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Every compact *aspherical* Riemannian manifold admits a canonical series of orbibundle structures with infrasolv fibers, which is called its *infrasolv tower*. The tower arises from the solvable radicals of isometry group actions on the universal covers. Its length and the geometry of its base measure the degree of continuous symmetry of an aspherical Riemannian manifold. We say that the manifold has *large local symmetry* if it admits a tower of orbibundle fibrations with locally homogeneous fibers whose base is a locally homogeneous space. We construct examples of aspherical manifolds with large local symmetry which do not support any locally homogeneous Riemannian metrics.

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1 Introduction

Let X be a contractible Riemannian manifold, and suppose there exists a discrete group Γ of isometries of X such that the quotient space

$$X/\,\Gamma$$

is compact. Then X is said to be *divisible*. If in addition Γ is torsion-free, the quotient space is a compact aspherical manifold. Let

$$\mathrm{Isom}(X)$$

denote the group of isometries of X. The action of the continuous part

$$\mathsf{G} = \mathrm{Isom}(X)^0$$

of the isometry group on the space X may be seen as describing the *local symmetry* of the quotient metric space $X/\,\Gamma$, which is a Riemannian orbifold.

Under the assumption that $X/\,\Gamma$ is a manifold, Farb and Weinberger [8] proved the fundamental fact that

$$X/\mathrm{Isom}(X)^0$$

is again a contractible manifold and gives rise to a *Riemannian orbibundle*

$$X/\,\Gamma \to X/\,\Gamma\mathrm{Isom}(X)^0,$$

where the fibers are locally homogeneous spaces modeled on

$$X_{\mathsf{G}} = \mathsf{G}/\mathsf{K},$$

with K a maximal compact subgroup of G.

In this setup, a basic observation is the fact that the continuous part of the isometry group of the Riemannian quotient

$$X/\mathrm{Isom}(X)^0$$

may be nontrivial, and thereby reveals *"hidden"* local symmetries of the space X and its quotient $X/\,\Gamma$. From this point of view, the local symmetry of $X/\,\Gamma$ will be encoded in an ensuing tower of orbifold fibrations with locally homogeneous fibers. And we can say that $X/\,\Gamma$ has *"maximal"* local symmetry if the tower finally stops over a locally homogeneous orbifold.

This motivates the following formal definition:

Definition 1.1 We say that $M = X/\,\Gamma$ has *large local symmetry* if there exists a tower of Riemannian orbibundles $M \to M_1 \to M_2 \to \cdots \to M_k = \{\mathrm{pt}\}$ with locally homogeneous fibers.

The precise notion of Riemannian orbibundle with locally homogeneous fibers which appears in Definition 1.1 and which is used in this paper may be found in Definition 5.4 below. It is modeled on the above geometric situation, where the bundle map is induced by taking the quotient of an isometric action of a Lie group.

The purpose of this article is to initiate the study of the *local symmetry of aspherical spaces* in terms of their naturally associated *towers of Riemannian orbifold fibrations*.

1.1 The radical quotient and divisibility

Our first step will be to associate to any compact aspherical Riemannian manifold a canonical Riemannian orbifold fibration whose fibers are modeled on a Riemannian homogeneous space of a solvable Lie group.

As a starting point, we are concerned with the action of the maximal connected normal solvable subgroup

$$\mathsf{R} \le \mathsf{G} = \mathrm{Isom}(X)^0,$$

which is called the *solvable radical* of G. We then call the quotient space

$$Y = X/\mathsf{R}$$

the *radical quotient* of X.

Divisibility of the radical quotient As our first main result we show that the radical quotient is a contractible Riemannian manifold, and that it is divisible by the image of Γ in $\mathrm{Isom}(X/\mathsf{R})$.

Theorem 1 (see Theorem 4.3) *The radical quotient X/R is a contractible Riemannian manifold, and the image Θ of Γ in $\mathrm{Isom}(X/\mathsf{R})$ acts properly discontinuously on X/R with compact quotient.*

Remark We may view this result as a parametrized version of the classical Bieberbach–Auslander–Wang theorem (see Proposition 2.5), which concerns lattices in Lie groups.

Note that Theorem 1 shows that the radical quotient gives rise to an *associated Riemannian orbibundle* of the form

$$X/\Gamma \to Y/\Theta \tag{1-1}$$

and this orbibundle has locally homogeneous fibers modeled on R. In the following, we will study the geometry of this fibration in more detail.

Before doing so, let us mention the following topological result, which is of independent interest and is used as a basic tool for the proof of Theorem 1. It asserts that there are no compact Lie group actions normalized by properly discontinuous actions on *acyclic smooth manifolds* with compact quotient:

Theorem 2 (see Theorem 3.5) *Let X be an orientable acyclic manifold and Γ a group which acts smoothly and properly discontinuously on X with compact quotient. Let κ be a compact Lie group acting faithfully and smoothly on X such that the action is normalized by Γ. Then $\kappa = \{1\}$.*

The proof of Theorem 2 is based mainly on *cohomology of groups acting on acyclic complexes and application of the Smith theorem*.

The first geometric application of Theorem 2 is the following.

Corollary 1 (see Corollary 3.7) *Let X be a contractible Riemannian manifold which is divisible. Then* $\mathrm{Isom}(X)$ *has no nontrivial compact normal subgroup.*

Remark A similar result was obtained in [8, Claim II] for connected compact normal subgroups of $\mathrm{Isom}(X)$ and is stated under the possibly stronger assumption that there exists Γ such that X/Γ is a manifold. The result also holds for divisible Riemannian manifolds *X*. Moreover, Corollary 1 strengthens the result to include possibly nonconnected groups.

Based on (the proof of) Theorems 1 and 2, we carry out a detailed analysis of the interaction of the smooth properly discontinuous action of the discrete group Γ on X with the radicals of $\mathrm{Isom}(X)^0$. Some of the additional results which we obtain are summarized in the following structure theorem:

Theorem 3 (see Theorem 4.13) *Let X be a contractible Riemannian manifold and $\Gamma \leq \mathrm{Isom}(X)$ a discrete subgroup such that X/Γ is compact. Let R denote the solvable radical of $\mathrm{Isom}(X)^0$. Then there exists a unique Riemannian metric on the quotient X/R such that the map*

$$X \to X/\mathsf{R}$$

is a Riemannian submersion. It follows further that:

1. $\mathrm{Isom}(X)/\mathsf{R}$ *acts properly on* X/R.
2. *The kernel of the action in* (1) *is the maximal compact normal subgroup of* $\mathrm{Isom}(X)/\mathsf{R}$.
3. *The image of* $\mathrm{Isom}(X)^0$ *in* $\mathrm{Isom}(X/\mathsf{R})$ *is a semisimple Lie group S of noncompact type without finite subgroups in its center. Moreover, it is a closed normal semisimple subgroup of* $\mathrm{Isom}(X/\mathsf{R})^0$ (*and therefore normal in a finite-index subgroup of* $\mathrm{Isom}(X/\mathsf{R})$).
4. *Moreover,* $\Theta \cap S$ *is a uniform lattice in S.*

Note that the Lie group S in (3) may have infinite (but discrete) center, and such examples do naturally occur (see Section 6.2).

The geometry of the fibers appearing in the Riemannian orbibundle (1-1) is determined by a Riemannian homogeneous space of a solvable Lie group. In fact, as we explain now the fibers of the orbibundle (1-1) naturally carry the geometry of an *infrasolv orbifold*:

Infrasolv orbifolds and orbibundles Let R be a simply connected solvable Lie group. We let

$$\mathrm{Aff}(R) = R \rtimes \mathrm{Aut}(R)$$

denote its group of affine transformations. Consider a discrete subgroup

$$\Delta \leq \mathrm{Aff}(R),$$

such that the homomorphic image of Δ in $\mathrm{Aut}(R)$ has compact closure. We may choose some left-invariant Riemannian metric on R such that Δ acts by isometries. Then the quotient space

$$R/\Delta$$

is an aspherical Riemannian orbifold. Compact orbifolds of this type are traditionally called *infrasolv orbifolds*.

Remark Infrasolv *manifolds* form an important class of aspherical locally homogeneous Riemannian manifolds, and play a role in various geometrical contexts. See Baues [1], Tuschmann [27] and Wilking [30] for review of infrasolv manifolds.

A Riemannian orbibundle will be called an *infrasolv bundle* if its fibers are compact infrasolv orbifolds (with respect to the induced metric) which are all modeled on the same Lie group R (see Definition 5.3).

We can then state our main result on the geometry of the Riemannian orbibundle (1-1) which is associated to the radical quotient of X. Recall that R denotes the solvable radical of $\mathrm{Isom}(X)$.

Theorem 4 (see Theorem 5.5) *There exists a simply connected solvable normal Lie subgroup R_0 of R such that $X/\,\Gamma$ has an induced structure of Riemannian infrasolv fiber space (modeled on R_0) over the compact aspherical Riemannian orbifold $Y/\,\Theta$.*

1.2 Infrasolv towers and manifolds of large symmetry

According to Theorem 1, the radical quotient $Y = X/\mathsf{R}$ is a contractible Riemannian manifold and it is divisible by the image of Γ in $\mathrm{Isom}(Y)$. In general, the isometry group $\mathrm{Isom}(Y)$ will have a nontrivial continuous part $\mathrm{Isom}(Y)^0$, and also the radical of $\mathrm{Isom}(Y)^0$ can be nontrivial. (We will discuss several types of examples below.)

We may thus repeat the process of taking radical quotients until the continuous part of the isometry group is semisimple (or trivial). This gives rise to a canonical tower of Riemannian orbibundles, which by Theorem 4 are infrasolv orbibundles, and filter the space $X/\,\Gamma$. Such a tower will be called an *infrasolv tower* for $X/\,\Gamma$:

Corollary 2 (see Corollary 5.6) *Every aspherical Riemannian orbifold $X/\,\Gamma$ gives rise to a canonical infrasolv tower*

$$X/\,\Gamma \to X_1/\,\Gamma_1 \to \cdots \to X_\ell/\,\Gamma_\ell, \tag{1-2}$$

where the solvable radical of $\mathrm{Isom}(X_\ell)$ is trivial.

Riemannian manifolds of large symmetry The length ℓ and structure of the tower (1-2) describe canonical invariants of Riemannian metrics on X and $X/\,\Gamma$. The following notion thus generalizes locally homogeneous Riemannian manifolds: if, in (1-2), $X_\ell = \{\mathrm{pt}\}$ or X_ℓ is a Riemannian homogeneous manifold of a semisimple Lie group, then X/Γ has *large local symmetry* in the sense of Definition 1.1.

We illustrate the concept by constructing in Section 6 several examples of aspherical manifolds with large symmetry. A simple example is a two-dimensional warped product of circles

$$M_f = S^1 \times_{\bar{f}} S^1$$

for a certain warping function $\bar{f}$. Here M_f is diffeomorphic to a 2–torus whose universal cover X satisfies $\mathrm{Isom}(X)^0 = \mathbb{R}$ (see Example 6.2). More generally, in this direction, we may also take *any* (compact) locally symmetric space of noncompact type

$$N = \Theta \backslash S/K,$$

and we can form the warped product (with fiber N and base S^1)

$$M_{N,f} = S^1 \times_{\bar{f}} N \to S^1$$

to obtain Riemannian manifolds of large symmetry which are not locally homogeneous.

The above kind of examples for metrics of large symmetry are built on spaces which admit locally homogeneous metrics from the beginning, and therefore may be seen as "merely" exhibiting symmetry properties of particular Riemannian metrics on these spaces. In the following we come to discuss the *topological significance* of the concept of large symmetry.

1.3 Riemannian orbibundles arising from group extensions

We introduce now a general method to construct infrasolv bundles, starting from abstract group extensions of the form

$$1 \to \Lambda \to \Gamma \to \Theta \to 1,$$

where the group Λ, in general, will be a virtually polycyclic group and Θ is a discrete group which divides a contractible Riemannian manifold Y.

The basic construction is partially based on the notion of *injective Seifert fiber spaces* (as developed by Lee and Raymond [17]). The details will be explained in Section 7. As an application of this method, we can derive:

Theorem 5 (see Theorem 7.5) *Let Θ be a torsion-free uniform lattice in the hyperbolic group* $\mathrm{PSO}(n,1)$*. Suppose*

$$1 \to \mathbb{Z}^k \to \Gamma \to \Theta \to 1$$

is a central group extension which has infinite order. Then there exists a compact aspherical manifold X/Γ such that:

(1) *X/Γ admits a metric of large symmetry with length $\ell = 1$.*

(2) *For $n \geq 3$, X/Γ does not admit a locally homogeneous Riemannian metric. In particular, Γ does not embed as a uniform lattice into a connected Lie group.*

(3) *For $n = 2$, X/Γ admits the structure of a locally homogeneous Riemannian manifold. In particular, Γ embeds as a uniform lattice into a connected Lie group.*

The space X/Γ, also called a Seifert fibering over the compact hyperbolic manifold $\mathbb{H}^n/\Theta$, will inherit a Riemannian orbifold bundle structure over $\mathbb{H}^n/\Theta$ with typical fiber a k–torus and exceptional fiber a Euclidean space form. This construction can be applied to any compact locally homogeneous symmetric manifold $\Theta\backslash G/K$ of real rank 1 such that $H^2(\Theta, \mathbb{Z}^k)$ has an element of infinite order.

Corollary 3 (see Corollary 7.6) *There exists a compact aspherical Riemannian manifold X/Γ of dimension four that admits a complete infrasolv tower of length one, which fibers over a three-dimensional hyperbolic manifold. Moreover, the manifold X/Γ does not admit any locally homogeneous Riemannian metric.*

The corollary shows the topological significance of the concept of large symmetry: *The class of aspherical smooth manifolds admitting a metric of large symmetry is strictly larger than the class of manifolds which can be presented as a Riemannian locally homogeneous space.*

1.4 Smooth rigidity problem for manifolds of large symmetry

In a closing result for this paper we briefly touch on a particular differential topological aspect of our topic. Namely, in the following we are concerned with the existence of metrics of large symmetry on smooth manifolds *homeomorphic* to the torus.

