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Examples of non-Kähler Calabi–Yau 3–folds with arbitrarily large b_2

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We construct non-Kähler simply connected Calabi–Yau 3–folds with arbitrarily large 2nd Betti numbers by smoothing normal crossing varieties with trivial dualizing sheaves.

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1 Introduction

In this paper, a *Calabi–Yau manifold* means a compact complex manifold whose canonical bundle is trivial and $H^i(X, \mathcal{O}_X) = H^0(X, \Omega_X^i) = 0$ for $0 < i < \dim X$. A projective Calabi–Yau manifold is often also called a strict Calabi–Yau manifold. Our main interest is a *Calabi–Yau* 3*–fold*, that is, a Calabi–Yau manifold of dimension 3.

Projective Calabi–Yau manifolds are one of the building blocks in the classification of algebraic varieties. Nevertheless, it is not known whether there are only finitely many topological types of projective Calabi–Yau 3–folds or not. The main purpose of this paper is to give infinitely many topological types of non-Kähler Calabi–Yau 3–folds.

Theorem 1.1 *Let $a > 0$ be any positive integer. Then there exists a simply connected Calabi–Yau 3–fold $X(a)$ with 2^{nd} Betti number $b_2(X(a)) = a + 3$, topological Euler number $e(X(a)) = -256a^2 + 32a - 224$ and algebraic dimension $a(X(a)) = 1$.*

As far as we know, our examples are the first examples of complex Calabi–Yau 3–folds with arbitrarily large b_2 in our sense. Their topological Euler number can be arbitrarily small and negative. For positive integers $a \neq a'$, we see that $X(a)$ and $X(a')$ are not bimeromorphic (Remark 3.13), thus showing bimeromorphic unboundedness of non-Kähler Calabi–Yau 3–folds. It is also remarkable that the Hodge to de Rham spectral sequence degenerates at E_1 on $X(a)$ and they have unobstructed deformations (Remark 3.8).

Friedman [16, Example 8.9] constructed infinitely many topological types of Calabi–Yau 3–folds with $b_2 = 0$ by deforming a Calabi–Yau 3–fold with ordinary double points based on Clemens' construction [8] of some quintic 3–folds with infinitely many smooth rational curves. There are also infinitely many examples of Calabi–Yau 3–folds of $b_2 = 1$ but with different cubic forms on H^2 as flops of a fixed Calabi–Yau 3–fold; see Friedman [16, Example 7.6] and Okonek and Van de Ven [28, Example 14]. Fine and Panov [14, Section 3] constructed simply connected compact complex 3–folds with trivial canonical bundle, arbitrarily large b_2 and nonzero holomorphic 2–forms.

Non-Kähler Calabi–Yau manifolds are also interesting from the point of view of string theory (the Strominger equations) and complex differential geometry; see for instance Fu, Li and Yau [18], Tseng and Yau [39] and Tosatti [38].

We shall construct the examples by smoothing simple normal crossing (SNC) varieties via the log deformation theory developed by Kawamata and Namikawa [24]. Lee [26] considered log deformations of SNC varieties consisting of two irreducible components, which are called Tyurin degenerations. We also use Tyurin degenerations to construct our examples. The new point in this paper is to consider gluing automorphisms of the intersection of irreducible components of SNC varieties. Tyurin degenerations are also studied in the context of mirror symmetry; see Tyurin [40], Doran, Harder and Thompson [10] and Kanazawa [23].

1.1 Sketch of the construction

First, we prepare an SNC variety $X_0(a) = X_1 \cup X_2$, where X_1 is the blowup of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ along some curves $f_1, \dots, f_a, C$ and $X_2 := \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. The curves $f_1, \dots, f_a$ are distinct smooth fibers of an elliptic fibration $S \to \mathbb{P}^1$ on a very general $(2, 2, 2)$–hypersurface S induced by the 1^{st} projection. We glue X_1 and X_2 along S and its strict transform to construct $X_0(a)$. Since S is an anticanonical member, we have $\omega_{X_0(a)} \simeq \mathcal{O}_{X_0(a)}$. In order to make $X_0(a)$ "d–semistable", we need to blow up $f_1, \dots, f_a$ and some curve C. The point is that we glue after twisting by a certain automorphism of S of infinite order. Because of this, the number of blowup centers for X_1 can be arbitrarily large.

Thus we obtain $X_0(a)$ which satisfies the hypothesis of Theorem 2.6 [24, Theorem 4.2] and can deform $X_0(a)$ to a Calabi–Yau 3–fold $X(a)$, which turns out to be non-Kähler. This $X(a)$ is the example in Theorem 1.1. Note that we can apply the smoothing result even when the SNC variety itself is not projective but the irreducible components are Kähler. (See Remark 2.7.)

We check that $X_0(a)$ and $X(a)$ are both non-Kähler if we twist by a nontrivial automorphism of S; see Proposition 3.18, also Remark 3.20. We use Lemma 3.14, which states that, under some conditions, an SNC variety which is a degeneration of a projective Calabi–Yau manifold admits a big line bundle whose restriction to each irreducible component still has a nonzero section. Moreover, we show that the algebraic dimension of X is 1 (Proposition 3.19). We also compute the topological Euler number of X (Claim 3.7) and check that X is simply connected (Proposition 3.10).

1.2 Notation

We work over the complex number field $\mathbb{C}$ throughout the paper. We call a complex analytic space X *a* (*proper*) *SNC variety* if X has only normal crossing singularities and its irreducible components are smooth (proper) varieties. We identify a proper scheme over $\mathbb{C}$ and its associated compact analytic space unless otherwise stated.

Let X be a proper SNC variety and $\phi\colon \mathcal{X} \to \Delta^1$ a proper flat morphism of analytic spaces over a unit disk Δ^1 such that $\phi^{-1}(0) \simeq X$, that is, ϕ is *a deformation of* X. We call ϕ *a semistable smoothing* of X if $\mathcal{X}$ is smooth and its general fiber $\mathcal{X}_t := \phi^{-1}(t)$ is smooth for $t \neq 0$.

2 Preliminaries

The following result guarantees the existence of a gluing of two schemes along their isomorphic closed subschemes.

Theorem 2.1 [1, Section 1.1], [13, Théorèmes 5.4 et 7.1], [35, Corollary 3.9] *Let Y, X_1, X_2 be schemes and $\iota_i\colon Y \hookrightarrow X_i$ be closed immersions for $i = 1, 2$. Then there exists a scheme X in the Cartesian diagram*

$$\begin{tikzcd} Y \arrow[r, hook, "\iota_1"] \arrow[d, hook, "\iota_2"] & X_1 \arrow[d, hook, "\phi_1"] \\ X_2 \arrow[r, hook, "\phi_2"] & X \end{tikzcd}$$

such that ϕ_1 and ϕ_2 are closed immersions and induce isomorphisms $X_1 \setminus Y \simeq X \setminus X_2$ and $X_2 \setminus Y \simeq X \setminus X_1$. (We say that X is the pushout of the morphisms ι_1 and ι_2.)

Proof We can show the proposition following [1, Section 1.1]. □

By this, we have the following corollary.

Corollary 2.2 *Let X_1, X_2 be smooth proper varieties and $D_i \subset X_i$ be smooth divisors for $i = 1, 2$ with an isomorphism $\phi\colon D_1 \xrightarrow{\simeq} D_2$. Let $i_1\colon D_1 \hookrightarrow X_1$ and $i_2\colon D_2 \hookrightarrow X_2$ be the given closed immersions and let Y be the pushout of two closed immersions $\iota_1 := i_1$ and $\iota_2 := i_2 \circ \phi$, which exists by Theorem 2.1.*

Then Y is a proper SNC variety with two irreducible components Y_1 and Y_2 such that $Y_i \simeq X_i$ and $Y_1 \cap Y_2 \simeq D_i$ for $i = 1, 2$. (We denote by $Y =: Y_1 \cup^{\phi} Y_2$ the pushout.)

