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Knot Floer homology and the unknotting number

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Given a knot $K \subset S^3$, let $u^-(K)$ (respectively, $u^+(K)$) denote the minimum number of negative (respectively, positive) crossing changes among all unknotting sequences for K. We use knot Floer homology to construct the invariants $\mathfrak{l}^-(K)$, $\mathfrak{l}^+(K)$ and $\mathfrak{l}(K)$, which give lower bounds on $u^-(K)$, $u^+(K)$ and the unknotting number $u(K)$, respectively. The invariant $\mathfrak{l}(K)$ only vanishes for the unknot, and satisfies $\mathfrak{l}(K) \geq \nu^+(K)$, while the difference $\mathfrak{l}(K) - \nu^+(K)$ can be arbitrarily large. We also present several applications towards bounding the unknotting number, the alteration number and the Gordian distance.

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1 Introduction

Given a knot $K \subset S^3$, by an *unknotting sequence* for K we mean a sequence of crossing changes for K which results in the unknot. The minimum length of an unknotting sequence for K is called the *unknotting number* of K and is denoted by $u(K)$. Let $u^-(K)$ denote the minimum number of negative crossing changes (ie changes of a negative crossing to a positive crossing) among all unknotting sequences for K and $u^+(K)$ denote the minimum number of positive crossing changes among all such sequences. It is then clear that $u(K) \geq u^+(K) + u^-(K)$, while the equality is not necessarily satisfied. The unknotting number is one of the simplest, yet most mysterious and intractable invariants of knots in S^3. The answer to several simple questions about the unknotting number is still not known. In particular, the following question is widely open:

Question 1.1 *If K and L are knots in S^3, is it true that $u(K \# L) = u(K) + u(L)$? How about the (weaker) inequality $u(K \# L) \geq \max\{u(K), u(L)\}$?*

Scharlemann proved that composite knots have unknotting number at least 2 [32]. However, no matter how large $u(K)$ and $u(L)$ are, it is not known in general whether $u(K \# L) \geq 3$; see Lackenby [17].

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Another example is Milnor's question about the unknotting number of the torus knot $T_{p,q}$, which remained open for a long time, until Kronheimer and Mrowka gave a positive answer to it using gauge theory [16]. Later, Ozsváth and Szabó reproved it using their invariant $\tau(K)$ [23] and Rasmussen gave a purely combinatorial proof by introducing his invariant $s(K)$ [31]. Both $|\tau(K)|$ and $\frac{1}{2}|s(K)|$, as well as classical lower bounds for the unknotting number coming from Levine–Tristram signatures [18; 33], are in fact lower bounds for the 4–ball genus $g_4(K)$. Since $g_4(K) \leq u(K)$, they also give lower bounds for the unknotting number. Nevertheless, lower bounds for $u(K)$ constructed by bounding the 4–ball genus fail to give effective data for many classes of knots. In particular, if $-K$ denotes the mirror image of the knot K, the knot $L = K \# -K$ is always slice and $\tau(L) = s(L) = 0$. It is thus interesting to construct lower bounds for $u(K)$, which do not come from bounds on $g_4(K)$. In this paper, we use knot Floer homology to construct the invariants $\mathfrak{l}^+(K)$, $\mathfrak{l}^-(K) = \mathfrak{l}^+(-K)$ and $\mathfrak{l}(K)$ associated with a knot $K \subset S^3$ and prove the following theorem:

Theorem 1.1 *For every knot $K \subset S^3$ we have:*

- $\mathfrak{l}^+(K) \leq u^+(K)$, $\mathfrak{l}^-(K) \leq u^-(K)$ *and* $\mathfrak{l}(K) \leq u(K)$.
- $\mathfrak{l}^+(K) \geq \nu^+(K) \geq \tau(K)$ *and* $\mathfrak{l}^-(K) \geq \nu^+(-K) \geq -\tau(K)$. *Therefore, for every* $0 \leq t \leq 1$ *we have* $t\mathfrak{l}^-(K) \geq \Upsilon_K(t) \geq -t\mathfrak{l}^+(K)$.
- $\mathfrak{l}(K) \geq \hat{\mathfrak{t}}(K)$, *where* $\hat{\mathfrak{t}}(K)$ *is the maximum order of* U*–torsion in* $\mathrm{HFK}^-(K)$.

Unlike most other lower bounds for the unknotting number, the torsion invariant $\hat{\mathfrak{t}}$ *resists* the connected sum operation; specifically:

Corollary 1.2 *If K and K' are knots in S^3 then*

$$u(K \# K') \geq \hat{\mathfrak{t}}(K \# K') = \max\{\hat{\mathfrak{t}}(K), \hat{\mathfrak{t}}(K')\}.$$

In particular, for the torus knot $T_{p,q}$ with $0 < p < q$, $\hat{\mathfrak{t}}(T_{p,q}) = p-1$ and, for every knot $K \subset S^3$,

$$u(K \# T_{p,q}) \geq p-1.$$

Therefore, for every coprime $0 < p < q$, $\mathfrak{l}(-T_{p,q} \# T_{p,q}) \geq p-1$, while the lower bounds ν^+, $|\tau|$ and $\frac{1}{2}|s|$ vanish, because $-T_{p,q} \# T_{p,q}$ is slice.

Theorem 1.1 naturally reproves the following corollary:

Corollary 1.3 *For every knot $K \subset S^3$, $\nu^+(K)$ is a lower bound for $u^+(K)$, while $\nu^+(-K)$ is a lower bound for $u^-(K)$. In particular, $u^+(T_{p,q}) = \frac{1}{2}(p-1)(q-1)$.*

Associated with a knot $K \subset S^3$, one can construct a Heegaard Floer chain complex $\mathrm{CF}(K)$, which is freely generated over $\mathbb{A} = \mathbb{F}[\mathsf{u},\mathsf{w}]$ by the intersection points associated with a Heegaard diagram for K. $\mathrm{CF}(K)$ is equipped with differential d, which is an $\mathbb{A}$–homomorphism defined by counting holomorphic disks; see Alishahi and Eftekhary [3]. Let $\mathbb{H}(K)$ denote the homology of $(\mathrm{CF}(K), d)$, which is again a module over $\mathbb{A}$. Let $\mathbb{T}(K)$ denote the torsion submodule of $\mathbb{H}(K)$, ie $\mathbb{T}(K)$ consists of $x \in \mathbb{H}(K)$ such that there exists a nonzero $a \in \mathbb{A}$ with $a \cdot x = 0$. Then, $\mathbb{H}(K)$ sits in a short exact sequence

$$0 \to \mathbb{T}(K) \to \mathbb{H}(K) \to \mathbb{A}(K) \to 0,$$

where the torsion-free part $\mathbb{A}(K)$ of the homology is isomorphic to an ideal in $\mathbb{A}$. Specifically, for every knot K, there is an *ideal sequence*

$$\imath(K) = (i_0 = 0 < i_1 < \cdots < i_n = \nu^+(K))$$

of some length $n = n(K)$ and a canonical identification

$$\mathbb{A}(K) = \langle \mathsf{u}^{i_k} \mathsf{w}^{i_{n-k}} \mid k = 0, 1, \dots, n \rangle_{\mathbb{A}} \le \mathbb{A}.$$

We define $\mathfrak{t}(K)$ as the smallest integer m such that w^m acts trivially on $\mathbb{T}(K)$ (ie maps $\mathbb{T}(K)$ to zero). For the unknot U, we have $\mathbb{T}(U) = 0$ and $\mathbb{A}(U) = \mathbb{A}$.

If K' is obtained from K by a sequence of m positive crossing changes and n negative crossing changes, we use the cobordism maps we constructed in [4] to show that $\mathsf{w}^n \mathbb{A}(K) \subset \mathbb{A}(K')$ and $\mathsf{w}^m \mathbb{A}(K') \subset \mathbb{A}(K)$, while $\mathsf{w}^{m+n} \mathbb{T}(K)$ may be embedded in $\mathbb{T}(K')$. This observation implies, in particular, that $\nu^+(K)$ is a lower bound for $u^+(K)$ and that $\mathfrak{t}(K)$ is lower bound for $u(K)$.

The above construction also gives lower bounds on the *Gordian distance* $u(K, K')$ from a knot K to another knot K', ie the minimum number of crossing changes required to get from K to K'. In particular, we give the following three lower bounds on the *alternation number* $\mathrm{alt}(K)$, defined as the least Gordian distance of an alternating knot from K:

Proposition 1.4 *The alternation number $\mathrm{alt}(K)$ of a knot $K \subset S^3$ satisfies the inequalities*

$$\mathrm{alt}(K) \ge \nu^+(K) - \mathfrak{a}(K), \quad \mathrm{alt}(K) \ge \hat{\mathfrak{t}}(K) - 1 \quad \textit{and} \quad \mathrm{alt}(K) \ge \min\{\mathfrak{t}^+(K) - 1, \nu^+(K)\},$$

where $\mathfrak{a}(K)$ is the minimum degree of a monomial in $\mathbb{A}(K)$.

In particular, as a corollary of this proposition we show that

$$\operatorname{alt}(T_{p,pn+1}) \geq n\left\lfloor \tfrac{1}{4}(p-1)^2 \right\rfloor,$$

which improves Theorem 1.3 of Jin, Lowrance, Polston and Zheng [14]. Specifically, in [14] the authors give the same lower bound for the dealternating number, which is bounded below by the alternation number.

A similar strategy is used by the first author in [1] to construct lower bounds for the unknotting number from Khovanov homology. The resulting invariants are used by Alishahi and Dowlin [2] to prove the *knight move conjecture* for knots with unknotting number at most 2.

In Section 2 we study the cobordism maps induced on knot chain complexes associated with a crossing change. These cobordism maps are used in Section 3 to construct lower bounds on the Gordian distance of knots, while simpler obstructions to the unknotting are extracted from these lower bounds in Section 4. We discuss several examples and applications in Section 5.

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2 Changing the crossings in knot diagrams

By a *crossing change* for an oriented knot (or link) $K \subset S^3$ we mean replacing a ball in S^3 in which K looks like a positive crossing to the ball in which K looks like a negative crossing (a positive crossing change), or the reverse of the above operation (a negative crossing change). Figure 1 illustrates how a band surgery on K can be used to do either of the following two changes (or their reverse):

- A positive crossing change and adding a positively oriented meridian for K as a new link component.
- A negative crossing change and adding a negatively oriented meridian for K as a new link component.

Let us assume that K' is obtained from K by a positive crossing change and that L is obtained from K' by adding a positively oriented meridian. As illustrated in Figure 1, one may then place a pair of markings p_1 and p_2 on K, and distinguish a band $\mathbb{I}$ with

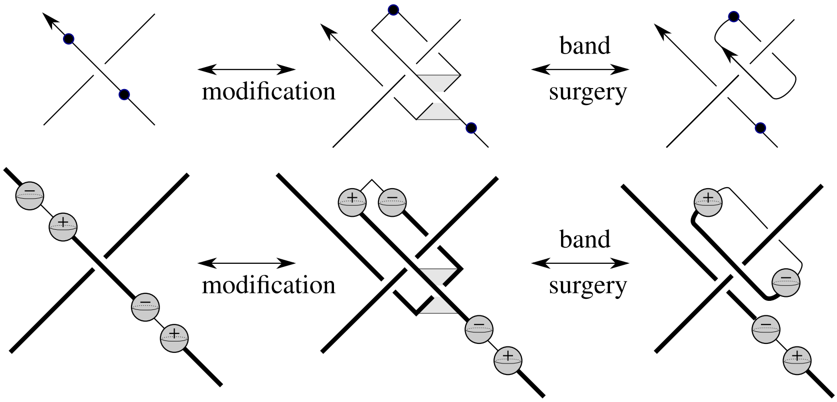

Figure 1: We may change a crossing in the expense of adding a meridian. The meridian can be positively or negatively oriented depending on whether the initial crossing is positive or negative, respectively.

endpoints on $K \setminus \{p_1, p_2\}$, such that the band surgery on $\mathbb{I}$ gives L, while p_1 lands on K' and p_2 lands on the positively oriented meridian.

Associated with the pointed knot (K, p_1, p_2), we may construct a tangle (equivalently, a sutured manifold) as follows. Fix an orientation on K and consider two disjoint small arcs on K which contain p_1 and p_2, respectively. Remove a small ball around each one of the four ends of these arcs to obtain a 3–manifold M with four sphere boundary components. Using the orientation on K we may orient these spheres so that two of them form $\partial^+ M$ and the other two form $\partial^- M$, as in Figure 1. Let T_1 and T_2 denote the remaining part of the two arcs around p_1 and p_2, respectively, which are now strands in M connecting $\partial^+ M$ to $\partial^- M$. The complement of the two arcs in K gives two other strands T_3 and T_4 which connect $\partial^- M$ to $\partial^+ M$. The 3–manifold M and the strands T_1, T_2, T_3 and T_4 then form a tangle associated with (K, p_1, p_2) (see [4]). Correspondingly, we also obtain a sutured manifold, which is constructed by removing a solid cylinder around each one of the strands and considering the boundary of these four solid cylinders as the set of sutures on the resulting 3–manifold. The construction of authors in [3], as well as the special case considered in [3, Section 8.2], may be used to associate a chain complex $\mathrm{CF}(K, p_1, p_2)$ with this tangle (or sutured manifold), which is a module over $\mathbb{A}' = \mathbb{F}[\mathsf{u}, \mathsf{v}, \mathsf{w}]$. The variables u and v are associated with the strands T_1 and T_2 (equivalently, with p_1 and p_2), while the variable w is associated with T_3 and T_4 (equivalently, with $K \setminus \{p_1, p_2\}$). Similarly, we can associate a chain complex $\mathrm{CF}(L, p_1, p_2)$ with the pointed link (L, p_1, p_2), which is

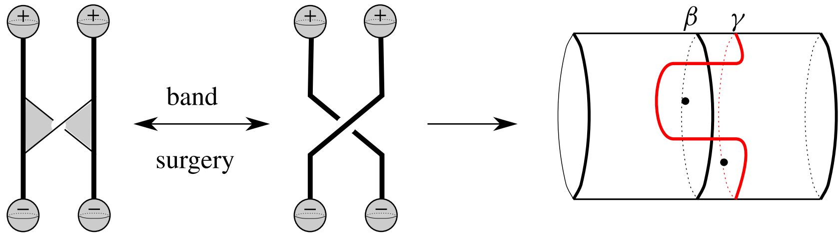

Figure 2: On the right are the two special β and γ curves corresponding to the band surgery on the Heegaard diagram subordinate to the band.

again a module over $\mathbb{A}'$. The generators of the two complexes all correspond to the unique Spin^c structure $\mathfrak{s}_0$ on S^3, which will be dropped from the notation.

