Kaledin's degeneration theorem and topological Hochschild homology

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We give a short proof of Kaledin's theorem on the degeneration of the noncommutative Hodge-to-de Rham spectral sequence. Our approach is based on topological Hochschild homology and the theory of cyclotomic spectra. As a consequence, we also obtain relative versions of the degeneration theorem, both in characteristic zero and for regular bases in characteristic p.

1 Introduction

Let X be a smooth and proper variety over a field k. A basic invariant of X arises from the *algebraic de Rham cohomology*, $\mathrm{H}^*_{\mathrm{DR}}(X)$, given as the hypercohomology of the complex Ω^*_X of sheaves of algebraic differential forms on X with the de Rham differential. Then $\mathrm{H}^*_{\mathrm{DR}}(X)$ is a finite-dimensional graded k–vector space, and is the abutment of the classical *Hodge-to-de Rham* spectral sequence $\mathrm{H}^i(X, \Omega^j_X) \Rightarrow \mathrm{H}^{i+j}_{\mathrm{DR}}(X)$ arising from the naive filtration of the complex of sheaves Ω^*_X. It is a fundamental fact in algebraic geometry that this spectral sequence degenerates when k has characteristic zero. When $k = \mathbb{C}$ and X is Kähler, the degeneration arises from Hodge theory.

After 2–periodization and in characteristic zero, the above invariants and questions have *noncommutative* analogs, ie they are defined more generally for differential graded (dg) categories rather than only for varieties. Let $\mathcal{C}$ be a smooth and proper dg category over a field k (eg $\mathcal{C}$ could be the derived category $D^b\mathrm{Coh}(X)$ of a smooth and proper variety X/k). In this case, a basic invariant of $\mathcal{C}$ is given by the *Hochschild homology* $\mathrm{HH}(\mathcal{C}/k)$, regarded as a noncommutative version of differential forms for $\mathcal{C}$ thanks to the classical Hochschild–Kostant–Rosenberg theorem.

Hochschild homology takes values in the derived category $D(k)$ of k–vector spaces; it produces a perfect complex equipped with an action of the circle S^1, the noncommutative version of the de Rham differential. As a result, one can take the S^1–Tate construction to form $\mathrm{HP}(\mathcal{C}/k) := \mathrm{HH}(\mathcal{C}/k)^{tS^1}$, called the *periodic cyclic homology*

of $\mathcal{C}$ and often regarded as a noncommutative version of de Rham cohomology. One has a general spectral sequence, arising from the Postnikov filtration of $\mathrm{HH}(\mathcal{C}/k)$, $\mathrm{HH}_*(\mathcal{C}/k)[u^{\pm1}] \Rightarrow \mathrm{HP}_*(\mathcal{C}/k)$, called the (noncommutative) *Hodge-to-de Rham spectral sequence*. When $\mathcal{C} = D^b\mathrm{Coh}(X)$ for X in characteristic zero, this reproduces a 2–periodic analog of the Hodge-to-de Rham spectral sequence.

The papers [31; 32] of Kaledin describe a proof of the following result, conjectured by Kontsevich and Soibelman [33, Conjecture 9.1.2].

Theorem 1.1 (Kaledin) *Let $\mathcal{C}$ be a smooth and proper dg category over a field k of characteristic zero. Then the Hodge-to-de Rham spectral sequence*

$$E_2 = \mathrm{HH}_*(\mathcal{C}/k)[u^{\pm1}] \Rightarrow \mathrm{HP}_*(\mathcal{C}/k)$$

degenerates at E_2.

An equivalent statement is that the S^1–action on $\mathrm{HH}(\mathcal{C}/k)$, considered as an object of the derived category $D(k)$, is trivial; thus we may regard the result as a type of *formality* statement. Using the comparison between 2–periodic de Rham cohomology and periodic cyclic homology in characteristic zero, one recovers the classical result that the (commutative) Hodge-to-de Rham spectral sequence $\mathrm{H}^i(X, \Omega_X^j) \Rightarrow \mathrm{H}_{\mathrm{dR}}^{i+j}(X)$ from Hodge cohomology to de Rham cohomology degenerates for a smooth and proper variety X in characteristic zero.

Kaledin's proof of Theorem 1.1 is based on reduction mod p. Motivated by the approach of Deligne and Illusie [17] in the commutative case, Kaledin proves a formality statement for Hochschild homology in characteristic p of smooth and proper dg categories which satisfy an amplitude bound on Hochschild cohomology and which admit a lifting mod p^2. Compare [32, Theorems 5.1 and 5.5].

In this paper, we will give a short proof of the following slight variant of Kaledin's characteristic p degeneration results. Analogous arguments as in [31; 32] show that this variant also implies Theorem 1.1.

Theorem 1.2 *Let $\mathcal{C}$ be a smooth and proper dg category over a perfect field k of characteristic $p > 0$. Suppose that*:

(1) *$\mathcal{C}$ has a lift to a smooth proper dg category over $W_2(k)$.*

(2) *$\mathrm{HH}_i(\mathcal{C}/k)$ vanishes for $i \notin [-p, p]$.*

Then the Hodge-to-de Rham spectral sequence $\mathrm{HH}_*(\mathcal{C}/k)[u^{\pm1}] \Rightarrow \mathrm{HP}_*(\mathcal{C}/k)$ degenerates at E_2.

We will deduce Theorem 1.2 from the framework of *topological* Hochschild homology and in particular the theory of cyclotomic spectra as recently reformulated by Nikolaus and Scholze [40]. We give an overview of this apparatus in Section 2. The idea of using cyclotomic spectra here is, of course, far from new, and is already indicated in the papers of Kaledin.

Given $\mathcal{C}$, one considers the topological Hochschild homology $\mathrm{THH}(\mathcal{C})$ as a module over the E_∞–ring $\mathrm{THH}(k)$, whose homotopy groups are given by $k[\sigma]$ for $|\sigma| = 2$. One has equivalences of spectra:

(1) $\mathrm{THH}(\mathcal{C})/\sigma \simeq \mathrm{HH}(\mathcal{C}/k)$.

(2) $\mathrm{THH}(\mathcal{C})[1/\sigma]^{(1)} \simeq \mathrm{HP}(\mathcal{C}/k)$ for smooth and proper $\mathcal{C}/k$. Here the superscript (1) denotes the Frobenius twist.

The first equivalence is elementary, while the second arises from the cyclotomic Frobenius and should be compared to the "noncommutative Cartier isomorphisms" studied by Kaledin. These observations imply that the difference between 2–periodic Hochschild homology and periodic cyclic homology (ie differentials in the spectral sequence) is controlled precisely by the presence of σ–torsion in $\mathrm{THH}_*(\mathcal{C})$. Under the above assumptions of liftability and amplitude bounds, the degeneration statement then follows from an elementary argument directly on the level of THH. We formulate this as a general formality statement in Proposition 3.9 below.

We also apply our methods to prove freeness and degeneration assertions in Hochschild homology for families of smooth and proper dg categories. We first review the commutative version. If S is a scheme of finite type over a field of characteristic zero and $f: X \to S$ a proper smooth map, then one knows by a classical theorem of Deligne [16] that the relative Hodge cohomology sheaves $R^i f_* \Omega^j_{X/S}$ form vector bundles on S, and that the relative Hodge-to-de Rham spectral sequence degenerates when S is affine. When S is smooth, this can be deduced by reduction mod p and a relative version of the Deligne–Illusie constructions as in [30].

There are noncommutative versions of these relative results, too. For example, in characteristic zero, one has the following result:

Theorem 1.3 *Let A be a commutative $\mathbb{Q}$–algebra and let $\mathcal{C}$ be a smooth proper dg category over A. Then*:

(1) *The Hochschild homology groups $\mathrm{HH}_i(\mathcal{C}/A)$ are finitely generated projective A–modules.*

(2) *The relative Hodge-to-de Rham spectral sequence degenerates.*

This result can be deduced from Kaledin's theorem. When A is smooth at least, the freeness of $\mathrm{HH}_i(\mathcal{C}/A)$ follows from the existence of a flat connection on periodic cyclic homology, due to Getzler [20], together with Theorem 1.1. Compare also [33, Remark 9.1.4] for a statement. We will give a short proof inspired by this idea, in the form of the nilinvariance of periodic cyclic homology in characteristic zero and a Künneth theorem.

In fact, we will formulate the argument as a general formality (and local freeness) criterion for S^1–actions via cyclotomic spectra. This includes the argument for Kaledin's theorem as well as additional input for the relative case.

Formality criterion   Let A be a commutative $\mathbb{Q}$–algebra and let $M \in \mathrm{Perf}(A)^{BS^1}$ be a perfect complex of A–modules equipped with an S^1–action. Suppose that there exists a finitely generated $\mathbb{Z}$–algebra $R \subset A$, a dualizable object M' in the ∞–category $\mathrm{Mod}_{\mathrm{THH}(R)}(\mathrm{CycSp})$ of $\mathrm{THH}(R)$–modules in cyclotomic spectra, and an equivalence $M \simeq M' \otimes_{\mathrm{THH}(R)} A$ in $\mathrm{Perf}(A)^{BS^1}$. Then the homology groups of M are finitely generated projective A–modules and the S^1–action on M is trivial.

In characteristic p, we can approach relative questions as well using the cyclotomic Frobenius, although our methods only apply when the base is smooth. Recent work of Petrov, Vaintrob and Vologodsky [41] has obtained related statements using the methods of Kaledin and the Gauss–Manin connection in periodic cyclic homology. In particular, within the range $[-(p-3), (p-3)]$, they obtain a Fontaine–Laffaille structure on periodic cyclic homology, which they observe implies projectivity and degeneration.

Theorem 1.4   (see also [41, Theorem 1])   *Let A be a regular noetherian $\mathbb{F}_p$–algebra such that the Frobenius map $A \to A$ is finite. Let $\widetilde{A}$ be a flat lift of A to $\mathbb{Z}/p^2$. Let $\mathcal{C}$ be a smooth and proper dg category over A. Suppose that*:

(1)   *$\mathcal{C}$ lifts to a smooth and proper dg category over $\widetilde{A}$.*

(2)   *$\mathrm{HH}_i(\mathcal{C}/A) = 0$ for $i \notin [-(p-1), p-1]$.*

Then the Hochschild homology groups $\mathrm{HH}_i(\mathcal{C}/A)$ are finitely generated projective A–modules and the relative Hodge-to-de Rham spectral sequence $\mathrm{HH}_*(\mathcal{C}/A)[u^{\pm 1}] \Rightarrow \mathrm{HP}_*(\mathcal{C}/A)$ degenerates at E_2.

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2 Topological Hochschild homology and cyclotomic spectra

Let $\mathcal{C}$ be a k–linear stable ∞–category over a perfect field k of characteristic $p > 0$. A basic invariant of $\mathcal{C}$ which we will use essentially in this paper is the *topological Hochschild homology* $\mathrm{THH}(\mathcal{C})$. The construction $\mathrm{THH}(\mathcal{C})$ is one of a general class of *localizing* invariants of stable ∞–categories, including algebraic K–theory, and about which there is a significant literature; compare for example [9].

The construction $\mathcal{C} \mapsto \mathrm{THH}(\mathcal{C})$ is naturally a functor to the homotopy theory of spectra, and can be refined substantially to the homotopy theory of *cyclotomic* spectra. By definition, $\mathrm{THH}(\mathcal{C})$ is the Hochschild homology of $\mathcal{C}$ relative to the sphere spectrum rather than to an ordinary ring. As we show below, $\mathrm{THH}(\mathcal{C})$ contains significant information about the Hochschild homology $\mathrm{HH}(\mathcal{C}/k)$ and the spectral sequence for $\mathrm{HP}(\mathcal{C}/k)$. We begin by giving a brief overview of the relevant structure in this case.

