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Eilenberg–Mac Lane spectra as equivariant Thom spectra

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We prove that the G–equivariant mod p Eilenberg–Mac Lane spectrum arises as an equivariant Thom spectrum for any finite, p–power cyclic group G, generalizing a result of Behrens and the second author in the case of the group C_2. We also establish a construction of $\mathrm{H}\underline{\mathbb{Z}}_{(p)}$, and prove intermediate results that may be of independent interest. Highlights include constraints on the Hurewicz images of equivariant spectra that admit norms, and an analysis of the extent to which the nonequivariant $\mathrm{H}\mathbb{F}_p$ arises as the Thom spectrum of a more than double loop map.

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1 Introduction

Both authors are fond of the following result of Mahowald [31]:

Theorem 1.1 (Mahowald) *The Thom spectrum of the unique nontrivial double loop map*

$$\Omega^2 S^3 \to BO \tag{1}$$

is $\mathrm{H}\mathbb{F}_2$.

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This result and several variants have enjoyed many subsequent proofs in the literature. Examples include Mahowald [32], Cohen, May and Taylor [12], Priddy [38], Mahowald, Ravenel and Shick [33], Antolín-Camarena and Barthel [1] and Mathew, Naumann and Noel [34]. In joint work with Mark Behrens [6], the second author proved a C_2–equivariant generalization of Mahowald's Theorem 1.1:

Theorem 1.2 (Behrens–Wilson) *Let ρ denote the real regular representation of the group C_2. Then there is a Ω^ρ–map*

$$\Omega^\rho S^{\rho+1} \to BO_{C_2}$$

with Thom spectrum $\mathrm{H}\underline{\mathbb{F}}_2$, the Eilenberg–Mac Lane object associated to the constant Mackey functor $\underline{\mathbb{F}}_2$.

A less well-known fact is that Mahowald's map (1) is in fact a triple loop map, as the authors first learned from Mike Hopkins:

Observation Mahowald's map (1) may be obtained by thrice looping the composite

$$\mathbb{H}P^\infty \simeq B\mathrm{Sp}(1) \to B\mathrm{Sp} \simeq B^5 O \xrightarrow{\eta} B^4 O.$$

The present work arose from the authors' attempt to understand how the above observation generalizes to the C_2–equivariant setting. We obtain the following theorem:

Theorem A *Let σ denote the sign representation of the group C_2. There is a C_2–action on the space $\mathbb{H}P^\infty$ and a $\Omega^{\rho+\sigma}$–map*

$$\Omega^{\rho+\sigma}\mathbb{H}P^\infty \to BO_{C_2},$$

with Thom spectrum $\mathrm{H}\underline{\mathbb{F}}_2$.

Since C_2 is the only group with a 2–dimensional real regular representation, the statement of Theorem 1.2 makes it unclear how the constant G–Mackey functor $\mathrm{H}\underline{\mathbb{F}}_2$ might be an equivariant Thom spectrum for any larger group G. On the other hand, the C_2–representation $\rho+\sigma = 2\sigma+1$ is naturally the restriction of a C_4–representation. In this paper, we in fact obtain Theorem A as a special case of the following more general result:

Theorem B *Fix an integer $n \geq 0$, and let $G = C_{2^n}$ denote the cyclic group of order 2^n. Let λ denote the standard representation of G on the complex plane, where the generator acts by $e^{2\pi i/2^n}$. Then there is a G–action on $\mathbb{H}P^\infty$, and a $\Omega^{\lambda+1}$–map*

$$\Omega^{\lambda+1}\mathbb{H}P^\infty \to BO_G,$$

with Thom spectrum $\mathrm{H}\underline{\mathbb{F}}_2$.

Remark 1.3 The group $G = C_2$ is special: only for this group does $\lambda = 2\sigma$ split as the sum of two smaller representations. This leads to a nonobvious equivalence of C_2–spaces

$$\Omega^\rho S^{\rho+1} = \Omega^{\sigma+1}S^{\sigma+2} \simeq \Omega^{2\sigma+1}\mathbb{H}P^\infty \simeq \Omega^{2\sigma}S^{2\sigma+1} = \Omega^\lambda S^{\lambda+1},$$

which implies a Ω^λ–analog of Theorem 1.2 (for details, see Section 3). A Borel equivariant version of this Ω^λ–analog previously appeared as Lemma 3.1 in Klang [27], where it is used in an essential manner to compute the factorization homology of Eilenberg–Mac Lane spectra.

We next turn to odd primes $p > 2$. So long as one is willing to contemplate Thom spectra of p–local spherical fibrations (as in eg Blumberg, Cohen and Schlichtkrull [8, Section 3.4]), Mahowald's Theorem 1.1 admits an analog due to Hopkins [34, Theorem 4.18]:

Theorem 1.4 (Hopkins) *Let*

$$S^3 \xrightarrow{1-p} B^3\mathrm{GL}_1(S^0_{(p)})$$

denote the class $1-p \in \pi_0(S_{(p)})^\times = \pi_3(B^3\mathrm{GL}_1(S_{(p)}))$. Then, applying Ω^2, one obtains a map

$$\Omega^2 S^3 \to B\mathrm{GL}_1(S^0_{(p)})$$

with Thom spectrum equivalent to $\mathrm{H}\mathbb{F}_p$.

In light of the situation at $p = 2$, it is natural to wonder if the map

$$1-p\colon S^3 \to B^3\mathrm{GL}_1(S^0_{(p)})$$

may be delooped. The authors were surprised to find that it *cannot*, which we record as our only nonequivariant result:

Theorem C *Let $S^0_{(p)}$ denote the p–local sphere spectrum, and suppose $p > 2$. Then there is no triple loop map*

$$X \to B\mathrm{GL}_1(S^0_{(p)}),$$

for any triple loop space X, with Thom spectrum $\mathrm{H}\mathbb{F}_p$. The same is true if $S^0_{(p)}$ is replaced by the p–completed sphere spectrum.

Remark 1.5 At the prime $p = 3$, the map

$$1 - p \colon S^3 \to B^3\mathrm{GL}_1(S^0_{(p)})$$

is not even an H–space map for the standard H–space structure on $S^3 = \mathrm{SU}(2)$. Crucially, we will see in Section 5 that the map *is* an H–space map for a certain exotic H–space structure on S^3. Determining the maximum amount of structure present on this map remains an interesting question.

Our equivariant generalization of Theorem 1.4 is as follows:

Theorem D (Theorem 8.1) *Fix an integer $n \geq 0$ and a prime number p, and let G denote the cyclic group C_{p^n}. Let λ denote the standard representation of G on the complex numbers, where a generator acts by $e^{2\pi i/p^n}$, and let $S^0_{(p)}$ denote the G–equivariant sphere spectrum. Finally, let*

$$\mu \colon \Omega^\lambda S^{\lambda+1} \to B\mathrm{GL}_1(S^0_{(p)})$$

denote the Ω^λ–map obtained by applying Ω^λ to the map

$$S^{\lambda+1} \to B^{\lambda+1}\mathrm{GL}_1(S^0_{(p)})$$

corresponding to $1 - p \in (\pi_0^G S^0_{(p)})^\times$. Then the Thom spectrum $(\Omega^\lambda S^{\lambda+1})^\mu$ of μ is $\mathrm{H}\underline{\mathbb{F}}_p$.

Remark 1.6 We will include a detailed discussion of Thom spectra of G–equivariant, p–local spherical fibrations in Section 4. This will include in particular a discussion of $B\mathrm{GL}_1(S^0_{(p)})$ as a G–infinite loop space, allowing us to form its λ–delooping $B^{\lambda+1}\mathrm{GL}_1(S^0_{(p)})$. The basic definitions are due to Lewis and May, and found in [29, Chapter X].

We also establish a (p–local) integral variant of Theorem D:

Theorem E (Theorem 9.1) *Let $S^{\lambda+1}\langle\lambda+1\rangle$ denote the fiber of the unit*

$$S^{\lambda+1} \to \Omega^\infty(\Sigma^{\lambda+1}\mathrm{H}\underline{\mathbb{Z}}).$$

Then there is an equivalence

$$\big(\Omega^\lambda(S^{\lambda+1}\langle\lambda+1\rangle)\big)^\mu \simeq \mathrm{H}\underline{\mathbb{Z}}_{(p)}.$$

Remark 1.7 Collections of little disks in the representation λ form an equivariant operad $\mathcal{O}_\lambda$, leading to the notion of an $\mathbb{E}_\lambda$–algebra; see Hauschild [19]. A nice modern discussion of $\mathbb{E}_\lambda$–algebras may be found in Hill [20].

If X is any pointed G–space, then $\Omega^\lambda X$ is naturally an $\mathbb{E}_\lambda$–algebra in G–spaces. If X is a G–connected, pointed G–space, then the free $\mathbb{E}_\lambda$–algebra on X is $\Omega^\lambda\Sigma^\lambda X$; see Rourke and Sanderson [40, Theorem 1]. Any G–$\mathbb{E}_\infty$–ring, such as the Eilenberg–Mac Lane object $\mathrm{H}\underline{\mathbb{Z}}_{(p)}$ — see Schwede [41, Section 5.7] — is naturally an $\mathbb{E}_\lambda$–algebra object in G–spectra. It was proved by Lewis [29, Remark X.6.4] that the Thom spectrum of a Ω^λ–map is naturally an $\mathbb{E}_\lambda$–algebra in G–spectra, and it follows from our proof that the equivalence of Theorem E is one of $\mathbb{E}_\lambda$–algebras.

Remark 1.8 Just as Mahowald's original Theorem 1.1 may be phrased as the fact that $\mathrm{H}\mathbb{F}_2$ is the free $\mathbb{E}_2$–algebra with a nullhomotopy $2 \simeq 0$ [34; 1], our Theorem D may be read as the statement that the free $\mathbb{E}_\lambda$–algebra with $p \simeq 0$ is $\mathrm{H}\underline{\mathbb{F}}_p$.

Conventions We freely use the language of ∞–categories (alias quasicategories, alias weak Kan complexes) in the form developed in Lurie [30] (though we give specific references within [30] whenever technical results are applied). We denote by Spaces the ∞–category of spaces, and append decorations to obtain the ∞–category of pointed spaces or of G–spaces for a finite group G, the latter being defined as the ∞–category of presheaves $\mathsf{Psh}(\mathcal{O}_G)$ on the 1–category of transitive G–sets. We denote by Sp^G the ∞–category of (genuine) G–spectra, and assume the reader is familiar with the standard notations of equivariant homotopy theory. We could not improve on the summary given in Hill, Hopkins and Ravenel [21, Sections 2–3], and recommend it to the reader. In particular, we will require the Eilenberg–Mac Lane spectrum associated to the constant Mackey functor $\underline{\mathbb{F}}_p$, and the notion of the geometric fixed points $X^{\Phi G}$ of a G–spectrum. Finally, we denote the G–space classifying equivariant stable spherical fibrations with fiber of type S^0 by $B\mathrm{GL}_1(S^0)$. (See Section 4 for more details).

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2 Outline of the proof of Theorem D

We fix, for the remainder of the paper, a prime p as well as a nonnegative integer n. We let $G = C_{p^n}$ denote the cyclic group of order p^n.

The Thom spectrum of the map

$$S^1 \xrightarrow{1-p} B\mathrm{GL}_1(S^0_{(p)})$$

is the mod p Moore spectrum $M(p)$, which admits a Thom class for $\mathrm{H}\underline{\mathbb{F}}_p$. It follows formally (see the argument for Proposition 5.3 in [6]) that there is a Thom class

$$\alpha\colon (\Omega^\lambda S^{\lambda+1})^\mu \to \mathrm{H}\underline{\mathbb{F}}_p.$$

Our goal is to show that this map is an equivalence of G–spectra. By induction on the order of the group (the base case being supplied by Hopkins and Mahowald), it will suffice to prove that the map on geometric fixed points

$$\alpha^{\Phi G}\colon ((\Omega^\lambda S^{\lambda+1})^\mu)^{\Phi G} \to \mathrm{H}\underline{\mathbb{F}}_p^{\Phi G}$$

is an equivalence.

The proof proceeds in several steps:

Step 1 Compute the homotopy groups of $((\Omega^\lambda S^{\lambda+1})^\mu)^{\Phi G}$.

Step 2 Compute the homotopy groups of $\mathrm{H}\underline{\mathbb{F}}_p^{\Phi G}$.

Step 3 Show that $\alpha^{\Phi G}$ is a ring map (for some ring structure on the source).