Associated with the band $\mathbb{I}$, the construction of [4] gives the $\mathbb{A}'$–cobordism maps

$$\mathfrak{g}^-\colon \mathrm{CF}(K, p_1, p_2) \to \mathrm{CF}(L, p_1, p_2) \quad \text{and} \quad \mathfrak{g}^+\colon \mathrm{CF}(L, p_1, p_2) \to \mathrm{CF}(K, p_1, p_2).$$

Moreover, $\mathfrak{g}^+ \circ \mathfrak{g}^-$ (resp. $\mathfrak{g}^- \circ \mathfrak{g}^+$) corresponds to the decorated cobordism from (K, p_1, p_2) (resp. (L, p_1, p_2)) to itself obtained from the product cobordism by adding a tube with feet on $T_3 \times [0,1]$ and $T_4 \times [0,1]$.

Lemma 2.1 *With the above notation fixed, the maps*

$$\mathfrak{g}^+ \circ \mathfrak{g}^-\colon \mathrm{CF}(K, p_1, p_2) \to \mathrm{CF}(K, p_1, p_2)$$

and

$$\mathfrak{g}^- \circ \mathfrak{g}^+\colon \mathrm{CF}(L, p_1, p_2) \to \mathrm{CF}(L, p_1, p_2)$$

are both multiplication by w*, up to chain homotopy.*

Proof For defining $\mathfrak{g}^+$ we may use a Heegaard triple

$$(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{z} = \{z_1, z_2, z_3, z_4\})$$

subordinate to the band $\mathbb{I}$, where z_i corresponds to the strand T_i. See Figure 2 for the special β and γ curves corresponding to a band $\mathbb{I}$ in a Heegaard triple subordinate to it. The corresponding $\mathbb{A}'$–coloring maps z_1 to u and z_2 to v, while z_3 and z_4 are mapped to w. If $\boldsymbol{\delta}$ is obtained by a small Hamiltonian isotopy from $\boldsymbol{\beta}$ which does not cross $\boldsymbol{z}$, then $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma}, \boldsymbol{\delta}, \boldsymbol{z})$ is subordinate to $\bar{\mathbb{I}}$, the reverse band surgery. Associated with the Heegaard quadruple $H = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}, \boldsymbol{z})$ we obtain

- the top generators $\Theta_{\beta\gamma}, \Theta_{\gamma\delta}$ and $\Theta_{\beta\delta}$ in $\mathbb{T}_\beta \cap \mathbb{T}_\gamma$, $\mathbb{T}_\gamma \cap \mathbb{T}_\delta$ and $\mathbb{T}_\beta \cap \mathbb{T}_\delta$, respectively;

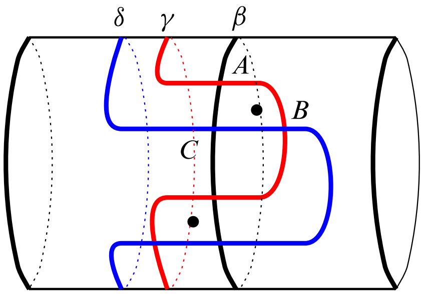

Figure 3: If $\boldsymbol{\delta}$ is obtained from $\boldsymbol{\beta}$ by a Hamiltonian isotopy supported away from the marked points, the domain of the distinguished triangle class in $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}, \boldsymbol{z})$ connecting $\Theta_{\beta\gamma}, \Theta_{\gamma\delta}$ and $\Theta_{\beta\delta}$ will contain one of the marked points corresponding to the strands connected by $\mathbb{I}$. The intersection of the domain of the triangle with the surface is the small triangle connecting A, B and C.

- the triangle maps $\mathfrak{f}_{\alpha\beta\gamma}$, $\mathfrak{f}_{\alpha\gamma\delta}$, $\mathfrak{f}_{\alpha\beta\delta}$ and $\mathfrak{f}_{\beta\gamma\delta}$, which are associated with the triples $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$, $(\boldsymbol{\alpha}, \boldsymbol{\gamma}, \boldsymbol{\delta})$, $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta})$ and $(\boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta})$, respectively, and the induced maps $\mathfrak{g}^- = \mathfrak{f}_{\alpha\beta\gamma}(- \otimes \Theta_{\beta\gamma})$ and $\mathfrak{g}^+ = \mathfrak{f}_{\alpha\gamma\delta}(- \otimes \Theta_{\gamma\delta})$; and
- the holomorphic square map $\mathfrak{S}$ which satisfies

$$d \circ \mathfrak{S} + \mathfrak{S} \circ d = \mathfrak{g}^+ \circ \mathfrak{g}^- + \mathfrak{f}_{\alpha\beta\delta}(- \otimes \mathfrak{f}_{\beta\gamma\delta}(\Theta_{\beta\gamma} \otimes \Theta_{\gamma\delta})).$$

The position of the curves in $\boldsymbol{\beta} \cup \boldsymbol{\gamma} \cup \boldsymbol{\delta}$, which is basically illustrated in Figure 3, implies that

$$\mathfrak{f}_{\beta\gamma\delta}(\Theta_{\beta\gamma} \otimes \Theta_{\gamma\delta}) = \mathsf{w}\Theta_{\beta\delta}.$$

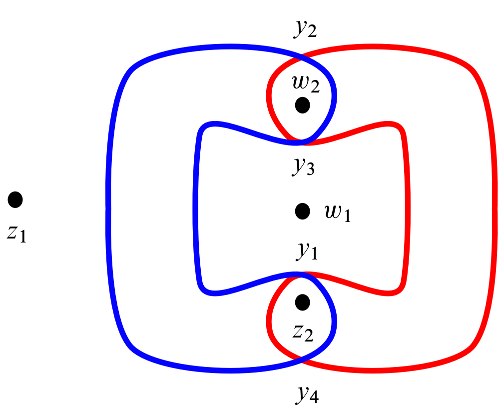

Figure 4: Heegaard diagram for the Hopf link, where the Heegaard surface is S^2, the red curve is an α curve and the blue curve is a β curve.

Since $\mathfrak{f}_{\alpha\beta\delta}(-\otimes\Theta_{\beta\delta})$ gives a map chain homotopic to the identity on $\mathrm{CF}(L,p_1,p_2)$, the above observation completes the proof for the composition $\mathfrak{g}^+\circ\mathfrak{g}^-$. A similar argument implies that $\mathfrak{g}^-\circ\mathfrak{g}^+$ is chain homotopic to multiplication by w. □

Removing p_2 from K, we obtain a knot with a single marked point on it. Correspondingly, we find a tangle with two strands and the standard knot chain complex $\mathrm{CF}(K)$ for K, which is a module over $\mathbb{A}=\mathbb{F}[\mathsf{u},\mathsf{w}]$. Similarly, there is a single marked point on K', and associated with it we obtain the chain complex $\mathrm{CF}(K')$, which is again an $\mathbb{A}$–module.

Lemma 2.2 *The chain homotopy types of* $\mathrm{CF}(K,p_1,p_2)$ *and* $\mathrm{CF}(L,p_1,p_2)$ *can be described in terms of* $\mathrm{CF}(K)$ *and* $\mathrm{CF}(K')$ *as the mapping cones*

$$\mathrm{CF}(K,p_1,p_2)\simeq(\mathrm{CF}(K)\otimes_{\mathbb{A}}\mathbb{A}'\xrightarrow{\mathsf{u}+\mathsf{v}}\mathrm{CF}(K)\otimes_{\mathbb{A}}\mathbb{A}')$$

and

$$\mathrm{CF}(L,p_1,p_2)\simeq\Big((\mathrm{CF}(K')\oplus\mathrm{CF}(K'))\otimes_{\mathbb{A}}\mathbb{A}'\xrightarrow{\left[\begin{smallmatrix}0&\mathsf{w}\\ \mathsf{u}+\mathsf{v}&\mathsf{u}\end{smallmatrix}\right]}(\mathrm{CF}(K')\oplus\mathrm{CF}(K'))\otimes_{\mathbb{A}}\mathbb{A}'\Big).$$

Proof The former is a direct corollary of [27, Lemma 6.1]. The latter is deduced from the identification $L=K'\#H$, where H denotes the right-handed Hopf link. More precisely, H is equipped with two marked points p_1' and p_2, located on distinct components, such that L is obtained by taking the connected sum on the component containing p_1'. Then, the connected sum formula implies

$$\mathrm{CF}(L,p_1,p_2)\simeq\mathrm{CF}(K')\otimes_{\mathbb{A}}\mathrm{CF}(H,p_1',p_2).$$

Here, $\mathrm{CF}(H,p_1',p_2)$ is the chain complex of $\mathbb{A}'$–modules that is defined similarly to $\mathrm{CF}(L,p_1,p_2)$. A Heegaard diagram for H is given in Figure 4.

This diagram is equipped with the $\mathbb{A}'$–coloring which maps z_1 and z_2 to u and v, respectively, while mapping w_1 and w_2 to w. Therefore,

$$\mathrm{CF}(H,p_1',p_2)\simeq\Big(\langle y_1+y_2,y_2\rangle_{\mathbb{A}'}\xrightarrow{\left[\begin{smallmatrix}0&\mathsf{w}\\ \mathsf{u}+\mathsf{v}&\mathsf{u}\end{smallmatrix}\right]}\langle y_3,y_4\rangle_{\mathbb{A}'}\Big),\tag{1}$$

and so the claim follows from the connected sum formula. □

Corresponding to the above chain homotopy equivalences, we may present $\mathfrak{g}^-$ and $\mathfrak{g}^+$ as 4×2 and 2×4 matrices $(\mathfrak{g}^-_{ij})_{ij}$ and $(\mathfrak{g}^+_{ji})_{ji}$, where

$$\mathfrak{g}^-_{ij}\colon\mathrm{CF}(K)\otimes_{\mathbb{A}}\mathbb{A}'\to\mathrm{CF}(K')\otimes_{\mathbb{A}}\mathbb{A}'\quad\text{and}\quad\mathfrak{g}^+_{ji}\colon\mathrm{CF}(K')\otimes_{\mathbb{A}}\mathbb{A}'\to\mathrm{CF}(K)\otimes_{\mathbb{A}}\mathbb{A}'.$$

Let us denote $\mathsf{u}+\mathsf{v}$ by σ and regard $\mathbb{A}'$ as $\mathbb{A}[\sigma]$. For each $1 \le i, j \le 2$, we decompose

$$\mathfrak{g}_{ij}^{-} = \mathfrak{g}_{ij,0}^{-} + \sigma \mathfrak{h}_{ij}^{-} \quad \text{and} \quad \mathfrak{g}_{ji}^{+} = \mathfrak{g}_{ji,0}^{+} + \sigma \mathfrak{h}_{ji}^{+},$$

where the maps $\mathfrak{g}_{ij,0}^{\pm}$ do not use the variable σ. We will find chain homotopies such that

$$\mathfrak{g}^{-} \simeq \begin{bmatrix} \mathfrak{g}_{11,0}^{-} & 0 \\ \mathfrak{g}_{21,0}^{-} & 0 \\ \mathfrak{g}_{31,0}^{-} & 0 \\ \mathfrak{g}_{41,0}^{-} & \mathfrak{g}_{11,0}^{-} \end{bmatrix} \quad \text{and} \quad \mathfrak{g}^{+} \simeq \begin{bmatrix} \mathfrak{g}_{24,0}^{+} & 0 & 0 & 0 \\ \mathfrak{g}_{21,0}^{+} & \mathfrak{g}_{22,0}^{+} & \mathfrak{g}_{23,0}^{+} & \mathfrak{g}_{24,0}^{+} \end{bmatrix}. \tag{2}$$

First, we deduce from $\mathfrak{g}^{-}$ and $\mathfrak{g}^{+}$ being chain maps that

$$\begin{aligned}
\sigma.\mathfrak{g}_{12}^{-} &= \mathfrak{g}_{11}^{-}\circ d + d\circ \mathfrak{g}_{11}^{-}, & \sigma.\mathfrak{g}_{11}^{+} &= \sigma.\mathfrak{g}_{24}^{+} + \mathfrak{g}_{21}^{+}\circ d + d\circ \mathfrak{g}_{21}^{+},\\
\sigma.\mathfrak{g}_{22}^{-} &= \mathfrak{g}_{21}^{-}\circ d + d\circ \mathfrak{g}_{21}^{-}, & \sigma.\mathfrak{g}_{12}^{+} &= \mathsf{w}.\mathfrak{g}_{23}^{+} + \mathsf{u}.\mathfrak{g}_{24}^{+} + \mathfrak{g}_{22}^{+}\circ d + d\circ \mathfrak{g}_{22}^{+},\\
\sigma.\mathfrak{g}_{32}^{-} &= \mathsf{w}.\mathfrak{g}_{21}^{-} + \mathfrak{g}_{31}^{-}\circ d + d\circ \mathfrak{g}_{31}^{-}, & \sigma.\mathfrak{g}_{13}^{+} &= \mathfrak{g}_{23}^{+}\circ d + d\circ \mathfrak{g}_{23}^{+},\\
\sigma.\mathfrak{g}_{42}^{-} &= \sigma.\mathfrak{g}_{11}^{-} + \mathsf{u}.\mathfrak{g}_{21}^{-} + \mathfrak{g}_{41}^{-}\circ d + d\circ \mathfrak{g}_{41}^{-}, & \sigma.\mathfrak{g}_{14}^{+} &= \mathfrak{g}_{24}^{+}\circ d + d\circ \mathfrak{g}_{24}^{+}.
\end{aligned}$$

The differentials d of the complexes do not use the variable v, hence are not in the image of $\sigma = \mathsf{u}+\mathsf{v}$ and the above equations imply

$$\begin{aligned}
\mathfrak{g}_{12}^{-} &= \mathfrak{h}_{11}^{-}\circ d + d\circ \mathfrak{h}_{11}^{-}, & \mathfrak{g}_{11}^{+} &= \mathfrak{g}_{24}^{+} + \mathfrak{h}_{21}^{+}\circ d + d\circ \mathfrak{h}_{21}^{+},\\
\mathfrak{g}_{22}^{-} &= \mathfrak{h}_{21}^{-}\circ d + d\circ \mathfrak{h}_{21}^{-}, & \mathfrak{g}_{12}^{+} &= \mathsf{w}.\mathfrak{h}_{23}^{+} + \mathsf{u}.\mathfrak{h}_{24}^{+} + \mathfrak{h}_{22}^{+}\circ d + d\circ \mathfrak{h}_{22}^{+},\\
\mathfrak{g}_{32}^{-} &= \mathsf{w}.\mathfrak{h}_{21}^{-} + \mathfrak{h}_{31}^{-}\circ d + d\circ \mathfrak{h}_{31}^{-}, & \mathfrak{g}_{13}^{+} &= \mathfrak{h}_{23}^{+}\circ d + d\circ \mathfrak{h}_{23}^{+},\\
\mathfrak{g}_{42}^{-} &= \mathfrak{g}_{11}^{-} + \mathsf{u}.\mathfrak{h}_{21}^{-} + \mathfrak{h}_{41}^{-}\circ d + d\circ \mathfrak{h}_{41}^{-}, & \mathfrak{g}_{14}^{+} &= \mathfrak{h}_{24}^{+}\circ d + d\circ \mathfrak{h}_{24}^{+}.
\end{aligned}$$

Then, it is easy to check that

$$H^{-} = \begin{bmatrix} 0 & \mathfrak{h}_{11}^{-} \\ 0 & \mathfrak{h}_{21}^{-} \\ 0 & \mathfrak{h}_{31}^{-} \\ 0 & \mathfrak{h}_{41}^{-} \end{bmatrix} \quad \text{and} \quad H^{+} = \begin{bmatrix} \mathfrak{h}_{21}^{+} & \mathfrak{h}_{22}^{+} & \mathfrak{h}_{23}^{+} & \mathfrak{h}_{24}^{+} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

are the chain homotopies for $\mathfrak{g}^{-}$ and $\mathfrak{g}^{+}$ which result in (2), respectively. Abusing the notation we keep denoting the new matrices by $\mathfrak{g}^{-} = (\mathfrak{g}_{ij}^{-})$ and $\mathfrak{g}^{+} = (\mathfrak{g}_{ij}^{+})$.