Proof We check the properness of Y by definition of properness. We also check that Y is normal crossing by a local computation. □

Remark 2.3 A proper SNC variety is nonprojective in general, even if its irreducible components are projective. Let $X = X_1 \cup X_2$ be a proper SNC variety such that X_1 and X_2 are projective varieties and $D := X_1 \cap X_2$. Then X is projective if and only if there are ample line bundles $\mathcal{L}_1$ on X_1 and $\mathcal{L}_2$ on X_2 such that $\mathcal{L}_1|_D \simeq \mathcal{L}_2|_D$.

Definition 2.4 Let X be an SNC variety and $X = \bigcup_{i=1}^N X_i$ be the decomposition into its irreducible components. Let $D := \operatorname{Sing} X = \bigcup_{i \neq j} (X_i \cap X_j)$ be the double locus and let $I_{X_i}, I_D \subset \mathcal{O}_X$ be the ideal sheaves of X_i and D on X. Let

$$\mathcal{O}_D(X) := \left(\bigotimes_{i=1}^N I_{X_i}/I_{X_i} I_D \right)^* \in \operatorname{Pic} D$$

be the *infinitesimal normal bundle* as in [15, Definition 1.9].

We say that X is *d–semistable* if $\mathcal{O}_D(X) \simeq \mathcal{O}_D$.

Remark 2.5 Let $X = \bigcup_{i=1}^N X_i$ and D be as in Definition 2.4. Friedman proved that, if X has a semistable smoothing, then X is d–semistable [15, Corollary 1.12].

When $N = 2$, we have $\mathcal{O}_D(X) \simeq \mathcal{N}_{D/X_1} \otimes_{\mathcal{O}_D} \mathcal{N}_{D/X_2}$ via a suitable identification of $D \subset X_1$ and $D \subset X_2$, where $\mathcal{N}_{D/X_i}$ is the normal bundle of $D \subset X_i$ for $i = 1, 2$.

We use the following theorem of Kawamata and Namikawa.

Theorem 2.6 [24, Theorem 4.2], [6, Corollary 7.4] *Let $n \geq 3$ and let X be an n–dimensional proper SNC variety such that*

(i) $\omega_X \simeq \mathcal{O}_X$,

(ii) $H^{n-1}(X, \mathcal{O}_X) = 0$, $H^{n-2}(X^\nu, \mathcal{O}_{X^\nu}) = 0$, *where $X^\nu \to X$ is the normalization,*

(iii) *X is d–semistable.*

Then there exists a semistable smoothing $\phi\colon \mathcal{X} \to \Delta^1$ of X over a unit disk.

Remark 2.7 In [24, Theorem 4.2], it is assumed that X is Kähler. However, we check that we only need to assume that X is a proper SNC variety (or each irreducible component is Kähler).

The essential tools are two spectral sequences. One is the log Hodge to de Rham spectral sequence in [24, Lemma 4.1], which follows from the construction of a cohomological mixed Hodge complex; see also [19, Theorem 3.12, Proposition 3.19]. Another spectral sequence is that in [15, Proposition 1.5(3)], which uses only the existence of a pure Hodge structure on the stratum of an SNC variety.

By using these spectral sequences, we show that log deformations of X are unobstructed as in [24, Theorem 4.2]. Then, by using the existence of the Kuranishi family of X and Artin's approximation, we can construct a semistable smoothing $\phi\colon \mathcal{X} \to \Delta^1$ of X; cf [24, Corollary 2.4].

However, if X is not projective, the general fiber of ϕ may not be an algebraic variety even when $H^2(X, \mathcal{O}_X) = 0$. Indeed, this happens in the examples in Theorem 3.4.

In [6, Corollary 7.4], the above result is shown without the assumption (ii) for a projective SNC variety. It is quite possible that we can remove the assumption when X is not projective but proper; cf [11; 12].

Remark 2.8 Let X be a proper SNC variety such that $X = X_1 \cup X_2$, with dualizing sheaf ω_X. If $\omega_X \simeq \mathcal{O}_X$, then $D := X_1 \cap X_2$ should satisfy $D \in |\omega_{X_i}^{-1}|$. The converse does not hold in general since D may not be connected. However, if D is connected and $D \in |\omega_{X_i}^{-1}|$, then we check that $\omega_X \simeq \mathcal{O}_X$.

In order to study a general fiber of a smoothing of an SNC variety, the following map of Clemens is useful.

Theorem 2.9 [7, Theorem 6.9; 41, Theorems 5.2, 5.4] *Let $\phi\colon \mathcal{X} \to \Delta$ be a proper flat surjective morphism from a complex manifold $\mathcal{X}$ onto a 1–dimensional open disk Δ. Assume that ϕ is smooth over $\Delta \setminus 0$ and $\phi^{-1}(0) \subset \mathcal{X}$ is an SNC divisor. Let $\mathcal{X}_t := \phi^{-1}(t)$ and, for $k \geq 0$, let $\mathcal{X}_0^{[k]} \subset \mathcal{X}_0$ be the locus where $k+1$ irreducible components of $\mathcal{X}_0$ intersect.*

Then we have a continuous map $c_t\colon \mathcal{X}_t \to \mathcal{X}_0$ for $t \neq 0$ such that we have a homeomorphism $c_t^{-1}(p) \approx (S^1)^k$ when $p \in \mathcal{X}_0^{[k]} \setminus \mathcal{X}_0^{[k+1]}$ and c_t induces a homeomorphism $c_t^{-1}(\mathcal{X}_0 \setminus \mathcal{X}_0^{[1]}) \xrightarrow{\approx} \mathcal{X}_0 \setminus \mathcal{X}_0^{[1]}$. (We call the map c_t the Clemens contraction.)

3 Construction of non-Kähler Calabi–Yau 3–folds with arbitrarily large b_2

3.1 Construction of the examples

First we explain the K3 surface which is essential in the construction of our Calabi–Yau 3–folds.

Let

$$P(3) := \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

and let $\mathbb{P}_i := \mathbb{P}^1$ be the i^{th} factor of $P(3)$ for $i = 1, 2, 3$. Let $\pi_i \colon P(3) \to \mathbb{P}_i$ be the i^{th} projection and $\mathcal{O}_{P(3)}(c_1, c_2, c_3) := \bigotimes_{i=1}^{3} \pi_i^* \mathcal{O}_{\mathbb{P}_i}(c_i)$ a line bundle on $P(3)$ for $c_1, c_2, c_3 \in \mathbb{Z}$.

Let $S \subset P(3)$ be a very general $(2,2,2)$–hypersurface, that is, a very general element of the linear system $|\mathcal{O}_{P(3)}(2,2,2)|$. Then S is a K3 surface. This surface is called a *Wehler surface* and studied in several articles; see eg [43; 3; 5]. We shall recall some of its properties.

For $1 \le i < j \le 3$, the surface S has a covering involution $\iota_{ij} \colon S \to S$ corresponding to the double cover $p_{ij} \colon S \to \mathbb{P}_i \times \mathbb{P}_j$ induced by the projection $\pi_{ij} \colon P(3) \to \mathbb{P}_i \times \mathbb{P}_j$. By the Noether–Lefschetz theorem (see [42, Proposition 2.27, Theorem 3.33] or [33, Theorem 1]), we see that $\operatorname{Pic} S = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3$, where e_i is the fiber class of the elliptic fibration $p_i \colon S \to \mathbb{P}_i$ for $i = 1, 2, 3$. By this, we see that S contains no (-2)–curve. Indeed, for $D = \sum_{i=1}^{3} a_i e_i \in \operatorname{Pic} S$, we have

$$D^2 = 2\Bigg(\sum_{1 \le i < j \le 3} a_i a_j (e_i \cdot e_j) \Bigg) = 4(a_1 a_2 + a_1 a_3 + a_2 a_3) \in 4\mathbb{Z}$$

by $e_i^2 = 0$ and $e_i \cdot e_j = 2$ for $i \ne j$. Hence the nef cone $\operatorname{Nef} S \subset \operatorname{Pic} S$ of S can be described as the positive cone

$$\operatorname{Nef} S = \{ D \in \operatorname{Pic} S \mid D^2 \ge 0, D \cdot H \ge 0 \},$$

where $H := e_1 + e_2 + e_3$. First we need the following claim on the action of the involution ι_{ij} on $\operatorname{Pic} S$.