Step 4 Show that $\alpha^{\Phi G}$ hits algebra generators in the target.

The computation in Step 2 is well known, and is stated as Lemma 8.2 below. At odd primes, one learns that

$$\pi_*\mathrm{H}\underline{\mathbb{F}}_p^{\Phi G} = \mathbb{F}_p[t] \otimes \Lambda(s), \quad |t| = 2,\ |s| = 1.$$

Step 1 is more difficult. In Section 3 we show that $(\Omega^\lambda S^{\lambda+1})^G = \Omega^2 S^3 \times \Omega S^2$, and, after developing some more properties of this Thom spectrum, we verify (Lemma 8.3) that the homotopy groups of $((\Omega^\lambda S^{\lambda+1})^\mu)^{\Phi G}$ are *additively* given by

$$\pi_*((\Omega^\lambda S^{\lambda+1})^\mu)^{\Phi G} = \mathbb{F}_p[x] \otimes \Lambda(y), \quad |x| = 2, \ |y| = 1.$$

There is a serious problem to be addressed in Step 3: by construction, $(\Omega^\lambda S^{\lambda+1})^\mu$ is an $\mathbb{E}_\lambda$–spectrum and the Thom class is represented by an $\mathbb{E}_\lambda$–map. When we take geometric fixed points, we are left without any obvious multiplicative structure on either the source or the map. We must do additional work to equip $\alpha^{\Phi G}$ with additional multiplicative structure. The weakest structure that suffices for our purposes is that of an $\mathbb{A}_2$–algebra; we recall the definition below:

Definition 2.1 An $\mathbb{A}_2$–algebra in a symmetric monoidal ∞–category $(\mathcal{C}, \otimes, \mathbf{1})$ consists of the following data:

(1) An object $X \in \mathcal{C}$.

(2) A *multiplication* map m: $X \otimes X \to X$.

(3) A *unit* map u: $\mathbf{1} \to X$.

(4) Choices of 2–simplices filling each of the diagrams

$$\begin{tikzcd} X \otimes 1 \arrow[r, "\simeq"] \arrow[d, "\mathrm{id}\otimes u"'] & X \\ X \otimes X \arrow[ur, "m"'] & \end{tikzcd} \qquad \begin{tikzcd} 1 \otimes X \arrow[r, "\simeq"] \arrow[d, "u\otimes \mathrm{id}"'] & X \\ X \otimes X \arrow[ur, "m"'] & \end{tikzcd}$$

Remark 2.2 An $\mathbb{A}_2$–algebra in pointed spaces (with the cartesian symmetric monoidal structure) is an H–space.

To accomplish Step 3, we show that $\alpha^{\Phi G}$ is an $\mathbb{A}_2$–algebra morphism for a certain $\mathbb{A}_2$–structure on the source. To do so requires a different argument at the prime 2 than at odd primes.

Step 3a In Section 3 we show that $S^{\lambda+1} = \Omega \mathbb{H}P^\infty$ for a certain G–action on $\mathbb{H}P^\infty$. We then show (Section 4) that, when $p = 2$, the map

$$S^{\lambda+1} \to B^\lambda B\mathrm{GL}_1(S^0_{(2)})$$

deloops once. Thus, $\alpha^{\Phi G}$ is an $\mathbb{A}_\infty$–map in this case. This section contains some material that may be of independent interest, such as a description of one of the spaces in the equivariant K–theory spectrum in terms of bundles of (twisted) G–$\mathbb{H}$–modules.

Step 3b In Section 5 we produce an exotic $\mathbb{A}_2$–structure on $S^{\lambda+1}$ at odd primes with respect to which the map μ is an $\mathbb{A}_2$–map in $\mathbb{E}_\lambda$–spaces. Our proof uses a small dose of unstable equivariant homotopy theory, in particular the EHP sequence.

Finally, we come to Step 4. The element $s \in \pi_1 \mathrm{H}\underline{\mathbb{F}}_p^{\Phi G}$ arises as a witness to the fact that the composite

$$S^0 \xrightarrow{\nabla} G_+ \xrightarrow{1} \mathrm{H}\underline{\mathbb{F}}_p$$

is null. Said differently, s witnesses the vanishing of the element $[G] \in \pi_0^G S^0$ in the Hurewicz image. Nonequivariantly, the zeroth homotopy group of $(\Omega^2 S^3)^\mu$ is already detected by the map

$$(S^1)^\mu = M(p) \to (\Omega^2 S^3)^\mu.$$

Equivariantly, this is no longer true. Recall [43; 15] that $\pi_0^G S^0 = A(G)$ is the Burnside ring of finite G–sets. While the element p dies in the Moore space $(S^1)^\mu$, the same is not true of the elements $[G/K]$ for $K \subsetneq G$. The proof that these elements vanish in the Hurewicz image of $(\Omega^\lambda S^{\lambda+1})^\mu$ is a consequence of the vanishing of p together with the existence of *norms* supplied by the $\mathbb{E}_\lambda$–structure. This is proved in Section 6 and the final pieces of the proof of Step 4 and the main result are then spelled out in Section 8.

We end in Section 9 with an explanation of how to produce $\mathrm{H}\underline{\mathbb{Z}}_{(p)}$ as a Thom spectrum as well as some unanswered questions.

3 Quaternionic projective space

We begin by examining a very natural action of S^1 on quaternionic projective space. It is in some ways analogous to the natural action of C_2 on $\mathbb{C}P^\infty$ by complex conjugation.

Construction 3.1 Consider the action of $\mathrm{Sp}(1)$ on the quaternions $\mathbb{H}$ by conjugation. The center $\{\pm 1\}$ acts trivially, so this produces an action of $\mathrm{Sp}(1)/\{\pm 1\} = \mathrm{SO}(3)$ on $\mathbb{H}$. This produces an action of $S^1 \subseteq \mathrm{SO}(3)$ on $\mathbb{H}$ which can be described in two equivalent ways:

- $z \in S^1$ acts on $q \in \mathbb{H}$ by $(\sqrt{z})q(\sqrt{z})^{-1}$;
- $z \in S^1$ acts on $q = u + vj$ by $u + (zv)j$, where $u, v \in \mathbb{C}$.

From the first description it's clear that $\mathbb{H}$ is an S^1–equivariant algebra. From the second description it's clear that $\mathbb{H} = 2 + \lambda$ as a representation.

Definition 3.2 Let $\mathbb{H}P^\infty$ denote the S^1–space obtained from $\mathbb{H}^\infty$ by imposing the relation

$$[q_0 : q_1 : \cdots] \sim [q_0 h : q_1 h : \cdots], \quad h \in \mathbb{H}^\times,$$

and acting by S^1 componentwise as in Construction 3.1.

We will, without comment, also denote by $\mathbb{H}P^\infty$ the G–space obtained by restricting the action to a finite subgroup $G \subseteq S^1$. We will assume that G is a fixed, *nontrivial* finite subgroup of S^1 for the remainder of this section.

Remark 3.3 It follows from the definition that the natural inclusion $\mathbb{C}P^\infty \subseteq \mathbb{H}P^\infty$ is equivariant for the *trivial* action on complex projective space and identifies the fixed points

$$\mathbb{C}P^\infty = (\mathbb{H}P^\infty)^G.$$

The reader might compare this to the equivalence $\mathbb{R}P^\infty = (\mathbb{C}P^\infty)^{C_2}$, where C_2 acts by complex conjugation on projective space.

We now study the loop spaces of $\mathbb{H}P^\infty$.

Proposition 3.4 $\Omega\mathbb{H}\mathbb{P}^\infty \simeq S^{\lambda+1}$.

Proof The usual inclusion $S^{\lambda+2} \simeq \mathbb{H}P^1 \to \mathbb{H}P^\infty$ is equivariant for the action above and induces a map $S^{\lambda+1} \to \Omega\mathbb{H}P^\infty$. This is an underlying equivalence, and on fixed points for nontrivial subgroups we have the standard map (see Remark 3.3) $S^1 \to \Omega\mathbb{C}P^\infty$, which is also an equivalence. □

Remark 3.5 As a corollary of the proof, we record that the map $\nu\colon S^{2\lambda+3} \to S^{\lambda+2}$ becomes $\eta\colon S^3 \to S^2$ upon passage to fixed points.

Proposition 3.6 *If $G = C_2$, then $\Omega^\sigma\mathbb{H}P^\infty \simeq S^{\rho+1}$.*

Proof Again, the usual inclusion $S^{2\sigma+2} \to \mathbb{H}P^\infty$ is equivariant and we get a map $S^{\sigma+2} = S^{\rho+1} \to \Omega^\sigma\mathbb{H}P^\infty$ which is an underlying equivalence. To compute the fixed points of the right-hand side, we use the fiber sequence of spaces

$$(\Omega^\sigma\mathbb{H}P^\infty)^{C_2} \to \mathbb{C}P^\infty \to \mathbb{H}P^\infty$$

arising from the cofiber sequence $C_{2+} \to S^0 \to S^\sigma$ of C_2–spaces. But the fiber of $BS^1 \to BS^3$ is $S^3/S^1 \simeq S^2$, by the Hopf fibration, which completes the proof. □

Corollary 3.7 *If $G = C_2$, then*

$$\Omega^\lambda S^{\lambda+1} \simeq \Omega^\rho S^{\rho+1}.$$

Moreover, this equivalence respects the inclusion of S^1 up to homotopy.

Proof There is a chain of equivalences

$$\Omega^\lambda S^{\lambda+1} \simeq \Omega^{2\sigma}\Omega\mathbb{H}P^\infty \simeq \Omega^\rho\Omega^\sigma\mathbb{H}P^\infty \simeq \Omega^\rho S^{\rho+1}. \qquad \square$$

Remark 3.8 If V is a representation of C_2, then the free $\mathbb{E}_V$–algebra on any C_2–connected, pointed C_2–space X is given by $\Omega^V\Sigma^V X$ [40, Theorem 1]. The above discussion therefore implies that the free $\mathbb{E}_{2\sigma}$–space and free $\mathbb{E}_\rho$–space on the pointed C_2–space S^1 coincide. A prototype of this result is that the free grouplike $\mathbb{E}_\sigma$–space and free grouplike $\mathbb{E}_1$–space on the pointed C_2–space S^0 also coincide, ie $\Omega S^1 \simeq \Omega^\sigma S^\sigma \simeq \mathbb{Z}$ (with trivial action). This can be proved in much the same way, using the equivalences

$$\Omega^\rho\mathbb{C}P^\infty \simeq \mathbb{Z}, \quad \Omega\mathbb{C}P^\infty \simeq S^\sigma \quad \text{and} \quad \Omega^\sigma\mathbb{C}P^\infty \simeq S^1,$$

where C_2 acts on $\mathbb{C}P^\infty$ by complex conjugation.

As a consequence of the equivalence $\Omega\mathbb{H}P^\infty \simeq S^{\lambda+1}$, we observe that $\Omega^\lambda S^{\lambda+1}$ inherits an $\mathbb{E}_{\lambda+1}$–structure. This gives the fixed points $(\Omega^\lambda S^{\lambda+1})^G$ an $\mathbb{E}_1$–algebra structure.

Proposition 3.9 *As an $\mathbb{E}_1$–space,*

$$(\Omega^{\lambda+1}\mathbb{H}P^\infty)^G \simeq \Omega^2 S^3 \times \Omega S^2.$$

Warning 3.10 At odd primes, the fixed points μ^G of the map in Theorem D is not an $\mathbb{E}_1$–map.

Proof of Proposition 3.9 Define $S^{\lambda/2}$ by the cofiber sequence

$$G_+ \to S^0 \to S^{\lambda/2},$$

and notice that we have a cofiber sequence

$$G_+ \wedge S^1 \to S^{\lambda/2} \to S^\lambda.$$

From the second cofiber sequence, we learn that there is a fiber sequence

$$\Omega^\lambda\mathbb{H}P^\infty \to \mathrm{map}_*(S^{\lambda/2}, \mathbb{H}P^\infty) \to \mathrm{map}_*(G_+ \wedge S^1, \mathbb{H}P^\infty).$$

Taking fixed points, we get

$$(\Omega^\lambda \mathbb{H}P^\infty)^G \to \mathrm{map}_*(S^{\lambda/2}, \mathbb{H}P^\infty)^G \to \Omega \mathbb{H}P^\infty \simeq S^3.$$

The first cofiber sequence identifies the middle term as the fiber of the inclusion of fixed points:

$$\mathrm{map}_*(S^{\lambda/2}, \mathbb{H}P^\infty)^G \to \mathbb{C}P^\infty \to \mathbb{H}P^\infty.$$

In other words, $\mathrm{map}_*(S^{\lambda/2}, \mathbb{H}P^\infty)^G \simeq S^2$ and we can identify our previous fiber sequence with

$$(\Omega^\lambda \mathbb{H}P^\infty)^G \to S^2 \to S^3.$$

The second map is null, so we learn that there is an equivalence

$$(\Omega^\lambda \mathbb{H}P^\infty)^G \simeq \Omega S^3 \times S^2,$$

and hence an equivalence of loop spaces

$$(\Omega^{\lambda+1} \mathbb{H}P^\infty)^G = \Omega(\Omega^\lambda \mathbb{H}P^\infty)^G \simeq \Omega(\Omega S^3 \times S^2). \qquad \square$$

We stress that the above result does not concern multiplicative structure on the Thom spectrum in question. This is the subject of the next section at the prime 2, and of the subsequent section at odd primes.