2.1 Review of stable ∞–categories

We will use the language of stable ∞–categories, following Lurie [35, Section 1.2]. Furthermore, we use the following notation, as in [9]:

Definition 2.1 (the ∞–category $\mathrm{Cat}_\infty^{\mathrm{perf}}$) We let $\mathrm{Cat}_\infty^{\mathrm{perf}}$ denote the ∞–category of idempotent-complete, small stable ∞–categories, where the morphisms are exact functors.

Recall [9, Section 3.1] that $\mathrm{Cat}_\infty^{\mathrm{perf}}$ itself acquires the structure of a symmetric monoidal ∞–category, via the Lurie tensor product [35, Section 4.8]. Given $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}_\infty^{\mathrm{perf}}$, the tensor product $\mathcal{C} \otimes \mathcal{D}$ is universal for the structure that one has a functor of ∞–categories $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$ which is biexact in each variable.

Alternatively, one can give an approach to $\mathrm{Cat}_\infty^{\mathrm{perf}}$ via the (point-set) theory of *spectrally enriched categories*, ie categories enriched over a good symmetric monoidal category of spectra, eg symmetric spectra [26] or orthogonal spectra [36]. The homotopy theory of spectrally enriched categories is studied in [43]. See [9, Theorem 4.23; 10, Theorem 4.6] for a comparison with $\mathrm{Cat}_\infty^{\mathrm{perf}}$.

Next, we recall the theory of R–linear ∞–categories. An example of an object in $\mathrm{Cat}_\infty^{\mathrm{perf}}$ is the stable ∞–category $\mathrm{Perf}(R)$ of perfect R–modules for R an E_∞–ring; in fact, via the R–linear tensor product this is a commutative algebra object in $\mathrm{Cat}_\infty^{\mathrm{perf}}$.

Definition 2.2 (linear ∞–categories) Given an E_∞–ring R, one has also a symmetric monoidal ∞–category of R–*linear* (idempotent-complete) stable ∞–categories (often abbreviated to "R–linear ∞–categories"). By definition, this is the ∞–category $\mathrm{Mod}_{\mathrm{Cat}_\infty^{\mathrm{perf}}}(\mathrm{Perf}(R))$ of modules in $\mathrm{Cat}_\infty^{\mathrm{perf}}$ over the commutative algebra object $\mathrm{Perf}(R)$ in $\mathrm{Cat}_\infty^{\mathrm{perf}}$.

See [34, Appendix D.1] for a treatment of the theory. We will mostly be interested in the case where $R = k$ is a field. In this case, one can also use the more classical theory of differential graded categories; see [15] for a comparison.

2.2 Topological Hochschild homology

We will use the theory of topological Hochschild homology for objects in $\mathrm{Cat}_\infty^{\mathrm{perf}}$. Treatments (which go through the language of spectral categories) appear in [11; 10; 1]; one can also formulate the construction purely ∞–categorically [6].

Construction 2.3 (cyclic bar construction and THH) Given a spectrally enriched category $\mathcal{C}$, one defines the *topological Hochschild homology* $\mathrm{THH}(\mathcal{C})$ as the geometric realization of the classical cyclic bar construction on $\mathcal{C}$.

The cyclic bar construction defines a symmetric monoidal functor from the category of spectrally enriched categories to the ∞–category of cyclic objects in Sp. After taking the geometric realization, it carries Morita equivalences to equivalences of spectra, and therefore descends to $\mathrm{Cat}_\infty^{\mathrm{perf}}$. It follows that we obtain a symmetric monoidal functor

$$\mathrm{THH}\colon \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathrm{Fun}(BS^1, \mathrm{Sp}).$$

A fundamental feature of THH (which, as we discuss below, is not shared by ordinary Hochschild homology) is that it acquires a lift to the ∞–category of *cyclotomic spectra*, studied by many authors including [12; 40; 6]. We follow the elegant definition of [40], which agrees with those of the others authors in the bounded-below case.

Definition 2.4 (cyclotomic spectra) A (p–typical) *cyclotomic spectrum* consists of an object $X \in \mathrm{Fun}(BS^1, \mathrm{Sp})$ together with a map $\varphi\colon X \to X^{tC_p}$ in $\mathrm{Fun}(BS^1, \mathrm{Sp})$, where we regard X^{tC_p} as a spectrum with an $S^1 \simeq S^1/C_p$–action. We let CycSp denote the presentably symmetric monoidal stable ∞–category of cyclotomic spectra.

The topological Hochschild homology of $\mathcal{C} \in \mathrm{Cat}_\infty^{\mathrm{perf}}$ can be refined to a cyclotomic spectrum. In fact, topological Hochschild homology yields a symmetric monoidal functor

$$\mathrm{THH} \colon \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathrm{CycSp}.$$

Construction 2.5 (THH of spectral categories and stable ∞–categories) We briefly sketch a construction of topological Hochschild homology of an object of $\mathrm{Cat}_\infty^{\mathrm{perf}}$ as a cyclotomic spectrum, following [40]; it is also possible to give a ∞–categorical construction as in [6]. Since $\mathrm{Cat}_\infty^{\mathrm{perf}}$ is obtained as a localization of spectrally enriched categories, it suffices to carry this construction out for a spectrally enriched category, where one has a well-defined set of objects.

Let $\mathfrak{C}$ be a spectrally enriched category, ie a category enriched over the category of orthogonal spectra, $\mathrm{Spectra}^{\mathrm{O}}$. In this case, one constructs the Hochschild–Mitchell cyclic nerve

$$N^{\mathrm{cyc}}(\mathfrak{C}) \colon \Lambda \to \mathrm{Spectra}^{\mathrm{O}},$$

for Λ the cyclic category. Let $\Lambda_p \to \Lambda$ be the edgewise subdivision (eg Appendix B of [40]). Its pth edgewise subdivision $\mathrm{sd}_p(N^{\mathrm{cyc}}(\mathcal{C}))$ yields a functor $\Lambda_p \to \mathrm{Spectra}^{\mathrm{O}}$. Unwinding the definitions and using the Tate diagonal as in [40], one obtains a map in the ∞–category of cyclic spectra, $N^{\mathrm{cyc}}(\mathfrak{C}) \to (N^{\mathrm{cyc}}(\mathfrak{C}) \circ \mathrm{sd}_p)^{tC_p}$. Taking geometric realizations, one obtains the cyclotomic structure $\varphi \colon \mathrm{THH}(\mathfrak{C}) \to \mathrm{THH}(\mathfrak{C})^{tC_p}$, and all constructions are lax symmetric monoidal.

We now specialize to the R–linear case, where R is an E_∞–ring. Since THH is a symmetric monoidal functor, it follows that if $\mathcal{C}$ is an R–linear stable ∞–category, then $\mathrm{THH}(\mathcal{C})$ is a module in CycSp over $\mathrm{THH}(R) = \mathrm{THH}(\mathrm{Perf}(R))$, which is an E_∞–algebra in CycSp. Moreover, THH defines a symmetric monoidal functor from R–linear stable ∞–categories to $\mathrm{Mod}_{\mathrm{CycSp}}(\mathrm{THH}(R))$.

Construction 2.6 (relative Hochschild homology) Let R be an E_∞–ring and let $\mathcal{C}$ be an R–linear ∞–category. We define the *relative Hochschild homology* $\mathrm{HH}(\mathcal{C}/R) \in \mathrm{Fun}(BS^1, \mathrm{Mod}(R))$ as the relative tensor product

$$(1) \qquad\qquad \mathrm{HH}(\mathcal{C}/R) = \mathrm{THH}(\mathcal{C}) \otimes_{\mathrm{THH}(R)} R,$$

where we use the canonical S^1–equivariant map $\mathrm{THH}(R) \to R$.

Remark 2.7 Suppose R arises from a commutative orthogonal ring spectrum R^o. For an R–linear ∞–category $\mathcal{C}$ presented via an R^o–spectral category $\mathfrak{C}$ (ie a category enriched over R^o–modules), it follows by comparing cyclic bar constructions and using the symmetric monoidality of geometric realizations that the above agrees with the usual definition of $\mathrm{HH}(\mathcal{C}/R)$. For example, when $R = k$, this agrees with the usual definition of Hochschild homology for a dg category.

Remark 2.8 (properties of THH) The primary focus of the paper is about the Hochschild homology of smooth and proper k–linear categories, for k a field. Nonetheless, the use of topological Hochschild homology appears for the following two reasons:

(1) THH is a more primitive invariant: according to (1), we can recover Hochschild homology from THH.

(2) THH has the additional structure given by the cyclotomic Frobenius φ, which does not exist on ordinary Hochschild homology.

2.3 Topological Hochschild homology over k

We will especially be interested in the topological Hochschild homology of a k–linear ∞–category for k a perfect field of characteristic $p > 0$, which exhibits some special features. A basic input here is the calculation in the case when $\mathcal{C} = \mathrm{Perf}(k)$, recalled below (see [25, Section 5]).

Theorem 2.9 (Bökstedt) $\mathrm{THH}_*(k) \simeq k[\sigma]$ *with* $|\sigma| = 2$.

Remark 2.10 Theorem 2.9 shows that THH can be controlled in a convenient manner. A more naive variant of the construction $\mathcal{C} \mapsto \mathrm{THH}(\mathcal{C})$ is to consider the Hochschild homology $\mathrm{HH}(\mathcal{C}/\mathbb{Z})$ over the integers. Since (by a straightforward calculation) $\mathrm{HH}_*(\mathbb{F}_p/\mathbb{Z}) \simeq \Gamma(\sigma)$ is a divided power algebra on a degree 2 class, the construction of THH should be regarded as an "improved" version of Hochschild homology over $\mathbb{Z}$.

As in (1), one has the relation

$$(2) \qquad\qquad \mathrm{THH}(\mathcal{C}) \otimes_{\mathrm{THH}(k)} k \simeq \mathrm{HH}(\mathcal{C}/k).$$

As a result of (2), $\mathrm{THH}(\mathcal{C})$ can be thought of as a one-parameter deformation of $\mathrm{HH}(\mathcal{C}/k)$ over the element σ.

Recall $\text{THH}(\mathcal{C})$ inherits an action of the circle S^1. The circle also acts on $\text{THH}(k)$ (considered as an $\boldsymbol{E}_\infty$–ring spectrum), and THH provides a symmetric monoidal functor

$$\{k\text{–linear stable } \infty\text{–categories}\} \to \text{Mod}_{\text{THH}(k)}(\text{Sp}^{BS^1}),$$

ie into the ∞–category of spectra with S^1–action equipped with a compatible $\text{THH}(k)$–action. Using this, one can define the following (which can be thought of as a noncommutative version of crystalline cohomology):

Definition 2.11 (Hesselholt [24]) The *periodic topological cyclic homology* of $\mathcal{C}$ is given by $\text{TP}(\mathcal{C}) = \text{THH}(\mathcal{C})^{tS^1}$.

A result of [8] (see also [4, Section 3]) shows that TP provides a lift to characteristic zero of the periodic cyclic homology $\text{HP}(\mathcal{C}/k)$. For example, $\text{TP}_*(k) \simeq W(k)[x^{\pm 1}]$ for $|x| = -2$, and in general one has a natural equivalence of $\text{TP}(k)$–modules

$$(3) \qquad \text{TP}(\mathcal{C}) \otimes_{\text{TP}(k)} \text{HP}(k) \simeq \text{TP}(\mathcal{C})/p \simeq \text{HP}(\mathcal{C}/k).$$

The construction $\mathcal{C} \mapsto \text{TP}(\mathcal{C})$ is another extremely useful invariant one can extract from this machinery. It naturally provides a lax symmetric monoidal functor

$$\{k\text{–linear stable } \infty\text{–categories}\} \to \text{Mod}_{\text{TP}(k)}.$$

At least for smooth and proper k–linear ∞–categories, the construction TP is actually symmetric monoidal, ie satisfies a Künneth theorem, by a result of Blumberg and Mandell [13] (see also [4]).

In (3), we saw that periodic cyclic homology can be recovered from TP by reducing mod p. Next, we show that we can reconstruct HP from THH in another way. Note first that there is a natural map of $\boldsymbol{E}_\infty$–rings $\text{TP}(k) \simeq \text{THH}(k)^{tS^1} \to \text{THH}(k)^{tC_p}$.

