



Geometry & Topology

Volume 27 (2023)

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We consider the connected component of the identity of G -character varieties of compact Riemann surfaces of genus $g > 0$ for connected complex reductive groups G of type A (eg SL_n and GL_n). We show that these varieties are \mathbb{Q} -factorial symplectic singularities and classify which admit symplectic resolutions. The classification reduces to the semisimple case, where we show that a resolution exists if and only if either $g = 1$ and G is a product of special linear groups of any rank and copies of the group PGL_2 , or $g = 2$ and $G = (SL_2)^m$ for some m .

14D20, 16D20; 16S80, 17B63

1 Introduction

The character varieties associated to the fundamental group of a topological space have long been objects of study for topologists, group theorists and algebraic geometers. The character varieties of reductive groups associated to the fundamental group of a Riemann surface play a particularly prominent role in this theory since it has been shown by Goldman that their smooth locus carries a natural symplectic structure (which is complex algebraic, and in particular holomorphic). The aim of this article is to study how this symplectic structure degenerates along the singular locus of the character variety. This is motivated by earlier work [9] of the authors, where the case of quiver varieties is considered. Our results for character varieties are based, in large part, on the ideas developed in [9] but the proofs given here are independent of the latter.

Before explaining the main results, we introduce some notation. Let Σ be a compact Riemann surface of genus $g > 0$ and π its fundamental group. Let G be a connected complex reductive group whose simple quotients all have type A (ie are all isomorphic to PGL_{n_i} for some n_i). The G -character variety of Σ is the affine quotient $\text{Hom}(\pi, G)//G$.

In general, by Li [37], $\text{Hom}(\pi, G)$ is not connected. We will only consider the connected component of the identity, $\text{Hom}(\pi, G)^\circ$, and set

$$\mathcal{Y}(G, g) := \text{Hom}(\pi, G)^\circ // G.$$

We will refer to $\mathcal{Y}(G, g)$ as the *connected G -character variety*. Note that, in the case that $G = \text{GL}_n$ or SL_n , we have $\text{Hom}(\pi, G) = \text{Hom}(\pi, G)^\circ$. We do not consider the case where Σ has punctures; in this case it is natural to impose conditions on the monodromy about the punctures and this situation is addressed by Schedler and Tirelli [44].

Goldman [27] shows that the symplectic structure on the smooth locus extends to a Poisson structure on the whole complex algebraic variety. An important (in both symplectic algebraic geometry and geometric representation theory) class of Poisson varieties with generically nondegenerate Poisson structure are those with symplectic singularities in the sense of Beauville [6]. To have symplectic singularities requires a strong compatibility between the symplectic form and resolutions of singularities. However, once one knows that a given space has symplectic singularities, it allows one to make strong statements about the quantizations and Poisson deformations of the space. For instance, Namikawa [42] tells us that the (formal) Poisson deformations of such a space are unobstructed, if it is an affine variety. Our first result shows that:

Theorem 1.1 *The irreducible variety $\mathcal{Y}(G, g)$ has symplectic singularities.*

Since, by [6, Proposition 1.3], all symplectic singularities are rational Gorenstein (meaning Gorenstein with rational singularities, and not meaning the varieties to be rational), we deduce:

Corollary 1.2 *The variety $\mathcal{Y}(G, g)$ has rational Gorenstein singularities.*

The fact that the character variety has rational singularities was established for large genus by Aizenbud and Avni [1]. **Corollary 1.2** shows that the bound on the genus g appearing in [1, Theorem IX] is not required, at least when G is of type A . Budur and Zordan [15; 16] show that $\text{Hom}(\pi, \text{GL}_n)$ and $\text{Hom}(\pi, \text{SL}_n)$ also have rational singularities.

Theorem 1.1 immediately raises the question of whether the character varieties admit (complex algebraic) symplectic resolutions; these are resolutions of singularities where the symplectic form on the smooth locus of the singularity extends to a symplectic form on the whole of the resolution. This is a very strong condition, and symplectic resolutions are correspondingly rare. Our main result is a complete classification of when these character varieties admit symplectic resolutions. As usual in these situations,

much of the effort is spent in proving that certain obstructions to the existence of symplectic resolutions exist in many examples. Such an obstruction is known to exist when the variety is singular, \mathbb{Q} -factorial and terminal.

Using Drézet's theorem [19] on (local) factoriality of GIT quotients, we show that:

Theorem 1.3 *The variety $\mathcal{Y}(G, g)$ is \mathbb{Q} -factorial. In the case of $g > 1$ and $(n, g) \neq (2, 2)$, it is moreover locally factorial for $G = \mathrm{GL}_n$ or SL_n .*

Here “locally factorial” means that the local rings are unique factorization domains, which is equivalent to saying that every Weil divisor is Cartier; \mathbb{Q} -factorial is the weaker condition that some multiple of every Weil divisor is Cartier.

To apply [Theorem 1.3](#) to the question of existence of symplectic resolutions, we need to estimate the codimension of the singular locus. Thanks to [Theorem 1.1](#) together with Kaledin [30, Theorem 2.3], the singular locus of $\mathcal{Y}(G, g)$ is a finite union of even-dimensional strata: the symplectic leaves. In the case of $G = \mathrm{GL}_n$, this is proved by Simpson [47] and by Sikora [45] when $g = 1$; by our methods, we can also reduce the general case to these ones. Note that, if G is abelian, ie a torus, then $\mathcal{Y}(G, g) \cong G^{2g}$, which is not interesting for our purposes.

Proposition 1.4 *Assume that G is nonabelian. Then $\mathcal{Y}(G, g)$ is singular. Its singular locus has codimension at least four if and only if $g > 1$ and, in the case $g = 2$, no simple quotient of G is isomorphic to PGL_2 .*

Thanks to Namikawa [41], when a symplectic singularity has singular locus of codimension at least four, it has terminal singularities. On the other hand, it is well known that a singular \mathbb{Q} -factorial terminal variety does not admit a proper crepant resolution, and hence not a symplectic one; see eg the proof of [9, Theorem 6.13]. Thus, [Theorem 1.3](#) and [Proposition 1.4](#) immediately imply:

Corollary 1.5 *Assume G is nonabelian, $g > 1$ and, if $g = 2$, then no simple quotient of G is isomorphic to PGL_2 . Then $\mathcal{Y}(G, g)$ does not admit a proper symplectic resolution. The same holds for any singular open subset.*

In particular, the above implies that, for $G = \mathrm{SL}_n, \mathrm{GL}_n$ or PGL_n , the character variety $\mathcal{Y}(G, g)$ does not admit a symplectic resolution when $n, g \geq 2$ and $(n, g) \neq (2, 2)$. In these cases we give a description of all the symplectic leaves, and hence the stratification of the singular locus into iterated singular loci (all even-dimensional).

Remark 1.6 In the case $G = \mathrm{GL}_n$, parallel to [9, Remark 1.14], we can give an alternative proof of the first statement of Corollary 1.5 using formal localization, reducing to the quiver variety case which is considered in that article. The formal neighborhood of the identity of $\mathcal{Y}(\mathrm{GL}_n, g)$ is well known to identify with the formal neighborhood of $(0, \dots, 0)$ in the quotient

$$\left\{ (X_1, Y_1, \dots, X_g, Y_g) \in \mathfrak{gl}_n \mid \sum_{i=1}^d [X_i, Y_i] = 0 \right\} // \mathrm{GL}_n.$$

See Proposition 3.14 for more details. The above quotient is the quiver variety associated to the quiver Q with one vertex and g arrows. However, we cannot directly conclude Theorem 1.3 using formal localization, and neither the stronger last statement of Corollary 1.5.

Remark 1.7 As in the discussion after Corollary 1.11 of [9], one can obtain singular open subsets $U \subseteq \mathcal{Y}(\mathrm{GL}_n, g)$ in the case $g > 1$ and $(n, g) \neq (2, 2)$ for which the formal neighborhood of every point does admit a resolution, even though the entire U does not admit one by Corollary 1.5. Indeed, as mentioned in the preceding remark, the formal neighborhoods identify with those of quiver varieties. Hence, one example is given analogously to the one in [9] (after Corollary 1.11 therein): for $n = 2$ and $g \geq 3$, take U to be the complement of the locus of representations of the form $Y^{\oplus 2}$ for Y one-dimensional (and hence irreducible). This locus is singular, terminal and locally factorial, but it is not formally locally \mathbb{Q} -factorial.

Remark 1.8 The question of whether the character variety

$$X(G, m) := \mathrm{Hom}(F_m, G) // G \cong G^m // G$$

associated to a free group (the “open surface” case) is Gorenstein or (locally) factorial has been considered by Lawton and Manon [35]. Namely they show that X is Gorenstein for all connected reductive complex G , and that X is locally factorial when $[G, G]$ is simply connected. It follows that X is \mathbb{Q} -factorial for all connected reductive G : as in Section 1.2 below, write $G \cong (T \times K) / Z$ for $Z < (T \times K)$ a finite central subgroup, and K simply connected semisimple; we may take K to be the universal cover of $[G, G]$. It follows that $X(G, m)$ is a finite quotient of $X(K, m) \times T^m$, the latter being locally factorial by Boissière, Gabber and Serman [13], and hence $X(G, m)$ is \mathbb{Q} -factorial. Note that, thanks to Boutot [14], $X(G, m)$ also has rational singularities (being a reductive quotient of a smooth variety), parallel to Corollary 1.2. Summarizing, for open surfaces, character varieties of connected reductive groups are \mathbb{Q} -factorial rational Gorenstein singularities, just as we show here for closed surfaces with groups of type A .

1.1 Construction of symplectic resolutions

As the preceding has made clear, in most cases there will not be a symplectic resolution. Let us explain how to construct the ones that do exist.

First suppose that the genus is one. For the general linear group, $\mathcal{Y}(\mathrm{GL}_n, 1)$ is well known to be isomorphic to the symmetric product $S^n(\mathbb{C}^\times \times \mathbb{C}^\times) := (\mathbb{C}^\times \times \mathbb{C}^\times)^n / S_n$ (see eg Joseph [29] and Gan and Ginzburg [24], as explained in Section 2.4 below). The Hilbert scheme produces a symplectic resolution, $\mathrm{Hilb}^n(\mathbb{C}^\times \times \mathbb{C}^\times) \rightarrow \mathcal{Y}(\mathrm{GL}_n, 1)$, by the Hilbert–Chow map. That $\mathrm{Hilb}^n(\mathbb{C}^\times \times \mathbb{C}^\times)$ is symplectic, owing to the fact that $\mathbb{C}^\times \times \mathbb{C}^\times$ is a holomorphic symplectic surface, was observed by Beauville [5]. For $\mathrm{SL}_n < \mathrm{GL}_n$, we have $\mathcal{Y}(\mathrm{SL}_n, 1) \subset \mathcal{Y}(\mathrm{GL}_n, 1)$ and the restriction of the Hilbert–Chow morphism to the preimage of $\mathcal{Y}(\mathrm{SL}_n, 1)$ is a symplectic resolution of the latter. This preimage is the smooth symplectic subvariety $\mathrm{Hilb}_0^n(\mathbb{C}^\times \times \mathbb{C}^\times) \subseteq \mathrm{Hilb}^n(\mathbb{C}^\times \times \mathbb{C}^\times)$, of schemes of length n in $\mathbb{C}^\times \times \mathbb{C}^\times$ such that the product of all pairs in the support is $(1, 1)$. See Section 2.4 for more details on these facts.

Similarly, we can produce a resolution for the group PGL_2 : here one obtains that $\mathcal{Y}(\mathrm{PGL}_2, 1) \cong \mathcal{Y}(\mathrm{SL}_2, 1) / \mathbb{Z}_2^2$, which allows us to produce a partial resolution via $\mathrm{Hilb}_0^2(\mathbb{C}^\times \times \mathbb{C}^\times) / \mathbb{Z}_2^2$, and finally blow it up to produce a symplectic resolution.

Finally, for genus at least two, there is one exceptional case where we have a resolution, namely for $\mathcal{Y}(\mathrm{GL}_2, 2)$, and hence also for $\mathcal{Y}(\mathrm{SL}_2, 2)$: in these cases a resolution was constructed by Lehn and Sorger [36] (in a slightly different context). Again, in this case it turns out one only needs to blow up the reduced singular locus. Put together, we obtain the following:

Theorem 1.9 (i) For $g = 1$, we have the projective symplectic resolutions

$$(1) \quad \mathrm{Hilb}^n(\mathbb{C}^\times \times \mathbb{C}^\times) \rightarrow \mathcal{Y}(\mathrm{GL}_n, 1), \quad \mathrm{Hilb}_0^n(\mathbb{C}^\times \times \mathbb{C}^\times) \rightarrow \mathcal{Y}(\mathrm{SL}_n, 1),$$

given by the Hilbert–Chow morphism, and

$$(2) \quad \widetilde{\mathrm{Hilb}_0^2(\mathbb{C}^\times \times \mathbb{C}^\times) / \mathbb{Z}_2^2} \rightarrow \mathcal{Y}(\mathrm{PGL}_2, 1),$$

given by blowing up the (reduced) singular locus (type A_1 surface singularities only).

(ii) For $g = 2$, we have the symplectic resolutions

$$(3) \quad \tilde{\mathcal{Y}}(\mathrm{GL}_2, 2) \rightarrow \mathcal{Y}(\mathrm{GL}_2, 2), \quad \tilde{\mathcal{Y}}(\mathrm{SL}_2, 2) \rightarrow \mathcal{Y}(\mathrm{SL}_2, 2),$$

obtained by blowing up the reduced singular locus.