4 Extra structure at the prime 2

Hopkins observed that the map $\mu\colon \Omega^2 S^3 \to BO$ admits a *triple* delooping as the composite

$$\mathbb{H}P^\infty = B\mathrm{Sp}(1) \to B\mathrm{Sp} \simeq B^5 O \xrightarrow{\eta} B^4 O.$$

We would like to establish an equivariant version of this result. The statement requires a few preliminaries.

The first results of this section hold for any finite subgroup $G \subseteq S^1$. We will indicate later when we must restrict attention to $G = C_{2^n}$.

Definition 4.1 A G–$\mathbb{H}$–*module* is a real G–representation V equipped with a G–equivariant algebra map $\mathbb{H} \to \mathrm{End}(V)$. Here $\mathrm{End}(V)$ is the G–representation of all endomorphisms and we use the G–action on $\mathbb{H}$ constructed in Construction 3.1. More generally, a G–$\mathbb{H}$–*bundle* on a G–space X is a G–equivariant, real vector bundle $E \to X$ together with a G–equivariant algebra map $\mathbb{H} \to \mathrm{End}(E)$.

Construction 4.2 For $F=\mathbb{R}$ or $\mathbb{H}$, let $\mathcal{U}_F$ be a *complete* G*–*F*–universe*. That is, $\mathcal{U}_F$ is a direct sum of infinitely many finite-dimensional G–F–modules which contains every finite-dimensional G–F–module as a summand (up to isomorphism). Let $\mathrm{Gr}^F(\mathcal{U}_F)$ denote the infinite grassmannian with its induced G–action. Then we define

$$BO_G = B\mathrm{GL}(\mathbb{R}) := \mathrm{Gr}^{\mathbb{R}}(\mathcal{U}_{\mathbb{R}}), \quad B\mathrm{GL}(\mathbb{H}) := \mathrm{Gr}^{\mathbb{H}}(\mathcal{U}_{\mathbb{H}}).$$

The G–space BO_G is well known, and there is an equivalence

$$\Omega^\infty \mathrm{KO}_G = \mathbb{Z} \times BO_G.$$

Warning 4.3 Since we have $BO_G \simeq B\mathrm{GL}(\mathbb{R})$ and the former is well known, we will use the notation BO_G. Beware, however, that the G–space $B\mathrm{GL}(\mathbb{H})$ is *not* the same as the space $B\mathrm{Sp}_G$ associated to equivariant, symplectic K–theory. The latter does not incorporate a nontrivial action of G on $\mathbb{H}$.

Warning 4.4 Neither BO_G nor $B\mathrm{GL}(\mathbb{H})$ is equivariantly connected when G is nontrivial. For example, $\pi_0^G BO_G$ is the group of virtual real representations of virtual dimension zero.

Theorem 4.5 (Karoubi) *We have*

$$\Omega^\infty \Sigma^{\lambda+2} \mathrm{KO}_G \simeq \mathbb{Z} \times B\mathrm{GL}(\mathbb{H}),$$

where $B\mathrm{GL}(\mathbb{H})$ is the G–space constructed above.

Proof sketch We indicate how to recover this result from the much more general work of Karoubi. First, if we endow λ with the standard negative definite quadratic form, then the Clifford algebra $\mathrm{Cl}(\lambda)$ is G–equivariantly isomorphic to $\mathbb{H}$ as an algebra. It follows from the "fundamental theorem" [25, Theorem 1.1] that, when X is compact, $\mathrm{KO}_G^{2+\lambda}(X)$ is given by Karoubi's (graded) K–theory of the graded Banach category of G–$\mathrm{Cl}(2+\lambda)$–bundles, in the sense of [24, Definition 2.1.6]. From the interpretation of this K–theory group explained on [25, page 192], we learn that a class in $\mathrm{KO}_G^{2+\lambda}(X)$ is specified by a G–$\mathrm{Cl}(2+\lambda)$–bundle E on X together with two extensions to a G–$\mathrm{Cl}(3+\lambda)$–bundle structure on E. Such a triple is declared trivial if the two extensions give isomorphic bundles, and two triples are equivalent if they become isomorphic after adding a trivial triple.

The naturality statement in [26, Theorem 3.10] produces equivariant isomorphisms

$$\mathrm{Cl}(2+\lambda) \simeq \mathrm{M}_2(\mathbb{H}), \quad \mathrm{Cl}(3+\lambda) \simeq \mathrm{M}_2(\mathbb{H}) \times \mathrm{M}_2(\mathbb{H}).$$

By Morita invariance, we may reinterpret elements in $\mathrm{KO}_G^{2+\lambda}(X)$ as equivalence classes of G–$\mathbb{H}$–bundles E equipped with two decompositions $\eta_1\colon E \simeq E_0 \oplus E_1$ and $\eta_2\colon E \simeq E_0' \oplus E_1'$. Arguing as in [26, Proposition 4.26; 42, Proposition 2.4], one can show that every such datum (E, η_1, η_2) is equivalent to one of the form $(X \times (M_0 \oplus M_1), \mathrm{id}, \eta)$, where M_0 and M_1 are G–$\mathbb{H}$–modules. After supplying a metric, we may replace η by the data of a sub-G–$\mathbb{H}$–module of $M_0 \oplus M_1$. For fixed M_0 and M_1, this data is equivalent to an equivariant map $X \to \coprod_{k \geq 0} \mathrm{Gr}_k^{\mathbb{H}}(M_0 \oplus M_1)$. Now the result follows by the definition of a complete G–$\mathbb{H}$–universe and the construction of $B\mathrm{GL}(\mathbb{H})$. $\square$

At this point we may form the composite

$$\mathbb{H}P^\infty \xrightarrow{\mathcal{O}(-1)-1} \mathbb{Z} \times B\mathrm{GL}(\mathbb{H}) \simeq \Omega^\infty \Sigma^{\lambda+2}\mathrm{KO}_G \xrightarrow{\eta} \Omega^\infty \Sigma^{\lambda+1}\mathrm{KO}_G,$$

where

- $\mathcal{O}(-1)$ is the tautological G–$\mathbb{H}$–bundle on $\mathbb{H}P^\infty$;
- $\eta \in \pi_1^G KO_G$ is the image of $\eta \in \pi_1 S^0 \subseteq \pi_1^G S^0$.

To complete the construction, we need an equivariant version of the J–homomorphism. The J–homomorphism and equivariant spherical fibrations have been studied previously (eg [43; 36; 46; 47]) and it is shown in [14; 44] that the classifying space of equivariant stable spherical fibrations is an equivariant infinite loop space. For the reader's convenience, we prove this here, as well as the corresponding notion and results regarding Picard spectra, which provides the target for the J–homomorphism. The construction below is natural from the point of view of [5], from which we draw inspiration.

Construction 4.6 (Picard G–spectrum) Bachmann and Hoyois [3, Section 9.2] refined the Hill–Hopkins–Ravenel norm construction to a product-preserving functor

$$\underline{\mathsf{Sp}}^G\colon \mathsf{A}^{\mathrm{eff}}(G) \to \mathsf{CAlg}(\mathsf{Cat}_\infty), \quad G/H \mapsto \mathsf{Sp}^H,$$

where the left-hand side denotes the (effective) Burnside $(2,1)$–category of finite G–sets and spans [4].

Given any symmetric monoidal ∞–category $\mathcal{C}$, define $\mathcal{P}\mathrm{ic}(\mathcal{C})$ to be the maximal subgroupoid of objects which are invertible under the tensor product. This is a grouplike $\mathbb{E}_\infty$–space and so deloops to a spectrum $\mathrm{pic}(\mathcal{C})$; moreover the formation of Picard spectra is product-preserving [35, Section 2.2]. We define $\underline{\mathrm{pic}}(S^0)$ as the composite functor

$$\underline{\mathrm{pic}}(S^0)\colon \mathsf{A}^{\mathrm{eff}}(G) \to \mathsf{CAlg}(\mathsf{Cat}_\infty) \to \mathsf{Sp}.$$

This is a spectral Mackey functor, and the ∞–category of spectral Mackey functors is equivalent to the ∞–category of genuine G–spectra [18; 37]. Thus we have produced a G–spectrum which we call the *Picard G–spectrum* of S^0. We denote by $\mathcal{P}\mathrm{ic}(S^0)$ the 0^{th} space of this spectrum, which is a grouplike G–$\mathbb{E}_\infty$–space (by which we mean, here and below, a G–commutative monoid in the sense of [37]). We note that this G–space, without extra structure, may be obtained directly from $\underline{\mathsf{Sp}}^G$ by assigning to the orbit G/H the maximal subgroupoid of the full subcategory of Sp^H consisting of invertible objects. If one further restricts to the full subcategory consisting of objects equivalent to S^0, this defines the G–space $B\mathrm{GL}_1(S^0)$. This subcategory is closed under the formation of norms and smash products and hence inherits the structure of a grouplike G–$\mathbb{E}_\infty$–space for which the inclusion

$$B\mathrm{GL}_1(S^0) \to \mathcal{P}\mathrm{ic}(S^0)$$

is a map of G–$\mathbb{E}_\infty$–spaces. In particular, the G–space $B\mathrm{GL}_1(S^0)$ is the zero space of a G–spectrum $\Sigma\mathrm{gl}_1(S^0)$.

More generally, given any virtual G–representation V, restricting, for each H, to the full subcategory of objects equivalent to $S^{\mathrm{res}_H(V)}$ produces a G–$\mathbb{E}_\infty$–space canonically equivalent to $B\mathrm{GL}_1(S^0)$.

Remark 4.7 The construction above works without any change for the category of p–local G–spectra to define G–infinite loop spaces $\mathcal{P}\mathrm{ic}(S^0_{(p)})$ and $B\mathrm{GL}_1(S^0_{(p)})$.

Warning 4.8 The space $\mathcal{P}\mathrm{ic}(S^0)$ does not decompose into a disjoint union of copies of $B\mathrm{GL}_1(S^0)$ when G is nontrivial. For example, take $G = C_2$ and let $Y \subseteq \mathcal{P}\mathrm{ic}(S^0)$ be the C_2–space with

- underlying space the subspace of nonequivariant spectra equivalent to S^0;
- fixed points the subspace of C_2–spectra equivalent to S^V, where V has virtual dimension zero.

Then Y splits off of $\mathcal{P}\mathrm{ic}(S^0)$, but does not decompose further. Indeed, $S^{\sigma-1}$ and S^0 lie in different components of Y^{C_2} but restrict to the same component on the underlying space. On the other hand, $B\mathrm{GL}_1(S^0)$ is G–connected, whence the claim.