We now set $\sigma = 0$, or equivalently $\mathsf{v} = \mathsf{u}$. Then, $\mathfrak{g}_{11}^{-}$ and $\mathfrak{g}_{11}^{+}$ induce chain maps

$$\mathfrak{g}_{11}^{-}\colon \mathrm{CF}(K) \to \mathrm{CF}(K') \quad \text{and} \quad \mathfrak{g}_{11}^{+}\colon \mathrm{CF}(K') \to \mathrm{CF}(K),$$

and we define $\mathfrak{f}^- = \mathfrak{g}^-_{11}$ and $\mathfrak{f}^+ = \mathfrak{g}^+_{11}$. Note that $(\mathfrak{g}^+ \circ \mathfrak{g}^-)_{11} = \mathfrak{g}^+_{11} \circ \mathfrak{g}^-_{11}$ and $(\mathfrak{g}^- \circ \mathfrak{g}^+)_{11} = \mathfrak{g}^-_{11} \circ \mathfrak{g}^+_{11}$. So, both $\mathfrak{f}^+ \circ \mathfrak{f}^-$ and $\mathfrak{f}^- \circ \mathfrak{f}^+$ are chain homotopic to multiplication by w.

If K' is obtained from K by a negative crossing change, a similar argument may be used to arrive at the same conclusion. The above discussion implies the following theorem:

Theorem 2.3 *If $K' \subset S^3$ is obtained from $K \subset S^3$ by a crossing change, there exist chain maps*

$$\mathfrak{f}^-\colon \mathrm{CF}(K) \to \mathrm{CF}(K') \quad \textit{and} \quad \mathfrak{f}^+\colon \mathrm{CF}(K') \to \mathrm{CF}(K)$$

such that $\mathfrak{f}^+ \circ \mathfrak{f}^-$ and $\mathfrak{f}^- \circ \mathfrak{f}^+$ are chain homotopic to multiplication by w.

Given a knot $K \subset S^3$, the knot Floer chain complex $\mathrm{CF}(K)$ (which is generated over $\mathbb{A} = \mathbb{F}[\mathsf{u}, \mathsf{w}]$) is $\mathbb{Z}$–bigraded. It has a Maslov grading μ and an Alexander grading A, as defined in [24]. Multiplication by u and w changes these gradings by

$$\mu(\mathsf{u}^a\mathsf{w}^b\boldsymbol{x}) = \mu(\boldsymbol{x}) - 2a \quad \text{and} \quad A(\mathsf{u}^a\mathsf{w}^b\boldsymbol{x}) = A(\boldsymbol{x}) - a + b.$$

Subsequently, we may write

$$\mathrm{CF}(K) = \bigoplus_{d,s\in\mathbb{Z}} \mathrm{CF}_d(K, s),$$

where d and s denote the Maslov and Alexander grading, respectively. For instance, for the unknot we obtain

$$\mathrm{CF}(\text{Unknot}) = \mathbb{F}[\mathsf{u}, \mathsf{w}] = \bigoplus_{s\in\mathbb{Z}} \mathbb{A}_0(s), \quad \text{where } \mathbb{A}_0(s) = \langle \mathsf{u}^a\mathsf{w}^b \mid b - a = s\rangle.$$

Proposition 2.4 *Both $\mathfrak{f}^+$ and $\mathfrak{f}^-$ are homogeneous maps. If K' is obtained from K by a positive crossing change then $\mathfrak{f}^-$ and $\mathfrak{f}^+$ have bidegree $(\mu, A) = (0, 0)$ and $(0, 1)$, respectively. Similarly, if K' is obtained from K by a negative crossing change then $\mathfrak{f}^-$ and $\mathfrak{f}^+$ have bidegree $(0, 1)$ and $(0, 0)$, respectively.*

Proof Suppose K' is obtained from K by a positive crossing change. In the situation of Lemma 2.1, the chain maps $\mathfrak{g}^-$ and $\mathfrak{g}^+$ are homogeneous [4, Lemma 7.8], and it follows from [34, Lemma 7.2] that $\mathfrak{g}^-$ and $\mathfrak{g}^+$ are homogeneous of bidegree $\left(0, \frac{1}{2}\right)$. Furthermore, considering the bigradings,

$$\mathrm{CF}(K, p_1, p_2) \otimes_{\mathbb{A}'} \mathbb{A} = \mathrm{CF}(K) \otimes_{\mathbb{A}} V,$$

where V is a free $\mathbb{A}$–module with two generators in bigradings $\left(\frac{1}{2},\frac{1}{2}\right)$ and $\left(-\frac{1}{2},-\frac{1}{2}\right)$. In addition,

$$\mathrm{CF}(L,p_1,p_2)\otimes_{\mathbb{A}'}\mathbb{A}=\mathrm{CF}(K')\otimes_{\mathbb{A}}W,$$

where $W=\mathrm{CF}(H,p_1,p_2)$ and H is the right-handed Hopf link. Specifically, it is the chain complex given in (1) with bigradings

$$(\mu(y_3),A(y_3))=\left(-\tfrac{3}{2},-1\right),\quad (\mu(y_4),A(y_4))=\left(\tfrac{1}{2},1\right),$$
$$(\mu(y_1),A(y_1))=(\mu(y_2),A(y_2))=\left(-\tfrac{1}{2},0\right).$$

Then, for every $a\in\mathrm{CF}(K)$ we have

$$\left(\mu(\mathfrak{g}^-\circ i_1(a)),A(\mathfrak{g}^-\circ i_1(a))\right)=\left(\mu(a)-\tfrac{1}{2},A(a)\right),$$

where $i_1\colon \mathrm{CF}(K)\to\mathrm{CF}(K,p_1,p_2)$ denotes the inclusion in the first summand. Since $(\mu(y_i),A(y_i))=\left(-\frac{1}{2},0\right)$ for $i=1,2$, this implies that $\mathfrak{f}^-$ preserves the Maslov and Alexander grading. On the other hand, if $a\in\mathrm{CF}(K')$, the above bigradings imply that

$$\left(\mu(\mathfrak{g}^+\circ i_4'(a)),A(\mathfrak{g}^+\circ i_4'(a))\right)=\left(\mu(a)+\tfrac{1}{2},A(a)+\tfrac{3}{2}\right)$$

and so $\mathfrak{f}^+$ has bidegree $(0,1)$. Here, $i_4'\colon \mathrm{CF}(K')\to\mathrm{CF}(L,p_1,2)$ is the inclusion in the fourth summand. The proof for a negative crossing change is analogous. □

Since the crossing change chain maps $\mathfrak{f}^+$ and $\mathfrak{f}^-$ do not change the Maslov index, we will drop it from the notation in the rest of the paper. Moreover, by degree of a homogeneous chain map f, denoted by $\deg(f)$, we mean the Alexander grading degree of f.

3 The depth of a knot and bounding the unknotting number

Let K and K' be knots in S^3 and $\mathbb{I}$ denote a sequence of crossing changes which modifies K to K'. We denote the length of $\mathbb{I}$ by $|\mathbb{I}|$, and the number of positive (resp. negative) crossing changes in $\mathbb{I}$ by $m^+(\mathbb{I})$ (resp. $m^-(\mathbb{I})$). For $\bullet\in\{+,-\}$, let $u^\bullet(K,K')$ denote the minimum of $m^\bullet(\mathbb{I})$ over all such sequences $\mathbb{I}$ of crossing changes. Further, the *Gordian distance* $u(K,K')$ between K and K' is defined as the minimum number of crossing changes required for modifying K to K'. Therefore,

$$u(K,K')\geq u^-(K,K')+u^+(K,K').$$

Define $u^\bullet(K) = u^\bullet(K, U)$, where U denotes the unknot. Note that in principle it is possible that $u^+(K)$ and/or $u^-(K)$ are realized in an unknotting sequence which does not have minimal length. The knot K' is called *Gordian adjacent* to K if there exists a minimal unknotting sequence for K containing K'. Equivalently, the Gordian distance $u(K, K')$ from K to K' is $u(K) - u(K')$. Based on Theorem 2.3 we make the following definition:

Definition 3.1 Given knots $K, K' \subset S^3$, consider all pairs of homogeneous chain maps

$$\mathfrak{f}^-\colon \mathrm{CF}(K) \to \mathrm{CF}(K') \quad \text{and} \quad \mathfrak{f}^+\colon \mathrm{CF}(K') \to \mathrm{CF}(K)$$

of degrees $m^- = \deg(\mathfrak{f}^-)$ and $m^+ = \deg(\mathfrak{f}^+)$ such that $\mathfrak{f}^- \circ \mathfrak{f}^+$ and $\mathfrak{f}^+ \circ \mathfrak{f}^-$ are chain homotopic to multiplication by w^m, where $m^- + m^+ = m$. Define $\mathfrak{l}^-(K, K')$, $\mathfrak{l}^+(K, K')$ and $\mathfrak{l}(K, K')$ as the least values for the integers $\deg(\mathfrak{f}^-)$, $\deg(\mathfrak{f}^+)$ and $m = \deg(\mathfrak{f}^-) + \deg(\mathfrak{f}^+)$, respectively, among all such pairs. In particular, define $\mathfrak{l}^\pm(K) = \mathfrak{l}^\pm(K, U)$ and $\mathfrak{l}(K) = \mathfrak{l}(K, U)$, where U denotes the unknot.

When $K' = U$, the chain complex $\mathrm{CF}(U)$ is chain homotopic to $\mathbb{A}$ (with trivial differentials). For defining $\mathfrak{l}^\pm(K)$ and $\mathfrak{l}(K)$, we are thus lead to consider all pairs of homogeneous chain maps

$$\mathfrak{f}^-\colon \mathrm{CF}(K) \to \mathbb{A} \quad \text{and} \quad \mathfrak{f}^+\colon \mathbb{A} \to \mathrm{CF}(K)$$

of degrees $m^- = \deg(\mathfrak{f}^-)$ and $m^+ = \deg(\mathfrak{f}^+)$ such that $\mathfrak{f}^- \circ \mathfrak{f}^+$ is multiplication by w^m and $\mathfrak{f}^+ \circ \mathfrak{f}^-$ is chain homotopic to multiplication by w^m. The discussion of the previous section, and in particular Theorem 2.3 and Proposition 2.4, imply the following theorem:

Theorem 3.1 *Given a pair of knots $K, K' \subset S^3$, $u^\bullet(K, K')$ is bounded below by $\mathfrak{l}^\bullet(K, K')$ for $\bullet \in \{-, +\}$, while $u(K, K')$ is bounded below by $\mathfrak{l}(K, K')$.*

Remark 3.2 Let K and K' be knots in S^3. Given chain maps $\mathfrak{f}^-$ and $\mathfrak{f}^+$ satisfying the assumptions of Definition 3.1, their adjoints are chain maps

$$\bar{\mathfrak{f}}^-\colon \mathrm{CF}(-K') \to \mathrm{CF}(-K) \quad \text{and} \quad \bar{\mathfrak{f}}^+\colon \mathrm{CF}(-K) \to \mathrm{CF}(-K')$$

of degrees m^- and m^+, respectively, satisfying

$$\bar{\mathfrak{f}}^- \circ \bar{\mathfrak{f}}^+ \simeq \mathsf{w}^m \quad \text{and} \quad \bar{\mathfrak{f}}^+ \circ \bar{\mathfrak{f}}^- \simeq \mathsf{w}^m.$$

Thus,

$$\mathfrak{l}^-(-K, -K') = \mathfrak{l}^+(K, K'), \quad \mathfrak{l}^+(-K, -K') = \mathfrak{l}^-(K, K'), \quad \mathfrak{l}(K, K') = \mathfrak{l}(-K, -K').$$

Let us denote the homology of $\mathrm{CF}(K,s)$ by $\mathbb{H}(K,s)$ for every $s\in\mathbb{Z}$, and set $\mathbb{H}(K)=\bigoplus_s \mathbb{H}(K,s)$. Then $\mathbb{H}(K)$ is a module over $\mathbb{A}=\mathbb{F}[\mathsf{u},\mathsf{w}]$. Let $\mathbb{T}(K)$ denote the torsion submodule of $\mathbb{H}(K)$, ie

$$\mathbb{T}(K)=\{x\in\mathbb{H}(K)\mid ax=0 \text{ for some } a\in\mathbb{A}-\{0\}\}.$$

It is clear that $\mathbb{T}(K)$ is a submodule of $\mathbb{H}(K)$, and there is a short exact sequence

$$0\to\mathbb{T}(K)\xrightarrow{\iota_K}\mathbb{H}(K)\xrightarrow{\pi_K}\mathbb{A}(K)\to 0,$$

where $\mathbb{A}(K)$, defined by the above exact sequence, is the torsion-free part of $\mathbb{H}(K)$. Fix a sequence $\mathbb{I}$ of crossing changes which modify K to the unknot. Correspondingly, we obtain the $\mathbb{A}$–homomorphisms $\mathfrak{f}_{\mathbb{I}}^-\colon \mathbb{H}(K)\to\mathbb{A}$ and $\mathfrak{f}_{\mathbb{I}}^+\colon\mathbb{A}\to\mathbb{H}(K)$. The map $\mathfrak{f}_{\mathbb{I}}^-$ induces a map $\mathfrak{f}_{\mathbb{I},\mathbb{T}}^-\colon\mathbb{T}(K)\to\mathbb{A}$, while $\mathfrak{f}_{\mathbb{I}}^+$ induces the map $\mathfrak{f}^+\colon\mathbb{A}\to\mathbb{A}(K)$.