An n–dimensional *exotic torus* is a compact smooth manifold homeomorphic to the standard n–torus T^n but not diffeomorphic to T^n. Such manifolds are known to exist by Wall [29] or Hsiang and Shaneson [12], for example. We prove that every smooth manifold homeomorphic to the torus which admits a metric of large symmetry must be diffeomorphic to the standard torus T^n. In other words:

Theorem 6 (see Theorem 8.2) *Let τ be an n–dimensional exotic torus. Then τ does not admit any Riemannian metric of* ***large symmetry****.*

This result is embedded in a wider context of smooth rigidity results which are known for certain classes of locally homogeneous manifolds (or orbifolds), for example locally symmetric spaces (by Mostow strong rigidity [21]) or infrasolv manifolds (as in [1]). In fact, since such spaces constitute the building blocks of manifolds of large symmetry, we would like to pose the following:

Question *Is the class of aspherical smooth manifolds admitting a metric of large symmetry smoothly rigid? That is, given any two aspherical manifolds of large symmetry with isomorphic fundamental group, are they diffeomorphic?*

Organization of the paper The paper is organized as follows. In Section 2 we briefly discuss proper actions of Lie groups, and properly discontinuous actions of discrete subgroups. In Section 3 we prove Theorems 2 and 1. Topological concepts such as group cohomology for infinite discrete groups and the Smith theorem for acyclic manifolds play a role here. In Section 4 we study the action of the solvable radical of the isometry group $\mathrm{Isom}(X)$ for divisible Riemannian manifolds X, and we prove Theorem 1. Also the structure theorem, Theorem 3, is proved here. In Section 5 we introduce the notion of infrasolv tower and the concept of large symmetry to prove our cornerstone results Theorem 4 and Corollary 2. In Sections 6 and 7 we give several examples of aspherical manifolds with large symmetry, as well as a general construction method related to Seifert fiber spaces, which gives Theorem 5 and Corollary 3. In Section 8 we prove the smooth rigidity theorem, Theorem 6, for topological tori with large symmetry.

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2 Preliminaries

2.1 Solvable Lie groups

Let R be a connected solvable Lie group and $T \leq R$ a maximal compact subgroup. Let N be the nilpotent radical of R, that is, its maximal connected normal nilpotent subgroup. The intersection $N \cap T$ is the maximal compact subgroup of N, which is central and characteristic in N, since N is nilpotent.

Lemma 2.1 *Assume that N is simply connected. Then there exists a characteristic simply connected subgroup R_0 of R, with $N \leq R_0$, such that $R = R_0 \rtimes T$.*

Proof Since the simply connected nilpotent Lie group N is contractible, R is a topological product of N and R/N. In particular, the maximal compact subgroups of R and R/N have the same dimension. Thus, dividing by N, we obtain a quotient map $R \xrightarrow{p} R/N = T_1 \times V_2$, where $T_1 = p(T)$ and V_2 is a vector group. Let $V = V_1 \times V_2$ be the universal covering group of $T_1 \times V_2$, where V_1 covers T_1. Let $\widetilde{R}$ be the universal covering of R. Then $V = \widetilde{R}/N$. Under the induced action of the automorphism group $\mathrm{Aut}(\widetilde{R})$ on $V = \widetilde{R}/N$ the identity component $\mathrm{Aut}(\widetilde{R})^0$ acts trivially (see [14, Chapter III, Theorem 7]). As $\mathrm{Aut}(\widetilde{R})$ (being isomorphic to the automorphism group of the Lie algebra of R) is an algebraic group, $\mathrm{Aut}(\widetilde{R})/\mathrm{Aut}(\widetilde{R})^0$ is finite. Note that $\mathrm{Aut}(R)$ is a subgroup of $\mathrm{Aut}(\widetilde{R})$. In particular, the image of $\mathrm{Aut}(R)$ in $\mathrm{GL}(V)$ is finite. As the factor V_1 is invariant under $\mathrm{Aut}(R)$, there is an invariant complementary subspace in V, which projects to an $\mathrm{Aut}(R)$–invariant vector subgroup V_2' of $T_1 \times V_2$. Then $R_0 = p^{-1}(V_2')$ is a subgroup of R which is invariant under $\mathrm{Aut}(R)$. It is obvious that R_0 is simply connected and that $R = R_0 \rtimes \mathsf{T}$. □

2.2 Discrete subgroups of Lie groups

Let G be a Lie group. We call a closed subgroup H of G uniform if G/H is compact. A discrete uniform subgroup is called a uniform lattice. We note the following elementary fact on uniform subgroups (compare [24, Theorem 1.13]):

Lemma 2.2 *Let H be a uniform subgroup of G and L a closed subgroup of G. If $L/L \cap H$ is compact then LH is closed in G. Moreover, if $H = \Gamma$ is discrete then $L/L \cap \Gamma$ is compact if and only if $L\Gamma$ is closed in G.*

Proof To prove the first claim, we consider the map $L/L \cap H \to G/H$. Now, if Γ is discrete, the map $L/L \cap \Gamma \to L\Gamma/\Gamma \subset G/\Gamma$ is a homeomorphism. This implies the second part of the lemma. $\square$

Here is a simple application:

Lemma 2.3 *Let Γ be a uniform subgroup of G. Let G^0 denote the identity component of G. Then $\Gamma \cap G^0$ is a uniform lattice in G^0.*

Proof Since G/G^0 is discrete, the image of Γ in G/G^0 is discrete. Therefore ΓG^0 is closed in G. By Lemma 2.2, $\Gamma \cap G^0$ is a uniform lattice of G^0. $\square$

Lemma 2.4 *Let Γ be a uniform lattice of G, L a closed normal subgroup of G and $\nu\colon G \to G/L$ the quotient homomorphism. Assume that the identity component $\overline{\nu(\Gamma)}^0$ of the closure of $\nu(\Gamma)$ is contained in a compact normal subgroup K of G/L. Then $\Gamma \cap \nu^{-1}(K)$ is a uniform lattice in $\nu^{-1}(K)$.*

Proof By Lemma 2.2, it is enough to show that $\nu^{-1}(K)\Gamma$ is closed in G. By assumption, $T = \overline{\nu(\Gamma)}^0$ is a subgroup of K. As $\overline{\nu(\Gamma)} = T\nu(\Gamma)$, we observe that $K\nu(\Gamma) = K\overline{\nu(\Gamma)}$. Since K is compact, so is $K/K \cap \overline{\nu(\Gamma)}$. Clearly, $\overline{\nu(\Gamma)}$ is a uniform subgroup of G/L. Hence, by the first part of Lemma 2.2, $K\overline{\nu(\Gamma)}$ is closed in G/L. Therefore, $\nu^{-1}(K)\Gamma = \nu^{-1}(K\nu(\Gamma))$ is closed in G. $\square$

2.2.1 Levi decomposition We will assume now that G is a connected Lie group. Let R be the solvable radical of G (that is, R is the maximal connected normal solvable subgroup of G). Then G admits a Levi decomposition

$$G = R \cdot S.$$

The Levi subgroup S is a semisimple (not necessarily closed) immersed subgroup of G and the intersection $R \cap S$ is a totally disconnected subgroup of R and central in S. We let K denote the maximal compact and *connected* normal subgroup of S. For the following important fact; see [28, Chapter 4, Theorem 1.7; 26]:

Proposition 2.5 (Bieberbach–Auslander–Wang theorem) *Let Γ be a uniform lattice in G. Then the intersection $(RK) \cap \Gamma$ is a uniform lattice in RK. In particular, in the associated exact sequence*

$$1 \to RK \to G \to S/K \to 1,$$

the group Γ projects to a uniform lattice in a semisimple Lie group S/K of noncompact type.

2.2.2 Straightening of embeddings Let Γ be a (uniform) lattice in a connected Lie group G. It is a basic technique to modify a lattice Γ in a controlled way to obtain another embedding of Γ to G which has possibly better properties. A useful result in this direction is the following Mostow deformation theorem:

Proposition 2.6 (deformation of lattices; see [20, Theorem 5.5(b)]) *Let G be a linear Lie group, D its maximal reductive normal connected subgroup, and F its maximal normal connected subgroup which does not contain any noncompact simple subgroups normal in G. Then a finite-index subgroup of Γ can be deformed into a uniform lattice Γ' of G, where*

$$\Gamma' = (\Gamma' \cap F) \cdot (\Gamma' \cap D).$$

2.3 Proper actions

Let G be a Lie group which acts on a locally compact Hausdorff space X. For any subset A of X, we put

$$\zeta_A(G) = \{g \in G \mid g \cdot A \cap A \neq \varnothing\}.$$

The action of G on X is called *proper* if, for all compact subsets $\kappa \subset X$, the set $\zeta_\kappa(G)$ is compact.

The following lemma is concerned with quotients of proper actions.

Lemma 2.7 *Let G be a Lie group which acts properly on a smooth manifold X. Let R be a closed normal subgroup of G. Then the quotient group $L = G/R$ acts properly on the quotient space $W = X/R$.*

Proof Choose an arbitrary compact subset $\bar{\kappa}$ from W and consider its preimage $\mathcal{K}$ in X. Since R acts properly on X, there exists a compact subset κ of X such that $\mathcal{K} = R \cdot \kappa$. (Use the slice theorem [23; 15] for the action of R on the manifold X.) It follows that

$$\zeta_{\mathcal{K}}(G) = \zeta_\kappa(G)R.$$

Note that $\zeta_{\mathcal{K}}(G)$ projects onto $\zeta_{\bar{\kappa}}(L)$ under the natural homomorphism $G \to L$. Then $\zeta_{\bar{\kappa}}(L)$ is compact as it is the image of the compact set $\zeta_\kappa(G)$. □

2.3.1 Isometries of Riemannian manifolds Let X be a Riemannian manifold. By a theorem of Myers and Steenrod [22], the group

$$G = \mathrm{Isom}(X)$$

of isometries of X acts properly on X. For $x \in M$, let

$$G_x = \{h \in G \mid hx = x\}$$

denote the isotropy group at x. Let Γ be a discrete subgroup of $\mathrm{Isom}(X)$.

Lemma 2.8 *Assume that X/Γ is compact. Then the group Γ is a uniform lattice in* $\mathrm{Isom}(X)$*. In particular,* $\Gamma \cap \mathrm{Isom}(X)^0$ *is a uniform lattice in the connected component* $\mathrm{Isom}(X)^0$.

Proof As $\mathrm{Isom}(X)$ acts properly, the quotient $X/\mathrm{Isom}(X)$ is Hausdorff. For the natural map $X/\Gamma \to X/\mathrm{Isom}(X)$, each fiber over a point $[x] \in X/\mathrm{Isom}(X)$ is homeomorphic to $\mathrm{Isom}(X)_x \backslash \mathrm{Isom}(X)/\Gamma$ and it is closed in X/Γ. As the stabilizer $\mathrm{Isom}(X)_x$ is always compact and X/Γ is compact, it follows that $\mathrm{Isom}(X)/\Gamma$ is compact. In view of Lemma 2.3, this shows that $\mathrm{Isom}(X)^0 \cap \Gamma$ is a uniform subgroup of $\mathrm{Isom}(X)^0$. □

3 Smooth crystallographic actions

Let X be a contractible smooth manifold and Γ a properly discontinuous group of diffeomorphisms of X. If the quotient space

$$X/\Gamma$$

is compact, we call the action of Γ on X crystallographic. In this section we are mostly concerned with the action of compact Lie groups on X/Γ.

3.1 Cohomology of groups acting on acyclic spaces

Let Γ be a group which has a properly discontinuous cellular action on a CW complex X. Let $R = \mathbb{Z}$ or $R = \mathbb{Z}_p$ for some prime p. The space X is called acyclic over R if $H_i(X, R) = \{0\}$ for $i \neq 0$ and $H_0(X, R) = R$. (If $R = \mathbb{Z}$ then X is called acyclic.) Let $R\Gamma$ denote the group ring for Γ with R–coefficients. We have the following important observation:

Proposition 3.1 (main lemma) *Assume that X is acyclic over R and X/Γ is compact; then $H^*(\Gamma, R\Gamma) = H_c^*(X, R)$.*

Here, $H^*(\Gamma, R\Gamma)$ denotes the group cohomology of Γ with $R\Gamma$–coefficients (see [5] for definition) and $H_c^*(X, R)$ denotes (cellular) cohomology of X with compact supports. This "main lemma" is the analogue of [7, Lemma F.2.2] (based on [5, Proposition 7.5, Exercise 4]), where both references deal with $\mathbb{Z}$–coefficients only.

Remark The isomorphism in Proposition 3.1 is actually an isomorphism of Γ–modules where the action on the left-side cohomology is induced from the right-action on the coefficient module $\mathbb{Z}_p[\Gamma]$. For any group Γ, the groups $H^i(\Gamma, \mathbb{Z}_p[\Gamma])$ are computed by taking the cohomology of the cochain complex $\mathrm{Hom}_\Gamma(F_*, \mathbb{Z}_p[\Gamma])$, where $F_* \to \mathbb{Z}$ is any $\mathbb{Z}[\Gamma]$–projective resolution; see [5, Chapter III, Section 1]. Note that, if Γ is a finite group, then $H^i(\Gamma, \mathbb{Z}_p[\Gamma]) = 0$, $i > 0$. Indeed, for a finite group Γ and any coefficient ring R, $R[\Gamma] = \mathrm{Map}(\Gamma, R) = \mathrm{Hom}(\mathbb{Z}[\Gamma], R]) = \mathrm{CoInd}_1^\Gamma(R)$, whence, according to Shapiro's lemma, $H^*(\Gamma, R[\Gamma]) = H^*(1, R)$. In view of this fact, the course of proof of the above main lemma runs through almost verbatim as in [7, Lemma F.2.2] over any coefficient ring R.

Every proper smooth action on a smooth manifold X has an invariant simplicial structure. In particular, if Γ acts properly discontinuously and smoothly on X, then X can be given the structure of a simplicial complex with simplicial action; see [13, Theorem II]. We deduce:

Theorem 3.2 *Let Γ be a group which acts smoothly and properly discontinuously with compact quotient on an n–dimensional R–orientable smooth manifold X. If X is acyclic over R, then $H^n(\Gamma, \Gamma R) = R$ and $H^i(\Gamma, \Gamma R) = \{0\}$ for $i \neq n$.*

Proof By Poincaré duality for noncompact manifolds (see [11, Theorem 26.6]), $H_c^i(X, R) = H_{n-i}(X, R)$. As X is acyclic, we deduce that $H_c^n(X, R) = R$ and $H_c^i(X, R) = \{0\}$ for $i \neq n$. By Illman's result [13], we may assume that X and the action of Γ are cellular. Therefore, Proposition 3.1 implies the theorem. □

3.1.1 Smooth actions on manifolds Let Γ be a group which acts properly discontinuously and smoothly with compact quotient on the smooth manifold X. Under the assumption that X is acyclic (over the integers) we show that there are no compact group actions on X which are normalized by Γ. Results of this type generalize the well-known fact (see [6]) that the only compact Lie groups which act on compact aspherical manifolds are tori.