Construction 4.9 (equivariant J–homomorphism) Let $\mathbf{Vect}_G$ denote the topological category of finite-dimensional G–representations. We use the same notation for the associated ∞–category. Consider the product-preserving functors

$$\underline{\mathbf{Vect}}_G, \underline{\mathsf{Spaces}}^G_*\colon \mathsf{A}^{\mathrm{eff}}(G) \to \mathsf{CAlg}(\mathsf{Cat}_\infty)$$

given by

- $\underline{\mathbf{Vect}}_G(G/H) := \mathbf{Vect}_H$ with direct sum, functoriality by restriction and coinduction;
- $\underline{\mathsf{Spaces}}^G_*(G/H) := \mathsf{Spaces}^H_* = \mathsf{Psh}(\mathcal{O}_H, \mathsf{Spaces})$ with $\wedge$, functoriality by restriction and norm defined by

$$N^G_H(X) := \mathrm{map}_H(G, X)/\{f : * \in f(G)\}.$$

The assignment $V \mapsto S^V$ produces a natural transformation

$$\underline{\mathbf{Vect}}_G \to \underline{\mathsf{Spaces}}^G_* \xrightarrow{\Sigma^\infty} \underline{\mathsf{Sp}}^G.$$

Restricting to maximal subgroupoids, and noting that each S^V is invertible, we get a natural transformation

$$\underline{\mathbf{Vect}}^{\simeq}_G \to \mathcal{P}\mathrm{ic}(S^0),$$

which we may regard as a map of G–$\mathbb{E}_\infty$–spaces. The target is grouplike, so this map factors through the group completion of the source. One can identify the underlying space of that group completion with $\mathbb{Z} \times BO_G$, so we have produced a G–$\mathbb{E}_\infty$–map

$$J\colon \mathbb{Z} \times BO_G \to \mathcal{P}\mathrm{ic}(S^0).$$

We also denote by J the restriction to $\{0\} \times BO_G = BO_G$ as well as any deloopings.

Warning 4.10 Unlike the classical case, the restriction to virtual dimension zero representations

$$BO_G \to \mathcal{P}\mathrm{ic}(S^0)$$

does *not* factor through $B\mathrm{GL}_1(S^0)$ when G is nontrivial. Again, an explicit example is given by the virtual representation $\sigma - 1$ when $G = C_2$.

Remark 4.11 Since $\Omega^{\lambda+1}\mathbb{H}P^\infty = \Omega^\lambda S^{\lambda+1}$ is equivariantly connected, the map $\mathbb{H}P^\infty \to \Omega^\infty\Sigma^{\lambda+1}\mathrm{KO}_G$ constructed above factors through $B^{\lambda+1}BO_G$.

Now we specialize to the case $G = C_{2^n}$.

Proposition 4.12 *Let g denote the composite*

$$\mathbb{H}P^\infty \to B^{\lambda+1}BO_G \to B^{\lambda+1}\mathcal{P}\mathrm{ic}(S^0)$$

Then $\Omega^{\lambda+1}g$ factors through $B\mathrm{GL}_1(S^0)$ and is homotopic to μ under the equivalence $\Omega^{\lambda+1}\mathbb{H}P^\infty \simeq \Omega^\lambda S^{\lambda+1}$.

Proof Since $\Omega^{\lambda+1}\mathbb{H}P^\infty$ is equivariantly connected, $\Omega^{\lambda+1}g$ automatically factors through $B\mathrm{GL}_1(S^0)$. To complete the proof, we need only identify the map

$$S^{\lambda+2} \to \mathbb{H}P^\infty \to B^{\lambda+1}\mathcal{P}\mathrm{ic}(S^0).$$

To begin, notice that the map

$$S^{\lambda+2} \to B^{\lambda+1}BO_G$$

corresponds to an element of KO_G^{-1}. By [2, page 17], this group is $RO(G)/R(G)$. For $G = C_{2^n}$ we have

$$RO(G)/R(G) = \begin{cases} \mathbb{F}_2\{1,\sigma\} & \text{if } n = 1, \\ \mathbb{F}_2 & \text{if } n > 1. \end{cases}$$

Even when $n = 1$, the bundle we started with was restricted from a bundle defined for $n > 1$, so the element in question is either 0 or 1. But we know that the *underlying* class is nonzero, so we must be looking at the nonzero element in $RO(G)/R(G)$. Moreover, this class corresponds precisely to the Möbius bundle on S^1, whence the claim. □

Proof of Theorem B assuming Theorem D Combine the above proposition with Theorem D. □

5 An equivariant H–space orientation

The purpose of this section is to prove the following theorem:

Theorem 5.1 *There is an $\mathbb{A}_2$–structure on $S^{\lambda+1}_{(p)}$ such that*:

(i) *The adjoint to $1 - p \in \pi_0(S^0_{(p)})^\times$ refines to an $\mathbb{A}_2$–map*

$$S^{\lambda+1}_{(p)} \to B^{\lambda+1}\mathrm{GL}_1(S^0_{(p)}).$$

(ii) *The composite*

$$S^{\lambda+1}_{(p)} \to B^{\lambda+1}\mathrm{GL}_1(S^0_{(p)}) \to B^{\lambda+1}\mathrm{GL}_1(\mathrm{H}\underline{\mathbb{F}}_p)$$

is nullhomotopic through $\mathbb{A}_2$–maps.

Before doing so, we record a corollary which follows from [29, Remark X.6.4] and the previous theorem:

Corollary 5.2 *The Thom class*

$$(\Omega^\lambda S^{\lambda+1}_{(p)})^\mu \to \mathrm{H}\underline{\mathbb{F}}_p$$

has the structure of a map of $\mathbb{A}_2$–algebras in $\mathbb{E}_\lambda$–algebras.

Remark 5.3 The $\mathbb{A}_2$–structure we define is necessarily p–local. We only use this structure to produce a multiplication on the homology of $(\Omega^\lambda S^{\lambda+1}_{(p)})^\mu$ which is preserved by the map induced by the Thom class. The localization map

$$\Omega^\lambda S^{\lambda+1} \to \Omega^\lambda S^{\lambda+1}_{(p)}$$

induces an isomorphism on mod p homology and using this one can show that the map

$$(\Omega^\lambda S^{\lambda+1})^\mu \to (\Omega^\lambda S^{\lambda+1}_{(p)})^\mu$$

is a p–local equivalence. On the other hand, the left-hand side is automatically p–local, being an (equivariant) homotopy colimit of p–local spectra by construction. We can transport the $\mathbb{A}_2$–structure from the target to the source and get a map

$$(\Omega^\lambda S^{\lambda+1})^\mu \to \mathrm{H}\underline{\mathbb{F}}_p$$

of $\mathbb{A}_2$–algebras in $\mathbb{E}_\lambda$, even though the $\mathbb{A}_2$–structure does not arise from applying the Thom construction to an $\mathbb{A}_2$–map of G–spaces.

We will need to recall a few facts about $\mathbb{A}_2$–monoid objects, mainly to establish notation.

Definition 5.4 Let $\mathcal{C}$ be an ∞–category which admits products. For $0 \le k \le \infty$, an $\mathbb{A}_k$*–monoid* X in $\mathcal{C}$ is a truncated simplicial object

$$\boldsymbol{B}_{\le k} X \colon \Delta^{\mathrm{op}}_{\le k} \to \mathcal{C}$$

such that:

- The object $(\boldsymbol{B}_{\le k} X)_0$ is final.
- If $k \ge 1$, then for $1 \le j \le k$, the maps

  $$(\boldsymbol{B}_{\le k} X)_j \to (\boldsymbol{B}_{\le k} X)_1,$$

  induced by $\{i, i+1\} \to [j]$, exhibit the source as the j–fold product of $(\boldsymbol{B}_{\le k} X)_1$.

In this case we denote $(\boldsymbol{B}_{\le k} X)_1$ by X. If $\mathcal{C}$ admits homotopy colimits, we denote the homotopy colimit of the diagram $\boldsymbol{B}_{\le k} X$ by $B_{\le k} X$.

The ∞–category of $\mathbb{A}_k$–monoids, $\mathsf{Mon}_{\mathbb{A}_k}(\mathcal{C})$, is the full subcategory of $\mathsf{Fun}(\Delta^{\mathrm{op}}_{\leq k}, \mathcal{C})$ spanned by the $\mathbb{A}_k$–monoids. Note that restriction defines forgetful functors

$$\mathsf{Mon}_{\mathbb{A}_k}(\mathcal{C}) \to \mathsf{Mon}_{\mathbb{A}_j}(\mathcal{C})$$

for $j \leq k$, and we have natural maps

$$B_{\leq j} X \to B_{\leq k} X.$$

Example 5.5 $(k=0)$ An $\mathbb{A}_0$–monoid is a final object of $\mathcal{C}$. The natural maps above provide each $B_{\leq j} X$ with a basepoint.

Example 5.6 $(k=1)$ An $\mathbb{A}_1$–monoid in $\mathcal{C}$ is specified by the data of an object X and a map $* \to X$, where $*$ is a final object in $\mathcal{C}$. The object $B_{\leq 1} X$ is computed as the colimit of

$$X \underset{\longrightarrow}{\overset{\longrightarrow}{\longleftarrow}} *$$

which is the suspension ΣX.

Example 5.7 $(k=2)$ Suppose $\mathcal{C}$ admits limits and colimits. By [30, A.2.9.16], extending a diagram $\boldsymbol{B}_{\leq 1} X$ to a diagram $\boldsymbol{B}_{\leq 2} X$ is equivalent to specifying a factorization

$$\begin{tikzcd} & C \arrow[dr, "(d_0, d_1, d_2)"] & \\ X \vee X \arrow[ur] \arrow[rr] & & X \times X \times X \end{tikzcd}$$

where $C \in \mathcal{C}$ is some object and $X \vee X$ denotes the pushout $X \amalg_* X$ where $* = (\boldsymbol{B}_{\leq 1} X)_0$ is a final object. In order for this extended diagram to define an $\mathbb{A}_2$–monoid, the maps d_0 and d_2 must give an equivalence $C \simeq X \times X$. Under this equivalence, the map $X \vee X \to X \times X$ is the standard one. The only additional data is the map $d_1\colon X \times X \simeq C \to X$. In summary, an $\mathbb{A}_2$–monoid in $\mathcal{C}$ is precisely the data of a pointed object X together with a map $m\colon X \times X \to X$ which extends the fold map $X \vee X \to X$. It follows that the cofiber of $B_{\leq 1} X \to B_{\leq 2} X$ is given by $\Sigma^2(X \wedge X)$.

Remark 5.8 If we endow $\mathcal{C}$ with the Cartesian monoidal structure, then we see that an $\mathbb{A}_2$–monoid object in $\mathcal{C}$ is the same data as an $\mathbb{A}_2$–algebra object as in Definition 2.1.

Example 5.9 (loop spaces) If $\mathcal{C} = \mathsf{Spaces}^G$ and Y is a pointed G–space, then ΩY has a natural $\mathbb{A}_\infty$–structure.

Now we will restrict attention to the ∞–category Spaces^G of G–spaces. The following is proved just as in the classical case, for which there are many references. The earliest appears to be [45, Proposition 3.5].

Lemma 5.10 *If X is an $\mathbb{A}_2$–algebra, then the map $X \to \Omega B_{\leq 2} X$ adjoint to*

$$\Sigma X = B_{\leq 1} X \to B_{\leq 2} X$$

extends to an $\mathbb{A}_2$–map.

We now return to the case of interest. We begin by establishing the existence of the $\mathbb{A}_2$–structure we need on $S^{\lambda+1}$.

Proposition 5.11 *For p an odd prime, there is an $\mathbb{A}_2$–structure on $S^{\lambda+1}_{(p)}$ with the property that the map $\Sigma X \to B_{\leq 2} X$ stably splits.*

We will deduce this proposition from the following calculation, which is an equivariant version of a classical result (see eg [23]). Here we use

$$E\colon [X, Y] \to [\Sigma X, \Sigma Y]$$

to denote the homomorphism on equivariant homotopy classes of maps given by suspending by S^1 equipped with the trivial G–action.

Proposition 5.12 *Denote by $\nu\colon S^{2\lambda+3} \to S^{\lambda+2}$ the attaching map for the inclusion $S^{\lambda+2} \simeq \mathbb{H}P^1 \to \mathbb{H}P^2$. Then, after localization at p,*

$$E(2\nu) \in E^2(\pi_{2\lambda+2} S^{\lambda+1}).$$

Proof of Proposition 5.11 assuming Proposition 5.12 Recall from that there is an equivalence $\Omega\mathbb{H}P^\infty \simeq S^{\lambda+1}$ and hence, by Example 5.9, we get an $\mathbb{A}_\infty$–structure on $S^{\lambda+1}$. This, in turn, determines an $\mathbb{A}_2$–structure on $S^{\lambda+1}$, and the attaching map for $B_{\leq 1} S^{\lambda+1} \to B_{\leq 2} S^{\lambda+1}$ is precisely

$$\nu\colon S^{2\lambda+3} \to S^{\lambda+2}.$$

In general, if one modifies an $\mathbb{A}_2$–structure on X by an element $d \in [X \wedge X, X]$, then the attaching map

$$\Sigma(X \wedge X) \to \Sigma X$$

for $B_{\leq 2} X$ is altered by $E(d)$. After inverting 2, Proposition 5.12 implies that $E(\nu) = E^2(x)$ for some $x \in \pi_{2\lambda+2} S^{\lambda+1}$. So alter the $\mathbb{A}_2$–structure above by x and the suspension of the attaching map for $B_{\leq 2} S^{\lambda+1}$ becomes null, proving the result. $\square$

Now we turn to the proof of Proposition 5.12. We will deduce this theorem from a slightly stronger result. Recall that, given classes $x \in [\Sigma A, X]$ and $y \in [\Sigma B, X]$, the Whitehead product $[x, y] \in [\Sigma(A \wedge B), X]$ is induced by the commutator of $\pi_A x$ and $\pi_B y$ in the group $[\Sigma(A \times B), X]$.