Claim 3.1 [43, Lemma 2.1] *Let $i, j, k \in \mathbb{Z}$ be integers such that $i < j$ and $\{i, j, k\} = \{1, 2, 3\}$. Then*

(i) $\iota_{ij}^*(e_i) = e_i$ *and* $\iota_{ij}^*(e_j) = e_j$,

(ii) $\iota_{ij}^*(e_k) = 2e_i + 2e_j - e_k$.

Proof of claim (i) This follows since ι_{ij} interchanges two points in a fiber $p_{ij}^{-1}(p)$ for a general $p \in \mathbb{P}_i \times \mathbb{P}_j$.

(ii) We shall show that $\iota_{12}^*(e_3) = 2e_1 + 2e_2 - e_3$. The others can be shown similarly.

Let $[S_0 : S_1], [T_0 : T_1], [U_0 : U_1]$ be the coordinates of $\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3$. Note that S can be described as

$$S = (U_0^2 F_1 + U_1^2 F_2 + U_0 U_1 F_3 = 0) \subset P(3)$$

for some very general $(2,2)$–polynomials $F_1, F_2, F_3 \in H^0(\mathbb{P}_1 \times \mathbb{P}_2, \mathcal{O}(2,2))$ on $\mathbb{P}_1 \times \mathbb{P}_2 = \mathbb{P}^1 \times \mathbb{P}^1$. Let $s := S_1/S_0$, $t := T_1/T_0$ and $u := U_1/U_0$ be the affine coordinates of $\mathbb{P}_1$, $\mathbb{P}_2$ and $\mathbb{P}_3$, respectively. Also, for $i = 1, 2, 3$, let $f_i \in \mathbb{C}[s,t]$ be the dehomogenization of F_i. Then the function field $K(S)$ of S can be described as

$$K(S) \simeq \mathbb{C}(s,t)[u] \Big/ \left(u^2 + \frac{f_3}{f_2} u + \frac{f_1}{f_2} \right)$$

and ι_{12} induces an element $\iota_{12}^\sharp \in \mathrm{Gal}(K(S)/K(\mathbb{P}_1 \times \mathbb{P}_2))$ determined by

$$\iota_{12}^\sharp(u) = -u - \frac{f_3}{f_2}.$$

By this description of $\iota_{12}^\sharp$, we see that

$$\iota_{12}^*(e_3) \neq e_3 \tag{1}$$

since F_1, F_2, F_3 are very general and f_3/f_2 is not constant on the fiber of $p_3 \colon S \to \mathbb{P}_3$. By (1) together with

$$\iota_{12}^*(e_1) \cdot \iota_{12}^*(e_3) = \iota_{12}^*(e_2) \cdot \iota_{12}^*(e_3) = 2, \quad \text{and} \quad \iota_{12}^*(e_3)^2 = 0,$$

we check that $\iota_{12}^*(e_3) = 2e_1 + 2e_2 - e_3$. □

Now let $\iota := \iota_{12} \circ \iota_{13}$, that is,

$$\iota \colon S \xrightarrow{\iota_{13}} S \xrightarrow{\iota_{12}} S.$$

Claim 3.2 *The automorphism ι induces the linear automorphism $\iota^* \in \mathrm{Aut}(\mathrm{Pic}\, S)$ corresponding to a matrix*

$$\begin{pmatrix} 1 & 2 & 6 \\ 0 & -1 & -2 \\ 0 & 2 & 3 \end{pmatrix};$$

that is, we have

$$\iota^*(e_1) = e_1, \quad \iota^*(e_2) = 2e_1 - e_2 + 2e_3, \quad \iota^*(e_3) = 6e_1 - 2e_2 + 3e_3.$$

Proof of claim Since we have $\iota_{12}^*(e_1) = e_1, \iota_{13}^*(e_1) = e_1$, we obtain $\iota^*(e_1) = e_1$.

We have $\iota_{13}^*(e_2) = 2e_1 - e_2 + 2e_3$ by Claim 3.1. By this and $\iota_{12}^*(e_2) = e_2$, we obtain $\iota^*(e_2) = 2e_1 - e_2 + 2e_3$.

By a similar computation, we obtain $\iota^*(e_3) = 6e_1 - 2e_2 + 3e_3$. Indeed, we have $\iota_{12}^*(e_3) = 2e_1 + 2e_2 - e_3$ and $\iota_{13}^*(2e_1 + 2e_2 - e_3) = 2e_1 + 2(2e_1 - e_2 + 2e_3) - e_3 = 6e_1 - 2e_2 + 3e_3$. □

By this claim, for $a \in \mathbb{Z}$, the a^{th} power $\iota^a \in \operatorname{Aut} S$ of ι induces

$$(\iota^a)^* = \begin{pmatrix} 1 & 4a^2 - 2a & 4a^2 + 2a \\ 0 & 1 - 2a & -2a \\ 0 & 2a & 1 + 2a \end{pmatrix} \in \operatorname{Aut}(\mathbb{Z}^3) \simeq \operatorname{Aut}(\operatorname{Pic} S)$$

with respect to the basis $e_1, e_2, e_3 \in \operatorname{Pic} S$ (by induction on a or use JCF).

Remark 3.3 In [5, Section 3.4], Cantat and Oguiso study $\operatorname{Aut}(S)$ in detail. They show that $\operatorname{Aut}(S)$ is a free product of 3 cyclic groups of order 2 generated by the three involutions ι_{12}, ι_{13} and ι_{23}.

Now we can construct our Calabi–Yau 3–folds as follows.

Theorem 3.4 *Let $a \in \mathbb{Z}$ be a positive integer. Then there exists a Calabi–Yau 3–fold $X := X(a)$ such that $b_2(X) = a + 3$ and $e(X) = -256a^2 + 32a - 224$, where $b_2(X)$ is the 2nd Betti number of X and $e(X)$ is the topological Euler number of X.*

Proof We first construct an SNC variety $X_0(a)$ by gluing two smooth projective varieties X_1 and X_2 as follows. For $c_1, c_2, c_3 \in \mathbb{Z}$, let $\mathcal{O}_S(c_1, c_2, c_3) := \mathcal{O}_{P(3)}(c_1, c_2, c_3)|_S$.

Construction of X_1 and X_2 Let $\mu'\colon X_1' \to P(3)$ be the blowup of $f_1, \dots, f_a$, where $f_1, \dots, f_a \in |\mathcal{O}_S(1, 0, 0)|$ are disjoint smooth fibers of the elliptic fibration $p_1\colon S \to \mathbb{P}^1$. Let $\nu\colon X_1 \to X_1'$ be the blowup along the strict transform of C_a, where $C_a \in |\mathcal{O}_S(16a^2 - a + 4, 4 - 8a, 4 + 8a)|$ is a general smooth member. Note that $\mathcal{O}_S(16a^2 - a + 4, 4 - 8a, 4 + 8a)$ is ample since we have

$$\begin{aligned}\mathcal{O}_S(16a^2 - a + 4, 4 - 8a, 4 + 8a)^2 &= 4(8(16a^2 - a + 4) + (4 - 8a)(4 + 8a)) \\ &= 4(64a^2 - 8a + 48) > 0\end{aligned}$$

and S contains no (-2)–curve. Then we see that $|\mathcal{O}_S(16a^2 - a + 4, 4 - 8a, 4 + 8a)|$ is free since there is no $\mathbb{P}^1$ on S; see eg [34, Proposition 8.1] and [22, Chapter 2, Corollary 3.15(ii)]. Thus we have $\mu := \mu' \circ \nu\colon X_1 \to P(3)$. Let $X_2 := P(3)$.