Proposition 2.12 *For $\mathcal{C}$ a k–linear stable ∞–category, one has an equivalence of* $\text{TP}(k)$*–module spectra* $\text{THH}(\mathcal{C})^{tC_p} \simeq \text{TP}(\mathcal{C}) \otimes_{\text{TP}(k)} \text{THH}(k)^{tC_p} \simeq \text{HP}(\mathcal{C}/k)$.

For future reference, we actually prove a more general statement:

Proposition 2.13 *Let X be an arbitrary object of the ∞–category $\text{Mod}_{\text{THH}(k)}(\text{Sp}^{BS^1})$ of modules over $\text{THH}(k)$ in the symmetric monoidal ∞–category of spectra equipped with an S^1–action.*[1]

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[1] Compare the discussion in [4] for a treatment.

Then the natural map of $\mathrm{TP}(k)$–modules

$$(4) \qquad X^{tS^1} \otimes_{\mathrm{TP}(k)} \mathrm{THH}(k)^{tC_p} \to X^{tC_p}$$

is an equivalence, and one has a natural equivalence of $\mathrm{TP}(k)$–modules

$$(5) \qquad X^{tS^1} \otimes_{\mathrm{TP}(k)} \mathrm{THH}(k)^{tC_p} \simeq (X \otimes_{\mathrm{THH}(k)} k)^{tS^1}.$$

Proof To see this, we note that there is an S^1–equivariant map of E_∞–rings $\mathbb{Z} \to \mathrm{THH}(\mathbb{F}_p)$, eg via the cyclotomic trace (see [40, Section IV.4]). One obtains a square of E_∞–rings

$$\begin{array}{ccc} \mathbb{Z}^{tS^1} & \longrightarrow & \mathbb{Z}^{tC_p} \\ \downarrow & & \downarrow \\ \mathrm{TP}(k) & \longrightarrow & \mathrm{THH}(k)^{tC_p} \end{array}$$

which one easily checks to be a pushout square. Now the equivalence (4) follows from [40, Lemma IV.4.12]. To see (5), we use the fact that $\mathrm{THH}(k)^{tC_p} \simeq \mathrm{TP}(k)/p$ as $\mathrm{TP}(k)$–modules. This implies the result via the formula $(X \otimes_{\mathrm{THH}(k)} k)^{tS^1} \simeq X^{tS^1} \otimes_{\mathrm{TP}(k)} k^{tS^1} \simeq X^{tS^1}/p$, which holds because $k = \mathrm{THH}(k)/\sigma$ belongs to the thick subcategory generated by the unit in $\mathrm{Mod}_{\mathrm{THH}(k)}(\mathrm{Sp}^{BS^1})$ (and which is a generalization of (3)). $\square$

2.4 The cyclotomic Frobenius over k

Example 2.14 (see [40, Section IV.4; 25]) Suppose $\mathcal{C} = \mathrm{Perf}(k)$. In this case, the map

$$\varphi \colon \mathrm{THH}(k) \to \mathrm{THH}(k)^{tC_p}$$

identifies the former with the connective cover of the latter, and $\pi_*(\mathrm{THH}(k)^{tC_p}) \simeq k[u^{\pm 1}]$ is a Laurent polynomial ring with $|u| = 2$. The map φ is given by the Frobenius on π_0 and sends $\sigma \mapsto u$. In particular, φ induces an equivalence

$$\mathrm{THH}(k)[1/\sigma] \simeq \mathrm{THH}(k)^{tC_p}.$$

This computation was originally done by Hesselholt and Madsen [25], and we refer to [40, Section IV.4] for a complete description of $\mathrm{THH}(k)$ as a cyclotomic spectrum.

Here $\mathrm{THH}(k) \in \mathrm{CAlg}(\mathrm{CycSp})$ is a commutative algebra object, and for $\mathcal{C}$ a k–linear stable ∞–category, $\mathrm{THH}(\mathcal{C})$ is a $\mathrm{THH}(k)$–module. The functor THH yields a symmetric

monoidal functor

$$\{k\text{--linear stable } \infty\text{--categories}\} \to \mathrm{Mod}_{\mathrm{THH}(k)}(\mathrm{CycSp}).$$

Note in particular that for a smooth and proper k--linear stable ∞--category (see [34, Chapter 11] for an account), THH is therefore a dualizable object of $\mathrm{Mod}_{\mathrm{THH}(k)}(\mathrm{CycSp})$. In this paper, all our degeneration arguments will take place in the latter ∞--category, and we will often state them in that manner.

We saw above that the cyclotomic Frobenius becomes an equivalence on connective covers for $\mathrm{THH}(k)$. More generally, one can show (see [23; 8, Corollary 8.18]) that for a smooth commutative k--algebra, the cyclotomic Frobenius is an equivalence in high enough degrees. For our purposes, we need a basic observation that in the smooth and proper case, the cyclotomic Frobenius becomes an equivalence after inverting σ. This is a formal dualizability argument once one knows both sides satisfy a Künneth formula.

Proposition 2.15 *Let $\mathcal{C}/k$ be a smooth and proper k--linear stable ∞--category. In this case, the cyclotomic Frobenius implements an equivalence*

$$\mathrm{THH}(\mathcal{C})[1/\sigma] \xrightarrow[\simeq]{\varphi} \mathrm{THH}(\mathcal{C})^{tC_p} \simeq \mathrm{HP}(\mathcal{C}/k).$$

More generally, if $X \in \mathrm{Mod}_{\mathrm{THH}(k)}(\mathrm{CycSp})$ is a dualizable object, then the cyclotomic Frobenius implements an equivalence

$$X[1/\sigma] \xrightarrow{\varphi} X^{tC_p} \simeq (X \otimes_{\mathrm{THH}(k)} k)^{tS^1}.$$

The first equivalence is φ--semilinear for the equivalence

$$\varphi \colon \mathrm{THH}(k)[1/\sigma] \simeq \mathrm{THH}(k)^{tC_p},$$

while the second equivalence is $\mathrm{TP}(k)$--linear.

Proof By Proposition 2.12, it suffices to prove that φ is an isomorphism. In fact, both the source and target of φ are symmetric monoidal functors from dualizable objects in $\mathrm{Mod}_{\mathrm{THH}(k)}(\mathrm{CycSp})$ to the ∞--category of $\mathrm{THH}(k)[1/\sigma] \simeq \mathrm{THH}(k)^{tC_p}$--module spectra (see [13; 4]) and the natural transformation is one of symmetric monoidal functors. Thus the map is an equivalence for formal reasons [4, Proposition 4.6]. $\square$

Let $\mathcal{C}$ be smooth and proper over k. On homotopy groups, it follows that one has isomorphisms of abelian groups $\pi_i \mathrm{THH}(\mathcal{C})[1/\sigma] \simeq \pi_i \mathrm{HP}(\mathcal{C}/k)$. Both sides are k--vector spaces, and the isomorphism is semilinear for the Frobenius. In particular, at the

level of k–vector spaces, one has a natural isomorphism

$$(\pi_i \operatorname{THH}(\mathcal{C})[1/\sigma])^{(1)} \simeq \operatorname{HP}_i(\mathcal{C}/k).$$

Remark 2.16   Suppose $\mathcal{C} = \operatorname{Perf}(A)$ for A a smooth commutative k–algebra. In this case, $\operatorname{HP}(\mathcal{C}/k)$ is related to 2–periodic de Rham cohomology of A (see [2]) while $\operatorname{THH}(\mathcal{C})[1/\sigma]$ is closely related to 2–periodic differential forms on $\mathcal{C}$ by [23] (and more precisely by [8, Corollary 8.18]). The relationship between differential forms and de Rham cohomology arising here is essentially the classical *Cartier isomorphism*, and is made precise in the work of Bhatt, Morrow and Scholze [8].

In addition, we expect that Proposition 2.15 can be compared with the "noncommutative Cartier isomorphism" studied by Kaledin [31; 32].

3   The degeneration argument

In this section, we give the main degeneration argument. We begin with the following basic observation and definition.

Let R be an E_∞–ring spectrum over $\mathbb{Z}$ (in this section, R will be a field), and let M be an R–module spectrum equipped with an S^1–action. Suppose the R–module M is graded projective. Then the following are equivalent:

  (1)   The S^1–Tate spectral sequence for $\pi_*(M^{tS^1})$ degenerates.

  (2)   The S^1–action on M (as an R–module) is trivial.

Clearly the second assertion implies the first. To see the converse, we observe that if the Tate spectral sequence degenerates, then by naturality, the homotopy fixed point spectral sequence for $\pi_*(M)$ must degenerate too, so that the map $\pi_*(M^{hS^1}) \to \pi_*(M)$ is surjective. Suppose M, as an underlying R–module, is obtained as the summand Fe associated to an idempotent endomorphism e of a free R–module F. If we give F the trivial S^1–action, the degeneration of the homotopy fixed point spectral sequence shows that we can realize the map $F \to M$ as an S^1–equivariant map. Restricting now to the summand Fe of F, we conclude that M is equivalent to Fe (with trivial action). This is the way in which we regard the degeneration of the S^1–Tate spectral sequence as a *formality* statement.

Definition 3.1   Let k be a field. Let $M \in \operatorname{Perf}(k)^{BS^1}$. We say that M is *formal* if the S^1–Tate spectral sequence for M^{tS^1} (or equivalently the homotopy fixed point

spectral sequence for M^{hS^1}) degenerates at E_2. This holds if and only if

(6) $\qquad \dim_k \pi_{\text{even}}(M) = \dim_k \pi_0 M^{tS^1}, \quad \dim_k \pi_{\text{odd}}(M) = \dim_k \pi_1 M^{tS^1}.$

For the rest of this section, k is a perfect field of characteristic $p > 0$. We will prove a formality criterion for objects of $\text{Perf}(k)^{BS^1}$. Our main interest, of course, is in the following example; in this section, we will state our arguments in the more general case of objects in $\text{Perf}(k)$ with S^1–action, though. Consider a smooth and proper k–linear stable ∞–category $\mathcal{C}/k$ and its Hochschild homology $\text{HH}(\mathcal{C}/k)$. One has that $\dim_k \text{HH}_*(\mathcal{C}/k) < \infty$ and that $\text{HH}(\mathcal{C}/k)$ inherits a circle action.

Definition 3.2 We say that the *Hodge-to-de Rham spectral sequence degenerates* for $\mathcal{C}/k$ if $\text{HH}(\mathcal{C}/k) \in \text{Perf}(k)^{BS^1}$ is formal. Equivalently, degeneration holds if and only if one has the numerical equalities $\text{HH}_{\text{even}}(\mathcal{C}/k) = \text{HP}_0(\mathcal{C}/k)$ and $\dim_k \text{HH}_{\text{odd}}(\mathcal{C}/k) = \text{HP}_1(\mathcal{C}/k)$.

One source of objects of $\text{Perf}(k)^{BS^1}$ is the ∞–category of dualizable objects of $\text{Mod}_{\text{THH}(k)}(\text{CycSp})$. Given $X \in \text{Mod}_{\text{THH}(k)}(\text{CycSp})$, we have $X \otimes_{\text{THH}(k)} k \in \text{Mod}_k^{BS^1}$ and if X is dualizable, then $X \otimes_{\text{THH}(k)} k$ is perfect as a k–module. For such objects, we will translate formality to a statement about $\text{THH}(k)$–modules. Note that $\text{HH}(\mathcal{C}/k) \in \text{Perf}(k)^{BS^1}$ arises in this way, via $X = \text{THH}(\mathcal{C})$.

First, we need the following observation about module spectra over $\text{THH}(k)$, which follows from the classification of finitely generated modules over a principal ideal domain:

Proposition 3.3 *Any perfect* $\text{THH}(k)$*–module spectrum is equivalent to a direct sum of copies of suspensions of* $\text{THH}(k)$ *and* $\text{THH}(k)/\sigma^n$ *for various* n.