Lemma 3.3 *The map $\mathfrak{f}^+\colon\mathbb{A}\to\mathbb{A}(K)$ induced by $\mathfrak{f}_{\mathbb{I}}^+$ is injective, while the map $\mathfrak{f}_{\mathbb{I},\mathbb{T}}^-\colon\mathbb{T}(K)\to\mathbb{A}$ is trivial. We thus have a map $\mathfrak{f}^-\colon\mathbb{A}(K)\to\mathbb{A}$ induced by $\mathfrak{f}_{\mathbb{I}}^-$, which is injective. The induced maps are homogeneous with respect to the Alexander grading.*

Proof Let $m^-=\deg(\mathfrak{f}_{\mathbb{I}}^-)$ and $m^+=\deg(\mathfrak{f}_{\mathbb{I}}^+)$. If $x\in\mathbb{T}(K)$ and $ax=0$ for $0\neq a\in\mathbb{A}$, it follows that $a\mathfrak{f}_{\mathbb{I}}^-(x)=0$ in $\mathbb{A}$, implying that $\mathfrak{f}_{\mathbb{I}}^-(x)=0$. Since the restriction $\mathfrak{f}_{\mathbb{I},\mathbb{T}}^-$ of $\mathfrak{f}_{\mathbb{I}}^-$ to $\mathbb{T}(K)$ is trivial, a map $\mathfrak{f}^-\colon\mathbb{A}(K)\to\mathbb{A}$ is induced by $\mathfrak{f}_{\mathbb{I}}^-$. Let us now assume that $x\in\mathbb{H}(K)$ is in the kernel of $\mathfrak{f}_{\mathbb{I}}^-$. Then $\mathsf{w}^m x=\mathfrak{f}_{\mathbb{I}}^+\circ\mathfrak{f}_{\mathbb{I}}^-(x)=0$, implying that $x\in\mathbb{T}(K)$. In particular, $\mathfrak{f}^-\colon\mathbb{A}(K)\to\mathbb{A}$ is injective. On the other hand, if $a\in\mathbb{A}$ and $x=\mathfrak{f}_{\mathbb{I}}^+(a)\in\mathbb{T}(K)$, it follows that $0=\mathfrak{f}_{\mathbb{I}}^-(x)=\mathsf{w}^m a$, implying that $a=0$. Thus $\mathfrak{f}^+\colon\mathbb{A}\to\mathbb{A}(K)$ is injective. This completes the proof of the lemma, as the statement about the Alexander grading follows immediately from our previous discussions. □

Proposition 3.4 *There is a sequence $0=i_0(K)<i_1(K)<\cdots<i_n(K)=\nu^+(K)$ associated with every knot $K\subset S^3$, and an identification*

$$\mathbb{A}(K)=\langle \mathsf{u}^{i_k(K)}\mathsf{w}^{i_{n-k}(K)}\mid k\in\{0,1,\dots,n\}\rangle_{\mathbb{A}}. \tag{3}$$

Moreover, the identification of (3) preserves the Alexander grading.

Proof For each s, $\mathrm{CF}(K,s)$ is a chain complex of $\mathbb{F}[U]$–modules for $U=\mathsf{uw}$ and multiplication by w and u induce chain maps

$$\mathsf{w}\colon\mathrm{CF}(K,s)\to\mathrm{CF}(K,s+1)\quad\text{and}\quad\mathsf{u}\colon\mathrm{CF}(K,s+1)\to\mathrm{CF}(K,s),$$

respectively. Moreover, for s sufficiently large (resp. small), $\mathrm{CF}(K,s)$ is isomorphic to $\mathrm{CF}^-(S^3)$ and multiplication by w (resp. u) is an isomorphism. Let $s_+ \gg 0$ (resp. $s_- \ll 0$) denote such a sufficiently large (resp. small) s.

For any $s < s_+$, let v_s denote the homomorphism induced by w^{s_+-s} from $\mathbb{H}(K,s)$ to $\mathbb{H}(K,s_+) \cong \mathbb{F}[U]$. The kernel of v_s is equal to $\mathbb{T}(K,s)$ and so the restriction of v_s to $\mathbb{A}(K,s) \cong \mathbb{F}[U]$ is injective.

The smallest $s \le s_+$ such that v_s is a surjective homomorphism, equivalently $v_s|_{\mathbb{A}(K,s)}$ is an isomorphism, is the invariant $\nu^+ = \nu^+(K) = \nu^-(K)$ defined by Hom and Wu [13], based on Rasmussen's work [29], and also Ozsváth and Szabó [22]. Note that this invariant is a lower bound for the slice genus. For all $s \ge \nu^+$, multiplication by w is an isomorphism from $\mathbb{A}(K,s)$ to $\mathbb{A}(K,s+1)$. Suppose $s < \nu^+$, and let $b \in \mathbb{A}(K,s)$ be the generator. Then, $\mathsf{w}^{\nu^+-s} b \in \mathbb{A}(K,\nu^+)$ and so $\mathsf{w}^{\nu^+-s} b = p(U)a$, where a is the generator of $\mathbb{A}(K,\nu^+)$ and $p(U) \in \mathbb{F}[U]$. Thus,

$$U^{\nu^+-s} b = p(U)\mathsf{u}^{\nu^+-s} a = p(U)p'(U)b,$$

and consequently $p(U) = U^{j_s}$ for some $0 \le j_s \le \nu^+ - s$. By definition, j_s is equal to the invariants V_s defined by Ni and Wu [21] and h_s defined by Rasmussen [28]. See Remark 3.5 for more details.

Additionally, $\mathbb{H}(K)$ is symmetric under exchanging the variables u and w, which gives an isomorphism between $\mathbb{H}(K,s)$ and $\mathbb{H}(K,-s)$. Thus, for all $s \le -\nu^+$ multiplication by u is an isomorphism from $\mathbb{A}(K,s)$ to $\mathbb{A}(K,s-1)$. Moreover, if $b \in \mathbb{A}(K,s)$ is the generator for some $s > -\nu^+$, then $\mathsf{u}^{s+\nu^+} b = U^{j'_s} a'$ where $j'_s = j_{-s}$ and a' is the generator of $\mathbb{A}(K,-\nu^+)$. By [12, Lemma 2.5],

$$j'_s = H_s = V_s + s = j_s + s.$$

Consequently, $j_{\nu^+} = j'_{-\nu^+} = 0$ implies $j_{-\nu^+} = j'_{\nu^+} = \nu^+$, ie

$$\mathsf{u}^{2\nu^+} a = U^{\nu^+} a' \quad \text{and} \quad \mathsf{w}^{2\nu^+} a' = U^{\nu^+} a.$$

Then, we define a grading-preserving $\mathbb{A}$–module homomorphism

$$\imath\colon \mathbb{A}(K) = \bigoplus_s \mathbb{A}(K,s) \to \mathbb{A}$$

by setting $\imath(b) = \mathsf{u}^{j_s}\mathsf{w}^{j'_s}$ for the generator $b \in \mathbb{A}(K,s)$. For instance, if $s \ge \nu^+$ then $\imath(b) = \mathsf{w}^s$, while if $s \le -\nu^+$ then $\imath(b) = \mathsf{u}^{-s}$. It is clear that $\imath$ is injective and it identifies $\mathbb{A}(K)$ with an ideal generated by at most $2\nu^+ + 1$ monomials of the form $\mathsf{u}^i \mathsf{w}^j$ with $0 \le i, j \le \nu^+$ in $\mathbb{A}$. This set of generators contains a unique minimal subset

$$\{\mathsf{u}^{i_k}\mathsf{w}^{j_k} \mid 0 = i_0 < i_1 < \cdots < i_n = \nu^+ \text{ and } \nu^+ = j_0 > j_1 > \cdots > j_n = 0\}$$

that generates the image of ι. The symmetry of $\mathbb{H}(K)$ implies that $j_k = i_{n-k}$ for all $k = 0, \dots, n$. $\square$

Definition 3.2 Under the identification of (3), for every knot $K \subset S^3$ the sequence

$$\iota(K) = (0 = i_0(K) < i_1(K) < \cdots < i_{n(K)}(K) = \nu^+(K))$$

is called the *ideal sequence* associated with the knot K. The ideal $\mathbb{A}(\iota)$ associated with a sequence $\iota = (0 = i_0 < i_1 < \cdots < i_n)$ is defined as

$$\mathbb{A}(\iota) = \langle \mathsf{u}^{i_k}\mathsf{w}^{i_{n-k}} \mid k \in \{0, 1, \dots, n\}\rangle_{\mathbb{A}},$$

and we identify $\mathbb{A}(K) = \mathbb{A}(\iota(K))$.

Remark 3.5 For every knot K in S^3, if we set $\mathsf{w} = 1$ and consider $\mathrm{CF}(K)$ as a chain complex filtered by the Alexander filtration, we obtain an identification of $\mathrm{CF}(K, s)$ and $C\{\max(i, j-s) \le 0\}$ as chain complexes of $\mathbb{F}[U]$–modules. Under this identification, the inclusion

$$C\{\max(i, j-s) \le 0\} \to C\{\max(i, j-s-1) \le 0\}$$

corresponds to the multiplication by w. Moreover, $\mathrm{CF}(K, s_+)$ is identified with $C\{i \le 0\}$. Consequently, ν^+ is the smallest s such that the map

$$v_s\colon H_\star\big(C\{\max(i, j-s) \le 0\}\big) \to H_\star(C\{i \le 0\}) = \mathbb{F}[U]$$

induced by inclusion is surjective, which by definition is $\nu^-(K) = \nu^+(K)$. Similarly, considering the definitions of V_s and H_s using HF^-, we have $j_s = V_s$ and $j_s' = H_s$. Consequently, the ideal sequence

$$\iota(K) = (i_0 < i_1 < \cdots < i_n) \subset \{V_s \mid -\nu^+ \le s \le \nu^+\}$$

is determined as follows. First, let $\{-\nu^+ = k_0 < k_1 < \cdots < k_m = \nu^+\}$ indicate the places of jumps in the sequence $\{V_s\}_{s=-\nu^-}^{\nu^+}$, ie

$$\{V_{k_i} \mid 0 \le i \le m\} = \{V_s \mid -\nu^+ \le s \le \nu^+\}$$

and $V_{k_i} < V_{k_i - 1}$ for all i. Then, $\iota(K)$ is the subset of $\{V_s\}$ consisting of every V_{k_i} such that $V_{-k_i} < V_{-k_{i+1}}$.

For finite increasing sequences $\imath$ and $\imath'$ of nonnegative integers as above define the *distance* $\ell(\imath,\imath')$ from $\imath$ to $\imath'$ as the smallest value for p such that $\mathsf{w}^p\mathbb{A}(\imath')\subset\mathbb{A}(\imath)$. Given knots $K,K'\subset S^3$, define the *positive distance* $\ell^+(K,K')$ as $\ell(\imath(K),\imath(K'))$. Define the *negative distance* by $\ell^-(K,K')=\ell^+(-K,-K')$, where $-K$ denotes the mirror image of K. Define the positive/negative *depth* of a knot K by $\ell^\pm(K)=\ell^\pm(K,U)$, where U denotes the unknot.

Note that under the identification of (3), the positive depth of K is equal to $\nu^+(K)$.

Proposition 3.6 *Let K and K' be knots in S^3. Then*

$$\mathfrak{l}^+(K,K')\geq\max\{\ell^+(K,K'),\ell^-(K',K)\}$$

and

$$\mathfrak{l}^-(K,K')\geq\max\{\ell^-(K,K'),\ell^+(K',K)\}.$$

Before proving this proposition, we need to make an algebraic observation.

Lemma 3.7 *Given integer sequences $\imath=(0=i_0<i_1<\cdots<i_n)$ and $\imath'=(0=i'_0<i'_1<\cdots<i'_m)$, every $\mathbb{A}$–homomorphism $f\colon\mathbb{A}(\imath)\to\mathbb{A}(\imath')$ is equal to the restriction of an $\mathbb{A}$–homomorphism from $\mathbb{A}$ to $\mathbb{A}$ and so it is defined by multiplication with some polynomial $p\in\mathbb{A}$.*

Proof Let $i\colon\mathbb{A}(\imath')\to\mathbb{A}$ be the inclusion map, and $F=i\circ f$. First,

$$F(\mathsf{u}^{i_n}\mathsf{w}^{i_n})=\mathsf{u}^{i_n}F(\mathsf{w}^{i_n})=\mathsf{w}^{i_n}F(\mathsf{u}^{i_n})$$

implies $F(\mathsf{u}^{i_n})=p\mathsf{u}^{i_n}$ and $F(\mathsf{w}^{i_n})=p\mathsf{w}^{i_n}$ for some $p\in\mathbb{A}$. Consequently,

$$\mathsf{u}^{i_k}\mathsf{w}^{i_n}p=F(\mathsf{u}^{i_k}\mathsf{w}^{i_n})=\mathsf{w}^{i_n-i_{n-k}}F(\mathsf{u}^{i_k}\mathsf{w}^{i_{n-k}}).$$

Therefore, $F(\mathsf{u}^{i_k}\mathsf{w}^{i_{n-k}})=\mathsf{u}^{i_k}\mathsf{w}^{i_{n-k}}p$ and we are done. □

Proof of Proposition 3.6 It is a straightforward corollary of the definition that $\mathfrak{l}^-(K,K')=\mathfrak{l}^+(K',K)$. So, Remark 3.2 implies that it suffices to show that $\mathfrak{l}^+(K,K')\geq\ell^+(K,K')$. By definition, there exist $\mathbb{A}$–homomorphisms

$$f\colon\mathbb{A}(K)\to\mathbb{A}(K')\quad\text{and}\quad g\colon\mathbb{A}(K')\to\mathbb{A}(K)$$

such that $f\circ g$ and $g\circ f$ are equal to multiplication by w^m, and $\deg(g)=\mathfrak{l}^+(K,K')$. Under the identification of (3), Lemma 3.7 implies that f and g are the restriction of $\mathbb{A}$–homomorphisms from $\mathbb{A}$ to $\mathbb{A}$ defined by multiplication with polynomials p and q in $\mathbb{A}$. Since, $pq = \mathsf{w}^m$ and $\deg(g) = \mathfrak{l}^+(K, K')$, we have $g = \mathsf{w}^{\mathfrak{l}^+(K,K')}$ and so $\mathfrak{l}^+(K, K') \geq \ell^+(K, K')$. □

Theorem 3.1 and the above proposition imply that $\ell^{\pm}(K, K') \leq u^{\pm}(K, K')$.