Let $S_2 := S \subset X_2$ and $S_1 \subset X_1$ be the strict transform of S and $i_j : S_j \hookrightarrow X_j$ be the inclusions for $j = 1, 2$, and let $\iota_a := \iota^a \circ \mu|_{S_1}$. By Corollary 2.2, we can construct the pushout $X_0(a)$ of two closed immersions i_1 and $i_2 \circ \iota_a$. For simplicity, we write $X_0 := X_0(a)$. Then X_0 is a proper SNC variety and fits in the diagram

$$\begin{tikzcd}
 & S_1 \arrow[r, hook, "i_1"] \arrow[dl, "\iota_a"] & X_1 \arrow[dd] \\
S_2 \arrow[d, hook, "i_2"] & & \\
X_2 \arrow[rr] & & X_0
\end{tikzcd}$$

The SNC variety X_0 satisfies the condition of Theorem 2.6 by the following claim.

Claim 3.5 (i) *X_0 is d–semistable.*

(ii) $\omega_{X_0} \simeq \mathcal{O}_{X_0}$.

(iii) $H^1(X_0, \mathcal{O}_{X_0}) = 0$ *and* $H^2(X_0^\nu, \mathcal{O}_{X_0^\nu}) = 0$, *where* $X_0^\nu \to X_0$ *is the normalization.*

Proof of claim (i) In order to check the d–semistability, we shall show that

$$\mathcal{O}_{S_1}(X_0) := \mathcal{N}_{S_1/X_1} \otimes (\iota_a)^* \mathcal{N}_{S_2/X_2} \simeq \mathcal{O}_{S_1}.$$

Let $\mu_1 := \mu|_{S_1} : S_1 \xrightarrow{\simeq} S$ and $\mathcal{O}_{S_1}(c_1, c_2, c_3) := \mu_1^* \mathcal{O}_S(c_1, c_2, c_3)$ for $c_1, c_2, c_3 \in \mathbb{Z}$. Since we have

$$\mathcal{N}_{S_1/X_1} \simeq \mathcal{O}_{S_1}(2,2,2) \otimes \mathcal{O}_{S_1}\biggl(-\biggl(\sum_{i=1}^{a} f_i + C_a\biggr)\biggr),$$
$$(\iota_a)^* \mathcal{N}_{S_2/X_2} \simeq (\iota_a)^* \mathcal{O}_S(2,2,2),$$

we obtain

$$\begin{aligned}
\mathcal{O}_{S_1}(X_0) &\simeq \mathcal{O}_{S_1}(2,2,2) \otimes \mathcal{O}_{S_1}\biggl(-\biggl(\sum_{i=1}^{a} f_i + C_a\biggr)\biggr) \otimes (\iota_a)^* \mathcal{O}_S(2,2,2) \\
&\simeq \mathcal{O}_{S_1}(16a^2+4, 4-8a, 4+8a) \otimes \mathcal{O}_{S_1}\biggl(-\biggl(\sum_{i=1}^{a} f_i + C_a\biggr)\biggr) \simeq \mathcal{O}_{S_1},
\end{aligned}$$

thus we obtain (i).

(ii) By $S_i \in |\omega_{X_i}^{-1}|$ for $i = 1, 2$ and Remark 2.8, we see that $\omega_{X_0} \simeq \mathcal{O}_{X_0}$; compare with [15, Remark 2.11].

(iii) The exact sequence

$$0 \to \mathcal{O}_{X_0} \to \mathcal{O}_{X_1} \oplus \mathcal{O}_{X_2} \to \mathcal{O}_{X_{12}} \to 0$$

implies $H^1(X_0, \mathcal{O}_{X_0}) = 0$, where $X_{12} := X_1 \cap X_2 \simeq S_i$ for $i = 1, 2$. We have $H^2(X_0^\nu, \mathcal{O}_{X_0^\nu}) = 0$ since X_1 and X_2 are rational. □

By the above and Theorem 2.6, there exists a semistable smoothing $\phi_a \colon \mathcal{X}(a) \to \Delta^1_\epsilon$ of X_0 over an open disk of a sufficiently small radius $\epsilon > 0$. Let $X(a)$ be a fiber of ϕ_a over $t \neq 0$. Note that we do not specify t and all such fibers are diffeomorphic. Let $\mathcal{X} := \mathcal{X}(a)$ and $X := X(a)$ for simplicity.

Then we have $\omega_X \simeq \mathcal{O}_X$ since we have $H^1(X_0, \mathcal{O}_{X_0}) = 0$, $H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = 0$ and, by the diagram

$$\begin{array}{ccc} H^1(\mathcal{X}, \mathcal{O}^*_{\mathcal{X}}) & \longrightarrow & H^2(\mathcal{X}, \mathbb{Z}) \\ \Big\downarrow {\scriptstyle i_0^*} & & \Big\downarrow {\scriptstyle \simeq} \\ H^1(X_0, \mathcal{O}^*_{X_0}) & \longrightarrow & H^2(X_0, \mathbb{Z}) \end{array}$$

we see that i_0^* is injective, where $i_0 \colon X_0 \hookrightarrow \mathcal{X}$ is the inclusion.

We also check that $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, 2$ by the upper semicontinuity theorem since ϵ is small. We have the following claim on the Betti numbers $b_i(X)$ of X for $i = 1, 2$.

Claim 3.6 (i) *We have $H^1(X, \mathbb{Z}) = 0$, thus $b_1(X) = 0$.*

(ii) $b_2(X) = a + 3$.

Proof of claim (i) By the exponential exact sequence, we have an exact sequence

$$H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X^*) \to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X).$$

This implies (i).

(ii) Note that $b_2(X_0) = \operatorname{rk} \operatorname{Pic} X_0$ since we can calculate that $H^i(X_0, \mathcal{O}_{X_0}) = 0$ for $i = 1, 2$ as in Claim 3.5(iii). Moreover, we see that $\operatorname{rk} \operatorname{Pic} X_0 = a + 4$ by the exact sequence

$$0 \to H^1(X_0, \mathcal{O}^*_{X_0}) \to H^1(X_1, \mathcal{O}^*_{X_1}) \oplus H^1(X_2, \mathcal{O}^*_{X_2}) \to H^1(X_{12}, \mathcal{O}^*_{X_{12}}) \to 0,$$

where the surjectivity follows from the explicit description. In order to compute $b_2(X)$, we use the Clemens contraction $c_t \colon X \to X_0$, which satisfies $c_t^{-1}(p) \approx S^1$ for $p \in X_{12}$

and $c_t^{-1}(p) = \{\mathrm{pt}\}$ for $p \notin X_{12}$ as in Theorem 2.9. We see that $R^1(c_t)_*\mathbb{Z}_X \simeq \mathbb{Z}_{X_{12}}$ since X_{12} is simply connected, and that $R^2(c_t)_*\mathbb{Z} = 0$. By this and the Leray spectral sequence

$$H^i(X_0, R^j(c_t)_*\mathbb{Z}) \Rightarrow H^{i+j}(X, \mathbb{Z}),$$

we see that $b_2(X) = b_2(X_0) - 1 = a + 3$. Indeed, we have

$$\begin{aligned} H^0(X_0, R^2(c_t)_*\mathbb{Z}) &= 0, \\ H^1(X_0, R^1(c_t)_*\mathbb{Z}) &= H^1(X_{12}, \mathbb{Z}) = 0, \\ H^2(X_0, (c_t)_*\mathbb{Z}) &\simeq H^2(X_0, \mathbb{Z}), \end{aligned}$$

and we see that the connecting homomorphism

$$\mathbb{Z} \simeq H^0(X_0, R^1(c_t)_*\mathbb{Z}) \to H^2(X_0, (c_t)_*\mathbb{Z}) \simeq H^2(X_0, \mathbb{Z})$$

is nonzero by $H^1(X, \mathbb{Z}) = 0$, and its cokernel is $H^2(X, \mathbb{Z})$. □

We compute the topological Euler number $e(X)$ as follows.