Proof A perfect $\text{THH}(k)$–module M yields a finitely generated $\text{THH}_*(k)$–module $\pi_*(M)$. Any finitely generated graded $\text{THH}_*(k) = k[\sigma]$–module is a direct sum of copies of shifts of $k[\sigma]$ and $k[\sigma]/\sigma^n$ for various n. It follows easily that M can be written as a direct sum as desired. $\qquad \square$

The following result now shows that degeneration is equivalent to a condition of torsion-freeness on THH:

Proposition 3.4 (1) *Let* $X \in \text{Mod}_{\text{THH}(k)}(\text{CycSp})$ *be dualizable. Then* $X \otimes_{\text{THH}(k)} k$ *in* $\text{Perf}(k)^{BS^1}$ *is formal if and only if* X *is (graded) free (equivalently,* σ– *torsion-free) as a* $\text{THH}(k)$*–module.*

(2) *If $\mathcal{C}$ is smooth and proper over k, the Hodge-to-de Rham spectral sequence for $\mathcal{C}$ degenerates if and only if $\mathrm{THH}(\mathcal{C})$ is free (equivalently, σ–torsion-free) as a $\mathrm{THH}(k)$–module.*

Proof Clearly, the second assertion is a special case of the first. It suffices to compare with (6). In fact, by the equivalence given by Proposition 2.15, one sees that $\pi_*(X)[1/\sigma]$ is a finitely generated graded free $\mathrm{THH}(k)_*[1/\sigma]$–module. Moreover, one has

$$\dim_k \pi_0((X \otimes_{\mathrm{THH}(k)} k)^{tS^1}) = \dim_k (\pi_0(X[1/\sigma])) = \mathrm{rank}_{k[\sigma^{\pm 1}]} \pi_{\mathrm{even}}(X)[1/\sigma],$$

and similarly for the odd terms. Thus, formality holds if and only if the ranks agree, ie

$$\mathrm{rank}_{k[\sigma^{\pm 1}]} \pi_{\mathrm{even}}(X)[1/\sigma] = \dim_k \pi_{\mathrm{even}}(X \otimes_{\mathrm{THH}(k)} k),$$
$$\mathrm{rank}_{k[\sigma^{\pm 1}]} \pi_{\mathrm{odd}}(X)[1/\sigma] = \dim_k \pi_{\mathrm{odd}}(X \otimes_{\mathrm{THH}(k)} k).$$

Note that $X \otimes_{\mathrm{THH}(k)} k \simeq X/\sigma$. It follows (eg using Proposition 3.3) that the ranks (over $\sigma = 0$ and σ invertible, respectively) agree if and only if X is (graded) free as a $\mathrm{THH}(k)$–module spectrum. $\square$

It thus follows that, in order to verify degeneration of the Hodge-to-de Rham spectral sequence, one needs criteria for testing σ–torsion-freeness in $\mathrm{THH}_*(\mathcal{C})$. We begin by observing that liftability to the sphere allows for a direct argument here. The general idea that liftability to the sphere should simplify the argument was well known, and we are grateful to N Rozenblyum for indicating it to us.

Example 3.5 Suppose[2] $k = \mathbb{F}_p$ and suppose $\mathcal{C}$ lifts to a stable ∞–category $\widetilde{\mathcal{C}}$ over the sphere S^0 (implicitly p–completed). Note that the map $S^0 \to \mathrm{THH}(\mathbb{F}_p)$ factors through the natural map $\mathbb{F}_p \to \mathrm{THH}(\mathbb{F}_p)$ given by choosing a basepoint in the circle S^1 via the equivalence $\mathrm{THH}(\mathbb{F}_p) \simeq S^1 \otimes \mathbb{F}_p$ in E_∞–rings [39]. Then, as $\mathrm{THH}(\mathbb{F}_p)$–module spectra, one has an equivalence

$$\mathrm{THH}(\mathcal{C}) \simeq \mathrm{THH}(\widetilde{\mathcal{C}}) \otimes_{S^0} \mathrm{THH}(\mathbb{F}_p) \simeq (\mathrm{THH}(\widetilde{\mathcal{C}}) \otimes_{S^0} \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathrm{THH}(\mathbb{F}_p).$$

Since every $\mathbb{F}_p$–module spectrum is (graded) free, this equivalence proves that $\mathrm{THH}(\mathcal{C})$ is free as a $\mathrm{THH}(\mathbb{F}_p)$–module. Thus, degeneration holds for $\mathcal{C}$.

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[2]Using the spectral version of the Witt vectors construction, one can replace k with any perfect field of characteristic p.

We will now give the argument for a lifting to $W_2(k)$. If a k–linear stable ∞–category $\mathcal{C}$ lifts to $W_2(k)$, then the THH(k)–module spectrum THH($\mathcal{C}$) lifts to THH($W_2(k)$). By considering the map THH($W_2(k)$) $\to$ THH(k), we will be able to deduce σ–torsion-freeness (and thus degeneration) in many cases. The argument will require a small amount of additional bookkeeping and rely on an amplitude assumption. The basic input is the following fact about the homotopy ring of THH($W_2(k)$). The entire computation is carried out in [14], at least additively, but we will only need it in low degrees. For the reader's convenience, we include a proof.

Proposition 3.6 (compare [14]) *Let k be a perfect field of characteristic p.*

(1) *We have*

$$\pi_* \tau_{\leq 2p-2}\text{THH}(W_2(k)) \simeq W_2(k)[u]/u^p \quad \text{with } |u| = 2.$$

(2) *The map* $\text{THH}_i(W_2(k)) \to \text{THH}_i(k)$ *is zero for* $0 < i \leq 2p-2$. *Furthermore, the map of E_∞–rings* $\text{THH}(W_2(k)) \to \text{THH}(k) \to \tau_{\leq 2p-2}\text{THH}(k)$ *factors through the map* $k \to \text{THH}(k) \to \tau_{\leq 2p-2}\text{THH}(k)$.

Proof We compare with Hochschild homology over the integers. The map $S^0_{(p)} \to \mathbb{Z}_{(p)}$ induces an equivalence on degrees $< 2p-3$. Thus, in the range stated in the theorem, we can compare THH with Hochschild homology over $\mathbb{Z}_{(p)}$ or over $W(k)$. We have

$$\text{HH}_*(W_2(k)/\mathbb{Z}_{(p)}) \simeq \Gamma^*_{W_2(k)}[u] \quad \text{with } |u| = 2,$$

ie the divided power algebra on a class in degree 2. Indeed, we have that

$$\text{HH}(W_2(k)/\mathbb{Z}_{(p)}) = \text{HH}(k/W_2(k)) = W_2(k) \otimes^L_{W_2(k)\otimes^L_{W(k)}W_2(k)} W_2(k).$$

Since $W_2(k) \otimes^L_{W(k)} W_2(k)$ is the free simplicial commutative ring over $W_2(k)$ on a class in degree 1 (equivalently, a square-zero extension on a class in degree 1), the Hochschild homology of $W_2(k)$ is the free simplicial commutative ring over $W_2(k)$ on a class in degree 2, which on homotopy yields a divided power algebra.

It remains to check that the map THH($W_2(k)$) $\to$ THH(k) vanishes on π_2. This, too, follows from the comparison with Hochschild homology over $\mathbb{Z}$. For a map of commutative rings $A \to B$, let $L_{B/A}$ denote the cotangent complex of B over A. Using the classical Quillen spectral sequence from the cotangent complex to Hochschild homology (see eg [40, Proposition IV.4.1]), we find that the only contributions to

$\pi_2\text{THH}(W_2(k)) = \pi_2\text{HH}(W_2(k)/\mathbb{Z}_{(p)})$ (resp. $\pi_2\text{THH}(k) = \pi_2\text{HH}(k/\mathbb{Z}_{(p)})$) come from the cotangent complex. Thus, one has to show that the following map vanishes:

$$(7) \qquad \pi_1 L_{W_2(k)/\mathbb{Z}_{(p)}} \to \pi_1 L_{k/\mathbb{Z}_{(p)}}.$$

Here one can replace the source $\mathbb{Z}_{(p)}$ with $W(k)$ since k is perfect. Recall also that if A is a ring and $a \in A$ a regular element, then one has a natural equivalence $L_{(A/a)/A} \simeq (a)/(a^2)[1]$. In our setting, one obtains for (7) the map of $W(k)$–modules

$$(p^2)/(p^4) \to (p)/(p^2),$$

which is zero. Finally, the factorization of the map of $\boldsymbol{E}_\infty$–rings follows because $\tau_{\leq 2p-2}\text{THH}(W_2(k))$ is the truncation of the free $\boldsymbol{E}_\infty$–ring over $W_2(k)$ on a class in degree 2. $\qquad\square$

We now give an argument that liftability together with a Tor–amplitude condition implies freeness. The observation is that if the Tor–amplitude is small, then any torsion has to occur in low homotopical degree.

Proposition 3.7 *Let M be a perfect $\text{THH}(k)$–module such that $\pi_i(M) = 0$ for $i < a$. Suppose that M lifts to a perfect module over $\text{THH}(W_2(k))$. Then multiplication by $\sigma\colon \pi_{i-2}(M) \to \pi_i(M)$ is injective for $i \leq a + 2p - 2$.*

Proof Without loss of generality, $a = 0$. By assumption, $M \simeq \widetilde{M} \otimes_{\text{THH}(W_2(k))} \text{THH}(k)$ for some connective and perfect $\text{THH}(W_2(k))$–module $\widetilde{M}$. Truncating, we find that there is a map of $\text{THH}(k)$–modules

$$(8) \qquad M \to \tau_{\leq 2p-2}\widetilde{M} \otimes_{\tau_{\leq 2p-2}\text{THH}(W_2(k))} \tau_{\leq 2p-2}\text{THH}(k),$$

which induces an isomorphism on degrees $\leq 2p - 2$. However, by Proposition 3.6 and the fact that any k–module spectrum is free, it follows that the right-hand side is a free module over $\tau_{\leq 2p-2}\text{THH}(k)$ on generators in nonnegative degrees. This shows that multiplication by σ is an injection in this range of degrees. $\qquad\square$

Proposition 3.8 *Let M be a perfect $\text{THH}(k)$–module with Tor–amplitude concentrated in $[-p, p]$. Suppose that M lifts to a perfect module over $\text{THH}(W_2(k))$. Then M is free.*

Proof M is a direct sum of $\text{THH}(k)$–modules each of which is either free or equivalent to $M_{i,j} = \Sigma^i \text{THH}(k)/\sigma^j$ for $-p \leq i \leq i + 2j + 1 \leq p$ as $M_{i,j}$ has Tor–amplitude

in $[i, i + 2j + 1]$. Note that $M_{i,j}$ has an element in π_{i+2j-2} annihilated by σ, so we find $i + 2j - 2 \geq p - 3$ and therefore $i + 2j + 1 \geq p$ by Proposition 3.7. Therefore, $i + 2j + 1 = p$.

In particular, we find that if $M_{i,j}$ occurs as a summand, then $i + 2j + 1 = p$. We observe now that if the hypotheses of the lemma apply to M, then they apply to the THH(k)–linear Spanier–Whitehead dual $\mathbb{D}M$; that is, $\mathbb{D}M$ is a perfect THH(k)–module with Tor–amplitude concentrated in $[-p, p]$, and such that $\mathbb{D}M$ lifts to a perfect module over THH($W_2(k)$). If $M_{i,j}$ occurs as a summand of M, then its dual, which is given by $\Sigma^{-i-2j-1}$THH(k)$/\sigma^j$, occurs as a summand of $\mathbb{D}M$. Applying the previous paragraph to $\mathbb{D}M$, it follows also that $-i = p$. Adding the two equalities, we find that $2j + 1 = 2p$, which is an evident contradiction. $\qquad\square$

Finally, we can state our general degeneration criterion in characteristic p, which will easily imply Theorem 1.2.

