $a_i \geq 0$ for all i, then the center diagram is an *almost-alternating* unknot; that is, switching the circled crossing makes the diagram alternating.

Figure 7 further shows a triple of arcs $c = c^{\alpha,m}$ in the 3–bridge sphere of J. Naming the arcs of c in Figure 7 as c_ν, c_μ, and c_λ from left to right, Figure 8 shows the result of the $(+1, -q/(p+q), -p/(p+q))$–rational tangle replacements on them followed by an isotopy eliminating two pairs of crossings. See Figure 6 for the conventions on rational tangles and the parametrizing 3–braid β. We refer to the link resulting from this tangle replacement as $J_c^{\alpha,m}(+1, -q/(p+q), -p/(p+q))$. Observe that if $p/q = [r_n, \dots, r_2, r_1]$ then

$$\frac{-p}{p+q} = -\frac{1}{1+q/p} = -\cfrac{1}{1-r_n-\cfrac{1}{-r_{n-1}-\cfrac{1}{\cdots-\cfrac{1}{-r_2-\cfrac{1}{-r_1}}}}}$$

$$= [1-r_n, -r_{n-1}, \dots, -r_2, -r_1]$$

and

$$\frac{-q}{p+q} = -\frac{1}{1+p/q} = -\cfrac{1}{1-\cfrac{1}{r_n-\cfrac{1}{\cdots-\cfrac{1}{r_2-\cfrac{1}{r_1}}}}}$$

$$= [1, r_n, \ldots, r_2, r_1] = [1, r_n, \ldots, r_2, r_1 + 1, 1],$$

so that the corresponding rational tangles may indeed be expressed as shown in Figure 8. When $m \geq 0$ and $a_i \geq 0$ for all i and $p/q > 0$, so that the continued fraction expansion $p/q = [-b_n, b_{n-1}, \ldots, b_2, -b_1]$ with n odd has $b_j \geq 0$ for all j, this resulting reduced diagram is alternating. That is, $J_c^{\alpha,m}(+1, -q/(p+q), -p/(p+q))$ is an alternating link.

Remark 3.1 If δ is a 3–braid, then we use $\overline{\delta}$ to denote the 3–braid obtained by rotating δ through the page along a vertical axis. We denote by $-\delta$ the 3–braid obtained by changing all crossings of δ and by δ^{-1} the inverse in the braid group.

Remark 3.2 The parametrization of the 3–braid β and the 3–braid α are different because of the assignment of a rational tangle to β and for efficiency in using α in the representation of the 3–bridge knot of Figure 7. If β corresponds to $b_n, b_{n-1}, \ldots, b_1$ and α corresponds to $a_n, a_{n-1}, \ldots, a_1$ where $b_i = a_i$, then $\alpha = \overline{\beta}$ as a 3–braid.

Since J is the unknot in S^3, the double branched cover of J is again S^3. In this cover, the 3–bridge sphere lifts to a genus 2 Heegaard surface Σ for S^3 and the triple of arcs $c = c^{\alpha,m}$ lifts to a triple of curves $\mathcal{C} = \mathcal{C}^{\alpha,m}$ that divide Σ into two pairs of pants. Label the components of $\mathcal{C}$ so that C_* is the lift of c_* for $* = \nu, \mu, \lambda$. Let $P = P^{\alpha,m}$ be one of the pairs of pants in Σ bounded by $\mathcal{C}$ (identified with the pair of pants P in Figure 2 by an orientation-preserving homeomorphism so that the labeling of ∂P by $\mathcal{C}$ agrees) and set $Q = Q^{\alpha,m}$ to be the other one.

Theorem 3.3 *For each 3–braid α that is a product of positive powers of σ_1^{-1} and σ_2, a nonnegative integer m, and nonnegative coprime integers p and q, the p/q–lashing of $P^{\alpha,m}$ admits a longitudinal alternating surgery. Consequently, the lashing is an L–space knot.*

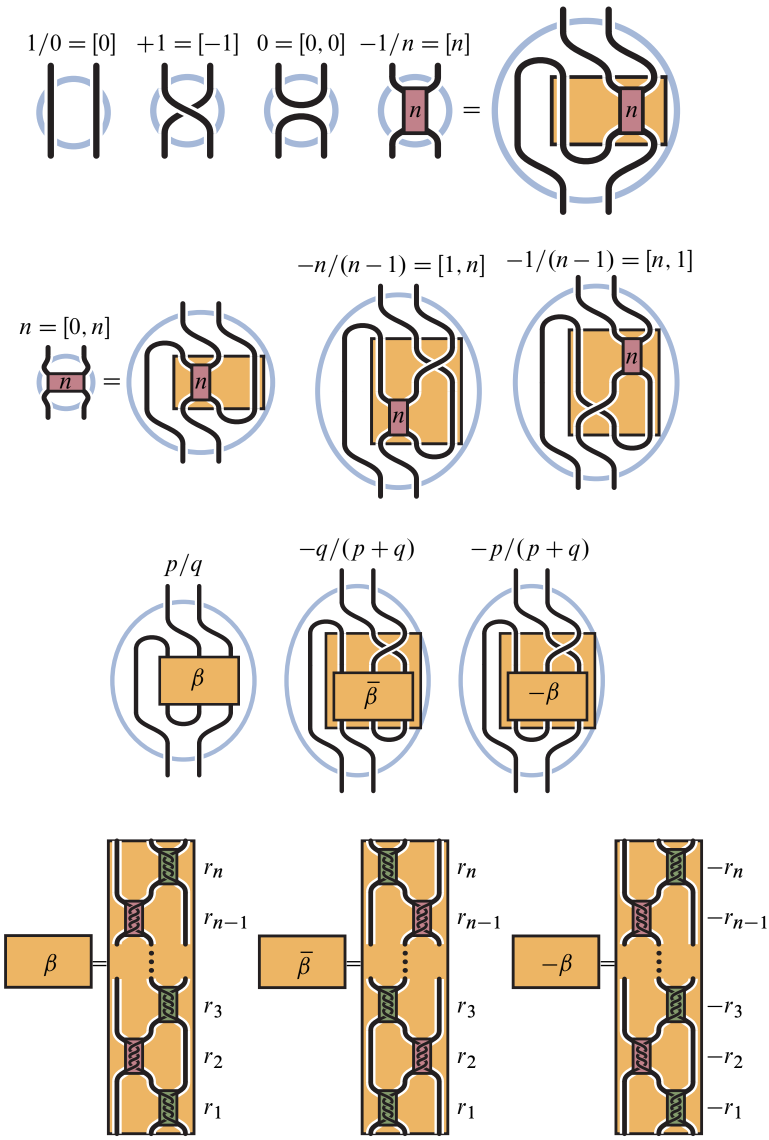


Figure 6: The top two rows show the convention for continued fractions and the corresponding rational tangles. (A twist box labeled n means n vertical right-handed twists when $n > 0$ and $|n|$ vertical left-handed twists when $n < 0$.) The bottom two rows indicate the general forms for the related rational tangles of slopes p/q, $-q/(p+q)$, and $-p/(p+q)$, where, for n odd, $p/q = [r_n, \dots, r_2, r_1]$, $-q/(p+q) = [1, r_n, r_{n-1}, \dots, r_2, r_1]$, and $-p/(p+q) = [1-r_n, -r_{n-1}, \dots, -r_2, -r_1]$. As illustrated, when $r_i \leq 0$ for i odd and $r_i \geq 0$ for i even, these tangles are alternating.

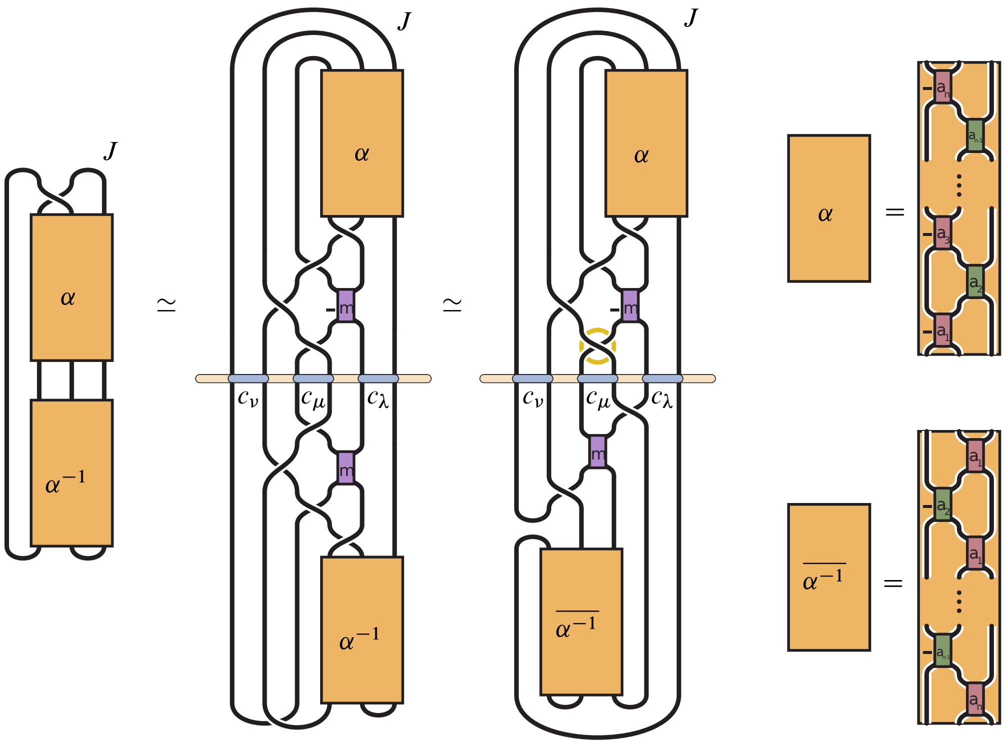


Figure 7: A diagram of the unknot $J = J^{\alpha,m}$ in a 3–bridge position so that it is almost-alternating when $m \geq 0$ and $a_i \geq 0$ for $i = 1, 2, \ldots, n$ is shown in the middle. The dealternating crossing is circled.

Proof Let $K = K^{\alpha,m}(p/q)$ be the framed knot that is the p/q–lashing of $P^{\alpha,m}$ in S^3, and let Z be the result of the framed surgery on K. By Lemma 2.2, Z is also obtained by $(+1, -q/(p+q), -p/(p+q))$–surgery on the link $C_\nu \cup C_\mu \cup C_\lambda = \mathcal{C} = \partial P^{\alpha,m}$ with respect to the framing by $P^{\alpha,m}$. By construction, the link $\mathcal{C}$ arises as the double branched cover of the arcs c and the framing of $\mathcal{C}$ by $P^{\alpha,m}$ is the lift of the framing of c by the bridge sphere. Hence, by the Montesinos trick, $(+1, -q/(p+q), -p/(p+q))$–surgery on $\mathcal{C}$ is the double branched cover of $J^* = J_c^{\alpha,m}(+1, -q/(p+q), -p/(p+q))$, the result of the $(+1, -q/(p+q), -p/(p+q))$–rational tangle replacement on the arcs $c = c^{\alpha,m}$ on the unknot $J = J^{\alpha,m}$. The result J^* of this rational tangle replacement is shown in Figure 8 and thus its double branched cover is the manifold Z. With the specified constraints on p/q, α, and m, this resulting link J^* has a connected alternating diagram, and therefore the link is nonsplit [29]. Thus Z is an L–space [35]. Since K has a nontrivial surgery to the L–space Z, it is an L–space knot by definition. □

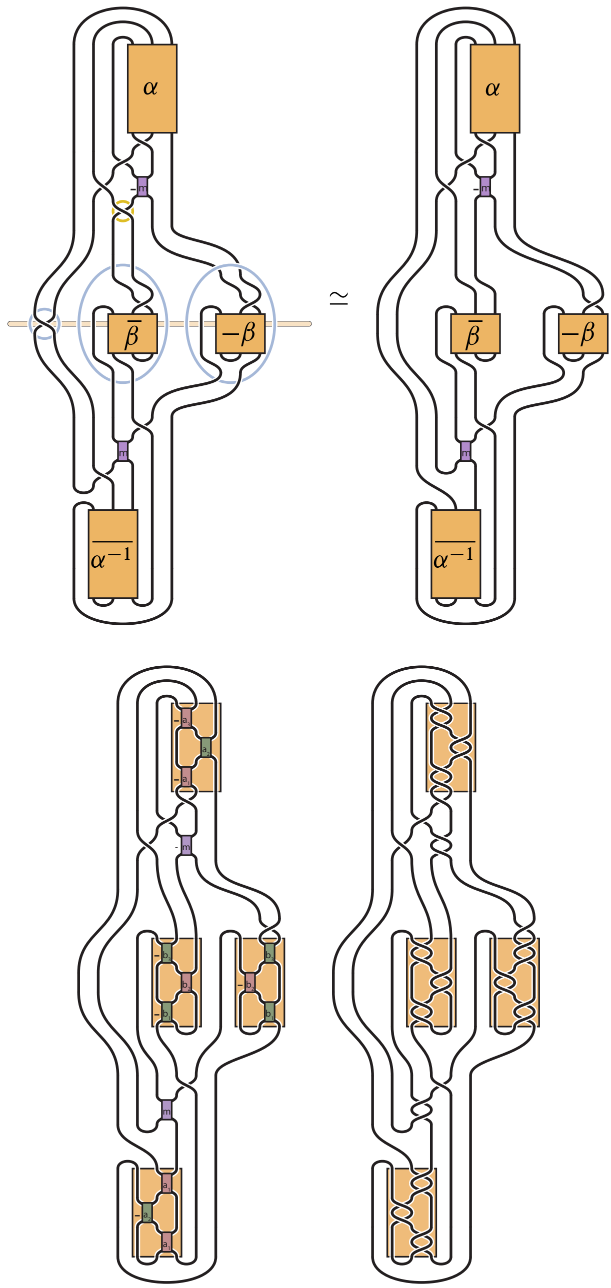


Figure 8: The rational tangle replacements on the link $J^{\alpha,m}$ are made along c to produce the link $J_c^{\alpha,m}(+1,-q/(p+q),-p/(p+q))$ and then the diagram is simplified to be alternating. Examples with α and β of length 3 and further with $m=a_1=a_2=a_3=b_1=b_2=b_3=2$ are shown.

4 A construction of asymmetric hyperbolic manifolds via lashings

4.1 Exteriors of lashings and basic twist families

Consider a lashing K of a pair of pants P embedded in a 3–manifold Y. Let H_P be the genus 2 handlebody $P \times I$ and set $M = Y \setminus H_P$ to be the exterior of this handlebody. Let K' be a properly embedded, essential arc in T that is disjoint from the curve K; see Figure 9, top left. Using the homeomorphism $H_P \cong T \times [-1,+1]$ with $K \subset T \times \{0\}$ (see Figure 3), the disk $D_K = K' \times [-1,1]$ is a meridional disk of H_P that is disjoint from K; see Figure 9, top right. Furthermore, cutting H_P along D_K yields a solid torus neighborhood of K; see Figure 9, bottom. A knot in a handlebody is a *core curve* if and only if there is a complete set of meridional disks of the handlebody such that all but one is disjoint from the knot and cutting the handlebody along these disjoint disks yields a solid torus neighborhood of the knot. The above shows that K is

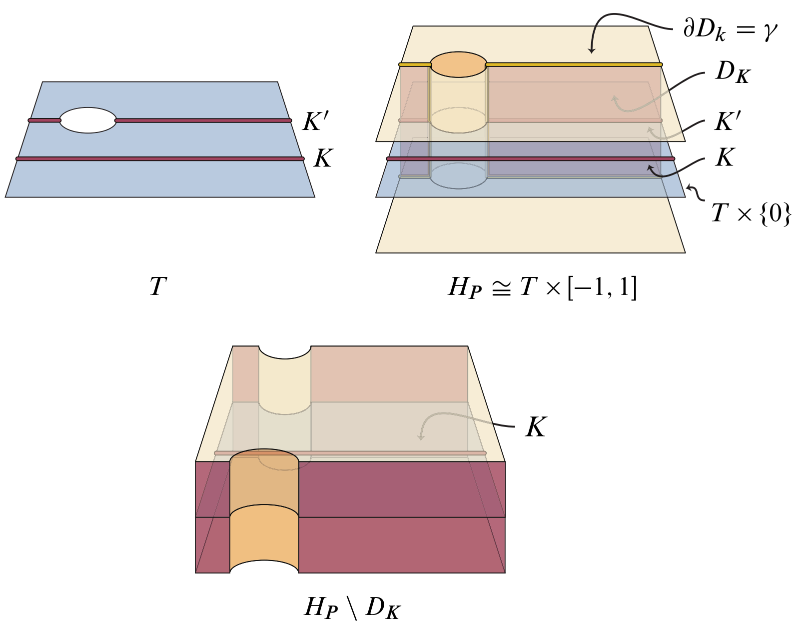


Figure 9: Up to homeomorphism, an essential curve K in T and its associated arc K' (top left); opposite sides of the figure are identified. In $H_P \cong T \times [-1,1]$, the arc K' defines a disk D_K with boundary γ (top right). Chopping H_P along the disk D_K produces a solid torus in which K is isotopic to the core (bottom); left and right sides of the figure are identified.

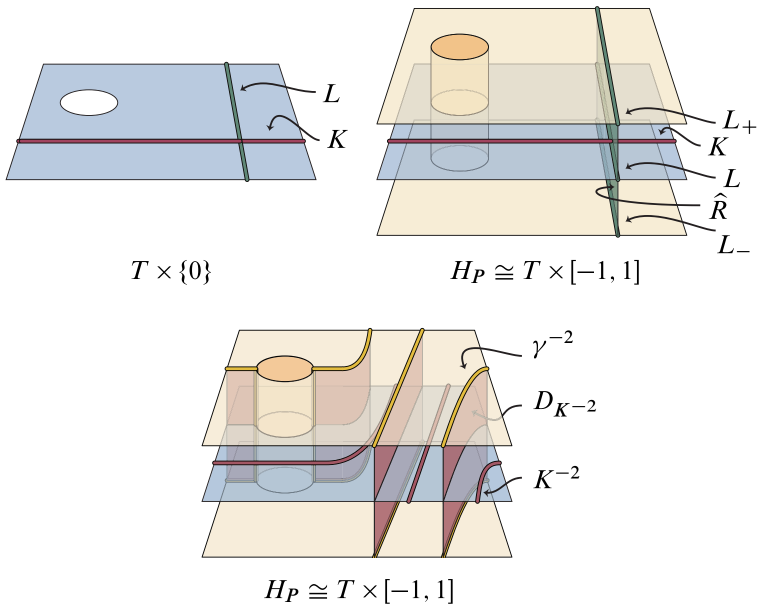


Figure 10: Up to homeomorphism, the curves K and L appear in T as shown (top left); opposite sides of the figure are identified. In $H_P \cong T \times [-1,1]$, the curve L defines an annulus $\widehat{R}$ with boundary $L_+ \cup L_-$ (top right). We twist K and D_K twice to the left along L and $\widehat{R}$ to produce the curve K^{-2} and the disk $D_{K^{-2}}$ (bottom).

a core curve of H_P. Let $\gamma = \partial D_K$. Then the exterior of K, which is $X_K = Y \setminus \mathcal{N}(K)$, may be written as $M[\gamma]$, the result of attaching a 2–handle to M along γ. That is, $X_K = M[\gamma]$.

Let T be the oriented once-punctured torus in H_P that contains the lashings of P. Let L be another lashing of P so that K and L transversally intersect once as simple closed curves in T; see Figure 10, top left. Let L_+ and L_- be push-offs of L to the positive and negative side of T so that they cobound an annulus $\widehat{R}$ in H_P which intersects T in the curve L and which K intersects once; see Figure 10, top right.

By construction, the link $\mathcal{L} = L_+ \cup L_- \cup K$ is contained in H_P, and H_P may be identified with the closure of $\mathcal{N}(K \cup \widehat{R})$. Observe that the link's exterior $X_{\mathcal{L}} = Y \setminus \mathcal{N}(\mathcal{L})$ contains the pair of pants $R = \widehat{R} \cap X_{\mathcal{L}}$ and we may further identify $X_{\mathcal{L}} \setminus \mathcal{N}(R)$ as the manifold M. This gives a decomposition of ∂M as the union of two pairs of pants R' and R'' that are push-offs to each side of R in $X_{\mathcal{L}}$ that are joined by the three annuli of $\mathcal{A}_{\mathcal{L}} = \partial X_{\mathcal{L}} \setminus \mathcal{N}(R)$.

Note that the core curves of the annuli of $\mathcal{A}_{\mathcal{L}}$ are each nonseparating in ∂M and they are collectively isotopic in $X_{\mathcal{L}}$ to $R \cap \partial X_{\mathcal{L}}$, a longitude of each L_+ and L_- and a meridian of K. In particular, the core curves of $\mathcal{A}_{\mathcal{L}}$ are the curves $L \times \{1\}$, $L \times \{-1\}$, and ∂D_L. Here $D_L = L' \times [-1, 1]$, where L' is a properly embedded, essential arc in T that is disjoint from the curve L. Because L intersects K once, L' may be taken to intersect K once too; hence ∂D_L is isotopic in $X_{\mathcal{L}}$ to a meridian of K.