Claim 3.7 *We have* $e(X) = -256a^2 + 32a - 224$.

Proof of claim We shall use the product formula of topological Euler numbers on an oriented fiber bundle (see [37, page 481, Theorem 1]) and also an additivity formula for the Euler number on a complex algebraic variety (see [20, page 95, Exercise]). Note that we have $e(f_i) = 0$ for $i = 1, \dots, a$,

$$e(C_a) = 2 - 2g(C_a) = -(C_a^2) = -256a^2 + 32a - 192,$$

and an exceptional divisor of a blowup along a curve is a $\mathbb{P}^1$–bundle. Thus we see that $e(X_1) = e(P(3)) - 256a^2 + 32a - 192 = -256a^2 + 32a - 184$ by the above two formulas. By this and the exact sequence

$$0 \to \mathbb{Z}_{X_0} \to \mathbb{Z}_{X_1} \oplus \mathbb{Z}_{X_2} \to \mathbb{Z}_{X_{12}} \to 0,$$

we see that

$$\begin{aligned} e(X_0) = e(X_1) + e(X_2) - e(X_{12}) &= (-256a^2 + 32a - 184) + 8 - 24 \\ &= -256a^2 + 32a - 200. \end{aligned}$$

Since $c_t^{-1}(X_{12}) \to X_{12}$ is an S^1–bundle over a K3 surface, we check that

$$e(X) = e(X_0) - e(X_{12}) = -256a^2 + 32a - 200 - 24 = -256a^2 + 32a - 224$$

by the Leray spectral sequence as in Claim 3.6(ii). Indeed, $H^i(X_0, R^j(c_t)_*\mathbb{Z}) = 0$ except when $j = 0, 1$. □

By these claims, we obtain X as described in the statement of Theorem 3.4. □

3.2 Some properties of $X(a)$

First, our examples have the following Hodge-theoretic property.

Remark 3.8 The Hodge to de Rham spectral sequence degenerates at E_1 on our Calabi–Yau 3–fold $X := X(a)$; see [29, Corollary 11.24]. By this and [2, Theorem 3.3], we check that:

Proposition 3.9 *X has unobstructed deformations.*

We also check that $\dim_{\mathbb{C}} H^i(X, \Omega_X^j) = \dim_{\mathbb{C}} H^j(X, \Omega_X^i)$ for any $i, j \in \mathbb{Z}$ as follows.

We calculate $H^0(X, \Omega_X^1) = 0$ by $H^1(X, \mathbb{C}) = 0$ (Claim 3.6(i)) and the E_1–degeneration. We also have $H^0(X, \Omega_X^2) = 0$ since we obtain $H^1(X, \mathcal{O}_X^*) \simeq H^2(X, \mathbb{Z})$ by considering the exponential exact sequence as in Claim 3.6. Thus, for $i = 1, 2$, since we have $H^i(X, \mathcal{O}_X) = 0$, we have the Hodge symmetry on $H^i(X, \mathbb{C})$.

On the direct summands of $H^3(X, \mathbb{C})$, we have

$$H^j(X, \Omega_X^{3-j}) \simeq H^{3-j}(X, \Omega_X^j)$$

by the Serre duality and $\omega_X \simeq \mathcal{O}_X$. By these, we have the required equality on $H^3(X, \mathbb{C})$. Thus we cannot judge the nonprojectivity of X from the Hodge numbers.

It might be possible to show the $\partial\bar{\partial}$–lemma on X as in [17].

It may be interesting to study the fundamental group $\pi_1(X)$, the second Chern class $c_2(X)$, etc. For the fundamental group, we have the following.

Proposition 3.10 *$X = X(a)$ is simply connected.*

Proof Let $V_i \subset X_i$ be a tubular neighborhood of X_{12} for $i = 1, 2$, which can be regarded as a Δ^1–bundle over X_{12}. Let $U_1 := X_1 \cup V_2$ and $U_2 := X_2 \cup V_1$. We check that $\pi_1(X_0) = \{1\}$ by applying van Kampen's theorem to the open covering $X_0 = U_1 \cup U_2$.

Note that $\widetilde{X}_{12} := c_t^{-1}(X_{12}) \to X_{12}$ is an S^1–fibration and, from the homotopy exact sequence, we see that $\pi_1(\widetilde{X}_{12})$ is a cyclic group generated by the S^1–fiber class. Let $\widetilde{X}_i := c_t^{-1}(X_i)$ for $i = 1, 2$ and consider a neighborhood $\widetilde{V}_i := c_t^{-1}(V_i) \subset \widetilde{X}_i$ of $\widetilde{X}_{12}$ for $i = 1, 2$. Let $\widetilde{U}_1 := \widetilde{X}_1 \cup \widetilde{V}_2$, $\widetilde{U}_2 := \widetilde{X}_2 \cup \widetilde{V}_1$ and $\widetilde{U}_{12} := \widetilde{U}_1 \cap \widetilde{U}_2$. We can regard $\widetilde{V}_1 \cup \widetilde{V}_2$ as an annulus bundle over X_{12}. By this, we see that $\widetilde{U}_i$ is homotopic to $X_i \setminus X_{12}$ for $i = 1, 2$. The following claim is important.

Claim 3.11 *Let $X'_i := X_i \setminus X_{12}$ for $i = 1, 2$. Then we have $\pi_1(X'_1) = \{1\}$ and $\pi_1(X'_2) \simeq \mathbb{Z}/2\mathbb{Z}$.*

Proof of claim We check that $\pi_1(X'_2)$ is abelian by Nori's result [27, Corollary 2.10], as follows. By [36, page 311, Proposition], we see that $\pi_1(X'_2) \simeq \pi_1(L^*)$, where L is the total space of $\mathcal{O}_{X_2}(X_{12})$ and $L^* \subset L$ is the complement of the zero section. Hence the homotopy exact sequence can be written as

$$\pi_1(\mathbb{C}^*) \to \pi_1(X'_2) \to \pi_1(X_2) \to 1,$$

and this implies that $\pi_1(X'_2)$ is abelian by $\pi_1(\mathbb{C}^*) \simeq \mathbb{Z}$ and $\pi_1(X_2) = \{1\}$.

Thus we compute $\pi_1(X'_2) \simeq H_1(X'_2, \mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$ by the Gysin long exact sequence

$$\cdots \to H_2(X_2, \mathbb{Z}) \to H_0(X_{12}, \mathbb{Z}) \to H_1(X'_2, \mathbb{Z}) \to H_1(X_2, \mathbb{Z}) \to \cdots$$

as in [9, page 46, (2.13)].

Let $E'_j := E_j \setminus X_{12}$ for $j = 1, \dots, a$ and $F' := F \setminus X_{12}$ be the open subsets of μ–exceptional divisors for the blowup $\mu\colon X_1 \to X_2$. Note that

$$(X'_1) \setminus (E'_2 \cup \cdots \cup E'_a \cup F') \simeq X'_2$$

since $X_1 \to X_2 = P(3)$ is the blowup along $f_1, \dots, f_a$ and the strict transform of C_a. Note also that E'_j and F' are $\mathbb{C}$–bundle over the blowup centers $f_1, \dots, f_a$ and C_a, respectively. Given these, we compute that $\pi_1(X'_1) = \{1\}$ by van Kampen's theorem, as follows. Let $W'_j \subset X'_1$ be a tubular neighborhood of E'_j for $j = 1, \dots, a$.