Proposition 3.9 (general formality criterion, characteristic p) *Let k be a perfect field of characteristic $p > 0$. Let $M \in \mathrm{Perf}(k)^{BS^1}$ be a perfect k–module with S^1–action whose amplitude is contained in $[-p, p]$. Suppose that there exists a dualizable object $M' \in \mathrm{Mod}_{\mathrm{THH}(k)}(\mathrm{CycSp})$ such that, as objects in $\mathrm{Perf}(k)^{BS^1}$, we have an equivalence $M \simeq M' \otimes_{\mathrm{THH}(k)} k$. Suppose furthermore that the underlying THH(k)–module of M' lifts to a perfect module over THH($W_2(k)$). Then M is formal.*

Proof Combine Propositions 3.4 and 3.8. $\qquad\square$

Proof of Theorem 1.2 Let $\mathcal{C}$ be a smooth and proper stable ∞–category over k satisfying the assumptions of the theorem. By assumption, there exists a smooth and proper lift $\widetilde{\mathcal{C}}$ over $W_2(k)$ such that $\mathcal{C} \simeq \widetilde{\mathcal{C}} \otimes_{W_2(k)} k$. Therefore, one has an equivalence of THH(k)–modules

$$\mathrm{THH}(\mathcal{C}) \simeq \mathrm{THH}(\widetilde{\mathcal{C}}) \otimes_{\mathrm{THH}(W_2(k))} \mathrm{THH}(k).$$

Furthermore, THH($\widetilde{\mathcal{C}}$) is a perfect THH($W_2(k)$)–module. Now, one can apply Proposition 3.9 with $M' = \mathrm{THH}(\mathcal{C})$. $\qquad\square$

Remark 3.10 The slight extension of the dimension range via duality goes back to the work of Deligne and Illusie [17] and appears in the recent work of Antieau and Vezzosi [5] on HKR isomorphisms in characteristic p. Note also that for a smooth and proper k–linear ∞–category $\mathcal{C}$, the Hochschild homology HH($\mathcal{C}/k$) is always

self-dual; see [42]. Hence, it is no loss of generality to assume that the interval in which the amplitude of Hochschild homology is concentrated is symmetric about the origin.

We now describe the deduction of Theorem 1.1 from Theorem 1.2, as in [31; 32]. We note that this is a standard argument and is also used in the commutative case [17]. We formulate the approach in the following formality criterion:

Theorem 3.11 (general formality criterion, field case) *Let K be a field of characteristic zero. Let $M \in \mathrm{Perf}(K)^{BS^1}$ be a perfect module equipped with an S^1- action. Suppose that there exists a finitely generated subring $R \subset K$ and a dualizable object $M' \in \mathrm{Mod}_{\mathrm{THH}(R)}(\mathrm{CycSp})$ such that we have an equivalence in $\mathrm{Perf}(K)^{BS^1}$, $M' \otimes_{\mathrm{THH}(R)} K \simeq M$. Then M is formal.*

Proof Any finitely generated field extension of $\mathbb{Q}$ is a filtered colimit of smooth $\mathbb{Z}$–algebras. Therefore, K is a filtered colimit of its finitely generated subalgebras which are smooth over $\mathbb{Z}$. Enlarging R, we can assume that R is smooth over $\mathbb{Z}$. Enlarging R further, we can assume that the homology groups of $M' \otimes_{\mathrm{THH}(R)} R$ (which is a perfect R–module spectrum) are finitely generated free R–modules and vanish for $i \notin [-p, p]$ for every prime p which is noninvertible in R.

Suppose that the S^1–action on M is nontrivial. Then the S^1–action on $M' \otimes_{\mathrm{THH}(R)} R$ is nontrivial too, and there exists a nontrivial differential in the Tate spectral sequence for $(M' \otimes_{\mathrm{THH}(R)} R)^{tS^1}$. Then we can find a maximal ideal $\mathfrak{m} \subset R$ such that the first differential (which is a map of finitely generated free R–modules) remains nontrivial after base-change along $R \to R/\mathfrak{m}$ and thus after base-change along $R \to k := \overline{R/\mathfrak{m}}$ (ie the algebraic closure of the residue field).

Let $M'_k = M' \otimes_{\mathrm{THH}(R)} \mathrm{THH}(k) \in \mathrm{Mod}_{\mathrm{THH}(k)}(\mathrm{CycSp})$, which is a dualizable object. Note that k is a perfect field of characteristic $p > 0$. Moreover, the map $R \to k$ lifts to the length 2 Witt vectors because R is smooth over $\mathbb{Z}$. It follows that the underlying $\mathrm{THH}(k)$–module of M'_k lifts to a perfect $\mathrm{THH}(W_2(k))$–module. It follows that $M' \otimes_{\mathrm{THH}(R)} k \in \mathrm{Perf}(k)^{BS^1}$ is formal by Proposition 3.9. This contradicts the statement that there is a nontrivial differential in the Tate spectral sequence for $(M' \otimes_{\mathrm{THH}(R)} k)^{tS^1}$ and proves the theorem. $\square$

Proof of Theorem 1.1 Let $\mathcal{C}$ be a smooth and proper stable ∞–category over a field K of characteristic zero. By the results of [45], there exists a smooth and proper stable ∞–category $\widetilde{\mathcal{C}}$ over a finitely generated subalgebra $R \subset K$ such that $\mathcal{C} \simeq \widetilde{\mathcal{C}} \otimes_R K$. Then one has the dualizable object $\mathrm{THH}(\widetilde{\mathcal{C}}) \in \mathrm{Mod}_{\mathrm{THH}(R)}(\mathrm{CycSp})$ and, by base-change,

one has an equivalence, in $\mathrm{Perf}(K)^{BS^1}$, $\mathrm{THH}(\widetilde{\mathcal{C}}) \otimes_{\mathrm{THH}(R)} K \simeq \mathrm{HH}(\mathcal{C}/K)$. Now apply Theorem 3.11. $\qquad\square$

We note that the above arguments actually enable a slight strengthening of Theorem 1.2. For example, Theorem 3.11 easily implies that if $F\colon \mathcal{C} \to \mathcal{D}$ is a functor of smooth and proper stable ∞–categories over K, then the S^1–action on the relative Hochschild homology $\mathrm{fib}(\mathrm{HH}(\mathcal{C}/K) \to \mathrm{HH}(\mathcal{D}/K))$ is also trivial. More generally, this would work for any appropriately finite diagram. We formulate this as follows.

Let K be a field of characteristic zero and let $\mathcal{N}\mathrm{Mot}_K$ denote the presentably symmetric monoidal ∞–category of noncommutative motives of K–linear stable ∞–categories introduced by Tabuada [44] (see also [9; 28]). Since Hochschild homology is an additive invariant, one has a symmetric monoidal, cocontinuous functor

$$\mathrm{HH}(\,\cdot\,/K)\colon \mathcal{N}\mathrm{Mot}_K \to \mathrm{Mod}_K^{BS^1},$$

from $\mathcal{N}\mathrm{Mot}_K$ into the ∞–category $\mathrm{Mod}_K^{BS^1}$ of K–module spectra equipped with an S^1–action. Let $\mathcal{N}\mathrm{Mot}_K^\omega \subset \mathcal{N}\mathrm{Mot}_K$ denote the thick subcategory generated by the smooth and proper stable ∞–categories. Recall that if $\mathcal{C}$ and $\mathcal{D}$ are smooth and proper K–linear stable ∞–categories, then we have associated objects $[\mathcal{C}], [\mathcal{D}] \in \mathcal{N}\mathrm{Mot}_K^\omega$, and the mapping spectrum is given as

$$\mathrm{Hom}_{\mathcal{N}\mathrm{Mot}_K}([\mathcal{C}], [\mathcal{D}]) \simeq \mathrm{K}(\mathrm{Fun}(\mathcal{C}, \mathcal{D})),$$

ie it is the connective algebraic K–theory spectrum of the ∞–category of (K–linear) functors $\mathcal{C} \to \mathcal{D}$. Note for instance that given a functor $F\colon \mathcal{C} \to \mathcal{D}$, one can form the fiber of the associated map $[\mathcal{C}] \to [\mathcal{D}]$ of noncommutative motives, so that relative Hochschild homology is given by Hochschild homology of an object of $\mathcal{N}\mathrm{Mot}_K^\omega$.

Corollary 3.12 *For any $X \in \mathcal{N}\mathrm{Mot}_K^\omega$, $\mathrm{HH}(X/K) \in \mathrm{Perf}(K)^{BS^1}$ is formal.*

Proof By the results of [45], and the fact that K–theory commutes with filtered colimits, it follows that $\mathcal{N}\mathrm{Mot}_K^\omega$ is the filtered colimit of the stable ∞–categories $\mathcal{N}\mathrm{Mot}_R^\omega$ of dualizable noncommutative motives of smooth and proper R–linear ∞–categories, as R ranges over the finitely generated subrings of K. Thus, there exists R such that X arises via base-change from a dualizable object $\widetilde{X}$ in the ∞–category $\mathcal{N}\mathrm{Mot}_R$. In this case, since THH is an additive invariant of R–linear stable ∞–categories into cyclotomic spectra (compare [11; 9; 6] for treatments), we can similarly form the dualizable object $\mathrm{THH}(\widetilde{X}) \in \mathrm{Mod}_{\mathrm{THH}(R)}(\mathrm{CycSp})$, which provides a lifting of $\mathrm{HH}(X/K)$. Now we can apply Theorem 3.11 as before. $\qquad\square$

4 Freeness results and degeneration in families

In this section, we will analyze Hodge-to-de Rham degeneration in families. In particular, we will give proofs of Theorems 1.3 and 1.4, showing that (under appropriate hypothesis) the relative Hodge-to-de Rham spectral sequence degenerates and that Hochschild homology is locally free. In characteristic zero, at least over a smooth base, this result follows from the existence of a connection [20] on periodic cyclic homology together with Theorem 1.1.

Throughout this section, we will need Künneth formulas, as in the form expressed in [4]. If $(\mathcal{C}, \otimes, \mathbf{1})$ is a symmetric monoidal stable ∞–category with biexact tensor product, then an object $X \in \mathcal{C}$ is called *perfect* if it belongs to the thick subcategory generated by the unit. Perfectness is extremely useful to control objects in $\mathcal{C}$ and their behavior. However, it can be tricky to check directly.

In [4], the main result is that if k is a perfect field of characteristic $p > 0$, in the ∞–category $\mathrm{Mod}_{\mathrm{THH}(k)}(\mathrm{Sp}^{BS^1})$ of modules over $\mathrm{THH}(k)$ in the ∞–category of spectra with an S^1–action, every dualizable object is perfect. This in particular implies the Künneth theorem for periodic topological cyclic homology proved by Blumberg and Mandell [13]. In this section, we will need variants of this result for nonregular rings in characteristic zero (Proposition 4.2) and in the perfect (but not necessarily field) case in characteristic p (Proposition 4.15). This will enable us to control Hochschild homology of stable ∞–categories over, respectively, local Artin rings in characteristic zero and large perfect rings in characteristic p.

4.1 Characteristic zero

In this subsection, we explain the deduction of Theorem 1.3, that the relative Hodge-to-de Rham spectral sequence degenerates for families of smooth and proper dg categories in characteristic zero, and that the relative Hochschild homology is locally free. We actually prove a result over connective E_∞–rings and give a strengthening of the general formality criterion, Theorem 3.11.

The strategy will be to reduce to the local Artinian case, as is standard. We use the following definition:

Definition 4.1 A connective E_∞–ring A is *local Artinian* if $\pi_0(A)$ is a local Artinian ring, each homotopy group $\pi_i(A)$ is a finitely generated $\pi_0(A)$–module, and $\pi_i(A) = 0$ for $i \gg 0$.

Fix a field k of characteristic zero. Let A be a local Artin E_∞–ring with residue field k. Note that $A \to k$ admits a section unique up to homotopy by formal smoothness — compare eg [37, Proposition 2.14] — and so we will consider A as an E_∞–algebra over k. Our first goal is to prove Künneth formulas for negative and periodic cyclic homology for smooth and proper stable ∞–categories over A.