Lemma 5.13 *Let $\iota_{\lambda+2} \in \pi_{2\lambda+3} S^{\lambda+2}$ be the fundamental class. Then, after localization at p,*

$$[\iota_{\lambda+2}, \iota_{\lambda+2}] \equiv 2\nu \mod E(\pi_{\lambda+2} S^{\lambda+1}).$$

Proof of Proposition 5.12 assuming Lemma 5.13 By Lemma 5.13, $2\nu - [\iota_{\lambda+2}, \iota_{\lambda+2}]$ lies in the image of E. Hence $E(2\nu) - E([\iota_{\lambda+2}, \iota_{\lambda+2}])$ lies in the image of E^2. The result now follows from the observation that suspensions of Whitehead products vanish. Indeed, with notation as in the definition of the Whitehead product above, $E([x, y])$ is computed as a commutator in $[\Sigma^2(A \times B), X]$. But this is an abelian group by the Eckmann–Hilton argument, so commutators vanish. □

In order to prove Lemma 5.13 we will establish an exact sequence of the form

$$\pi_{2\lambda+2} S^{\lambda+1} \xrightarrow{E} \pi_{2\lambda+3} S^{\lambda+2} \xrightarrow{H} \pi_{2\lambda+3} S^{2\lambda+3}$$

and then identify the image of the Whitehead product in the last group. To that end, we note that the James splitting

$$\Sigma\Omega\Sigma X \simeq \Sigma\big(\textstyle\bigvee_{k\geq 1} X^{\wedge k}\big)$$

holds in Spaces_*^G (see [28]). This provides a natural transformation

$$H\colon \Omega\Sigma X \to \Omega\Sigma X^{\wedge 2},$$

which induces a map

$$H\colon \pi_{\star+1}\Sigma X \to \pi_{\star+1}\Sigma X^{\wedge 2}$$

for any X.

Lemma 5.14 *The sequence*

$$S^{\lambda+1} \xrightarrow{E} \Omega S^{\lambda+2} \xrightarrow{H} \Omega S^{2\lambda+3}$$

is a fiber sequence when localized at p.

Proof Let F denote the homotopy fiber of H, so that we have a natural map $S^{\lambda+1} \to F$. We would like to show this is an equivalence. Since restriction to underlying spaces and fixed points preserves homotopy limits and colimits, we are reduced to the nonequivariant statement that

$$S^{2n+1} \xrightarrow{E} \Omega S^{2n+2} \xrightarrow{H} \Omega S^{4n+3}$$

is a fiber sequence when localized at p for $n = 0, 1$. In fact, it is a classical result of James [23] that this is a p–local fiber sequence for any $n \geq 0$. □

We will need some control over the last term in this sequence, which is provided by an equivariant version of the Brouwer–Hopf degree theorem. For us, the only fact we need is that the homomorphism

$$\pi_{2\lambda+3} S^{2\lambda+3} \to \bigoplus_{K \subseteq G} \mathbb{Z},$$

recording each of the degrees of a map on K–fixed points, is an injection. See eg [16, Lemma 8.4.1]. We now prove the only remaining lemma necessary for producing the exotic H–space structure on $S^{\lambda+1}$.

Proof of Lemma 5.13 The formation of Whitehead products commutes with passage to fixed points and restriction to underlying classes, as does the map H. From the remarks above, it then suffices to check the nonequivariant formulas

$$H([\iota_4, \iota_4]) = 2H(\nu) \qquad \text{and} \qquad H([\iota_2, \iota_2]) = 2H(\nu^K), \quad K \neq \{e\}.$$

But ν and $\nu^K = \eta$ (see Remark 3.5) have Hopf invariant 1, while $[\iota_{2n}, \iota_{2n}]$ has Hopf invariant 2 for any $n \geq 1$, whence the result. □

Since the attaching map in $B_{\leq 2} S^{\lambda+1}_{(p)}$ is stably null, the following lemma is immediate:

Lemma 5.15 *There exists a dotted map making the diagram below commute up to homotopy in* Sp^G:

$$\begin{tikzcd} S^{\lambda+2}_{(p)} \arrow[r, "1-p"] \arrow[d] & \Sigma^{\lambda+2} \mathrm{gl}_1 S^0_{(p)} \\ \Sigma^\infty B_{\leq 2} S^{\lambda+1}_{(p)} \arrow[ur, dashed] & \end{tikzcd}$$

We will eventually need to produce a Thom isomorphism in mod p cohomology which respects our extra structure. For that we require the next lemma.

Lemma 5.16 *Choose a dotted map $\widetilde{f}$ as in the previous lemma. Then the composite*

$$\Sigma^\infty B_{\le 2} S^{\lambda+1}_{(p)} \xrightarrow{\widetilde{f}} \Sigma^{\lambda+2}\mathrm{gl}_1(S^0_{(p)}) \to \Sigma^{\lambda+2}\mathrm{gl}_1(\mathrm{H}\underline{\mathbb{F}}_p)$$

is null.

Proof The composite

$$S^{\lambda+2}_{(p)} \to \Sigma^\infty B_{\le 2} S^{\lambda+1}_{(p)} \xrightarrow{\widetilde{f}} \Sigma^{\lambda+2}\mathrm{gl}_1(S^0_{(p)}) \to \Sigma^{\lambda+2}\mathrm{gl}_1(\mathrm{H}\underline{\mathbb{F}}_p)$$

vanishes since $1-p=1 \in \pi_0^G \mathrm{gl}_1(\mathrm{H}\underline{\mathbb{F}}_p)$ is the basepoint component. So the map $\widetilde{f}$ factors through some map

$$S^{2\lambda+4} \to \Sigma^{\lambda+2}\mathrm{gl}_1(\mathrm{H}\underline{\mathbb{F}}_p).$$

But

$$\pi^G_{\lambda+2}\mathrm{gl}_1(\mathrm{H}\underline{\mathbb{F}}_p) \simeq \pi^G_{\lambda+2}\mathrm{H}\underline{\mathbb{F}}_p = 0$$

since $S^{\lambda+2}$ is 2–connective, whence the claim. □

Finally, we arrive at the proof of the main theorem of the section.

Proof of Theorem 5.1 Choose a dotted map as in Lemma 5.15 and let f be its adjoint,

$$f\colon B_{\le 2} S^{\lambda+1}_{(p)} \to B^{\lambda+2}\mathrm{GL}_1(S^0_{(p)}).$$

Then the map $1-p\colon S^{\lambda+1}_{(p)} \to B^{\lambda+1}\mathrm{GL}_1(S^0_{(p)})$ factors as a composite

$$S^{\lambda+1}_{(p)} \to \Omega B_{\le 2} S^{\lambda+1}_{(p)} \xrightarrow{\Omega f} \Omega B^{\lambda+2}\mathrm{GL}_1(S^0_{(p)}) \to B^{\lambda+1}\mathrm{GL}_1(S^0_{(p)}),$$

each of which is an H–map. This proves part (i) of the theorem.

To prove part (ii), consider the diagram

$$\begin{tikzcd}
S^{\lambda+1}_{(p)} \arrow[r] & \Omega B_{\le 2} S^{\lambda+1}_{(p)} \arrow[r, "\Omega f"] & \Omega B^{\lambda+2}\mathrm{GL}_1(S^0_{(p)}) \arrow[r] \arrow[d, "\Omega B^{\lambda+2}\mathrm{GL}_1(\iota)"'] & B^{\lambda+1}\mathrm{GL}_1(S^0_{(p)}) \arrow[d] \\
 & & \Omega B^{\lambda+2}\mathrm{GL}_1(\mathrm{H}\underline{\mathbb{F}}_p) \arrow[r] & B^{\lambda+1}\mathrm{GL}_1(\mathrm{H}\underline{\mathbb{F}}_p)
\end{tikzcd}$$

where $\iota\colon S^0_{(p)} \to \mathrm{H}\underline{\mathbb{F}}_p$ is the unit map.

The composite

$$B_{\le 2} S^{\lambda+1}_{(p)} \xrightarrow{f} B^{\lambda+2}\mathrm{GL}_1(S^0_{(p)}) \xrightarrow{B^{\lambda+2}\mathrm{GL}_1(\iota)} B^{\lambda+2}\mathrm{GL}_1(\mathrm{H}\underline{\mathbb{F}}_p)$$

is null by Lemma 5.16. The loop of this composite is then null through $\mathbb{A}_\infty$–maps and the result follows. □

6 Computing the zeroth homotopy Mackey functor

In this section, we establish that the zeroth homotopy Mackey functor of our Thom spectrum is as expected. That is to say, we give a proof that

$$\underline{\pi}_0(\Omega^\lambda S^{\lambda+1})^\mu = \underline{\mathbb{F}}_p.$$

By construction, $(\Omega^\lambda S^{\lambda+1})^\mu$ receives a map from the mod p Moore spectrum $M(p) = (S^1)^\mu$. This is enough to guarantee that $p = 0$ in $\underline{\pi}_0(\Omega^\lambda S^{\lambda+1})^\mu$. However, $\underline{\pi}_0 S^0$ is the Burnside Mackey functor $\underline{A}$, and $\underline{A}/(p)$ is not $\underline{\mathbb{F}}_p$. For example, when $G = C_p$, we have

$$\underline{A}/(p) = \begin{tikzcd} \mathbb{F}_p\{[C_p]\} \arrow[d, bend right] \\ \mathbb{F}_p \arrow[u, bend right] \end{tikzcd}$$

We will need to use some extra structure on $(\Omega^\lambda S^{\lambda+1})^\mu$ to show that $[C_p]$ also vanishes. More generally, we must show that $[C_{p^n}/C_{p^k}]$ vanishes in the Hurewicz image for all k.

For the remainder of this section we write $G = C_{p^n}$ for a cyclic group of prime power order.

Definition 6.1 We say that a G–spectrum X is *weakly normed* if it is equipped with a map $S^0 \to X$, and, for each $H \subseteq G$, a map of H–spectra $\mathrm{N}^H X \to X$ such that the diagram

$$\begin{tikzcd} \mathrm{N}^H(S^0) \arrow[r, equal] \arrow[d] & S^0 \arrow[d] \\ \mathrm{N}^H(X) \arrow[r] & X \end{tikzcd}$$

commutes in the homotopy category.

Remark 6.2 This is the weakest structure necessary to run the arguments below, but it is perhaps not the most natural definition. In most examples one at least has compatibility between the norms as H varies, and the map $S^0 \to X$ acts as a unit for an underlying multiplication.

The Thom spectrum $(\Omega^\lambda S^{\lambda+1})^\mu$ is weakly normed, as we now show. This result is well known; compare [20, Theorem 2.12], for example.

Lemma 6.3 *If X is an $\mathbb{E}_\lambda$–algebra then it is canonically weakly normed.*

Proof In order to conform with the existing literature we will present a proof within the point-set model of orthogonal G–spectra as in [21, Appendix B]. In particular, we will model X by a positively cofibrant orthogonal G–spectrum.

Since the restriction of an $\mathbb{E}_\lambda$–algebra is still an $\mathbb{E}_\lambda$–algebra, it will suffice to construct the norm $\mathrm{N}^G X \to X$.