We compute that

$$\pi_1(X'_1 \setminus (E'_2 \cup \cdots \cup E'_a \cup F')) \simeq \pi_1(X'_2) *_{\pi_1(W'_1 \setminus E'_1)} \pi_1(W'_1) = \{1\}$$

as follows: W'_1 and $W'_1 \setminus E'_1$ can be regarded as a Δ^1–bundle and a $(\Delta^1)^*$–bundle over E'_1, where $(\Delta^1)^* := \Delta^1 \setminus \{0\}$. We check that $\pi_1(W'_1 \setminus E'_1) \to \pi_1(W'_1)$ is surjective and its kernel $K \simeq \mathbb{Z}$ maps surjectively to $\pi_1(X'_2)$ by $\mu_*\colon \pi_1(W'_1 \setminus E'_1) \to \pi_1(X'_2)$. The latter surjectivity follows from a commutative diagram

$$\begin{tikzcd}
H_0(X_{12}, \mathbb{Z}) \arrow[r] & H_1(X'_2, \mathbb{Z}) \arrow[r] & 0 \\
H_0(E'_1, \mathbb{Z}) \arrow[r] \arrow[u] & H_1(W'_1 \setminus E'_1, \mathbb{Z}) \arrow[u] &
\end{tikzcd}$$

as in [9, page 46, (2.13)], since a generator of $H_0(E'_1, \mathbb{Z})$ is sent to that of $H_1(X'_2, \mathbb{Z})$. Hence we see that $X'_1 \setminus (E'_2 \cup \cdots \cup E'_a \cup F')$ is simply connected.

Similarly, we check that the fundamental group does not change if we add divisors $E'_2, \dots, E'_a, F' \subset X'_1$. In particular, we have $\pi_1(X'_1) = \{1\}$. □

By Claim 3.11 and the isomorphism

$$\pi_1(X) \simeq \pi_1(\widetilde{U}_1) *_{\pi_1(\widetilde{U}_{12})} \pi_1(\widetilde{U}_2) \simeq \pi_1(X_1') *_{\pi_1(\widetilde{X}_{12})} \pi_1(X_2'),$$

we obtain $\pi_1(X) = \{1\}$, since we have the following claim:

Claim 3.12 *The map $\pi_1(\widetilde{U}_{12}) \to \pi_1(\widetilde{U}_2)$ is surjective.*

Proof of claim Since we have $\widetilde{U}_2 = \widetilde{U}_{12} \cup (\widetilde{X}_2 \setminus \widetilde{X}_{12})$, we have

$$\pi_1(\widetilde{U}_2) \simeq \pi_1(\widetilde{U}_{12}) *_{\pi_1(\widetilde{V}_2 \setminus \widetilde{X}_{12})} \pi_1(\widetilde{X}_2 \setminus \widetilde{X}_{12}).$$

Since $\widetilde{X}_2 \setminus \widetilde{X}_{12} \simeq X_2 \setminus X_{12} = X_2'$ and $\widetilde{V}_2 \setminus \widetilde{X}_{12} \simeq V_2 \setminus X_{12} =: V_2'$, it is enough to show the surjectivity of

$$(\iota_{V_2'})_* \colon \pi_1(V_2') \to \pi_1(X_2').$$

V_2' is a $(\Delta^1)^*$–bundle over X_{12} and $\pi_1(V_2')$ is a cyclic group, thus $\pi_1(X_2')$ and $\pi_1(V_2')$ are abelian. Hence the surjectivity of $(\iota_{V_2'})_*$ follows from the commutative diagram

$$\begin{tikzcd}
H_2(V_2,\mathbb{Z}) \arrow[r] \arrow[d] & H_0(X_{12},\mathbb{Z}) \arrow[r] \arrow[d] & H_1(V_2 \setminus X_{12},\mathbb{Z}) \arrow[r] \arrow[d] & 0 \\
H_2(X_2,\mathbb{Z}) \arrow[r] & H_0(X_{12},\mathbb{Z}) \arrow[r] & H_1(X_2 \setminus X_{12},\mathbb{Z}) \arrow[r] & 0
\end{tikzcd}$$

with exact rows as in [9, page 46, (2.13)]. □

This completes the proof of Proposition 3.10. □

An anonymous referee pointed out the following bimeromorphic unboundedness of our examples.

Remark 3.13 This is based on the referee's comment. Let $a \neq a'$ be positive integers. Then $X(a)$ and $X(a')$ are not bimeromorphic, which we prove as follows.

Suppose that they are bimeromorphic and let $\phi \colon X(a) \dashrightarrow X(a')$ be a bimeromorphic map. Let $\nu \colon \widetilde{X} \to X(a)$ be a resolution of the indeterminacy of ϕ which induces a bimeromorphic morphism $\mu \colon \widetilde{X} \to X(a')$. We see that ϕ is an isomorphism in codimension one since $\omega_{X(a)}$ and $\omega_{X(a')}$ are trivial. Then we have the pushforward homomorphism $\phi_* := \nu_* \circ \mu^* \colon \operatorname{Pic} X(a) \to \operatorname{Pic} X(a')$. We see that ϕ_* is an isomorphism since ϕ is an isomorphism in codimension one. Hence $\operatorname{rk}\operatorname{Pic} X(a) = \operatorname{rk}\operatorname{Pic} X(a')$, and this is a contradiction.

Hence our examples show the bimeromorphic unboundedness of non-Kähler Calabi–Yau 3–folds. We are not sure whether the examples of Clemens and Friedman are bimeromorphically unbounded or not. (We do not know whether a bimeromorphic map preserves the Betti number of non-Kähler Calabi–Yau 3–folds.)

3.3 On the nonprojectivity of X

In this section, we check the nonprojectivity of the SNC variety X_0 and the Calabi–Yau 3–fold X, which are constructed in Theorem 3.4.

Hironaka [21] constructed a degeneration of a projective manifold to a proper manifold which is nonprojective. Thus we cannot judge nonprojectivity of a general fiber from nonprojectivity of a central fiber. We use the following lemma to see the nonprojectivity of a general fiber of the smoothing.

Lemma 3.14 *Let $\phi\colon \mathcal{X} \to \Delta^1$ be a semistable smoothing of a proper SNC variety $\mathcal{X}_0$ with $\omega_{\mathcal{X}_0} \simeq \mathcal{O}_{\mathcal{X}_0}$ such that some fiber $\mathcal{X}_t$ of ϕ over $t \neq 0$ is a projective Calabi–Yau n–fold. Assume that $\mathcal{X}_0$ has only two projective irreducible components X_1 and X_2, and that $X_{12} := X_1 \cap X_2$ is a simply connected Calabi–Yau $(n-1)$–fold.* (Note that $\mathcal{X}_0$ may not be projective.)

Then there exists a big line bundle $\mathcal{L}_0$ on $\mathcal{X}_0$ such that $h^0(X_i, \mathcal{L}_0|_{X_i}) > 0$ for $i = 1, 2$.

Proof We first need the following.