Lemma 3.3 *Let T be a compact torus acting faithfully and smoothly on X, where X is an orientable acyclic manifold. Assume that the action is normalized by Γ. Then $T = \{1\}$.*

Proof By Theorem 3.2, $H^n(\Gamma, \Gamma\mathbb{Z}) = \mathbb{Z}$, where $n = \dim X$, and $H^i(\Gamma, \Gamma\mathbb{Z}) = \{0\}$ for $i \neq n$. Let X^T be the fixed-point set of T. Since X is an acyclic manifold and the action of T is smooth, Smith theory implies that the fixed-point set X^T is also acyclic. (See [4, Chapter IV, Corollary 1.5].) Note in particular that X^T is nonempty and a connected manifold. By [4, Chapter IV, Theorem 2.1] the manifold X^T is also orientable. Since Γ normalizes T, it acts on the acyclic manifold X^T with compact quotient. Again, by Theorem 3.2, we also have $H^r(\Gamma, \Gamma\mathbb{Z}) = \mathbb{Z}$, where $r = \dim X^T$. This implies $\dim X = \dim X^T$ and, hence, $T = \{1\}$. □

Closely related is the following fact (which is well known in the case that Γ acts freely; compare eg [17, Lemma 3.1.13]):

Lemma 3.4 *Let $C \leq \mathrm{Diff}(X)$ be a finite group of diffeomorphisms of the smooth manifold X. Assume further that C is normalized by Γ. If X is acyclic and orientable over $\mathbb{Z}_p$, where p is prime, then p does not divide the order of C. In particular, if X is acyclic and orientable, then $C = \{1\}$.*

Proof Let $C(p)$ be a p–Sylow subgroup of C. Since X is acyclic over $\mathbb{Z}_p$, the Smith theorem (see [4, Chapter III, Theorems 5.2 and 7.11]) implies that the fixed-point set $X^{C(p)}$ is also acyclic over $\mathbb{Z}_p$. In particular, $X^{C(p)}$ is a connected manifold. It is also orientable over $\mathbb{Z}_p$, by the proof of [4, Chapter IV, Theorem 2.1]. As Γ normalizes C, it acts on the set of p–Sylow subgroups of C. Therefore, we may assume (after going down to a subgroup of finite index in Γ) that Γ normalizes $C(p)$. In particular, Γ leaves $X^{C(p)}$ invariant. Since $X^{C(p)}$ is a connected manifold which is acyclic and orientable over $\mathbb{Z}_p$, Theorem 3.2 implies $\dim X = \dim X^{C(p)}$. Hence, $C(p) = \{1\}$ □

From the above, we deduce that properly discontinuous actions on acyclic *smooth manifolds* do not normalize compact Lie groups:

Theorem 3.5 *Let X be an orientable acyclic manifold and let Γ be a group which acts smoothly and properly discontinuously on X with compact quotient. Let κ be a compact Lie group acting faithfully and smoothly on X such that the action is normalized by Γ. Then $\kappa = \{1\}$.*

Proof Since Γ normalizes also the center of κ, which is an extension of a toral group by a finite group, we may in light of Lemmas 3.3 and 3.4 assume from the beginning that κ is connected and semisimple with trivial center. Moreover, since κ is semisimple, the group of outer automorphisms $\mathrm{Out}(\kappa)$ is finite. Therefore, by going down to a finite-index subgroup of Γ if necessary, we may assume that conjugation with the elements of Γ induces inner automorphisms of κ. As Γ normalizes κ, the group $\mathsf{D} = \kappa\Gamma$ is a Lie group of diffeomorphisms which acts properly on X. In view of the fact that the center of κ is trivial, this implies that there exists a closed subgroup $\Gamma' \leq \kappa\Gamma$, which *centralizes* κ, such that

$$\mathsf{D} = \kappa \cdot \Gamma = \kappa \cdot \Gamma'.$$

(Indeed, given $\ell \in \mathsf{D}$, define $\mu(\ell)\colon \kappa \to \kappa$ by $\mu(\ell)(k) = \ell k \ell^{-1}$ for all $k \in \kappa$. For each $\gamma \in \Gamma$, let $k_\gamma \in \kappa$ be the unique element, such that $\mu(\gamma) = \mu(k_\gamma)$. Since the center of κ is trivial, the map $\gamma \mapsto k_\gamma^{-1}\gamma$ is a homomorphism with kernel $\Gamma \cap \kappa$. Hence, the subgroup

$$\Gamma' = \{\gamma' = k_\gamma^{-1}\gamma \mid \gamma \in \Gamma\}$$

has the required properties.) Since Γ' centralizes κ, κ must be trivial, by Lemma 3.3. □

We next turn to a related auxiliary result, which plays a role in Section 4.2.

Lemma 3.6 *Let $\mathsf{p}\colon X \to Y$ be a fiber bundle of contractible manifolds. Let Γ act properly discontinuously on X with compact quotient and assume that the action descends equivariantly to a smooth action on Y. Then any compact torus T acting on Y which is normalized by the image of Γ acts trivially.*

Proof The fixed-point set Y^T is an acyclic orientable manifold (see Lemma 3.3). Since the fibers of the bundle are contractible (implicitly using the long exact homotopy sequence of the fiber bundle and Whitehead's theorem), the preimage $\mathsf{p}^{-1}(Y^T)$ is also acyclic (by application of the Leray–Serre spectral sequence of a fibration). The latter is acted upon by Γ, since the action of T is normalized by the image of Γ. Now, as in the proof of Lemma 3.3, $H^r(\Gamma, \Gamma\mathbb{Z}) = \mathbb{Z}$, where $r = \dim X$, and, by Proposition 3.1, $H^i(\Gamma, \Gamma\mathbb{Z}) = \{0\}$ for $i > \dim \mathsf{p}^{-1}(Y^T)$, so that $r = \dim \mathsf{p}^{-1}(Y^T)$. Therefore, $Y = Y^T$, which implies that T acts trivially on Y. □

3.2 First application to Riemannian manifolds

It was proved in [8, Claim II] that $\mathrm{Isom}(X)^0$ contains no compact connected factor under the somewhat stronger assumption that X/Γ is a manifold. By the above, this generalizes to divisible Riemannian manifolds X and simultaneously strengthens the statement to include possibly nonconnected compact normal groups:

Corollary 3.7 *Let X be a contractible Riemannian manifold which is divisible. Then $\mathrm{Isom}(X)$ has no nontrivial compact normal subgroup.*

4 Isometry groups with radical

Let X be contractible Riemannian manifold which is *divisible*; that is, there exists a discrete group Γ of isometries such that the quotient space

$$X/\,\Gamma$$

is compact. In this section, we are concerned with the properties of the action of the continuous part

$$\mathsf{G} = \mathrm{Isom}(X)^0$$

of $\mathrm{Isom}(X)$ on X. In particular, we study the action of the maximal connected normal solvable subgroup

$$\mathsf{R} \leq \mathsf{G},$$

which is called the *solvable radical* of G. One main goal here is to show that the *radical quotient*

$$X/\mathsf{R}$$

is a Riemannian manifold which is divisible by the image of Γ in $\mathrm{Isom}(X/\mathsf{R})$. To this end, a detailed analysis for the action of $\mathrm{Isom}(X)$ on X/R is required.

4.1 Principal bundle structure of the radical quotient

As our starting point, we show here that the quotient

$$\mathsf{q}\colon X \to X/\mathsf{R}$$

inherits a natural structure of a *principal bundle*, where the base is a contractible manifold and the structure group is a simply connected solvable Lie group R_0 which is contained in R.

Let N be the nilpotent radical of R. Then N is a closed and characteristic subgroup of $\mathrm{Isom}(X)$.

Lemma 4.1 *N is simply connected. In particular, N has no (nontrivial) compact subgroup and N acts properly and freely on X.*

Proof Let T be the maximal compact subgroup of N. Then T is a compact torus which acts effectively on X. As N is nilpotent, T is central and characteristic in N, and also characteristic in G. Therefore, Γ normalizes T. Lemma 3.3 implies that $T = \{1\}$. □

Therefore, X/N is a contractible manifold and

$$X \to X/\mathsf{N}$$

is a principal bundle with group N.

We continue with our study of the action of R on X. Let T be a maximal compact subgroup of R (which is a compact torus). By Lemma 4.1, T intersects N only trivially. Hence, Lemma 2.1 asserts that we may *choose* a simply connected solvable characteristic subgroup R_0 of R, with $\mathsf{N} \leq \mathsf{R}_0$, such that

$$\mathsf{R} = \mathsf{R}_0 \cdot \mathsf{T} \quad \text{and} \quad \mathsf{R}_0 \cap \mathsf{T} = \{1\}. \tag{4-1}$$

Since the maximal compact subgroup of R_0 is trivial, R_0 acts freely on X. Indeed, the geometry of the R–orbits on X is controlled by R_0, and, as the following proposition shows, the quotient map

$$\mathsf{q}\colon X \to X/\mathsf{R}$$

is a principal bundle with group R_0.

Proposition 4.2 *The compact torus* T *acts trivially on* X/N. *Consequently*, *we have*:

(1) R_0 *acts simply transitively on each fiber of* q.

(2) X/R *is a contractible manifold.*

(3) T *is faithfully represented on* N *by conjugation.*

(4) *For every* $x \in X$, *the stabilizer* R_x *at* x *is conjugate to* T *by an element of* N.

Proof The image of T is the maximal compact subgroup of the abelian group R/N (see the proof of Lemma 2.1), and it is a characteristic subgroup of R/N. Since $\mathrm{Isom}(X)$ normalizes R, it follows that the induced action of $\mathrm{Isom}(X)$ on

$$Z = X/\mathsf{N}$$

normalizes the induced action of T. We choose the unique Riemannian metric on Z which makes the map $X \to Z$ a Riemannian submersion. Since N is characteristic in $\mathrm{Isom}(X)$, there is a homomorphism

$$\mathrm{Isom}(X) \to \mathrm{Isom}(Z).$$

With respect to the induced actions, Γ normalizes T. Therefore, Lemma 3.6 implies that T acts trivially on Z. This implies that $X/\mathsf{R} = X/(\mathsf{R}_0\mathsf{T}) = X/\mathsf{R}_0$. In particular, (1) and (2) hold.

Let $Z_{\mathsf{T}}(\mathsf{N}) \leq \mathsf{T}$ denote the centralizer of N in T. Since T acts trivially on the space $Z = X/\mathsf{N}$, T and therefore also its subgroup $Z_{\mathsf{T}}(\mathsf{N})$ act on each orbit $\mathsf{N} \cdot x$, the latter centralizing the simply transitive action of N. Since N is simply connected nilpotent, N does not contain any nontrivial compact subgroups. The same is true for the centralizer of the simply transitive action of N (which is diffeomorphic to N). This implies that the compact group $Z_{\mathsf{T}}(\mathsf{N})$ acts trivially on each orbit of N. We deduce that $Z_{\mathsf{T}}(\mathsf{N}) = \{1\}$, which proves (3).

As R acts properly on X, the stabilizer R_x is compact. Since $\dim \mathsf{R}_x = \dim \mathsf{R} - \dim \mathsf{R}_0 = \dim \mathsf{T}$, R_x is maximal compact in R, as well. By the conjugacy of maximal compact subgroups of R, R_x is conjugate to T, and, in fact, the conjugation is given by an element of the nilradical N (since T and R_x are contained in the center of respective Cartan subgroups of R, and the latter are conjugate by N — see [3, Chapter VII, Section 3, Theorem 3]) — thus proving (4). □

4.2 Divisibility of the radical quotient

Since $\mathsf{q}\colon X \to Y$ is a principal bundle, we may equip the radical quotient

$$Y = X/\mathsf{R}$$

with the unique Riemannian structure such that $\mathsf{q}\colon X \to Y$ is a Riemannian submersion. Since R is normal in $\mathrm{Isom}(X)$, we also have a well-defined homomorphism

$$\phi\colon \mathrm{Isom}(X) \to \mathrm{Isom}(Y). \tag{4-2}$$

Let

$$\Theta = \phi(\Gamma)$$

denote the image of Γ in $\mathrm{Isom}(Y)$.

The main goal of this subsection is to show:

Theorem 4.3 *Θ is a discrete subgroup of $\mathrm{Isom}(Y)$. In particular, Θ acts properly discontinuously on Y and Y/Θ is compact.*

Hence, in particular, the Riemannian manifold Y is divisible.

Remark We may view Theorem 4.3 as a parametrized version of Proposition 2.5.

To prepare the proof of Theorem 4.3, we consider the kernel of the homomorphism ϕ in (4-2) (additional information on $\ker \phi$ will be derived in Proposition 4.11 below).

By construction of Y, the action of $\mathrm{Isom}(X)$ on Y factors over the quotient $\mathrm{Isom}(X)/\mathsf{R}$. Then we note:

Lemma 4.4 *The image of* $\ker\phi$ *in* $\mathrm{Isom}(X)/\mathsf{R}$ *is compact. In particular*, $\ker\phi$ *has finitely many connected components.*

Proof Note that, by Lemma 2.7, $\mathrm{Isom}(X)/\mathsf{R}$ acts properly on Y. In particular, the kernel of this action is compact. This shows that $\ker\phi$ has compact image in $\mathrm{Isom}(X)/\mathsf{R}$. □

Recall that, by Lemma 2.8,

$$\Gamma_0 = \mathsf{G} \cap \Gamma$$

is a uniform lattice in $\mathsf{G} = \mathrm{Isom}(X)^0$ and consider the projection homomorphism

$$\psi : \mathsf{G} \to \mathsf{G}/\mathsf{R}.$$

Note that by the proof of Proposition 2.5, the identity component

$$T = \overline{\psi(\Gamma_0)}^0 \le \mathsf{G}/\mathsf{R}$$

is a compact torus.

Lemma 4.5 *T acts trivially on Y.*

Proof By Proposition 4.2, Y is a contractible manifold. The action of the compact torus T on Y is normalized by the induced action of Γ on Y. Hence, Lemma 3.6 implies that T acts trivially on Y. □

Proposition 4.6 (radical kernel is solvable lattice) *The group*

$$\Delta = \Gamma \cap \ker\phi$$

is a uniform lattice in $\ker\phi$. *In particular*, Δ *is a virtually polycyclic group.*

Proof The image of $\ker\phi$ is a normal subgroup of $\mathrm{Isom}(X)/\mathsf{R}$ and it is compact by Lemma 4.4. By Lemma 4.5, it also contains

$$T = \overline{\psi(\Gamma_0)}^0.$$

Hence, by Lemma 2.4, $\Gamma \cap \ker\phi$ is a uniform lattice in $\ker\phi$. Since $\ker\phi$ is a Lie group, which is an extension of a compact group by a solvable group, it follows that the Lie group $\ker\phi$ is amenable. As is well known, the discreteness of Γ together with amenability of $\ker\phi$ implies that $\Gamma \cap \ker\phi$ is virtually polycyclic. (See for example [19, Lemma 2.2(b)].) □

We are ready for the:

Proof of Theorem 4.3 Since $\Gamma \cap \ker\phi$ is a uniform lattice in $\ker\phi$, the image Θ of Γ in $\phi(\mathrm{Isom}(X)) = \mathrm{Isom}(X)/\ker\phi$ is a uniform lattice, see Lemma 2.2. Since $\mathrm{Isom}(X)/\mathsf{R}$ acts properly on Y, $\mathrm{Isom}(X)/\ker\phi$ is a closed subgroup in $\mathrm{Isom}(Y)$. We conclude that Θ is discrete in $\mathrm{Isom}(Y)$. Hence, Θ acts properly discontinuously on Y, and, in particular, Y/Θ is Hausdorff. The natural surjective map $X/\Gamma \to Y/\Theta$ shows that Y/Θ is compact. □

4.3 Action of $\mathrm{Isom}(X)$ on the radical quotient

We choose a Levi decomposition (see Section 2.2.1)

$$\mathrm{Isom}(X)^0 = \mathsf{R}\cdot\mathsf{S},$$

where S is a semisimple subgroup and R is the solvable radical of $\mathrm{Isom}(X)^0$. Moreover, let

$$\mathcal{K} \le \mathsf{S}$$

denote the maximal compact *normal* subgroup of S.