The theorem will be proved in Sections 2.3 and 2.4. In Section 3.1, we explain the closest approximation to such a resolution that exists in general.

1.2 Classification of symplectic resolutions

Many of the preceding results (in particular Theorems 1.1 and 1.3) reduce easily to the case of general and special linear groups, handled in Section 2. We now explain how to strengthen these results to a full classification of character varieties $\mathcal{Y}(G, g)$ admitting symplectic resolutions for G a (necessarily connected) reductive group whose simple quotients are all of type A . The first step is to reduce to the case where G is semisimple. Here “semisimple” means reductive with finite center.

Note that a general group G as above has the form

$$1 \rightarrow Z \rightarrow H \times K \rightarrow G \rightarrow 1,$$

where H is a torus, K is a semisimple connected group and $Z < H \times Z(K)$ is finite. Moreover, K can be chosen so that the composition $Z \rightarrow H \times K \rightarrow H$ is injective. Since H is central, this implies that Z^{2g} acts freely on $\mathcal{Y}(H \times K, g) = \mathcal{Y}(H, g) \times \mathcal{Y}(K, g)$. We have $\mathcal{Y}(G, g) \cong \mathcal{Y}(H \times K, g)/Z^{2g}$ (see Section 3.2), and the two have the same singularities. So, for most of our purposes, we can replace G by K .

In these terms our main result on symplectic resolutions takes the following general form:

Theorem 1.10 *Assume that G is nonabelian. The following are equivalent:*

- (a) *The character variety $\mathcal{Y}(G, g)$ admits a projective symplectic resolution.*
- (b) *The character variety $\mathcal{Y}(K, g)$ admits a projective symplectic resolution.*
- (c) *One of the following two conditions holds:*
 - (i) *$g = 1$ and $K \cong \prod_j \mathrm{SL}_{n_j} \times \mathrm{PGL}_2^m$ for some $m \geq 0$.*
 - (ii) *$g = 2$ and $K \cong \mathrm{SL}_2^m$ for some $m \geq 1$.*

In the cases in part (c), a resolution can be constructed thanks to Theorem 1.9. Namely, by taking products, we get a resolution $\tilde{\mathcal{Y}}(K, g) \rightarrow \mathcal{Y}(K, g)$, and then

$$(\mathcal{Y}(H, g) \times \tilde{\mathcal{Y}}(K, g))/Z^{2g}$$

produces a resolution of $\mathcal{Y}(G, g)$.

We first prove these results for $G = \mathrm{GL}_n$ or SL_n , and then in Section 3 we deduce the results for arbitrary type A reductive groups. Similar techniques are applicable to

Hitchin's moduli spaces of semistable Higgs bundles over smooth projective curves; see Tirelli [48].

Our classification of character varieties of type A admitting symplectic resolutions has recently been used by Felisetti and Mauri [20] to prove the $P = W$ conjecture in all cases where a symplectic resolution exists.

Conventions

Throughout, a variety will mean a reduced, quasiprojective scheme of finite type over \mathbb{C} . If X is a (quasiprojective) variety equipped with the action of a reductive algebraic group G , then $X//G$ will denote the good quotient (when it exists). In this case, let $\xi: X \rightarrow X//G$ denote the quotient map. Then each fiber $\xi^{-1}(x)$ contains a unique closed G -orbit. Following Luna, this closed orbit is denoted $T(x)$.

Acknowledgments

Bellamy was partially supported by EPSRC grant EP/N005058/1. Schedler was partially supported by NSF grant DMS-1406553. The authors are grateful to the University of Glasgow for the hospitality provided during the workshop *Symplectic representation theory*, where part of this work was done, and Schedler to the 2015 Park City Mathematics Institute as well as to the Max Planck Institute for Mathematics for excellent working environments.

We would like to thank Victor Ginzburg for suggesting we consider character varieties. We would also like to thank David Jordan, Johan Martens and Ben Martin for their many explanations regarding character varieties.

Finally, we would like to thank the referee for a detailed reading of the paper and several suggestions that have improved the exposition.

2 Character varieties for general and special linear groups

Recall from the introduction that Σ is a compact Riemannian surface of genus $g > 0$ and π is its fundamental group. For a connected reductive group G we have defined the character varieties

$$\mathfrak{Y}(G, g) = \text{Hom}(\pi, G)^\circ // G,$$

where $\text{Hom}(\pi, G)^\circ \subset \text{Hom}(\pi, G)$ is the connected component containing the trivial representation. These are affine varieties. By [37], $\text{Hom}(\pi, G)^\circ = \text{Hom}(\pi, G)$ for both GL_n and SL_n .

2.1 Quasi-Hamiltonian reduction

In this section we will only consider $G = \mathrm{GL}_n$ or SL_n . If $g > 1$ then the (complex) dimension of $\mathcal{Y}(\mathrm{SL}_n, g)$ is $2(g-1)(n^2-1)$, and when $g = 1$, it has dimension $2(n-1)$. On the other hand $\dim \mathcal{Y}(\mathrm{GL}_n, g) = \dim \mathcal{Y}(\mathrm{SL}_n, g) + 2g$ always. We begin by recalling the basic properties of the affine varieties $\mathrm{Hom}(\pi, \mathrm{GL})$ and $\mathcal{Y}(\mathrm{GL}_n, g)$:

Theorem 2.1 [47, Section 11; 43] *Assume $g > 1$.*

- (1) *Both $\mathrm{Hom}(\pi, \mathrm{GL}_n)$ and $\mathcal{Y}(\mathrm{GL}_n, g)$ are reduced, irreducible and normal.*
- (2) *$\mathrm{Hom}(\pi, \mathrm{GL}_n)$ is a complete intersection in GL_n^{2g} .*
- (3) *The generic points of $\mathrm{Hom}(\pi, \mathrm{GL}_n)$ and $\mathcal{Y}(\mathrm{GL}_n, g)$ correspond to irreducible representations of the fundamental group π .*

As shown originally by Goldman [26], the variety X has a natural Poisson structure. This Poisson structure becomes clear in the realization of these spaces as quasi-Hamiltonian reductions; see [2], where it is shown that the symplectic structure defined by Goldman on the smooth locus of X agrees with the Poisson structure of X as a quasi-Hamiltonian reduction. In particular, if $C_{(1,n)}$ denotes the dense open subset of X parametrizing simple representations of π , then it is shown in [2] that the Poisson structure on $C_{(1,n)}$ is nondegenerate.

It will be useful for us to reinterpret the quasi-Hamiltonian reduction $\mathcal{Y}(\mathrm{GL}_n, g)$ as a moduli space of semisimple representations of the multiplicative preprojective algebra. Let Q be the quiver with a single vertex and g loops, labeled a_1, \dots, a_g . Let a_i^* denote the loop dual to a_i in the doubled quiver \bar{Q} . Associated to Q is the multiplicative preprojective algebra $\Lambda(Q)$, as defined in [18]. Namely, $\mathbb{C}\bar{Q} \rightarrow \Lambda(Q)$ is the universal homomorphism such that each $1 + a_i a_i^*$ and $1 + a_i^* a_i$ is invertible and

$$\prod_{i=1}^g (1 + a_i a_i^*)(1 + a_i^* a_i)^{-1} = 1.$$

Here the product is ordered. Following [17], let $\Lambda(Q)'$ denote the universal localization of $\Lambda(Q)$, where each a_i is also required to be invertible. Let $(T^* \mathrm{Rep}(Q, n))^\circ$ denote the space of all n -dimensional representations (A_i, A_i^*) of $\mathbb{C}\bar{Q}$ such that $1 + A_i A_i^*$, $1 + A_i^* A_i$ and A_i are invertible for all i . It is an open, GL_n -stable affine subset of $T^* \mathrm{Rep}(Q, n)$. The action of GL_n on $(T^* \mathrm{Rep}(Q, n))^\circ$ is quasi-Hamiltonian, with multiplicative moment map

$$\Psi: \mathrm{Rep}(\Lambda(Q)', n) \rightarrow \mathrm{GL}_n, \quad (A_i, A_i^*) \mapsto \prod_{i=1}^g (1 + A_i A_i^*)(1 + A_i^* A_i)^{-1}.$$

As noted in [17, Proposition 2], the category $\Lambda(Q)'$ -mod of finite dimensional $\Lambda(Q)'$ -modules is equivalent to π -mod, in such a way that we have a GL -equivariant identification

$$\Psi^{-1}(1) \xrightarrow{\sim} \mathrm{Hom}(\pi, \mathrm{GL}_n), \quad (A_i, A_i^*) \mapsto (A_i, B_i) = (A_i, A_i^{-1} + A_i^*).$$

Hence, we have an identification of Poisson varieties

$$\Psi^{-1}(1) // \mathrm{GL}_n = \mathcal{Y}(\mathrm{GL}_n, g).$$

See [10] for further details.

2.2 Symplectic singularities

We recall from [6, Definition 1.1] that a variety X is said to be a symplectic singularity if it is normal, its smooth locus has a symplectic 2-form ω , and, for any resolution of singularities $f: Y \rightarrow X$, the rational 2-form $f^*\omega$ is regular. Moreover, f is said to be a symplectic resolution if the 2-form $f^*\omega$ is also nondegenerate. In particular, this makes Y an algebraic symplectic manifold.

The space $\mathcal{Y}(\mathrm{GL}_n, g)$ has a stratification by representation type, which is also the stratification by stabilizer type; see [39, Theorem 5.4]. We say that a *weighted partition* ν of n is a sequence $(\ell_1, \nu_1; \dots; \ell_k, \nu_k)$, where each ℓ_i and ν_i is a positive integer and

$$\nu_1 \geq \nu_2 \geq \dots, \quad \sum_{i=1}^k \ell_i \nu_i = n.$$

Lemma 2.2 *Assume $n, g > 1$.*

(1) *The strata C_ν of $\mathcal{Y}(\mathrm{GL}_n, g)$ are labeled by weighted partitions of n such that*

$$\dim C_\nu = 2 \left(k + (g-1) \sum_{i=1}^k \nu_i^2 \right).$$

(2) *If $(n, g) \neq (2, 2)$, then $\dim \mathcal{Y}(\mathrm{GL}_n, g) - \dim C_\nu \geq 4$ for all $\nu \neq (1, n)$.*

(3) *If $(n, g) \neq (2, 2)$ and $\nu \neq (1, n)$, then $\dim \mathcal{Y}(\mathrm{GL}_n, g) - \dim C_\nu \geq 8$ unless either*

- (i) *$(n, g) = (3, 2)$ and $\nu = (1, 2; 1, 1)$, or*
- (ii) *$(n, g) = (2, 3)$ and $\nu = (1, 1; 1, 1)$.*

Proof By [Theorem 2.1](#), the set of points $C_{(1,n)}$ in $\mathcal{Y}(\mathrm{GL}_n, g)$ parametrizing irreducible representations of π is a dense open subset contained in the smooth locus. Therefore, $\dim C_{(1,n)} = 2(1 + n^2(g - 1))$. An arbitrary semisimple representation of π of dimension n has the form $x = x_1^{\oplus \ell_1} \oplus \cdots \oplus x_k^{\oplus \ell_k}$, where the x_i are pairwise nonisomorphic irreducible π -modules of dimension v_i and $n = \sum_{i=1}^k \ell_i v_i$. Thus, the representation type strata correspond to weighted partitions of n . Let C_ν denote the locally closed subvariety of all such representations. Though the x_i are pairwise distinct, there will generally exist i and j for which $v_i = v_j$. Reordering if necessary, we write the multiset $\{\{v_1, \dots, v_k\}\}$ as $\{\{m_1 \cdot v_1, \dots, m_r \cdot v_r\}\}$, with $v_i \neq v_j$ and $r \leq k$. Then

$$C_\nu \cong S^{m_1, \circ} C_{(1, v_1)} \times \cdots \times S^{m_r, \circ} C_{(1, v_r)},$$

where $S^{n, \circ} X$ is the open subset of $S^n X$ consisting of n pairwise distinct points. Thus,

$$\dim C_\nu = \sum_{i=1}^r 2(1 + v_i^2(g - 1))m_i = 2\left(k + (g - 1) \sum_{i=1}^k v_i^2\right).$$

For the second part, notice that

$$\begin{aligned} (4) \quad \dim \mathcal{Y}(\mathrm{GL}_n, g) - \dim C_\nu &= 2(n^2(g - 1) + 1) - 2 \sum_{i=1}^k (v_i^2(g - 1) + 1) \\ &= 2(g - 1) \sum_{i,j=1}^k (\ell_i \ell_j - \delta_{i,j}) v_i v_j - 2(k - 1). \end{aligned}$$

Since $\sum_{i,j=1}^k (\ell_i \ell_j - \delta_{i,j}) v_i v_j - (k - 1) \geq 1$, we clearly have $\dim \mathcal{Y}(\mathrm{GL}_n, g) - \dim C_\nu \geq 4$ when $g > 2$. When $g = 2$, a simple computation shows that $\dim \mathcal{Y}(\mathrm{GL}_n, g) - \dim C_\nu = 2$ if and only if $n = 2$ and $\nu = (1, 1; 1, 1)$.