Following [4], we translate this into the following statement. As in Section 2, $\mathrm{HH}(A/k)$ defines a commutative algebra object in the ∞–category Sp^{BS^1} of spectra with an S^1–action[3] and we can consider the symmetric monoidal ∞–category of modules $\mathrm{Mod}_{\mathrm{HH}(A/k)}(\mathrm{Sp}^{BS^1})$. Given an A–linear stable ∞–category $\mathcal{C}$, the Hochschild homology $\mathrm{HH}(\mathcal{C}/k)$ defines an object in $\mathrm{Mod}_{\mathrm{HH}(A/k)}(\mathrm{Sp}^{BS^1})$. The homotopy fixed points $\mathrm{HH}(\mathcal{C}/k)^{hS^1}$ are written as $\mathrm{HC}^-(\mathcal{C}/k)$ and called the *negative cyclic homology* of $\mathcal{C}$ (over k). See also [27] for comparisons with more classical definitions.

Proposition 4.2 *Any dualizable object in the symmetric monoidal ∞–category* $\mathrm{Mod}_{\mathrm{HH}(A/k)}(\mathrm{Sp}^{BS^1})$ *is perfect.*

Proof Let $M \in \mathrm{Mod}_{\mathrm{HH}(A/k)}(\mathrm{Sp}^{BS^1})$ be a dualizable object. We have a lax symmetric monoidal functor

$$F \colon \mathrm{Mod}_{\mathrm{HH}(A/k)}(\mathrm{Sp}^{BS^1}) \to \mathrm{Mod}_{\mathrm{HC}^-(A/k)}, \quad N \mapsto N^{hS^1}.$$

By [38, Section 7], $\mathrm{Mod}_k(\mathrm{Sp}^{BS^1})$ is identified (via the functor $X \mapsto X^{hS^1}$) with the ∞–category of $C^*(BS^1; k)$–modules complete with respect to the augmentation $C^*(BS^1; k) \to k$. Taking modules over $\mathrm{HH}(A/k)$, we conclude that the functor F is fully faithful. Equivalently, the left adjoint functor

$$\mathrm{Mod}_{\mathrm{HC}^-(A/k)} \to \mathrm{Mod}_{\mathrm{HH}(A/k)}(\mathrm{Sp}^{BS^1})$$

is a symmetric monoidal localization.

To check the claim, it suffices to prove that the functor F is strong symmetric monoidal on dualizable objects by [38, Lemma 7.18]. That is, for dualizable objects $M, N \in \mathrm{Mod}_{\mathrm{HH}(A/k)}(\mathrm{Sp}^{BS^1})$, one needs the map

$$(9) \qquad\qquad F(M) \otimes_{\mathrm{HC}^-(A/k)} F(N) \to F(M \otimes N)$$

to be an equivalence of $\mathrm{HC}^-(A/k)$–module spectra. Note that we have an element $x \in \pi_{-2}\mathrm{HC}^-(A/k)$ (ie a generator of $\pi_{-2}\mathrm{HC}^-(k/k) \simeq \pi_{-2}C^*(BS^1; k)$) such that $\mathrm{HC}^-(A/k)/x \simeq \mathrm{HH}(A/k)$ and one has an equivalence of $\mathrm{HH}(A/k)$–module spectra

---

[3]One could work in the derived ∞–category $D(k)$ in this subsection.

$F(M)/x \simeq M$ for any $M \in \mathrm{Mod}_{\mathrm{HH}(A/k)}(\mathrm{Sp}^{BS^1})$ (see [38, Section 7]). It thus follows that (9) becomes an equivalence after base-change $\mathrm{HC}^-(A/k) \to \mathrm{HH}(A/k)$.

It thus suffices to show that (9) becomes an equivalence after inverting x. Now we have

$$(F(M) \otimes_{\mathrm{HC}^-(A/k)} F(N))[1/x] \simeq M^{tS^1} \otimes_{\mathrm{HP}(A/k)} N^{tS^1},$$

$$F(M \otimes N)[1/x] \simeq (M \otimes_{\mathrm{HH}(A/k)} N)^{tS^1}.$$

In other words, it suffices to show that the functor

$$F' \colon \mathrm{Mod}_{\mathrm{HH}(A/k)}(\mathrm{Sp}^{BS^1}) \to \mathrm{Mod}_{\mathrm{HP}(A/k)}, \quad N \mapsto N^{tS^1},$$

is strong symmetric monoidal on dualizable objects.

However, by Lemma 4.3 below, it follows that F' can be identified with the functor $M \mapsto (M \otimes_{\mathrm{HH}(A/k)} k)^{tS^1}$, ie F' factors through the symmetric monoidal functor $\mathrm{Mod}_{\mathrm{HH}(A/k)}(\mathrm{Sp}^{BS^1}) \to \mathrm{Mod}_k(\mathrm{Sp}^{BS^1})$ given by base-change $\mathrm{HH}(A/k) \to k$. Furthermore, $\mathrm{HP}(A/k) \simeq k^{tS^1}$. Since dualizable objects in $\mathrm{Mod}_k(\mathrm{Sp}^{BS^1})$ are perfect, it follows that F' satisfies a Künneth formula. This implies the result. $\qquad \square$

Lemma 4.3 *If M is an object of $\mathrm{Mod}_{\mathrm{HH}(A/k)}(\mathrm{Sp}^{BS^1})$ such that M is bounded below, then the natural map $M \to M \otimes_{\mathrm{HH}(A/k)} k$ induces an equivalence on S^1–Tate constructions.*

Proof Now $M \simeq \varprojlim \tau_{\leq n} M$ and $M \otimes_{\mathrm{HH}(A/k)} k \simeq \varprojlim (\tau_{\leq n} M \otimes_{\mathrm{HH}(A/k)} k)$. Both of these inverse limits become constant in any given range of dimensions. Therefore, they commute with S^1–Tate constructions. Therefore, it suffices to assume that M is n–truncated. By a further dévissage, we can assume that M is actually a discrete k–module, considered as an $\mathrm{HH}(A/k)$–module via the augmentation. We are thus reduced to showing that if N is a discrete k–module, then the map

$$N \to N \otimes_{\mathrm{HH}(A/k)} k \simeq N \otimes_k (k \otimes_{\mathrm{HH}(A/k)} k) \simeq N \otimes_k \mathrm{HH}(k \otimes_A k/k)$$

induces an equivalence on S^1–Tate constructions.

However, since the homology of $k \otimes_A k$ forms a connected graded, commutative Hopf algebra, it follows that $\pi_*(k \otimes_A k)$ is the tensor product of polynomial algebras on even-dimensional classes and exterior algebras on odd-dimensional classes. Therefore, $k \otimes_A k$ is a free E_∞–k–algebra $\mathrm{Sym}^* V$ for some k–module spectrum V with $\pi_i(V) = 0$ for $i \leq 0$. Furthermore, $\mathrm{HH}(k \otimes_A k/k) \simeq \mathrm{Sym}^*(S^1_+ \otimes V)$. The desired equivalence now follows because for $i > 0$, $\mathrm{Sym}^i(S^1_+ \otimes V)$ is a free module over the group ring $k[S^1]$, and so the terms for $i > 0$ (as a direct sum of graded free

$k[S^1]$–modules, and hence a graded free $k[S^1]$–module) do not contribute to the Tate construction. □

Corollary 4.4 *Let A be a local Artin E_∞–ring and let $\mathcal{C}$ be a smooth and proper stable ∞–category over A. Then the map $\mathrm{HP}(\mathcal{C}/k) \to \mathrm{HP}(\mathcal{C} \otimes_A k/k)$ is an isomorphism.*

Note that when $A = k$ itself, this recovers certain cases of the classical theorem of Goodwillie [21, Theorem II.5.1; 22, Lemma I.3.3] about the nilinvariance of periodic cyclic homology. The corollary follows from Lemma 4.3 because one has an equivalence

$$\mathrm{HH}(\mathcal{C} \otimes_A k/k) \simeq \mathrm{HH}(\mathcal{C}/k) \otimes_{\mathrm{HH}(A/k)} k.$$

Corollary 4.5 *Let A be a local Artin E_∞–ring. Let $M \in \mathrm{Mod}_{\mathrm{HH}(A/k)}(\mathrm{Sp}^{BS^1})$ be dualizable, and let $M_A = M \otimes_{\mathrm{HH}(A/k)} A \in \mathrm{Mod}_A(\mathrm{Sp}^{BS^1})$ and $M_k \in M \otimes_{\mathrm{HH}(A/k)} k \in \mathrm{Perf}(k)^{BS^1}$. Then:*

(1) *$M_A \in \mathrm{Mod}_A(\mathrm{Sp}^{BS^1})$ belongs to the thick subcategory generated by the unit.*

(2) *$M_A^{tS^1} \otimes_A k \simeq M_k^{tS^1}$.*

(3) *$M_A^{tS^1}$ is a graded free A^{tS^1}–module.*

Proof By Proposition 4.2, M belongs to the thick subcategory generated by the unit in $\mathrm{Mod}_{\mathrm{HH}(A/k)}(\mathrm{Sp}^{BS^1})$. It follows that $M_A \in \mathrm{Mod}_A(\mathrm{Sp}^{BS^1})$ belongs to the thick subcategory generated by the unit. Thus, we obtain the first claim. The second claim is implied by the first, as for any perfect object $X \in \mathrm{Mod}_A(\mathrm{Sp}^{BS^1})$, one has $(X \otimes_A k)^{tS^1} \simeq X^{tS^1} \otimes_A k$ by a thick subcategory argument.

Finally, one has natural maps

$$M^{tS^1} \to M_A^{tS^1} \to M_A^{tS^1} \otimes_A k \simeq M_k^{tS^1}$$

such that the composite is an equivalence by Lemma 4.3. Thus, the map

$$M_A^{tS^1} \to (M_A^{tS^1}) \otimes_A k$$

has a section of k–module spectra. Lifting a basis, this implies that $M_A^{tS^1}$ is free as an A^{tS^1}–module. □

Lemma 4.6 *Let A be an augmented local Artin E_∞–ring with residue field k. Let M be a perfect A–module. Then*

(10) $$\dim_k(\pi_*(M)) \leq (\dim_k \pi_*(A))(\dim_k \pi_*(M \otimes_A k)),$$

and if equality holds then M is free.

Proof Since A has a filtration (in A–modules) by copies of k, the inequality is evident. If equality holds, suppose that $i \in \mathbb{Z}$ is minimal such that $\pi_i(M) \neq 0$. Then also $\pi_i(M \otimes_A k) = \pi_i(M) \otimes_{\pi_0(A)} k \neq 0$ by Nakayama's lemma. Choose $x \in \pi_i(M)$ whose image in $\pi_i(M \otimes_A k)$ is nonzero. Form a cofiber sequence $\Sigma^i A \xrightarrow{x} M \to N$ of A–modules. It follows that

$$\dim_k(\pi_*(N \otimes_A k)) = \dim_k(\pi_*(M \otimes_A k)) - 1,$$
$$\dim_k(\pi_*(M)) \leq \dim_k(\pi_*(N)) + \dim_k \pi_*(A).$$

Combining this with (10), we find that $\dim_k \pi_*(N) = (\dim_k \pi_*(A))(\dim_k \pi_*(N \otimes_A k))$. By an evident induction, N is free as an A–module. The long exact sequence in homotopy, which must reduce to a short exact sequence, now shows that M is also free as an A–module. □

We can now prove the main freeness and degeneration theorems of this section, which provides a substantial strengthening of Theorem 3.11. Let $\mathrm{CAlg}(\mathrm{Sp}_{\geq 0})$ denote the ∞–category of connective E_∞–rings. In the following argument, one could also work with simplicial commutative rings. We will use the notion of a compact or finitely presented object in the ∞–category of connective E_∞–rings. Recall that this in particular implies that, for any such R, $\pi_0(R)$ is finitely generated as a ring, and $\pi_i(R)$ is a finitely generated $\pi_0(R)$–module for each $i \geq 0$; see [35, Section 7.2.4].