Now Theorem 4.3 combined with Theorem 3.5 implies:

Proposition 4.7 *The action of $\mathcal{K}$ on the quotient Y which is induced by*

$$\phi\colon \mathrm{Isom}(X) \to \mathrm{Isom}(Y)$$

is trivial. In particular, every finite subgroup in the center of S acts trivially on Y.

Proof Since $\mathcal{R}\cdot\mathcal{K}$ is characteristic in $\mathrm{Isom}(X)^0$, it is normalized by Γ. Therefore, the image of $\mathcal{K}$ is normalized by $\Theta = \phi(\Gamma)$ in $\mathrm{Isom}(Y)$, with $Y = X/\mathsf{R}$. We may now apply Theorem 3.5 to deduce that $\mathcal{K}$ acts trivially on Y. □

That is, we have shown that $\mathcal{K}$ is contained in $\ker\phi$.

Strengthening Lemma 4.4, we add the following observation:

Proposition 4.8 *The image of $\ker\phi$ in $\mathrm{Isom}(X)/\mathsf{R}$ is the unique maximal compact normal subgroup of $\mathrm{Isom}(X)/\mathsf{R}$. In particular, $\ker\phi$ has finitely many connected components.*

Proof Let $\overline{\mathcal{K}}_1$ be the image in $\mathrm{Isom}(Y)$ of any compact normal subgroup $\mathcal{K}_1$ of $\mathrm{Isom}(X)/\mathsf{R}$. As before, let Θ be the image of Γ. By Theorem 4.3, Θ acts properly discontinuously with compact quotient on Y and Θ normalizes the compact group $\overline{\mathcal{K}}_1$. Therefore Theorem 3.5 implies $\overline{\mathcal{K}}_1 = \{1\}$, which implies that $\mathcal{K}_1$ is contained in the image of $\ker\phi$. On the other hand, Lemma 4.4 asserts that the image of $\ker\phi$ is compact. Since the image is also a normal subgroup of $\mathrm{Isom}(X)/\mathsf{R}$, our claim follows. □

Consider $\mathsf{R}_0 \leq \mathsf{R}$ as in Proposition 4.2.

Lemma 4.9 *Let D be a compact Lie group of diffeomorphisms of X which centralizes R and acts trivially on Y. Then $\mathsf{D} = \{1\}$.*

Proof Since D acts trivially on Y, D centralizes the simply transitive action of R_0 on each fiber of the map $X \to Y$. Since, R_0 is a simply connected solvable Lie group, its centralizer in the diffeomorphism group of the fiber (being isomorphic to R_0) does not contain any nontrivial compact subgroup. Hence, the induced action of D on each fiber must be trivial. Therefore, D acts trivially on X. This implies $\mathsf{D} = \{1\}$. □

In the view of Proposition 4.7, we deduce:

Proposition 4.10 *Any compact subgroup of $\ker\phi$ acts faithfully on R by conjugation. In particular, this holds for the maximal compact normal subgroup $\mathcal{K}$ of S.*

We derive some additional observations on the kernel of

$$\phi\colon \mathrm{Isom}(X) \to \mathrm{Isom}(Y).$$

Proposition 4.11 *The following hold:*

(1) *$\ker\phi$ acts faithfully on each fiber of $\mathsf{q}\colon X \to Y$.*

(2) $(\ker\phi)^0 = \mathsf{R}\cdot\mathcal{K}^0$.

(3) *$\mathcal{K}$ is contained in $\ker\phi$, and $\mathsf{R}\cdot\mathcal{K}$ is of finite index in $\ker\phi \cap \mathsf{G}$.*

Proof Recall that R_0 acts simply transitively on the fibers of q and let $\mathsf{R}_0 \cdot p$ for $p \in X$ be a fiber. Since R_0 is characteristic in $\mathrm{Isom}(X)$, $\ker\phi$ normalizes R_0. Consider the subgroup T_p of elements in $\ker\phi$, which act trivially on the orbit $\mathsf{R}_0 \cdot p$. Since R_0 acts freely, T_p centralizes R_0 in $\mathrm{Isom}(X)$. Also T_p is compact, since $\ker\phi$ acts properly on X. Therefore, with Proposition 4.10 we have (1).

Clearly, $\mathsf{S} \cap \ker\phi$ is a normal subgroup of S, and, by Proposition 4.8, the homomorphism

$$\ker\phi \to \mathrm{Isom}(X)/\mathsf{R}$$

projects $\mathrm{Isom}(X)^0 \cap \ker\phi$ (and therefore also $\mathsf{S} \cap \ker\phi$) onto the maximal compact normal subgroup

$$\bar{\kappa} \leq \mathsf{G}/\mathsf{R} = \mathsf{S}/\mathsf{R} \cap \mathsf{S}.$$

The induced surjective homomorphism

$$\mathsf{S} \cap \ker\phi \to \bar{\kappa}$$

has kernel $\mathsf{R} \cap \mathsf{S}$, which is central in S.

Since $(\ker\phi)^0$ is a normal subgroup of $\mathsf{G} = \mathrm{Isom}(X)^0$, $\mathsf{R} \subseteq \ker\phi$ is also the solvable radical of $(\ker\phi)^0$. Let H be a Levi subgroup of $(\ker\phi)^0$ with $H \leq \mathsf{S}$. (Any Levi subgroup of $(\ker\phi)^0$ is contained in a Levi subgroup of G. Therefore, $H \leq \mathsf{S}$ exists since all Levi subgroups are conjugate by an element of R. See [14, Chapter III, Section 9].) Since

$$(\ker\phi)^0 = \mathsf{R} \cdot H,$$

the Levi subgroup $H = (\mathsf{S} \cap \ker\phi)^0$ projects onto $\bar{\kappa}^0$, and is normal in S. Since the projection has discrete kernel, $\bar{\kappa}^0$ is semisimple (and also compact). Therefore, $\pi_1(\bar{\kappa}^0)$ is finite. This shows that the covering group H is compact. Since $\mathcal{K}^0$ is the maximal compact connected normal subgroup in S, $H \subseteq \mathcal{K}^0$. On the other hand, by Proposition 4.7, $\mathcal{K} \subseteq \ker\phi$. Therefore, $\mathcal{K}^0 \subseteq H$. This proves (2).

Since $\bar{\kappa}$ is compact, it has only finitely many connected components. Since $\ker\phi \cap \mathsf{S}$ projects surjectively on the image $\bar{\kappa}$ of $\ker\phi$, using (2), it follows that $\mathcal{K}^0(\mathsf{S} \cap \mathsf{R})$ is of finite index in $\ker\phi \cap \mathsf{S}$. Hence, also $\mathcal{K}(\mathsf{S} \cap \mathsf{R})$ is of finite index in $\ker\phi \cap \mathsf{S}$. Therefore, (3) follows. □

The following example shows that, in general, the maximal finite normal subgroup C in the center of S can be nontrivial, even if S has no *connected* compact normal subgroup. Of course, this can only happen provided that the isometry group has a nontrivial connected radical on which S, or C, acts faithfully. (Indeed, by Proposition 4.7, the induced action of C on the radical quotient Y is always trivial.) However, even if the radical R of $\mathrm{Isom}(X)$ is trivial, S can have infinite center as in Example 6.4.

Example 4.12 (compact locally homogeneous manifolds with radical) Let S be a semisimple Lie group of noncompact type which is faithfully represented on a vector space V. Since S is linear, its center C is finite. Let K be a maximal compact subgroup of S. Since C is finite, C is contained in K. Put

$$G = V \rtimes S.$$

Now consider the contractible manifold

$$X = V \times_K S = G/K.$$

Then G acts properly and faithfully on X. Choose a G–invariant Riemannian metric such that $G = \mathrm{Isom}(X)^0$. Since V is the solvable radical, the radical quotient

$$Y = X/V = S/K$$

is a *Riemannian symmetric space* of noncompact type. Now choose V and a uniform subgroup Θ of S such that Θ is arithmetic with respect to a lattice Λ in V. (By a classical result of Borel [2] such Θ and V always exist.) Then the group $\Gamma = \Lambda \rtimes \Theta$ is a discrete uniform subgroup of $\mathrm{Isom}(X)$. This constructs a corresponding compact locally homogeneous quotient X/Γ with $G = \mathrm{Isom}(X)^0$.

4.4 Divisible Riemannian manifolds

Summarizing the above we arrive at the following structure theorem for the continuous part of the isometry group of a contractible Riemannian manifold which is divisible.

Theorem 4.13 (main theorem on radicals in isometry groups) *Let X be a contractible Riemannian manifold and $\Gamma \leq \mathrm{Isom}(X)$ a discrete subgroup such that X/Γ is compact. Let R denote the solvable radical of $\mathrm{Isom}(X)^0$. Then the following hold*:

1. *The maximal compact normal subgroup of $\mathrm{Isom}(X)^0$ is trivial.*
2. *The nilpotent radical N of $\mathrm{Isom}(X)^0$ is simply connected and $\Gamma \cap \mathsf{N}$ is a uniform lattice in N.*
3. *The radical quotients X/N and X/R are contractible Riemannian manifolds.*

With respect to the Riemannian quotient metric on X/R and the induced homomorphism

$$\phi\colon \mathrm{Isom}(X) \to \mathrm{Isom}(X/\mathsf{R}),$$

it follows further that:

4. *The image Θ of Γ in $\mathrm{Isom}(X/\mathsf{R})$ is discrete (and a uniform lattice in $\mathrm{Isom}(X/\mathsf{R})$).*
5. *$\mathrm{Isom}(X)/\mathsf{R}$ acts properly on X/R.*
6. *The kernel of the action in (5) is the maximal compact normal subgroup of $\mathrm{Isom}(X)/\mathsf{R}$.*

(7) *The image* S *of* $\mathrm{Isom}(X)^0$ *in* $\mathrm{Isom}(X/\mathsf{R})$ *is a semisimple Lie group of noncompact type without finite subgroups in its center, and it is a closed normal subgroup of* $\mathrm{Isom}(X/\mathsf{R})^0$. (*In particular, it is normal in a finite-index subgroup of* $\mathrm{Isom}(X/\mathsf{R})$.)

(8) *Moreover,* $\Theta \cap S$ *is a uniform lattice in S.*

Proof Recall that (1) is a consequence of Corollary 3.7, and (3) is proved in Proposition 4.2. Next, (4) and (5) are contained in Theorem 4.3 and Lemma 2.7, whereas (6) is Proposition 4.8. For (2), observe that by (1), $\mathrm{Isom}(X)^0$ has no connected compact normal semisimple subgroup. Since $\Gamma \cap \mathrm{Isom}(X)^0$ is a lattice in $\mathrm{Isom}(X)^0$, a result of Mostow [20, Lemma 3.9] shows that $\Gamma \cap \mathsf{N}$ is a uniform lattice in N (which is simply connected by Lemma 4.1).

To prove (7), we first observe that $S = \phi(\mathrm{Isom}(X)^0)$ acts properly on X/R, by (5). In particular, S must be a closed subgroup of $\mathrm{Isom}(X/\mathsf{R})$. By Proposition 4.11, the maximal compact normal subgroup of S, where S is a Levi subgroup of $\mathsf{G} = \mathrm{Isom}(X)^0$, is contained in $\ker\phi$. Therefore, $S = \phi(\mathsf{S})$, is semisimple of noncompact type. By (4), X/R is divisible by $\Theta = \phi(\Gamma)$, and S is normalized by this action. By Theorem 3.5, S must be without finite normal subgroups.

Next we show that S is centralized by the nilpotent radical $\mathsf{N}_{X/\mathsf{R}}$ of $\mathrm{Isom}(X/\mathsf{R})^0$. Let $\Theta_1 = \mathsf{N}_{X/\mathsf{R}} \cap \Theta$, where $\Theta = \phi(\Gamma)$. Since Θ normalizes S, $[S, \Theta_1] \subseteq S \cap \mathsf{N}_{X/\mathsf{R}}$. The latter is a disconnected subgroup and therefore connectedness of S implies that $[S, \Theta_1] = \{1\}$. Therefore, S centralizes Θ_1.

Moreover, since Θ divides X/R, $\Theta_0 = \Theta \cap \mathrm{Isom}(X/\mathsf{R})^0$ is a uniform lattice in $\mathrm{Isom}(X/\mathsf{R})^0$. By (2), Θ_1 is thus a uniform lattice in the simply connected nilpotent group $\mathsf{N}_{X/\mathsf{R}}$. Since S centralizes Θ_1 (which is Zariski-dense with respect to the natural algebraic structure of $\mathsf{N}_{X/\mathsf{R}}$; see [24, Theorem 2.1]), it also centralizes $\mathsf{N}_{X/\mathsf{R}}$.

By the Maltsev Harish-Chandra theorem [14, Chapter III, page 92], all Levi subgroups of a Lie group are conjugate by elements of its nilpotent radical. Note that the intersection L_o of all Levi subgroups of $\mathrm{Isom}(X/\mathsf{R})^0$ is a semisimple normal — in fact, characteristic — subgroup of $\mathrm{Isom}(X/\mathsf{R})$. Being centralized by $\mathsf{N}_{X/\mathsf{R}}$, we conclude that S is contained in L_o.

Since L_o is characteristic in $\mathrm{Isom}(X/\mathsf{R})^0$, it is normalized by Θ. The same holds for the maximal compact factor of L_o. As a consequence of Theorem 3.5, any compact factor of L_o is trivial. Now, as follows from [20, Lemma 3.1(1)], the adjoint image of L_o, acting on the Lie algebra of $\mathrm{Isom}(X/\mathsf{R})^0$, is contained in the Zariski closure of the adjoint image of the uniform lattice Θ_0. Since Θ_0 normalizes S, its adjoint image stabilizes the Lie subalgebra belonging to S, and the same is true for the Zariski closure of the adjoint image of Θ_0. In particular, this holds for the adjoint image of L_o. This shows that S is a normal semisimple subgroup of L_o. As a semisimple factor of L_0, it is also normal in $\mathrm{Isom}(X/\mathsf{R})^0$. Similarly, the simple factors of L_o are permuted by $\mathrm{Isom}(X/\mathsf{R})$. Hence, a finite-index subgroup of $\mathrm{Isom}(X/\mathsf{R})$ normalizes S. This completes (7).