Corollary 3.8 *For any knot $K \subset S^3$, we have*

$$u^+(K) \geq \mathfrak{l}^+(K) \geq \nu^+(K) \geq \tau(K) \quad \text{and} \quad u^-(K) \geq \mathfrak{l}^-(K) \geq \nu^+(-K) \geq -\tau(K).$$

Therefore, for $0 \leq t \leq 1$, we have $-t\mathfrak{l}^+(K) \leq \Upsilon_K(t) \leq t\mathfrak{l}^-(K)$.

Proof The first two claims follow from Proposition 3.6 and the inequality $\nu^+(K) \geq \tau(K)$ from [13, Proposition 2.3]. The last claim follows from the inequality $-t\nu^+(K) \leq \Upsilon_K(t)$ from [22, Proposition 4.7]. □

4 The torsion obstruction

Let us assume that a sequence $\mathbb{I}$ of crossing changes is used to unknot $K \subset S^3$. Let us further assume that $m^+ = m^+(\mathbb{I})$ and $m^- = m^-(\mathbb{I})$, while $m = m^+ + m^- = |\mathbb{I}|$. The argument of Lemma 3.3 then implies that multiplication by w^m trivializes all of $\mathbb{T}(K)$. This observation gives a weaker obstruction to the unknotting number.

Definition 4.1 Define the *positive torsion depth* $\mathfrak{t}^+(K)$ of a knot $K \subset S^3$ to be the smallest integer m such that multiplication by w^m is trivial on $\mathbb{T}(K)$. Let $\mathfrak{t}^-(K) = \mathfrak{t}^+(-K)$. Then $\mathfrak{t}(K) = \max\{\mathfrak{t}^-(K), \mathfrak{t}^+(K)\}$ is called the *torsion depth* of K.

Consider the homomorphism $\hat{\phi}\colon \mathbb{A} \to \mathbb{F}[\mathsf{w}]$ defined by $\hat{\phi}(\mathsf{u}) = 0$ and $\hat{\phi}(\mathsf{w}) = \mathsf{w}$. This homomorphism makes $\mathbb{F}[\mathsf{w}]$ into an $\mathbb{A}$–module. We define

$$\widehat{\mathrm{CF}}(K) = \mathrm{CF}(K) \otimes_{\hat{\phi}} \mathbb{F}[\mathsf{w}] \quad \text{and} \quad \widehat{\mathbb{H}}(K) = H_\star(\widehat{\mathrm{CF}}(K)).$$

Note that after replacing w with U, $\widehat{\mathrm{CF}}(K)$ and $\widehat{\mathbb{H}}(K)$ are isomorphic to $\mathrm{CFK}^-(K)$ and $\mathrm{HFK}^-(K)$, respectively. Thus, $\widehat{\mathbb{H}}(K)$ is a $\mathbb{F}[\mathsf{w}]$–module, with a free summand isomorphic to $\mathbb{F}[\mathsf{w}]$ and a torsion summand denoted by $\widehat{\mathbb{T}}(K)$. Define $\hat{\mathfrak{t}}(K)$ as the smallest m such that multiplication by w^m is trivial on $\widehat{\mathbb{T}}(K)$. The following proposition is a straightforward corollary of previous definitions and discussions:

Proposition 4.1 *For any knot $K \subset S^3$, the torsion classes $\hat{\mathfrak{t}}(K)$, $\hat{\mathfrak{t}}(-K)$ and $\mathfrak{t}(K)$ are lower bounds for $\mathfrak{l}(K)$, and thus for the unknotting number $u(K)$.*

Proposition 4.2 *If the genus $g(K)$ of a knot $K \subset S^3$ is strictly bigger than $\tau(K)$ then $\mathbb{T}(K) \neq 0$, and, in particular, $\mathfrak{t}^+(K) \geq 1$.*

Proof The differential d of the chain complex $\mathrm{CF}(K)$ may be written as $d = \sum_{i,j\geq 0} \mathsf{u}^i \mathsf{w}^j d^{i,j}$. Using a spectral sequence determined by $(\mathrm{CF}(K), d)$, we can replace $\mathrm{CF}(K)$ with page 1 of the aforementioned spectral sequence and assume that $d^{0,0} = 0$. Let x denote a generator of $\widehat{\mathrm{HFK}}(K, g(K))$. If a generator y appears in $d^{i,0}(x)$ (where $i > 0$), it follows that

$$g(K) = A(x) = A(\mathsf{u}^i y) = A(y) - i < A(y).$$

Since $\widehat{\mathrm{HFK}}(K,s) = 0$ for $s > g(K)$, the above observation implies that $d^{i,0}(x) = 0$. In particular, $d(x) = \mathsf{w}^p z$ for some $p > 0$ and some z representing a class $[z] \in \mathbb{H}(K)$. Clearly, $\mathsf{w}^p[z] = 0$ in $\mathbb{H}(K)$. If $z = d(x')$ for some $x' \in \mathrm{CF}(K)$, then $d(x + \mathsf{w}^p x') = 0$. Since $\tau(K) < g(K)$, the image of $x + \mathsf{w}^p x'$ under the chain map $\mathrm{CF}(K) \to \widehat{\mathrm{CF}}(K)$ represents a trivial homology class. Thus, x appears in $d^{0,i}(y)$ (where $i > 0$) for some generator $y \in \widehat{\mathrm{HFK}}(K)$. So,

$$A(y) = A(\mathsf{w}^i x) = A(x) + i > g(K),$$

which is a contradiction. In particular, $[z]$ is nonzero in $\mathbb{T}(K)$. □

Corollary 4.3 *If K is a nontrivial knot then both $\hat{\mathfrak{t}}(K)$ and $\mathfrak{t}(K)$ are strictly bigger than zero.*

Proof First $\hat{\mathfrak{t}}(K) > 0$ is a trivial consequence of the definition and genus and unknot detection of knot Floer homology. Second, either $\tau(K) < g(K)$ or $\tau(-K) = -g(K) < g(-K)$. Thus, $\mathfrak{t}(K) > 0$ follows from Proposition 4.2. □

Proposition 4.4 *Suppose K and K' are knots in S^3. Then,*

$$\hat{\mathfrak{t}}(K \# K') = \max\{\hat{\mathfrak{t}}(K), \hat{\mathfrak{t}}(K')\}.$$

Proof By the Künneth theorem for homology, there is a split exact sequence

$$0 \to \widehat{\mathbb{H}}(K) \otimes \widehat{\mathbb{H}}(K') \to \widehat{\mathbb{H}}(K \# K') \to \mathrm{Tor}_{\mathbb{F}[\mathsf{w}]}(\widehat{\mathbb{H}}(K), \widehat{\mathbb{H}}(K')) \to 0.$$

Thus, $\hat{\mathfrak{t}}(K \# K')$ is equal to the maximum order of w–torsions in $\widehat{\mathbb{H}}(K) \otimes \widehat{\mathbb{H}}(K')$ and $\mathrm{Tor}_{\mathbb{F}[\mathsf{w}]}(\widehat{\mathbb{H}}(K), \widehat{\mathbb{H}}(K'))$, which is equal to $\max\{\hat{\mathfrak{t}}(K), \hat{\mathfrak{t}}(K')\}$. □

Remark 4.5 One can construct a similar lower bound $\mathfrak{t}_{p/q}$ by sending u and w to v^p and v^q in $\mathbb{F}[\mathsf{v}]$, respectively, which satisfy a statement similar to Proposition 4.4.

5 Examples and applications

Example 5.1 Let $K = T_{p,q}$ be the (p,q) torus knot with $0 < p < q$. The chain homotopy type of $\mathrm{CF}(K)$ is specified by the Alexander polynomial of K [26]. Specifically, the symmetrized Alexander polynomial of K is equal to

$$\Delta_K(t) = t^{-(p-1)(q-1)/2}\frac{(t^{pq}-1)(t-1)}{(t^p-1)(t^q-1)} = (-1)^n + \sum_{i=1}^{n}(-1)^{n-i}(t^{a_i}+t^{-a_i}),$$

where $0 < a_1 < a_2 < \cdots < a_n = \frac{1}{2}(p-1)(q-1)$. The chain complex $\mathrm{CF}(K)$ which is determined by the Alexander polynomial is a staircase, freely generated over $\mathbb{A}$ with generators $\{\boldsymbol{x}_i\}_{i=-n}^{n}$ and differential

$$d\boldsymbol{x}_i = \begin{cases} \mathsf{w}^{a_i-a_{i-1}}\boldsymbol{x}_{i-1} + \mathsf{u}^{a_{i+1}-a_i}\boldsymbol{x}_{i+1} & \text{if } n-i \text{ is odd,} \\ 0 & \text{if } n-i \text{ is even,} \end{cases}$$

where $a_0 = 0$ and $a_{-i} = -a_i$; see Figure 5. Consequently, $\mathbb{T}(K) = 0$ and $\mathbb{A}(K) = \mathbb{H}(K)$ is generated by $[\boldsymbol{x}_{n-2i}]$ for $i = 0, \dots, n$. Moreover, for each $j = n-2i$,

$$\mathsf{u}^{a_{j+2}-a_{j+1}}[\boldsymbol{x}_{j+2}] = \mathsf{w}^{a_{j+1}-a_j}[\boldsymbol{x}_j].$$

Let i_j denote the first coordinate of $\boldsymbol{x}_{n-2j}$, so

$$i_j = \sum_{k=0}^{2j-1}(-1)^k a_{n-k}. \tag{4}$$

It is easy to check that the second coordinate of $\boldsymbol{x}_{n-2j}$ is equal to i_{n-j}. Moreover, mapping $[\boldsymbol{x}_{n-2j}]$ to $\mathsf{u}^{i_j}\mathsf{w}^{i_{n-j}}$ gives an isomorphism between $\mathbb{H}(T_{p,q}) = \mathbb{A}(T_{p,q})$ and $\mathbb{A}(i_0 = 0 < i_1 < i_2 < \cdots < i_n = g)$.

Additionally, we may describe $\imath(K)$ in terms of the Ni–Wu invariants $V_s(K)$ [21] (or Rasmussen's local h–invariants [28]). Using Remark 3.5, we will show that $i_j = V_{a_{n-2j}}$ and so

$$\imath(K) = (0 = i_0 < i_1 < \cdots < i_n = g) = (0 = V_{a_n} < V_{a_{n-2}} < V_{a_{n-4}} < \cdots < V_{a_{-n}} = g).$$

First, it follows from the computations in [25] that $V_s = 0$ for $s \geq a_n$ and

$$V_{s-1} - V_s = \begin{cases} 0 & \text{if } a_{n-2i} < s \leq a_{n-2i+1} \text{ for } 1 \leq i \leq n, \\ 1 & \text{otherwise.} \end{cases}$$

In particular, if $V_{s-1} - V_s = 1$ then $V_{-s} - V_{-s+1} = 0$. Suppose $V_s \in \imath(K)$ and $V_s < V_{s-1}$. If $s \neq a_{n-2i}$ for any i, then $V_{s+1} < V_s$ and thus $V_{-s} = V_{-s-1}$. Note that by Remark 3.5, this contradicts $V_s \in \imath(K)$.

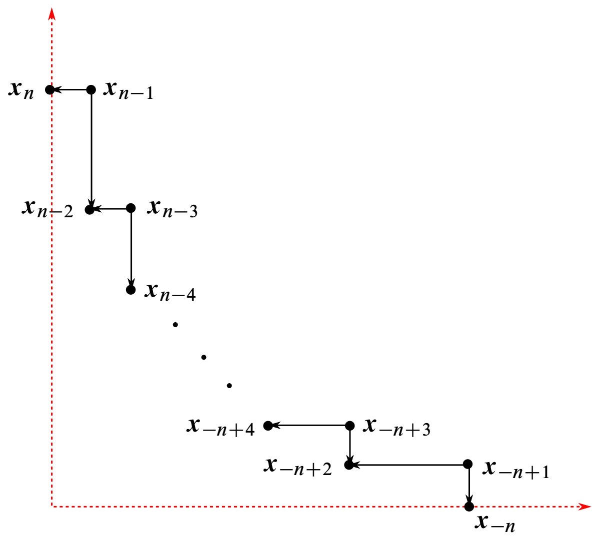

Figure 5: Each horizontal arrow from x_i to x_{i+1} has length $a_{i+1}-a_i$, while each vertical arrow from x_i to x_{i-1} has length a_i-a_{i-1}.

Consider $\widehat{\mathrm{CF}}(K)=\mathrm{CF}(K)\otimes_{\hat\phi}\mathbb{F}[\mathsf{w}]$, where as before $\hat\phi\colon\mathbb{A}\to\mathbb{F}[\mathsf{w}]$ is the homomorphism defined by $\hat\phi(\mathsf{u})=0$ and $\hat\phi(\mathsf{w})=\mathsf{w}$. Recall that $\widehat{\mathbb{H}}(K)$ is isomorphic to $\mathrm{HFK}^-(K)$ after switching w with U. By the above discussion, $\widehat{\mathbb{H}}(K)$ has a free summand generated by $[x_n]$. Moreover, for each $j=n-2i$, $[x_j]$ is a torsion class of order $a_{j+1}-a_j$. It is easy to check that

$$a_{n-1}-a_{n-2}=p-1 \quad\text{and}\quad a_{j+1}-a_j\le p-1 \quad\text{for every } j=n-2i \text{ and } 2\le i\le n.$$

Therefore, $\hat{\mathfrak{t}}(T_{p,q})=p-1$.