For the third part, we use again (4), noticing the following points: the right-hand side of (4) is increasing in g ; the right-hand side is increased if we replace (ℓ_i, n_i) by $(\ell_i - 1, n_i; 1, n_i)$; the right-hand side is increased if we replace $(1, a)$ and $(1, b)$ by $(1, a + b)$ (when $a + b < n$); and for $a > b > 1$, the right-hand side is increased if we replace $(1, a)$ and $(1, b)$ by $(1, a + 1)$ and $(1, b - 1)$. Since it suffices to prove the inequality after performing operations that increase the right-hand side, the result follows once we observe that the inequality holds in the following cases:

- (i) for $\nu = (1, n - 1; 1, 1)$ whenever $n \geq 4$ as well as $(1, 1; 1, 1; 1, 1)$;
- (ii) for $\nu = (1, 1; 1, 1)$ whenever $g \geq 4$, as well as $\nu = (2, 1)$ for $g = 3$. □

Recall that $\mathcal{Y}(\mathrm{SL}_n, g)$ is the character variety associated to the compact Riemann surface Σ , of genus g , with values in SL_n . Let $T \cong (\mathbb{C}^\times)^{2g}$ denote the $2g$ -torus. Let $\varrho: \mathrm{Hom}(\pi, \mathrm{GL}) \rightarrow T$ be the map sending (A_i, B_i) to $(\det(A_i), \det(B_i))$. This map is GL_n -equivariant, where the action on T is trivial. Moreover, it fits into a commutative diagram of GL_n -varieties

$$(5) \quad \begin{array}{ccc} \mathrm{Hom}(\pi, \mathrm{SL}_n) \times_{\mathbb{Z}_n^{2g}} T & \xrightarrow{\sim} & \mathrm{Hom}(\pi, \mathrm{GL}_n) \\ & \searrow \mathrm{pr} & \swarrow \varrho \\ & T & \end{array}$$

where \mathbb{Z}_n^{2g} acts freely on T , and the map $\mathrm{Hom}(\pi, \mathrm{SL}_n) \times T \rightarrow \mathrm{Hom}(\pi, \mathrm{GL}_n)$ sends $((A_i, B_i), (t_i, s_i))$ to $(t_i A_i, s_i B_i)$. Therefore it descends to a commutative diagram

$$(6) \quad \begin{array}{ccc} \mathcal{Y}(\mathrm{SL}_n, g) \times_{\mathbb{Z}_n^{2g}} T & \xrightarrow{\sim} & \mathcal{Y}(\mathrm{GL}_n, g) \\ & \searrow \mathrm{pr} & \swarrow \varrho \\ & T & \end{array}$$

where \mathbb{Z}_n^{2g} acts freely on $\mathcal{Y}(\mathrm{SL}_n, g) \times T$. As in [22, Corollary 2.6] (see also [46, Proposition 5]), we deduce that:

Lemma 2.3 *The variety $\mathcal{Y}(\mathrm{GL}_n, g)$ is an étale locally trivial fiber bundle over T with fiber $\mathcal{Y}(\mathrm{SL}_n, g)$.*

Lemma 2.3 implies that the dimension estimates in Lemma 2.2(2)–(3) hold for $\mathcal{Y}(\mathrm{SL}_n, g)$ too.

Proof of Theorem 1.1 for $\mathcal{Y}(\mathrm{SL}_n, g)$ and $\mathcal{Y}(\mathrm{GL}_n, g)$ When $g = 1$ the claim follows from Proposition 2.8. The case $(n, g) = (2, 2)$ is dealt with in Corollary 2.6 below. The case $n = 1$ is trivial.

Therefore, we assume $n, g > 1$ and $(n, g) \neq (2, 2)$. Recall from Theorem 2.1 that $\mathcal{Y}(\mathrm{GL}_n, g)$ is normal. First, we deduce from this that $\mathcal{Y}(\mathrm{SL}_n, g)$ is also normal. Lemma 2.3 implies that $\mathcal{Y}(\mathrm{SL}_n, g)$ is an irreducible variety of dimension $2(g-1)(n^2-1)$ since $\dim \mathcal{Y}(\mathrm{GL}_n, g) = 2n^2(g-1)+2$. If $\mathcal{Y}(\mathrm{SL}_n, g)$ were not normal, then $\mathcal{Y}(\mathrm{SL}_n, g) \times T$ would also not be normal. But the fact that $\mathcal{Y}(\mathrm{SL}_n, g) \times_{\mathbb{Z}_n^{2g}} T \cong \mathcal{Y}(\mathrm{GL}_n, g)$ is normal and the map $\mathcal{Y}(\mathrm{SL}_n, g) \times T \rightarrow \mathcal{Y}(\mathrm{GL}_n, g)$ is étale implies by [40, Proposition 3.17] that $\mathcal{Y}(\mathrm{SL}_n, g) \times T$ is normal. Thus, $\mathcal{Y}(\mathrm{SL}_n, g)$ is normal.

By Theorem 2.1, the Poisson structure on the dense open subset $C_{(1,n)}$ of $\mathcal{Y}(\mathrm{GL}_n, g)$ is nondegenerate. This implies that the Poisson structure on the whole of the smooth locus

is nondegenerate since the complement to $C_{(1,n)}$ in $\mathcal{Y}(\mathrm{GL}_n, g)$ has codimension at least four. The identification $\mathcal{Y}(\mathrm{SL}_n, g) \times_{\mathbb{Z}_n^{2g}} T \cong \mathcal{Y}(\mathrm{GL}_n, g)$ of [Lemma 2.3](#) is Poisson, where we equip $\mathcal{Y}(\mathrm{SL}_n, g) \times T$ with the product Poisson structure. We deduce that the Poisson structure on the smooth locus of $\mathcal{Y}(\mathrm{SL}_n, g)$ is nondegenerate, and the singular locus of $\mathcal{Y}(\mathrm{SL}_n, g)$ has codimension at least four when $(n, g) \neq (2, 2)$.

Since $\mathcal{Y}(\mathrm{SL}_n, g)$ and $\mathcal{Y}(\mathrm{GL}_n, g)$ are normal and their singular locus has codimension at least four, it follows from Flenner's theorem [\[21\]](#) that they have symplectic singularities. \square

In the proof of [Theorem 1.1](#), we have only used the fact that $C_{(1,n)}$ is contained in the smooth locus. As shown by Goldman [\[26\]](#), the singular locus is precisely the complement to $C_{(1,n)}$.

Lemma 2.4 *The singular locus of $\mathcal{Y}(\mathrm{SL}_n, g)$ and $\mathcal{Y}(\mathrm{GL}_n, g)$ is precisely the complement to the open stratum $C_{(1,n)}$.*

Proof Let $G = \mathrm{GL}_n$ or SL_n and $PG := G/Z(G)$. The explicit description of the strata given in [Lemma 2.2](#) shows that if $\rho \in \mathrm{Hom}(\pi, G)$ is a semisimple representation that is not simple then its stabilizer under PG has strictly positive dimension. Therefore, it follows from the formula for the dimension of $T_{[\rho]}X$ given in Section 1.5 of [\[26\]](#) that $[\rho]$ belongs to the singular locus of X . \square

As for quiver varieties, the symplectic leaves of the character variety X are the stabilizer type strata C_v . For $g > 1$ and $G = \mathrm{GL}_n$, this result is contained in [\[47\]](#) and it is in [\[45\]](#) when $g = 1$. The same statement holds for $\mathcal{Y}(\mathrm{SL}_n, g)$ using [Lemma 2.3](#).

2.3 The case $(n, g) = (2, 2)$

The case $(n, g) = (2, 2)$ can be thought of as a ‘‘local model’’ for the moduli space M_{2v} of semistable sheaves with Mukai vector $2v$ on an abelian or $K3$ surface, where v is primitive with $\langle v, v \rangle = 2$. Therefore, we are able to emulate the arguments of Lehn and Sorger [\[36\]](#) and Kaledin and Lehn [\[32\]](#) in this case. [Lemma 2.2\(1\)](#) says that $\mathcal{Y}(\mathrm{GL}_2, 2)$ has three strata: $C_{(1,2)}$, consisting of simple representations E ; $C_{(1,1;1,1)}$, consisting of semisimple representations $E = F_1 \oplus F_2$, where F_1 and F_2 are a pair of nonisomorphic one-dimensional representations of π ; and $C_{(2,1)}$, the stratum of semisimple representations $E = F^{\oplus 2}$, where F is a one-dimensional representation. By [Lemma 2.4](#), the singular locus of $\mathcal{Y}(\mathrm{GL}_2, 2)$ equals $\bar{C}_{(1,1;1,1)} = C_{(1,1;1,1)} \sqcup C_{(2,1)}$.

Theorem 2.5 *The blowup $\sigma: \tilde{\mathcal{Y}}(\mathrm{GL}_2, 2) \rightarrow \mathcal{Y}(\mathrm{GL}_2, 2)$ along the reduced ideal defining the singular locus of $\mathcal{Y}(\mathrm{GL}_2, 2)$ defines a semismall resolution of singularities.*

Here, a map $X \rightarrow Y$ is called *semismall* if $\dim X \times_Y X = \dim X$.

Proof Fix points $E \in C_{(1,1;1,1)}$ and $E' \in C_{(2,1)}$. The character variety $\mathcal{Y}(\mathrm{GL}_n, g)$ is the moduli space of representations of the group algebra $\mathbb{C}[\pi]$ of the fundamental group π of a genus g surface. The group algebra $\mathbb{C}[\pi]$ is a two-dimensional Calabi–Yau algebra. Then [12, Theorems 6.3 and 6.6]¹ and [34, Corollary 5.21] imply that the formal neighborhoods of E and E' in $\mathcal{Y}(\mathrm{GL}_n, g)$ can be identified with the formal neighborhood of 0 in the quiver varieties associated to the corresponding ext quiver of these points. In the case of E , this is the quiver with two vertices, an arrow in each direction and a loop at each vertex; the dimension vector is $(1, 1)$. For E' this is the quiver with one vertex and two loops; the dimension vector is (2) .

Let $\mathcal{O} = \{B \in \mathfrak{sp}(4) \mid B^2 = 0, \mathrm{rk} B = 2\}$. The set \mathcal{O} is a six-dimensional nilpotent adjoint $\mathrm{Sp}(4)$ -orbit in $\mathfrak{sp}(4)$. We define \mathcal{N} to be the closure of \mathcal{O} in $\mathfrak{sp}(4)$; this variety is a union of three nilpotent orbits. The quiver varieties associated to the above ext quivers are then isomorphic to $\mathbb{C}^8 \times (\mathbb{C}^2/\mathbb{Z}_2)$ and $\mathbb{C}^4 \times \mathcal{N}$, respectively; see [9, Theorem 5.1]. By Artin approximation [3, Corollary 1.6], these identification of formal neighborhoods lift to isomorphisms of analytic germs

$$(\mathcal{Y}(\mathrm{GL}_2, 2), E) \cong (\mathbb{C}^8 \times (\mathbb{C}^2/\mathbb{Z}_2), 0), \quad (\mathcal{Y}(\mathrm{GL}_2, 2), E') \cong (\mathbb{C}^4 \times \mathcal{N}, 0).$$

Clearly, blowing up $\mathbb{C}^8 \times (\mathbb{C}^2/\mathbb{Z}_2)$ along the singular locus gives a semismall resolution of singularities. The key result [32, Remark 5.4] — see also [36, Théorème 2.1] — says that blowing up along the reduced ideal defining the singular locus in $\mathbb{C}^4 \times \mathcal{N}$ also produces a semismall resolution of singularities. \square

Corollary 2.6 *The blowup $\tilde{\mathcal{Y}}(\mathrm{GL}_2, 2)$ of $\mathcal{Y}(\mathrm{GL}_2, 2)$ along the reduced ideal defining the singular locus of $\mathcal{Y}(\mathrm{GL}_2, 2)$ is a smooth symplectic variety and $\mathcal{Y}(\mathrm{GL}_2, 2)$ has symplectic singularities.*

Proof Let $\sigma: \tilde{\mathcal{Y}}(\mathrm{GL}_2, 2) \rightarrow \mathcal{Y}(\mathrm{GL}_2, 2)$ denote the blowup map. The singularities of $\mathcal{Y}(\mathrm{GL}_2, 2)$ in an analytic neighborhood of a point in $C_{(1,1;1,1)}$ are equivalent to an A_1 singularity. Therefore the pullback $\sigma^*\omega$ of the symplectic 2-form ω on the smooth locus of $\mathcal{Y}(\mathrm{GL}_2, 2)$ extends to a symplectic 2-form on $\sigma^{-1}(U)$, where U is the open set $C_{(1,2)} \cup C_{(1,1;1,1)}$. Since σ is semismall, $\sigma^{-1}(C_{(2,1)})$ has codimension at least

¹Thanks to Raf Bocklandt for pointing out these results.

three in $\tilde{\mathcal{Y}}(\mathrm{GL}_2, 2)$. Therefore, $\sigma^*\omega$ extends to a symplectic 2-form on the whole of $\tilde{\mathcal{Y}}(\mathrm{GL}_2, 2)$. Since $\mathcal{Y}(\mathrm{GL}_2, 2)$ is normal and σ is birational, it is a symplectic resolution. In particular, $\mathcal{Y}(\mathrm{GL}_2, 2)$ has symplectic singularities. \square

Corollary 2.7 *The blowup $\tilde{\mathcal{Y}}(\mathrm{SL}_2, 2)$ of $\mathcal{Y}(\mathrm{SL}_2, 2)$ along the reduced ideal defining the singular locus of $\mathcal{Y}(\mathrm{SL}_2, 2)$ is a smooth symplectic variety and $\mathcal{Y}(\mathrm{SL}_2, 2)$ has symplectic singularities.*

Proof Let I denote the reduced ideal in $\mathbb{C}[\mathcal{Y}(\mathrm{SL}_2, 2)]$ defining the singular locus. Since the singular locus is stable under the action of \mathbb{Z}_n^{2g} , so too is I . Therefore, the action of \mathbb{Z}_n^{2g} lifts to the blowup $\tilde{\mathcal{Y}}(\mathrm{SL}_2, 2)$ making $\sigma: \tilde{\mathcal{Y}}(\mathrm{SL}_2, 2) \rightarrow \mathcal{Y}(\mathrm{SL}_2, 2)$ equivariant. [Theorem 2.5](#), together with the fact that

$$\tilde{\mathcal{Y}}(\mathrm{GL}_2, 2) \cong \tilde{\mathcal{Y}}(\mathrm{SL}_2, 2) \times_{\mathbb{Z}_n^{2g}} T,$$

implies that $\tilde{\mathcal{Y}}(\mathrm{SL}_2, 2)$ is smooth. Moreover, the fact that $\tilde{\mathcal{Y}}(\mathrm{GL}_2, 2) \rightarrow \mathcal{Y}(\mathrm{GL}_2, 2)$ is semismall implies that $\sigma: \tilde{\mathcal{Y}}(\mathrm{SL}_2, 2) \rightarrow \mathcal{Y}(\mathrm{SL}_2, 2)$ is semismall. The argument that this implies that σ is a symplectic resolution is identical to the first part of the proof of [Corollary 2.6](#). \square

Looking at the proofs of the preceding corollaries, we see that the above blowup provides symplectic resolutions of $\mathcal{Y}(\mathrm{GL}_2, 2)$ and $\mathcal{Y}(\mathrm{SL}_2, 2)$. This verifies [Theorem 1.9\(ii\)](#).