Definition 4.1 (basic twist family) For each integer n, the knot $K^n = \phi^n_L(K)$ resulting from n Dehn twists of K along L in T is also a lashing of P. We say a collection of lashings such as $\{K^n\}$ is a *basic twist family* of lashings of P; see [2, Definition 2.6]. See Figure 10, top right and bottom for an example with $n = -2$. The lashing K^n may be obtained as the image of K upon $(-1/n, 1/n)$–Dehn surgery on the link $L_+ \cup L_-$ framed with respect to $\widehat{R}$, and hence the exterior of K^n, $X^n = Y \setminus \mathcal{N}(K^n)$, may be obtained as $X_{\mathcal{L}}(-1/n, 1/n, *)$. Here the single $*$ indicates that no filling is being done on the boundary component corresponding to K. We let L^n_+ and L^n_- be the core curves of the two fillings in X^n. Since the link $L_+ \cup L_-$ is isotopic into $\partial M = \partial H_P$ with the framing given by $\widehat{R}$, the exterior X^n may be regarded as $M[\gamma^n]$ (that is, $X^n \cong M[\gamma^n]$) where $\gamma^n = \partial D_{K^n}$ is the result of Dehn twisting γ along L_+ and L_- in ∂M. See Figure 10, bottom for an example with $n = -2$.

Remark 4.2 When $\partial M = \partial H_P$ is viewed from outside H_P, $\gamma^n = \phi^{-n}_{L_-}(\phi^n_{L_+}(\gamma))$; when ∂M is viewed from outside M, $\gamma^n = \phi^n_{L_-}(\phi^{-n}_{L_+}(\gamma))$.

Lemma 4.3 *For suitably large n, γ^n minimally intersects each component of ∂P a distinct nonzero number of times. Furthermore, γ^n intersects some component of ∂P an odd number of times.*

Proof We use the following criterion to determine minimal intersection number: Let α and β be two nontrivial simple closed curves which are not isotopic in a surface F. Then α and β realize the minimal intersection number under isotopy if and only if there is no disk $D \subset F$ such that $D \cap (\alpha \cup \beta) = a \cup b$, where a is a subarc of α, b is a subarc of β, and $a \cap b = \partial a = \partial b$ (ie D is a bigon guiding an isotopy of α that reduces its intersection with β).

We denote the minimal intersection number between α and β in F by $\Delta_F(\alpha, \beta)$. Note that the above criterion implies that if α and β intersect coherently (ie upon orienting, all intersections are the same sign), then α and β intersect minimally.

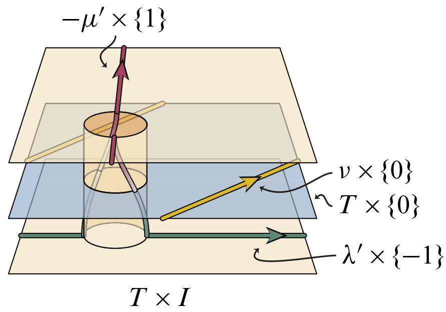


Figure 11: The oriented curve C_ν is a union of the arcs $-\mu' \times \{1\}$ and $\lambda' \times \{-1\}$ and two arcs in $\partial T \times I$; it is isotopic to $\nu = \nu \times \{0\}$ (left). The projection of the curve C_ν to T is isotopic to the curve $\nu = K(\frac{-1}{1})$ and disjoint from the arc ν' (right).

Recall that $H_P = P \times [-1,1] = T \times [-1,1]$.

Any pair of an essential simple closed curve τ and an essential properly embedded arc α' in T may be isotoped to intersect coherently, so that $|\tau \cap \alpha'| = \Delta_T(\tau, \alpha)$ where α is the essential simple closed curve in T that is disjoint from α'. Then for the product disk $D_\alpha = \alpha' \times [-1,1]$, we have $\Delta_{\partial H_P}(\tau \times \{1\}, \partial D_\alpha) = \Delta_{\partial H_P}(\tau \times \{-1\}, \partial D_\alpha) = \Delta_T(\tau, \alpha)$. Recall the notation from Section 2 where lashings are parametrized by a basis of curves μ and λ in T, and ∂P is the triple of curves C_μ, C_λ, and C_ν. Since $C_\mu = \mu \times \{1\}$, $C_\lambda = \lambda \times \{-1\}$ (Figure 3), and γ^n is the boundary of $D_{K^n} = (K^n)' \times [-1,1]$, we therefore have that $\Delta_{\partial H_P}(C_\mu, \gamma^n) = \Delta_T(\mu, K^n)$ and $\Delta_{\partial H_P}(C_\lambda, \gamma^n) = \Delta_T(\lambda, K^n)$.

The curve C_ν, however, is the union of arcs $\mu' \times \{1\}$ and $\lambda' \times \{-1\}$ and a pair of arcs in $(\partial T) \times [-1,1]$ such that C_ν projects to an embedded curve in $T = T \times \{0\}$ that is isotopic to ν as illustrated in Figure 11. Figure 12 indicates minimal intersection presentations between C_ν and ∂D_α using this projection. If $[\alpha] = a_1[\mu] + a_2[\lambda]$ in $H_1(T)$ so that $\alpha = K(a_1/a_2)$, then Figure 12, bottom right corresponds to the minimal presentation when a_1 and a_2 are of the same sign, and Figure 12, bottom left corresponds to when a_1 and a_2 are of opposite sign. Note that ∂D_α contains two copies of α' oriented oppositely. One checks that there are no bigons of intersection between C_ν and ∂D_α, thereby verifying the minimal intersection. In Figure 12, bottom right this follows because all intersections are of the same sign on ∂H_P. In Figure 12, bottom left, one checks directly that the criterion holds. Figure 12, bottom right then shows that $\Delta_{\partial H_P}(C_\nu, \partial D_\alpha) = |a_1| + |a_2|$ when a_1 and a_2 are of the same sign. Figure 12, bottom left shows that $\Delta_{\partial H_P}(C_\nu, \partial D_\alpha) = |a_1| + |a_2| - 2$ when a_1 and a_2 are of opposite sign.

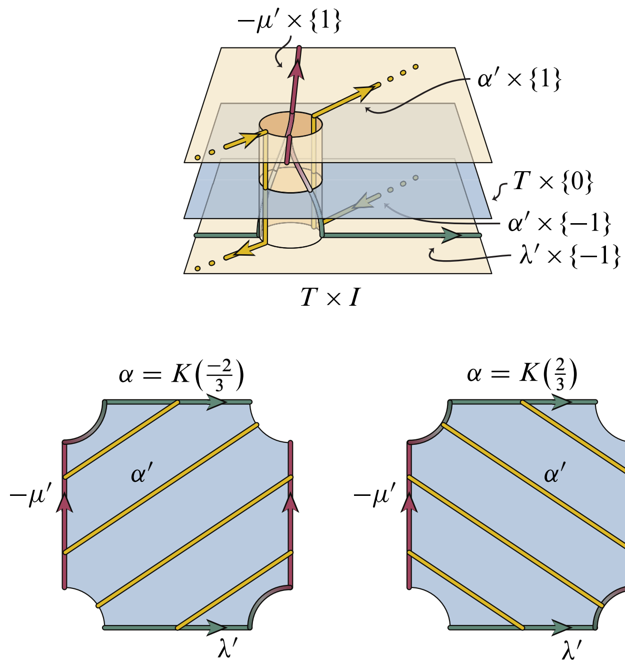


Figure 12: The curve ∂D_α is the union of the two arcs $\alpha' \times \{\pm 1\}$ and two vertical arcs in $\partial T \times I$ since $D_\alpha = \alpha' \times I$; in the cylinder $\partial T \times I$, the curves C_ν and ∂D_α are either disjoint or intersect twice; with α of negative slope as shown, they are disjoint (top). The projection of C_ν and ∂D_α to T where $\alpha = K(a_1/a_2)$ is illustrated for negative slopes with $a_1/a_2 = -\frac{2}{3}$ (bottom left) and for positive slopes with $a_1/a_2 = \frac{2}{3}$ (bottom right).

As curves in T, we may choose orientations on K and L so that $L \cdot K = +1$ with $[K] = p[\mu] + q[\lambda]$ and $[L] = r[\mu] + s[\lambda]$ where $rq - ps = 1$. Then $[K^n] = [K] + n[L] = (p + rn)[\mu] + (q + sn)[\lambda]$. Hence

- $\Delta_{\partial H_P}(C_\mu, \gamma^n) = \Delta_T(\mu, K^n) = |q + sn|$,
- $\Delta_{\partial H_P}(C_\lambda, \gamma^n) = \Delta_T(\lambda, K^n) = |p + rn|$,
- $\Delta_{\partial H_P}(C_\nu, \gamma^n) = |p + rn| + |q + sn|$ when $q + sn$ and $p + rn$ are of the same sign, and
- $\Delta_{\partial H_P}(C_\nu, \gamma^n) = |p + rn| + |q + sn| - 2$ when $q + sn$ and $p + rn$ are of opposite sign.

The fact that (p, q) and (r, s) represent different slopes of T guarantees that $|p+rn| \neq |q + sn|$ and $|p + rn| + |q + sn| - 2 > \max(|p + rn|, |q + sn|)$ for all but finitely many n, thereby verifying that γ^n intersects each component of ∂P a different number of times. Finally, since $[K^n] = (p+rn)[\mu] + (q+sn)[\lambda]$ is the homology class of a simple closed curve in T, the integers $|p+rn|$ and $|q+sn|$ must be relatively prime, and hence one must be odd. □

Lemma 4.4 *Unless L is isotopic to μ or λ in T, each component of ∂P is intersected by some core curve of the annuli $\mathcal{A}_{\mathcal{L}}$.*

Proof Recall that the core curves of $\mathcal{A}_{\mathcal{L}}$ are the curves $L \times \{1\}$, $L \times \{-1\}$, and ∂D_L while ∂P is the triple of curves C_ν, C_μ and C_λ which are isotopic in ∂H_P to a union of the arcs $-\mu' \times \{1\}$ and $\lambda' \times \{-1\}$ and two arcs in $\partial T \times I$ (as shown in Figure 11), $\mu \times \{1\}$, and $\lambda \times \{+1\}$ respectively. Continuing as in the proof of Lemma 4.3, if $[L] = r[\mu] + s[\lambda]$ in T, then we may calculate

- $\Delta_{\partial H_P}(C_\mu, L \times \{1\}) = \Delta_{\partial H_P}(\mu \times \{1\}, L \times \{1\}) = \Delta_T(\mu, L) = |s|$,
- $\Delta_{\partial H_P}(C_\lambda, L \times \{-1\}) = \Delta_{\partial H_P}(\lambda \times \{-1\}, L \times \{-1\}) = \Delta_T(\lambda, L) = |r|$,
- $\Delta_{\partial H_P}(C_\nu, L \times \{1\}) = \Delta_T(\mu', L) = \Delta_T(\mu, L) = |s|$, and
- $\Delta_{\partial H_P}(C_\nu, L \times \{-1\}) = \Delta_T(\lambda', L) = \Delta_T(\lambda, L) = |r|$.

Hence we have the desired conclusion as long as both r and s are nonzero, ie as long as L is not μ or λ. □

4.2 Hyperbolicity of $\mathcal{L}$ and uniqueness of R, given the simplicity of M

For this section, we continue to assume that we have a basic twist family $\{K^n\}$ corresponding to a link $\mathcal{L}$ in H_P.

Lemma 4.5 *If F' is a ∂–parallel disk or annulus in M intersecting each component of $\mathcal{A}_{\mathcal{L}}$ in a collection of spanning arcs, then each collection has an even number of arcs.*

Proof Regarding each component of $\mathcal{A}_{\mathcal{L}}$ as represented by a core curve, we may then regard $\partial F'$ as transverse to $\mathcal{A}_{\mathcal{L}}$. Since F' is ∂–parallel, $\partial F'$ bounds a surface in ∂M. Elementary mod 2 intersection theory then gives the result. □

Definition 4.6 (simple 3–manifold) A *simple* 3–manifold is irreducible, ∂–irreducible, atoroidal, and acylindrical. That is, a manifold is simple if and only if every 2–sphere bounds a ball, every properly embedded disk is boundary parallel, and every properly embedded, incompressible annulus or torus is boundary parallel.

A link in a 3–manifold is hyperbolic if its complement admits a complete hyperbolic metric of finite volume (and this complement is said to be hyperbolic).

Proposition 4.7 *If M is simple, then $\mathcal{L}$ is hyperbolic.*

Proof By geometrization for Haken manifolds [42], to show $\mathcal{L}$ is hyperbolic it is sufficient to show that if F is a properly embedded sphere, disk, annulus, or torus in $X_{\mathcal{L}}$ then F is not essential — ie F is a sphere that bounds a ball or F is a disk, annulus, or torus which is either compressible or boundary parallel.

So assume F is an essential properly embedded sphere, disk, annulus, or torus in $X_{\mathcal{L}}$, isotoped to intersect R minimally. If $F \cap R = \varnothing$, then $(F, \partial F) \subset (M, \mathcal{A}_{\mathcal{L}})$. Since M is simple (by hypothesis), F must then be ∂–parallel in M and hence must be a disk or annulus. However, since any disk or annulus in ∂M bounded by curves in $\mathcal{A}_{\mathcal{L}}$ must itself be contained in $\mathcal{A}_{\mathcal{L}}$, it would follow that F is ∂–parallel in $X_{\mathcal{L}}$, a contradiction. Therefore $F \cap R \neq \varnothing$. Since F is essential, any simple closed curve of $F \cap R$ must be ∂–parallel in R and any arc of $F \cap R$ must either separate two components of ∂R or connect two components of ∂R.

If $F \cap R$ contains simple closed curves that bound disks in F, then an innermost such curve bounds a disk in F with interior disjoint from R. Since this curve must be ∂–parallel in R and the boundary components of R are all essential curves in ∂M, the manifold M is boundary reducible. But then M is not simple, a contradiction.

If $F \cap R$ contains arcs that are ∂–parallel in F, then an outermost such arc bounds a disk in F with interior disjoint from R. Thus this disk is properly embedded in M and its boundary is a curve in ∂M that intersects the three annuli $\mathcal{A}_{\mathcal{L}}$ of $\partial X_{\mathcal{L}} \setminus R$ in a single spanning arc. Since the disk must be ∂–parallel in M because M is simple, this contradicts Lemma 4.5.

The previous two paragraphs show that F cannot be a sphere or a disk. Hence F is either an annulus or a torus.

If F is an annulus, then $F \cap R$ is either a collection of spanning arcs in F or a collection of essential simple closed curves. In the former case, each component of $F \setminus R$ is a properly embedded disk in M that crosses $\mathcal{A}_{\mathcal{L}}$ twice. Yet this implies that the arcs of $F \cap R$ must be ∂–parallel in R, a contradiction. In the latter case, an outermost component in F cuts off a subannulus F' in F with interior disjoint from R and a subannulus R' of R. Since together the annulus $F' \cup R'$ may be nudged off R to be a properly embedded annulus in M, by the simplicity of M the annulus $F' \cup R'$ must be ∂–parallel. This parallelism then guides an isotopy of F through R' that reduces $|F \cap R|$, contradicting the assumed minimality.

If F is a torus, then the components of $F \setminus R$ are all annuli. Since these annuli are all properly embedded in M and M is simple, they all must be ∂–parallel. Hence their boundary components are all isotopic in R to the same component of ∂R. By the minimality of $|F \cap R|$ they cannot be ∂–parallel into one side of R but rather must be ∂–parallel across a component of $\mathcal{A}_{\mathcal{L}}$. Thus any component $F \setminus R$ is an annulus that runs between opposite sides of R and may be joined together by a subannulus of R (or just a single curve of $F \cap R$) to form a torus that is ∂–parallel in $X_{\mathcal{L}}$. Since F is an embedded closed compact surface, $F \cap R$ must in fact be a single curve and hence F itself is a ∂–parallel torus. Yet this means F is not essential. □

Proposition 4.8 *Assume M is simple. If R' is a properly embedded pair of pants in $X_{\mathcal{L}}$ with a component of $\partial R'$ in each component of $\partial X_{\mathcal{L}}$, then either*

(1) *there is a properly embedded pair of pants in M that is incompressible and not ∂–parallel whose boundary is the set of core curves of $\mathcal{A}_{\mathcal{L}}$, or*

(2) *R' is isotopic to R.*

Proof The pair of pants R' may be isotoped to intersect R transversally and minimally. Then $R' \setminus R$ is a collection of properly embedded surfaces in $M = X_{\mathcal{L}} \setminus \mathcal{N}(R)$.

Assume $R \cap R' = \varnothing$. Thus R' is a properly embedded pair of pants in M whose boundary is the core curves of $\mathcal{A}_{\mathcal{L}}$. If R' is neither compressible nor ∂–parallel, then we have our first conclusion. If R' is compressible in M, then by the simplicity of M some component of $\partial R'$ bounds a disk in ∂M; but this is contrary to the cores of $\mathcal{A}_{\mathcal{L}}$ being nonseparating curves in ∂M. If R' is ∂–parallel, then R' is isotopic to R, giving our second conclusion.

So assume $R \cap R' \neq \varnothing$. Following the arguments of Proposition 4.7 with R' in the stead of F, the intersection $R \cap R'$ is a nonempty set of arcs, essential in each of R and R' and with no two parallel. (Use the argument for when F is an annulus to show $R \cap R'$ contains no simple closed curves that are essential in both R and R'.) Consequently, $R \cap R'$ is one, two, or three essential arcs in R' where at most one is separating. The possible configurations are illustrated in Figure 13. In order to not violate Lemma 4.5 each component of $R' \setminus R$ must be incident to each component

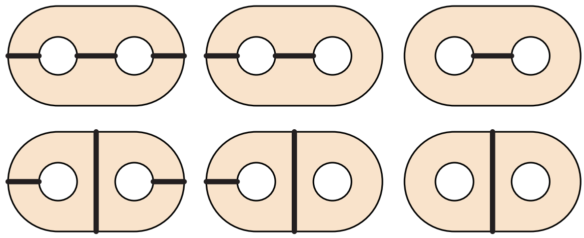


Figure 13: Enumerated are the six possible configurations, up to homeomorphism, of the nonempty set of arcs $R \cap R'$ in the pair of pants R' such that the arcs are essential in R' and no two are parallel.

of $\partial R'$ an even number of times. (This is because each component of $\partial R'$ is in its own component of $\partial X_{\mathcal{L}}$ and each component of $(\partial R') \setminus R$ is a spanning arc of $\mathcal{A}_{\mathcal{L}}$.) An examination of Figure 13 shows this does not occur. □

4.3 Twisting K, hyperbolicity and short geodesics

Let $\{K^n\}$ be a *basic twist family* of lashings of P, where K^n is the n–fold Dehn twist in T of K along L; see Definition 4.1. Recall that $\mathcal{L}$ is the link $L_+ \cup L_- \cup K$, and let $\mathring{X}_K$ and $\mathring{X}_{\mathcal{L}}$ denote the complements of K and $\mathcal{L}$ in Y. The complement, $\mathring{X}^n$, of the lashing K^n in Y may then be obtained as the $(-1/n, 1/n)$–Dehn surgery on the link (L_+, L_-) in $\mathring{X}_K$. Let L_+^n and L_-^n be the closed curves in $\mathring{X}^n$ gotten by taking the cores of the attached solid tori in that Dehn surgery.

Lemma 4.9 *Assume $\mathring{X}_{\mathcal{L}}$ is hyperbolic. For large n, $\mathring{X}^n$ is hyperbolic and L_+^n and L_-^n are the two shortest geodesics in $\mathring{X}^n$.*

Proof This is a well-known strong version of Thurston's hyperbolic Dehn surgery theorem and follows from the proof thereof by Benedetti and Petronio; see [6, Theorem E.5.1]. We apply this to our sequence of manifolds $\{\mathring{X}^n\}$. Their proof shows that for large n, there is a sequence of structures $\{z_n\}$ in $\mathrm{Def}(\mathring{X}_{\mathcal{L}})$ (the incomplete hyperbolic structures on $\mathring{X}_{\mathcal{L}}$ in a neighborhood of the complete structure, z_0) that complete to $\mathring{X}^n$, and $z_n \to z_0$ in $\mathrm{Def}(\mathring{X}_{\mathcal{L}})$. By [6, Proposition E.6.29], the corresponding complete hyperbolic structures $\mathring{X}^n$ converge to the complete structure on $\mathring{X}_{\mathcal{L}}$ in the geometric topology. Furthermore, by Neumann and Zagier [31], the cores of the attached solid tori, L_+^n and L_-^n, are geodesics whose lengths go to 0 in $\mathring{X}^n$ as $n \to \infty$.

Because $\{\mathring{X}^n\}$ converges in the geometric topology to the complete structure on $\mathring{X}_{\mathcal{L}}$, [6, Theorem E.2.4] says that, for small enough $\epsilon > 0$ and large enough n, the ϵ–thin part of $\mathring{X}^n$ must be the tubular neighborhoods of two simple geodesics along with a cusp neighborhood. By [6, Proposition D.3.11], the core curves are the unique geodesics in these tubular neighborhoods. For large enough n, the geodesics L_+^n and L_-^n have lengths less than ϵ and consequently lie in the ϵ–thin part of X^n. Thus L_+^n and L_-^n must be the core curves of these tubes and the shortest geodesics in $\mathring{X}^n$. □

4.4 Asymmetric hyperbolic lashings

Theorem 4.10 *Consider a connected, closed, compact, oriented 3–manifold Y that contains an embedded genus 2 Heegaard surface decomposed into two pairs of pants P and Q such that $P \cap Q = \partial P = \partial Q$. Assume the following:*

(1) *The manifold $M = Y \setminus H_P$ is simple.*

(2) *The pair of pants Q is a properly embedded, incompressible, boundary incompressible, separating surface in M, dividing M into handlebodies H_+ and H_-.*

(3) *The pairs (H_+, Q) and (H_-, Q) are not homeomorphic.*

(4) *Any properly embedded pair of pants in M is either compressible, ∂–parallel, isotopic to Q, or nonseparating and can be properly isotoped in M to be disjoint from some component of ∂Q.*

Let $\{K^n\}$ be a basic twist family of lashings of P in which the twisting curve L is neither μ nor λ. Then K^n, for sufficiently large n, is a hyperbolic knot with asymmetric complement.