By definition, X comes equipped with a map

$$\widetilde{\mathrm{Conf}}_{p^n}(\lambda)_+ \wedge_{\Sigma_{p^n}} X^{\wedge p^n} \to X,$$

where $\widetilde{\mathrm{Conf}}_{p^n}(\lambda)$ denotes the G–space of configurations of p^n ordered points in λ. Consider the inclusion $G \hookrightarrow \Sigma_{p^n}$ which sends a generator to the standard p^n–cycle $(1, 2, \dots, p^n)$ and let Γ denote the graph of this inclusion. Let $\zeta = e^{2\pi i/p^n}$. Then the ordered tuple $(1, \zeta, \zeta^2, \dots, \zeta^{p^n-1}) \in \widetilde{\mathrm{Conf}}_{p^n}(\lambda)$ produces a $G\times\Sigma_{p^n}$–equivariant inclusion

$$\frac{G \times \Sigma_{p^n}}{\Gamma} \to \widetilde{\mathrm{Conf}}_{p^n}(\lambda).$$

This, in turn, gives us a map

$$\left(\frac{G \times \Sigma_{p^n}}{\Gamma}\right)_+ \wedge_{\Sigma_{p^n}} X^{\wedge p^n} \to X.$$

To complete the proof, we note that [9, Proposition 6.2], for any G–spectrum Y, we have

$$\left(\frac{G \times \Sigma_{p^n}}{\Gamma}\right)_+ \wedge_{\Sigma_{p^n}} Y^{\wedge p^n} \simeq \mathrm{N}^G Y. \qquad \square$$

Proposition 6.4 *Suppose X is weakly normed. Suppose further that $p = 0 \in \pi_0^G X$. Then $[H/K] = 0 \in \pi_0^H X$ for all $K \subseteq H \subseteq G$.*

Corollary 6.5 $$\underline{\pi}_0(\Omega^\lambda S^{\lambda+1})^\mu = \mathbb{F}_p.$$

Proof of Proposition 6.4 Recall that $G = C_{p^n}$. If the result is proved for $C_{p^{n-1}} \subseteq G$, then the classes

$$p, \mathrm{tr}^G_{C_{p^{n-1}}}([C_{p^{n-1}}]) = [G], \mathrm{tr}^G_{C_{p^{n-1}}}([C_{p^{n-1}}/C_p]) = [G/C_p], \dots, [G/C_{p^{n-2}}]$$

all vanish in $\pi_0^G X$. The result now follows from the next lemma. $\square$

Lemma 6.6 *If X is weakly normed, then*

$$\mathrm{N}^G(p) \equiv -[G/C_{p^{n-1}}] \mod (p, [G/K] : K \subsetneq C_{p^{n-1}}).$$

Proof It suffices to prove this formula when $X = S^0$, ie for the Burnside Mackey functor $\underline{A}$. The norm of p is the class of the G–set $\operatorname{map}(G, \{1, \dots, p\})$, by [21, Lemma A.36]. By recording the size of the image of a map, we get an equality in $A(G)$,

$$[\operatorname{map}(G, \{1, \dots, p\})] = \sum_{0<k\le n} \binom{p}{k} [\operatorname{surj}(G, \{1, \dots, k\})],$$

where $\operatorname{surj}(G, \{1, \dots, k\})$ denotes the G–set of surjective maps $G \to \{1, \dots, k\}$. So we have

$$\mathrm{N}^G(p) \equiv [\operatorname{surj}(G, \{1, \dots, p\})] \mod p.$$

We are only concerned with the orbits in $\operatorname{surj}(G, \{1, \dots, p\})$ with isotropy $C_{p^{n-1}}$ or G. There are $(p-1)!$ orbits with isotropy $C_{p^{n-1}}$, namely the orbit of the quotient map $G \to G/C_{p^{n-1}} \simeq \{1, \dots, p\}$ and the orbits of the maps obtained from this one by reordering $\{2, \dots, p\}$. There are p orbits with isotropy G, namely the p constant maps. This completes the proof. □

7 Toward the first homotopy groups of the fixed points

The tom Dieck splitting [15; 43],

$$(S^0)^{C_{p^n}} = \bigoplus_{0\le k\le n} S^0_{h\operatorname{Aut}_{C_{p^n}}(C_{p^n}/C_{p^k})}$$

induces, upon taking π_1, a map

$$\alpha\colon \pi_1^{C_{p^n}}(S^0) \to \operatorname{Aut}(C_{p^n}/C_{p^{n-1}}) \cong \mathbb{Z}/p.$$

This section is devoted to a proof of the following proposition:

Proposition 7.1 *Let X denote the Thom spectrum $(\Omega^\lambda S^{\lambda+1})^\mu$. Then there is an element $x \in \pi_1^{C_{p^n}} S^0$ such that:*

(i) $\alpha(x) \neq 0$.

(ii) *x is sent to zero under the unit map $\pi_1^{C_{p^n}}(S^0) \to \pi_1(X^{C_{p^n}})$.*

The key observation necessary to prove Proposition 7.1 is the following:

Lemma 7.2 *Let*

$$\widetilde{1+p}\colon \Omega^{\lambda+1}S^{\lambda+1} \to \mathrm{GL}_1(S^0_{(p)})$$

denote the unique $\Omega^{\lambda+1}$ *map extending* $1+p \in \pi_0^{C_{p^n}}(\mathrm{GL}_1(S^0_{(p)}))$*. Then there is an element* $y \in \pi_1^{C_{p^n}}(\Omega^{\lambda+1}S^{\lambda+1})$ *whose image,* x*, under the map* $\widetilde{1+p}$ *has* $\alpha(x)$ *nonzero.*

Proof of Proposition 7.1 assuming Lemma 7.2 First observe that the Thom spectrum associated to the map

$$B(\widetilde{1+p})\colon \Omega^{\lambda}S^{\lambda+1} \to B\mathrm{GL}_1(S^0)$$

is equivalent to X. By Lemma 7.2, we have a map $S^2 \to \Omega^{\lambda}S^{\lambda+1}$ such that the induced map on Thom spectra becomes

$$S^0/x \to X.$$

In particular, the element $x \in \pi_1^{C_{p^n}}S^0$ maps to zero in $\pi_1^{C_{p^n}}X$, which proves (ii). □

Now we turn to the proof of Lemma 7.2. Observe that we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Free}_{\mathbb{E}_{\lambda+1}}(*) \arrow[r] \arrow[d] & \mathrm{Free}_{\mathbb{E}_{\infty\rho}}(*) \arrow[d] & \\
\Omega^{\lambda+1}S^{\lambda+1} \arrow[r] & \Omega^{\infty}S^0 \arrow[r, "\widetilde{1+p}"] & \mathrm{GL}_1(S^0_{(p)})
\end{tikzcd}$$

Here, $\mathbb{E}_{\infty\rho}$ denotes the union of the operads of little disks in the representations $m\rho$, where ρ is the regular representation (see [19; 20]). We recall that the C_{p^n}–space $\mathbb{E}_{\infty\rho}(k)/\Sigma_k$ is a model for $B_{C_{p^n}}\Sigma_k$, the classifying C_{p^n}–space for principal Σ_k–bundles.

Construction 7.3 Consider the diagram

$$\begin{tikzcd}
S(\lambda) \arrow[r] \arrow[d] & S^1 \times \mathbb{R}^{\lambda+1} \arrow[dl, "\mathrm{proj}_{S^1}"] \\
S^1 &
\end{tikzcd}$$

where $S(\lambda) \to S^1$ is the quotient map and $S(\lambda) \subseteq \mathbb{R}^{\lambda} \subseteq \mathbb{R}^{\lambda+1}$ is the inclusion. This gives an equivariant S^1–family of p^n points in $\mathbb{R}^{\lambda+1}$, and hence is classified by a map

$$S^1 \to \mathbb{E}_{\lambda+1}(p^n)/\Sigma_{p^n}.$$

This defines an element $y \in \pi_1 \mathrm{Free}_{\mathbb{E}_{\lambda+1}}(*)$.

The image of y in $\mathrm{Free}_{\mathbb{E}_{\infty\rho}}(*)$ corresponds to the map $S^1 \to B_{C_{p^n}} \Sigma_{p^n}$ classifying the cover $S(\lambda) \to S^1$. We will denote this class also by y.

Combining the above construction with the diagram

$$\begin{tikzcd} & & \Omega^\infty S^0 \arrow[d] \\ \mathrm{Free}_{\mathbb{E}_{\infty\rho}}(*) \arrow[r, "\widetilde{1+p}"'] \arrow[urr, "\widetilde{1+p}"] & \mathrm{GL}_1(S^0_{(p)}) \arrow[r] & \Omega^\infty S^0_{(p)} \end{tikzcd}$$

we see then that Lemma 7.2 follows from:

Lemma 7.4 *The image, x, of y under the map*

$$\mathrm{Free}_{\mathbb{E}_{\infty\rho}}(*) \xrightarrow{\widetilde{1+p}} \Omega^\infty S^0$$

has $\alpha(x)$ nonzero. (Here we are using the *multiplicative* monoid structure on $\Omega^\infty S^0$ to define $\widetilde{1+p}$).

In order to prove Lemma 7.4, we will model the map $\widetilde{1+p}$ using finite C_{p^n}–sets. Let $\mathrm{Fin}^{C_{p^n}}$ denote the groupoid of finite C_{p^n}–sets, so that we have an equivalence

$$\mathrm{Fin}^{C_{p^n}} \simeq (\mathrm{Free}_{\mathbb{E}_{\infty\rho}}(*))^{C_{p^n}},$$

and the group completion of this space (under disjoint union) gives $(\Omega^\infty S^0)^{C_{p^n}}$ [43]. Unwinding the definitions, the map $\widetilde{1+p}$, on fixed points, arises from the functor

$$\mathrm{Fin}^{C_{p^n}} \to \mathrm{Fin}^{C_{p^n}},\ J \mapsto \mathrm{map}(J, \{1, \dots, p+1\}).$$

Remark 7.5 The invariant α can be described in terms of finite C_{p^n}–sets as follows. Given a finite C_{p^n}–set K and an automorphism θ, let K' denote the summand with isotropy equal to $C_{p^{n-1}}$. Then $\theta|_{K'}$ can be written as a permutation of K'/C_{p^n} followed by an automorphism of each individual summand of K'. The composition of these automorphisms is then an element in $\mathrm{Aut}(C_{p^n}/C_{p^{n-1}}) \simeq \mathbb{Z}/p$. This describes $\alpha((K,\theta))$, where we view (K,θ) as specifying an element in $\pi_1(\Omega^\infty S^0)^{C_{p^n}}$.

Notice that the class y arose from the cover $S(\lambda) \to S^1$, which has fibers isomorphic to C_{p^n} and monodromy given by the generator $\gamma \in C_{p^n}$. It follows from the previous remark that Lemma 7.4 is equivalent to the following:

Lemma 7.6 $\alpha\big(\mathrm{map}(C_{p^n}, \{1, \dots, p+1\}), \gamma^*\big)$ *is nonzero.*

Proof We have a decomposition of pairs

$$\big(\mathrm{map}(C_{p^n},\{1,\ldots,p+1\}),\gamma^*\big)\simeq\coprod_{1\le k\le p+1}\binom{p+1}{k}\big(\mathrm{Surj}(C_{p^n},\{1,\ldots,k\}),\gamma^*\big).$$

The invariant α vanishes mod p, so we need only consider the summands with $k=1$, p and $p+1$. When $k=1$, the isotropy groups are all of C_{p^n}, so α vanishes for these summands. Notice that a surjection $C_{p^n}\to\{1,\ldots,k\}$ has isotropy containing $C_{p^{n-1}}$ if and only if it factors through $C_{p^n}/C_{p^{n-1}}$; this rules out the case $k=p+1$. For $k=p$, we are then left with the $(p-1)!$ different orbits obtained from the orbit of the canonical map

$$C_{p^n}\to C_{p^n}/C_{p^{n-1}}\simeq\{1,\ldots,p\}$$

by reordering the elements $\{2,\ldots,p\}$. The value of α on this orbit is -1 (since γ^* is the automorphism given by precomposition with γ on $\mathrm{map}(C_{p^n},\{1,\ldots,p+1\})$, while the action is given by precomposing with γ^{-1}). It follows that

$$\alpha\big(\big(\mathrm{map}(C_{p^n},\{1,\ldots,p+1\}),\gamma^*\big)\big)=-(p-1)!\cdot\binom{p+1}{p}=1\in\mathbb{Z}/p.$$

This completes the proof of Lemma 7.4 and hence of Proposition 7.1. □

8 Proof of the main theorem

We are now ready to prove the main theorem, which we recall here for convenience.

Theorem 8.1 *Let $G=C_{p^n}$ and let $S^1\to B\mathrm{GL}_1(S^0_{(p)})$ be adjoint to $1-p\in\pi_0^G S^0_{(p)}$. Denote by μ: $\Omega^\lambda S^{\lambda+1}\to B\mathrm{GL}_1(S^0_{(p)})$ the extension of this map over the λ–loop space. Then the Thom class*

$$(\Omega^\lambda S^{\lambda+1})^\mu\to\mathrm{H}\underline{\mathbb{F}}_p$$

is an equivalence of G–spectra.