Since the quotient of X/R by Θ is compact, Θ is a uniform lattice in $\phi(\mathrm{Isom}(X))$ which acts properly on X/R by (5). By Lemma 2.3, Θ intersects S as a uniform lattice. So (8) is proved. □

5 Infrasolv towers

5.1 Review of infrasolv spaces

Infra *R*–geometry Let R be a connected Lie group and $\mathrm{Aff}(R)$ its group of affine transformations. By definition, $\mathrm{Aff}(R)$ is precisely the normalizer of the left translation action of R on itself in the group of all diffeomorphisms of R. The group $\mathrm{Aut}(R)$ of continuous automorphisms of R is then naturally a subgroup of $\mathrm{Aff}(R)$. Identifying R with its group of *left* translations gives rise to a semidirect product decomposition

$$\mathrm{Aff}(R) = R \rtimes \mathrm{Aut}(R).$$

The associated projection homomorphism

$$\mathrm{Aff}(R) \to \mathrm{Aut}(R),$$

is called the *holonomy homomorphism*. Let

$$\Delta \leq \mathrm{Aff}(R)$$

be a *discrete* subgroup which acts properly discontinuously on R. Then the quotient space

$$R/\Delta$$

is called an *orbifold with R–geometry*.

Infrasolv spaces We assume now that R is a simply connected solvable Lie group. Traditionally, a space with R–geometry is called infrasolv if it admits an underlying compatible Riemannian geometry.

Definition 5.1 A *compact* space R/Δ with R–geometry is called an *infrasolv orbifold* if the closure of the holonomy image of Δ in $\mathrm{Aut}(R)$ is compact. If the infrasolv orbifold R/Δ is a manifold, it is called an *infrasolv* manifold.

Equivalently, we can say that R/Δ *is infrasolv if and only if the closure of* $R\Delta$ *in* $\mathrm{Aff}(R)$ *acts properly on* R. In particular, if R/Δ is infrasolv, there exist left-invariant Riemannian metrics on R preserved by Δ (these are not necessarily unique) and any such compatible metric gives rise to an associated *infrasolv Riemannian structure* on R/Δ.

Remark (equivalence of infrasolv structures) A presentation of orbifolds $M = R/\Delta$, where R and Δ are as above, is called an infrasolv structure for the space M. Two infrasolv structures on M are considered equivalent if they are related by an affine map of R.

Lemma 5.2 *Any Riemannian space X with a simply transitive isometric action of R, and a cocompact properly discontinuous subgroup Δ of isometries of X, which normalizes this action, defines an* ***infrasolv orbifold structure with R–geometry*** *on X/Δ. This infrasolv structure is unique up to a right multiplication of R.*

Proof Let $\mathrm{Aff}(X, R)$ denote the normalizer of R in $\mathrm{Diff}(X)$. Fixing a point $p \in X$ defines identifications

$$R = X \quad \text{and} \quad \mathrm{Aff}(X, R) = R \rtimes \mathrm{Aut}(R),$$

and hence a corresponding embedding $\Delta \leq \mathrm{Aff}(R)$. This gives X/Δ the structure of an orbifold with R–geometry.

Because the identifications depend on the choice of basepoint $p \in X$, the embedding of Δ is defined up to conjugation with a right multiplication of R. This already proves the uniqueness statement for the structure of orbifold with R–geometry.

Furthermore, with these identifications in place, $\mathrm{Aut}(R)$ corresponds to the stabilizer of p in $\mathrm{Aff}(X, R)$. For $\delta \in \Delta$, let

$$\delta = r\phi, \quad r \in R,\ \phi \in \mathrm{Aut}(R),$$

be its corresponding holonomy decomposition. Observe that ϕ is an isometry, since δ and r are. Furthermore, the linear isotropy representation of $\mathrm{Aut}(R)$ on the tangent space $T_p X$ at p is an isomorphism of $\mathrm{Aut}(R)$ onto a closed subgroup of $\mathrm{GL}(T_p X)$.

Since ϕ is an isometry, its image under the linear isotropy map is contained in a compact subgroup of $\mathrm{GL}(T_p X)$. This shows that the holonomy image of Δ has compact closure in $\mathrm{Aut}(R)$. Therefore, the structure of orbifold with R–geometry on X/Δ is infrasolv (and the metric induced from X is a compatible Riemannian metric). □

5.2 Infrasolv fiber spaces

We introduce now a notion of orbibundles whose fibers carry an affine geometry modeled on the solvable Lie group R. We use a characterization in terms of group actions on the universal cover.

Let X be a simply connected manifold on which R acts properly and freely. Then let $\mathrm{Diff}(X, R)$ denote the normalizer of R in $\mathrm{Diff}(X)$. Consider a subgroup

$$\Gamma \leq \mathrm{Diff}(X, R)$$

which acts properly discontinuously on X. Let $\Delta \leq \Gamma$ be the normal subgroup of Γ which acts trivially on the quotient manifold

$$Y = X/R.$$

Put

$$\Theta = \Gamma/\Delta$$

and assume further that Θ *acts properly discontinuously on* Y. Thus, in this case, Y/Θ is a Hausdorff space; in fact, it is an orbifold.

We consider the projection map $\mathsf{p}\colon X/\Gamma \to Y/\Theta$ induced by $\mathsf{q}\colon X \to Y$. Since Θ acts properly discontinuously, the fiber stabilizers

$$\Delta_y = \{\gamma \in \Gamma \mid \gamma y = y\}, \quad y \in Y,$$

are finite extension groups of Δ. Since $\Gamma \leq \mathrm{Diff}(X, R)$ normalizes R, the restriction of Δ_y to $\mathsf{q}^{-1}(y)$ acts by affine transformations with respect to the simply transitive action of R on $\mathsf{q}^{-1}(y)$. Therefore, the fibers

$$\mathsf{p}^{-1}(\bar{y}) = R/\Delta_y$$

carry the structure of a space with R–geometry.

Definition 5.3 The projection map

$$\mathsf{p}\colon X/\Gamma \to Y/\Theta$$

is called a *fiber bundle with R–geometry* over the base Y/Θ. It is called an *infrasolv fiber space* if the fibers of p are (compact) infrasolv orbifolds.

Given an infrasolv fiber space p and, in addition, Riemannian metrics on X and Y, invariant by R, Γ and Θ, respectively, such that

$$X \to Y$$

is a Riemannian submersion, we call p a *Riemannian infrasolv bundle* (or fiber space) modeled on the group R.

5.2.1 Riemannian orbibundles More generally we define the notion of Riemannian orbibundle with *locally homogeneous fibers* as follows.

Let X be a Riemannian manifold and consider a closed subgroup $L \leq \mathrm{Isom}(X)$ such that $Y = X/L$ is a Riemannian manifold with the induced Riemannian metric (meaning that the projection map $\mathsf{p}\colon X \to Y$ is a Riemannian submersion).

Definition 5.4 Given a properly discontinuous subgroup Γ of isometries of X that normalizes L such that the image Θ of Γ in $\mathrm{Isom}(Y)$ is discrete, then the map $X/\Gamma \to Y/\Theta$ induced by p is called a *Riemannian orbibundle* with locally homogeneous fibers.

With this definition, the fibers of such an orbibundle map are locally homogeneous Riemannian orbifolds modeled on a homogenous space of L. Riemannian infrasolv bundles appearing above are thus Riemannian orbibundles whose fibers are infrasov manifolds.

5.3 Structure theorems

Applying the results in Sections 4.1 and 4.2, we can state now:

Theorem 5.5 *Let X be a contractible Riemannian manifold which is divisible and let $\Gamma \leq \mathrm{Isom}(X)$ be a discrete subgroup such that X/Γ is compact. Let R be the solvable radical of $\mathrm{Isom}(X)$ and put $Y = X/\mathsf{R}$. Let Θ denote the homomorphic image of Γ in $\mathrm{Isom}(Y)$. Then X/Γ has an induced structure of Riemannian infrasolv fiber space over the compact aspherical Riemannian orbifold Y/Θ.*

Proof By Proposition 4.2(1), there exists a simply connected characteristic subgroup R_0 of R, acting properly and freely on X, such that $Y = X/\mathsf{R}_0$ is a contractible manifold. Moreover, the metric on X descends to Y such that the map $X \to Y$ is a Riemannian submersion. Since R_0 is characteristic in $\mathrm{Isom}(X)$, we have $\Gamma \leq \mathrm{Diff}(X, \mathsf{R}_0)$. Consider the associated homomorphism $\phi\colon \mathrm{Isom}(X) \to \mathrm{Isom}(Y)$. Then the image $\Theta = \phi(\Gamma)$ acts properly discontinuously, by Theorem 4.3. Hence, the map $\mathsf{p}\colon X/\Gamma \to Y/\Theta$ is a fiber bundle with R_0–geometry over the base orbifold Y/Θ.

Obviously, since X/Γ is compact, the fibers of p are compact. It remains to show that the bundle is *infrasolv*. That is, we have to show that the holonomy of every fiber has compact closure.

For this, note that the metric of X restricts to a $R\Delta_y$–invariant Riemannian metric on the R–orbit $\mathsf{q}^{-1}(y)$. As remarked in Lemma 5.2, this implies that the fibers are infrasolv. □

Remark (infrasolv structure on the fibers) The infrasolv structure on the fibers depends on the choice of group R_0, which is not necessarily unique. However, once R_0 is fixed the structures are defined up to equivalence (compare Lemma 5.2).

Remark (addition to Theorem 5.5) The geometry of the fibers of the bundle p constructed in the proof of Theorem 5.5 is determined by the holonomy image of Δ in the fibers of the map $X \to Y$. Since $\Delta \leq \ker\phi$, Proposition 4.11(3) shows that, *up to finite index*, the holonomy of the fibers is contained in the holonomy image of the subgroup $\mathsf{R}\cdot\mathcal{K} \leq \mathrm{Isom}(X)$.

The theorem applies in particular to compact aspherical Riemannian *manifolds*

$$M = X/\Gamma.$$

Since M is a manifold, Γ acts freely on X, and $\Gamma \leq \mathrm{Isom}(X)$ is a discrete torsion-free subgroup isomorphic to the fundamental group $\pi_1(M)$. Then M inherits the structure of an infrasolv fiber space

$$M \to Y/\Theta$$

over the base Y/Θ which, in general, is an *orbifold*. However, the fibers of the bundle map are always infrasolv manifolds, since Γ acts freely on X.

5.4 Towers of infrasolv fiber spaces

Using Theorem 5.5 we construct a sequence of Riemannian submersions

$$\mathsf{q}_i\colon X \to X_i,$$

subsequently dividing by the solvable radical R_i of $\mathrm{Isom}(X_i)$. That is, assuming that q_i is constructed, we put

$$X_{i+1} = X_i/\mathsf{R}_i$$

and define $\mathsf{q}_{i+1}\colon X \to X_{i+1}$ as the composition of q_i with the projection $X_i \to X_{i+1}$. In this way, we obtain a tower of Riemannian submersions

$$X \to X_1 \to \cdots \to X_k, \tag{5-1}$$

and the induced maps $\mathrm{Isom}(X_j) \to \mathrm{Isom}(X_{j+1})$ for $j+1 \le i$ compose to homomorphisms

$$\phi_i\colon \mathrm{Isom}(X) \to \mathrm{Isom}(X_i)$$

such that

$$\Gamma_i = \phi_i(\Gamma)$$

acts properly discontinuously and with compact quotient

$$X_i/\,\Gamma_i.$$

If, for some ℓ, $\mathrm{Isom}(X_\ell)$ has trivial solvable radical, the process terminates, and we call ℓ the *length* of the tower.

In view of Theorems 4.13 and 5.5, the following properties are satisfied:

Corollary 5.6 (infrasolv tower for X) (1) *X_i is a contractible Riemannian manifold and the projection $X_i \to X_{i+1}$ is a principal bundle with structure group a simply connected solvable Lie group.*

(2) *The maps $X_i/\,\Gamma_i \to X_{i+1}/\,\Gamma_{i+1}$ for $i+1 \le \ell$ are Riemannian infrasolv fiber spaces.*

(3) $\mathrm{Isom}(X_\ell)^0$ *is a semisimple* (*or trivial*) *Lie group of noncompact type which has no nontrivial finite subgroups in its center.*

Any sequence of maps

$$X/\,\Gamma \to X_1/\,\Gamma_1 \to \cdots \to X_k/\,\Gamma_k \tag{5-2}$$

such that, at each step, $X_i/\,\Gamma_i \to X_{i+1}/\,\Gamma_{i+1}$ is a Riemannian infrasolv bundle will be called an *infrasolv tower* for X over the *base*

$$X_k/\,\Gamma_k.$$

We call the tower *complete* if the solvable radical of $\mathrm{Isom}(X_k)^0$ is trivial. Thus, for a complete tower,

$$\mathsf{S} = \mathrm{Isom}(X_k)^0$$

is semisimple and the center of S is a finitely generated abelian group (which is torsion-free, by Corollary 3.7.) Let r_k denote the rank of $\mathcal{Z}$, where $\mathcal{Z}$ is the center of S. Set $X = X_0$.

Definition 5.7 The *solvable rank* of the complete infrasolv tower is the integer

$$r = \sum_{i=0}^{k-1} (\dim X_i - \dim X_{i+1}) + r_k = \dim X - \dim X_k + r_k.$$

The definition is motivated by Example 6.4.

Corollary 5.8 *The group Γ contains a normal polycyclic subgroup of rank equal to r.*

Proof Indeed, we have $\operatorname{rank} \Gamma_i / \Gamma_{i+1} = \dim X_i - \dim X_{i+1}$. (Recall that any virtually polycyclic group which acts properly discontinuously with compact quotient on a contractible manifold X has virtual cohomological dimension $\operatorname{vcd} \Gamma = \dim X$; see Theorem 3.2. Furthermore, $\operatorname{rank} \Gamma = \operatorname{vcd} \Gamma$, see [5].) □

6 Aspherical manifolds with large symmetry

We give some examples of Riemannian metrics on aspherical manifolds which exhibit various types of local symmetry.