Observe that $\mathcal{C} = \partial Q = \partial P$ also decomposes ∂M into two pairs of pants $P_+ \subset \partial H_+$ and $P_- \subset \partial H_-$ that are both isotopic to P in H_P. Schematically, the three pairs of pants P_+, P_-, and Q decompose Y as in Figure 14.

Remark 4.11 Our proof of Theorem 4.10 uses the work of Oertel on homeomorphisms of handlebodies [33]. Nonetheless, we expect the theorem to continue to hold when the submanifolds H_+ and H_- are not necessarily handlebodies and thus the genus 2 surface $P \cup Q$ is not necessarily a Heegaard surface.

Proof of Theorem 4.10 Let $K = K^0$ and L be the pair of lashings such that the twist family $\{K^n\}$ is obtained by twisting K along L, and use the notation of Section 4.1. In particular, K with two push-offs of L forms the link $\mathcal{L} = K \cup L_+ \cup L_-$ in Y which

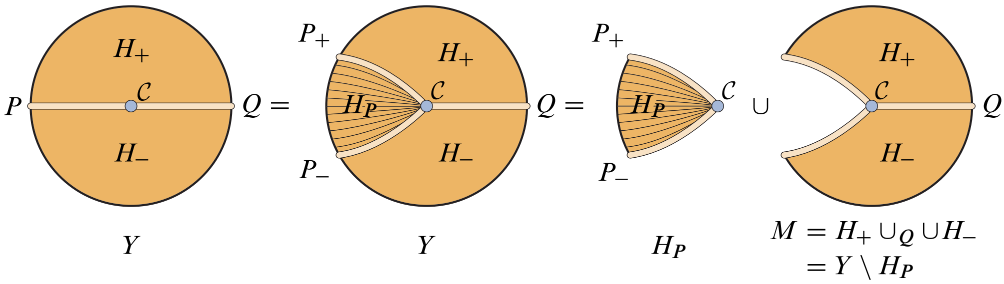


Figure 14: The schematic decomposition of Y along the three pairs of pants P_+, P_-, and Q.

has exterior $X_{\mathcal{L}} = Y \setminus \mathcal{N}(\mathcal{L})$ and complement $\mathring{X}_{\mathcal{L}} = Y - \mathcal{L}$. We denote the exterior of K^n in Y as $X^n = Y \setminus \mathcal{N}(K^n)$, and the complement of K^n in Y as $\mathring{X}^n = Y - K^n$.

Because M is simple by (1), Proposition 4.7 implies $\mathring{X}_{\mathcal{L}}$ is hyperbolic. Then for $n \gg 0$, $\mathring{X}^n$, the interior of $X^n = M[\gamma^n]$, is hyperbolic and the core curves of the fillings L^n_+ and L^n_- are the two shortest simple geodesics in $\mathring{X}^n$ by Lemma 4.9. Also, γ^n, for suitably large n, minimally intersects each component of $\partial Q = \partial P$ a distinct nonzero number of times and at least one component an odd number of times by Lemma 4.3.

For economy of notation, set $K = K^n$ where we choose n large enough that

(a) $\mathring{X}_K = \mathring{X}^n$ is hyperbolic,

(b) the curves $L_+ = L^n_+$ and $L_- = L^n_-$ are the two shortest geodesics in $\mathring{X}_K$, and

(c) $\gamma = \gamma^n$ minimally intersects each component of ∂Q a distinct nonzero number of times, and it intersects at least one component of ∂Q an odd number of times.

We show that K is asymmetric, verifying the theorem.

Let h be a diffeomorphism of $\mathring{X}_K$. Since $\mathring{X}_K$ is a (complete, finite volume) hyperbolic manifold by (a), h is isotopic to an isometry (see the discussion preceding Theorem 6.2 of [8]). So we may begin by taking h to be an isometry of $\mathring{X}_K$. Our goal is then to show that h is isotopic to the identity diffeomorphism, implying that the isometry group of $\mathring{X}_K$ must be trivial and the hyperbolic knot K is asymmetric.

Given that h is an isometry, since L_+ and L_- are the two shortest geodesics in $\mathring{X}_K$ by (c), $h(L_+ \cup L_-) = L_+ \cup L_-$. Therefore h restricts to a diffeomorphism $h_{\mathcal{L}}$ on the link exterior $X_{\mathcal{L}}$ that preserves the original boundary torus T_K and the pair of new boundary tori $T_+ \cup T_-$ obtained by drilling the geodesics $L_+ \cup L_-$ (for example, the proof of Lemma 4.9 shows that the thick part of the complement, which is preserved by an isometry, is the exterior $X_{\mathcal{L}}$). We will now think of h as a diffeomorphism on X_K and show that it is isotopic to the identity. This implies that the original diffeomorphism on $\mathring{X}_K$ is isotopic to the identity.

By Lemma 4.4, each component of $\partial Q = \partial P$ is intersected by some core curve of $\mathcal{A}_{\mathcal{L}}$. Hence, with hypothesis (4), Proposition 4.8 then implies that $h_{\mathcal{L}}(R)$ is isotopic to R, keeping $\partial h_{\mathcal{L}}(R)$ in $\partial X_{\mathcal{L}}$. Hence the diffeomorphism $h_{\mathcal{L}}$ may be isotoped so that R is invariant.

Since $\widehat{R} \cap \mathcal{N}(L_+ \cup L_-)$ is just a pair of annuli, one each in the solid tori $\mathcal{N}(L_+)$ and $\mathcal{N}(L_-)$ from the boundary to the core, the isotopy of $h_{\mathcal{L}}(R)$ to R extends across $\mathcal{N}(L_+ \cup L_-)$ to give an isotopy of $h_{\mathcal{L}}(\widehat{R} \cap X_K)$ to $\widehat{R} \cap X_K$. Hence the diffeomorphism h may be isotoped so that this punctured annulus $\widehat{R} \cap X_K$ is invariant. A further isotopy of h ensures that a regular neighborhood $\mathcal{N}(\widehat{R} \cap X_K)$ is invariant under h. Since the closure of $\mathcal{N}(\widehat{R} \cup K)$ is the handlebody H_P, h now restricts to a diffeomorphism $h|_M$ of the submanifold $M = Y \setminus \mathcal{N}(\widehat{R} \cup K) = Y \setminus H_P$.

Because Q is an incompressible, boundary incompressible pair of pants in M by hypothesis (2), its image $h|_M(Q)$ must be as well. Similarly, since Q is separating in M, so too must be $h|_M(Q)$. So by hypothesis (4), $h|_M(Q)$ must be isotopic to Q in M. Therefore h may be isotoped to preserve the pair (M, Q). Then, by hypothesis (3), $h|_M$ must preserve the sides of Q. That is, h may be isotoped so that there is an h–invariant product neighborhood $Q \times I$ of Q in M (with $\partial Q \times I \subset \partial M$) so that h restricted to $Q \times I$ acts as identity on the I factor.

We may view $X_K = M[\gamma]$ as M in union with a compression body

$$W = (\partial M \times I) \cup (2\text{–handle}),$$

where the 2–handle is attached along the nonseparating curve γ (since W is the exterior of a core curve of the genus 2 handlebody H_P). Thus γ is the unique isotopy class in ∂W of the boundary of a nonseparating, boundary-reducing disk of W (see [2, Lemma 2.8]). Thus $h(\gamma)$ is isotopic to γ in ∂W and hence in the component ∂M of ∂W.

Place a hyperbolic structure on ∂M. Isotope ∂Q and γ to geodesics. By Lemma 2.6 of [13] (applied to $C_1 = h(\partial Q)$ and $C_2 = h(\gamma)$), and the uniqueness of geodesic representatives, h can be isotoped so that $h(\partial Q) = \partial Q$ and $h(\gamma) = \gamma$. Thus $h|_{\partial M}$ is a graph homeomorphism of the graph $\partial Q \cup \gamma$ to itself.

Claim 4.12 *$h|_{\partial M}$ fixes the vertices of the graph $\partial Q \cup \gamma$.*

Proof Since geodesic representatives intersect minimally in their isotopy classes, γ intersects each component of ∂Q a distinct nonzero number of times by (c). Thus h takes each component of ∂Q to itself. As h restricted to $Q \times I$ is the identity on the I–factor, $h|_{\partial M}$ preserves the sides of each component of ∂Q.

First assume that $h|_{\partial M}$ reverses orientation. Then as $h|_{\partial M}$ preserves sides of ∂Q, it must take each component of ∂Q to itself reversing orientation. Furthermore, as γ is transverse to each component of ∂Q and is invariant under $h|_{\partial M}$, $h|_{\partial M}$ takes γ to itself preserving orientation. Thus $h|_{\partial M}$ acts as a rotation along γ on the vertices $\gamma \cup \partial Q$. In particular, if $h|_{\partial M}$ fixes one vertex, it fixes all. But by (c) above, γ intersects some component of ∂Q an odd number of times. Since $h|_{\partial M}$ takes this component to itself reversing orientation, it must fix some point of $\gamma \cap \partial Q$. This proves the claim in this case.

So assume $h|_{\partial M}$ is orientation-preserving. Because it preserves the sides of each component of ∂Q, $h|_{\partial M}$ is orientation-preserving on each component of ∂Q. Thus $h|_{\partial M}$ rotates the vertices of $\gamma \cup \partial Q$ along each component of ∂Q (possibly trivially). Using the facts that $h|_{\partial M}$ takes arcs of $\gamma - \partial Q$ to arcs of $\gamma - \partial Q$ connecting the same components of ∂Q and that γ intersects each component of ∂Q, one can see that h restricted to $\partial Q \cup \gamma$ fixes vertices along each component of ∂Q. □

Since $h|_{\partial M}$ fixes vertices, it can be isotoped to be the identity on the full graph. As the complementary regions of this graph in ∂M are all disks (γ intersects each component of ∂Q), the Alexander isotopy trick then allows us to extend this to an isotopy of all of $h|_{\partial M}$ to the identity.

We next claim that, after a further isotopy of h with support in the interior of M, h is the identity on Q. Recall that the pair of pants Q divides the manifold M into the two genus 2 handlebodies H_+ and H_-. Since $h(Q) = Q$ and h preserves the sides of Q, it restricts to a diffeomorphism of each of these handlebodies. Moreover, since $h|_{\partial M}$ is the identity, $h|_{\partial H_+}$ and $h|_{\partial H_-}$ are each a composition of Dehn twists along a collection of disjoint curves in Q. By [33, Theorem 1.11] and its proof, for each H_+ and H_-, this collection of curves is the boundary of a collection of disjoint meridional disks and incompressible, non-∂–parallel annuli (so that twists along the disks and annuli produce the diffeomorphism of the handlebody). However, since Q is a pair of pants, each of these curves are isotopic to a component of ∂Q, and so these disks and annuli may be isotoped in their respective handlebodies to have boundary in ∂M. Yet unless these collections of disks and annuli are empty, this now contradicts that M is simple. Hence we may further isotope h to also be the identity on Q.

Since h is the identity on $\partial M \cup Q$, it must be isotopic to the identity on each handlebody H_+ and H_-, and hence M.

Finally, since γ bounds a nonseparating disk D in the compression body $W = X_K \setminus M$, we may further isotope h in the interior of W so that D is invariant under h. Because h is the identity on $\gamma = \partial D$, it may be isotoped to be the identity on all of D. Thus, h may be further isotoped in the compression body to be the identity on a collar of $\partial M \cup D$, and thus to be the identity on X_K. Hence the diffeomorphism h is isotopic to the identity. □

5 The pair of pants Q in the submanifold $M = H_+ \cup_Q H_-$

We continue with the notation set in the statement of Theorem 4.10 and proceed to develop conditions that ensure the hypotheses of the theorem are met. In particular, in this section we only need that M is a 3–manifold with genus 2 boundary obtained as the union of two genus 2 handlebodies H_+ and H_- glued together along a pair of pants Q. Then Lemmas 5.2, 5.3, and 5.4 respectively demonstrate that the requirements (2), (1), and (4) on M and Q of Theorem 4.10 are implied by conditions on the disk-busting and annulus-busting nature of ∂Q in the two handlebodies H_+ and H_-. For our application in the special case that M is the exterior of the pair of pants $P = P^{\alpha,m} \subset S^3$ (as set in Section 3), these conditions are then checked in Section 8.

Definition 5.1 (disk-busting and annulus-busting) Let $\mathcal{C}$ be a collection of simple closed curves embedded in the boundary of an orientable 3–manifold H. We say $\mathcal{C}$ is *k–disk-busting* (*in* H) for a positive integer k if any properly embedded disk in H that $\mathcal{C}$ intersects fewer than k times is ∂–parallel in H. When $k = 1$, we simply say $\mathcal{C}$ is *disk-busting*. Similarly, we say $\mathcal{C}$ is *annulus-busting* if any properly embedded annulus in H that is disjoint from $\mathcal{C}$ is either compressible or ∂–parallel in H.

Lemma 5.2 *Assume ∂Q is 3–disk-busting in each of H_+ and H_-. Then Q is incompressible and boundary incompressible in M.*

Proof Since Q separates M into H_+ and H_-, any compressing or ∂–compressing disk for Q would lie in either H_+ or H_-. However, since such disks must be meridional in the handlebodies and must meet ∂Q either zero or two times at most, ∂Q could not be 3–disk-busting. Thus Q is incompressible and ∂–incompressible. □

Lemma 5.3 *If ∂Q is 6–disk-busting and annulus-busting in each of H_+ and H_-, then M is simple.*

Consequently, M is a hyperbolic 3–manifold with geodesic boundary.

Proof Among 2–spheres that do not bound 3–balls and properly embedded incompressible, non-∂–parallel disks, annuli, or tori that are transverse to Q, choose F to be one that intersects Q minimally. Note that Q is incompressible and boundary incompressible by Lemma 5.2. We may assume no simple closed curve component of $F \cap Q$ is trivial in Q, as otherwise surgery of F along the disk bounded by an innermost such curve will produce a new essential surface intersecting Q fewer times. Thus any simple closed curve of $F \cap Q$ must be isotopic in Q to a component of ∂Q. Similarly no arc component of $F \cap Q$ can be boundary parallel in Q, else surgery would find a new essential surface intersecting Q fewer times. No simple closed curve of $F \cap Q$ can bound a disk in F, for an innermost such disk would be a compressing disk for Q. No arc component of $F \cap Q$ can be boundary parallel in F as such would give rise to a boundary compression of Q.

***F* is a sphere** If F is a sphere, then since every simple closed curve in F bounds a disk, F must be disjoint from Q. But then the sphere F is contained in a handlebody. Since handlebodies are irreducible, F must bound a ball, a contradiction.

***F* is a disk** If F is a disk, then since every simple closed curve in F bounds a disk and every arc in F is ∂–parallel, F must be disjoint from Q. Thus ∂F is contained in one of the pairs of pants P_+ or P_- and is therefore isotopic to a component of ∂P_+ or ∂P_-. Since $\partial P_+ = \partial P_- = \partial Q$, it must be that ∂F is isotopic to a component of ∂Q. Hence F would be a compressing disk for Q, a contradiction.

***F* is an annulus** If F is an annulus, then $F \cap Q$ consists of either only spanning arcs of F or only curves isotopic into ∂F. Because Q is annulus-busting, $F \cap Q$ is nonempty.

Because Q is separating, if there is one arc of $F \cap Q$ then there must be another and Q chops F into rectangles contained in either H_+ or H_-. Since these rectangles are disks in H_+ or H_- that cross ∂Q exactly four times, they must be ∂–parallel because Q is 6–disk-busting. Such a boundary parallelism guides an isotopy of F that reduces the number of intersections of F with Q, a contradiction.

If $F \cap Q$ is a collection of simple closed curves, they are all ∂–parallel in each of F and Q. Let γ be one that is outermost in Q, cutting off a subannulus $A \subset Q$

with a component of ∂Q. Let F' and F'' be the two annuli formed by surgering F along A. Let A^* be the "dual" annulus in ∂M so that surgering $F' \cup F''$ along A^* recovers F. Since each F' and F'' intersect Q fewer times than F, they must be boundary parallel. Any compression of F' or F'' would give a compression of F. Let V' and V'' be the two solid tori in X_P through which F' and F'' are ∂–parallel; they are either disjoint or nested. If V' and V'' are disjoint, then A is also disjoint from them and $V' \cup \mathcal{N}(A^*) \cup V''$ is a solid torus giving a ∂–parallelism of F. If, say, V' is contained in V'', then $V'' \setminus (\mathcal{N}(A^*) \cup V')$ is a solid torus giving a ∂–parallelism of F.

***F* is a torus** If F is a torus, then because handlebodies contain no embedded closed incompressible surfaces, $F \cap Q$ is nonempty. Since $Q \cap F$ is a collection of simple closed curves, they must all be essential in F and hence parallel in F. Thus Q chops F into annuli in H_+ and H_- each with boundary disjoint from ∂Q. Since Q is annulus-busting in each of H_+ and H_-, these annuli must be boundary parallel or compressible. These annuli must be incompressible since otherwise they would induce a compression of Q, contrary to Lemma 5.2. Hence they are ∂–parallel. Since no two components of ∂Q are parallel in ∂H_+ or ∂H_-, these annuli must all be parallel into Q. Hence F is isotopic into $\mathcal{N}(Q)$ and is therefore compressible, a contradiction. □

Lemma 5.4 *Assume ∂Q is 6–disk-busting and annulus-busting in each of H_+ and H_- and 8–disk-busting in either H_+ or H_-.*

Then any properly embedded pair of pants in M is either compressible, ∂–parallel, isotopic to Q, or nonseparating and can be isotoped to be disjoint from some component of ∂Q.

Proof Assume there is a properly embedded, incompressible, non-∂–parallel pair of pants in M that is not isotopic to Q. Among such surfaces that are transverse to Q but not isotopic to Q, choose F to be one that intersects Q minimally. As at the beginning of the proof of Lemma 5.3, any simple closed curve of $F \cap Q$ must be isotopic in Q to a component of ∂Q and isotopic in F to a component of ∂F because both Q and F are incompressible. Similarly, no arc of $Q \cap F$ is ∂–parallel in F because Q is ∂–incompressible.

First we show that F cannot be ∂–compressible. Assume F admits a ∂–compression along a disk δ. Compressing F along δ produces one or two annuli, properly embedded in X_P. Let δ^* be the arc in ∂M dual to δ so that surgering these annuli along δ^* recovers F. These annuli must be incompressible since they share a boundary component with the incompressible surface F. Therefore by Lemma 5.3 these annuli must be ∂–parallel. Using the arrangements of the ∂–parallelisms and δ^*, it follows that F must either compress or be ∂–parallel, contrary to assumption. (If F surgers along δ to give one annulus F' with ∂–parallelism through the solid torus V', then either δ^* is inside V' and so F is compressible or δ^* is outside V' and F is ∂–parallel. If F surgers along δ to give two annuli F' and F'' with ∂–parallelism through the solid tori V' and V'' joined by δ^*, then either V' and V'' are disjoint so that F is ∂–parallel or V' and V'' are nested with F' and F'' parallel so that F compresses.)

Thus any component of $F \cap Q$ is either an arc that is essential in both F and Q or a simple closed curve that is ∂–parallel in both F and Q.

$F \cap Q$ is empty If $F \cap Q$ is empty, then F is contained in either H_+ or H_-, say H_+. Then ∂F is contained in P_+. Since F is incompressible, each component of ∂F is isotopic to some component of $\partial P_+ = \partial Q$.

If F is nonseparating, then we have the final conclusion. (Though note that because handlebodies do not contain closed nonseparating surfaces, at least two components of ∂F are isotopic in P_+.)

If F is separating in M, it is separating in H_+. Then ∂F is separating in ∂H_+ and so ∂F is isotopic to ∂Q. If not, then two of the isotopy classes of ∂Q contain an even number of components of ∂F and the remaining contains an odd number of components of ∂F. Since each component of ∂Q is nonseparating in ∂H_+, there is a loop in ∂H_+ intersecting this third component of ∂Q an odd number of times. Therefore this loop must intersect ∂F an odd number of times, a contradiction.

So now isotope F in H_+ so that $\partial F = \partial Q$. Then let D be a meridional disk of H_+ intersecting F minimally. Because F is not ∂–compressible in M, an arc of $F \cap D$ that is outermost in D must cut off a ∂–compressing disk δ for F in H_+ so that $\partial\delta$ is the union of an essential arc in F and an essential arc in Q. The ∂–compression of F along δ produces one or two annuli properly embedded in H_+ whose boundaries cobound annuli in Q. These annuli must be incompressible since F is incompressible in H_+. Then, since ∂Q is annulus-busting in H_+, these annuli must be ∂–parallel. Thus they are ∂–parallel into Q. Such parallelisms taken together with δ give an isotopy of F to Q.

$F \cap Q$ contains a simple closed curve If there is a simple closed curve of $F \cap Q$, let γ be one that is outermost in Q, cutting off a subannulus $A \subset Q$ with a component of ∂Q. Any arc of $F \cap Q$ in A would bound a disk in A giving a ∂–compression of F, so the interior of A is disjoint from F. Let F' and F'' be the annulus and pair of pants respectively formed by surgering F along A. Let A^* be the "dual" annulus in ∂M so that surgering $F' \cup F''$ along A^* recovers F. Both F' and F'' must be incompressible since their boundary components are all isotopic to boundary components of incompressible surfaces. By Lemma 5.3 the annulus F' must then be ∂–parallel. Since F'' intersects Q fewer times than F, by the assumed minimality of $|F \cap Q|$, the pair of pants F'' must be ∂–parallel (if F'' were isotopic to Q then F would be also). Let V' and V'' be the solid torus and genus 2 handlebody of parallelisms for F' and F''; since F' and F'' are disjoint, V' and V'' are either disjoint or nested. If V' and V'' are disjoint, then A is also disjoint from them and $V' \cup \mathcal{N}(A^*) \cup V''$ is a handlebody giving a ∂–parallelism of F. If V' is contained in V'', then $V'' \setminus (\mathcal{N}(A^*) \cup V')$ is a handlebody giving a ∂–parallelism of F. Since F'' is incompressible, V'' cannot be contained in V'. Thus $F \cap Q$ contains no simple closed curves.