Theorem 4.7 (general formality criterion, relative case) *Let A be a connective E_∞–algebra over $\mathbb{Q}$. Let $M \in \mathrm{Perf}(A)^{BS^1}$. Suppose that there exists a compact object $R \in \mathrm{CAlg}(\mathrm{Sp}_{\geq 0})$ with a map $R \to A$, a dualizable object $M'_R \in \mathrm{Mod}_{\mathrm{THH}(R)}(\mathrm{CycSp})$, and an equivalence $M'_R \otimes_{\mathrm{THH}(R)} A \simeq M \in \mathrm{Perf}(A)^{BS^1}$. Then M is a finitely generated graded projective A–module and the S^1–action on M is formal.*

Proof We first treat the case where A is a local Artin E_∞–ring with residue field k. To see that M is free, it suffices to show that equality holds in (10). Our assumptions show that M lifts to a dualizable object of $\mathrm{Mod}_{\mathrm{HH}(A/k)}(\mathrm{Sp}^{BS^1})$. Using the Tate spectral sequence, one obtains

$$(11) \qquad \dim_k \pi_0(M^{tS^1}) + \dim_k \pi_1(M^{tS^1}) \leq \dim_k \pi_*(M).$$

Moreover, by Corollary 4.5, we know that M^{tS^1} is a free A^{tS^1}–module and that $M^{tS^1} \otimes_A k \simeq (M \otimes_A k)^{tS^1}$. Note that $\pi_0 R$ is a finitely generated $\mathbb{Z}$–algebra. Thus

we can apply Theorem 3.11, and we find that $M \otimes_A k$ is formal in $\mathrm{Perf}(k)^{BS^1}$. We obtain

$$
\dim_k \pi_0(M^{tS^1}) + \dim_k \pi_1(M^{tS^1})
$$
$$
= \left(\dim_k \pi_0((M \otimes_A k)^{tS^1}) + \dim_k \pi_1((M \otimes_A k)^{tS^1})\right) \dim_k \pi_*(A)
$$
$$
= \dim_k \pi_*(M \otimes_A k) \dim_k \pi_*(A).
$$

Combining the above two inequalities, we obtain $\dim_k \pi_*(M \otimes_A k) \dim_k \pi_*(A) \leq \dim_k \pi_*(M)$, which shows that the converse of (10) holds and M is free. Moreover, equality holds in (11), so that the S^1–Tate spectral sequence for M degenerates and M is formal.

We now treat the general case. Clearly it suffices to treat the case where A is a compact object of the ∞–category of connective E_∞–algebras over $\mathbb{Q}$. In this case, $\pi_0(A)$ is noetherian (as a finitely generated $\mathbb{Q}$–algebra) and the homotopy groups $\pi_i(A)$ are finitely generated $\pi_0(A)$–modules. We thus suppose A is of this form.

To check the above statements, it suffices to replace A by its localization at any prime ideal of $\pi_0(A)$. Thus, we may assume that $\pi_0(A)$ is local. Let $x_1, \ldots, x_n \in \pi_0(A)$ be a system of generators of the maximal ideal. For each $r > 0$, we let $A'_r = A/(x_1^r, \ldots, x_n^r)$. Note moreover that $A'_r \simeq \varprojlim \tau_{\leq m} A'_r$ and that $\varprojlim_r A'_r$ is the completion of A, which is in particular faithfully flat over A. By the above analysis, $M \otimes_A \tau_{\leq m} A'_r$ is a free $\tau_{\leq m} A'$–module for each m and r and the Tate spectral sequence degenerates. Now we can let $m, r \to \infty$. Since M is perfect as an A–module, it follows that M is free, as desired, and the S^1–action is formal. $\qquad\square$

Let A be a connective E_∞–algebra over $\mathbb{Q}$. Similarly, one can construct [28] the ∞–category $\mathcal{N}\mathrm{Mot}_A$ of noncommutative motives of A–linear ∞–categories. We let $\mathcal{N}\mathrm{Mot}_A^\omega$ denote the thick subcategory generated by the motives of smooth and proper A–linear ∞–categories. We have, again, a Hochschild homology functor $\mathrm{HH}(\cdot/A) \to \mathrm{Mod}_A^{BS^1}$. The next result gives a basic formality property of this functor; for smooth and proper A–linear ∞–categories, it includes the degeneration of the relative Hodge-to-de Rham spectral sequence.

Corollary 4.8 *Let $X \in \mathcal{N}\mathrm{Mot}_A^\omega$. Then $\mathrm{HH}(X/A) \in \mathrm{Mod}_A^{BS^1}$ is a finitely generated projective A–module and the S^1–action is formal.*

Proof Here we use a refinement of the results of [45] for E_∞–algebras. Namely, we claim that the functor which assigns to an E_∞–ring spectrum R the ∞–category of

smooth and proper R–linear ∞–categories commutes with filtered colimits in R. Now, smooth and proper R–linear ∞–categories are compact; in fact, combine [3, Propositions 3.5 and 3.11]. Therefore, it suffices to see that if R is a filtered colimit of E_∞–algebras R_i, then any smooth and proper R–linear ∞–category $\mathcal{C}$ descends to some R_i. To see this, we observe that $\mathcal{C}$ is equivalent to $\mathrm{Perf}(B)$ for an associative A–algebra B which is compact [3, Proposition 3.11] and we can descend the algebra to a compact algebra over some R_i thanks to [34, Lemma 11.5.7.17]. Moreover, by compactness we can also descend the duality datum to some finite stage.

In view of this, we conclude that given $X \in \mathcal{N}\mathrm{Mot}_A^\omega$, there exists a compact object $R \in \mathrm{CAlg}(\mathrm{Sp}_{\geq 0})$ mapping to A and a smooth and proper R–linear ∞–category $\widetilde{\mathcal{C}}$ such that $\mathcal{C} \simeq \widetilde{\mathcal{C}} \otimes_R A$. Using Theorem 4.7, we can now conclude the proof as before. $\square$

4.2 Characteristic p

The characteristic zero assertion essentially amounts to the idea that periodic cyclic homology should form a crystal over the base which is also coherent, and any such is necessarily well known to be locally free. In characteristic p, one can appeal to an analogous argument: given a smooth algebra R in characteristic p, any finitely generated R–module M isomorphic to its own Frobenius twist is necessarily locally free [19, Proposition 1.2.3]. In this subsection, we prove Theorem 1.4 from the introduction. In doing so, we essentially use the Frobenius semilinearity of the cyclotomic Frobenius.

We first discuss what we mean by liftability. Let A be a regular (noetherian) $\mathbb{F}_p$–algebra. Recall that A is F–*finite* if the Frobenius map $\varphi\colon A \to A$ is a finite morphism. We refer to [18, Section 2.2] for a general discussion of F–finite rings.

Definition 4.9 Given an F–finite regular noetherian ring A, a *lift* of A to $\mathbb{Z}/p^2$ will mean a flat $\mathbb{Z}/p^2$–algebra $\widetilde{A}$ with an isomorphism $\widetilde{A} \otimes_{\mathbb{Z}/p^2} \mathbb{F}_p \simeq A$.

Let A be a regular noetherian $\mathbb{F}_p$–algebra. By Popescu's smoothing theorem (see [7, Tag 07GC] for a general reference), A is a filtered colimit of smooth $\mathbb{F}_p$–algebras. It follows that the cotangent complex $L_{A/\mathbb{F}_p}$ is concentrated in degree 0 and identified with the Kähler differentials; in addition, they form a flat A–module. If A is in addition F–finite, then the Kähler differentials are finitely generated and therefore projective as an A–module. Recall that the cotangent complex controls the infinitesimal deformation theory of A [29, Chapter III, Section 2]. Therefore, A is formally smooth as an $\mathbb{F}_p$–algebra, and a lift to $\mathbb{Z}/p^2$ exists. Given a lift $\widetilde{A}$ to $\mathbb{Z}/p^2$, it follows that $\widetilde{A}$

is formally smooth over $\mathbb{Z}/p^2$. In particular, it follows that any two lifts to $\mathbb{Z}/p^2$ are (noncanonically) isomorphic. Moreover, if $A \to B$ is a map of F–finite regular noetherian $\mathbb{F}_p$–algebras and $\widetilde{A}$ and $\widetilde{B}$ are respective lifts to $\mathbb{Z}/p^2$, then the map lifts to a map $\widetilde{A} \to \widetilde{B}$.

Let A be a regular F–finite $\mathbb{F}_p$–algebra. Then the Frobenius $\varphi \colon A \to A$ is a finite, flat morphism. We let A_{perf} denote the *perfection* of A, ie the colimit of copies of A along the Frobenius map. Then we have inclusions

$$A \subset A^{1/p} \subset A^{1/p^2} \subset \cdots \subset A_{\mathrm{perf}}$$

such that all maps are faithfully flat and the colimit is A_{perf}. Our strategy will essentially be descent to A_{perf}. Unfortunately, A_{perf} is not noetherian. Thus, we will need the following result:

Proposition 4.10 *Let A be a regular F–finite $\mathbb{F}_p$–algebra.*

(1) *Then the ring A_{perf} is coherent, ie the finitely presented modules form an abelian category.*

(2) *Let $I \subset A$ be an ideal. Given a finitely presented A_{perf}–module M, the submodule $M' \subset M$ consisting of those elements annihilated by a power of I is also coherent and its annihilator in A_{perf} is finitely generated.*

Proof The first assertion follows because A_{perf} is the filtered colimit of copies of the noetherian ring A along the Frobenius map, which is flat by regularity. If M is a coherent A_{perf}–module, then M descends to A^{1/p^n} for some n, ie there exists a finitely generated module M_n over A^{1/p^n} such that $M \simeq A_{\mathrm{perf}} \otimes_{A^{1/p^n}} M_n$. Then M_n has an A^{1/p^n}–submodule M'_n consisting of the I–power torsion, which is also finitely generated (and hence finitely presented), and such that the quotient has no I–power torsion. It follows from flatness that $M'_n \otimes_{A^{1/p^n}} A_{\mathrm{perf}} = M'$, which is thus coherent. Since M' is coherent, its annihilator ideal is also coherent. $\qquad\square$

We will also need to observe that analogs of Bökstedt's calculation of $\mathrm{THH}(k)$ hold when k is any perfect $\mathbb{F}_p$–algebra, not only a field. Similarly, analogs of Propositions 3.6 and 3.7 hold with analogous arguments.

Proposition 4.11 *Let k be a perfect $\mathbb{F}_p$–algebra. Suppose M is a perfect $\mathrm{THH}(k)$–module and $\pi_i(M/\sigma) = 0$ for $i \notin [a, b]$ for $b - a \leq 2p - 2$. Suppose M lifts to a perfect $\mathrm{THH}(W_2(k))$–module. Then, as $\pi_* \mathrm{THH}(k) \simeq k[\sigma]$–modules, one has $\pi_*(M) \simeq \pi_*(M/\sigma) \otimes_k k[\sigma]$.*

Proof   Without loss of generality, $a = 0$. For each j, we need to argue that multiplication by σ is a *split* injection of k–modules $\pi_{j-2}(M) \to \pi_j(M)$. Using the long exact sequence and the assumption on M/σ, we find that multiplication by σ is an isomorphism for $j \geq 2p - 1$. For $j \leq 2p - 2$, the equivalence in the range $[0, 2p - 2]$ as in (8) implies the result.   □

We can now state and prove the main formality statement in characteristic p over a regular F–finite base.