6.1 Warped product metrics

A special case of Riemannian submersions $X \to Y$ are warped products $X = Y \times_f F$ of Riemannian manifolds Y and F, where $f\colon Y \to \mathbb{R}^{>0}$ denotes the warping function. The manifolds $(y \times F)$ for $y \in Y$ are called the fibers of the warped product. The following lemma describes fiber preserving warped product isometries:

Lemma 6.1 *Assume that the function f is bounded. Then every isometry Φ of X which maps fibers to fibers is of the form $\Phi = \psi \times \phi$, where $\phi \in \operatorname{Isom}(F)$ and $\psi \in \operatorname{Isom}(Y)$ satisfies $f \circ \psi = f$.*

Proof Indeed, since Φ is an isometry which preserves the fibers, it induces an isometry ψ of the base Y. It also respects the horizontal distribution of the warped product, which is tangent to the horizontal leaves $Y \times x$ for $x \in F$. Therefore, $\Phi = \psi \times \phi$ for some map $\phi\colon F \to F$. Writing $g = g_Y \times f g_F$ for the warped product metric, we see that Φ is an isometry if and only if, for all $y \in Y$ and $v \in T_x F$,

$$f(\psi(y)) g_{F,\phi(x)}(d\phi_x(v), d\phi_x(v)) = f(y) g_{F,x}(v, v).$$

Therefore, keeping x fixed, we deduce that there exists a unique $\lambda = \lambda(\psi) > 0$ such that the relation

$$f(\psi(y)) = \lambda f(y)$$

holds for all $y \in Y$. Consequently, ϕ is a homothety for g_F with factor λ^{-1}. Clearly, the map $k \mapsto \lambda(\psi^k)$ defines a homomorphism from a cyclic group to $\mathbb{R}^{>0}$. Hence, if f is bounded, it follows that $\lambda = 1$. $\square$

Compact aspherical manifolds of large symmetry are easily constructed using warped product metrics. The following simple example exhibits a compact aspherical Riemannian surface M_f, diffeomorphic to the two-torus, whose metric has large symmetry, in the sense of Definition 1.1. Its associated solvtower is of length two. (In particular, the metric is not locally homogeneous.)

Example 6.2 (torus of revolution) Consider the warped product of circles $M_f = S^1 \times_{\bar{f}} S^1$ which is covered by $X = \mathbb{R} \times_f \mathbb{R}$, where

$$f(x) = 2 + \sin x. \tag{6-1}$$

Accordingly, X has metric $g = dx^2 + f\, dy^2$. Then $\mathrm{Isom}(X)$ contains the translation group $V = \mathbb{R} \times 2\pi\mathbb{Z}$, and, putting $\Lambda = \mathbb{Z} \times 2\pi\mathbb{Z}$, we have that

$$M_f = X/\Lambda.$$

We claim that $V^0 = \mathbb{R}$ must be the nilpotent radical of $\mathrm{Isom}(X)^0$. Note that $\mathrm{Isom}(X)^0$ has no semisimple part (otherwise X is isometric to the hyperbolic plane, which is absurd since T^2 has no metric of negative curvature). Therefore, $\mathrm{Isom}(X)^0$ is solvable. We can infer from Lemma 6.1 that the nilpotent radical is one-dimensional. Indeed, since the nilradical acts freely, it is at most two-dimensional. In particular, it must be abelian, and therefore respects the fibers of the warped product. Therefore, the radical quotient for X coincides with the warped product projection

$$X \to (\mathbb{R}, dx^2).$$

It gives a Riemannian submersion over the real line, whose fibers are the orbits of $\mathrm{Isom}(X)^0 = V^0$, which are permuted by $\mathrm{Isom}(X)$.

More generally, for *any* Riemannian manifold N, we can form the warped product

$$M_{N,f} = S^1 \times_{\bar{f}} N,$$

where $\bar{f}$ *is as in* (6-1). Take, for example, a (compact) locally symmetric space of noncompact type

$$N = \Theta \backslash S/K$$

as fiber. Put

$$X = B \times_f S/K$$

for the lifted warped product on the universal cover, where $B = \mathbb{R}$ is the real line. By Theorem 5.5 and Proposition 4.11, S acts locally faithfully on the radical quotient of X. It follows that the nilpotent radical of $\mathrm{Isom}(X)^0$ is at most one-dimensional, and it is therefore centralized by S. In particular, the nilradical of $\mathrm{Isom}(X)^0$ acts on the fibers of the warped product, which are the orbits of S. We deduce from Lemma 6.1 that the nilradical of $\mathrm{Isom}(X)^0$ must be trivial. Hence,

$$\mathrm{Isom}(X)^0 = S$$

is semisimple, and the base of the lifted warped product

$$X = B \times_f S/K$$

is a Euclidean space. We proved:

Proposition 6.3 *The compact aspherical Riemannian warped product*

$$M_{\Theta \backslash S/K, \bar{f}} \to S^1$$

with f *as in* (6-1) *satisfies* $\mathrm{Isom}(X)^0 = S$ *is semisimple. Moreover, the isometry group of the base of the warped product metric on* X *is*

$$\mathrm{Isom}(B) = \mathrm{Isom}(\mathbb{R}).$$

6.2 Metrics on $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ and its compact quotients

Let

$$\widetilde{\mathrm{SL}}(2, \mathbb{R})$$

denote the universal covering group of $\mathrm{SL}(2, \mathbb{R})$.

Example 6.4 Consider the group manifold $X = \widetilde{\mathrm{SL}}(2, \mathbb{R})$ with any left-invariant Riemannian metric. Then, by the left action on itself,

$$\widetilde{\mathrm{SL}}(2, \mathbb{R}) \leq \mathrm{Isom}(X)$$

is represented as a subgroup of the isometry group. Let R denote the solvable radical of $\mathrm{Isom}(X)^0$ and $\mathcal{Z}$ the center of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$. Two principal cases do occur:

(I) The radical R is a one-dimensional vector group and acts freely on X. The Riemannian quotient X/R is (up to scaling) isometric to the hyperbolic plane $\mathbb{H}^2$, so that the associated fibering

$$X_{\widetilde{\mathrm{SL}}(2,\mathbb{R})} \to \mathbb{H}^2$$

is a Riemannian submersion, and

$$\mathrm{Isom}(X)^0 = \mathsf{R}\widetilde{\mathrm{SL}}(2,\mathbb{R}), \quad \mathsf{R} \cap \widetilde{\mathrm{SL}}(2,\mathbb{R}) = \mathcal{Z}.$$

(II) $\mathrm{Isom}(X)^0 = \widetilde{\mathrm{SL}}(2,\mathbb{R})$.

Proof Observe that $\mathrm{Isom}(X)^0$ is reductive with Levi subgroup $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. In fact, let S be a Levi subgroup containing $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. Then

$$X = X_{\mathsf{S}} = \mathsf{S}/\mathsf{K},$$

where K is maximal compact in S and $\mathsf{S} = \widetilde{\mathrm{SL}}(2,\mathbb{R})\mathsf{K}$. Assuming $\mathsf{K} \neq \{1\}$, either $\mathsf{K} = \mathrm{SO}(3)$ or $\mathsf{K} = S^1$. In the first case, X is of constant curvature. In particular, $\mathsf{S} = \mathrm{Isom}(X)^0$ is a linear group. This contradicts $\widetilde{\mathrm{SL}}(2,\mathbb{R}) \leq \mathsf{S}$. Similarly, in the latter case, S is a four-dimensional simple group. Such a group doesn't exist. Hence, $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ is a Levi subgroup of $\mathrm{Isom}(X)^0$. Similarly, we see that the radical R of $\mathrm{Isom}(X)^0$ is at most one-dimensional and $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ must intersect R in a cocompact lattice (because R is simply connected and noncompact).

Assume that the radical R is nontrivial. Since R is abelian and centralizing $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, it arises as a one-parameter group of right-translations on $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. Since $\mathrm{Isom}(X)^0$ acts properly, the adjoint representation of this one-parameter group defines a compact subgroup of inner automorphisms of the Lie algebra of $\mathrm{SL}(2,\mathbb{R})$. It also preserves the scalar product on the tangent space of the identity, which defines the left-invariant Riemannian metric on X. In particular, R arises from the subgroup $\widetilde{\mathrm{SO}}(2,\mathbb{R})$ in $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, which covers (a conjugate of) $\mathrm{SO}(2)$. Moreover, $X/\mathsf{R} = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2) = \mathbb{H}^2$. This is type (I). □

Remark The second case actually occurs and is the generic one. For example, take a standard basis $\{X, Y, H\}$ of the Lie algebra of $\mathrm{SL}(2,\mathbb{R})$ with $[H, X] = 2X$, $[H, Y] = -2Y$ and $[X, Y] = H$. Define it as orthonormal basis for the scalar product at the tangent space at the identity. We verify that this scalar product is not preserved by any compact subgroup in the adjoint image of $\mathrm{SL}(2,\mathbb{R})$. Hence, this defines a left-invariant metric on $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, which is of type (II).

Proof We show that the scalar product does not admit a one parameter group of inner isometries. Let $B = aH + bX + cY \in \mathfrak{sl}(2,\mathbb{R})$, where a, b and c are real numbers. Then $\mathrm{ad}(B)\colon \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{sl}(2,\mathbb{R})$ is represented, with respect to the basis $\{H, X, Y\}$, by a matrix of the form

$$\begin{pmatrix} 0 & -c & b \\ -2b & 2a & 0 \\ 2c & 0 & -2a \end{pmatrix}.$$

Now this matrix is skew if and only if $B = 0$. □

Associated principal circle bundle over Kähler manifolds Now let $\Gamma \le \widetilde{\mathrm{SL}}(2,\mathbb{R})$ be a (uniform) lattice. Then $\Delta = \Gamma \cap \mathcal{Z}$ has finite index in $\mathcal{Z}$ (by [24, Corollary 5.17]) and Γ projects to a uniform lattice

$$\Theta = \Gamma/\Delta \le \mathrm{PSL}_2\,\mathbb{R} = \mathrm{Isom}(\mathbb{H}^2)^0.$$

Let Γ act by left multiplication on $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ and put

$$M = X/\Gamma = \Gamma\backslash\widetilde{\mathrm{SL}}(2,\mathbb{R}).$$

Therefore:

(I) **The Sasakian case** Since $\mathrm{Isom}(X)^0$ has a radical R isomorphic to the real line, there is an induced infrasolv tower of length $k = 1$ over a compact hyperbolic orbifold:

$$M \to \mathbb{H}^2/\Theta,$$

where $X/\mathsf{R} = \mathbb{H}^2$. The map is actually a principal circle bundle and a Riemannian submersion. These manifolds M and their geometry play a prominent role in the classification of three manifolds. Compare [9].

(II) $\mathrm{Isom}(X)^0 = \mathsf{S} = \widetilde{\mathrm{SL}}(2,\mathbb{R})$ is semisimple with infinite cyclic center $\mathcal{Z}$. The Riemannian manifold X therefore does not admit an infrasolv-fibering.

In both cases, M is locally homogeneous (in particular, M has large symmetry) and the solvable rank of the metrics (see Definition 5.7) satisfies $r = 1$.

7 Constructing Riemannian manifolds from group extensions

In this section we introduce a method which allows us to construct examples of aspherical Riemannian orbibundles

$$\mathsf{p}\colon X/\Gamma \to Y/\Theta$$

which arise from associated group extensions of the form

$$1 \to \Lambda \to \Gamma \to \Theta \to 1.$$

In our setup, Λ, in general, will be a virtually polycyclic group. We will work out the details only in a specific, particularly simple case. Our main purpose here is to construct an aspherical manifold, which supports Riemannian metrics of large symmetry, but at the same time does not admit any locally homogeneous Riemannian metric; see Corollary 7.6. The method though generalizes considerably. The basic construction is partially based on the notion of *injective Seifert fiber spaces* (see [17] for an extensive account).

7.1 Fiberings over hyperbolic Riemannian orbifolds

Setup As a base space of our tentative fibration, we choose a hyperbolic orbifold (that is, a space of constant negative curvature)

$$\mathbb{H}^n/\Theta.$$

Here, $\Theta \le \mathrm{PSO}(n,1)$ is a discrete uniform subgroup, and

$$\mathrm{PSO}(n,1) = \mathrm{Isom}(\mathbb{H}^n)^0$$

denotes the identity component of the group of isometries of real hyperbolic space $\mathbb{H}^n$. In addition, we consider a central group extension

$$1 \to \mathbb{Z}^k \to \Gamma \to \Theta \to 1. \tag{7-1}$$

Up to isomorphism, (7-1) is determined by its extension class

$$[f] \in H^2(\Theta, \mathbb{Z}^k),$$

where f is a two-cocycle with values in $\mathbb{Z}^k$. Recall that (by the Borel density theorem) the lattice Θ does not contain any solvable or finite normal subgroup. Therefore, we remark:

($*$) The image of $\mathbb{Z}^k$ in Γ is the maximal virtually solvable normal subgroup of Γ.

First construction step Consider the standard inclusion $\mathbb{Z}^k \subset \mathbb{R}^k$, where $\mathbb{Z}^k$ is a lattice in $\mathbb{R}^k$. Using the cocycle f for (7-1), we obtain a pushout diagram of group extensions,

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}^k & \longrightarrow & \Gamma & \longrightarrow & \Theta & \longrightarrow & 1 \\ & & \cap & & \cap & & \| & & \\ 1 & \longrightarrow & \mathbb{R}^k & \longrightarrow & \mathbb{R}^k \cdot \Gamma & \longrightarrow & \Theta & \longrightarrow & 1 \end{array} \tag{7-2}$$

Second construction step The Seifert construction shows that there exists a proper action of the pushout $\mathbb{R}^k \cdot \Gamma$ on the product manifold

$$X = \mathbb{R}^k \times \mathbb{H}^n. \tag{7-3}$$

This action extends the translation action of $\mathbb{R}^k$ on the left factor of X, and induces the original action of Θ on $\mathbb{H}^n$. Thus, the quotient X/Γ is a compact aspherical orbifold and

$$T^k \to X/\Gamma \to \mathbb{H}^n/\Theta \tag{7-4}$$

is a *Seifert fibering* with typical fiber a k–torus T^k. See [17, Theorem 7.2.4, Sections 7.3–7.4].

Third construction step The total space X/Γ of the fibration (7-4) carries a compatible Riemannian metric of large local symmetry:

Proposition 7.1 (metric of large symmetry on X/Γ) *There exists a Riemannian metric g on X such that*

(1) $\mathbb{R}^k \cdot \Gamma$ *acts properly by isometries, and*
(2) *the projection map $X \to \mathbb{H}^n$ is a Riemannian submersion.*

Proof Since $\mathbb{R}^k$, Θ and also (according to [17, Theorem 7.2.4]) Γ act properly, we infer in light of the fact that $\Gamma \cap \mathbb{R}^k$ is lattice in $\mathbb{R}^k$ that the pushout $\mathbb{R}^k \cdot \Gamma$ acts properly on X. Thus, there exists an $\mathbb{R}^k \cdot \Gamma$–invariant Riemannian metric g' on X (see [15], for example). Let $\mathcal{H}$ be the horizontal distribution orthogonal to the orbits of $\mathbb{R}^k$ with respect to this metric. If $\mathsf{p}\colon X \to \mathbb{H}^n$ is the projection onto the second factor of X, the induced bundle map $\mathsf{p}_*\colon \mathcal{H} \to T\mathbb{H}^n$ identifies horizontal spaces in X with the respective tangent spaces of $\mathbb{H}^n$.