$F \cap Q$ contains only arcs Since $F \cap Q$ is nonempty but contains no simple closed curves, $F \cap Q$ may contain only arcs that are essential in each of F and Q. In particular, this means that no disk component of $F \setminus Q$ can be ∂–parallel in either H_+ or H_-. Since Q is separating, either $F \cap Q$ contains a pair of arcs that are parallel in F cutting off a rectangle component of $F \setminus Q$, or $F \cap Q$ is a set of three nonisotopic arcs chopping F into two hexagonal disks, or $F \cap Q$ is a single arc chopping F into two annuli. In the first case, since ∂Q is 6–disk-busting in each of H_+ and H_-, any such rectangle must be ∂–parallel, a contradiction. In the second case, since neither of the two hexagonal disks $F \cap H_+$ and $F \cap H_-$ is ∂–parallel, ∂Q cannot be 8–disk-busting in either H_+ or H_-, a contradiction. Therefore Q cuts F along a single arc into two annuli, one in each of H_+ and H_-.

Consider the annulus $F_+ = F \cap H_+$. One component of ∂F_+ is disjoint from Q and thus isotopic in P_+ to a component of ∂Q. The other component of ∂F_+ intersects Q in an essential arc. Therefore if F_+ were ∂–parallel in H_+, this second component of ∂F_+ would intersect P_+ in a ∂–parallel arc, so that F_+ would be isotopic to a component of $Q \setminus F$ in Q. Yet then F could be isotoped to be disjoint from Q, contrary to assumption. Hence F_+ is not ∂–parallel in H_+.

Assume F_+ is separating in H_+. Because F_+ is a properly embedded, incompressible, separating annulus in a genus 2 handlebody H_+, the two components of ∂F_+ must be isotopic in ∂H_+ (eg see [3, Section 6.3]). Thus, since one component of ∂F_+ is disjoint from Q, the other may be isotoped in ∂H_+ to also be disjoint from Q. Since ∂Q is annulus-busting, F_+ must be ∂–parallel. However we have already concluded that this cannot be the case.

So F_+ must be nonseparating in H_+. Since a component of ∂F_+ is in P_+ and isotopic to a component of ∂Q, F_+ is disjoint from that component of ∂Q. Thus F is nonseparating and disjoint from that component of ∂Q, giving the final conclusion. □

6 Busting disks and annuli in genus 2 handlebodies

6.1 Basics of disk-busting and primitive curves

Lemma 6.1 (eg [2, Lemma 5.3]) *Let $\mathcal{C}$ be a collection of curves in a genus g boundary component of a 3–manifold H. If $\mathcal{C}$ is disk-busting, then either $g = 1$ or $\mathcal{C}$ is 2–disk-busting.*

Proof Assume D is a properly embedded disk in H intersecting $\mathcal{C}$ just once. Then two copies of D can be banded together along the component of $\mathcal{C}$ that intersects D to form a new disk D' that is disjoint from $\mathcal{C}$. If $g \geq 2$, then $\partial D'$ is essential in ∂H and so D' cannot be ∂–parallel. But this contradicts that $\mathcal{C}$ is disk-busting. □

A single simple closed curve C in the boundary of a handlebody H is called *primitive* if there is some meridional disk of H that it intersects only once.

Let $\mathcal{C}$ be a collection of simple closed curves embedded in the boundary of an orientable 3–manifold H. Let $H[\mathcal{C}]$ be the 3–manifold obtained by attaching 2–handles to H along the components of $\mathcal{C}$ and then attaching 3–handles to any sphere components of the resulting boundary.

Lemma 6.2 *Let C be a simple closed curve in the boundary of a handlebody H of genus $g \geq 2$. The curve C is primitive if and only if $H[C]$ is a genus $g-1$ handlebody. The curve C is disk-busting if and only if $H[C]$ is ∂–irreducible.*

Proof If C is primitive in H, then in $H[C]$ the core disk of the attached 2–handle cancels a meridional disk of H intersecting C once. Hence $H[C]$ is a handlebody. Conversely, if $H[C]$ is a handlebody, then C is primitive by [20].

If C is disk-busting, then $\partial H - C$ is incompressible in H. Since H is irreducible and ∂H is compressible, the handle addition lemma [17, Lemma 2.1.1] implies that $H[C]$ is irreducible and ∂–irreducible. If C is not disk-busting, then there is a compressing disk D for $\partial H - C$ in H. We may take D so that it is nonseparating in H, so that D is nonseparating in $H[C]$. (If D were separating, then the component of $H \setminus D$ disjoint from C would be a handlebody of genus at least one containing a nonseparating compressing disk.) Hence D is a compressing disk of $H[C]$. □

6.2 The busting nature of curves in a genus 2 handlebody

Throughout this subsection, assume $\mathcal{C}$ is a triple of simple closed curves in the boundary of a genus 2 handlebody H bounding two pairs of pants in ∂H.

Lemma 6.3 *If each curve in $\mathcal{C}$ is disk-busting, then $\mathcal{C}$ is both 6–disk-busting and annulus-busting.*

Proof By Lemma 6.1, each curve in $\mathcal{C}$ is actually 2–disk-busting. Since each curve of $\mathcal{C}$ intersects the boundary of an essential disk of H at least twice, together they intersect this disk at least six times. Hence $\mathcal{C}$ is 6–disk-busting.

Now let A be a properly embedded, incompressible annulus in H that is not ∂–parallel. Assume A is disjoint from $\mathcal{C}$. Because $\partial H \setminus \mathcal{C}$ is two pairs of pants and no component of ∂A bounds a disk in ∂H, the components of ∂A are each isotopic to some component of $\mathcal{C}$. Since A is an incompressible annulus in a handlebody, it must admit a ∂–compression. The result of such a ∂–compression is a properly embedded disk D whose boundary is disjoint from ∂A. Furthermore, because A is incompressible and not ∂–parallel, this disk D cannot be ∂–parallel; it must be a meridional disk of H. Each component of $\mathcal{C}$ intersects ∂D by hypothesis, and yet ∂A is disjoint from ∂D. This contradicts that each component of ∂A is isotopic to some component of $\mathcal{C}$. □

Lemma 6.4 *If two of the curves in $\mathcal{C}$ are disk-busting and the third is primitive, then $\mathcal{C}$ is 6–disk-busting.*

Proof Assume $\mathcal{C}$ is not 6–disk-busting. Since $\mathcal{C}$ is separating in ∂H, any curve in ∂H transversely intersects $\mathcal{C}$ an even number of times. Hence if $\mathcal{C}$ were 5–disk-busting, it would be 6–disk-busting. So we may assume $\mathcal{C}$ is not 5–disk-busting.

Let B and B' be the disk-busting curves of $\mathcal{C}$ and let C be the primitive curve. Let P and Q be the two pairs of pants in ∂H bounded by $\mathcal{C}$. Since any disk-busting curve is 2–disk-busting (Lemma 6.1), there is a meridional disk D of H that is disjoint from C and intersects each B and B' exactly twice. Therefore $\partial D \cap Q$ is a pair of arcs, essential and parallel in Q, that join B and B'. Similarly, $\partial D \cap P$ is a pair of arcs, essential and parallel in P, that join B and B'.

If D is separating, then $\partial H \setminus \partial D$ is two once-punctured tori. Then the component containing C is the union of the two annulus components of $P \setminus \partial D$ and $Q \setminus \partial D$ along the curve C and an arc in each of B and B'. This union however must have $\chi = -2$ and thus is not a once-punctured torus, a contradiction.

If D is nonseparating, then it becomes a meridional disk in the solid torus $H[C]$ which is disjoint from the dual arc C^* (the cocore of the attached 2–handle). Because C is primitive in H, there is a disk D' in H intersecting C in a single point; furthermore, this disk D' can be chosen to be disjoint from D. Thus C^* is a trivial arc in the ball $H[C] - \mathcal{N}(D)$. The pair of pants Q closes off to an annulus $A = Q[C]$ in $\partial H[C]$ that crosses D twice and contains only one endpoint of the arc C^*. The core of the annulus A winds twice around the solid torus $H[C]$. Since A contains only one endpoint of C^*, the arc C^* is isotopic to an arc in the boundary of $H[C]$ intersecting some component of ∂A exactly once. The trace of this isotopy can be taken to be a disk properly embedded in H and intersecting B or B' exactly once, a contradiction. □

Lemma 6.5 *If two of the curves in $\mathcal{C}$ are disk-busting and the third is primitive, then $\mathcal{C}$ is annulus-busting.*

Proof As in the proof of Lemma 6.3, let A be a properly embedded, incompressible annulus in H that is not ∂–parallel. Assume A is disjoint from $\mathcal{C}$. Because $\partial H \setminus \mathcal{C}$ is two pairs of pants and no component of ∂A bounds a disk in ∂H, the components of ∂A are each isotopic to some component of $\mathcal{C}$. Since A is an incompressible annulus in a handlebody, it must admit a ∂–compression. The result of such a ∂–compression is a properly embedded disk D whose boundary is disjoint from ∂A. Furthermore, because A is incompressible and not ∂–parallel, this disk D cannot be ∂–parallel; it must be a meridional disk of H.

If A is nonseparating, then the two components of ∂A are nonisotopic. Hence they are isotopic to distinct components of $\mathcal{C}$. However this contradicts that only one component of $\mathcal{C}$ is not disk-busting. Therefore A is separating. Hence the two components of ∂A are isotopic and must therefore be isotopic to the primitive component of $\mathcal{C}$. Since both components of ∂A are isotopic to a primitive curve, there is a meridional disk D' of H that intersects A in a single spanning arc. Therefore $D' \setminus A$ is a pair of ∂–compression disks for A in H, one to each side of A. But this implies that A is ∂–parallel in H; eg see [3, Section 6.3]. □

Lemma 6.6 *If each of the curves in $\mathcal{C}$ is disk-busting and $H[\mathcal{C}]$ is not a lens space (including $S^1 \times S^2$), $\mathbb{R}P^3 \# \mathbb{R}P^3$, or a prism manifold, then $\mathcal{C}$ is 8–disk-busting.*

Proof Assume each of the curves in $\mathcal{C}$ is disk-busting and that $\mathcal{C}$ is not 8–disk-busting. Since $\mathcal{C}$ is separating, its intersection number with any curve in ∂H is even; hence $\mathcal{C}$ cannot be 7–disk-busting. Since each component of $\mathcal{C}$ is disk-busting, $\mathcal{C}$ is necessarily 6–disk-busting (Lemma 6.3). So assume D is a meridional disk of H that $\mathcal{C}$ intersects six times. By Lemma 6.1 each component intersects D twice.

Observe that ∂H is a Heegaard surface of $H[\mathcal{C}]$: the attached 2–handles together with the 3–handles filling up the sphere boundary components define a genus 2 handlebody $H_{\mathcal{C}}$ with $\partial H_{\mathcal{C}} = \partial H$ in which the curves $\mathcal{C}$ are meridians. Since the curves of $\mathcal{C}$ each intersect ∂D twice, there is a homeomorphism from $H_{\mathcal{C}}$ to an interval bundle over F, where F is a surface with boundary, in which an annular collar of ∂D in ∂H maps to the corresponding interval bundle over ∂F. Here F must be a connected surface with one boundary component and $\chi(F) = -1$. Thus F is either a once-punctured torus or a once-punctured Klein bottle depending on whether ∂D is separating or not in $\partial H_{\mathcal{C}}$.

If ∂D is separating, then $H_{\mathcal{C}} = F \times [-1, 1]$, and $D \cup F \times \{0\}$ is a torus whose exterior in $H[\mathcal{C}]$ is $H \setminus \mathcal{N}(D)$. Since ∂D is separating in ∂H, the disk D must be separating in H. Hence $H \setminus \mathcal{N}(D)$ must be two solid tori. Therefore $H[\mathcal{C}]$ is a lens space.

If ∂D is nonseparating, then capping the 0–section of the interval bundle $H_{\mathcal{C}}$ gives a Klein bottle whose exterior in $H[\mathcal{C}]$ is $H \setminus \mathcal{N}(D)$. Since ∂D is nonseparating in ∂H, D must be nonseparating in H. Hence $H \setminus \mathcal{N}(D)$ must be one solid torus. Thus $H[\mathcal{C}]$ is a Dehn filling of the twisted I–bundle over the Klein bottle. In other words, $H[\mathcal{C}]$ is a (generalized) Seifert fibered space $S^2\bigl(0; \alpha, \frac{1}{2}, \frac{1}{2}\bigr)$ for some α, that is, a prism manifold (including $\mathbb{R}P^3 \# \mathbb{R}P^3$ and $S^1 \times S^2$). □

7 Tangles

For $n \geq 1$, an *n–strand tangle* (in a ball) is a pair $\tau = (B, t)$ of a 3–ball B with a properly embedded 1–manifold t that is homeomorphic to n arcs and some number of circles. The n–strand tangle $\tau = (B, t)$ is called *rational* (or also *trivial*) if t has no closed components and is isotopic rel ∂ into ∂B. Here we take a 0–strand tangle to be (the pair of) a link t in $B = S^3$.

Let $\tau = (B, t)$ be a n–strand tangle. If any properly embedded disk or sphere in B that is disjoint from t cuts off a ball disjoint from t, then τ is *nonsplit*. If any properly embedded disk in B that t transversally intersects just once cuts off a trivial 1–strand tangle, then τ is *indivisible*. If τ is nonsplit, indivisible, and not the trivial 1–strand tangle, then τ is *essential*.

If any embedded sphere in B that intersects t in two points cuts out a trivial 1–strand tangle from B, then τ is *locally trivial*. If τ is nonsplit, indivisible, and locally trivial, then τ is *prime*. So if τ is prime and not the trivial 1–strand tangle, then it is essential. (Tangles with closed components can be essential without being prime.)

The tangle τ is *toroidal* if some embedded torus in $B-\mathcal{N}(t)$ is neither compressible in $B-\mathcal{N}(t)$ nor isotopic to a boundary component of $B-\mathcal{N}(t)$. A *Conway sphere* for τ is a sphere S embedded in B that meets t transversally in four points so that $S-\mathcal{N}(t)$ is incompressible in $B-\mathcal{N}(t)$ and $S-\mathcal{N}(t)$ is not isotopic rel ∂ into $\partial(B-\mathcal{N}(t))$. A *Conway disk* for τ is a disk D properly embedded in B that meets t transversally in two points so that $D-\mathcal{N}(t)$ is incompressible in $B-\mathcal{N}(t)$ and $D-\mathcal{N}(t)$ is not isotopic rel ∂ into $\partial(B-\mathcal{N}(t))$.

A diagram of the tangle τ is a projection of B to a disk D so that ∂t projects to distinct points in ∂D while the rest of t projects to the interior of D with over/under information at the crossings as in an ordinary link diagram. A diagram of τ is *alternating* if when following along any arc of τ, the crossings alternate between over and under. It is *locally trivial* if any loop in D meeting the arcs of the diagram bounds a disk in which the diagram consists of a single arc. It is *reduced* if it contains no nugatory crossings.

7.1 Disk-busting and primitive curves in genus 2 handlebodies via tangle quotients

Since a genus 2 handlebody H is the double branched cover of a 3–strand rational tangle τ, a collection of disjoint, nonseparating, pairwise nonparallel, simple closed curves $\mathcal{C}$ in ∂H can be described as the preimage of a set of arcs c in the boundary τ. When presented in this manner, it can be convenient to determine the disk-busting and primitive nature of the components of $\mathcal{C}$ in H in terms of the arc components of c in τ.

Let c be collection of simple arcs in the boundary sphere of a tangle $\tau = (B, t)$ such that $\partial c = c \cap t$. Let $\tau_1 = (B_1, t_1)$ be the trivial 1–strand tangle. The *arc-closure* of τ along c (or simply the *c–closure* of τ) is the tangle $\tau[c] = (B', t')$ obtained by attaching a copy of τ_1 to τ along each disk component of a regular neighborhood of c in ∂B so that the endpoints of the arcs t_1 correspond to the endpoints $\partial c \subset t$, t' is the union of t with the arcs t_1, and B' is the union of B with the balls B_1 with its boundary further filled with a ball (ie S^3) if $\partial t' = \varnothing$. Equivalently, $\tau[c] = (B', t')$, where the properly embedded 1–manifold t' is obtained from $c \cup t$ by pushing the interior of $c \cup t$ into the interior of B, and B' is either B or B with its boundary filled with a ball (ie S^3) if $\partial t' = \varnothing$.

If H is the double branched cover of τ, then c lifts to a curve C in ∂H so that $H[C]$ is the double branched cover of $\tau[c]$. When τ is a rational tangle, so that H is a handlebody, Lemma 6.2 allows us to determine if the curve C in ∂H is primitive or disk-busting in terms of the tangle $\tau[c]$.

Lemma 7.1 *Let c be an arc in the boundary of a rational 3–strand tangle τ. Let H be the genus 2 handlebody that is the double branched cover of τ, and let the curve $C \subset \partial H$ be the lift of c. Then*

- *C is primitive in H if and only if $\tau[c]$ is a rational 2–strand tangle, and*
- *C is disk-busting in H if and only if $\tau[c]$ is an essential tangle.*

Proof Since the involution of H given by the double cover is the hyperelliptic involution, every simple closed curve in ∂H and properly embedded disk in H can be made invariant under the involution. Thus this lemma follows directly from Lemma 6.2. (A disk indicating the failure of $\tau[c]$ to be nonsplit or indivisible will lift to a boundary-reducing disk of $H[C]$. If $H[C]$ is boundary reducible, there is a boundary-reducing disk that lies in H by the handle addition lemma [17, Lemma 2.1.1] which may then be made equivariant under the involution by an isotopy. Either this disk transversally intersects the fixed set once or it contains an arc of the fixed set. In the former case, the disk itself quotients to disk indicating the failure of $\tau[c]$ to be indivisible. In the latter case an equivariant pair of push-offs of the disk descends to show that $\tau[c]$ fails to be nonsplit.) □

7.2 Nonsimple manifolds via tangle quotients

In this section, we collect some well-known results that will allow us to recognize the nonsimplicity of a double branched cover from its tangle quotient.

Lemma 7.2 *Let X be the double branched cover of an n–strand tangle χ in a ball B (or the 3–sphere B when $n = 0$). If X is reducible then χ is not prime.*

Proof Let ι be the involution on X that is the deck transformation of the branched covering $p\colon X \to \chi$, and let $\mathrm{fix}(\iota)$ be its fixed set.

Assume X contains a reducing sphere. By the equivariant sphere theorem [19; 28], there exists a reducing sphere S such that either $S \cap \iota(S) = \varnothing$ or S is transverse to $\mathrm{fix}(\iota)$ and $S = \iota(S)$. So first assume S is disjoint from $\mathrm{fix}(\iota)$. Then S projects to a sphere that doesn't bound a ball in the tangle complement, though it must bound a ball in B. Hence χ must be split and therefore not prime. So now if instead S is not disjoint from $\mathrm{fix}(\iota)$, then an Euler characteristic argument shows that they intersect twice. Hence S projects to a sphere $p(S)$ intersected twice by the tangle. Since $p(S)$ bounds a ball in B, it bounds a 1–strand tangle. However since S does not bound a ball in X, this 1–strand tangle cannot be trivial. Hence χ is not locally trivial and therefore not prime. □

Lemma 7.3 *Let X be the double branched cover of a 2–strand tangle χ in a ball B. If X is toroidal, then either χ contains an essential Conway sphere or χ is toroidal. If X is annular but not toroidal, then χ is the tangle sum of two rational tangles.*

Proof Let ι be the involution on X that is the deck transformation of the branched covering $X \to \chi$, and let $\mathrm{fix}(\iota)$ be its fixed set.

Assume X contains an essential torus. Then [22, Corollary 4.6] shows that there is an essential torus T such that either $T \cap \iota(T) = \varnothing$, or T is transverse to $\mathrm{fix}(\iota)$ and $T = \iota(T)$. So first assume that T is disjoint from $\mathrm{fix}(\iota)$. Then T projects to an incompressible, non-∂–parallel torus in the tangle complement; ie χ is toroidal. Now if instead T is not disjoint from $\mathrm{fix}(\iota)$, then an Euler characteristic argument shows its quotient must be a Conway sphere. Since T is essential, this Conway sphere must be essential.

Assume X contains an essential annulus but no essential torus. Therefore X is a Seifert fibered space over the disk with two exceptional fibers. Taking X with such a Seifert fibration, the main theorem of [43] shows that ι may be taken to be fiber-preserving. The involution on X then induces an involution on the orbit surface D that leaves the set of singular points invariant. Thus it restricts to an involution ι' on the 3–punctured sphere D' that is the exterior of these singular points. Moreover, we know that the involution takes the "outside" boundary $\partial D \subset \partial D'$ of this 3–punctured sphere to itself. By taking an essential arc with endpoints on this outside boundary that intersects itself minimally under the involution on the surface, we find one that is either invariant or taken off itself. That is, there is such an arc $\alpha \subset D'$ with either $\alpha = \iota'(\alpha)$ or $\alpha \cap \iota'(\alpha) = \varnothing$.