2.4 The genus one case

Recall that G equals either GL or SL . Let \mathbb{T} be a maximal torus in G . The following is well known. It can be deduced from the corresponding statement for the commuting variety in $\mathfrak{g} \times \mathfrak{g}$, where $\mathfrak{g} = \mathrm{Lie} G$; see [[29](#); [24](#), Sections 2.7 and 2.8].

Proposition 2.8 *Fix $g = 1$. As symplectic singularities, the G -character variety of Σ is isomorphic to $(\mathbb{T} \times \mathbb{T})/\mathfrak{S}_n$.*

Unlike the case $g > 1$, it is not clear whether $\mathrm{Hom}(\pi, G)$ is reduced, but it is shown in [[24](#)] that the corresponding G -character variety is reduced. This is the main difficulty: the above statement on the level of the reduced character variety was proved earlier by Joseph [[29](#)]. In the case $G = \mathrm{GL}$, the Hilbert–Chow morphism defines a symplectic resolution $\pi: \mathrm{Hilb}^n(\mathbb{C}^\times \times \mathbb{C}^\times) \rightarrow (\mathbb{T} \times \mathbb{T})/\mathfrak{S}_n$. Similarly, the preimage $\mathrm{Hilb}_0^n(\mathbb{C}^\times \times \mathbb{C}^\times) \subset \mathrm{Hilb}^n(\mathbb{C}^\times \times \mathbb{C}^\times)$ of $\mathcal{Y}(\mathrm{SL}_n, 1) \subset \mathcal{Y}(\mathrm{GL}_n, 1)$ under π defines a symplectic resolution of $\mathcal{Y}(\mathrm{SL}_n, 1)$. Notice that the case $n = 1$ is trivial since $\mathcal{Y}(\mathrm{GL}_1, 1) = \mathbb{C}^\times \times \mathbb{C}^\times$ with its standard symplectic structure.

We also need one additional case in genus one, aside from general and special linear groups, which is best treated here:

Proposition 2.9 *The character variety $\mathcal{Y}(\mathrm{PGL}_2, 1)$ admits a projective resolution of singularities, given by the blowup of $\mathrm{Hilb}_0^2(\mathbb{C}^\times \times \mathbb{C}^\times)/\mathbb{Z}_2^2$ along the singular locus.*

Proof Observe that $\mathrm{PGL}_2 = \mathrm{SL}_2/\mathbb{Z}_2$. Hence, $\mathcal{Y}(\mathrm{PGL}_2, 1) = \mathcal{Y}(\mathrm{SL}_2, 1)/\mathbb{Z}_2^2$. Now, the \mathbb{Z}_2^2 -action lifts to the resolution $\mathrm{Hilb}_0^2(\mathbb{C}^\times \times \mathbb{C}^\times) \rightarrow \mathcal{Y}(\mathrm{SL}_2, 1)$, acting by negation in each of the factors of \mathbb{C}^\times . Then we have the birational projective Poisson morphism $\mathrm{Hilb}_0^2(\mathbb{C}^\times \times \mathbb{C}^\times)/\mathbb{Z}_2^2 \rightarrow \mathcal{Y}(\mathrm{PGL}_2, 1)$.

We claim that the singularities on the source are all type A_1 surface singularities, and hence reduced. This is because, writing $\mathbb{Z}_2 = \{-I, I\} \in \mathrm{SL}_2$, the fixed locus of $(-I, I) \in \mathrm{SL}_2^2$ consists of the two points $\{(i, 1), (-i, 1)\}, \{(i, -1), (-i, -1)\} \in \mathrm{Hilb}^2(\mathbb{C}^\times \times \mathbb{C}^\times)$, which do not include any of the fixed points for $(I, -I)$ (namely, the two points $\{(\pm 1, i), (\pm 1, -i)\} \in \mathrm{Hilb}^2(\mathbb{C}^\times \times \mathbb{C}^\times)$) or for $(-I, -I)$ (the two points $\{(i, \pm i), (-i, \mp i)\} \in \mathrm{Hilb}^2(\mathbb{C}^\times \times \mathbb{C}^\times)$). Thus, the nontrivial isotropy groups of points on $\mathrm{Hilb}_0^2(\mathbb{C}^\times \times \mathbb{C}^\times)$ are all of size two, at which we get type A_1 surface singularities. This is also a priori clear since \mathbb{Z}_2^2 acts symplectically on the surface $\mathrm{Hilb}_0^2(\mathbb{C}^\times \times \mathbb{C}^\times)$, and so the linearization of the isotropy group of a point is isomorphic to a subgroup of $\mathrm{Sp}_2 = \mathrm{SL}_2$, which cannot be \mathbb{Z}_2^2 .

Now, since $\mathrm{Hilb}_0^2(\mathbb{C}^\times \times \mathbb{C}^\times)/\mathbb{Z}_2^2$ only has type A_1 surface singularities, the blowup of the singular locus is a symplectic resolution, which by composition gives a symplectic resolution of $\mathcal{Y}(\mathrm{PGL}_2, 1)$. \square

This immediately implies [Theorem 1.9\(i\)](#).

2.5 Factoriality

In this section we show that the character variety is locally factorial when $n, g \geq 2$ and $(n, g) \neq (2, 2)$. We begin with the GL -character variety. Recall that $\xi: \mathrm{Hom}(\pi, \mathrm{GL}_n) \rightarrow \mathcal{Y}(\mathrm{GL}_n, g)$ is the quotient map. We begin with the following dimension estimates of Crawley-Boevey and Shaw:

Theorem 2.10 [[18](#), [Theorem 7.2](#) and [Corollary 7.3](#)] *Consider a stratum C_ν in $\mathcal{Y}(\mathrm{GL}_n, g)$ of representation type $(\ell_1, \nu_1; \dots; \ell_k, \nu_k)$. Then, for all $z \in C_\nu$, the fiber $\xi^{-1}(z) \subseteq \mathrm{Hom}(\pi, \mathrm{GL}_n)$ has dimension at most $n^2g - \sum_t (\nu_t^2(g-1) + 1)$, so the dimension of $\xi^{-1}(C_\nu)$ is at most $n^2g + \sum_t (\nu_t^2(g-1) + 1)$.*

The action of GL on $\text{Hom}(\pi, GL_n)$ factors through PGL_n . Let $\text{Hom}(\pi, GL_n)_{\text{free}}$ denote the open subset of $\text{Hom}(\pi, GL_n)$ where PGL_n acts freely.

Lemma 2.11 *Assume that $n, g > 1$ and $(n, g) \neq (2, 2)$. The variety $\text{Hom}(\pi, GL_n)$ is normal and locally factorial. The complement to $\text{Hom}(\pi, GL_n)_{\text{free}}$ in $\text{Hom}(\pi, GL_n)$ has codimension at least four.*

Proof By Theorem 2.1(2), $\text{Hom}(\pi, GL_n)$ is a complete intersection and hence Cohen–Macaulay. Thus, it satisfies (S_2) . By a theorem of Grothendieck [33, Theorem 3.12], in order to show that $\text{Hom}(\pi, GL_n)$ is locally factorial, it suffices to check that it satisfies (R_3) too. We need to check the second claim that the complement to $\text{Hom}(\pi, GL_n)_{\text{free}}$ in $\text{Hom}(\pi, GL_n)$ has codimension at least four.

Note that the free locus is the same as the locus of representations whose endomorphism algebra has dimension one. These are the “bricks”. We represent $\text{Hom}(\pi, GL_n)$ as the union of preimages of the (finitely many) strata, and consider over each such preimage the nonfree locus.

If the preimage of the stratum has codimension at least four, it can be ignored. Thus, we just need to show that the complement to $\xi^{-1}(C)_{\text{free}}$ has codimension at least four for those strata C with $\text{codim} \overline{\xi^{-1}(C)} \leq 3$. Since we are explicitly excluding the case $(n, g) = (2, 2)$, Theorem 2.10, together with Lemma 2.2(3), implies that we are reduced to considering the cases $(n, g) = (3, 2)$ and $\nu = (1, 2; 1, 1)$, or $(n, g) = (2, 3)$ and $\nu = (1, 1; 1, 1)$.

The semisimple part of a fiber $\xi^{-1}(M)$ is the GL -orbit of M , which has dimension $n^2 - \dim \text{End}(M)$. If $\xi^{-1}(C)_{\text{ss}}$ denotes the semisimple part of the preimage of the stratum C then

$$\text{codim}_{\text{Hom}(\pi, GL_n)} \xi^{-1}(C)_{\text{ss}} = \dim \text{Hom}(\pi, GL_n) - \dim C - (n^2 - \dim \text{End}(M))$$

and the difference $\text{codim}_{\text{Hom}(\pi, GL_n)} \xi^{-1}(C)_{\text{ss}} - \text{codim}_{\mathcal{Y}(GL_n, g)} C$ equals

$$\begin{aligned} 2n^2g - (n^2 - 1) - \dim C - (n^2 - \dim \text{End}(M)) - (2n^2(g - 1) + 2 - \dim C) \\ = \dim \text{End}(M) - 1. \end{aligned}$$

That is, the semisimple part of the preimage $\xi^{-1}(C)$ has codimension (in $\text{Hom}(\pi, GL_n)$) at least the codimension of C itself (in $\mathcal{Y}(GL_n, g)$). Thus, if C has codimension at least four (which is the case for us), then we can ignore the semisimple part of $\xi^{-1}(C)$.

Next, if we consider a stratum C of type $(1, a; 1, b)$, note that every representation in this stratum is either semisimple or an indecomposable extension of two nonisomorphic representations. The latter type is a brick, since there is a unique simple quotient and a unique simple subrepresentation and the two are nonisomorphic. Therefore, applying the previous paragraph together with [Lemma 2.2\(2\)](#) shows that we can ignore $\xi^{-1}(C)$ (the nonfree locus has overall codimension at least four). This proves the final assertion. \square

If $\text{Hom}(\pi, \text{SL})_{\text{free}}$ denotes the open subset of $\text{Hom}(\pi, \text{SL})$ where PGL acts freely, then [Lemma 2.3](#) implies that the statement of [Lemma 2.11](#) holds for $\text{Hom}(\pi, \text{SL})_{\text{free}}$ too.

Proof of Theorem 1.3 for $\mathcal{Y}(\text{SL}_n, g)$ and $\mathcal{Y}(\text{GL}_n, g)$ Let $X := \mathcal{Y}(\text{SL}_n, g)$ or $\mathcal{Y}(\text{GL}_n, g)$. First we suppose that $n, g > 1$ and $(n, g) \neq (2, 2)$. Let X_s denote the dense open subset of X consisting of simple representations and $\text{Hom}(\pi, G)_s$ its preimage in $\text{Hom}(\pi, G)$. Then $\xi: \text{Hom}(\pi, G) \rightarrow X_s$ is a principal PGL -bundle. Moreover, by [Lemma 2.2\(2\)](#), the complement to X_s has codimension at least four in X . We are therefore in the situation where we can apply the results of Drézet's theorem to X .

The stratum C_ρ of type $\rho = (n, 1)$ is contained in the closure of all other strata in X . If $y \in T(x)$ is a lift in $\text{Hom}(\pi, G)$ of a point x of C_ρ then y corresponds to the representation $\mathbb{C}^{\oplus n}$, where \mathbb{C} denotes here the trivial π -module. Therefore, $\text{PGL}_y = \text{PGL}_n$ has no nontrivial characters. In particular, PGL_y will act trivially on L_y for any PGL -equivariant line bundle on $\text{Hom}(\pi, G)$. Hence, we deduce from Drézet's theorem [[19](#), théorème A] that X is factorial at every point of the closed stratum C_ρ .

Now consider an arbitrary stratum C_τ in X . If X is factorial at one point of the stratum then it will be factorial at every point in the stratum. On the other hand, the main result of [[13](#)] says that the subset of factorial points of X is an open subset. Since this open subset is a union of strata and contains the unique closed stratum, it must be the whole of X . This completes the proof in these cases.