Put $u = T + A$ and $v = S + B$ for $u, v \in T_xX$, S and T tangent to the fibers, and $A, B \in \mathcal{H}$. Let $g_{\mathbb{H}}$ denote the hyperbolic metric and define a Riemannian metric g on X by putting

$$g(u, v) = g'(T, S) + g_{\mathbb{H}}(\mathsf{p}_*A, \mathsf{p}_*B). \tag{7-5}$$

Therefore, the metric g has the same horizontal spaces $\mathcal{H}$ as g', and also $\mathsf{p}\colon X \to \mathbb{H}^n$ is a Riemannian submersion, by construction. Also $\mathbb{R}^k \cdot \Gamma$ acts isometrically on the fibers, and the horizontal distribution $\mathcal{H}$ for g is invariant by $\mathbb{R}^k \cdot \Gamma$. Since $\mathbb{R}^k$ acts trivially on $\mathcal{H}$, the metric g is clearly invariant by $\mathbb{R}^k$. Moreover, since p is Γ–equivariant and Θ acts by isometries on the base, also Γ acts by isometries with respect to g. □

Corollary 7.2 *The compact orbifold X/Γ admits a metric which has a complete infrasolv tower of length one and of solvable rank k* (see Definition 5.7)*. In particular, X/Γ carries a metric of large symmetry.*

Suppose g is any Riemannian metric on X which satisfies Proposition 7.1(1)–(2). Then the situation is sufficiently rigid that the radical projection

$$\mathsf{q}\colon X \to X/\mathsf{R},$$

as defined in Section 4.2, actually coincides with the projection map onto the second factor of (7-3). More precisely:

Proposition 7.3 (rigidity of the projection) *For any metric on X satisfying Proposition 7.1*(1)–(2):

(1) *The image of $\mathbb{R}^k$ in* $\mathrm{Isom}(X)$ *is the maximal simply connected normal solvable subgroup R_0 of* $\mathrm{Isom}(X)$*.*

(2) *The fibers of q are Euclidean spaces on which $\mathbb{R}^k$ acts simply transitively by translations.*

(3) *The radical quotient X/R is isometric to $\mathbb{H}^n$, and the projection map onto the second factor of* (7-3) *corresponds to the radical projection q.*

Proof Write $\mathrm{Isom}(X)^0 = \mathsf{G} = \mathsf{R}\cdot\mathsf{S}$, as in Section 4.3. Put $\Gamma_0 = \Gamma \cap \mathsf{G}$. Then Γ_0 is a uniform lattice in G. Consider the projection homomorphism

$$\phi\colon \mathrm{Isom}(X) \to \mathrm{Isom}(X/\mathsf{R})$$

corresponding to the Riemannian submersion q. If the image

$$S_0 = \phi(\mathrm{Isom}(X)^0) = \phi(\mathsf{S})$$

is not the trivial group, it is a semisimple Lie group of noncompact type and $\phi(\Gamma_0)$ is a uniform lattice in S_0. (See Theorem 4.13(7).)

Denote with V the vector subgroup of $\mathrm{Isom}(X)$ which arises by the isometric action of $\mathbb{R}^k$ on X. By construction, V is normalized by Γ_0 and the image $\phi(V)$ in S_0 is normalized by the uniform lattice $\phi(\Gamma_0)$. By Borel's density theorem (applied to the adjoint form of S_0), S_0 does not contain any abelian connected subgroup normalized by $\phi(\Gamma_0)$. This shows that V is contained in $\ker\phi$. Consequently, V acts by isometries on each fiber of the projection q, and by construction the action is free.

Next, observe that the fibers of $\mathsf{q}\colon X \to X/\mathsf{R}$ are at most k–dimensional. Indeed, by Proposition 4.6,

$$\mathrm{rad}_{\mathsf{G}}(\Gamma) = \Gamma_0 \cap \ker\phi$$

is a virtually polycyclic normal subgroup, and it acts with compact quotient on each fiber $\mathsf{q}^{-1}(y)$. This shows that $\dim \mathsf{q}^{-1}(y) = \operatorname{rank}\mathrm{rad}_{\mathsf{G}}(\Gamma)$; see, for example, the argument given in the proof of Corollary 5.8. As observed in $(*)$ (following (7-1)), we must have $\mathrm{rad}_{\mathsf{G}}(\Gamma) \subseteq \mathbb{Z}^k$, and consequently $\operatorname{rank}\mathrm{rad}_{\mathsf{G}}(\Gamma) \le k$.

We deduce that V is a simply transitive group of isometries on each fiber $\mathsf{q}^{-1}(y)$. Since any left-invariant metric on a vector space is flat (and unique up to an affine transformation), this implies that $\mathsf{q}^{-1}(y)$ is isometric to an Euclidean space $\mathbb{E}^k$ on which V acts by translations. In particular, (2) holds.

Recall from Proposition 4.11 that $\ker\phi$ acts faithfully on $\mathsf{q}^{-1}(y)$ by isometries, so $\ker\phi$ embeds as a subgroup of the Euclidean group $\mathrm{Isom}(\mathbb{E}^k)$. It follows that the above translation group V is normal in $\ker\phi$, and it is the nilpotent radical of $(\ker\phi)^0$. Recall that a maximal simply connected normal subgroup R_0 of R, containing the nilpotent radical of R, acts simply transitively on the fibers. We conclude that $V = \mathsf{R}_0$. In particular (1), holds.

So far it is shown that the Riemannian submersions q and the projection to $\mathbb{H}^n$ have the same fibers. Clearly, any two Riemannian submersions $f_i\colon X \to B_i$ with the same fibers give rise to an isometry $h\colon B_1 \to B_2$ such that $f_2 = h \circ f_1$. Therefore, (3) holds. □

Next we turn to describe the continuous symmetries of arbitrary Γ–invariant metrics on X. We will show that these are mainly determined by the group extension (7-1).

7.2 Symmetry and rigidity of fiberings

A Riemannian submersion $\mathsf{q}\colon X \to Y$ will be called Γ–*compatible* if the following hold: $\Gamma \le \mathrm{Isom}(X)$ permutes the fibers of q, the image of the induced map $\phi\colon \Gamma \to \mathrm{Isom}(Y)$ is a discrete uniform subgroup, and the kernel of ϕ is virtually solvable.

Proposition 7.4 *Let $\mathsf{q}\colon X \to Y$ be a Γ–compatible Riemannian submersion such that Y is a homogeneous Riemannian space, with $\mathrm{Isom}(Y)^0$ semisimple of noncompact type. Then the adjoint form of $\mathrm{Isom}(Y)^0$ is isomorphic to $\mathrm{PSO}(n,1)$.*

Proof Put $S = \mathrm{Isom}(Y)^0$, $\Theta_\phi = \phi(\Gamma) \cap S$ and $\Gamma_0 = \phi^{-1}(\Theta_\phi)$. Note that Γ_0 is a finite-index normal subgroup in Γ. It follows that $\Lambda_0 = \Gamma_0 \cap \mathbb{Z}^k$ is the maximal solvable normal subgroup of Γ_0. In particular, Γ_0 satisfies an exact sequence $1 \to \Lambda_0 \to \Gamma_0 \to \Theta_0 \to 1$, analogous to (7-1), where Θ_0 is (isomorphic to) a uniform lattice in $\mathrm{PSO}(n,1)$.

Let $\widehat{\phi}\colon \Gamma_0 \to \widehat{S}$ denote the induced map, where $\widehat{S}$ is the adjoint form of S. Since the image $\phi(\Gamma)$ divides Y, $\phi(\Gamma) \cap S$ is a uniform lattice in S, and $\widehat{\phi}(\Gamma_0)$ is a uniform lattice in $\widehat{S}$.

Observe that $\ker\widehat{\phi}$ is an extension of an abelian group (the center of S intersected with $\phi(\Gamma_0)$) by $\ker\phi \cap \Gamma_0$, and the latter is a normal subgroup of Γ_0, and virtually solvable by assumption. By the remark $(*)$ following (7-1), $\ker\phi \cap \Gamma_0$ is contained in the characteristic central subgroup Λ_0 of Γ_0. In particular, $\ker\phi \cap \Gamma_0$ is abelian. Reiterating the argument, we conclude that, in fact, the image of $\ker\widehat{\phi}$ in Θ_0 must be trivial.

A symmetric argument with respect to the homomorphism $\widehat{\phi}\colon \Gamma_0 \to \widehat{\phi}(\Gamma_0)$ shows that $\ker\widehat{\phi} = \Lambda_0$. We conclude that there exists an isomorphism of uniform lattices

$$\bar{\phi}\colon \Theta_0 \to \widehat{\phi}(\Gamma_0).$$

If $n \geq 3$, the Mostow strong rigidity theorem [21, Theorem A$'$] states that $\bar{\phi}$ extends to an isomorphism $\mathrm{PSO}(n,1) \to \widehat{S}_0$. In the case $n = 2$, Θ_0 is a surface group, and so is $\widehat{\phi}(\Gamma_0)$. Therefore, $\widehat{S}_0$ is isomorphic to $\mathrm{PSO}(2,1)$. □

The symmetry properties of Γ–invariant metrics on X are tightly coupled to the group extension (7-1):

Theorem 7.5 (isometry group and extension class) *Let Θ be a torsion-free lattice in $\mathrm{PSO}(n,1)$. Assume that the extension class for Γ, defined by (7-1), has infinite order. Then the following hold for the manifold X/Γ:*

(1) *X/Γ admits a metric of large symmetry.*

(2) *For $n \geq 3$, X/Γ does not admit a locally homogeneous Riemannian metric.*

(3) *For $n = 2$, X/Γ admits the structure of a locally homogeneous Riemannian manifold. In particular, Γ embeds into a connected Lie group.*

Proof Now (1) is just a special case of Corollary 7.2.

For (2), suppose that there exists a Γ–invariant homogeneous Riemannian metric on X. In particular, $\mathsf{G} = \mathrm{Isom}(X)^0$ acts transitively on X. By Theorem 4.13, the radical projection $\mathsf{q}\colon X \to Y$ is Γ–compatible and the image $S_0 = \phi(\mathsf{G}) \leq \mathrm{Isom}(Y)$ is semisimple of noncompact type, and normal in $\mathrm{Isom}(Y)^0$. Since Γ is not solvable, it is clear that Y and S_0 are nontrivial. Since G acts transitively on X, S_0 acts transitively on Y. It is clear by now that S_0 is a Levi subgroup of $\mathrm{Isom}(Y)^0$. If necessary (see Example 6.4), we may repeat the process of dividing out the radical. In any case, there exists a Γ–compatible Riemannian submersion $\mathsf{q}\colon X \to Y$, where $S_0 = \phi(\mathsf{G}) = \mathrm{Isom}(Y)^0$ is semisimple of noncompact type and acts transitively on Y. Note that, by (possibly repeated) application of Proposition 4.6, the kernel of the natural map $\Gamma \to \mathrm{Isom}(Y)$ is virtually polycyclic (see [25, Chapter 1, Proposition 2]); in particular, it is virtually solvable. By Proposition 7.4, S_0 is locally isomorphic to $\mathrm{PSO}(n,1)$. So is the noncompact type semisimple part S of a Levi subgroup of G.

Now we are assuming $n \geq 3$. Therefore, S is, in fact, a finite cover of $\mathrm{PSO}(n,1)$, and from the beginning the image S_0 of the radical projection in $\mathrm{Isom}(Y) = \mathrm{Isom}(X/\mathsf{R})$ is $\mathrm{PSO}(n,1)$. As before, we write

$$\mathsf{G} = \mathrm{Isom}(X)^0 = \mathsf{R}\cdot\mathsf{S} = \mathsf{R}\mathcal{K}^0 S,$$

where $\mathcal{K}^0$ is compact semisimple. Also, by going down to a finite-index subgroup, we may assume that $\Gamma \subset \mathrm{Isom}(X)^0$. The inclusion of Γ then satisfies $\Gamma \cap (\mathsf{R}\cdot\mathcal{K}^0) = \mathbb{Z}^k$.

Since the nilpotent radical N of R is simply connected and S is a finite cover of $\mathrm{PSO}(n,1)$, it is not difficult to see that (up to dividing a finite group in the center), G is a linear group (see for example [10, Theorems 3(1) and 7]). Recall further that, by Corollary 3.7, the maximal compact normal semisimple subgroup of G is trivial. Moreover, by Theorem 4.13(2), the intersection of Γ with N is a lattice in N. Since the extension (7-1) is central, the Borel density theorem implies that N is centralized by S. Hence, S is a factor of G. Therefore, the deformation theorem of Mostow, Proposition 2.6, states that (a finite-index subgroup of) the lattice Γ can be deformed into a lattice Γ', where $\Gamma' = (\Gamma' \cap R\mathcal{K}^0)\cdot(\Gamma' \cap S)$.

It follows that a finite-index subgroup of Γ splits as a direct product with factor $\mathbb{Z}^k$. However, this is impossible, since the extension class for Γ is assumed to have infinite order. The contradiction shows that (2) holds.

For (3) recall that any torsion-free lattice $\widetilde{\Gamma}$ in $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ admits an exact sequence of the form $1 \to \mathbb{Z} \to \widetilde{\Gamma} \to \Theta \to 1$, where Θ is a lattice in $\mathrm{PSL}(2,\mathbb{R})$. The extension class of the exact sequence, is determined by the index of $\tilde{\Gamma} \cap Z$, where Z is the center of $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$. From another point of view, this identifies with the Euler-number of the associated circle bundle

$$\widetilde{\mathrm{PSL}}(2,\mathbb{R})/\tilde{\Gamma} \to \mathbb{H}^2/\Theta.$$

Such bundles are studied in detail in [16, Theorem 8.5(b)].

Since $\mathbb{H}^2/\Theta$ is an orientable surface, we have

$$H^2(\mathbb{H}^2/\Theta,\mathbb{Z}) = H^2(\Theta,\mathbb{Z}) \cong \mathbb{Z}.$$

In particular, in the case $k=1$, $n=2$, there exists for any extension class of infinite order a locally homogeneous orbifold X/Γ, where $X = \widetilde{\mathrm{PSL}}(2,\mathbb{R})$ and Γ embeds as a lattice in $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$. This proves (3) in the case $k=1$.