Before the proof, we record a well-known computation (the $p=2$ case is proven in [21, Proposition 3.18] and the odd primary proof is much the same).

Lemma 8.2 *For $G=C_{p^n}$ and p odd, we have*

$$\pi_*\mathrm{H}\underline{\mathbb{Z}}^{\Phi G}=\mathbb{F}_p[t],\quad |t|=2,\qquad \pi_*\mathrm{H}\underline{\mathbb{F}}_p^{\Phi G}=\mathbb{F}_p[t]\otimes\Lambda(s),\quad |s|=1.$$

When $p=2$, the second computation becomes

$$\pi_* \mathrm{H}\underline{\mathbb{F}}_2^{\Phi G} = \mathbb{F}_2[s], \quad |s| = 1.$$

We also will need the corresponding result about our Thom spectrum.

Lemma 8.3 *Let X denote the Thom spectrum $(\Omega^\lambda S^\lambda)^\mu$. Then the homotopy group $\pi_k(X^{\Phi G})$ of the geometric fixed points is isomorphic to $\mathbb{F}_p$ for each $k \geq 0$.*

Proof As explained in Remark 5.3, we have equipped X with the structure of an $\mathbb{A}_2$–algebra in $\mathbb{E}_\lambda$–algebras and hence the norm map

$$\mathrm{N}^G(X) \to X$$

is a map of $\mathbb{A}_2$–algebras. In particular, $X^{\Phi G}$ is a module over $(\mathrm{N}^G(X))^{\Phi G} \simeq (\mathrm{N}^G(\mathrm{H}\mathbb{F}_p))^{\Phi G} \simeq \mathrm{H}\mathbb{F}_p$. Since $\mathrm{H}\mathbb{F}_p$ is a field spectrum, $X^{\Phi G}$ splits as a wedge of suspensions of $\mathrm{H}\mathbb{F}_p$. The homotopy groups of $X^{\Phi G}$ are then determined by the homology groups of $X^{\Phi G}$. By the Thom isomorphism, we have $\mathrm{H}_*(X^{\Phi G}) \simeq \mathrm{H}_*(\Omega^2 S^3 \times \Omega S^2)$, and the result follows. □

Finally, let $\mathcal{P}$ denote the family of proper subgroups of G. For any G–spectrum E, we let $E_{h\mathcal{P}}$ denote the spectrum $(E \wedge \mathrm{E}\mathcal{P}_+)^G$ (see for example [21, Section 2.5.2]), and we note the natural isotropy separation sequence

$$E_{h\mathcal{P}} \to E^G \to E^{\Phi G}.$$

We will need the following final lemma before proving the main theorem:

Lemma 8.4 *Let p be an odd prime, so that $\pi_1 S^0_{h\mathcal{P}} \to \pi_1^{C_{p^n}} S^0$ is a p–local equivalence. Let $\alpha\colon \pi_1^G S^0_{(p)} \to \mathbb{Z}/p$ denote the function defined at the beginning of Section 7. Then an element $z \in \pi_1(S^0_{h\mathcal{P}})$ maps to a generator in $\pi_1(\underline{\mathbb{F}}_p)_{h\mathcal{P}}$ if and only if $\alpha(z)$ is nonzero.*

Proof If Y is any C_{p^n}–spectrum, then we have

$$Y_{h\mathcal{P}} \simeq Y^{C_{p^{n-1}}}_{h\mathrm{Aut}(C_{p^n}/C_{p^{n-1}})},$$

since every proper subgroup is contained in $C_{p^{n-1}}$. The map

$$\bigoplus_{0 \leq k \leq n-1} S^0_{h\mathrm{Aut}(C_{p^{n-1}}/C_{p^k})} \simeq (S^0)^{C_{p^{n-1}}} \to \underline{\mathbb{F}}_p^{C_{p^{n-1}}} = \mathbb{F}_p$$

sends every summand to zero except the summand corresponding to the trivial $C_{p^{n-1}}$–orbit, $[C_{p^{n-1}}/C_{p^{n-1}}]$, which maps to the generator. It follows that the induced map on homotopy orbits factors through

$$S^0_{h\mathrm{Aut}(C_{p^n}/C_{p^{n-1}})} \to (\mathbb{F}_p)_{h\mathrm{Aut}(C_{p^n}/C_{p^{n-1}})}.$$

This is an isomorphism on $\pi_1^{C_{p^n}}$, which completes the proof. $\square$

Proof of the main theorem We prove the theorem by induction on n. When $n=0$, this is the nonequivariant result of Hopkins and Mahowald. For the induction hypothesis we assume that the map

$$\alpha\colon X := (\Omega^\lambda S^{\lambda+1})^\mu \to \mathrm{H}\underline{\mathbb{F}}_p$$

is an equivalence after restriction to $C_{p^{n-1}}$, and we assume that $n \geq 1$ from now on. We must prove that the induced map

$$X^{\Phi C_{p^n}} \to \underline{\mathbb{F}}_p^{\Phi C_{p^n}}$$

is an equivalence.

The homotopy groups of the target are

$$\pi_* \underline{\mathbb{F}}_p^{\Phi C_{p^n}} = \begin{cases} \Lambda(s) \otimes \mathbb{F}_p[t] & \text{if } p \text{ is odd,} \\ \mathbb{F}_2[s] & \text{if } p = 2, \end{cases}$$

where $|s|=1$ and $|t|=2$. Additively, these agree with the homotopy groups of $X^{\Phi C_{p^n}}$. So we need only show that

$$\pi_1 X^{\Phi C_{p^n}} \to \pi_1 \underline{\mathbb{F}}_p^{\Phi C_{p^n}}$$

is surjective and, at odd primes, that

$$\pi_2 X^{\Phi C_{p^n}} \to \pi_2 \underline{\mathbb{F}}_p^{\Phi C_{p^n}}$$

is surjective. From the isotropy separation sequence and the induction hypothesis, we have a diagram of exact sequences, for $i \geq 1$,

$$\begin{tikzcd}
\pi_i X^{\Phi C_{p^n}} \arrow[r] \arrow[d] & \pi_{i-1} X_{h\mathcal{P}} \arrow[r] \arrow[d, "\cong"] & \pi_{i-1} X^{C_{p^n}} \arrow[r] \arrow[d] & \pi_{i-1} X^{\Phi C_{p^n}} \arrow[d] \\
\pi_i \underline{\mathbb{F}}_p^{\Phi C_{p^n}} \arrow[r, "\cong"'] & \pi_{i-1} (\underline{\mathbb{F}}_p)_{h\mathcal{P}} \arrow[r, "0"] & \pi_{i-1} \underline{\mathbb{F}}_p \arrow[r] & \pi_{i-1} \underline{\mathbb{F}}_p^{\Phi C_{p^n}}
\end{tikzcd}$$

Applying this to the case $i = 1$, and using that $\pi_0 X^{C_{p^n}} \to \mathbb{F}_p$ is an isomorphism (Corollary 6.5), we deduce that the map $\pi_0 X_{h\mathcal{P}} \to \pi_0 X^{C_{p^n}}$ is zero. Hence the map

$$\mathbb{Z}/p = \pi_1 X^{\Phi C_{p^n}} \to \pi_0 X_{h\mathcal{P}} = \mathbb{Z}/p$$

is surjective, and so it is an isomorphism. It follows that $\pi_1 X^{\Phi C_{p^n}} \to \pi_1 \underline{\mathbb{F}}_p^{\Phi C_{p^n}}$ is an isomorphism, which completes the proof when $p = 2$. When p is odd, we continue as follows. First, the fact that $\pi_1 X^{\Phi C_{p^n}} \to \pi_0 X_{h\mathcal{P}}$ is an isomorphism means that

$$\mathbb{Z}/p = \pi_1(\underline{\mathbb{F}}_p)_{h\mathcal{P}} \cong \pi_1 X_{h\mathcal{P}} \to \pi_1 X^{C_{p^n}}$$

is surjective.

By Lemma 8.4 and Proposition 7.1, the generator of the source maps to zero in the target, so we deduce that $\pi_1 X^{C_{p^n}} = 0$, and hence that in the diagram

$$\begin{tikzcd} \pi_2 X^{\Phi C_{p^n}} \arrow[r] \arrow[d] & \pi_1 X_{h\mathcal{P}} \arrow[d, "\cong"] \\ \pi_2 \underline{\mathbb{F}}_p^{\Phi C_{p^n}} \arrow[r, "\cong"'] & \pi_1(\underline{\mathbb{F}}_p)_{h\mathcal{P}} \end{tikzcd}$$

the top horizontal arrow is surjective (hence an isomorphism). Thus, the left vertical arrow is surjective (hence an isomorphism). □

9 Concluding remarks

The integral Eilenberg–Mac Lane spectrum Define $S^{\lambda+1}\langle\lambda+1\rangle$ as the fiber of the unit map

$$S^{\lambda+1} \to K(\underline{\mathbb{Z}}, \lambda+1) := \Omega^\infty(\Sigma^{\lambda+1}\mathrm{H}\underline{\mathbb{Z}}).$$

Then we have the following result:

Theorem 9.1 *There is an equivalence of $\mathbb{E}_\lambda$–algebras*

$$\big(\Omega^\lambda(S^{\lambda+1}\langle\lambda+1\rangle)\big)^\mu \simeq \mathrm{H}\underline{\mathbb{Z}}_{(p)}.$$

Proof We argue as in Antolín-Camarena and Barthel [1, Section 5.2], though we need not develop all the technology present there. We have a fiber sequence

$$\Omega^\lambda S^{\lambda+1}\langle\lambda+1\rangle \to \Omega^\lambda S^{\lambda+1} \to S^1.$$

Decomposing S^1 into a 0–cell and a 1–cell, and trivializing the fibration on each cell, produces a decomposition of the Thom spectrum $(\Omega^\lambda S^{\lambda+1})^\mu$ as a cofiber

$$(\Omega^\lambda S^{\lambda+1}\langle\lambda+1\rangle)^\mu \xrightarrow{x} (\Omega^\lambda S^{\lambda+1}\langle\lambda+1\rangle)^\mu \to (\Omega^\lambda S^{\lambda+1})^\mu \simeq \mathrm{H}\underline{\mathbb{F}}_p.$$

Each of these Thom spectra came from bundles classified by $\mathbb{A}_2$–maps, which is enough to ensure that the map x induces a map $\pi_*(\Omega^\lambda S^{\lambda+1}\langle\lambda+1\rangle)^\mu \to \pi_*(\Omega^\lambda S^{\lambda+1}\langle\lambda+1\rangle)^\mu$ of modules over $\pi_0(\Omega^\lambda S^{\lambda+1}\langle\lambda+1\rangle)^\mu$. In particular, on homotopy the map corresponds to multiplication by some element $x \in \pi_0(\Omega^\lambda S^{\lambda+1}\langle\lambda+1\rangle)^\mu$. Arguments similar to those in the proof of the main theorem show that

$$\underline{\pi}_0(\Omega^\lambda S^{\lambda+1}\langle\lambda+1\rangle)^\mu \simeq \underline{\mathbb{Z}}_{(p)},$$

so we must have $x = p$. The result follows from Nakayama's lemma once one argues that the genuine fixed point spectra have finitely generated homotopy groups in each degree. (For example, isotropy separation reduces us to the corresponding statement on geometric fixed points, where it follows from the Thom isomorphism.) □

Remark 9.2 The map $S^{\lambda+1} \to K(\underline{\mathbb{Z}}, \lambda+1)$ deloops to a map $\mathbb{H}P^\infty \to K(\underline{\mathbb{Z}}, \lambda+2)$. It follows from Theorem B that the equivalence above is one of $\mathbb{E}_{\lambda+1}$–algebras when $p = 2$.

Remark 9.3 Unlike the classical case, it is unclear whether the statement globalizes to a construction of $\mathrm{H}\underline{\mathbb{Z}}$. Our methods do not construct $\mathrm{H}\underline{\mathbb{F}}_\ell$ as a Thom spectrum when ℓ does not divide the order of G.

Questions We conclude with a few open-ended questions.

Question 9.4 Is $\mathrm{H}\underline{\mathbb{F}}_p$ a Thom spectrum for any group G that is not cyclic of p–power order? It seems plausible that this is so for dihedral groups, as was suggested to the authors by Stefan Schwede. Can obstructions be found for other G?