Claim 3.15 *The restriction homomorphism $\gamma\colon H^2(\mathcal{X}, \mathbb{Z}) \to H^2(\mathcal{X}_t, \mathbb{Z})$ is surjective for $t \neq 0$.*

Proof of claim By the Clemens contraction $c_t\colon \mathcal{X}_t \to \mathcal{X}_0$ as in Theorem 2.9, we may regard γ as

$$c_t^*\colon H^2(\mathcal{X}_0, \mathbb{Z}) \to H^2(\mathcal{X}_t, \mathbb{Z}).$$

This is surjective since we have

$$H^1(\mathcal{X}_0, R^1(c_t)_*\mathbb{Z}) = 0 \quad \text{and} \quad H^0(\mathcal{X}_0, R^2(c_t)_*\mathbb{Z}) = 0.$$

Indeed, we have $R^2(c_t)_*\mathbb{Z} = 0$ since $\mathcal{X}_0$ has no triple point, and since X_{12} is simply connected we see that $R^1(c_t)_*\mathbb{Z} \simeq \mathbb{Z}_{X_{12}}$. Thus we can use the Leray spectral sequence as in Claim 3.6. □

Since we have $h^i(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = 0$ and $h^i(\mathcal{X}_t, \mathcal{O}_{\mathcal{X}_t}) = 0$ for $i = 1, 2$, we have

$$\operatorname{Pic} \mathcal{X} \simeq H^2(\mathcal{X}, \mathbb{Z}) \quad \text{and} \quad \operatorname{Pic} \mathcal{X}_t \simeq H^2(\mathcal{X}_t, \mathbb{Z})$$

by the exponential exact sequence. Let $\mathcal{L}_t$ be a very ample line bundle on $\mathcal{X}_t$. By the above and Claim 3.15, there exists a line bundle $\mathcal{L}$ on $\mathcal{X}$ such that $\mathcal{L}|_{\mathcal{X}_t} \simeq \mathcal{L}_t$. We can lift sections of $\mathcal{L}_t$ to $\mathcal{L}$ as follows.

Claim 3.16 *The restriction $H^0(\mathcal{X}, \mathcal{L}) \to H^0(\mathcal{X}_t, \mathcal{L}_t)$ is surjective.*

Proof of claim Since we have an exact sequence

$$(2) \quad H^0(\mathcal{X}, \mathcal{L}) \to H^0(\mathcal{X}_t, \mathcal{L}_t) \to H^1(\mathcal{X}, \mathcal{L} \otimes \mathcal{O}_{\mathcal{X}}(-\mathcal{X}_t)) \xrightarrow{\Phi} H^1(\mathcal{X}, \mathcal{L}) \to H^1(\mathcal{X}_t, \mathcal{L}_t),$$

it is enough to show that Φ is injective. We see that Φ is surjective by $H^1(\mathcal{X}_t, \mathcal{L}_t) = 0$. We also see that $H^1(\mathcal{X}, \mathcal{L})$ is finite-dimensional. Indeed, $H^1(\Delta^1, \phi_* \mathcal{L}) = 0$ and $H^0(\Delta^1, R^1 \phi_* \mathcal{L})$ is finite-dimensional since $R^1 \phi_* \mathcal{L}$ is coherent and supported on the origin. By these and $\mathcal{O}_{\mathcal{X}}(-\mathcal{X}_t) \simeq \mathcal{O}_{\mathcal{X}}$, we see that Φ is an isomorphism, and therefore injective. □

By Claim 3.16, we can choose sections $s_0, \dots, s_M \in H^0(\mathcal{X}, \mathcal{L})$ which lift a basis of $H^0(\mathcal{X}_t, \mathcal{L}_t)$. Let $Z(s_j) \subset \mathcal{X}$ be the divisor defined by s_j for $j = 0, \dots, M$. Let

$$m_i := \min_{j=0,\dots,M} \{\operatorname{mult}_{X_i}(Z(s_j))\} \quad \text{for } i = 1, 2,$$
$$\mathcal{L}' := \mathcal{L} \otimes \mathcal{O}_{\mathcal{X}}(-m_1 X_1 - m_2 X_2).$$

Then we obtain sections $s'_0, \dots, s'_M \in H^0(\mathcal{X}, \mathcal{L}')$ induced by $s_0, \dots, s_M$ whose base locus does not contain X_1 and X_2. Hence there exists $s' \in H^0(\mathcal{X}, \mathcal{L}')$ which does not vanish identically on each X_i.

Now let $\mathcal{L}_0 := \mathcal{L}'|_{\mathcal{X}_0}$. Then we have nonzero sections $s'|_{X_i} \in H^0(X_i, \mathcal{L}_0|_{X_i})$ for $i = 1, 2$. We also see that $\mathcal{L}_0$ is big since we have $\mathcal{L}'|_{\mathcal{X}_t} \simeq \mathcal{L}_t$, and we check that $H^0(\mathcal{X}, \mathcal{L}'^{\otimes m}) \to H^0(\mathcal{X}_t, \mathcal{L}_t^{\otimes m})$ is surjective for $m > 0$, as in Claim 3.16. Thus $\mathcal{L}_0$ has the required property. □

Remark 3.17 There is a conjecture which states that any smooth degeneration of a projective manifold is Moishezon; see [30, Conjecture 1.1] and also [31; 32]. We can also ask whether a semistable degeneration of a projective manifold admits a big line bundle as in Lemma 3.14.

We can conclude that X_0 and X are both nonprojective by the following result.

Proposition 3.18 *Let $X_0 := X_0(a)$ and $X := X(a)$ be the SNC variety and the Calabi–Yau 3–fold constructed in Theorem 3.4 for $a > 0$. Let $\mathcal{L}_0 \in \operatorname{Pic} X_0$ be a line bundle such that $h^0(X_i, \mathcal{L}_i) > 0$ for $i = 1, 2$, where $\mathcal{L}_i := \mathcal{L}_0|_{X_i}$. Also let $E_j \subset X_1$ for $j = 1, \dots, a$ be the exceptional divisor over the elliptic curve f_j.*

Then we have

$$\mathcal{L}_1 \simeq \mu^* \mathcal{O}_{P(3)}(a_1, 0, 0) - \sum_{j=1}^{a} b_j E_j \quad \text{and} \quad \mathcal{L}_2 \simeq \mathcal{O}_{P(3)}\Bigl(a_1 - \sum_{j=1}^{a} b_j, 0, 0\Bigr)$$

for some $a_1 \geq 0$ and $b_j \in \mathbb{Z}$. In particular, X_0 does not admit a line bundle as in Lemma 3.14, thus X_0 and X are not projective.

Proof Recall that $\mu\colon X_1 \to P(3)$ is the blowup of $f_1, \dots, f_a, C_a$ and $X_2 = P(3)$, where $f_1, \dots f_a \in |\mathcal{O}_S(1,0,0)|$ and $C_a \in |\mathcal{O}_S(16a^2 - a + 4, 4 - 8a, 4 + 8a)|$. Let $F \subset X_1$ be the μ–exceptional divisor over C_a. Then we can write

$$\mathcal{L}_1 = \mu^* \mathcal{O}_{P(3)}(a_1, a_2, a_3) \otimes \mathcal{O}_{X_1}\Bigl(-\sum_{j=1}^{a} b_j E_j - cF\Bigr)$$

for some integers $a_1, a_2, a_3, b_1, \dots, b_a, c$. We can also write

$$\mathcal{L}_2 = \mathcal{O}_{P(3)}(a_1', a_2', a_3')$$

for some integers a_1', a_2', a_3'. We see that $a_i, a_i' \geq 0$ for all i since $\mathcal{L}_1$ and $\mathcal{L}_2$ are effective.