Assume there is an essential arc $\alpha \subset D'$ with $\partial\alpha \subset \partial D$ such that $\alpha \cap \iota'(\alpha) = \varnothing$. Then there is product region Δ in D' between the arc α and its image $\iota'(\alpha)$ that is invariant under the involution. Since the involution ι' switches the arcs, it is either conjugate to a rotation on the disk or to a reflection across the axis in Δ between the two arcs. The first can't happen because then $\operatorname{fix}(\iota)$ would be disjoint from ∂X, which we know is not the case (since χ is a 2–strand tangle in a ball). So the latter must occur. Let X_Δ be the S^1–bundle over Δ as a Seifert fibered submanifold of X. Then ι restricted to X_Δ must be reflection across the annulus A over this axis followed by reflection in the S^1 factor since ι is orientation-preserving. Therefore $\operatorname{fix}(\iota) \cap X_\Delta$ consists of two spanning arcs of A. Since $\operatorname{fix}(\iota)$ is a properly embedded 1–manifold in X meeting ∂X in four points, any remaining components of $\operatorname{fix}(\iota)$ must be closed components in the two solid tori of $X - X_\Delta$. However because ι exchanges these two solid tori, the fixed set cannot meet their interior. Hence $\operatorname{fix} \iota$ consists of only these two arcs of A. Thus, the annulus A projects to a rectangle giving a parallelism between the two strands of χ. Furthermore the exterior of this rectangle in the tangle ball is a solid torus. Since χ is a tangle in a ball, χ is a trivial tangle. Hence X is a solid torus and therefore contains no essential annulus, a contradiction.

So we assume there is an invariant essential arc α. The involution on α is either the identity or conjugate to reflection in the midpoint of α. Look at the essential annulus A sitting above this arc. Assume $\operatorname{fix}(\iota)$ intersects A. If the involution is the identity on α then it must be conjugate to reflection in the circle fibers. The involution must switch sides of the annulus and we arrive at the contradiction above. So the involution on α must be conjugate to reflection along the midpoint. Since $\operatorname{fix}(\iota)$ intersects ∂X, the involution ι must preserve the sides of A. Since ι is orientation-preserving it must reverse the orientation of the circle fibers in A. Hence ι must be conjugate to reflection in the circle fiber over the midpoint of α. So the quotient of A is a Conway disk in χ. This Conway disk must be essential since A is. Since each side of A in X is a solid torus, the Conway disk must split χ into two rational tangles.

Assume $\operatorname{fix}(\iota)$ is disjoint from A. Since $\operatorname{fix}(\iota)$ is not disjoint from ∂X, the involution ι must fix the sides of A. Thus ι restricts to an involution on each of the solid tori $X' \setminus A = X_1 \cup X_2$ on the two sides of the annulus. Since $\operatorname{fix}(\iota)$ intersects ∂X in four points but is disjoint from A, the pair of its intersection numbers with the boundaries of these two solid tori is either $\{2,2\}$ or $\{0,4\}$. Since ι further restricts to an involution on each of ∂X_1 and ∂X_2, the pair $\{2,2\}$ cannot occur; an involution on a torus cannot have a fixed set consisting of just two points. Therefore $\mathrm{fix}(\iota)$ intersects ∂X_1, say, in four points, and it does so in the annular complement A_1 of A. Since A is invariant under ι, so is A_1. Thus ι restricts to an involution on the annulus A_1 with exactly four fixed points, which cannot occur. □

8 Construction of asymmetric L–space knots

Throughout this section we continue with the notation used for the construction of L–space knots in Section 3 and its further development in the statement of Theorem 4.10. In particular, we consider the presentation $J^{\alpha,m}$ of the unknot as shown in the center of Figure 7 and its decomposition into tangles $\tau_+ = (B^3, t_+)$ (above) and $\tau_- = (B^3, t_-)$ (below) by the bridge sphere containing arcs $c = c^{\alpha,m} = \{c_\nu, c_\mu, c_\lambda\}$. For convenience, these are shown again in Figure 15.

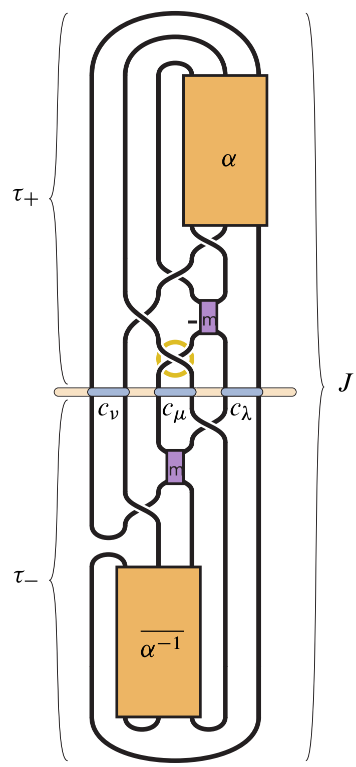


Figure 15: The knot $J = J^{\alpha,m}$ from the center of Figure 7 is shown again, divided into the tangles τ_+ and τ_-.

The double branched cover of this unknot $J = J^{\alpha,m}$ is the manifold $Y = S^3$. Let Σ be the genus 2 Heegaard surface that is the lift of the bridge sphere, let the curves $\mathcal{C} = \mathcal{C}^{\alpha,m} = \{C_\nu, C_\mu, C_\lambda\} \subset \Sigma$ be the lift of the arcs c, and let $P = P^{\alpha,m}$ and $Q = Q^{\alpha,m}$ be the two pairs of pants in Σ bounded by $\mathcal{C}$. The exterior of P is the manifold $M = H_+ \cup_Q H_- = Y \setminus H_P$ and $\partial M \cap H_\pm = P_\pm$.

Since P_+ and P_- are isotopic in H_P, the handlebodies H_+ and H_- may be expanded through H_P to P to be the handlebodies of the Heegaard splitting by Σ. Hence the tangles τ_+ and τ_- are also the quotients of handlebodies H_+ and H_- by the hyperelliptic involution, and the arcs c in each of their boundary spheres are the quotients of the curves $\partial Q = \mathcal{C}$ in the boundaries of these handlebodies.

To be explicit, let us take α to be the alternating 3–braid

$$\alpha = \prod_{i=n}^{1} \sigma_{\bar{i}}^{\epsilon_i a_i} = \sigma_{\bar{n}}^{\epsilon_n a_n} \cdots \sigma_1^{-a_3} \sigma_2^{a_2} \sigma_1^{-a_1},$$

where $n \geq 1$ is an integer, $\epsilon_i = (-1)^i$, $\bar{i}$ equals 1 or 2 according to the parity of i, and $a_i \geq 1$ for $i = 1, \dots, n$.

Lemma 8.1 *Take α as above with $n \geq 3$ and also $m \geq 3$. In H_+, each component of $\mathcal{C}$ is disk-busting, and cumulatively $\mathcal{C}$ is 8–disk-busting. In H_-, two components of $\mathcal{C}$ are disk-busting, and one component of $\mathcal{C}$ is primitive.*

Remark 8.2 As one may observe from the proof of Lemma 8.1, regardless of choice of α, the component C_ν of $\mathcal{C}$ will always be primitive in H_-. One may further check that C_λ will be primitive in H_- when $m = 1$ and primitive in H_+ when $n = 2$.

Proof We use Lemma 7.1, the tangle form of Lemma 6.2, to determine when the curves of $\mathcal{C}$ are disk-busting or primitive in H_+ and H_-. Let c_ν, c_μ, and c_λ be the three arcs of c.

The bottom of the top left of Figure 16 shows that $\tau_-[c_\nu]$ is a rational tangle. (The diagrams shown for $\tau_-[c_\nu]$ are not reduced. Reductions would yield a crossingless diagram since the tangle is rational.) Hence C_ν is primitive in H_-. The rest of Figure 16 shows that the other five tangles, $\tau_+[c_\nu]$, $\tau_+[c_\mu]$, $\tau_+[c_\lambda]$, $\tau_-[c_\mu]$, and $\tau_-[c_\lambda]$, have reduced (because $m \geq 2$), connected, alternating diagrams. Direct inspection further shows that these five diagrams are each locally trivial and indivisible as well. (While Figure 16 is drawn with $n = 3$, one may observe that the result continues to hold for

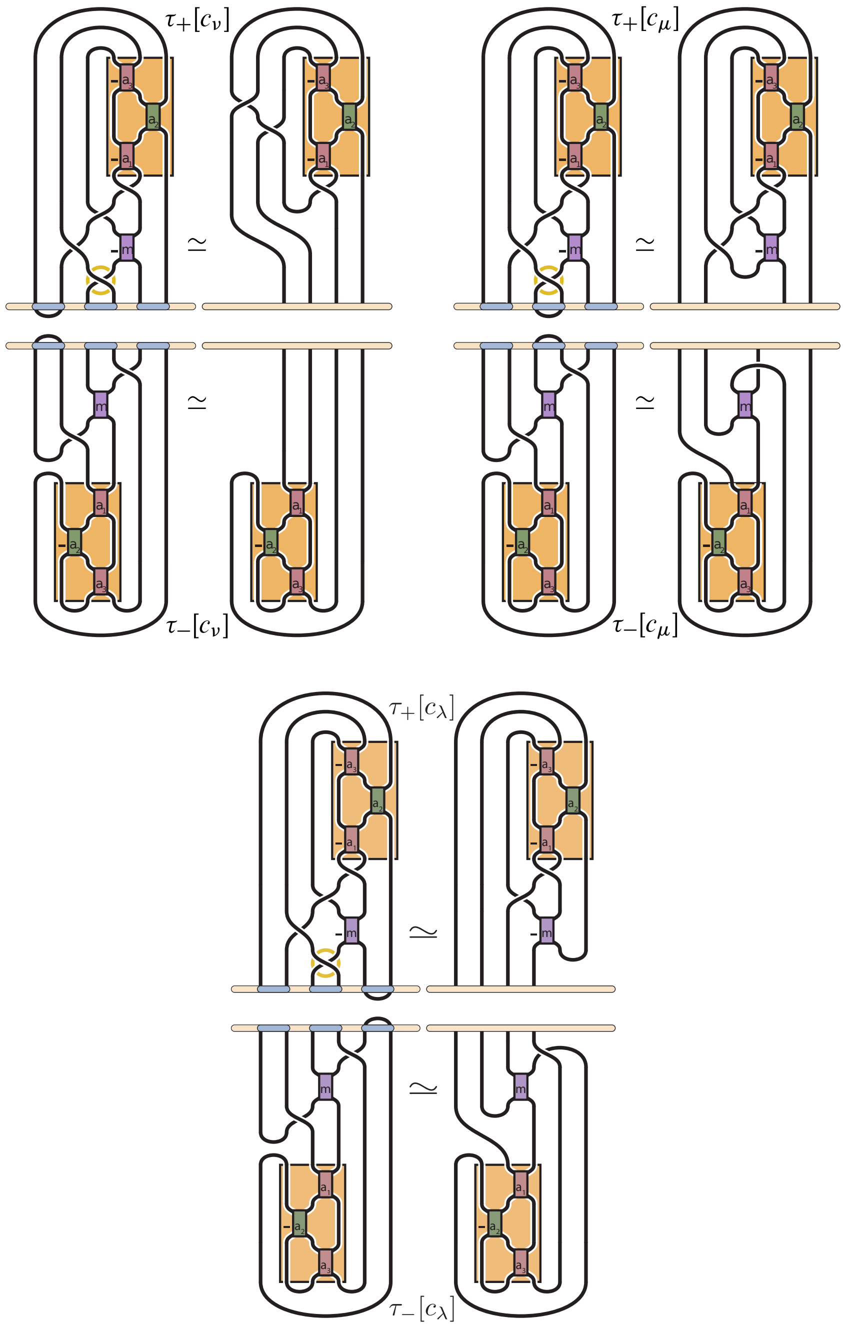


Figure 16: Shown in the case that $n = 3$, the tangles resulting from the arc-closures of the tangles τ_+ and τ_- along the components of c individually are isotoped to simpler, alternating diagrams.

larger integers n.) Then according to [40, Theorem 1.2] the tangles are prime. Hence they are essential. By Lemma 7.1, the corresponding components of $\mathcal{C}$ in H_+ and H_- are disk-busting.

We next show that $H_+[\mathcal{C}]$ is neither a lens space (including $S^1 \times S^2$), $\mathbb{R}P^3 \# \mathbb{R}P^3$, nor a prism manifold.

Figure 17, top shows the link $\tau_+[c]$ with an isotopy to a simpler configuration in which the diagram is reduced, alternating, and prime. Since an alternating diagram of a nonprime link is nonprime [29], $\tau_+[c]$ is a prime link. By Lemma 7.2 (for 0–tangles), $H_+[\mathcal{C}]$ cannot be reducible and hence is neither $\mathbb{R}P^3 \# \mathbb{R}P^3$ nor $S^1 \times S^2$.

Figure 17, bottom left shows a rational tangle replacement (by setting $m = 0$) of distance $\Delta = m$ that produces a reduced, nonsplit alternating diagram of a 2–bridge link. Because the double branched cover of a 2–bridge link is a lens space, the Montesinos trick shows that $H_+[\mathcal{C}]$ contains a knot with a Dehn surgery to a lens space (other than $S^1 \times S^2$). In particular the exterior X of this knot is the double branched cover of the exterior tangle χ of the $-m$–tangle in $\tau_+[c]$ shown in Figure 17, bottom right.

Observe that this diagram of the tangle χ in Figure 17, bottom right is reduced, connected, locally trivial, indivisible, and alternating. By [40, Theorem 1.2] again, χ is a prime tangle. Since it is not crossingless, χ is not a rational tangle [41, Corollary 3.2]. Inspection shows there is no "visible" essential Conway disk in this diagram of χ, so χ does not represent the sum of two rational tangles; see the last paragraph of [41, Section 3]. Theorem 4.2 of [39] shows that the tangle χ is atoroidal. Further inspection also reveals that there are no "visible" or "hidden" essential Conway spheres in this diagram of χ, so the tangle has no essential Conway sphere; see [41, Section 3] or consider alternating rational tangle fillings of χ and apply [29]. Therefore, with Lemmas 7.2 and 7.3, this implies that X, the double branched cover of χ, is neither a solid torus, a Seifert fibered space over the disk, a toroidal manifold, nor a reducible manifold. (Recall that a Seifert fibered space over the disk is either toroidal, annular, or a solid torus.)

On the other hand, applying the cyclic surgery theorem [17] and the finite surgery theorems [10], if $H_+[\mathcal{C}]$ were a lens space or a prism manifold, X should either be a solid torus, a Seifert fibered space over the disk with two exceptional fibers, or a union of a cable space and a Seifert fibered space over the disk with at most two exceptional fibers. In the last case, if the constituent Seifert fibered space has two exceptional fibers then X is toroidal. If it has at most one, then either X is a Seifert fibered space over a

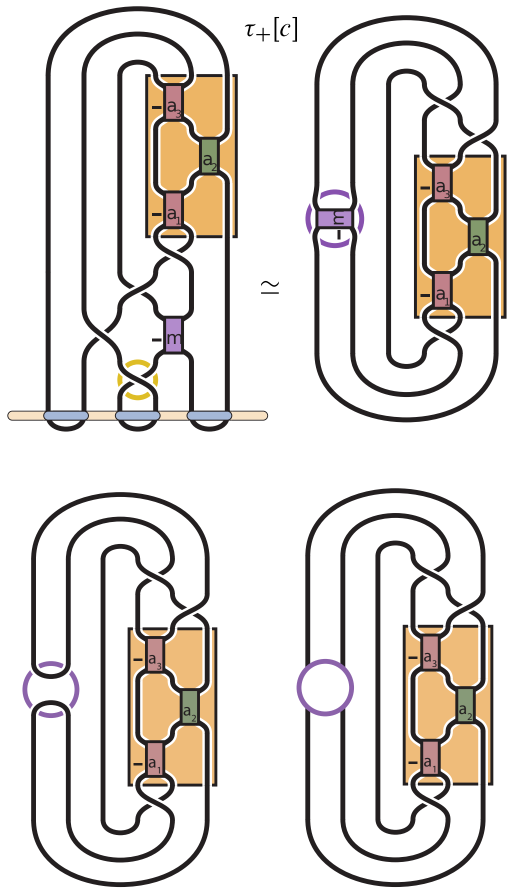


Figure 17: The c–closure of τ_+ is simplified to a reduced, prime, alternating link (top). A rational tangle replacement of distance $\Delta = m$ produces a 2–bridge link with a nonsplit reduced alternating diagram (bottom left). The exterior of the $-m$–twist tangle is the tangle χ, shown with a reduced alternating diagram (bottom right).

disk or is a connected sum of a lens space and a solid torus. This however contradicts our previous determination about X.

The above goes to imply that $H_+[\mathcal{C}]$, the double branched cover of $\tau_+[c]$, is not a lens space or a prism manifold. Since we have already shown that the components of $\mathcal{C}$ are each disk-busting in H_+, Lemma 6.6 implies that $\mathcal{C}$ is then 8–disk-busting in H_+. $\square$

Lemma 8.3 *The manifold* $M = H_+ \cup_Q H_-$ *is*

(1) *a simple* 3*–manifold*

in which

(2) (H_+, Q) *and* (H_-, Q) *are not homeomorphic,*

(3) Q *is incompressible and boundary incompressible, and*

(4) *any properly embedded pair of pants in* M *is either compressible,* ∂*–parallel, isotopic to* Q*, or nonseparating and can be isotoped to be disjoint from some component of* ∂Q*.*

Proof By Lemma 8.1, one component of ∂Q is primitive in H_- while no components of ∂Q are primitive in H_+. Hence (H_+, Q) and (H_-, Q) are not homeomorphic.

Because of Lemma 8.1, Lemma 6.3 shows that ∂Q is 6–disk-busting and annulus-busting in H_+ while Lemmas 6.4 and 6.5 show that ∂Q is 6–disk-busting and annulus-busting in H_-. Therefore Lemma 5.2 shows that Q is incompressible and boundary incompressible in M, and Lemma 5.3 shows that M is simple.

Finally because Lemma 8.1 also shows that ∂Q is 8–disk-busting in H_+, Lemma 5.4 gives the final desired property. □

Theorem 8.4 *Take* α *as above with* $n \geq 3$ *and also* $m \geq 3$*. Take integers* $p, q \geq 0$ *and* $p', q' \geq 1$ *such that* $|pq' - p'q| = 1$*. Then for suitably large integers* N*, the* $(p+Np')/(q+Nq')$*–lashing of* $P = P^{\alpha,m}$ *(with respect to* $\partial P = \mathcal{C} = \{C_\nu, C_\mu, C_\lambda\}$*) is an asymmetric hyperbolic knot with an longitudinal surgery to the double branched cover of a nonsplit alternating link. In particular, such a lashing is an asymmetric L–space knot.*

Proof Observe that for fixed integers p, q, p' and q' satisfying $|pq' - p'q| = 1$ and $N \in \mathbb{Z}$, the knot K of slope p/q and the knot L of slope p'/q' intersect once in the once-punctured torus T. Therefore the knots K^N of slope $(p+Np')/(q+Nq')$ in T are obtained by twisting K along L and hence form a basic twist family as in Section 4.1.

For $N \geq 0$, we have that $(p+Np')/(q+Nq') > 0$ since $p, p', q, q' \geq 0$. Hence the lashing K^N has a longitudinal surgery to the double branched cover of a nonsplit alternating link and is thus an L–space knot by Theorem 3.3.

Lemma 8.3 ensures the four numbered hypotheses of Theorem 4.10 are satisfied. Since $p', q' \geq 1$, the knot L is not isotopic to μ or λ, so that the final hypothesis of Theorem 4.10 is satisfied. Thus the lashing K^N is an asymmetric hyperbolic knot for suitably large N. □

9 Asymmetric L–space knots in lens spaces and $S^1 \times S^2$

Here we explain how to adapt the above construction of asymmetric L–space knots in S^3 to produce asymmetric L–space knots in any lens space, including $S^1 \times S^2$. Since this ends up being a mild modification, we will only discuss the necessary changes and impacts on relevant lemmas and theorems above and present the result in Theorem 9.2.

We may generalize Figure 7, by using the 3–braid ω in the stead of α^{-1} to form a link diagram $J^{\alpha,\omega,m}$ as depicted in Figure 18, top right. Here we take 3–braids of the form

$$(*)\qquad \begin{aligned} \alpha &= \prod_{i=n}^{1} \sigma_{\bar{i}}^{\epsilon_i a_i} = \sigma_{\bar{n}}^{\epsilon_n a_n} \cdots \sigma_1^{-a_3} \sigma_2^{a_2} \sigma_1^{-a_1}, \\ \omega &= \prod_{i=1}^{r} \sigma_{\bar{i}}^{-\epsilon_i z_i} = \sigma_1^{z_1} \sigma_2^{-z_2} \sigma_1^{z_3} \cdots \sigma_{\bar{r}}^{-\epsilon_r z_r}, \end{aligned}$$

where $n, r \geq 1$ are integers, $\epsilon_j = (-1)^j$, and $\bar{j}$ is 1 or 2 according to the parity of j. The braid $\bar{\omega}$ is obtained from ω by swapping σ_1 and σ_2. When the integers a_i and z_j are nonnegative, α and $\bar{\omega}$ are each alternating 3–braids with negative twist boxes on the left and positive on the right. Examples of braids α and $\bar{\omega}$ for odd n and r are shown in Figure 18, bottom.

Then for any integer m the diagram $J^{\alpha,\omega,m}$ depicts a 2–bridge link that is a plat closure of the 3–braid $\sigma_1^{-1}\alpha\omega$ as illustrated by the isotopies in Figure 18, top right from its right to left. (Here we take the closure of any 3–braid η as in Figure 18, top left.) Observe that if we further take all the integers a_i and z_j and the integer m to be nonnegative, then $J^{\alpha,\omega,m}$ is an almost-alternating diagram of this 2–bridge link.