In the case that $(n, g) = (2, 2)$, as observed in [[32](#), Remark 4.6] — see also [[33](#)] and [[23](#), Proposition 7.3] — the varieties $\mathcal{Y}(\text{GL}_n, g)$ and $\mathcal{Y}(\text{SL}_n, g)$ are \mathbb{Q} -factorial (but not locally factorial). In the case $g = 1$, by [Proposition 2.8](#), it follows that $\mathcal{Y}(\text{GL}_n, g)$ and $\mathcal{Y}(\text{SL}_n, g)$ are \mathbb{Q} -factorial, being finite quotients of smooth (hence locally factorial) varieties (see eg [[8](#), Section 2.4]). \square

Remark 2.12 A similar analysis has been performed in [[44](#)] in order to classify which moduli spaces of semisimple representations of an arbitrary multiplicative deformed preprojective algebra admit symplectic resolutions.

3 Arbitrary type A groups

Now let G be connected reductive, all of whose simple quotients are of type A . We begin this section by stating (but not yet proving) a refined version of [Theorem 1.10](#), giving the closest thing to a symplectic resolution that can exist in general. Then we explain a general procedure to reduce statements about G to those about a finite covering, and hence to the case of SL_n . This is already enough, thanks to [Section 2](#), to prove [Theorems 1.1](#) and [1.3](#), which we do in [Section 3.3](#). Next, we explain how to estimate codimension of fixed loci under translation by finite-order central elements of G . This, together with the preceding reduction, is enough to prove [Proposition 1.4](#). Finally, we put all these results together and prove [Theorems 3.1](#) and [1.10](#).

3.1 Relative minimal models

Most of our main results are consequences of the more general [Theorem 3.1](#), which describes for *all* character varieties of type A the closest thing to a symplectic resolution that can exist. Namely, a \mathbb{Q} -factorial terminalization $f : X \rightarrow Y$ of Y is a \mathbb{Q} -factorial variety X with terminal singularities and a crepant projective birational morphism $f : X \rightarrow Y$. Such pairs can always be constructed as a special type of *relative minimal model* thanks to [[11](#), Corollary 1.4.3]; hence, we will sometimes use this terminology. Note that f is actually *maximal* crepant in the sense that any further crepant proper birational morphism $X' \rightarrow X$ is an isomorphism.

Given a connected reductive group G , we write, as in [Section 1.2](#), $G = (H \times K)/Z$, where K is semisimple and $Z < H \times Z(K)$ is a finite subgroup with $Z \rightarrow H \times K \rightarrow H$ injective. Then $\mathcal{Y}(G, g) = \mathcal{Y}(H \times K, g)/Z^{2g}$, where the Z^{2g} -action on the product $\mathcal{Y}(H \times K, g) = \mathcal{Y}(H, g) \times \mathcal{Y}(K, g)$ is free because of the injectivity of $Z \rightarrow H$. It will be convenient also to write $K = K_0/C_0$, where $K_0 = \prod_{i=1}^m \mathrm{SL}_{n_i}$ is simply connected and $C_0 < Z(K_0)$. Then G is a finite quotient of the product $\mathbb{G} := H \times K_0$ of a torus with special linear groups.

Finally, in the situation of genus one, we will also need the intermediate group $K_1 = \prod_{i=1}^{m-m_1} \mathrm{SL}_{n_i} \times \mathrm{PGL}_2^{m_1}$, which fits into a chain $K_0 \twoheadrightarrow K_1 \twoheadrightarrow K$ such that m_1 is maximal. In other words, K_1 is the quotient of K_0 by the subgroup of C_0 generated by elements which are $-I_2$ in some factor SL_2 and the identity in other factors. Let C_1 be the quotient of C_0 such that $K = K_1/C_1$.

Note that $\mathcal{Y}(K_0, g) = \prod_i \mathcal{Y}(\mathrm{SL}_{n_i}, g)$; each factor $\mathcal{Y}(\mathrm{SL}_{n_i}, g)$ either admits a symplectic resolution (when $g = 1$ or $(n, g) = (2, 2)$ by [Theorem 1.9](#)), or is \mathbb{Q} -factorial terminal

(by Theorem 1.3 and Proposition 1.4 for the SL_n case). Similarly, in the case $g = 1$,

$$\mathcal{Y}(K_1, 1) = \prod_{i=1}^{m-m_1} \mathcal{Y}(SL_{n_i}, 1) \times \mathcal{Y}(PGL_2, 1)^{m_1}.$$

In this case each factor admits a symplectic resolution by Theorem 1.9. Thus, by taking symplectic resolutions of factors admitting them, we obtain a \mathbb{Q} -factorial terminalization $\tilde{\mathcal{Y}}(K_0, g) \rightarrow \mathcal{Y}(K_0, g)$ for $g > 1$ and $\tilde{\mathcal{Y}}(K_1, 1) \rightarrow \mathcal{Y}(K_1, 1)$ for $g = 1$. To see that the result is indeed \mathbb{Q} -factorial, we note that it is a product of \mathbb{Q} -factorial varieties, and apply the main result of [13].

In general, automorphisms do not lift to \mathbb{Q} -factorial terminalizations. However, we claim that a \mathbb{Q} -factorial terminalization $\tilde{\mathcal{Y}}(G, g)$ of $\mathcal{Y}(G, g)$ may be obtained from the aforementioned one by taking a finite quotient:

- Theorem 3.1**
- (i) If $g = 1$, a \mathbb{Q} -factorial terminalization of $\mathcal{Y}(K, 1)$ is given by the quotient of $\tilde{\mathcal{Y}}(K_1, 1)$ by a (unique) lift of the action of C_1^2 .
 - (ii) If $g \geq 2$, a \mathbb{Q} -factorial terminalization of $\mathcal{Y}(K, g)$ can be obtained as the quotient of $\tilde{\mathcal{Y}}(K_0, g)$ by a (unique) lift of the action of C_0^{2g} .

In both cases, given a \mathbb{Q} -factorial terminalization $\tilde{\mathcal{Y}}(K, g) \rightarrow \mathcal{Y}(K, g)$, a \mathbb{Q} -factorial terminalization $\tilde{\mathcal{Y}}(G, g)$ of $\mathcal{Y}(G, g)$ can be obtained as the étale quotient

$$(\mathcal{Y}(H, g) \times \tilde{\mathcal{Y}}(K, g)) / Z^{2g}$$

(with the same singularities as $\tilde{\mathcal{Y}}(K, g)$).

3.2 Reduction to SL case

There exists a product of special linear groups and a torus H such that G is the quotient of $\mathbb{G} := H \times \prod_i SL(n_i, \mathbb{C})$ by a finite central subgroup, call it Z_0 (to distinguish from the group Z appearing in Section 1.2, which is the quotient of Z_0 by C_0).

We now have the following general lemma:

Lemma 3.2 *Let G_0 be an algebraic (or topological) group and $\phi: G_0 \rightarrow G_1$ a surjective homomorphism with central finite (or discrete) kernel Z_0 . Then we have a G_1 -equivariant isomorphism $\text{Hom}(\pi, G_1)^\circ \cong \text{Hom}(\pi, G_0)^\circ / Z_0^{2g}$.*

Proof Let F be the free group on $2g$ generators. Then $\text{Hom}(\pi, Z_0) = \text{Hom}(F, Z_0) = Z_0^{2g}$ because the abelianizations of π and F are both \mathbb{Z}^{2g} . Any homomorphism $\pi \rightarrow G_0$ can be lifted arbitrarily on generators x_i and y_i of F , so that the image of the relation $\prod [x_i, y_i]$ is some element $z \in Z_0$. This defines a connected component R_z

of $R := \text{Hom}(\pi, G_0)$, namely the image of the subset of $\text{Hom}(F, G_0) \cong G_0^{2g}$ where $\prod_i [x_i, y_i] \mapsto z$, under composition with ϕ . Note that composition with ϕ identifies with the quotient map by the free action of Z_0^{2g} on G_0^{2g} . The connected component of the trivial homomorphism of $\text{Hom}(\pi, G_0)$ is then $R_1 = \text{Hom}(\pi, G_0)^\circ / Z_0^{2g}$. Finally, since Z_0 is central in G_0 , we have that the adjoint action of G_0 on $\text{Hom}(\pi, G_0)$ commutes with the action of Z_0^{2g} . On the quotient $\text{Hom}(\pi, G_0) / Z_0^{2g}$, this factors to an adjoint action of G_1 . Then, by construction, the adjoint actions of G_1 on both sides of the isomorphism match. \square

Specializing to our situation we have:

Corollary 3.3 *With notation as above, $\mathcal{Y}(G, g) \cong \mathcal{Y}(\mathbb{G}, g) / Z_0^{2g}$. The action of Z_0^{2g} on $\mathcal{Y}(\mathbb{G}, g)$ is symplectic.*

Proof Note that $\text{Hom}(\pi, \mathbb{G})$ is connected. By [Lemma 3.2](#), we have established $\text{Hom}(\pi, G)^\circ \cong \text{Hom}(\pi, \mathbb{G}) / Z_0^{2g}$, compatibly with the adjoint actions. Therefore, taking the adjoint quotients, we obtain the desired isomorphism. The fact that the action of Z_0^{2g} is symplectic follows from the construction of [\[26\]](#), since the adjoint action of Z_0 on the Lie algebra of \mathbb{G} is trivial. \square

Remark 3.4 More generally, the proof of [Lemma 3.2](#) above shows that, for every connected component $X \subseteq \text{Hom}(\pi, G_0)$, the quotient X / Z_0^{2g} is a connected component of $\text{Hom}(\pi, G_1)$, which is the image of X under composition with the surjection $\phi: G_0 \rightarrow G_1$. Still more generally, we can replace π by a quotient F/K of a free group F on ℓ generators by any subgroup $K < [F, F]$ generated by commutators. Then every connected component of $\text{Hom}(F/K, G_0)$ maps, under composition with ϕ , onto a connected component of $\text{Hom}(F/K, G_1)$, and this map is the quotient map by the free Z_0^ℓ -action.

Remark 3.5 An alternative proof of [Lemma 3.2](#) can be given using group cohomology: the exact sequence $1 \rightarrow Z_0 \rightarrow G_0 \rightarrow G_1 \rightarrow 1$ can be viewed as an exact sequence of nonabelian π -modules with trivial π -action. Since Z_0 is abelian, the long exact sequence for nonabelian cohomology extends to $H^2(\pi, Z_0)$. Since the action is trivial, on zeroth cohomology this includes the original exact sequence. As $H^1(\pi, M) = \text{Hom}(\pi, M)$ for M a module with trivial action, we obtain the exact sequence

$$1 \rightarrow \text{Hom}(\pi, Z_0) \rightarrow \text{Hom}(\pi, G_0) \rightarrow \text{Hom}(\pi, G_1) \rightarrow H^2(\pi, Z_0).$$

Since $H^2(\pi, Z_0)$ is discrete, the image of $\text{Hom}(\pi, G_1)^\circ$ under the connecting morphism $\text{Hom}(\pi, G_1) \rightarrow H^1(\pi, Z_0)$ is trivial. Thus, $\text{Hom}(\pi, G_1)^\circ = \text{Hom}(\pi, G_0)^\circ / Z_0^{2g}$.

3.3 Proof of Theorems 1.1 and 1.3

In Section 2, we proved these theorems for the cases $\mathcal{Y}(\text{GL}_n, g)$ and $\mathcal{Y}(\text{SL}_n, g)$. They therefore hold for the product $\mathbb{G} = H \times \prod_i \mathcal{Y}(\text{SL}_{n_i}, g)$ as follows. The product of symplectic singularities is clearly symplectic. For the \mathbb{Q} -factorial property, this not obvious but proved in [13]. In the special case when $g > 2$, or the case $g = 2$ and \mathbb{G} has no factors of the form SL_2 , we can also use the argument given in Section 2 — deducing factoriality from a quotient — verbatim to deduce the property for \mathbb{G} .

By [6, Proposition 2.4], the finite symplectic quotient $\mathcal{Y}(G, g)$ of $\mathcal{Y}(\mathbb{G}, g)$ also has symplectic singularities. This result is also known for the \mathbb{Q} -factorial property: see eg [8, Section 2.4].

3.4 Codimension of fixed points: genus $g \geq 2$

We consider the action of Z_0^{2g} on the open subset $\mathcal{Y}(\mathbb{G}, g)_{\text{irr}}$ of $\mathcal{Y}(\mathbb{G}, g)$ consisting of products of irreducible representations. Since $\mathcal{Y}(\mathbb{G}, g)$ is a product of the character varieties of its factors, it is irreducible by Theorem 2.1(1), Lemma 2.3 and Proposition 2.8. Thus, if nonempty, $\mathcal{Y}(\mathbb{G}, g)_{\text{irr}}$ is dense. It is nonempty if $g > 1$ by Theorem 2.1(3): given an irreducible representation in $\mathcal{Y}(\text{GL}_n, g)$, by rescaling generators we get an irreducible representation in $\mathcal{Y}(\text{SL}_n, g)$, ie a representation of $\mathcal{Y}(\text{GL}_n, g)$ is irreducible if and only if the point of its fiber of $\mathcal{Y}(\text{GL}_n, g) \rightarrow T$ in Lemma 2.3 is irreducible as an element of $\mathcal{Y}(\text{SL}_n, g)$.

Note that, for $g = 1$, $\mathcal{Y}(\mathbb{G}, g)_{\text{irr}}$ is nonempty if and only if \mathbb{G} is abelian, otherwise the generic representation of $\mathbb{Z}^2 \rightarrow \text{SL}_n$ for $n > 1$ is a direct sum of one-dimensional representations.

Proposition 3.6 *The set of points in $\mathcal{Y}(\mathbb{G}, g)_{\text{irr}}$ with nontrivial stabilizer under the action of Z_0^{2g} has codimension at least $4(g - 1)$.*

In particular, for $g \geq 2$, the codimension is at least four. In the proof we find the precise codimension; see Remark 3.7.