For $k \geq 2$, we sketch the proof as follows: Let $[f] \in H^2(\Theta,\mathbb{Z}^k)$ be an extension class. By the natural coefficient isomorphism

$$H^2(\Theta,\mathbb{Z}^k) = H^2(\Theta,\mathbb{Z})^k,$$

the class $[f]$ corresponds to extension classes $[f_i] \in H^2(\Theta,\mathbb{Z})$, and there exist corresponding central group extensions $\mathbb{Z} \to \Gamma_i \to \Theta$ for $i = 1,\dots,k$. We may now form the fibered product with respect to the maps $\Gamma_i \to \Theta$ to obtain an extension

$$1 \to \mathbb{Z}^k \to \Gamma \to \Theta \to 1 \tag{7-6}$$

representing the extension class $[f]$. For this recall that the fibered product $\Gamma \to \Theta$ is constructed as the preimage of the diagonal via the induced map

$$\Gamma_1 \times \cdots \times \Gamma_k \to \Theta^k. \tag{7-7}$$

In the same way we construct an associated aspherical manifold X/Γ which is a torus bundle over $\mathbb{H}^2/\Theta$ as the fibered product of spaces

$$\widetilde{\mathrm{PSL}}(2,\mathbb{R})/\tilde{\Gamma}_i \to \mathbb{H}^2/\Theta$$

occurring in the case $k=1$. From Example 6.4(I) recall that the groups Γ_i each embed as a uniform lattice into a Lie group of the form

$$\mathbb{R} \cdot_{\mathcal{Z}} \widetilde{\mathrm{PSL}}(2,\mathbb{R}),$$

which is an $\mathbb{R}$–bundle over $\mathrm{PSL}(2,\mathbb{R})$. Thus Γ embeds as a uniform lattice into the corresponding k–fold fibered product

$$1 \to \mathbb{R}^k \to \mathsf{G} \to \mathrm{PSL}(2,\mathbb{R}) \to 1 \tag{7-8}$$

and, moreover, X is a homogenous space for the connected Lie group G. □

Groups which satisfy the assumption of Theorem 7.5 are obtained by finding a discrete cocompact hyperbolic group Θ whose representative cocycle $[f] \in H^2(\Theta, \mathbb{Z}^k)$ is of infinite order. For example, there exists a compact hyperbolic 3–manifold $\mathbb{H}^3/\Theta$, whose Betti number b_1 is not zero in $H_1(\Theta, \mathbb{Z})$; see [18]. As $H_1(\Theta, \mathbb{Z}) \otimes \mathbb{Z}^k = H_1(\Theta, \mathbb{Z}^k) \cong H^2(\Theta; \mathbb{Z}^k)$, the fundamental group Θ has representative cocycles of infinite order in $H^2(\Theta, \mathbb{Z}^k)$.

Corollary 7.6 *There exists a compact aspherical Riemannian manifold X/Γ of dimension 4 that admits a complete infrasolv tower of length one which fibers over a three-dimensional hyperbolic manifold. Moreover, the manifold X/Γ does not admit any locally homogeneous Riemannian metric.*

8 Application to tori

An n–dimensional *exotic torus* τ is a compact smooth manifold homeomorphic to the standard n–torus T^n but not diffeomorphic to T^n. In this section we shall prove that *an exotic torus has no large symmetry* (compare Definition 1.1). In the proof we replace the infrasolv tower (5-2) with its associated *infranil tower*, which is obtained by naturally dissecting infrasolv orbibundles into a composition of infranil orbibundles.

8.1 Infrasolv orbibundle fiber over infranil orbibundles

Let

$$\mathsf{p}\colon X/\Gamma \to Y/\Theta$$

be an infrasolv fiber bundle with associated group extension

(8-1) $$1 \to \Delta \to \Gamma \to \Theta \to 1,$$

as in Definition 5.3, where $Y = X/\mathsf{R}$. The induced homomorphism

$$\phi\colon \mathrm{Isom}(X) \to \mathrm{Isom}(Y)$$

satisfies $\Delta = \ker\phi \cap \Gamma$. Recall that the bundle p is modeled on the maximal simply connected normal subgroup R_0 of R, which acts freely on X. If N is the nilpotent radical of $\mathsf{R} = \mathsf{R}_0\mathsf{T}$, then it is a simply connected characteristic subgroup with $\mathsf{N} \leq \mathsf{R}_0$. As N acts freely on X by Lemma 4.1, we may put

$$Z = X/\mathsf{N}, \quad \mathsf{V} = \mathsf{R}_0/\mathsf{N},$$

where Z carries the Riemannian quotient metric from X. Then the homomorphism ϕ factors over $\mathrm{Isom}(Z)$ as

$$\begin{tikzcd} \mathrm{Isom}(X) \geq \Gamma \arrow[rr, "\phi"] \arrow[dr, "\phi_1"'] & & \Theta \leq \mathrm{Isom}(Y) \\ & Q \leq \mathrm{Isom}(Z) \arrow[ur, "\phi_2"'] & \end{tikzcd} \tag{8-2}$$

As $\Gamma \leq \mathrm{Diff}(X, \mathsf{R}_0)$, note that $\phi_1(\Gamma) = Q \leq \mathrm{Diff}(Z, \mathsf{V})$. (Recall from Section 5.2 that $\mathrm{Diff}(X, \mathsf{R}_0)$ specifies the normalizer of R_0 in the diffeomorphism group of X.)

Proposition 8.1 *The infrasolv fiber bundle* $\mathsf{p}\colon X/\Gamma \to Y/\Theta$ *decomposes into a composition of an infranil bundle and an infra-abelian bundle*:

$$\begin{tikzcd} X/\Gamma \arrow[rr, "\mathsf{p}"] \arrow[dr, "\mathsf{p}_1"'] & & Y/\Theta \\ & Z/Q \arrow[ur, "\mathsf{p}_2"'] & \end{tikzcd}$$

Proof Let $\mathsf{N} \to X \xrightarrow{\mathsf{q}_1} Z$ be the principal bundle associated to the action of N on X. As $\Gamma \leq \mathrm{Diff}(X, \mathsf{R}_0)$, $\ker\phi_1 \cap \Gamma$ normalizes N and may be seen as a group of affine transformations of the fibers of q_1. This induces an embedding

$$\ker\phi_1 \cap \Gamma \leq \mathrm{Aff}(\mathsf{N})$$

such that $\ker\phi_1 \cap \Gamma$ acts properly discontinuously on N. If we note that $\mathsf{N} \cap \Gamma$ is a uniform lattice in N from Theorem 4.13(2), then $\mathsf{N} \cap \Gamma$ is a finite-index subgroup of $\ker\phi_1 \cap \Gamma$. Thus the quotient

$$\mathsf{N}/(\ker\phi_1 \cap \Gamma)$$

is an infranil orbifold. Also $G = \mathsf{N}\Gamma$ is a closed subgroup of $\mathrm{Isom}(X)$. As N is normal in G, we apply Lemma 2.7 to deduce that

$$G/\mathsf{N} = \Gamma/\,\Gamma \cap \mathsf{N}$$

acts properly (discontinuously) on $X/\mathsf{N} = Z$. In particular,

$$Q = \Gamma/(\ker\phi_1 \cap \Gamma)$$

acts properly discontinuously on Z with compact quotient. Thus,

$$\mathsf{p}_1\colon X/\Gamma \to Z/Q$$

is an infranil fiber bundle (compare the proof of Theorem 5.5).

Next observe that $\mathsf{p}_2\colon Z/Q \to Y/\Theta$ is a fiber bundle with V–geometry in the sense of Definition 5.3. By the commutative diagram (8-2), it is easy to see that $\phi_1(\Delta) = \ker\phi_2 \cap Q$. In particular, $\phi_1(\Delta)$ acts properly discontinuously on V by affine transformations, and the quotient $\mathsf{V}/\phi_1(\Delta)$ is a compact Hausdorff space.

From Proposition 4.11(2), $(\ker\phi)^0 = \mathsf{R}_0(\mathsf{T}\mathcal{K}^0)$. As $\Delta \le \ker\phi$ and $\ker\phi$ has finitely many components, $\Delta \cap (\ker\phi)^0$ is a finite-index subgroup of Δ. On the other hand, if we apply Proposition 4.7 to $\phi_1 : \mathrm{Isom}(X) \to \mathrm{Isom}(Z)$, then it is noted that $\mathsf{T}\mathcal{K} \le \ker\phi_1$. Since $\Delta \cap (\ker\phi)^0 \le \mathsf{R}_0(\mathsf{T}\mathcal{K}^0)$ and $\phi_1(\mathsf{R}_0) = \mathsf{V}$,

$$\phi_1(\Delta \cap (\ker\phi)^0) \le \mathsf{V}$$

is a discrete uniform subgroup of V. Hence, $\mathsf{V}/\phi_1(\Delta)$ is a compact Euclidean orbifold. As a consequence, $\mathsf{p}_2\colon Z/Q \to Y/\Theta$ is an infra-abelian bundle. □

8.2 Exotic tori

We come now to the proof of:

Theorem 8.2 *Let τ be an n–dimensional exotic torus. Then τ does not admit any Riemannian metric of* ***large symmetry****.*

Proof Put $\Gamma = \pi_1(\tau) \cong \mathbb{Z}^n$ and let X be the universal covering space of τ. Suppose that τ has *large symmetry*. This implies that there exists a tower of Riemannian submersions with locally homogeneous fibers

$$X/\Gamma \to X_1/\Gamma_1 \to \cdots \to X_{\ell-1}/\Gamma_{\ell-1} \to \{\mathrm{pt}\}. \tag{8-3}$$

As Γ_i divides X_i, applying Lemma 3.4 to each $\Gamma_i \le \mathrm{Isom}(X_i)$ shows that Γ_i is *torsion-free*; that is, Γ_i is a free abelian subgroup for $i = 0, \dots, \ell$. By Lemma 2.8, each $\Gamma_i \cap \mathrm{Isom}(X_i)^0$ is a uniform abelian subgroup in $\mathrm{Isom}(X_i)^0$ and therefore Proposition 2.5 implies

$$\mathrm{Isom}(X_i)^0 = \mathsf{R}_i\mathsf{K}_i,$$

where R_i is the solvable radical and K_i is a compact connected semisimple group. Also, since $\Gamma_i \cap \mathrm{Isom}(X_i)^0$ is torsion-free, the radical R_i is nontrivial.

By the definition of Riemannian orbifold fibration (Definition 1.1), for each i, there exists a connected subgroup $\mathsf{L}_i \le \mathrm{Isom}(X_i)^0$ normalized by Γ_i such that $X_{i+1} = X_i/\mathsf{L}_i$ and $\Gamma_i \cap \mathsf{L}_i$ is a uniform lattice in L_i. As above, $\mathsf{L}_i = \mathsf{R}'_i\mathsf{K}'_i$ must be an extension of a compact group by a nontrivial solvable Lie group. In view of Theorem 4.13(3), (6), $X_{i+1} = X_i/\mathsf{L}_i = X_i/\mathsf{R}_i'$ is contractible and a radical quotient. Now Theorem 5.5 implies that each map $X_i/\,\Gamma_i \to X_{i+1}/\,\Gamma_{i+1}$ is an infrasolv fiber space in the sense of Definition 5.3. Therefore, (8-3) is an *infrasolv tower*.

By Proposition 8.1, each infrasolv bundle $X_i/\,\Gamma_i \to X_{i+1}/\,\Gamma_{i+1}$ dissects into infranil bundles

$$X_i/\,\Gamma_i \to Z_i/Q_i \to X_{i+1}/\,\Gamma_{i+1}. \tag{8-4}$$

Since Q_i acts properly discontinuously and Z_i/Q_i is compact, note that K_i acts trivially on $Z_i = X_i/\mathsf{N}_i$ as in the proof of Proposition 4.7.

Inserting Z_i/Q_i to the sequence (8-3), we may assume from the beginning that the tower (8-3) is an infranil tower.

With this assumption in place, we put $\Delta_i = \ker\phi_i \cap \Gamma_i$, where $\phi_i\colon \mathrm{Isom}(X_i) \to \mathrm{Isom}(X_{i+1})$ denotes the natural map for $i = 0, \dots, \ell-1$. Since the tower is infranil, Δ_i is contained in $\mathrm{Aff}(\mathsf{N}_i)$, and the holonomy image of Δ_i has compact closure in $\mathrm{Aut}(\mathsf{N}_i)$. On the other hand, $\mathsf{N}_i \cap \Gamma_i$ is a uniform lattice in N_i by Theorem 4.13(2). Since $\mathsf{N}_i \cap \Gamma_i$ is free abelian, N_i is isomorphic to the vector space $\mathbb{R}^{n_i}$ for $n_i = \dim \mathsf{N}_i$. Then Δ_i is a Bieberbach group in the Euclidean group $\mathrm{E}(\mathbb{R}^{n_i})$. As Δ_i is free abelian, we have $\Delta_i = \mathbb{R}^{n_i} \cap \Gamma_i$.

As Γ_i *normalizes* $\mathbb{R}^{n_i}$, associated with the group extension

$$1 \to \Delta_i \to \Gamma_i \to \Gamma_{i+1} \to 1$$

there is an *injective Seifert fibering* (see [17])

$$\mathbb{R}^{n_i}/\Delta_i \to X_i/\,\Gamma_i \xrightarrow{p_i} X_{i+1}/\,\Gamma_{i+1}, \tag{8-5}$$

where $\mathbb{R}^{n_i}/\Delta_i$ is the standard n_i–torus.

Since $X_{\ell-1}/\,\Gamma_{\ell-1} \to X_\ell = \{\mathrm{pt}\}$ is also a Seifert fibering, $\mathbb{R}^{n_{\ell-1}}/\Delta_{n_{\ell-1}} = X_{\ell-1}/\,\Gamma_{\ell-1}$ is an $n_{\ell-1}$–torus. Assume inductively from (8-5) that $X_1/\,\Gamma_1$ is diffeomorphic to $T^{n_1} = \mathbb{R}^{n_1}/\mathbb{Z}^{n_1}$. Let $\bar\phi\colon \Gamma_1 \to \mathbb{Z}^{n_1}$ be an isomorphism induced by an equivariant diffeomorphism $\bar\varphi\colon X_1 \to \mathbb{R}^{n_1}$. Then we have an isomorphism $\phi\colon \Gamma \to \Delta_0 \times \mathbb{Z}^{n_1} \cong \mathbb{Z}^{n_0+n_1}$ which makes the diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta_0 & \longrightarrow & \Gamma & \longrightarrow & \Gamma_1 & \longrightarrow & 1 \\
 & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle \phi} & & \Big\downarrow{\scriptstyle \bar\phi} & & \\
1 & \longrightarrow & \Delta_0 & \longrightarrow & \Delta_0 \times \mathbb{Z}^{n_1} & \longrightarrow & \mathbb{Z}^{n_1} & \longrightarrow & 1
\end{array} \tag{8-6}$$

commutative. By the *Lee–Raymond–Seifert rigidity* for abelian fiber [17], there exists a fiber-preserving equivariant diffeomorphism

$$(\phi, \varphi)\colon (\Gamma, X) \to (\mathbb{Z}^n, \mathbb{R}^n).$$

Hence, $X/\,\Gamma$ is diffeomorphic to the torus T^n $(n = n_0 + n_1)$. This proves the induction step and finishes the proof. □

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