To calculate $\mathfrak{t}(T_{p,q})$ we need to find $\mathfrak{t}^+(-T_{p,q})$. For any knot K, $\mathrm{CF}(-K)\simeq\mathrm{CF}(K)^\star$. So, for $-T_{p,q}=T_{p,-q}$, the above discussion implies that $\mathrm{CF}(-T_{p,q})$ is chain homotopic to the chain complex freely generated over $\mathbb{A}$ with generators $\{x_i\}_{i=0}^{2n}$ and differential

$$dx_i=\begin{cases}0 & \text{if } i \text{ is odd,}\\ \mathsf{u}^{a_{i-1}-a_i}x_{i-1}+\mathsf{w}^{a_i-a_{i+1}}x_{i+1} & \text{if } i \text{ is even.}\end{cases}$$

Thus, $\mathbb{A}(-T_{p,q})\cong\mathbb{A}$ is generated by $\sum_{k=0}^{n}\mathsf{u}^{i_{n-k}}\mathsf{w}^{i_k}x_{2k}$, while $[x_{2k+1}]$ is torsion of order i_{k+1} for $k=0,\dots,n-1$. Therefore,

$$\mathfrak{t}(T_{p,q})=\mathfrak{t}^+(T_{p,q})=i_n=\nu^+(T_{p,q})=\tfrac{1}{2}(p-1)(q-1).$$

Special case $p = 2$, $q = 2n+1$ For the torus knot $T_{2,2n+1}$ we have

$$\Delta_{T_{2,2n+1}}(t) = (-1)^n + \sum_{i=1}^{n} (-1)^{n-i}(t^i + t^{-i}).$$

So, $a_i = i$ for $-n \le i \le n$, and thus $V_{n-2i} = V_{n-2i+1} = i$ for $0 \le i \le n$. Therefore,

$$\iota(T_{2,2n+1}) = (0 < 1 < 2 < \cdots < n) \quad \text{and} \quad \mathbb{A}(T_{2,2n+1}) = \langle \mathsf{u}^i \mathsf{w}^j \mid i + j \ge n \rangle_{\mathbb{A}}.$$

Special case $p = 3$, $q = 3k \pm 1$ Suppose $q = 3k+1$. First, we compute the symmetrized Alexander polynomial of $T_{3,3k+1}$:

$$\begin{aligned}
\Delta_{T_{3,3k+1}}(t) &= t^{-3k} \frac{(t^{3(3k+1)} - 1)(t-1)}{(t^{3k+1}-1)(t^3-1)} = t^{-3k} \frac{t^{2(3k+1)} + t^{3k+1} + 1}{t^2 + t + 1} \\
&= t^{-3k} \frac{t^{3k+2}(t^{3k} - 1) + t^{3k}(t^2 + t + 1) + 1 - t^{3k}}{t^2 + t + 1} \\
&= \sum_{i=1}^{k} (t^{3i} - t^{3i-1}) + 1 + \sum_{i=-k}^{-1} (t^{3i} - t^{3i+1}).
\end{aligned}$$

Therefore, $\iota(K) = (0 = V_{3k} < V_{3k-3} < \cdots < V_{-3k} = 3k)$. Furthermore,

$$V_{3k-3i} = \begin{cases} i & \text{for } 0 \le i \le k, \\ 2i - k & \text{for } k < i \le 2k. \end{cases}$$

Consequently,

$$\mathbb{A}(T_{3,3k+1}) = \langle \mathsf{u}^i \mathsf{w}^j \mid 2i + j \ge 3k \text{ and } i + 2j \ge 3k \rangle_{\mathbb{A}}.$$

For $q = 3k-1$, an analogous argument implies that

$$\mathbb{A}(T_{3,3k-1}) = \langle \mathsf{u}^i \mathsf{w}^j \mid 2i + j \ge 3k - 2 \text{ and } i + 2j \ge 3k - 2 \rangle_{\mathbb{A}}.$$

More generally, the ideal sequence for the torus knot $T_{p,pn+1}$ takes the explicit form

$$\iota(T_{p,pn+1}) = \left(0 < 1 < \cdots < n < n+2 < \cdots < 3n < 3n+3 < \cdots < \binom{p}{2} n\right)$$

or, equivalently,

$$i_k = \left(k - \frac{n}{2}\left\lfloor \frac{k}{n} \right\rfloor\right)\left(\left\lfloor \frac{k}{n} \right\rfloor + 1\right) \quad \text{for } k = 0, 1, \ldots, n(p-1).$$

Let $\mathfrak{a}(K)$ denote the minimum degree of a monomial in $\mathbb{A}(K)$. Then, for $T_{p,pn+1}$ we have

$$
\begin{aligned}
(5)\quad \mathfrak{a}(T_{p,pn+1}) &= i_{\lfloor n(p-1)/2\rfloor} + i_{\lceil n(p-1)/2\rceil}\\
&= \Big(\big\lfloor \tfrac12 n(p-1)\big\rfloor - \tfrac12 n\Big\lfloor \frac{1}{n}\big\lfloor \tfrac12 n(p-1)\big\rfloor\Big\rfloor\Big)\Big(\Big\lfloor \frac{1}{n}\big\lfloor \tfrac12 n(p-1)\big\rfloor\Big\rfloor + 1\Big)\\
&\quad + \Big(\big\lceil \tfrac12 n(p-1)\big\rceil - \tfrac12 n\Big\lfloor \frac{1}{n}\big\lceil \tfrac12 n(p-1)\big\rceil\Big\rfloor\Big)\Big(\Big\lfloor \frac{1}{n}\big\lceil \tfrac12 n(p-1)\big\rceil\Big\rfloor + 1\Big)\\
&= n\big\lfloor \tfrac14 p^2\big\rfloor.
\end{aligned}
$$

In general, $\mathfrak{a}(K)$ is determined by the concordance homomorphism upsilon [22], and one can derive this invariant for the torus knot $T_{p,q}$ from the ideal $\mathbb{A}(T_{p,q})$ as follows:

Lemma 5.1 *For every torus knot $T_{p,q}$ with $0<p<q$ we have*

$$\Upsilon_{T_{p,q}}(t) = \Upsilon_{p,q}(t) = -2\min\big\{\tfrac12 ti + \big(1-\tfrac12 t\big)j \mid \mathsf{u}^i\mathsf{w}^j \in \mathbb{A}(T_{p,q})\big\}.$$

Proof Note that

$$\min\big\{\tfrac12 ti + \big(1-\tfrac12 t\big)j \mid \mathsf{u}^i\mathsf{w}^j \in \mathbb{A}(T_{p,q})\big\} = \min\big\{\tfrac12 ti_j + \big(1-\tfrac12 t\big)i_{n-j} \mid 0\le j\le n\big\}.$$

The claim follows from the description of $\Upsilon_{p,q}(t)$ in [19] and the fact that (i_j, i_{n-j}) are the coordinates of $\boldsymbol{x}_{n-2j}$ in Figure 5. □

Corollary 5.2 *For any pair of coprime integers $0<p<q$, we have $\mathfrak{a}(T_{p,q}) = -\Upsilon_{p,q}(1)$ and so*

$$\mathfrak{a}(T_{p,q}) \ge \frac{q-1}{p}\big\lfloor \tfrac14 p^2\big\rfloor.$$

Proof The first part follows from Lemma 5.1 by setting $t=1$. For proving the inequality we use induction with the inductive formula

$$\Upsilon_{p,q}(t) = \Upsilon_{p,q-p}(t) + \Upsilon_{p,p+1}(t)$$

from [7]. First, it follows from (5) that $-\Upsilon_{p,p+1}(1) = \big\lfloor \tfrac14 p^2\big\rfloor$ for all p. Assume the inequality holds for $T_{p,q-p}$. If $p<q-p$, then

$$-\Upsilon_{p,q}(1) = -\Upsilon_{p,q-p}(1) - \Upsilon_{p,p+1}(1) \ge \frac{q-p-1}{p}\big\lfloor \tfrac14 p^2\big\rfloor + \big\lfloor \tfrac14 p^2\big\rfloor = \frac{q-1}{p}\big\lfloor \tfrac14 p^2\big\rfloor.$$

Otherwise, $p>q-p$ and

$$-\Upsilon_{p,q}(1) \ge \frac{p-1}{q-p}\big\lfloor \tfrac14 (q-p)^2\big\rfloor + \big\lfloor \tfrac14 p^2\big\rfloor \ge \frac{q-p-1}{p}\big\lfloor \tfrac14 p^2\big\rfloor + \big\lfloor \tfrac14 p^2\big\rfloor = \frac{q-1}{p}\big\lfloor \tfrac14 p^2\big\rfloor,$$

where the second inequality follows from

$$\frac{p-1}{q-p}\cdot\left(\tfrac{1}{4}(q-p)^2-1\right)\geq\frac{q-p-1}{p}\cdot\left(\tfrac{1}{4}p^2\right).$$

This will finish the proof. □

Proposition 5.3 *If a torus knot $K=T_{p,p'}$ with $0<p<p'$ is Gordian adjacent to a torus knot $K'=T_{q,q'}$ with $0<q<q'$, then*

$$\mathbb{A}(T_{q,q'})\leq\mathbb{A}(T_{p,p'})\quad\textit{and}\quad \mathsf{w}^u\mathbb{A}(T_{p,p'})\leq\mathbb{A}(T_{q,q'}),$$

where $u=u(K')-u(K)=\frac{1}{2}(q-1)(q'-1)-\frac{1}{2}(p-1)(p'-1)$. In particular, $\mathfrak{a}(T_{q,q'})\geq\mathfrak{a}(T_{p,p'})$.

Proof Since $T_{p,p'}$ is Gordian adjacent to $T_{q,q'}$, there exist $\mathbb{A}$–homomorphisms

$$f\colon \mathbb{H}(T_{q,q'})\to\mathbb{H}(T_{p,p'})\quad\text{and}\quad g\colon \mathbb{H}(T_{p,p'})\to\mathbb{H}(T_{q,q'})$$

such that $f\circ g=g\circ f=\mathsf{w}^u$. Note that $\mathbb{H}(T_{p,p'})=\mathbb{A}(T_{p,p'})$ and $\mathbb{H}(T_{q,q'})=\mathbb{A}(T_{q,q'})$. So, f and g are defined by multiplication with polynomials $p,q\in\mathbb{A}=\mathbb{F}[\mathsf{u},\mathsf{w}]$. Thus, $f\circ g=g\circ f=\mathsf{w}^u$ implies that $f=\mathsf{w}^{m^-}$ and $g=\mathsf{w}^{m^+}$ such that $m^++m^-=u$. On the other hand, by Corollary 3.8, a minimal unknotting sequence for a torus knot only consists of positive crossing changes. Thus, $\deg f=m^-=0$ and $\deg g=m^+=u$. Therefore, f is multiplication by 1, g is multiplication by w^u and $\mathbb{A}(T_{q,q'})\leq\mathbb{A}(T_{p,p'})$ and $\mathsf{w}^u\mathbb{A}(T_{p,p'})\leq\mathbb{A}(T_{q,q'})$. □

The computations in Example 5.1 and Proposition 5.3 have a number of quick consequences. One outcome is the following corollary, which was suggested to us by Jennifer Hom. This result was first proved by Borodzik and Livingston in [5].

Corollary 5.4 *If a torus knot $T_{p,p'}$ with $0<p<p'$ is Gordian adjacent to a torus knot $T_{q,q'}$ with $0<q<q'$, then $p\leq q$.*

Proof Assume that

$$\imath(T_{p,p'})=(i_0<\cdots<i_n)\quad\text{and}\quad \imath(T_{q,q'})=(j_0<\cdots<j_m).$$

Proposition 5.3 implies that $\mathsf{w}^{j_m-i_n}\mathbb{A}(T_{p,p'})\leq\mathbb{A}(T_{q,q'})$. Thus,

$$\mathsf{w}^{j_m-i_n+i_{n-1}}\mathsf{u}^{i_1}=\mathsf{w}^{u(K')-u(K)}\mathsf{w}^{i_{n-1}}\mathsf{u}^{i_1}\in\mathbb{A}(T_{q,q'}).$$

Note that $i_1 = j_1 = 1$ and

$$i_n - i_{n-1} = \sum_{k=0}^{2n-1}(-1)^k a_{n-k} - \sum_{k=0}^{2n-3}(-1)^k a_{n-k} = a_{-n+2} - a_{-n+1} = a_{n-1} - a_{n-2} = p-1.$$

Thus, $j_m - j_{m-1} = q-1$, and

$$j_m - i_n + i_{n-1} \geq j_{m-1} \iff \tfrac{1}{2}(q-1)(q'-1) - p + 1 \geq \tfrac{1}{2}(q-1)(q'-1) - q + 1 \iff q \geq p,$$

completing the proof. □

We also obtain a proof of the following corollary. The second statement of the corollary was first proved by Peter Feller in [6].

Corollary 5.5 *If the torus knot $T_{p,pn+1}$ is Gordian adjacent to the torus knot $T_{q,qm+1}$, then*

$$n\left\lfloor \tfrac{1}{4}p^2 \right\rfloor \leq m\left\lfloor \tfrac{1}{4}q^2 \right\rfloor.$$

If $T_{2,n}$ is Gordian adjacent to $T_{3,m}$, where n is odd and m is not a multiple of 3, then $n \leq \frac{4}{3}m + \frac{1}{3}$.

Proof Proposition 5.3 implies that $\mathfrak{a}(T_{p,pn+1}) \leq \mathfrak{a}(T_{q,qm+1})$. So, following the computations of Example 5.1 we have

$$n\left\lfloor \tfrac{1}{4}p^2 \right\rfloor \leq m\left\lfloor \tfrac{1}{4}q^2 \right\rfloor.$$

Moreover, from the same example we know that $\mathbb{A}(T_{3,m}) \leq \mathbb{A}(T_{2,n})$ if and only if, for any pair (i,j) such that $i+2j \geq m-1$ and $j+2i \geq m-1$, we have $i+j \geq \frac{1}{2}(n-1)$. It is clear that

$$\min\{i+j \mid i+2j \geq m-1 \text{ and } 2i+j \geq m-1\} = \left\lceil \tfrac{2}{3}(m-1) \right\rceil = \left\lfloor \tfrac{2}{3}m - \tfrac{1}{3} \right\rfloor.$$

Thus, $\frac{1}{2}(n-1) \leq \frac{2}{3}m - \frac{1}{3}$ and $n \leq \frac{4}{3}m + \frac{1}{3}$. □

Example 5.2 An interesting example is the case of the figure 8 knot, where the chain complex is generated by five generators X, Y, Z, W and B, where $d(B) = d(X) = 0$ while $d(W) = \mathsf{u}Z + \mathsf{w}Y$, $d(Y) = \mathsf{u}X$ and $d(Z) = \mathsf{w}X$. Thus, $\mathbb{T}(K)$ is generated by $\boldsymbol{x} = [X]$ and $\mathsf{u}\boldsymbol{x}$ and $\mathsf{w}\boldsymbol{x}$ are both zero. Moreover, $\mathbb{A}(K)$ is generated by $[B]$ and is isomorphic with $\mathbb{A}$. In particular, $\nu^+(K) = \nu^+(-K) = 0$, while $\mathfrak{t}(K) = \hat{\mathfrak{t}}(K) = 1$. The subcomplex generated by X, Y, Z and W will be referred to as a *square*.

Example 5.3 Alternating knots are known to have simple knot Floer chain complexes. The restriction on the Alexander and Maslov grading of generators (that their difference is a constant number) implies that the chain complex decomposes as the (shifted) direct sum of a copy of $\mathrm{CF}(\pm T_{2,2n+1})$ and several squares. In particular, if K is an alternating knot with $\tau(K) > 0$ then

$$\iota(K) = \iota(T_{2,2n+1}) = (0 < 1 < 2 < \cdots < \tau(K)),$$

while $\mathfrak{t}^+(K) \leq 1$. Moreover, $\mathfrak{t}^-(K) = \mathfrak{t}^+(-K) = \mathfrak{t}^+(-T_{2,2n+1}) = n$. So $\mathfrak{t}(K) = n$.