Lemma 9.1 *Any 2–bridge link may be expressed as the plat closure of the 3–braid $\sigma_1^{-1}\alpha\omega$ as shown in Figure 18, top left with α and ω of the form $(*)$, both n and r greater than or equal to 3, and all the integers a_i and z_j positive. In particular, $J^{\alpha,\omega,m}$ is an almost-alternating diagram of this 2–bridge link.*

Proof First, take a 3–braid $\alpha' = \prod_{i=n'}^{1} \sigma_{\bar{i}}^{\epsilon_i a'_i}$, for some integer $n' \geq 3$, with positive coefficients a'_i and where $\epsilon_i = (-1)^i$.

Now, any 2–bridge link L may be expressed as the plat closure of some alternating 3–braid as shown in Figure 18, top left using its Conway normal form [14]. In particular L is the plat closure of an alternating 3–braid ξ. Because the 3–braids η and $\sigma_2^N \eta$

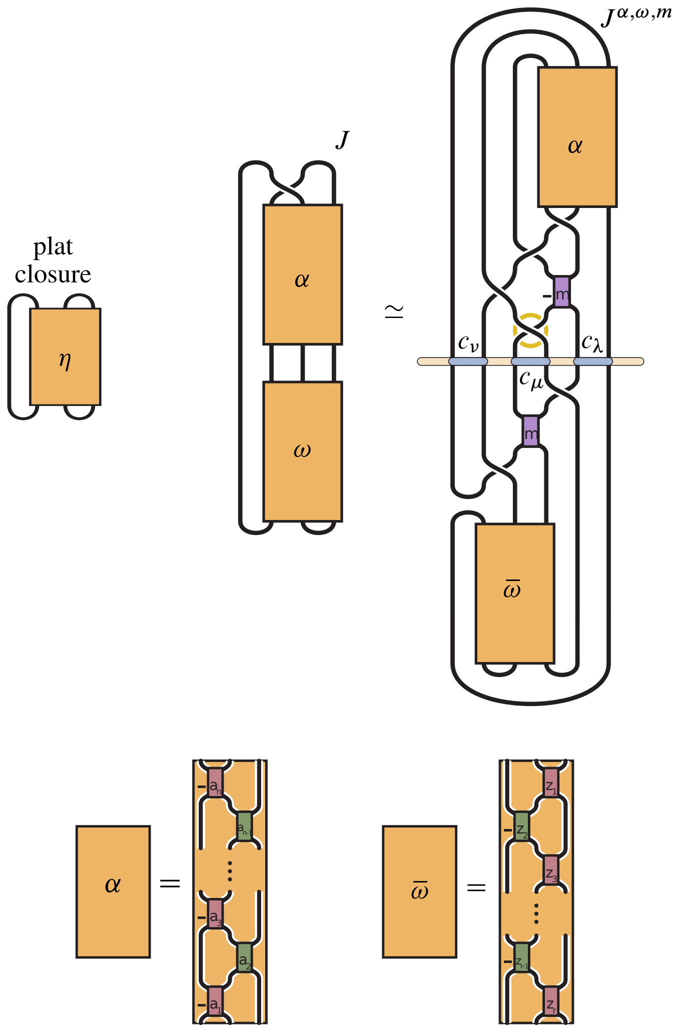


Figure 18: The plat closure of a 3–braid η (top left). Replacing α^{-1} in Figure 7 with another braid ω enables the production of almost-alternating diagrams $J^{\alpha,\omega,m}$ of other 2–bridge links, analogous to our almost-alternating unknot diagrams (top right). The braids α and $\overline{\omega}$ are shown for α and ω in form $(*)$ with n and r odd (bottom).

have plat closures (as in Figure 18, top left) giving isotopic links for any integer N, the 3–braid ξ may be taken to begin with a nonzero power of σ_1 unless ξ is the trivial braid.

If ξ begins with a negative power of σ_1, let ξ' be the alternating 3–braid such that $\xi = \sigma_1^{-1}\xi'$. Then set $\alpha = \xi'\alpha'$ and $\omega = (\alpha')^{-1}$.

If ξ begins with a positive power of σ_1, set $\alpha = \alpha'$ and $\omega = (\alpha')^{-1}\sigma_1\xi$.

If ξ is the trivial braid, then set $\alpha = \alpha'$ and $\omega = (\alpha')^{-1}\sigma_1$.

In each of these cases $\sigma_1^{-1}\alpha\omega = \xi$ as needed. □

Theorem 9.2 *Let Y be a lens space, including $S^1 \times S^2$. Then Y contains infinitely many asymmetric hyperbolic L–space knots with a nontrivial alternating surgery.*

Proof A lens space Y is the double branched cover of some 2–bridge link. By Lemma 9.1, this 2–bridge link may be taken to be the plat closure (as in Figure 18, top left) of an alternating 3–braid $\sigma_1^{-1}\alpha\omega$ where α and ω each have at least 3 twist regions. Figure 18, top right shows this link then has an almost-alternating 3–bridge presentation $J^{\alpha,\omega,m}$ for any choice of integer m. Choose $m \geq 2$. The three arcs $c = \{c_\nu, c_\mu, c_\lambda\}$ lift to the triple of curves $\mathcal{C} = \{C_\nu, C_\mu, C_\lambda\}$ in the genus 2 Heegaard surface Σ that is the lift of the bridge sphere. Let $P = P^{\alpha,\omega,m}$ be one of the pairs of pants in Σ bounded by $\mathcal{C}$, and let Q be the complementary pair of pants. Theorem 3.3 now extends directly to show that for any nonnegative slope p/q, framed surgery on the p/q–lashing of P (with respect to $\partial P = \mathcal{C} = \{C_\nu, C_\mu, C_\lambda\}$ as shown in Figure 2) produces the double branched cover of an alternating link. Hence a nonnegatively sloped p/q–lashing is an L–space knot.

As above take the exterior of P, that is, $Y \setminus H_P$, to be the manifold $M = H_+ \cup_Q H_-$ where Q separates M into the handlebodies H_+ and H_-. Above and below the bridge sphere of the 2–bridge link $J^{\alpha,\omega,m}$ as shown in Figure 18 are two 3–strand rational tangles $\tau_+ = (B^3, t_+)$ and $\tau_- = (B^3, t_-)$. The handlebodies H_+ and H_- are the double branched covers of these tangles and the curves $\mathcal{C} = \partial Q$ in their boundaries are the lifts of the arcs c in the bridge sphere. Observe that the proof of Lemma 8.1 applies equally well with ω in place of α^{-1} as long as $r \geq 3$ (so that ω also has at least three twist regions) to give the same results about $\mathcal{C}$ in H_-. Therefore Lemma 8.3 also continues to hold.

Now we follow Theorem 8.4. Let p, q, p', and q' be nonnegative integers with $p', q' \geq 1$ such that $|pq' - p'q| = 1$, and let K^N be the $(p + Np')/(q + Nq')$–lashing of $P = P^{\alpha,\omega,m}$ (with respect to $\partial P = \mathcal{C} = \{C_\nu, C_\mu, C_\lambda\}$). Then for any $N \geq 0$ the knot K^N is an L–space knot. Since Lemma 8.3 ensures the four numbered hypotheses of Theorem 4.10 are satisfied, for each suitably large integer N the lashing K^N is also an asymmetric hyperbolic knot with a surgery to the double branched cover of an alternating link. □

10 Tunnel number

Proposition 10.1 *Let K be a p/q–lashing of a genus 2 Heegaard splitting. Then the tunnel number of K is at most 3.*

Proof Let $\Sigma = P \cup Q$ be the genus 2 Heegaard surface in which K is a lashing of P. Since K is a core of the handlebody $P \times I$, it follows that K is a core of the genus 4 handlebody $\Sigma' \times I$, where Σ' is the Heegaard surface Σ punctured once. The complement of $\Sigma' \times I$ is then the boundary connect sum of the two genus 2 handlebodies H_+ and H_-. This gives a genus 4 splitting of the exterior of K. Hence the tunnel number of K is at most 3. □

Proposition 10.2 *Let K be an asymmetric hyperbolic p/q–lashing as constructed here. Then the tunnel number of K is 2.*

Proof Recall the construction of our family of p/q–lashings in S^3 from Section 3. The proof of Lemma 8.1 shows that the curve C_v of ∂Q is primitive in H_-. This

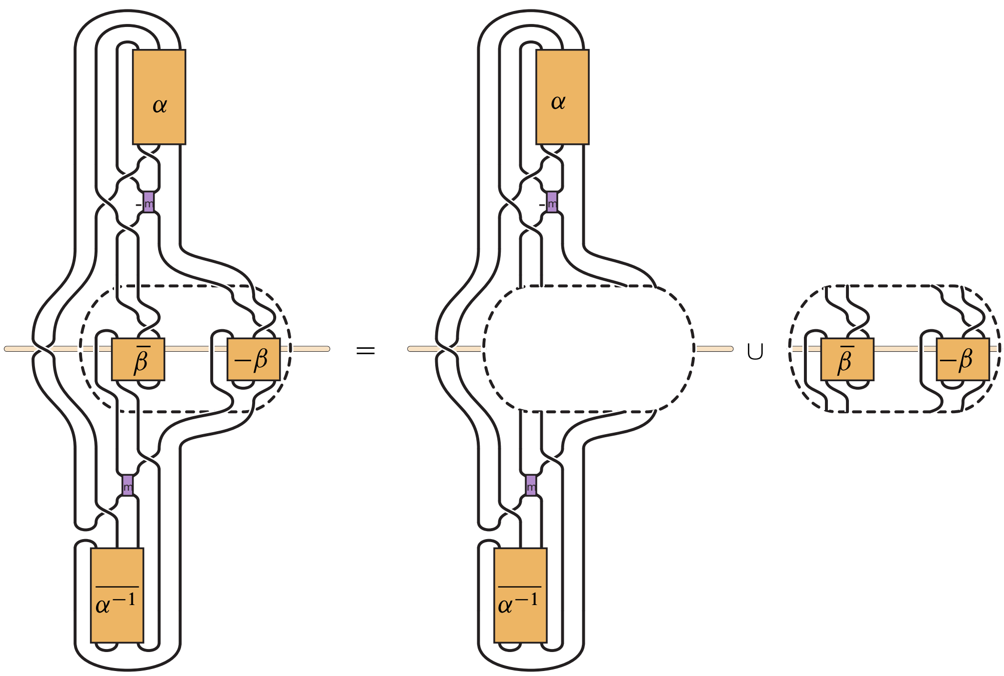


Figure 19: The link $J_c^{\alpha,m}(+1, -q/(p+q), -p/(p+q))$ splits into two 4–strand tangles as shown. One fairly easily sees that each of these two tangles is freely isotopic to the trivial 4–strand tangle.

implies the manifold $M = H_+ \cup_{\mathcal{N}(C_\nu)} H_-$ has Heegaard genus 3. As above, since K is a core of the handlebody $P \times I$, it follows that K is a core of the genus 3 handlebody $(\Sigma - \mathcal{N}(C_\nu)) \times I$. Thus we have a genus 3 splitting of the exterior of K. Hence its tunnel number is at most 2. Since any knot with tunnel number 1 is strongly invertible, the asymmetry of K implies its tunnel number is exactly 2. □

Proposition 10.3 *The double branched cover of $J_c^{\alpha,m}(+1, -q/(p+q), -p/(p+q))$ has Heegaard genus at most 3. When $p/q = +1$, the double branched cover of $J_c^{\alpha,m}\big(+1, \frac{-1}{2}, \frac{-1}{2}\big)$ has Heegaard genus at most 2.*

Proof The link $J_c^{\alpha,m}(+1, -q/(p+q), -p/(p+q))$ decomposes as the union of two trivial 4–strand tangles as indicated in Figure 19. Hence its bridge number is at most 4. Thus its double branched cover has Heegaard genus at most 3.

Because the $+1$–tangle and the $\frac{-1}{2}$–tangle are only vertical twists, one can easily see that the link $J_c^{\alpha,m}\big(+1, \frac{-1}{2}, \frac{-1}{2}\big)$ is 3–bridge. Hence its double branched cover has Heegaard genus at most 2. □

11 Surgery descriptions and examples

Here we first obtain a surgery description of the lashings constructed in Section 3, specifically for when α is of length at most 3. (One may use this as a model for obtaining surgery descriptions when α has length greater than 3.) These lashings contain a family of asymmetric L–space knots that each have a surgery to the double branched cover of an alternating link. The surgery description allows us to input these lashings into SnapPy.

Then, in a different manner, we obtain a Dehn twist presentation of the pair of pants $P^{\alpha,m}$ in the genus 2 Heegaard surface for S^3. This allows us to obtain a framed oriented train track that carries our L–space knot lashings for any α as in Section 3 and $m \geq 0$. Then we isotope this train track into a form that carries closed positive braids. Using this we are able to calculate the genus and alternating surgery slope of any positive p/q–lashing of $P^{\alpha,m}$. The explicit presentation as a closed braid also gives another method for inputting these lashings into SnapPy.

Using SnapPy along with Regina [11] and Sage, both the surgery and closed braid descriptions enable us to confirm that many small examples not covered by Theorem 8.4 are also asymmetric hyperbolic L–space knots with the expected surgery to the double branched cover of the correct alternating link. SnapPy allows us to also confirm that our two surgery descriptions and positive braid descriptions agree for these small examples. Furthermore we explicitly present two small asymmetric examples.

11.1 Surgery description of lashings

Consider the case where α is of length at most three, $\alpha = \sigma_1^{-a_3}\sigma_2^{a_2}\sigma_1^{-a_1}$. Figures 20, 21, and 22 illustrate the passage from the tangle presentation of c of Figure 7 to a surgery presentation of their lift C and finally a surgery presentation of a family of lashings of the pair of pants P that they bound.

This surgery presentation allows us to easily calculate the homology of the result of the framed surgery on the lashings. Observe that the link on the bottom of Figure 22 is a union of unknots. If we orient them all clockwise in the diagram, shown with $b_i = 0$ for $i \geq 3$, then we obtain the linking matrix for the r–surgery on the knot,

$$\left(\begin{array}{ccccccccccc|ccccc}
A & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & m & 0 & -1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 & -1 \\
-1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & -a_1-1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & -a_3 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & \frac{1}{m} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & a_2 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{b_2} & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{b_1} & 1 & 0 & 1 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & r & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{b_1} & 1 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & \frac{1}{b_2}
\end{array}\right),$$

where $A = [0, a_1, -a_2, a_3] = (a_1a_2a_3 + a_1 + a_3)/(a_2a_3 + 1)$.

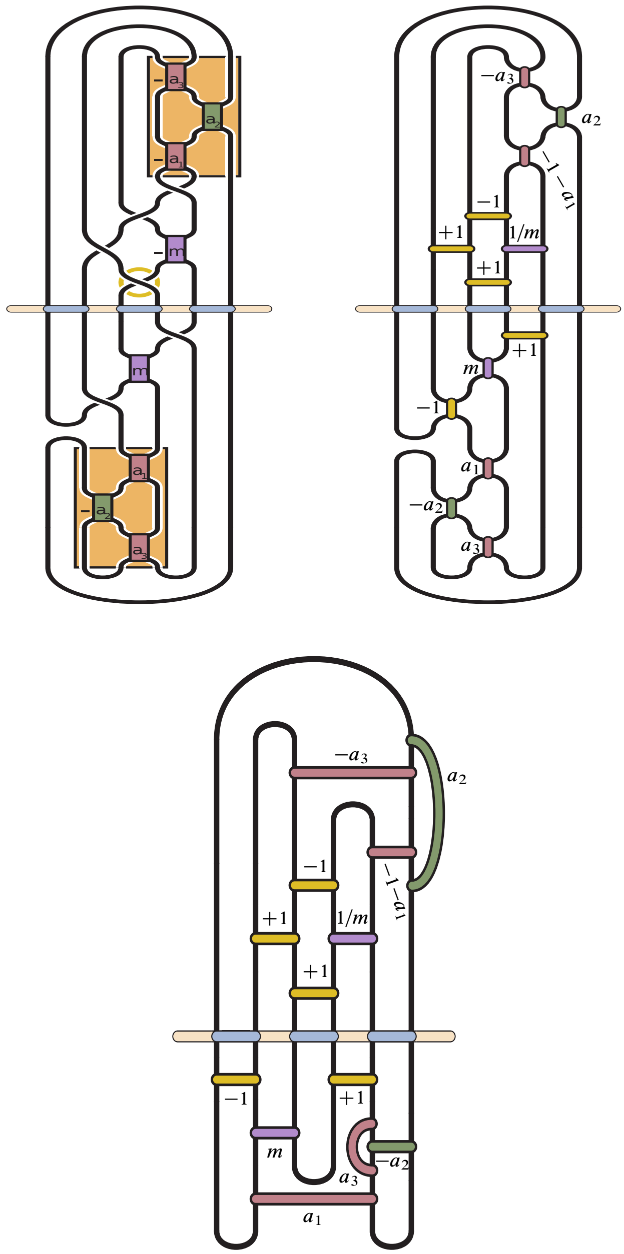


Figure 20: Twist regions (top left) of Figure 7 with the choice $\alpha = \sigma_1^{-a_3}\sigma_2^{a_2}\sigma_1^{-a_1}$ are traded (top right) for blackboard framed arcs with rational tangle replacement instructions. The diagram is then simplified by a planar isotopy that more clearly exhibits the resulting link as a planar unknot while retaining a sense of the original structure (bottom).

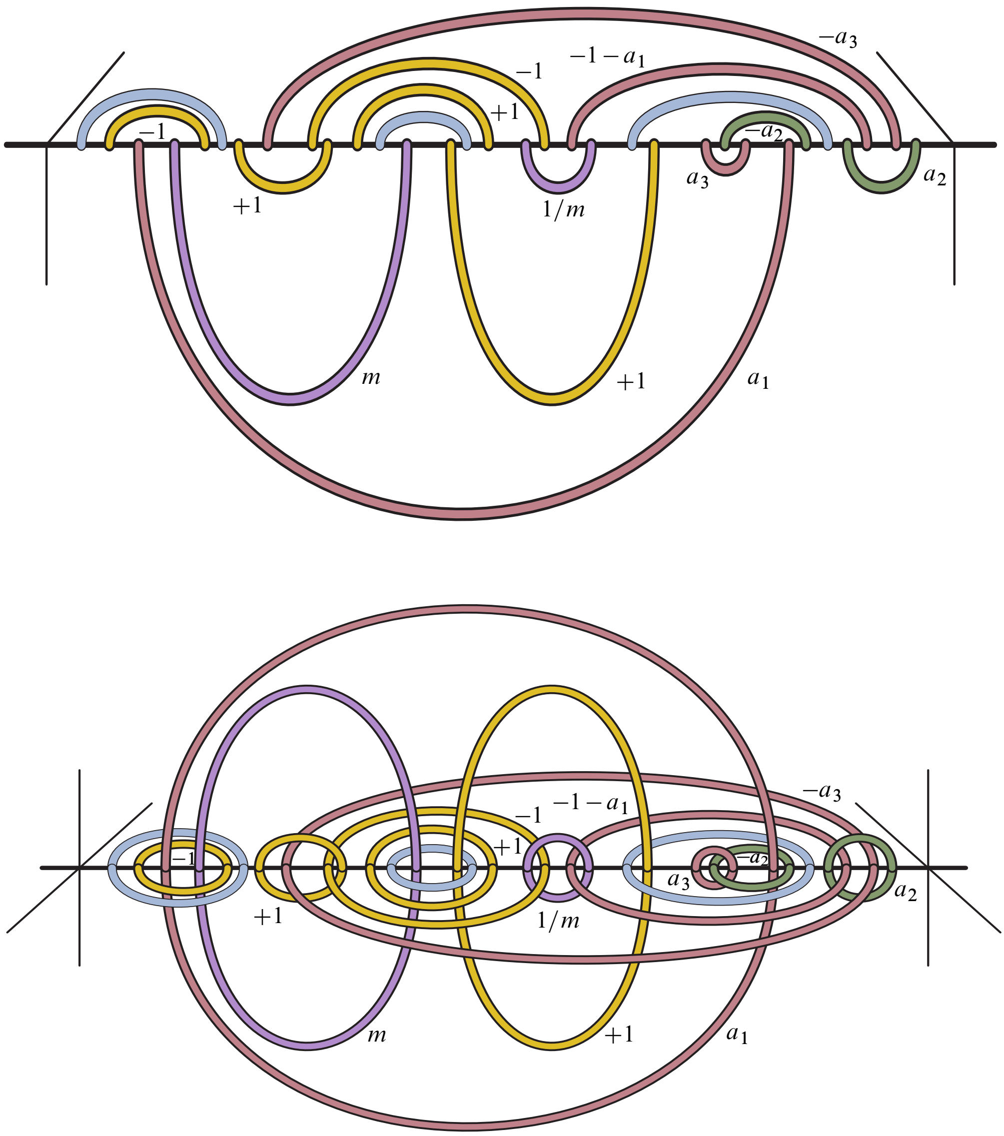


Figure 21: The planar unknot at the end of Figure 20 is mapped to a horizontal axis (with a point at infinity) so that its inside becomes a horizontal half-plane into the page while its outside becomes a downward vertical half-plane (top). This facilitates the visualization of the lifts of the arcs in the double cover of S^3 branched over the unknot (bottom). The rational tangle replacement instructions lift to Dehn surgery instructions and the arcs c_ν, c_μ, and c_λ lift to the knots C_ν, C_μ, and C_λ.