Note that X_0 is the union of X_1 and X_2 glued along anticanonical members $S_i \in |-K_{X_i}|$ via an isomorphism

$$\iota_a := \iota^a \circ \mu|_{S_1}\colon S_1 \to S_2.$$

Then we have

$$\mathcal{L}_1|_{S_1} \simeq (\iota_a)^* \mathcal{L}_2|_{S_2}$$

and both sides can be written as

$$\mathcal{L}_1|_{S_1} \simeq \mathcal{O}_{S_1}\Bigl(a_1 - \sum_{j=1}^{a} b_j - c(16a^2 - a + 4), a_2 - c(4 - 8a), a_3 - c(4 + 8a)\Bigr),$$

$$\begin{aligned}(\iota_a)^* \mathcal{L}_2|_{S_2} \simeq \mathcal{O}_{S_1}\bigl(&a_1' + a_2'(4a^2 - 2a) + a_3'(4a^2 + 2a),\\ &a_2'(1 - 2a) + a_3'(-2a), a_2'(2a) + a_3'(1 + 2a)\bigr).\end{aligned}$$

By comparing the 2nd and 3rd coordinates, we obtain

$$\begin{aligned} a_2 + c(8a - 4) &= a_2'(1 - 2a) + a_3'(-2a), \\ a_3 - c(8a + 4) &= a_2'(2a) + a_3'(1 + 2a). \end{aligned}$$

These imply that

$$c(8a-4) = -a_2 + a_2'(-2a+1) + a_3'(-2a), \tag{3}$$

$$c(8a+4) = a_3 + a_2'(-2a) + a_3'(-2a-1). \tag{4}$$

Now suppose that one of a_2, a_3, a_2', a_3' is positive. By equation (3), we obtain $c \le 0$; $c = 0$ is possible only when $a_2 = a_2' = a_3' = 0$. Then we have $a_3 > 0$, and this contradicts (4). Hence we obtain $c < 0$. Moreover, by (3) and (4) we obtain

$$0 > 4c = c(8a+4) - \frac{2a}{2a-1}c(8a-4) = a_3 + \frac{2a}{2a-1}a_2 + \frac{1}{2a-1}a_3' \ge 0.$$

This is a contradiction, and we see that $a_2 = a_3 = a_2' = a_3' = 0$. This implies that $c = 0$ and that $\mathcal{L}_1$ and $\mathcal{L}_2$ are of the form stated. □

We compute the algebraic dimension of the very general fiber X as follows.

Proposition 3.19 *Let $X = X(a) = \mathcal{X}_t$ be a smooth fiber of a semistable smoothing $\mathcal{X}(a) \to \Delta^1$ over $t \in \Delta^1 \setminus \{0\}$, as in Theorem 3.4. Let $a(X)$ be its algebraic dimension.*

(i) *X admits a surjective morphism $\psi\colon X \to \mathbb{P}^1$ whose general fibers are K3 surfaces.*

(ii) *X is not projective.*

(iii) *We have $a(X) = 1$ for a very general $t \in \Delta^1$.*

Proof (i) Let $\mathcal{H}_1 := \mu^*\mathcal{O}(1,0,0)$ on X_1 and $\mathcal{H}_2 := \mathcal{O}(1,0,0)$ on X_2. These glue to give a line bundle $\mathcal{H}_0 \in \operatorname{Pic} X_0$, which induces a morphism $X_0 \to \mathbb{P}^1$. We calculate that $H^1(X_0, \mathcal{H}_0) = 0$.

Since we have $H^2(\mathcal{X}, \mathbb{Z}) \simeq H^2(X_0, \mathbb{Z})$, there exists $\mathcal{H} \in \operatorname{Pic} \mathcal{X}$ such that $\mathcal{H}|_{X_0} \simeq \mathcal{H}_0$. We check that $H^1(\mathcal{X}, \mathcal{H}) = 0$ by $H^1(\mathcal{X}_0, \mathcal{H}_0) = 0$ and the upper-semicontinuity theorem. Thus we see that $H^0(\mathcal{X}, \mathcal{H}) \to H^0(\mathcal{X}_t, \mathcal{H}_t)$ is surjective for $t \in \Delta^1$ sufficiently close to 0. Hence the line bundle $\mathcal{H}_t$ also induces a surjective morphism $\psi_t\colon X := \mathcal{X}_t \to \mathbb{P}^1$.

We check that the general fiber X_λ of ψ_t at $\lambda \in \mathbb{P}^1$ is a K3 surface as follows. For $i = 0, 1, 2$, let $X_{i,\lambda}$ be the general fiber of the morphism $X_i \to \mathbb{P}^1$ induced by $\mathcal{H}_i$. We see that $X_{1,\lambda}$ is isomorphic to a blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at 16 points, and $X_{2,\lambda} \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Thus we compute that $H^1(X_{0,\lambda}, \mathcal{O}) = 0$, and this implies that $H^1(X_\lambda, \mathcal{O}) = 0$ by the upper-semicontinuity. Hence, X_λ is a K3 surface.

(ii) Suppose that some $\mathcal{X}_t$ is a projective Calabi–Yau 3–fold. By Lemma 3.14, there exists a big line bundle $\mathcal{L}_0$ on $\mathcal{X}_0$ such that $h^0(\mathcal{L}_0|_{X_i}) > 0$ for $i = 1, 2$. However, this does not exist on $X_0(a)$ by Proposition 3.18. This is a contradiction, and $\mathcal{X}_t$ is not projective.

(iii) For $\mathcal{M} \in \operatorname{Pic} \mathcal{X}$, the dimension $h^0(\mathcal{X}_t, \mathcal{M}_t)$ for $\mathcal{M}_t := \mathcal{M}|_{\mathcal{X}_t}$ is constant for very general $t \in \Delta^1$ and $h^0(\mathcal{X}_t, \mathcal{M}_t) \le h^0(\mathcal{X}_0, \mathcal{M}_0)$. This follows from the upper semicontinuity theorem and the fact that $\operatorname{Pic} X_0$ is countable.

Suppose that $a(X) \ge 2$. Then X admits an effective line bundle L with Kodaira dimension $\kappa(L) \ge 2$. Since the restriction homomorphism $H^2(\mathcal{X}, \mathbb{Z}) \to H^2(\mathcal{X}_t, \mathbb{Z})$ is surjective, there exists $\mathcal{L} \in \operatorname{Pic} \mathcal{X}$ such that $\mathcal{L}_t := \mathcal{L}|_{\mathcal{X}_t} \simeq L$. Then we see that $\kappa(\mathcal{L}_{t'}) \ge 2$ for very general $t' \in \Delta^1$. Hence we obtain $\kappa(\mathcal{L}_0) \ge 2$ and $\phi_* \mathcal{L} \ne 0$. Thus we see that $H^0(\mathcal{X}, \mathcal{L}) = H^0(\Delta^1, \phi_* \mathcal{L}) \ne 0$, since $\phi_* \mathcal{L}$ is coherent. Now, for $i = 1, 2$, let

$$m_i := \min\{\operatorname{ord}_{X_i}(s) \mid s \in H^0(\mathcal{X}, \mathcal{L}) \setminus \{0\}\}$$

and $\mathcal{L}' := \mathcal{L} \otimes \mathcal{O}_{\mathcal{X}}(-m_1 X_1 - m_2 X_2)$. Then $\mathcal{L}'$ admits a nonzero section which does not vanish entirely on both X_1 and X_2. Thus we see that $\mathcal{L}'_0 := \mathcal{L}'|_{X_0}$ satisfies the property as in Proposition 3.18, and $\kappa(\mathcal{L}'_0) \le 1$. This is a contradiction, since $\mathcal{L}'_0$ should also satisfy $\kappa(\mathcal{L}'_0) \ge 2$.

Hence we obtain $a(X) \le 1$. By this and (i), we obtain $a(X) = 1$. □

Remark 3.20 Let us also check that $X = \mathcal{X}_t$ is not of class $\mathcal{C}$ for a very general t, that is, X is not bimeromorphic to a Kähler manifold. Suppose that X is of class $\mathcal{C}$ and has a proper bimeromorphic map $\widetilde{X} \to X$ from a Kähler manifold $\widetilde{X}$. Since we also have $H^2(\widetilde{X}, \mathbb{C}) \simeq H^1(\widetilde{X}, \Omega^1_{\widetilde{X}})$ by $H^2(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = 0 = H^0(\widetilde{X}, \Omega^2_{\widetilde{X}})$, we see that $\widetilde{X}$ is projective by the Kodaira embedding theorem. Thus X is Moishezon and this contradicts Proposition 3.19.

We do not know whether a Calabi–Yau 3–fold of algebraic dimension ≥ 2 appears as some fiber of the smoothing. Note that the Moishezon (or class $\mathcal{C}$) property is not stable under deformation [4; 25].

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