Proof It suffices to show that for each nontrivial $\sigma \in Z_0^{2g}$, the set of points in $\mathcal{Y}(\mathbb{G}, g)_{\text{irr}}$ fixed by σ has codimension at least $2(g - 1)$. Let $\rho: \pi \rightarrow \mathbb{G}$ with $[\rho] \in \mathcal{Y}(\mathbb{G}, g)_{\text{irr}}$ and assume $[\rho]$ is fixed by σ . Then $\sigma \cdot \rho \cong \rho$. This implies that there exists $A \in \mathbb{G}$ such that $A^{-1}\rho(f)A = \sigma(f)\rho(f)$ for all $f \in \pi$.

Let $\ell \geq 2$ be the order of σ . Note that, since $\sigma^\ell = 1$, the element A^ℓ is an endomorphism of ρ . By assumption this implies that A^ℓ is a scalar matrix in each factor of \mathbb{G} . As a result, up to rescaling A in each factor, we can assume $A^\ell = 1$, and hence is semisimple with eigenvalues ℓ^{th} roots of unity.

The element σ acts on the tangent space of $\mathcal{Y}(\mathbb{G}, g)_{\text{irr}}$ at $[\rho]$ and it suffices to show that the codimension of $(T_{[\rho]}\mathcal{Y}(\mathbb{G}, g)_{\text{irr}})^\sigma$ is at least four in $T_{[\rho]}\mathcal{Y}(\mathbb{G}, g)_{\text{irr}}$. To compute the action of σ on $T_{[\rho]}\mathcal{Y}(\mathbb{G}, g)_{\text{irr}}$, we twist the action so that $\rho \in \text{Hom}(\pi, \mathbb{G})$ is a fixed point. Namely, define $\tilde{\sigma}$ acting by $(\tilde{\sigma}\tau)(x) = \sigma(x)A\tau(x)A^{-1}$ for all $\tau \in \text{Hom}(\pi, \mathbb{G})$. Then $\tilde{\sigma}\rho = \rho$.

Let \mathfrak{g} be the Lie algebra of \mathbb{G} . Let \mathfrak{g}_ρ denote the representation of π obtained by the action $\gamma \cdot x := \text{Ad}(\rho(\gamma))(x)$. As noted in [26], we can identify $T_{[\rho]}\mathcal{Y}(\mathbb{G}, g)_{\text{irr}} = H^1(\pi, \mathfrak{g}_\rho)$. Let $[u] \in H^1(\pi, \mathfrak{g}_\rho)$ be represented by the 1-cocycle u . As in [26, (1.1)], the tangent vector $[u]$ defines an infinitesimal curve through ρ by $\rho_t(x) = \exp(tu(x))\rho(x)$. Then

$$\begin{aligned} (\tilde{\sigma}\rho_t)(x) &= \sigma(x)A \exp(tu(x))\rho(x)A^{-1} \\ &= \sigma(x)A \exp(tu(x))A^{-1}A\rho(x)A^{-1} \\ &= \exp(tAu(x)A^{-1})\sigma(x)A\rho(x)A^{-1} \\ &= \exp(tAu(x)A^{-1})\rho(x), \end{aligned}$$

where we have used the fact that $\sigma(x)$ is central. Thus, the action of σ on $H^1(\pi, \mathfrak{g}_\rho)$ is given by conjugation by A on \mathfrak{g} . We use the ingenious argument in [26, Section 1.5] to estimate $\dim H^1(\pi, \mathfrak{g}_\rho)^A$. Since Σ is $K(\pi, 1)$, \mathfrak{g}_ρ defines a local system on Σ of rank $\dim \mathfrak{g}$ with Euler characteristic

$$\sum_{i=0}^2 (-1)^i \dim H^i(\pi, \mathfrak{g}_\rho) = -2(g-1) \dim \mathfrak{g}.$$

This formula follows from the fact that, as a “virtual” vector space,

$$\sum_{i=0}^2 (-1)^i H^i(\pi, \mathfrak{g}_\rho) = \sum_{i=0}^2 (-1)^i C^i \otimes_{\mathbb{C}} \mathfrak{g},$$

where C^\bullet are the terms of the (finite-dimensional) simplicial cochain complex computing the cohomology of π with coefficients in \mathbb{C} . It follows that

$$\sum_{i=0}^2 (-1)^i H^i(\pi, \mathfrak{g}_\rho)^A = \sum_{i=0}^2 (-1)^i C^i \otimes_{\mathbb{C}} \mathfrak{g}^A$$

and hence

$$\sum_{i=0}^2 (-1)^i \dim H^i(\pi, \mathfrak{g}_\rho)^A = -2(g-1) \dim \mathfrak{g}^A.$$

As noted in [26, Section 1.4], $H^0(\pi, \mathfrak{g}) = \text{End}_{\mathfrak{g}}(\mathfrak{g}_\rho)$ is the infinitesimal stabilizer of ρ . The fact that V is irreducible implies that $H^0(\pi, \mathfrak{g}) \cong \mathfrak{z}(\mathfrak{g})$, the center of \mathfrak{g} . This is the Lie algebra of H^{2g} , of dimension $2g \dim H$. By Poincaré duality, $\dim H^2(\pi, \mathfrak{g}) = 2g \dim H$ too. Thus,

$$\begin{aligned} \dim H^1(\pi, \mathfrak{g})^A &= 2(g-1) \dim \mathfrak{g}^A + \dim H^0(\pi, \mathfrak{g})^A + \dim H^2(\pi, \mathfrak{g})^A \\ &\leq 2(g-1) \dim \mathfrak{g}^A + 4g \dim H; \end{aligned}$$

in fact we have equality since $\mathfrak{z}(\mathfrak{g})^A = \mathfrak{z}(\mathfrak{g})$ (but we do not need this). Thus, the codimension of $(T_{[\rho]} \mathcal{Y}(\mathbb{G}, g)_{\text{irr}})^\sigma$ in $(T_{[\rho]} \mathcal{Y}(\mathbb{G}, g)_{\text{irr}})^\sigma$ is (at least) $2(g-1)(\dim \mathfrak{g} - \dim \mathfrak{g}^A)$. Conjugation by A is nontrivial on at least one simple summand. Such a summand is of the form \mathfrak{sl}_m for some $m \geq 2$. Therefore, for the dimension estimate we may assume $\mathfrak{g} = \mathfrak{sl}_m$. If the multiplicities of the ℓ^{th} roots appearing on the diagonal of A are m_1, \dots, m_ℓ , where $m = m_1 + \dots + m_\ell$, then an easy induction shows that

$$\dim \mathfrak{sl}_m - \dim \mathfrak{sl}_m^A = m^2 - \sum_{i=1}^{\ell} m_i^2 \geq 2(t-1),$$

where t is the number of m_i not equal to 0. Since A is not the identity matrix, $t \geq 2$. This implies that $2(g-1)(\dim \mathfrak{sl}_m - \dim \mathfrak{sl}_m^A) \geq 2(g-1)$, as required. \square

The argument of the proof above does not make use of the product decomposition for \mathbb{G} , only for its Lie algebra \mathfrak{g} . Hence, it is valid with \mathbb{G} replaced by any connected reductive group with type A quotients and Z any finite central subgroup, replacing the irreducible locus by the locus of representations whose endomorphisms identify with $\mathfrak{z}(\mathfrak{g})$.

Remark 3.7 We can actually compute the precise codimension as follows. Let $\sigma \in Z_0^{2g}$ be nontrivial. If σ has a nontrivial projection to H , then it is clear that σ has no fixed points, as H is abelian. So assume that this is not the case. Let σ_i be the projection of σ to each factor $\text{SL}(n_i, \mathbb{C})$, and let $\ell_i \geq 1$ be the minimum positive integer such that $\sigma_i^{\ell_i}$ is a scalar (ie the minimum order of a scalar multiple of σ_i). Then we claim that

$$\text{codim}_{\mathcal{Y}(\mathbb{G}, g)_{\text{irr}}} \mathcal{Y}(\mathbb{G}, g)_{\text{irr}}^\sigma = 2(g-1) \sum_i n_i^2 \left(1 - \frac{1}{\ell_i}\right).$$

We prove this claim below. In particular, this is at least $4(g-1)$. The codimension of the nonfree locus under all of Z_0^{2g} is the minimum of the above codimensions; this recovers the statement of [Proposition 3.6](#).

To prove the claim, note that it suffices to consider the case of a single factor $\mathrm{SL}(m, \mathbb{C})$. Up to scaling, as in the proof of the proposition, assume that $\sigma = A \in \mathrm{SL}_n$ has order ℓ . Let ζ be a primitive ℓ^{th} root of unity. Conjugating by a permutation matrix, we can assume that, for some $m_0, \dots, m_{\ell-1}$, the first m_0 diagonal entries of A are $\zeta^0 = 1$, the next m_1 are ζ^1 , the next m_2 are ζ^2 , and so on. Now let $X \in \mathrm{GL}_n(\mathbb{C})$ be a matrix such that $AXA^{-1} = \lambda X$ for some $\lambda \in \mathbb{C}$. Then $\lambda = \zeta^k$ for some $k \mid \ell$. Then X must be a block permutation matrix with respect to the decomposition $m = m_0 + \dots + m_{\ell-1}$, expressing the permutation $i \mapsto i+k \pmod{\ell}$. For this to be invertible, the blocks must be square, which implies $m_i = m_{i+k}$ for all i , with addition taken modulo ℓ . Now, if X_1, \dots, X_{2g} are the generating matrices, then we claim that the powers k_1, \dots, k_{2g} appearing in this calculation must have gcd equal to one. Otherwise, their least common multiple ℓ' would be a proper factor of ℓ , and $A^{\ell'}$ would be a nonscalar automorphism of the representation — impossible by Schur's lemma. Thus, since $m_i = m_{i+k_j}$ for all i and j , we get that $m_0 = m_1 = \dots = m_{\ell-1}$. We thus conclude

$$\dim \mathfrak{sl}_m - \dim \mathfrak{sl}_m^A = m^2 - \ell \left(\frac{m}{\ell} \right)^2 = m^2 \left(1 - \frac{1}{\ell} \right).$$

Taking the sum of codimensions over each factor, we obtain the claim.

The above allows us to prove [Proposition 1.4](#):

Proof of Proposition 1.4 Let us first assume that $g = 1$ and G is nonabelian. We claim that the singular locus has codimension at most two; since $\mathcal{Y}(G, g)$ is a symplectic singularity, this implies immediately that the codimension is exactly two. Thanks to [Corollary 3.3](#), it suffices to assume that $G = \mathbb{G}$. In turn this reduces to the case of $\mathcal{Y}(\mathrm{SL}_n, 1)$ for $n \geq 2$. Now the statement follows from [Proposition 2.8](#).

Next assume that $g = 2$ and that G has PGL_2 as a quotient. We claim again that the singular locus has codimension at most two (and hence exactly two). As before, this reduces to the case of $\mathcal{Y}(\mathrm{SL}_2, 2)$. Then the statement follows from [Lemma 2.2\(1\)](#) (see also [Section 2.3](#)).

Finally, assume that $g \geq 2$ and, if $g = 2$, then there are no simple quotients of G of the form PGL_2 . Then the singular locus of $\mathcal{Y}(\mathbb{G}, g)$ has codimension at least four by [Lemma 2.2](#). Its complement, the smooth locus of $\mathcal{Y}(\mathbb{G}, g)$, equals $\mathcal{Y}(\mathbb{G}, g)_{\mathrm{irr}}$ by

Lemma 2.4. Therefore, the singular locus of $\mathcal{Y}(G, g) = \mathcal{Y}(\mathbb{G}, g)/Z_0^{2g}$ is the image under the quotient by Z_0^{2g} of the union of the singular locus of $\mathcal{Y}(\mathbb{G}, g)$ and the nonfree locus. Thus we can conclude by [Proposition 3.6](#) that the singular locus has codimension at least four. \square

3.5 Genus one case

We estimate the codimension of the singular locus in genus one:

Proposition 3.8 *Let $\sigma \in Z_0^2$ be nontrivial. The set of points in $\mathcal{Y}(\mathbb{G}, 1)$ with nontrivial stabilizer under the action of σ has codimension at least four unless σ is equal to minus the identity in a factor of the form SL_2 and the identity in other factors, in which case this locus has codimension two.*

Proof The nonfree locus of σ on $\mathcal{Y}(\mathbb{G}, 1)$ is the product of such loci on each factor $\mathcal{Y}(H, 1)$ and $\mathcal{Y}(SL_{n_i}, 1)$. Therefore, the codimension is the sum of these codimensions. Note that these codimensions are all even. So, in order to not have codimension four, σ must have trivial projection to all factors but one, and there the nonfree locus must have codimension exactly two. This reduces the question to the case of a single factor. Note that, if H is this factor, the nonfree locus is empty, so cannot have codimension two. We have reduced to the statement that the nonfree locus of the action of a nontrivial element $\sigma \in Z(SL_n)^2$ on $\mathcal{Y}(SL_n, 1)$ has codimension two if and only if $n = 2$.

To prove this, let $\sigma = (\zeta I_n, \eta I_n) \in SL_n^2$, for $\zeta^n = \eta^n = 1$. Let $T < SL_n$ be the maximal torus of diagonal elements. Via the isomorphism $\mathcal{Y}(SL_n, 1) \cong (T \times T)/S_n$ of [Proposition 2.8](#), we see that the action of σ is by multiplication by (ζ, η) . Let us think of an element of $(T \times T)/S_n$ as a multiset of n elements $(a_i, b_i) \in \mathbb{C}^\times \times \mathbb{C}^\times$, with $\prod_i a_i = \prod_i b_i = 1$. Then $(\zeta, \eta) \cdot (a_i, b_i) = (\zeta a_i, \eta b_i)$.