On the other hand, Since $\mathfrak{l}(T_{2,2n+1}) = \mathfrak{l}^+(T_{2,2n+1}) = u^+(T_{2,2n+1}) = n$, there are chain maps

$$\mathfrak{f}^-\colon \mathrm{CF}(T_{2,2n+1}) \to \mathbb{A} \quad \text{and} \quad \mathfrak{f}^+\colon \mathbb{A} \to \mathrm{CF}(T_{2,2n+1})$$

such that both $\mathfrak{f}^+ \circ \mathfrak{f}^-$ and $\mathfrak{f}^- \circ \mathfrak{f}^+$ are chain homotopic to multiplication by w^n. Further, for each square summand, it is easy to check that multiplication by w is nullhomotopic. Thus, we can extend $\mathfrak{f}^-$ to a map from $\mathrm{CF}(K)$ by defining it to be zero on all square summands, and compose $\mathfrak{f}^+$ with the inclusion of $\mathrm{CF}(T_{2,2n+1})$ in $\mathrm{CF}(K)$ and both $\mathfrak{f}^+ \circ \mathfrak{f}^-$ and $\mathfrak{f}^- \circ \mathfrak{f}^+$ remain chain homotopic to multiplication by w^n. Therefore,

$$n \geq \mathfrak{l}(K) \geq \mathfrak{l}^+(K) \geq \nu^+(K) = n$$

and so $\mathfrak{l}(K) = \mathfrak{l}^+(K) = \nu^+(K) = n$.

Example 5.3 gives interesting bounds on the *alternation number* $\mathrm{alt}(K)$ of a knot K, defined as the minimum Gordian distance between K and an alternating knot. The first bound is very similar to, yet different from, the bound constructed in [8, Corollary 2.2].

Proposition 5.6 *The alternation number* $\mathrm{alt}(K)$ *of a knot* $K \subset S^3$ *satisfies*

$$\mathrm{alt}(K) \geq \nu^+(K) - \mathfrak{a}(K), \quad \mathrm{alt}(K) \geq \hat{\mathfrak{t}}(K) - 1 \quad \textit{and} \quad \mathrm{alt}(K) \geq \min\{\mathfrak{t}^+(K) - 1, \nu^+(K)\}.$$

Proof Let us assume that K is modified to an alternating knot K' using a sequence of m^+ positive crossing changes and m^- negative crossing changes and that $\mathrm{alt}(K) = m^+ + m^-$. It follows that $\nu^+(K') \geq \nu^+(K) - m^+$. Since $\mathsf{w}^{m^-}\mathbb{A}(K)$ is a subset of $\mathbb{A}(K')$, it follows that $\mathbb{A}(K')$ includes a monomial of degree $m^- + \mathfrak{a}(K)$. However, every monomial in $\mathbb{A}(K')$ has degree at least $\nu^+(K')$. This means that

$$\mathfrak{a}(K) + m^- \geq \nu^+(K') \geq \nu^+(K) - m^+ \implies m^+ + m^- \geq \nu^+(K) - \mathfrak{a}(K),$$

and completes the proof of the first inequality. The second and third inequalities are easier. For the second equality, note that in the above situation,

$$u(K,K') \geq \hat{\mathfrak{t}}(K) - \hat{\mathfrak{t}}(K') = \hat{\mathfrak{t}}(K) - 1.$$

For the third inequality, we have

$$\nu^+(K) \leq \nu^+(K') + m^+ \quad \text{and} \quad \mathfrak{t}^+(K) \leq \mathfrak{t}^+(K') + m^+ + m^-.$$

If $\nu^+(K') = 0$ then $\nu^+(K) \leq m^+ \leq \mathrm{alt}(K)$. Otherwise, $\tau(K') = \nu^+(K') > 0$ and $\mathbb{T}(K')$ can only include torsion elements trivialized by w. In particular, $\mathfrak{t}^+(K') \leq 1$ and $\mathrm{alt}(K) = m^+ + m^- \geq \mathfrak{t}^+(K) - 1$. □

For torus knots, we obtain the following corollary from our computations in Example 5.1. Similar bounds may also be obtained using upsilon invariants; see [8] for the case $p < 5$.

Corollary 5.7 *The alternation number of the torus knot $T_{p,pn+1}$ is $\geq n\lfloor \frac{1}{4}(p-1)^2 \rfloor$.*

Proof Using the first inequality in Proposition 5.6, we have

$$\mathrm{alt}(T_{p,pn+1}) \geq \nu^+(T_{p,pn+1}) - \mathfrak{a}(T_{p,pn+1}) = n\binom{p}{2} - n\lfloor \tfrac{1}{4}p^2 \rfloor = n\lfloor \tfrac{1}{4}(p-1)^2 \rfloor.$$

This completes the proof. □

Example 5.4 The knot $12n_{404}$, which is a $(1,1)$ knot, is illustrated in Figure 6. Using Rasmussen's notation [30, page 14], it is given by the quadruple $[29, 7, 14, 1]$. The corresponding chain complex $\mathrm{CF}(12n_{404})$ may be computed combinatorially, eg using Krcatovich's computer program [15]. After a straightforward change of basis, we arrive at the chain complex illustrated in Figure 7.

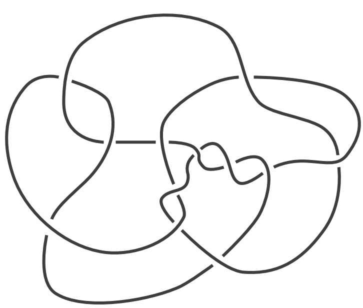

Figure 6: The knot $12n_{404}$.

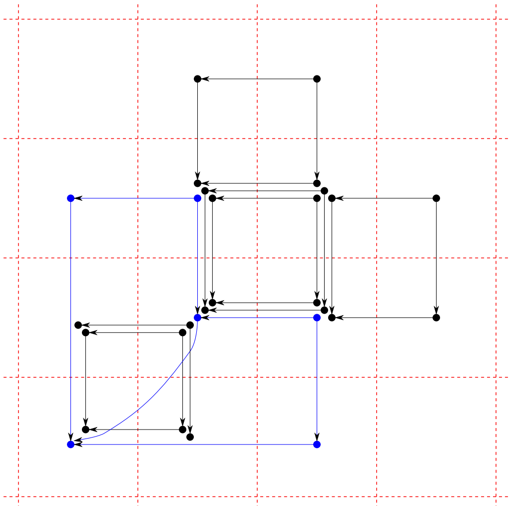

Figure 7: The chain complex associated with the knot $12n_{404}$.

Each dot represents a generator of $\mathrm{CF}(12n_{404})$. An arrow which connects a dot corresponding to a generator $\boldsymbol{x}$ to a dot representing a generator $\boldsymbol{y}$ and cuts i vertical lines and j horizontal lines corresponds to the contribution of $\mathsf{u}^i\mathsf{w}^j\boldsymbol{y}$ to $d(\boldsymbol{x})$. The blue dots and the black dots in the diagram generate subcomplexes C and C' of the knot chain complex, respectively, and we obtain a decomposition $\mathrm{CF}(12n_{404})=C\oplus C'$. We may then identify

$$C=\langle X,Y_0,Y_1,Y_2,Z_0,Z_1\rangle_{\mathbb{A}},\quad d(Y_i)=\mathsf{u}^i\mathsf{w}^{2-i}X\quad\text{and}\quad d(Z_i)=\mathsf{u}Y_i+\mathsf{w}Y_{i+1}.$$

The homology of C is generated by $\boldsymbol{x}=[X]$, with $\mathsf{w}^2\boldsymbol{x}=\mathsf{wu}\boldsymbol{x}=\mathsf{u}^2\boldsymbol{x}=0$.

For C', the same argument as in Example 5.3 implies that there are chain maps

$$f^-\colon C'\to\mathbb{A}\quad\text{and}\quad f^+\colon\mathbb{A}\to C'$$

such that $\deg f^-=0$, $\deg f^+=1$ and both $f^-\circ f^+$ and $f^+\circ f^-$ are chain homotopic to multiplication by w. Thus, $\mathfrak{t}^+(12n_{404})=2$. In fact, it is straightforward from the above presentation of chain complex to conclude that $\mathfrak{t}(12n_{404})=\widehat{\mathfrak{t}}(12n_{404})=2$.

On the other hand, the map $h\colon C\to C$ defined as

$$h(X)=Y_0,\quad h(Y_1)=\mathsf{w}Z_0,\quad h(Y_2)=\mathsf{u}Z_0+\mathsf{w}Z_1,\quad h(Y_0)=h(Z_0)=h(Z_1)=0$$

gives a homotopy between multiplication with w^2 and 0. So we can extend $\mathsf{w}f^-$ trivially to a map from $\mathrm{CF}(12n_{404})$ and compose f^+ with the inclusion map to get

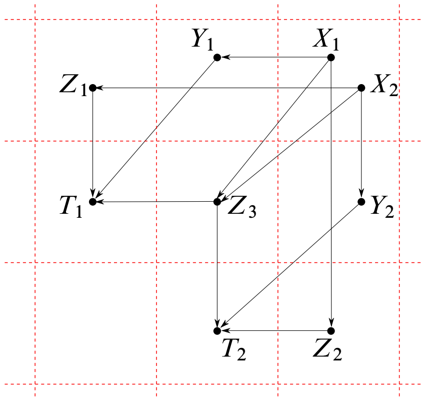

Figure 8: The chain complex associated with the knot $T_{2,3;2,-1}$.

chain maps

$$\mathfrak{f}^-\colon \mathrm{CF}(12n_{404}) \to \mathbb{A} \quad \text{and} \quad \mathfrak{f}^+\colon \mathbb{A} \to \mathrm{CF}(12n_{404})$$

such that $\mathfrak{f}^+ \circ \mathfrak{f}^-$ and $\mathfrak{f}^- \circ \mathfrak{f}^+$ are chain homotopic to multiplication by w^2. Therefore,

$$\mathfrak{l}^+(12n_{404}) = \nu^+(12n_{404}) = 1 \quad \text{and} \quad \mathfrak{l}(12n_{404}) = \mathfrak{t}(12n_{404}) = 2.$$

Similarly, we may use f^- and $\mathsf{w} f^+$ to construct another pair $\mathfrak{f}^-$ and $\mathfrak{f}^+$. This implies

$$\mathfrak{l}^-(12n_{404}) = 0.$$

The knot $12n_{404}$ may be unknotted by changing three crossings. It is not known, however, whether $u(12n_{404})$ is equal to 3 or not [20]. The alternation number $\mathrm{alt}(12n_{404})$ is 1, which matches the lower bound given by the last two inequalities in Proposition 5.6.

Example 5.5 Consider the $(2,-1)$ cable of the torus knot $T_{2,3}$, which is denoted by $T_{2,3;2,-1}$. The chain complex associated with this knot is computed in [9] and is illustrated in Figure 8.

The chain complex is generated over $\mathbb{F}[\mathsf{u},\mathsf{w}]$ by the nine generators X_1, X_2, Y_1, Y_2, Z_1, Z_2, Z_3, T_1 and T_2. The differential is given by $d(T_i) = 0$, $d(Y_i) = \mathsf{uw}T_i$ for $i = 1, 2$ and

$$d(Z_1) = \mathsf{w}T_1, \qquad d(X_1) = \mathsf{u}Y_1 + \mathsf{uw}Z_3 + \mathsf{w}^2 Z_2,$$
$$d(Z_3) = \mathsf{u}T_1 + \mathsf{w}T_2, \qquad d(X_2) = \mathsf{u}^2 Z_1 + \mathsf{uw}Z_3 + \mathsf{w}Y_2.$$
$$d(Z_2) = \mathsf{u}T_2,$$

The generators of homology may then be specified as $\boldsymbol{t}_1 = [T_1], \boldsymbol{t}_2 = [T_2], \boldsymbol{y}_1 = [Y_1 + \mathsf{u}Z_1]$ and $\boldsymbol{y}_2 = [Y_2 + \mathsf{w}Z_2]$, where we have

$$\mathsf{u}\boldsymbol{t}_1 = \mathsf{w}\boldsymbol{t}_2, \quad \mathsf{w}\boldsymbol{t}_1 = \mathsf{u}\boldsymbol{t}_2 = 0 \quad \text{and} \quad \mathsf{u}\boldsymbol{y}_1 = \mathsf{w}\boldsymbol{y}_2.$$

It thus follows that

$$\mathbb{H}(T_{2,3;2,-1}) = \mathbb{A}(T_{2,3;2,-1}) \oplus \mathbb{T}(T_{2,3;2,-1}) = \langle \mathsf{u}, \mathsf{w}\rangle_{\mathbb{A}} \oplus \frac{\langle \mathsf{u}, \mathsf{w}\rangle_{\mathbb{A}}}{\langle \mathsf{u}^2, \mathsf{w}^2\rangle_{\mathbb{A}}}.$$

In particular, $\mathfrak{t}^+(T_{2,3;2,-1}) = \hat{\mathfrak{t}}(T_{2,3;2,-1}) = 2$, and $\nu^+(T_{2,3;2,-1}) = 1$. Therefore, $\mathfrak{l}(T_{2,3;2,-1}) \geq 2$. Let

$$\mathfrak{f}^-\colon \mathrm{CF}(T_{2,3;2,-1}) \to \mathbb{F}[\mathsf{u}, \mathsf{w}] \quad \text{and} \quad \mathfrak{f}^+\colon \mathbb{F}[\mathsf{u}, \mathsf{w}] \to \mathrm{CF}(T_{2,3;2,-1})$$

be the chain maps defined by

$$\mathfrak{f}^-(Y_1) = \mathsf{w}^2, \quad \mathfrak{f}^-(Z_3) = \mathsf{w}, \quad \mathfrak{f}^-(Y_2) = \mathsf{uw},$$
$$\mathfrak{f}^-(X_1) = \mathfrak{f}^-(X_2) = \mathfrak{f}^-(Z_1) = \mathfrak{f}^-(Z_2) = \mathfrak{f}^-(T_1) = \mathfrak{f}^-(T_2) = 0$$

and $\mathfrak{f}^+(1) = Y_1 + \mathsf{u}Z_1$. Then, $\mathfrak{f}^- \circ \mathfrak{f}^+ = \mathsf{w}^2$ and $\mathfrak{f}^+ \circ \mathfrak{f}^- \simeq \mathsf{w}^2$ where the chain homotopy is given by

$$h(T_1) = \mathsf{w}Z_1, \quad h(T_2) = Y_1 + \mathsf{w}Z_3, \quad h(Z_2) = X_1, \quad h(Y_2) = \mathsf{w}X_2,$$
$$h(X_1) = h(X_2) = h(Y_1) = h(Z_1) = h(Z_3) = 0.$$

Thus, $\mathfrak{l}(T_{2,3;2,-1}) = 2$. Since the torsion invariant $\mathfrak{t}^+(T_{2,3})$ is zero, it follows that the Gordian distance between $T_{2,3;2,-1}$ and the trefoil $T_{2,3}$ is at least 2.