Theorem 4.12   (formality criterion, relative characteristic p case)   *Let A be a regular F–finite $\mathbb{F}_p$–algebra. Let $\widetilde{A}$ be a flat lift to $\mathbb{Z}/p^2$. Let $M \in \mathrm{Perf}(A)^{BS^1}$. Suppose that*:

   (1)   *There exists a dualizable object $M' \in \mathrm{Mod}_{\mathrm{THH}(A)}(\mathrm{CycSp})$ such that*

$$M' \otimes_{\mathrm{THH}(A)} A \in \mathrm{Perf}(A)^{BS^1}.$$

   (2) $\pi_i(M)$ *vanishes for $i \notin [a, b]$ for some a and b with $b - a \leq 2p - 2$.*

   (3)   *The underlying $\mathrm{THH}(A)$–module of M' lifts to a perfect $\mathrm{THH}(\widetilde{A})$–module.*

Then M is a finitely generated (graded) projective A–module, and the S^1–action on M is formal.

Proof   First, we can reduce to the case where A is an F–finite regular local ring with maximal ideal $\mathfrak{m}$. In this case, we can induct on the Krull dimension d of A. We can assume that the result holds for all F–finite regular local rings of Krull dimension less than d. When $d = 0$, the claim is of course Theorem 1.2.

To verify the claims for A, we can now replace A by its $\mathfrak{m}$–adic completion $\widehat{A}$, which is faithfully flat over A. Note that $\widehat{A}$ is also an F–finite regular local ring of Krull dimension d. Since $\widehat{A}$ is complete, it contains a copy of its residue field k and is identified with $\widehat{A} \simeq k[\![x_1, \ldots, x_n]\!]$. We can consider the faithfully flat map $\widehat{A} \to k_{\mathrm{perf}}[\![x_1, \ldots, x_d]\!]$. Replacing A with $k_{\mathrm{perf}}[\![x_1, \ldots, x_d]\!]$, we will now simply assume that A is in addition complete and has perfect residue field. By the inductive hypothesis, all the differentials in the Hodge-to-de Rham spectral sequence are $\mathfrak{m}$–power torsion and that $\mathrm{HH}(\mathcal{C}/A)$ is locally free away from $\mathfrak{m}$.

Let A_{perf} denote the (colimit) perfection of A, so one has a faithfully flat map $A \to A_{\mathrm{perf}}$. We form the base-changes $M'_{\mathrm{perf}} := M' \otimes_{\mathrm{THH}(A)} \mathrm{THH}(A_{\mathrm{perf}}) \in \mathrm{Mod}_{\mathrm{THH}(A_{\mathrm{perf}})}(\mathrm{CycSp})$ (which is a dualizable object) and $M_{\mathrm{perf}} = M \otimes_A A_{\mathrm{perf}} \in \mathrm{Perf}(A_{\mathrm{perf}})^{BS^1}$. We claim

that the cyclotomic Frobenius

$$\varphi \colon M'_{\mathrm{perf}}[1/\sigma] \to (M'_{\mathrm{perf}})^{tC_p} \simeq (M_{\mathrm{perf}})^{tS^1}$$

is an equivalence. This follows using the same arguments as in [4, Section 4]; again, one needs to know that both sides are symmetric monoidal functors in M'_{perf}. For this, it suffices to show that M'_{perf} belongs to the thick subcategory generated by the unit in $\mathrm{Mod}_{\mathrm{THH}(A_{\mathrm{perf}})}(\mathrm{Sp}^{BS^1})$. We will check this in Proposition 4.15 below.

Note that M'_{perf} is a $\mathrm{THH}(A_{\mathrm{perf}})$–module, and $M'_{\mathrm{perf}}/\sigma \simeq M_{\mathrm{perf}}$. Under the liftability hypotheses, we conclude, using Proposition 4.11, that there is an isomorphism of $A_{\mathrm{perf}}[\sigma]$–modules

$$\pi_*(M'_{\mathrm{perf}}) \simeq \pi_*(M_{\mathrm{perf}})[\sigma].$$

Combining, we find an isomorphism of A_{perf}–modules

$$(12) \qquad \pi_*(M_{\mathrm{perf}})[\sigma^{\pm 1}]^{(1)} \simeq \pi_*(M_{\mathrm{perf}}^{tS^1}).$$

In addition, we have the Tate spectral sequence, which shows that $\pi_0(M_{\mathrm{perf}}^{tS^1})$ is a subquotient of $\pi_{\mathrm{even}}(M_{\mathrm{perf}})$ and is a coherent A_{perf}–module. Since the differentials are $\mathfrak{m}$–power torsion, it follows that the $\mathfrak{m}$–power torsion in $\pi_0(M_{\mathrm{perf}}^{tS^1})$ is a subquotient of the $\mathfrak{m}$–power torsion in $\pi_{\mathrm{even}}(M_{\mathrm{perf}})$.

Let I be the annihilator of the $\mathfrak{m}$–power torsion in $\pi_{\mathrm{even}}(M_{\mathrm{perf}})$, which, by Proposition 4.10, is a finitely generated ideal. Then, combining the above observations and (12), we find that $I^{[p]}$ (ie the ideal generated by p^{th} powers of elements in I) is the annihilator of the $\mathfrak{m}$–power torsion in $\pi_0(M_{\mathrm{perf}}^{tS^1})$. Since this is a subquotient of $\pi_{\mathrm{even}}(M_{\mathrm{perf}})$, it follows that $I \subset I^{[p]}$, which is only possible for a finitely generated proper ideal if $I = (0)$. Therefore, $\pi_{\mathrm{even}}(M_{\mathrm{perf}})$ (and similarly for the odd-dimensional Hochschild homology) is torsion-free.

Finally, it suffices to prove freeness. We have proved that $\pi_*(M)$ consists of finitely generated, torsion-free A–modules. Let $x \in \mathfrak{m} \setminus \mathfrak{m}^2$, so that A/x is a regular local ring too. It follows that $\pi_*(M/x)$ is x–torsion-free and that, by induction on the Krull dimension, $\pi_*(M)/(x)$ is a free $A/(x)$–module. This easily implies that $\pi_*(M)$ is free as an A–module. By comparing with the base-change from A to the perfection of its fraction field, it also follows that M is formal. □

In the course of the above argument, we had to check a perfectness statement. In [4], such results are proved when A_{perf} is a field, but they depend on noetherianness hypotheses. One can carefully remove the noetherianness hypotheses in this case, but

for simplicity, we verify this by using the technique of relative THH (also discussed in [4, Section 3]). The starting point is a relative version of Bökstedt's calculation. We denote by $S^0[q_1, \ldots, q_n]$ the E_∞–ring $\Sigma^\infty_+(\mathbb{Z}^n_{\geq 0})$. The idea of considering THH relative to such E_∞–rings is known to experts, and plays an important role in [8].

Proposition 4.13 *Let A be an F–finite regular local ring with system of parameters $t_1, \ldots, t_n$ and perfect residue field k. Consider the map of E_∞–rings*

$$S^0[q_1, \ldots, q_n] \to A, \quad q_i \mapsto t_i.$$

Then

$$\mathrm{THH}(A/S^0[q_1, \ldots, q_n])_* \simeq A[\sigma] \quad \text{with } |\sigma| = 2,$$

where σ is the image of the Bökstedt element under the natural map $\mathrm{THH}(\mathbb{F}_p) \to \mathrm{THH}(A/S^0[q_1, \ldots, q_n])$.

Proof Compare also the treatment in [4, Section 3]. Since A is F–finite and regular, the cotangent complex $L_{A/\mathbb{F}_p}$ is a finitely generated free module in degree 0. By the transitivity sequence, $L_{A/\mathbb{Z}_p[t_1,\ldots,t_n]}$ is a perfect A–module. Thus, by the Quillen spectral sequence, the homotopy groups of $\mathrm{HH}(A/\mathbb{Z}[q_1, \ldots, q_n])$ and thus $\mathrm{THH}(A/\mathbb{Z}[q_1, \ldots, q_n])$ are finitely generated A–modules. Compare also [18] for general finite generation results.

Moreover, after the base-change $S^0[q_1, \ldots, q_n] \to S^0$ sending $q_i \mapsto 0$, one obtains Bökstedt's calculation $\mathrm{THH}(k)_* \simeq k[\sigma]$. Since the homotopy groups of

$$\mathrm{THH}(A/S^0[q_1, \ldots, q_n])$$

are finitely generated A–modules, and A is local, the result follows. $\square$

Let A be as above. Given a smooth and proper A–linear stable ∞–category $\mathcal{C}$, one can consider the invariant $\mathrm{THH}(\mathcal{C}/S^0[q_1, \ldots, q_n])$, which naturally takes values in the symmetric monoidal ∞–category $\mathrm{Mod}_{\mathrm{THH}(A/S^0[q_1,\ldots,q_n])}(\mathrm{Sp}^{BS^1})$. This produces a one-parameter deformation of Hochschild homology over A, and it is particularly well behaved (at least for smooth and proper A–linear stable ∞–categories) by the following result:

Proposition 4.14 *Let A be an F–finite regular local ring with system of parameters $t_1, \ldots, t_n$ and perfect residue field k. Any dualizable object in*

$$\mathrm{Mod}_{\mathrm{THH}(A/S^0[q_1,\ldots,q_n])}(\mathrm{Sp}^{BS^1})$$

is perfect.

Proof This follows by regularity from [4, Theorem 2.15]. □

Proposition 4.15 *Let A be an F–finite regular local ring. Let $N \in \mathrm{Mod}_{\mathrm{THH}(A)}(\mathrm{Sp}^{BS^1})$ be dualizable. Then $N \otimes_{\mathrm{THH}(A)} \mathrm{THH}(A_{\mathrm{perf}})$ is perfect.*

Proof In fact, we have a factorization of E_∞–rings with S^1–action

$$\mathrm{THH}(A) \to \mathrm{THH}(A/S^0[q_1, \ldots, q_n])$$
$$\to \mathrm{THH}(A_{\mathrm{perf}}/S^0[q_1^{1/p^\infty}, \ldots, q_n^{1/p^\infty}]) \simeq \mathrm{THH}(A_{\mathrm{perf}}).$$

Here we use the general observation that for an $E_\infty - S^0[q_1^{1/p^\infty}, \ldots, q_n^{1/p^\infty}]$–algebra R, the relative Hochschild homology $\mathrm{THH}(R/S^0[q_1^{1/p^\infty}, \ldots, q_n^{1/p^\infty}])$ and the absolute $\mathrm{THH}(R)$ agree after p–adic completion; this follows easily from

$$\mathrm{THH}(S^0[q_1^{1/p^\infty}, \ldots, q_n^{1/p^\infty}])/p \simeq S^0[q_1^{1/p^\infty}, \ldots, q_n^{1/p^\infty}]/p$$

by perfectness (see [8, Proposition 11.7]).

We have just seen that $N \otimes_{\mathrm{THH}(A)} \mathrm{THH}(A/S^0[q_1, \ldots, q_n])$ is perfect in

$$\mathrm{Mod}_{\mathrm{THH}(A/S^0[q_1,\ldots,q_n])}(\mathrm{Sp}^{BS^1});$$

base-changing up to $\mathrm{THH}(A_{\mathrm{perf}})$, the result follows. □

Once more, we make the statement for Hochschild homology of categories, or more generally for noncommutative motives. Let A be an F–finite regular noetherian ring with lift $\widetilde{A}$ to $\mathbb{Z}/p^2$. We use, again, the ∞–category $\mathcal{N}\mathrm{Mot}_A$, its subcategory $\mathcal{N}\mathrm{Mot}_A^\omega$ generated by the motives of smooth and proper A–linear ∞–categories, and the Hochschild homology functor $\mathrm{HH}(\,\cdot\,/A)\colon \mathcal{N}\mathrm{Mot}_A \to \mathrm{Mod}_A^{BS^1}$.

Corollary 4.16 *Let $X \in \mathcal{N}\mathrm{Mot}_A^\omega$. Suppose that X lifts to an object of $\mathcal{N}\mathrm{Mot}_{\widetilde{A}}^\omega$ and that $\mathrm{HH}_i(X/A)$ vanishes for $i \notin [a, b]$ for $b - a \leq 2p - 2$. Then the Hochschild homology groups $\mathrm{HH}_i(X/A)$ are finitely generated projective A–modules and the S^1–action is formal.*

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