For σ to fix the multiset, it must be partitioned into orbits in $\mathbb{C}^\times \times \mathbb{C}^\times$ under the action of σ . Each orbit is uniquely determined by any one of its elements, which can be arbitrary so long as the overall product of pairs is $(1, 1)$. The latter condition can be interpreted to mean that all orbits except for one are arbitrarily chosen, with the final one having only finitely many possible choices. Therefore, the dimension of the fixed locus is $2(m - 1)$, where m is the number of orbits. Hence, the codimension of the fixed locus is $2(n - m)$. For this to be two, we need $m = n - 1$. However, every orbit is nontrivial, since $T \times T$ itself has no fixed points under σ . Therefore, this happens precisely if $n = 2$ and $m = 1$. \square

3.6 Proof of Theorem 3.1

In general, automorphisms of a symplectic singularity do not lift to a symplectic resolution. However, this is the case in our situation when the automorphisms in question come from the (symplectic) action of Z^{2g} . More precisely, we have seen that there are three situations where the character variety associated to an almost simple group of type A admits a symplectic resolution:

- (a) $\mathcal{Y}(\mathrm{SL}_2, 2)$,
- (b) $\mathcal{Y}(\mathrm{SL}_n, 1)$, and
- (c) $\mathcal{Y}(\mathrm{PGL}_2, 1)$.

In (a), any automorphism of $\mathcal{Y}(\mathrm{SL}_2, 2)$ lifts to the blowup of the singular locus, since the latter is always fixed by the automorphism. In case (b), every automorphism of $\mathcal{Y}(\mathrm{SL}_n, 1)$ coming from $Z(\mathrm{SL}_n)^2 \cong \mathbb{Z}_n^2$ lifts to an automorphism of the resolution, by the universal properties of the Hilbert–Chow morphism. This is because the action of \mathbb{Z}_n^2 is induced from an action on $\mathbb{C}^\times \times \mathbb{C}^\times$; see the proof of [Proposition 3.8](#). In both cases, since the resolution is birational the lifts are necessarily unique. Finally, in case (c), the symplectic resolution is given by blowing up the A_1 surface singularities as in [Proposition 2.9](#), but the group Z^{2g} is trivial in this case.

Let Y be the variety occurring in one of (a)–(c) and $f : X \rightarrow Y$ the symplectic resolution described above. Let σ be an automorphism coming from the action of a central (ie scalar) element of SL_n (or $\sigma = I_2$ when the group is PGL_2). Then σ lifts to X with f equivariant.

Lemma 3.9 *Let $C \subseteq X^\sigma$ be a connected component. At generic $x \in C$, the restriction $f|_C : C \rightarrow f(C)$ is an immersion.*

Since $f(X^\sigma) \subseteq Y^\sigma$, we obtain the following immediate consequence:

Corollary 3.10 $\dim X^\sigma \leq \dim Y^\sigma$.

Proof of Lemma 3.9 Let $y = f(x)$. We induct on the codimension of the symplectic leaf through $y \in Y$. Over the open leaf (codimension zero), f is an isomorphism, so the statement is clear. We proceed to the inductive step. Note that our base variety, $Y := \mathcal{Y}(\mathrm{SL}_n, g)$ (for $g = 1$ or $(n, g) = (2, 2)$), has finitely many symplectic leaves. This is a general fact about symplectic varieties [[30](#), Theorem 2.3], but in our case can also be seen explicitly (see the end of [Section 2.2](#)). Thus, Y^σ intersects only finitely

many symplectic leaves. Now suppose inductively that $y \in Y^\sigma$ lies in a symplectic leaf L of codimension $2k > 0$. Since X^σ is smooth symplectic, its codimension is locally constant. By genericity of x , we can assume that nearby points in C do not map to a symplectic leaf in Y of smaller codimension. Hence, nearby points also map to L . That is, a neighborhood of x in C maps entirely to L . Now, $f|_{f^{-1}(L)}: f^{-1}(L) \rightarrow L$ is a fibration, thanks to the infinitesimal automorphisms of $f: X \rightarrow Y$ given by Hamiltonian vector fields. Moreover, the fibers of f are all isotropic (this is actually true for an arbitrary projective symplectic resolution thanks to [25, Proposition 1.2.2], based on [49]). Since $C \cap f^{-1}(L)$ is a symplectic manifold, as is its image — a component of L^σ — we conclude that its fibers are in fact zero-dimensional. Thus, f is an immersion at x , as desired. \square

Remark 3.11 The proof shows that f is an immersion everywhere on $C \cap f^{-1}(L)$, so one can replace “generic x ” in the statement of the lemma with “for $x \in C$ whose image in Y has maximal rank under the Poisson structure”.

We can construct a \mathbb{Q} -factorial terminalization $\tilde{\mathcal{Y}}(K_0, g)$ (resp. $\tilde{\mathcal{Y}}(K_1, 1)$) of $\mathcal{Y}(K_0, g)$ (resp. of $\mathcal{Y}(K_1, 1)$), by taking the symplectic resolutions described above for the factors of type (a)–(c), and leaving other factors fixed. We have shown that the action of C_j^{2g} on $\mathcal{Y}(K_j, g)$ lifts (uniquely) to the \mathbb{Q} -factorial terminalization $\tilde{\mathcal{Y}}(K_j, g)$.

We must check that $\tilde{\mathcal{Y}}(K_j, g)/C_j^{2g}$ is a \mathbb{Q} -factorial terminalization of $\mathcal{Y}(K_j, g)/C_j^{2g} = \mathcal{Y}(K, g)$. By Corollary 3.10, the codimension of the singular locus of $\tilde{\mathcal{Y}}(K_j, g)/C_j^{2g}$ is the minimum of the codimension of the nonfree locus in $\mathcal{Y}(K_j, g)$ and the codimension of the singular locus of the partial resolution $\tilde{\mathcal{Y}}(K_j, g)$ (which is at least four). It follows from our codimension estimates, Propositions 3.6 and 3.8, that this codimension is at least four (the final situation described in Proposition 3.8 does not occur precisely because of the definition of K_1). Hence, the partial resolution $\tilde{\mathcal{Y}}(K_j, g)/C_j^{2g}$ has terminal singularities by [41]. By construction and Theorem 1.3, it is a finite quotient of a \mathbb{Q} -factorial variety, and hence \mathbb{Q} -factorial; see eg [8, Section 2.4].

The final statement is clear, since a quotient by a free action of a finite group preserves the properties of being \mathbb{Q} -factorial and of being terminal. This completes the proof of Theorem 3.1.

3.7 Proof of Theorem 1.10

This proof is based on showing that the character variety is formally locally conical. (Note that, thanks to [4], we could replace “formal” by “étale” if desired.)

First, the fact that (a) and (b) of [Theorem 1.10](#) are equivalent follows immediately from the final statement of [Theorem 3.1](#). Moreover, if we are in one of the cases in (c) then it is a consequence of [Theorem 1.9](#) that $\mathcal{Y}(G, g)$ admits a (projective) symplectic resolution. Therefore, it remains to show that if (K, g) is not listed in (c) then $\mathcal{Y}(K, g)$ does not admit a projective symplectic resolution.

Definition 3.12 Let a *symplectic cone* mean a conical symplectic singularity, where the symplectic structure has positive weight under dilations (where the functions on the cone, by definition, have nonnegative weight). We say that a variety is *formally locally symplectically conical* at a point if the formal neighborhood of a point is isomorphic to a formal neighborhood of a symplectic cone.

It has been conjectured by Kaledin [[31](#), Conjecture 1.8] that every symplectic singularity is formally locally symplectically conical. Therefore, [Proposition 3.14](#) below can be interpreted as saying that Kaledin’s conjecture holds for all character varieties of type A and their relative minimal models.

Recall that $\mathbb{G} = H \times \prod_i \mathrm{SL}_{n_i}$ and $Z_0 \subset \mathbb{G}$ a finite central subgroup such that $G = \mathbb{G}/Z_0$. Let $\rho \in \mathrm{Hom}(\pi, \mathbb{G})$ be a semisimple representation. The stabilizer \mathbb{G}_ρ is reductive.

Lemma 3.13 *There exists a homogeneous formal cone \widehat{Y}_0 such that:*

- (i) \mathbb{G}_ρ acts on \widehat{Y}_0 preserving the dilation filtration and $\widehat{Y}_0//\mathbb{G}_\rho$ is a formal symplectic cone.
- (ii) *There is a \mathbb{G} -equivariant isomorphism*

$$(7) \quad \widehat{\mathrm{Hom}}(\pi, \mathbb{G})_{\mathbb{G}_\rho} \cong \widehat{Y}_0 \times_{\mathbb{G}_\rho} \mathbb{G}$$

of homogeneous cones inducing an isomorphism of symplectic cones

$$\widehat{\mathcal{Y}}(\mathbb{G}, g)_{[\rho]} \cong \widehat{\mathrm{Hom}}(\pi, \mathbb{G})_{\mathbb{G}_\rho} // \mathbb{G} \cong (\widehat{Y}_0 \times_{\mathbb{G}_\rho} \mathbb{G}) // \mathbb{G} \cong \widehat{Y}_0 // \mathbb{G}_\rho.$$

Proof We begin by noting, as in the proof of [Theorem 2.5](#), that the character variety $\mathcal{Y}(\mathrm{GL}_n, g)$ is the moduli space of representations of the group algebra $\mathbb{C}[\pi]$ of the fundamental group π of a genus g surface. Therefore, by [[12](#), Theorems 6.3 and 6.6; [34](#), Corollary 5.21], the formal neighborhood of $[\rho] \in \mathcal{Y}(\mathrm{GL}_n, g)$ is isomorphic as a Poisson variety to the formal neighborhood of 0 in a Nakajima quiver variety $\mathfrak{M}_0(\alpha, 0)$, since the group algebra $\mathbb{C}[\pi]$ is a two-dimensional Calabi–Yau algebra. Moreover, the

statement of the lemma, but with $\mathbb{G} = \mathrm{GL}_n$, is implied by [34, Theorem 5.16] applied to our particular situation. Diagram (5) implies that

$$\widehat{\mathrm{Hom}}(\pi, \mathrm{GL}_n)_{\mathrm{GL}_n \cdot \rho} = \widehat{\mathrm{Hom}}(\pi, \mathrm{SL}_n)_{\mathrm{GL}_n \cdot \rho_0} \widehat{\times} (\widehat{\mathbb{C}^{2g}})_x,$$

where $(\rho_0, x) \in \mathrm{Hom}(\pi, \mathrm{SL}_n) \times T$ is a point such that $\mathbb{Z}_n^{2g} \cdot (\rho_0, x) = \rho$. Restricting to the formal subscheme $\widehat{\mathrm{Hom}}(\pi, \mathrm{SL}_n)_{\mathrm{GL}_n \cdot \rho_0} \times \{x\}$ in $\widehat{\mathrm{Hom}}(\pi, \mathrm{GL}_n)_{\mathrm{GL}_n \cdot \rho}$ shows that the statement of the lemma also holds for $\mathbb{G} = \mathrm{SL}_n$.

Finally, for $\mathbb{G} = H \times \prod_i \mathrm{SL}_{n_i}$, the character variety $\mathcal{Y}(\mathbb{G}, g)$ equals $H^{2g} \times \prod_i \mathcal{Y}(\mathrm{SL}_{n_i}, g)$ and hence the statement of lemma follows by taking the product over all SL_{n_i} . \square

Proposition 3.14 *Both $\mathcal{Y}(G, g)$ and its minimal model $\widehat{\mathcal{Y}}(G, g)$ are formally locally symplectically conical at every point.*

Proof Let $x \in \mathcal{Y}(G, g) = \mathcal{Y}(\mathbb{G}, g) / Z_0^{2g}$. We wish to describe the formal neighborhood $\widehat{\mathcal{Y}}(G, g)_x$ of x in $\mathcal{Y}(G, g)$. We would like to apply Theorem A.3 for the action of Z_0^{2g} on $\mathcal{Y}(\mathbb{G}, g)$, but we can't immediately do so since it isn't a priori clear that the stabilizer at the point $[\rho]$ preserves the filtration generated by dilations.

Choose $\rho \in \mathrm{Hom}(\pi, \mathbb{G})$, as in Lemma 3.13, with $Z_0^{2g} \cdot [\rho] = x$. Let $S = \mathrm{Stab}_{Z_0^{2g}}[\rho]$ denote the stabilizer of the point $[\rho]$. Since Z_0^{2g} is a finite group, whose action on $\mathrm{Hom}(\pi, \mathbb{G})$ commutes with the conjugation action of \mathbb{G} , we have

$$\widehat{\mathcal{Y}}(G, g)_x \cong (\widehat{\mathcal{Y}}(\mathbb{G}, g)_{[\rho]}) // S \cong (\widehat{\mathrm{Hom}}(\pi, \mathbb{G})_{\mathbb{G} \cdot \rho} // \mathbb{G}) // S \cong \widehat{\mathrm{Hom}}(\pi, \mathbb{G})_{\mathbb{G} \cdot \rho} // (\mathbb{G} \times S),$$

where the second identification is Lemma 3.13(ii).