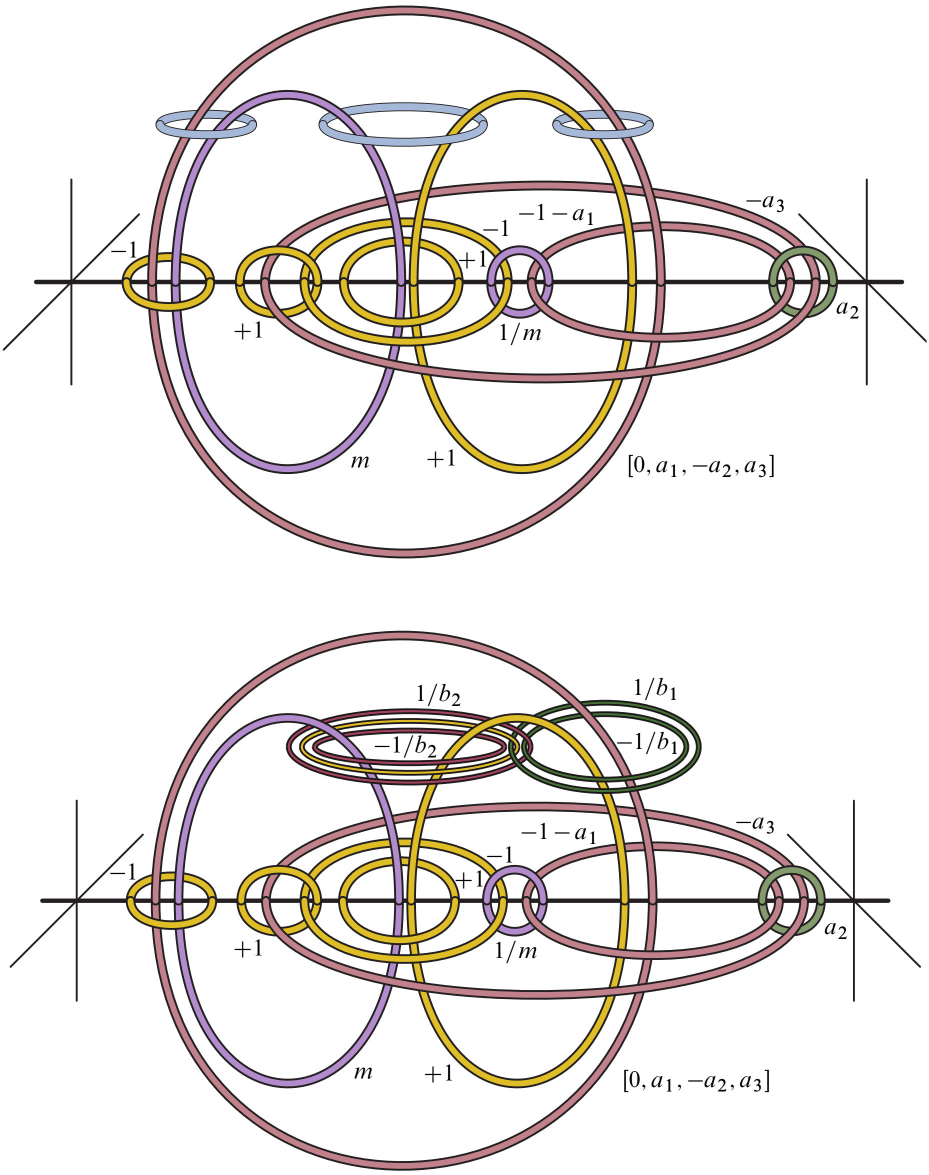


Figure 22: The genus 2 Heegaard surface containing the knots C_ν, C_μ, and C_λ can be traced through its lift of the bridge sphere at the beginning of Figure 20 (top). Here we lift these curves upwards to recognize the punctured horizontal plane they bound (with the point at infinity) as the pair of pants P in which our lashings occur. This triple of knots is then traded for the interlaced stack of knots as in Figure 5 to give surgery instructions for a p/q–lashing (bottom). Recall the convention $p/q = [-b_n, b_{n-1}, \dots, b_2, -b_1]$ with $b_j \geq 0$.

Setting $a_1 = a_2 = a_3 = m = 1$ and taking $r \in \mathbb{Z}$, the surgered manifold has first homology of order

$$|-389 - r - b_1(563 + 778b2) - b_1^2(204 + 563b_2 + 389b_2^2)|$$

(this may be calculated from the associated framing matrix; see eg [38, Section 1.1.6]).

With $b_2 = 0,\ b_1 = n > 0$, and $r \in \mathbb{Z}_+$, this has homology of order

$$389 + r + 563n + 204n^2.$$

Taking $r = 0$ corresponds to the framed surgery of the lashing.

11.2 Framed train tracks

Beginning again from the presentation of $J = J^{\alpha,m}$ of Figure 7, we isotope J so that the bridge sphere containing the arcs c nearly mirrors the tangle above to the tangle below. We coalesce the twistings of α and m into rational tangle replacements along horizontal arcs, red and green for the left and right twisting of α and purple for m. Then we flatten J by a height-preserving isotopy at the expense of twisting the arcs of c in the bridge sphere. This is done in Figure 23.

With J flattened and the twisting arcs for α and m slid into the bridge sphere along with c, we take the double branched cover of J and lift all these arcs to the simple closed curves in the genus 2 Heegaard surface that is the lift of the bridge sphere. This is shown on the top of Figure 24. On the bottom, we show the result of an isotopy of $\mathcal{C}$ (the lift of c) that more obviously bounds a pair of pants P_0, along with a purple curve in an annulus (to emphasize its framing) and darker once-punctured torus containing the red and green lifts of the red and green arcs as a basis. The basis of the once-puncture torus for the lashings from Figure 2 is shown on the bottom left and in the pair of pants P_0.

The pair of pants $P = P^{\alpha,m}$ is then obtained by first performing m left-handed Dehn twists of P_0 along the purple curve and then left-handed and right-handed Dehn twists along the red and green curves according to α. In Figure 25 we show the result of these Dehn twists as an oriented train track for a p/q–lashing of P with $p, q \geq 0$. On the top, we retain the sense of the pair of pants P_0, the purple annulus and the once-punctured torus. On the bottom, we retain just the framing of this train track and its embedding in the Heegaard surface as well as the weights of the branches in the case $m \geq 0$ and $\alpha = \sigma_1^{-a_3}\sigma_2^{a_2}\sigma_1^{-a_1}$ for $a_1, a_2, a_3 \geq 0$.

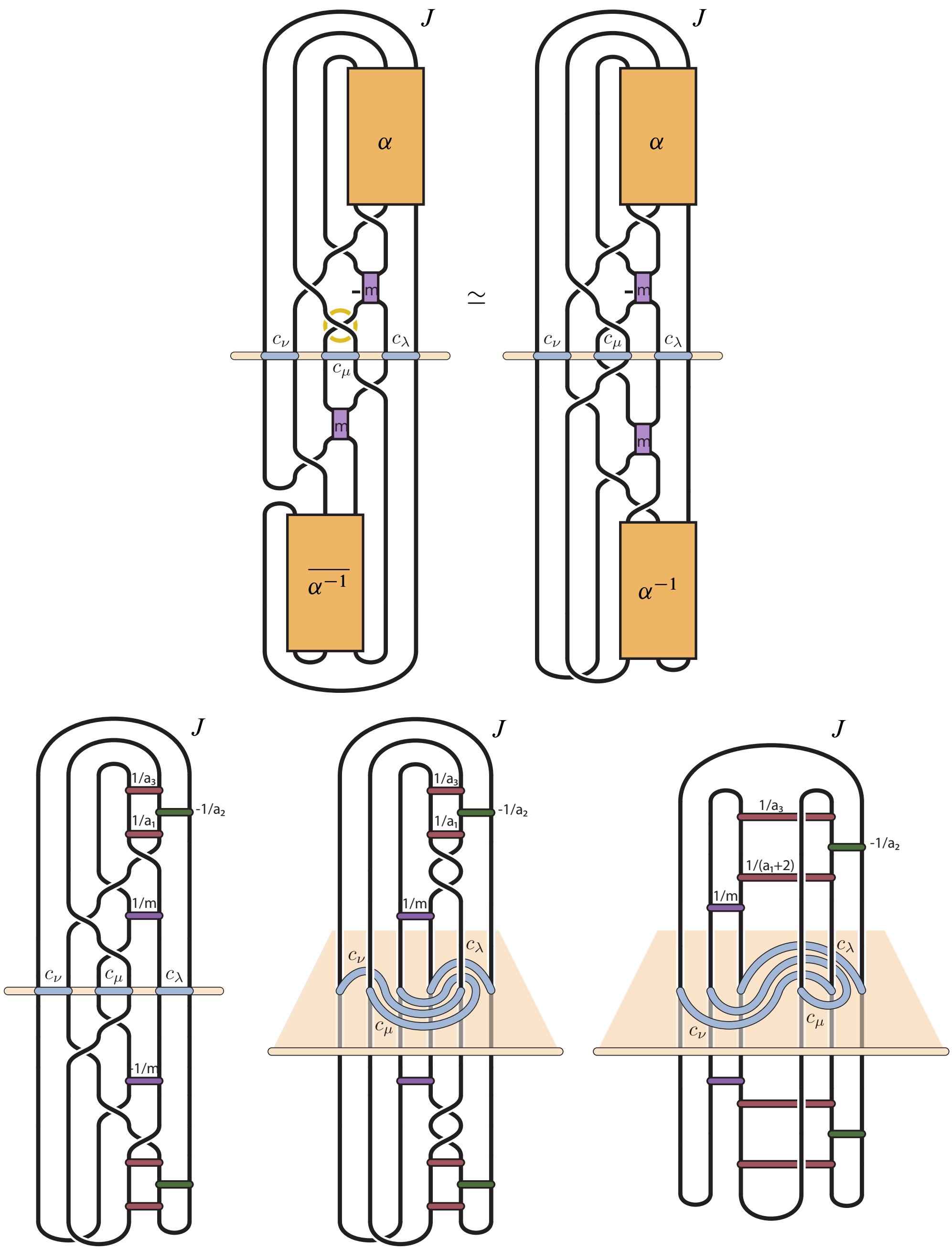


Figure 23: Isotopy, pairs of tangle replacements, and twisting transform the almost-alternating unknot $J^{\alpha,m}$ into a planar 3–bridge unknot where the arcs c_ν, c_μ, and c_λ are twisted along the bridge sphere.

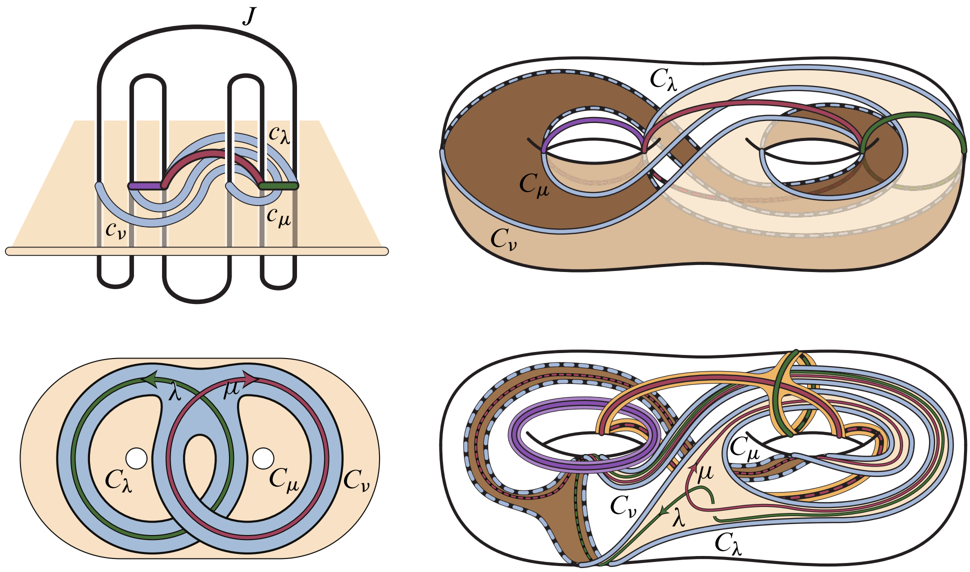
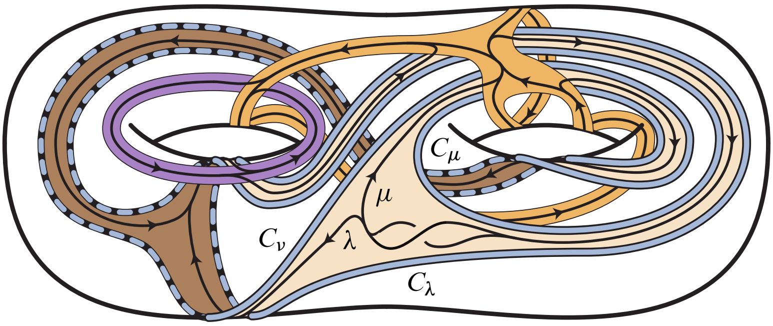


Figure 24: The double branched cover of the planar 3–bridge unknot gives a genus 2 Heegaard surface containing the lifts of the arcs in the bridge sphere (top, left and right). After an isotopy, the curves C_ν, C_μ, and C_λ more clearly bound the pair-of-pants P (bottom right). The basis of curves μ and λ in the once-punctured torus T are shown in P. Figure 2 (bottom left) is included for reference.

More generally, consider using $\alpha = \sigma_{\overline{n}}^{\epsilon_n a_n} \cdots \sigma_1^{-a_3}\sigma_2^{a_2}\sigma_1^{-a_1}$ for positive integers n and $a_1, \dots, a_n$ (ie α having the form of $(*)$ from the beginning of Section 9). Then, as one may check, the weights on the train track on the bottom of Figure 25 labeled x_3 and y_3 become x_n and y_n according to the following recursive formula for n equal to $2k$ or $2k+1$ with $k \geq 1$:

$$(**) \qquad \begin{cases} x_0 = 0, \\ y_0 = 0, \end{cases} \quad \begin{cases} x_1 = (a_1+2)(p+(m+1)(p+q)), \\ y_1 = 0, \end{cases} \quad \begin{cases} x_{2k} = x_{2k-1}, \\ y_{2k} = y_{2k-1} + a_{2k}(x_{2k-1} - 2p - q), \end{cases} \quad \begin{cases} x_{2k+1} = x_{2k} + a_{2k+1}(y_{2k} + p + (m+1)(p+q)), \\ y_{2k+1} = y_{2k}. \end{cases}$$

Note that the switching in the train track of Figure 25 requires that $x_n \geq m(p+q)$, which one easily checks; see the proof of Theorem 11.1.



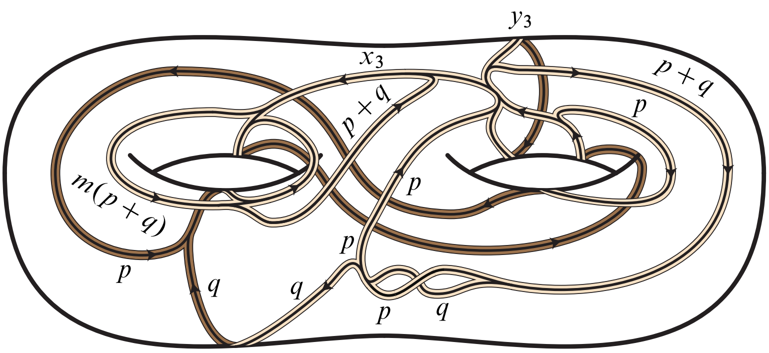


Figure 25: The p/q–lashing of P (for $p,q>0$) as a framed train track in the Heegaard surface. Here $m\ge 0$ and $\alpha=\sigma_1^{-a_3}\sigma_2^{a_2}\sigma_1^{-a_1}$, where $a_1,a_2,a_3\ge 0$ are such that $x_3=\big((a_3a_2(a_1+2)+a_3+a_1+2)(m+2)-2a_3a_2\big)p+\big((a_3a_2(a_1+2)+a_3+a_1+2)(m+1)-a_3a_2\big)q$ and such that $y_3=\big(a_2(a_1+2)(m+2)-2a_2\big)p+\big(a_2(a_1+2)(m+1)-a_2\big)q$.

11.3 Positive braids

Theorem 11.1 *Assume that p, q, m, $a_1,\dots,a_n$ are nonnegative integers for some positive integer n and that p and q are coprime. Then the p/q–lashings of $P^{\alpha,m}$ are positive braids. Furthermore, the p/q–lashing has genus*

$$g=1+(5+2m)p^2+(1+2m)q^2-2x_n+x_n^2+x_ny_n+(4+m-4x_n-mx_n-2y_n)p \\ +(1+m-x_n-mx_n-y_n)q+(4+4m)pq.$$

and an alternating surgery of integral slope

$$\lambda_{\mathrm{alt}}=(3+m)p^2+mq^2+x_n^2+x_ny_n+(-4x_n-mx_n-2y_n)p+(-x_n-mx_n-y_n)q \\ +(1+2m)pq.$$

Proof Assuming that $x_n \geq (m+2)(p+q)+2p$, Figure 26 shows a sequence of isotopies that transforms the framed train track of Figure 25 into one that carries closed positive braids.

The recursive formula $(**)$ implies that $\{x_i\}$ is an increasing sequence where $x_1 \geq (m+2)(p+q)+2p$, with equality only when $m = a_1 = 0$. Thus the isotopies of Figure 26 apply to show that the lashing is a closed positive braid. The resulting braided train track is shown again at the top of Figure 27.

Using this presentation as a closed positive braid, we can easily count the Euler characteristic of the fiber of the knot as the braid index minus the length of the positive braid word. From this one calculates that the lashing has the genus stated.

We can further flatten this presentation of the train track, as shown in the bottom of Figure 27, to have the blackboard framing while all crossings are still positive crossings. The slope of the framing then equals the crossing number of this diagram. One also computes it to be as stated. □

11.4 Small examples in S^3

Let $K(a_3, a_2, a_1, m, b_1)$ denote the $1/b_1$–lashings of $P^{\alpha,m}$ constructed here where $\alpha = \sigma_1^{-a_3}\sigma_2^{a_2}\sigma_1^{-a_1}$ for nonnegative integers a_3, a_2, a_1, m, and b_1. For Theorem 8.4 to apply, we need each of a_1, a_2, and a_3 to be positive, $m \geq 3$, and to instead take, for instance, the $(1+1/b_1)$–lashings where b_1 is "sufficiently large". (The b_1–lashings correspond to taking $(p', q') = (0, 1)$, and hence Theorem 8.4 does not apply.) In practice, we find by computer verification that these conditions may be relaxed to still produce many asymmetric hyperbolic L–space knots. Indeed, in general it appears considering the $1/b_1$–lashings with simply $b_1 > 0$ is suitable to find asymmetric hyperbolic L–space knots among the knots $K(a_3, a_2, a_1, m, b_1)$. Furthermore we may also use $a_3 = 0$ and $m = 1$ to produce examples.

For small parameter values, we use the surgery description above to input the knot into SnapPy, find its hyperbolic structure, and calculate its hyperbolic invariants.[2] These calculations are presented in Table 1. We calculate the knot genus and slope of the alternating surgery using the formulae in Theorem 11.1. (Using SnapPy within SageMath, we can also determine the genus of the lashing K from its Alexander

[2] In the table, #tetr. is the number of tetrahedra reported after using the `canonize()` function. We have observed occurrences where this has increased the number of tetrahedra.

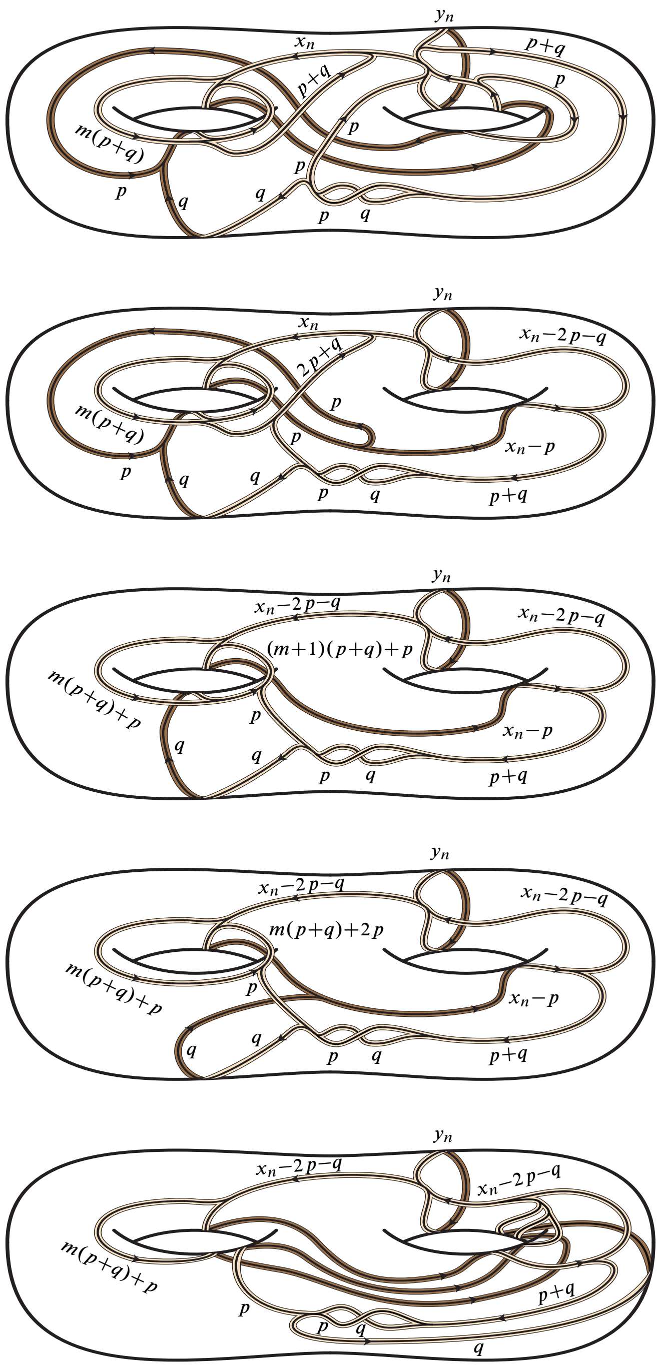


Figure 26a: A sequence of isotopies (continued in Figure 26b) shows the framed train track of Figure 25 carries closed positive braids.