Example 5.6 Let us now consider the $(2,-3)$ cable of the torus knot $T_{2,3}$, which is denoted by $T_{2,3;2,-3}$. We focus on the mirror image $K = -T_{2,3;2,-3}$ of the aforementioned knot. The chain complex associated with K is illustrated in Figure 9 (see [11]).

The chain complex is generated over $\mathbb{F}[\mathsf{u}, \mathsf{w}]$ by 11 generators T_1, T_2, X_1, X_2, X_3, Y_1, Y_2, Z_1, Z_2, Z_3 and Z_4. The differential is given by $d(T_1) = d(T_2) = 0$ and

$$\begin{aligned} d(Y_1) &= \mathsf{u}T_1, & d(Z_1) &= \mathsf{w}^2 T_1, & d(X_1) &= \mathsf{u}Z_1 + \mathsf{w}Z_3,\\ d(Y_2) &= \mathsf{w}T_2, & d(Z_2) &= \mathsf{u}^2 T_2, & d(X_2) &= \mathsf{w}Z_2 + \mathsf{u}Z_4,\\ & & d(Z_3) &= \mathsf{uw}T_1, & d(X_3) &= \mathsf{u}Z_3 + \mathsf{w}Z_4 + \mathsf{uw}(Y_1 + Y_2).\\ & & d(Z_4) &= \mathsf{uw}T_2, & & \end{aligned}$$

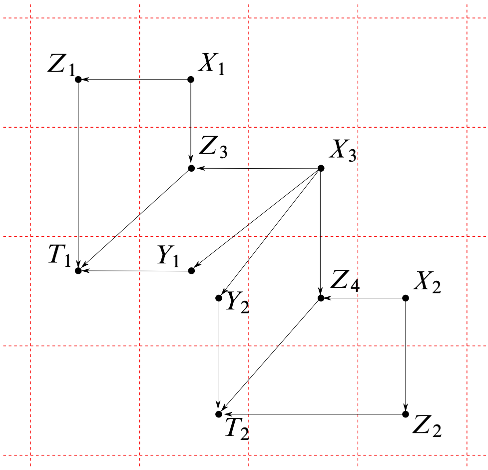

Figure 9: The chain complex associated with the knot $-T_{2,3;2,-3}$.

The homology of the above chain complex is generated by $\boldsymbol{t}_1 = [T_1]$, $\boldsymbol{t}_2 = [T_2]$, $\boldsymbol{y}_1 = [Z_3 + \mathsf{w}Y_1]$ and $\boldsymbol{y}_2 = [Z_4 + \mathsf{u}Y_2]$, while we also have

$$\mathsf{u}\boldsymbol{t}_1 = \mathsf{w}^2\boldsymbol{t}_1 = \mathsf{w}\boldsymbol{t}_2 = \mathsf{u}^2\boldsymbol{t}_2 = 0 \quad \text{and} \quad \mathsf{u}\boldsymbol{y}_1 = \mathsf{w}\boldsymbol{y}_2.$$

It thus follows that

$$\begin{aligned}\mathbb{H}(-T_{2,3;2,-3}) &= \mathbb{A}(-T_{2,3;2,-3}) \oplus \mathbb{T}(-T_{2,3;2,-3}) \\ &= \langle \mathsf{u}, \mathsf{w}\rangle_{\mathbb{A}} \oplus \left(\frac{\mathbb{A}}{\langle \mathsf{u}, \mathsf{w}^2\rangle_{\mathbb{A}}} \oplus \frac{\mathbb{A}}{\langle \mathsf{u}^2, \mathsf{w}\rangle_{\mathbb{A}}} \right).\end{aligned}$$

By considering the dual complex, one can show that

$$\mathbb{H}(T_{2,3;2,-3}) = \mathbb{A} \oplus \frac{\langle \mathsf{u}, \mathsf{w}\rangle_{\mathbb{A}}}{\langle \mathsf{u}^2, \mathsf{w}^2\rangle_{\mathbb{A}}}.$$

In particular, we have $\nu^+(-T_{2,3;2,-3}) = 1$ and $\nu^+(T_{2,3;2,-3}) = 0$, while the torsion invariants are nontrivial:

$$\mathfrak{t}^+(T_{2,3;2,-3}) = \mathfrak{t}^-(T_{2,3;2,-3}) = \hat{\mathfrak{t}}(T_{2,3;2,-3}) = \hat{\mathfrak{t}}(-T_{2,3;2,-3}) = 2.$$

Example 5.7 This example illustrates that $\mathbb{H}(K)$ is not necessarily the direct sum of $\mathbb{A}(K)$ and $\mathbb{T}(K)$. Let $K = T_{4,5} \# -T_{2,3;2,5} \# T_{2,3}$. The chain complex $\mathrm{CF}(T_{2,3;2,5})$ is given in [10], and using that one could show $\mathrm{CF}(K) = C \oplus C'$, where C is as illustrated in Figure 10 and C' is a direct sum of acyclic pieces. Moreover, the homology of C' is freely generated by torsion elements $\boldsymbol{t}_i$ such that $\mathsf{u}\boldsymbol{t}_i = \mathsf{w}\boldsymbol{t}_i = 0$.

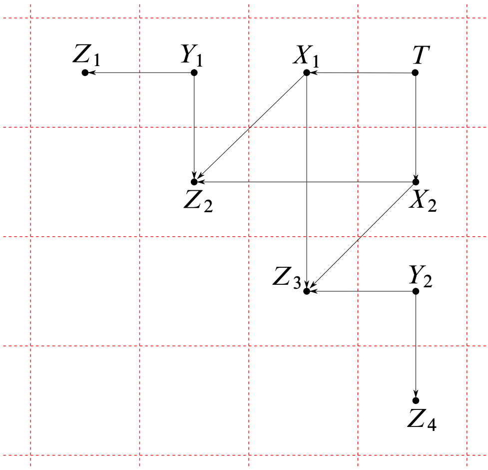

Figure 10: The chain complex C associated with the knot $T_{4,5}\#-T_{2,3;2,5}\#T_{2,3}$.

The chain complex C is generated over $\mathbb{F}[\mathsf{u},\mathsf{w}]$ by the nine generators X_1, X_2, Y_1, Y_2, Z_1, Z_2, Z_3, Z_4 and T. The differential is given by $d(Z_i)=0$ for $i=1,2,3,4$ and

$$\begin{aligned} d(Y_1)&=\mathsf{u}Z_1+\mathsf{w}Z_2, \\ d(Y_2)&=\mathsf{u}Z_3+\mathsf{w}Z_4, \\ d(T)&=\mathsf{u}X_1+\mathsf{w}X_2, \end{aligned} \qquad \begin{aligned} d(X_1)&=\mathsf{uw}Z_2+\mathsf{w}^2Z_3, \\ d(X_2)&=\mathsf{u}^2Z_2+\mathsf{uw}Z_3. \end{aligned}$$

The homology of C is then generated by the classes $\boldsymbol{z}_i=[Z_i]$ for $i=1,2,3,4$, while we have

$$\mathsf{u}\boldsymbol{z}_1=\mathsf{w}\boldsymbol{z}_2,\quad \mathsf{u}\boldsymbol{z}_3=\mathsf{w}\boldsymbol{z}_4,\quad \mathsf{uw}\boldsymbol{z}_2=\mathsf{w}^2\boldsymbol{z}_3\quad\text{and}\quad \mathsf{u}^2\boldsymbol{z}_2=\mathsf{uw}\boldsymbol{z}_3.$$

In particular, $\boldsymbol{t}=\mathsf{u}\boldsymbol{z}_2-\mathsf{w}\boldsymbol{z}_3$ is a torsion element, and $\mathsf{u}\boldsymbol{t}=\mathsf{w}\boldsymbol{t}=0$. We then have a short exact sequence

$$0\to\frac{\mathbb{A}}{\langle\mathsf{u},\mathsf{w}\rangle_{\mathbb{A}}}\to H_*(C)\to\mathbb{A}(K)=\langle\mathsf{u}^3,\mathsf{u}^2\mathsf{w},\mathsf{uw}^2,\mathsf{w}^3\rangle_{\mathbb{A}}\to 0,$$

which does not split. The chain complex C is an illustration of pieces which may appear in a knot chain complex and make the homology and the unknotting invariants interesting. The next *virtual* example gives another instance of this phenomenon.

Example 5.8 Let $C=C_{i,j}$ denote the chain complex generated over $\mathbb{A}$ by the generators X_1, X_2, Y_1, Y_2 and Z with

$$A(X_1)=-A(X_2)=i,\quad A(Y_1)=-A(Y_2)=j\quad\text{and}\quad A(Z)=0.$$

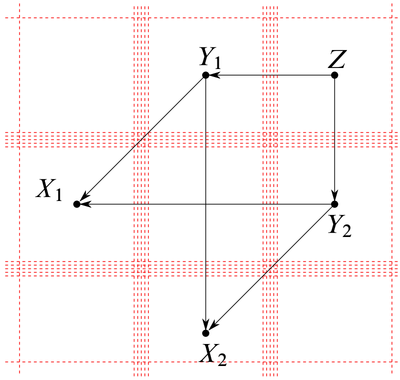

Figure 11: The chain complex $C_{i,j}$.

The differential $d=d_{i,j}$ of C is defined by setting $d(X_1)=d(X_2)=0$ and

$$d(Y_1)=\mathsf{u}^i\mathsf{w}^j X_1+\mathsf{w}^{i+j}X_2,\quad d(Y_2)=\mathsf{u}^{i+j}X_1+\mathsf{u}^j\mathsf{w}^i X_2,\quad d(Z)=\mathsf{u}^j Y_1+\mathsf{w}^j Y_2.$$

Figure 11 illustrates this chain complex. We treat $C_{i,j}$ as a direct summand in a knot chain complex, or a *virtual* knot chain complex.

It is then not hard to check that the homology group $\mathbb{H}=\mathbb{H}_{i,j}$ of C is generated by the homology classes $\boldsymbol{x}_1=[X_1]$ and $\boldsymbol{x}_2=[X_2]$. Furthermore, $\boldsymbol{t}=\mathsf{u}^i\boldsymbol{x}_1+\mathsf{w}^i\boldsymbol{x}_2$ is a torsion element in $\mathbb{H}$. In fact,

$$\mathsf{w}^j\boldsymbol{t}=[dY_1]=0\quad\text{and}\quad\mathsf{u}^j\boldsymbol{t}=[dY_2]=0.$$

Thus, $\mathfrak{t}^+(C)=j$. Assume $\mathfrak{f}^-\colon\mathbb{H}\to\mathbb{A}$ and $\mathfrak{f}^+\colon\mathbb{A}\to\mathbb{H}$ are homogeneous maps of degrees m^- and m^+, respectively, so that $\mathfrak{f}^-\circ\mathfrak{f}^+$ and $\mathfrak{f}^+\circ\mathfrak{f}^-$ are multiplication by $\mathsf{w}^{m^++m^-}$. Let $m=m^++m^-$, and

$$\mathfrak{f}^+(1)=f_1^+\boldsymbol{x}_1+f_2^+\boldsymbol{x}_2,\quad\mathfrak{f}^-(\boldsymbol{x}_1)=f_1^-\quad\text{and}\quad\mathfrak{f}^-(\boldsymbol{x}_2)=f_2^-,$$

where $f_i^+,f_i^-\in\mathbb{A}$ for $i=1,2$. Then, $f_1^-f_1^++f_2^-f_2^+=\mathsf{w}^m$ and so

$$\mathfrak{f}^+\circ\mathfrak{f}^-(\boldsymbol{x}_1)=\mathsf{w}^m\boldsymbol{x}_1+f_2^+(f_2^-\boldsymbol{x}_1+f_1^-\boldsymbol{x}_2)$$

and

$$\mathfrak{f}^+\circ\mathfrak{f}^-(\boldsymbol{x}_2)=\mathsf{w}^m\boldsymbol{x}_2+f_1^+(f_2^-\boldsymbol{x}_1+f_1^-\boldsymbol{x}_2).$$

Consequently, $f_2^-\boldsymbol{x}_1+f_1^-\boldsymbol{x}_2$ is a multiple of $\boldsymbol{t}$, which implies that $f_2^-=\mathsf{w}^{m^-}\mathsf{u}^i g$ and $f_1^-=\mathsf{w}^{m^-+i}g$ for some $g\in\mathbb{A}$. On the other hand, since $f_1^-f_1^++f_2^-f_2^+=\mathsf{w}^m$ we have $g(0,0)=1$ and $f_1^+ = \mathsf{w}^{m^+-i}h$ for some $h \in \mathbb{A}$ such that $h(0,0)=1$. Therefore,

$$f_1^+(f_2^- \boldsymbol{x}_1 + f_1^- \boldsymbol{x}_2) = \mathsf{w}^{m^+-i}h(\mathsf{w}^{m^-} g\cdot \boldsymbol{t}) = (\mathsf{w}^{m-i}hg)\boldsymbol{t} = 0,$$

which implies that $m-i \geq j$ and so $m \geq i+j$. In other words, $\mathfrak{l}(C) \geq i+j$.

Moreover, we show that $\mathfrak{l}(C) = i+j$ by finding explicit chain maps. Let $\mathfrak{f}^-\colon C \to \mathbb{A}$ and $\mathfrak{f}^+\colon \mathbb{A} \to C$ be chain maps defined as

$$\mathfrak{f}^-(X_1) = \mathsf{w}^i, \quad \mathfrak{f}^-(X_2) = \mathsf{u}^i, \quad \mathfrak{f}^-(Y_1) = \mathfrak{f}^-(Y_2) = \mathfrak{f}^-(Z) = 0, \qquad \mathfrak{f}^+(1) = X_1.$$

It is easy to check that $\mathsf{w}^j(\mathfrak{f}^+ \circ \mathfrak{f}^-)$ is chain homotopic to multiplication by w^{i+j}. In fact, chain homotopy is given by

$$h(X_2) = Y_1, \quad h(Y_2) = \mathsf{w}^i Z \quad \text{and} \quad h(X_1) = h(Y_1) = h(Z) = 0.$$

Further, $\mathsf{w}^j(\mathfrak{f}^- \circ \mathfrak{f}^+)$ is equal to multiplication by w^{i+j}. Therefore, $\mathfrak{l}(C) = i+j$, while, considering the pairs $(\mathsf{w}^j\mathfrak{f}^-, \mathfrak{f}^+)$ and $(\mathfrak{f}^-, \mathsf{w}^j\mathfrak{f}^+)$, we have $\mathfrak{l}^+(C) = \nu^+(C) = i$ and $\mathfrak{l}^-(C) = 0$, respectively. Moreover, $\mathfrak{t}(C) = j$. Thus, C gives an example with $\mathfrak{l}(C) = \nu^+(C) + \mathfrak{t}(C)$. It is interesting to note that in this example, $\hat{\mathfrak{t}}(C) = i+j$.

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