The group $\mathbb{G} \times S$ automatically preserves the adic filtration on $\widehat{\mathrm{Hom}}(\pi, \mathbb{G})_{\mathbb{G} \cdot \rho}$ since $\mathbb{G} \cdot \rho$ is stable under $\mathbb{G} \times S$. Therefore, under the isomorphism (7), the group $\mathbb{G} \times S$ acts on $\widehat{Y}_0 \times_{\mathbb{G}_\rho} \mathbb{G}$ preserving the adic filtration. Identify \widehat{Y}_0 with a closed subscheme of $\widehat{Y}_0 \times_{\mathbb{G}_\rho} \mathbb{G}$ in the obvious way and let S_0 denote the stabilizer of $(0, 1)$ in $\widehat{Y}_0 \times_{\mathbb{G}_\rho} \mathbb{G}$. Under the identification, S_0 acts on \widehat{Y}_0 preserving the adic filtration. Since the cone \widehat{Y}_0 is homogeneous, the adic filtration equals the dilation filtration. Thus, S_0 acts on \widehat{Y}_0 preserving the dilation filtration. Theorem A.3(a) implies that this action is conjugate to a homogeneous action. Moreover, as noted in Remark A.4, we may assume it also preserves the Poisson structure. Therefore, as in Theorem A.3(b), we deduce that $\widehat{Y}_0 // S_0$ is a symplectic cone.

Finally, since

$$\widehat{\mathrm{Hom}}(\pi, \mathbb{G})_{\mathbb{G} \cdot \rho} // (\mathbb{G} \times S) \cong (\widehat{Y}_0 \times_{\mathbb{G}_\rho} \mathbb{G}) // (\mathbb{G} \times S) \cong \widehat{Y}_0 // S_0,$$

we deduce that $\widehat{\mathcal{Y}}(G, g)_x$ is a symplectic cone.

Next we consider $\tilde{\mathcal{Y}}(G, g)$. Note first that an étale quotient induces isomorphisms on formal local rings, so we may assume by the final statement of [Theorem 3.1](#) that $G = H \times K$. In fact, we may assume $G = K$ is semisimple. For clarity, we will treat the cases $g = 1$ and $g > 1$ separately. In the case $g = 1$, we write $K = K_1/C_1$ as in [Theorem 3.1\(i\)](#). Then $\tilde{\mathcal{Y}}(K, 1) = \tilde{\mathcal{Y}}(K_1, 1)/C_1^2$ with $\tilde{\mathcal{Y}}(K_1, 1)$ a smooth symplectic variety. It follows from [Theorem A.3\(b\)](#) that $\tilde{\mathcal{Y}}(K, 1)$ is formally locally symplectically conical at every point; see [Remark A.4](#) for the symplectic part.

Similarly, if $g \geq 2$ then we write $K = K_0/C_0$ as in [Theorem 3.1\(ii\)](#). Then $\tilde{\mathcal{Y}}(K, g) = \tilde{\mathcal{Y}}(K_0, g)/C_1^{2g}$ and $\tilde{\mathcal{Y}}(K_0, g)$ is a product of varieties (one for each simple factor of K_0), where each factor is either a smooth symplectic variety or has the form $\mathcal{Y}(\mathrm{SL}_n, g)$. It follows from the first part of the proof of the proposition that every point in $\tilde{\mathcal{Y}}(K, g)$ is formally locally symplectically conical; again, see [Remark A.4](#) for the symplectic part. \square

Lemma 3.15 (i) *Let $g = 1$ and suppose that the minimal model of [Theorem 3.1](#) is not smooth. Then the local cone at every point of that model is \mathbb{Q} -factorial and terminal.*

(ii) *Let $g \geq 2$ and suppose that $x \in \tilde{\mathcal{Y}}(G, g)$ is a point having a preimage in $\tilde{\mathcal{Y}}(\mathbb{G}, g)$ whose projection to each factor $\tilde{\mathcal{Y}}(\mathrm{SL}_n, g)$ is either a smooth point or, for $(g, n) \neq 2$, a point in the minimal stratum $C_{(1,n)}$ of $\mathcal{Y}(\mathrm{SL}_n, g)$. Then the local cone at x is \mathbb{Q} -factorial and terminal.*

Proof Note that the terminal properties are immediate thanks to [\[41\]](#), since the codimension of the singular locus of a local cone must be at least that in the ambient space. So we only have to show that these local cones are \mathbb{Q} -factorial. In the case of (i), this is a finite quotient of a smooth variety, so this is standard; for instance, see [\[8, Section 2.4\]](#). For part (ii), we have to handle a finite quotient of $\tilde{\mathcal{Y}}(\mathbb{G}, g)$, which is a product of smooth varieties and of factors $\mathcal{Y}(\mathrm{SL}_n, g)$. The proof of [Theorem 1.3](#) in [Section 2.5](#) shows that the formal neighborhood of the trivial local system in $\mathcal{Y}(\mathrm{SL}_n, g)$ is locally factorial for $g \geq 2$ and $(n, g) \neq (2, 2)$; the same then also follows for every point of the minimal stratum. Thus, our local cone is a finite quotient of a product of locally factorial cones. Thanks to [\[13\]](#), the product is locally factorial and, since we take a finite quotient, again by [\[8, Section 2.4\]](#), the result is \mathbb{Q} -factorial. \square

[Lemma 3.15](#) implies that, in the situation of [Theorem 1.10](#) where part (c) is not satisfied, there exists a point x of $\tilde{\mathcal{Y}}(G, g)$ whose formal neighborhood does not admit a projective symplectic resolution: this follows by [\[9, Lemma 6.16\]](#) (replacing the variety there

by an arbitrary symplectic cone) since the closed point of this formal neighborhood is singular yet \mathbb{Q} -factorial terminal. Let y denote the image of x in $\mathcal{Y}(G, g)$. By [Proposition 3.14](#), the formal neighborhood of y is a conic symplectic singularity. Therefore, we deduce from [\[7, Theorem 2.2\]](#) (based on [\[42\]](#)) that $\mathcal{Y}(G, g)$ does not admit a (projective) symplectic resolution. This completes the proof of [Theorem 1.10](#).

Appendix Reductive group actions on local cones

In this appendix, we prove a generalized version of the following folklore result (see eg [\[28\]](#)). We work over the complex numbers, but one could replace this with any algebraically closed field of characteristic zero (and perhaps the algebraic closure requirement is unnecessary).

Theorem A.1 *Let X be a smooth scheme over \mathbb{C} and G a reductive group acting on X with fixed point $x \in X$. Then the action of G can be linearized in a formal neighborhood of x .*

Note that a linearization here means that, for some choice of formal coordinates $\hat{\mathcal{O}}_{X,x} \cong \mathbb{C}[[x_1, \dots, x_n]]$, the action of G is via a linear representation $G \rightarrow \mathrm{GL}_n$.

To generalize this, note that a linearization is the same thing as a *homogeneous* action of G on the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$ whose induced action on the completion $\mathbb{C}[[x_1, \dots, x_n]]$ coincides with the original one, up to isomorphism. We generalize this to cones:

Definition A.2 A cone is an affine variety $X = \mathrm{Spec} A$ whose coordinate ring A is nonnegatively graded and connected ($A_0 = \mathbb{C}$ and $A_{<0} = 0$). It is called *homogeneous* if A is generated in a single positive degree $m > 0$ (without loss of generality $m = 1$).

In general, if X is a cone then it is often called *quasihomogeneous*.

Example 1 The type A_n Du Val singularity is $\{xy + z^n = 0\} \subseteq \mathbb{C}^3$. This is quasihomogeneous for the grading $|x| = |y| = n$ and $|z| = 2$, and hence a cone. It is homogeneous if and only if $n = 2$.

Given a cone $X = \mathrm{Spec} A$, there is a canonical action of the multiplicative group by dilations. On a homogeneous function $f \in A_n$ it is given by $\lambda \star f := \lambda^{-n} f$ for $\lambda \in \mathbb{C}^\times$. Conversely, we can recover the grading on $\mathcal{O}(X) = A$ from this action. We

say therefore that the grading, and the resulting filtration $\mathcal{O}(X)_{\leq n} := \bigoplus_{i \leq n} \mathcal{O}(X)_i$, are the ones induced by dilations on X .

Example 2 Let X be any scheme and $x \in X$. Equip the local ring $\mathcal{O}_{X,x}$ with the $\mathfrak{m}_{X,x}$ -adic filtration (for $\mathfrak{m}_{X,x}$ the maximal ideal). The tangent cone $C_{X,x} = \text{Spec gr } \mathcal{O}_{X,x}$ is then homogeneous. In general, X is a homogeneous cone based at x if and only if it is isomorphic to $C_{X,x}$.

For a general cone X , the local ring $\mathcal{O}_{X,x}$, and its completion, are equipped with two in general distinct filtrations: the $\mathfrak{m}_{X,x}$ -adic filtration, and the one coming from dilations on X (ie the descending filtration generated by homogeneous global function on X).

Example 3 Let X be a scheme and $x \in X$. If G is any reductive group acting on X fixing x , then the categorical quotient $C_{X,x} // G = \text{Spec gr } \mathcal{O}_{X,x}^G$ is also a cone (but in general not homogeneous). In general, $C_{X,x} // G$ and $X // G$ need not be formally isomorphic at the image \bar{x} of x . If they are, then $X // G$ is formally locally a cone at \bar{x} .

In view of these examples, [Theorem A.1](#) can be restated as the following: if X is *smooth* at x , then there is an isomorphism of formal neighborhoods of x of the two pairs (X, G) and $(C_{X,x}, G)$. Thus, $X // G$ is formally locally a cone at \bar{x} .

Our main result gives a generalization of this to the case where X need not be smooth and x need not be a fixed point:

Theorem A.3 *Let G be a reductive group acting on a scheme X and let $x \in X$ be a point such that the formal neighborhood \widehat{X}_x is a cone, ie isomorphic to \widehat{Y}_y for Y a cone with cone point $y \in Y$. Assume moreover that either*

- (i) Y is homogeneous (ie $Y \cong C_{X,x}$); or
- (ii) the induced action of the stabilizer G_x on \widehat{Y}_y preserves the filtration coming from dilations.

Then the following hold:

- (a) If G_x is reductive, then the action of G_x on \widehat{Y}_y is conjugate to a homogeneous (ie dilation-equivariant) action;
- (b) If X is an affine variety and the orbit $G \cdot x$ is closed, then $X // G$ is formally locally a cone at \bar{x} .

Note that, by Matsushima's criterion, the hypotheses of (b) imply those of (a). Also, note that hypothesis (i) implies hypothesis (ii), since in this case the filtration coming from dilations is the $\mathfrak{m}_{X,x}$ -adic filtration. It is clear that this hypothesis is satisfied in the situation of [Theorem A.1](#), so that it is a special case of [Theorem A.3](#).

Proof of Theorem A.3 As observed, it is enough to assume condition (ii). We first prove (a). Let H be the group of all automorphisms of \widehat{Y}_y (ie of $\widehat{\mathcal{O}}_{Y,y}$) preserving the filtration coming from dilations (this is just the entire automorphism group in the homogeneous case). Let $H_0 < H$ be the subgroup of homogeneous automorphisms of Y (ie automorphisms commuting with dilations). We have a splitting of the inclusion, $\pi_0: H \rightarrow H_0$, given by taking the associated graded automorphism. Let U be the kernel. Observe that U is pro-unipotent. We have a semidirect product decomposition $H = H_0 \ltimes U$.

Now, the action of G_x can be expressed as a homomorphism $\rho: G_x \rightarrow H_0 \ltimes U$. Statement (a) amounts to saying that this is conjugate to the homomorphism $\rho_0 = \pi_0 \circ \rho: G_x \rightarrow H_0$. Letting $\pi': H_0 \ltimes U \rightarrow U$ be the (set-theoretic) projection to the second factor, ρ defines a one-cocycle $\rho' = \pi' \circ \rho: G \rightarrow U$, with respect to the action given by ρ_0 . Then (a) is equivalent to the statement that ρ' is a coboundary. Since G_x is reductive, $H^1(G_x, \mathbb{C}) = 0$ as there are no nontrivial homomorphisms from a reductive group to a connected unipotent one. Since U is pro-unipotent, the long exact sequences on group cohomology imply also that $H^1(G_x, U) = 0$. This proves (a).

For part (b), we apply Luna's slice theorem. The construction of the formal slice to $G \cdot x$ at $x \in X$ can be constructed in the formal neighborhood \widehat{X}_x so as to be homogeneous with respect to dilations (by finding a section of the Zariski tangent space $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$ in $\widehat{\mathcal{O}}_{X,x}$ which is equivariant for dilations and the action of G_x , and similarly for a complement to the Zariski tangent space of $G \cdot x$; see [\[38, Section III\]](#)). The formal neighborhood of $X//G$ at \bar{x} is isomorphic to the quotient of this formal slice by G_x , so the result follows from (a). \square

Remark A.4 If X is a Poisson scheme and G preserves the Poisson structure then the formal cone \widehat{Y}_y of [Theorem A.3](#) inherits a Poisson structure from X . By restricting the group H in the proof of the theorem to automorphisms preserving the Poisson structure, we see that the action of G_x on \widehat{Y}_y is conjugate to a homogeneous action preserving the Poisson structure. In this case, [Theorem A.3\(b\)](#) says that $X//G$ is formally locally a Poisson cone at \bar{x} .

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Proposed: Ciprian Manolescu

Received: 26 September 2019

Seconded: Frances Kirwan, Richard P Thomas

Revised: 10 July 2021

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Geometry & Topology (ISSN 1465-3060 printed, 1364-0380 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

GT peer review and production are managed by EditFLOW[®] from MSP.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing
<http://msp.org/>

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GEOMETRY & TOPOLOGY

Volume 27

Issue 1 (pages 1–415)

2023

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