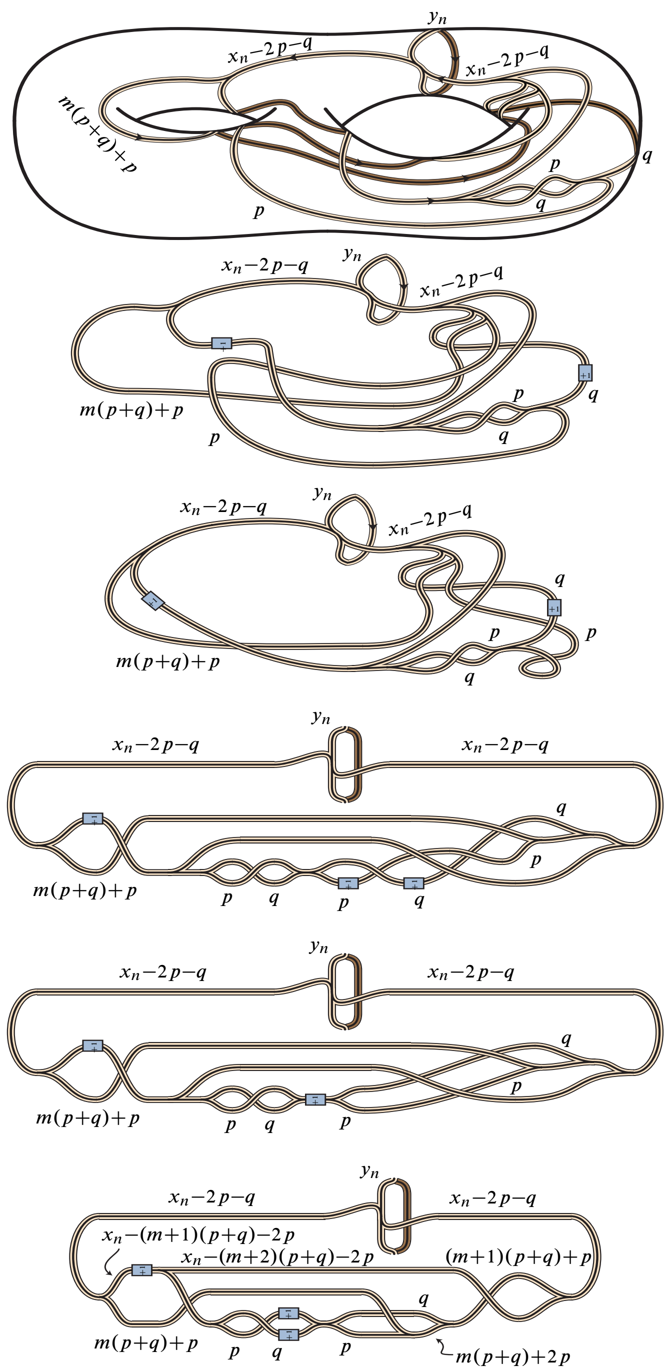


Figure 26b: A sequence of isotopies (continued from Figure 26a) shows the framed train track of Figure 25 carries closed positive braids.

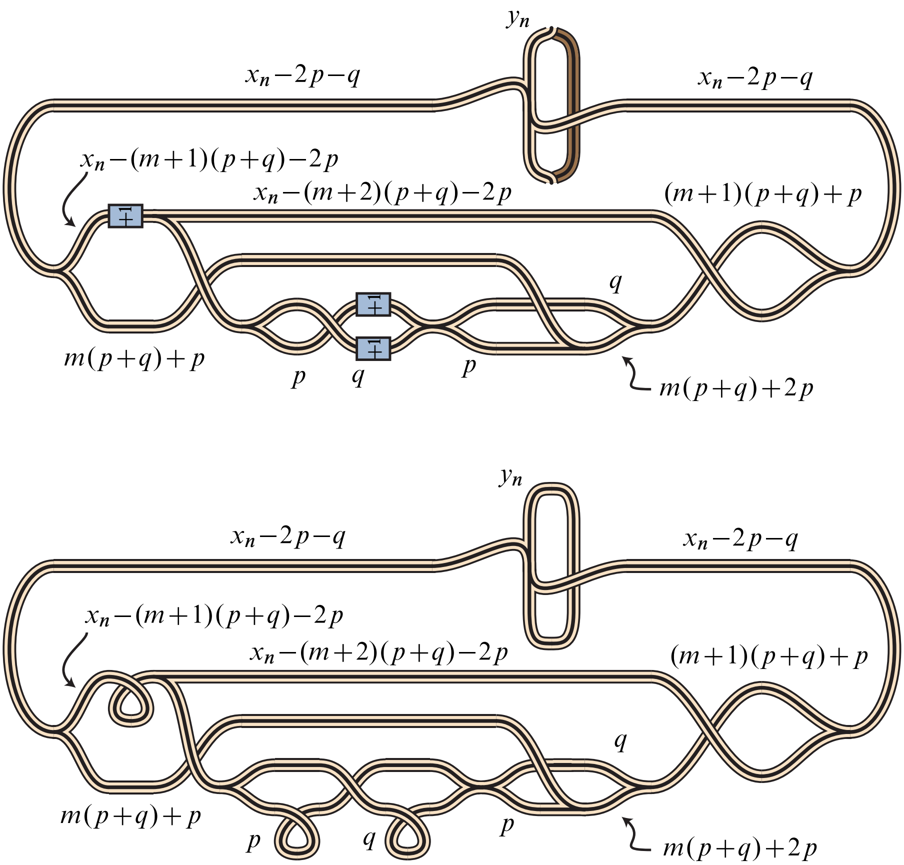


Figure 27: The final braided train track (top) of Figure 26b along with a flattening of its twists (bottom).

polynomial — being an L–space knot, K is fibered. The calculation of homology of the alternating surgery given at the end of Section 11.1 also determines the surgery slope of the alternating surgery.)

Letting $J = J^{\alpha,m}(1, -b_1/(1+b_1), -1/(1+b_1))$ be the associated alternating link, by doing $(2,0)$ orbifold filling on all components of J followed by taking the 2–fold cover, we should obtain the result of the alternating surgery on K. Indeed, in each of these examples SnapPy (with some help from Regina) finds a hyperbolic structure on the resulting 2–fold cover and identifies it as isometric to the alternating surgery on K.

We now explicitly present the examples $K(0,1,1,1,1)$ and $K(1,1,1,1,1)$ as closures of positive braids using the train track of Figure 26.

parameters					top data		hyperbolic data			
a_3	a_2	a_1	m	b_1	genus	alt. surg.	Sym	volume	#tetr.	cusp shape
0	1	1	1	1	119	$\mathbb{Z}/272$	$\mathbb{1}$	10.20098	13	$0.41433+1.19820i$
1	1	0	1	1	214	$\mathbb{Z}/471$	$\mathbb{1}$	14.76163	21	$0.34127+1.50327i$
0	1	1	1	2	253	$\mathbb{Z}/555$	$\mathbb{1}$	12.39382	16	$0.47070+1.06723i$
0	1	1	2	1	269	$\mathbb{Z}/588$	$\mathbb{1}$	11.26105	15	$0.28229+1.21171i$
1	1	0	2	1	501	$\mathbb{Z}/1067$	$\mathbb{1}$	17.24796	22	$0.28244+1.43145i$
1	1	1	1	1	544	$\mathbb{Z}/1156$	$\mathbb{1}$	16.05972	22	$0.36009+1.53081i$
0	1	1	2	2	583	$\mathbb{Z}/1239$	$\mathbb{1}$	13.59287	20	$0.38818+1.09450i$
1	1	1	1	2	1117	$\mathbb{Z}/2331$	$\mathbb{1}$	18.24257	22	$0.47741+1.38316i$
0	0	1	2	2	258	$\mathbb{Z}/563$	$\mathbb{Z}/2$	9.12009	17	$0.07382+1.16144i$
1	1	1	0	2	274	$\mathbb{Z}/597$	$\mathbb{Z}/2$	7.47528	13	$0.31371+0.90262i$

Table 1: Examples of hyperbolic L–space knots $K(a_3, a_2, a_1, m, b_1)$ in S^3 with alternating surgeries.

The knot $K(0,1,1,1,1)$ (where $p = q = 1$, $m = 1$, and $a_1 = a_2 = 1$) is the closure of the positive 12–strand braid of length 249 shown in Figure 1.

The knot $K(1,1,1,1,1)$ (where $p = q = 1$, $m = 1$, and $a_1 = a_2 = a_3 = 1$) is the closure of the positive 29–strand braid of length 1116 shown in Figure 28.

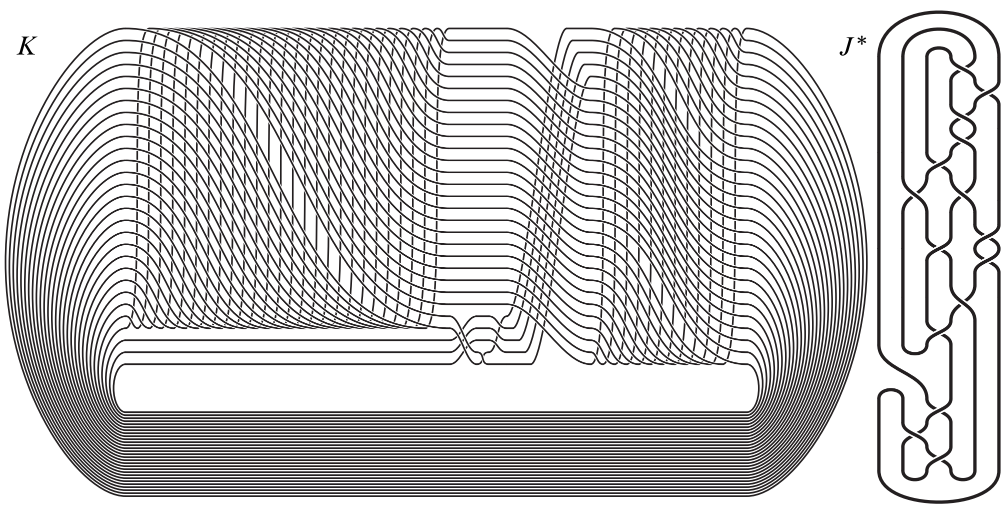


Figure 28: The knot $K = K(1,1,1,1,1)$ (left) is the closure of the 29–strand positive braid shown. It is an asymmetric hyperbolic L–space knot. The result of $\frac{1156}{1}$–surgery on the knot K produces the double branched cover of the 15–crossing alternating link J^* (right).

11.5 Strongly invertible examples and bandings of unknots

Table 1 shows that the two small examples $K(1,1,1,0,2)$ and $K(0,0,1,2,2)$ of our lashings are strongly invertible hyperbolic L–space knots in S^3 with alternating surgeries. Using the surgery description of these knots, and simplifications given by SnapPy, one can write the quotient of these surgeries as a band surgery to the unknot of the corresponding alternating knots. In Figures 29 and 30 we show the alternating knots $J^*(1,1,1,0,2)$ and $J^*(0,0,1,2,2)$ whose double branched covers are obtained by the framed surgeries on $K(1,1,1,0,2)$ and $K(0,0,1,2,2)$. In each of these alternating diagrams a blackboard framed arc is also shown followed by a banding along the arc that produces an almost-alternating diagram of the unknot. Observe that a flype of the alternating diagram $J^*(0,0,1,2,2)$ to another alternating diagram of the knot is needed so that the banding produces an almost-alternating diagram. As shown, a further flype in fact allows the arc to be isotopic to a proper arc in a region of the alternating diagram, and the banding is dual to a smoothing of the dealternation crossing in an almost-alternating unknot diagram. We propose the following modification of Conjecture 1.5.

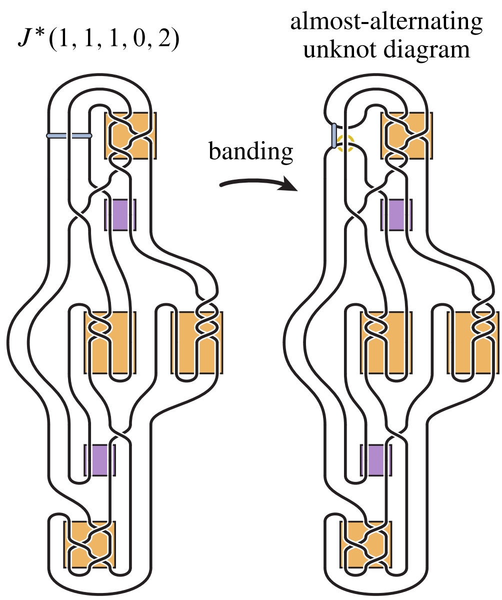


Figure 29: The alternating knot $J^*(1,1,1,0,2)$ admits a banding along an arc that crosses an alternating diagram once to produce an almost-alternating diagram of the unknot.

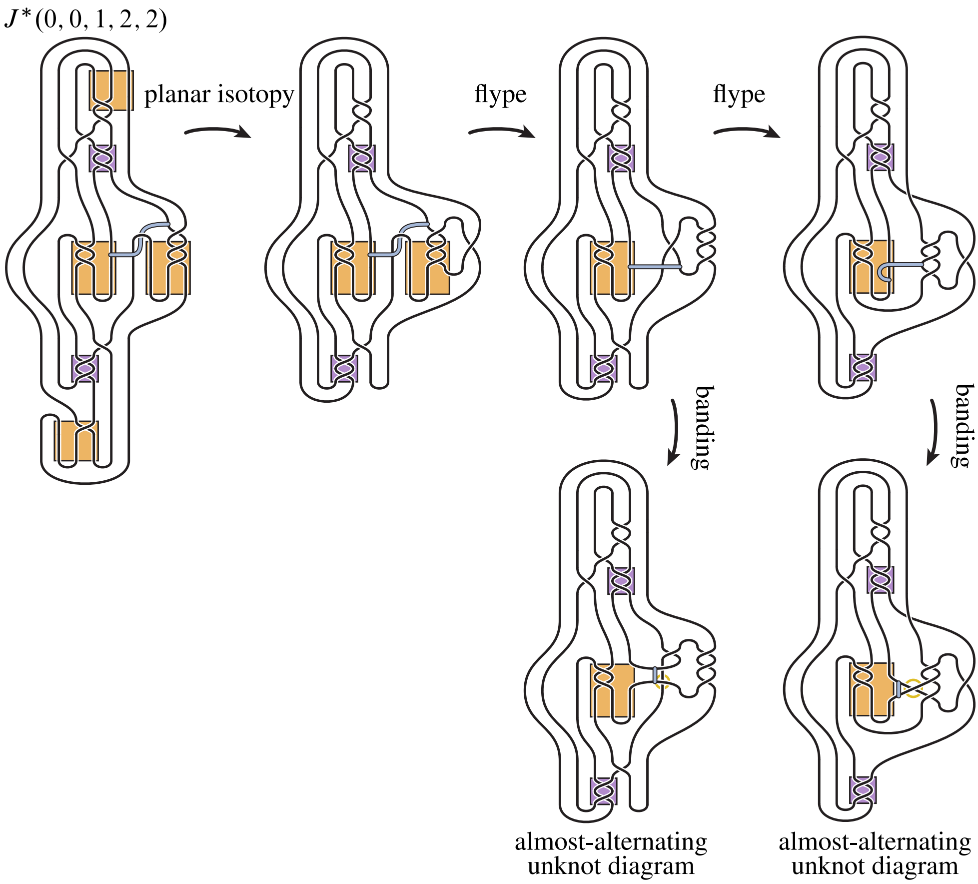


Figure 30: The alternating knot $J^*(0,0,1,2,2)$, after planar isotopy and a flype, admits a banding along a flat arc crossing an alternating diagram once to produce an almost-alternating diagram of the unknot. After a further flype, this banding is dual to the smoothing of the dealternation crossing in an almost-alternating diagram of the unknot.

Conjecture 11.2 *If a banding of the unknot produces an alternating link, then there is an almost-alternating diagram of the unknot in which the banding is either*

(1) *a smoothing of the dealternation crossing, or*

(2) *a flat banding from the dealternation crossing to an adjacent crossing* (see Figure 31, left).

Observe that if the first case of Conjecture 11.2 occurs (see Figure 31, right), then the other smoothing of the dealternation crossing is obtained by a banding along the same

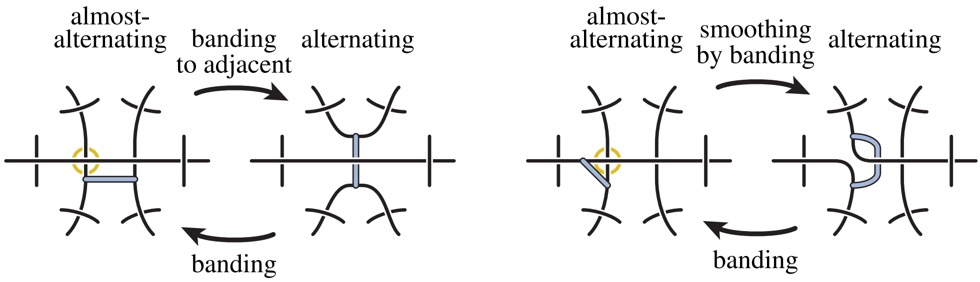


Figure 31: In an almost-alternating diagram, a banding of the dealternation crossing to an adjacent crossing produces an alternating diagram (left). The dual blackboard framed arc in the alternating diagram crosses the diagram once. In an almost-alternating diagram, a smoothing of the dealternation crossing by a banding is shown, though the intermediate removal of a nugatory crossing is not shown (right). The dual blackboard framed arc in the alternating diagram is shown.

core arc but with a framing that is twisted by half. In particular, via the Montesinos trick, this case corresponds to a knot in S^3 with two consecutive integral alternating surgeries. Indeed this knot will have infinitely many alternating surgeries corresponding to alternating rational tangle replacements of the dealternation crossing. However in the second case it is not diagramatically apparent that the knot arising from the Montesinos trick would have any other alternating surgeries.

The links resulting from the two bandings of $J^*(1,1,1,0,2)$ that differ from the unknotting banding of Figure 29 by a half-twist both have Jones polynomials (as one may calculate) with a span of 15. Thus if either were alternating, then its crossing number would be 15 [24]. However both of these links appear to have crossing number 18. We do not know if either of these links is alternating. Crowell's condition [15] that (reduced) Alexander polynomials of alternating links are alternating gives no obstruction. Furthermore, since for each of the two links the two smoothings of the crossing arising from the half-twisted band produce the unknot and the alternating knot $J^*(1,1,1,0,2)$ of determinant 597, the link with determinant 598 is necessarily *quasi-alternating*; see [35]. We do not know if the one with determinant 596 is quasi-alternating.

11.6 Small examples in $S^1 \times S^2$

Let $\alpha = \sigma_1^{-a_3}\sigma_2^{a_2}\sigma_1^{-a_1}$ and $\omega = \sigma_1^{a_1}\sigma_2^{-a_2}\sigma_1^{a_3+1}$. Then for any nonnegative integers a_1, a_2, a_3, and m, $J^{\alpha,\omega,m}$ is an almost-alternating diagram of the 2–component unlink. Therefore, for nonnegative slopes p/q, the p/q–lashing of $P^{\alpha,\omega,m}$ is a

parameters					top data		hyperbolic data			
a_3	a_2	a_1	m	b_1	braid index	alt. surg.	Sym	volume	#tetr.	cusp shape
0	1	0	1	1	16	$\mathbb{Z}/256$	$\mathbb{1}$	10.76121	13	$0.38759+1.27100i$
0	2	0	1	1	23	$\mathbb{Z}/529$	$\mathbb{1}$	12.58840	17	$0.42497+1.42566i$
0	1	0	1	2	23	$\mathbb{Z}/23+\mathbb{Z}/23$	$\mathbb{1}$	12.96886	17	$0.44340+1.15926i$
0	1	1	1	1	26	$\mathbb{Z}/2+\mathbb{Z}/338$	$\mathbb{1}$	11.29252	15	$0.38693+1.26481i$
1	1	0	1	1	28	$\mathbb{Z}/784$	$\mathbb{1}$	15.20006	23	$0.35616+1.54654i$
1	1	0	1	2	40	$\mathbb{Z}/2+\mathbb{Z}/800$	$\mathbb{1}$	17.47078	23	$0.46933+1.38764i$

Table 2: Examples of hyperbolic L–space knots $K'(a_3, a_2, a_1, m, b_1)$ in $S^1 \times S^2$ with alternating surgeries.

knot in $S^1 \times S^2$ with an alternating surgery. Let $K'(a_3, a_2, a_1, m, b_1)$ denote the $1/b_1$–lashing in $S^1 \times S^2$. Using that $\omega = \alpha^{-1}\sigma_1$, one is able to readily adapt the surgery description obtained in Section 11.1 to produce a surgery description of the knots $K'(a_3, a_2, a_1, m, b_1)$: change the surgery coefficient of the largest red circle in Figure 22 from $[0, a_1, -a_2, a_3]$ to $[0, a_1, -a_2, a_3 + 1]$. Table 2 collects the results of calculations using SnapPy of basic hyperbolic data of some of these knots and the homology of their alternating surgery. According to [32], these knots in $S^1 \times S^2$ are spherical braids, and thus they may be isotoped to be transverse to the S^2 fibers. So in lieu of the knot genus, we also include the braid index in Table 2. This is quickly calculated since the square of the winding number (with respect to the S^1 factor of $S^1 \times S^2$) of a knot in $S^1 \times S^2$ that is not null-homologous is the order of the first homology of its framed surgeries.

The knot $K'(0, 1, 0, 1, 1)$ has surgery to the double branched cover of an alternating link which SnapPy identifies as the 3–bridge link $L12a1091$.

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