Question 9.5 Calculations indicate that the spectrum $(\Omega^\lambda S^{\lambda+1})^\mu$ is not $\mathrm{H}\underline{\mathbb{F}}_p$ for the groups C_n when n is not a power of p. What can be said about $(\Omega^\lambda S^{\lambda+1})^\mu$ as an S^1 or $O(2)$–equivariant spectrum? At least one expects an interesting C_{p^∞}–spectrum.

Question 9.6 One of Mahowald's motivations for proving the equivalence $(\Omega^2 S^3)^\mu \simeq \mathrm{H}\mathbb{F}_2$ is that the left-hand side carries a natural filtration due to Milgram and May. This produces a filtration of $\mathrm{H}\mathbb{F}_2$ by spectra which turn out to be the Brown–Gitler spectra of [10] (see [11; 13; 22]). The G–space $\Omega^\lambda S^{\lambda+1}$ also carries the arity filtration from the $\mathbb{E}_\lambda$–operad, so we could *define* equivariant Brown–Gitler spectra using this filtration. It would be interesting to know if these spectra are of any use.

In the case $G = C_2$ there are two different operadic filtrations of $\Omega^{2\sigma} S^{2\sigma+1} \simeq \Omega^\rho S^{\rho+1}$. This leads to two different notions of Brown–Gitler spectra. How are they related?

Question 9.7 What are the Thom spectra obtained by killing other natural elements in the Burnside ring in a highly structured manner? What is the free $\mathbb{E}_\lambda$–algebra in C_p–spectra with $[C_p] = 0$?

Question 9.8 Can the C_{p^n}–equivariant dual Steenrod algebra $\mathrm{H}\underline{\mathbb{F}}_p \wedge \mathrm{H}\underline{\mathbb{F}}_p$ be profitably studied via its equivalence with $\mathrm{H}\underline{\mathbb{F}}_p \wedge \Sigma^\infty_+ \Omega^\lambda S^{\lambda+1}$?

Appendix Proof of Theorem C

For convenience, we recall the statement of Theorem C, which this appendix is devoted to proving. The result is entirely nonequivariant.

Theorem *Let S^0_p denote the p–complete sphere spectrum, and suppose that $p > 2$. Then there is no triple loop map*

$$X \to B\mathrm{GL}_1(S^0_p),$$

for any triple loop space X, that makes $\mathrm{H}\mathbb{F}_p$ as a Thom spectrum.

Remark A.1 It follows also that $\mathrm{H}\mathbb{F}_p$ is not a triple loop Thom spectrum over the p–local sphere spectrum. If it were, then the composition

$$X \to B\mathrm{GL}_1(S^0_{(p)}) \to B\mathrm{GL}_1(S^0_p)$$

would provide a counterexample to the above.

Proof Suppose, for the sake of contradiction, that such a triple loop map

$$X \to B\mathrm{GL}_1(S^0_p)$$

exists. The Thom isomorphism then implies that

$$\mathrm{H}\mathbb{F}_p \wedge \Sigma^\infty_+ X \simeq \mathrm{H}\mathbb{F}_p \wedge \mathrm{H}\mathbb{F}_p$$

as $\mathrm{H}\mathbb{F}_p$–$\mathbb{E}_3$–algebras.

In particular, by Theorem 1.4,

$$\mathrm{H}\mathbb{F}_p \wedge \Sigma^\infty_+ X \simeq \mathrm{H}\mathbb{F}_p \wedge \Sigma^\infty_+ \Omega^2 S^3$$

as $\mathrm{H}\mathbb{F}_p$–$\mathbb{E}_2$–algebras, and the latter object is the free $\mathrm{H}\mathbb{F}_p$–$\mathbb{E}_2$–algebra on a class in degree 1.

The Hurewicz theorem gives a map $S^1 \to X$, which extends to a double loop map $\Omega^2 S^3 \to X$, and the above discussion implies that this double loop map is a homology isomorphism. Thus, the p–completion of X is the p–completion of $\Omega^2 S^3$, as a double loop space.

Transporting the $\mathbb{E}_3$–algebra structure on X yields an $\mathbb{E}_3$–algebra structure on the p–completion of $\Omega^2 S^3$, extending the usual $\mathbb{E}_2$–algebra structure. The theorems of Dwyer, Miller and Wilkerson [17] show that there is a unique such $\mathbb{E}_3$–algebra structure, and so the p–completion of $B^3 X$ must be the p–completion of $\mathbb{H}P^\infty$.

Now, the composite

$$X \to B\mathrm{GL}_1(S_p^0) \to B\mathrm{GL}_1(\mathrm{H}\mathbb{F}_p)$$

is null, and it follows that there is a factorization through the fiber F of $B\mathrm{GL}_1(S_p^0) \to B\mathrm{GL}_1(\mathrm{H}\mathbb{F}_p)$. The equivalence $\mathbb{Z}_p^\times \cong \mu_{p-1} \times \mathbb{Z}_p$ implies that the homotopy groups of F are p–complete. Thus, with $\mathbb{H}P_p^\infty$ denoting the p–completion of $\mathbb{H}P^\infty$, there is a commuting diagram

$$\begin{tikzcd}
 & B^3 X \arrow[r] \arrow[d] & B^4\mathrm{GL}_1(S_p^0) \\
\mathbb{H}P^\infty \arrow[r] & \mathbb{H}P_p^\infty \arrow[ur] &
\end{tikzcd}$$

In particular, there is a triple loop map

$$\Omega^2 S^3 \to \Omega^3 \mathbb{H}P^\infty \to B\mathrm{GL}_1(S_p^0)$$

with Thom spectrum equivalent (at least after p–completion) to $\mathrm{H}\mathbb{F}_p$.

The underlying double loop map is determined by a class in

$$1 + p\alpha \in \pi_3(B^3\mathrm{GL}_1(S_p^0)) \cong \mathbb{Z}_p^\times.$$

Our original assumption, made for the sake of contradiction, is reduced to the assertion that a dashed arrow exists as in the diagram

$$\begin{tikzcd}
S^4 \arrow[r, "1+p\alpha"] \arrow[d] & B^4\mathrm{GL}_1(S_p^0) \\
\mathbb{H}P^\infty \arrow[ur, dashed] &
\end{tikzcd}$$

We will show this to be impossible by proving the nonexistence of a solution to the weaker lifting problem

$$\begin{tikzcd} S^4 \arrow[r, "1+p\alpha"] \arrow[d] & \Sigma^\infty B^4\mathrm{GL}_1(S^0_p) \arrow[r, "\ell"] & \Sigma^4 L_{K(1)} S^0 \\ \Sigma^\infty \mathbb{H}P^\infty \arrow[urr, dashed] & & \end{tikzcd}$$

where $L_{K(1)}S^0$ is $K(1)$–local sphere spectrum and ℓ is the Rezk logarithm [39]. We first calculate the composite

$$S^4 \xrightarrow{1+p\alpha} B^4\mathrm{GL}_1(S^0_p) \xrightarrow{\ell} \Sigma^4 L_{K(1)} S^0,$$

using Rezk's formula [39, Theorem 1.9] for the logarithm at odd primes:

$$\ell(1+p\alpha) = \log(1+p\alpha) - \frac{1}{p}\log(1+p\alpha).$$

If α were not a p–adic unit, then the composite $\Omega^2 S^3 \to B\mathrm{GL}_1(S^0) \to B\mathrm{GL}_1(\mathrm{H}\mathbb{Z}/p^2)$ would be null as a 2–fold loop map, providing a ring map $\mathrm{H}\mathbb{F}_p \to \mathrm{H}\mathbb{Z}/p^2$. Since this is absurd, α must be a p–adic unit, and we learn that $\ell(1+p\alpha)$ is also a p–adic unit.

Without loss of generality, then, we are reduced to showing the impossibility of the lifting problem

$$\begin{tikzcd} S^4 \arrow[r, "1"] \arrow[d] & \Sigma^4 L_{K(1)} S^0 \\ \Sigma^\infty \mathbb{H}P^\infty \arrow[ur, dashed] & \end{tikzcd}$$

where 1 is the unit of the ring spectrum $L_{K(1)}S^0$.

Let KU_p denote p–complete complex K–theory. Recall that the composite

$$L_{K(1)}S^0 \to \mathrm{KU}_p \xrightarrow{\psi^q - 1} \mathrm{KU}_p$$

is null for any Adams operation ψ^q with q relatively prime to p. Since p is odd, to finish the problem it will suffice for us to show that no element of $\mathrm{KU}_p^4(\mathbb{H}P^\infty)$ simultaneously

(1) restricts to the unit in $\mathrm{KU}_p^4(S^4)$;

(2) is invariant under the action of ψ^2.

Now,

$$\mathrm{KU}_p^*(\mathbb{H}P^\infty) \cong \mathbb{Z}_p[\![e]\!][\beta^\pm],$$

where $|e| = 0$ and β is the Bott class in degree -2. Of course, $\psi^2(\beta) = 2\beta$, and it will be necessary also to understand $\psi^2(e)$.

Remembering that $\mathbb{H}P^\infty$ is $B\mathrm{SU}(2)$, we may calculate $\psi^2(e)$ by determining the restriction of e along the inclusion of the maximal torus $BS^1 \to B\mathrm{SU}(2)$. Indeed, $\mathrm{KU}_p^*(BS^1) \cong \mathbb{Z}_p[\![x]\!][\beta^{\pm 1}]$, where $x = L-1$. On the other hand, $e = V-2$, where V is the standard representation of $\mathrm{SU}(2)$ on $\mathbb{C}^2$. The restriction of e is thus $L+L^{-1}-2$, where

$$L^{-1} = (x+1)^{-1} = 1 - x + x^2 - x^3 + \cdots.$$

Since

$$\begin{aligned}\psi^2(L+L^{-1}-2) = L^2 + L^{-2} - 2 &= (x+1)^2 + \frac{1}{(x+1)^2} - 2 \\ &= \Big(x+1+\frac{1}{x+1}-2\Big)^2 + 4\Big(x+1+\frac{1}{x+1}-2\Big),\end{aligned}$$

we calculate that

$$\psi^2(e) = e^2 + 4e.$$

An element of $\mathrm{KU}^4(\mathbb{H}P^\infty)$ is of the form $\beta^{-2}P(e)$, where $P(e)$ is a power series in $\mathbb{Z}_p[\![e]\!]$. The lifting problem in question is equivalent to finding a power series $P(e) = e + c_2e^2 + \cdots$ such that

$$P(e) = 2^{-2}P(\psi^2(e)).$$

Using the calculations above, this can be rewritten as the relation

$$4P(e) = P(e^2+4e).$$

The relation

$$4(e + c_2e^2 + c_3e^3 + \cdots) = (e^2+4e) + c_2(e^2+4e)^2 + c_3(e^2+4e)^3 + \cdots$$

inductively determines each c_i, given $c_2, \ldots, c_{i-1}$, according to the formula

$$c_i = \frac{2}{(2i)!}\prod_{j=2}^{i}(-(j-1)^2).$$

In particular, this formula does not yield a p–adic integer for $i = \frac{1}{2}(p+1)$, implying that there is no lift through $\mathbb{H}P^{(p+1)/2}$. □

Remark A.2 The Adams conjecture provides a map from the connective cover of the $K(1)$–local sphere spectrum into $\mathrm{gl}_1(S^0_{(p)})$. Using a variant of this due to Bhattacharya and Kitchloo [7], it is possible to construct maps $\mathbb{H}P^k \to B^4\mathrm{GL}_1(S^0_{(p)})$ for small values of k. Indeed, [7] employs arguments very similar to the ones above in order to produce multiplicative structures on Moore spectra. The authors believe, but have not verified, that it is possible to equip the map $S^3 \xrightarrow{1-p} B^3\mathrm{GL}_1(S^0_{(p)})$ with an $\mathbb{A}_{(p-1)/2}$–algebra structure in this manner.

Remark A.3 It is well known that the integral Eilenberg–Mac Lane spectrum $\mathrm{H}\mathbb{Z}_{(p)}$ is the Thom spectrum of a double loop composite

$$\Omega^2(S^3\langle 3\rangle) \to \Omega^2 S^3 \to B\mathrm{GL}_1(S^0_p).$$

One could attempt to refine this to a triple loop map, using the equivalence $\Omega\mathbb{H}P^\infty\langle 4\rangle \simeq S^3\langle 3\rangle$. The same obstruction as above proves that this strategy cannot work at odd primes, because the map

$$\mathbb{H}P^\infty\langle 4\rangle \to \mathbb{H}P^\infty$$

is a $K(1)$–local equivalence.

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