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d_p–convergence and ϵ–regularity theorems for entropy and scalar curvature lower bounds

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Consider a sequence of Riemannian manifolds (M_i^n, g_i) whose scalar curvatures and entropies are bounded from below by small constants $R_i, \mu_i \geq -\epsilon_i$. The goal of this paper is to understand notions of convergence and the structure of limits for such spaces. As a first issue, even in the seemingly rigid case $\epsilon_i \to 0$, we will construct examples showing that from the Gromov–Hausdorff or intrinsic flat points of view, such a sequence may converge wildly, in particular to metric spaces with varying dimensions and topologies and at best a Finsler-type structure. On the other hand, we will see that these classical notions of convergence are the incorrect ones to consider. Indeed, even a metric space is the wrong underlying category to be working on.

Instead, we will introduce a weaker notion of convergence called d_p–convergence, which is valid for a class of rectifiable Riemannian spaces. These rectifiable spaces will have a well-behaved topology, measure theory and analysis. This includes the existence of gradients of functions and absolutely continuous curves, though potentially there will be no reasonably associated distance function. Under this d_p notion of closeness, a space with almost nonnegative scalar curvature and small entropy bounds must in fact always be close to Euclidean space, and this will constitute our ϵ–regularity theorem. In particular, any sequence (M_i^n, g_i) with lower scalar curvature and entropies tending to zero must d_p–converge to Euclidean space.

More generally, we have a compactness theorem saying that sequences of Riemannian manifolds (M_i^n, g_i) with small lower scalar curvature and entropy bounds $R_i, \mu_i \geq -\epsilon$ must d_p–converge to such a rectifiable Riemannian space X. In the context of the examples from the first paragraph, it may be that the distance functions of M_i are degenerating, even though in a well-defined sense the analysis cannot be. Applications for manifolds with small scalar and entropy lower bounds include an L^∞–Sobolev embedding and a priori L^p scalar curvature bounds for $p < 1$.

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1 Introduction

It is a well-known theme that understanding the structure of a manifold (M^n, g) under restrictions on curvature is essentially equivalent to understanding the structure of singular limits $M_i^n \to X$. During the early days of studying manifolds with bounded curvature operator, it was sufficient to restrict the study to manifold limits X under $C^{k,\alpha}$–convergence; see Cheeger [14; 15]. When the analysis of spaces with lower and bounded Ricci curvature began, it became necessary to expand this point of view to general metric space limits X, and to discuss convergence in the Gromov–Hausdorff sense; see Gromov [26]. This allowed for the necessary formation of singularities in possible limit spaces. It also became quite important at this stage to distinguish between *collapsed* and *noncollapsed* limits, where noncollapsing of the sequence M_i^n can be understood as the existence of a uniform lower bound on the volumes of balls. A key result in this context, and indeed the beginning point for the regularity theory, is an ϵ–regularity theorem. This says that if the volume of a unit ball is close to that of the Euclidean ball, then that ball must be close both topologically and geometrically to a Euclidean ball.

In this paper we study manifolds and sequences M_i^n under lower bounds on scalar curvature. The correct replacement for *noncollapsing* in this context is a lower bound on the entropy μ of the manifold, or almost equivalently one could ask for bounds on the L^1–Sobolev constant, though we will see there are unnatural aspects to that assumption. Our goal is then to prove and understand the corresponding ϵ–regularity in this context: a statement which should say that if the scalar and entropy lower bounds are small, then a ball should be close to a Euclidean ball.

It is already understood from the work of Sormani [48] that there is an immediate problem when dealing with lower scalar curvature bounds, namely the notion of Gromov–Hausdorff closeness cannot be the correct one. The examples in [48] mimic those from minimal surface theory, and show that small volume tentacles may appear when only a lower scalar curvature bound is assumed. One possible fix for issues like this is the intrinsic flat distance; see Sormani and Wenger [49]. We will see in this paper that the problem is actually much worse. We will build examples — see Theorems 1.12–1.14 and Section 9 — which show that even under small lower bounds on scalar curvature and entropy, the Gromov–Hausdorff and intrinsic flat limits may be completely wild. Wild here can include jumps in topology, dimension and the formation of Finsler or worse types of geometries. Fundamentally, the issue at hand is that distance functions simply do not behave well under lower scalar curvature and entropy bounds, and therefore any notion of convergence which is based on the distance function must correspondingly fail. From the correct perspective, this should not be surprising as the distance function is closely related to the $W^{1,\infty}$ behavior of functions, and it may simply be too much to ask that this remains uniformly controlled in such a sequence. Indeed, it is now well understood from the study of RCD spaces[1] that, said correctly, $W^{1,\infty}$–control on the analysis is essentially equivalent to lower bounds on *Ricci* curvature, and therefore one should almost expect distance functions to break down in the context of only scalar curvature bounds.

In order to solve this problem we will introduce in this paper a new notion of convergence, d_p–convergence. The effect of this will be to take the required $W^{1,\infty}$–control needed for convergence of distance functions, and reduce it to a required $W^{1,p}$–control for this weaker notion of convergence. The notion of d_p–convergence is based on associating to a manifold, or more generally a rectifiable space, a natural family of distance functions d_p. As we will see, d_p understands and controls the behavior of the Sobolev space $W^{1,p}$, with $d_\infty = d$ becoming the standard distance function. Let us begin with a definition.

Definition 1.1 (d_p–distance on manifolds) Given a Riemannian manifold (M^n, g) and a real number $p \in (n, \infty]$, we define the $d_{p,g}$–distance between any $x, y \in M$ by

$$d_{p,g}(x,y) = d_p(x,y) = \sup\left\{ |f(x) - f(y)| : \int_M |\nabla f|^p \, d\mathrm{vol}_g \le 1 \right\}. \tag{1-1}$$

[1] See Ambrosio, Gigli and Savaré [6], Bakry and Émery [9], Lott and Villani [38] and Sturm [50].

Remark 1.2 The concept of the d_p–distance has made an appearance in the literature previously for distinct reasons; for instance in [20], De Cecco and Palmer used it in the study of Lipschitz n–manifolds and showed that $\delta(x,y) := \lim_{p\to\infty} d_p(x,y)$ defined a distance function on such spaces and coincides with the geodesic distance on smooth Riemannian manifolds.

We will discuss this more precisely in Section 2.2, but let us observe that d_p does not need an underlying metric structure in order to be defined. A rectifiable structure, which gives the ability to differentiate functions and integrate them, will be sufficient. In particular, the functions d_p are well-defined on rectifiable Riemannian spaces (X,g). These are precisely defined in Definition 2.2, but roughly are topological measure spaces with a compatible rectifiable structure and Riemannian metric on the rectifiable charts.

For such a rectifiable space X, it may be that $d_p(x,y)=0$ for p sufficiently large, and thus d_p only defines a weak distance function. This will be possible even for limits of manifolds under small lower scalar curvature and entropy bounds $R,\mu\ge-\epsilon$; see Example 9.5. We will say X is d_p–complete when it defines an honest distance function whose topology is that of X; see Section 2.2 for a larger account of the subtle points which arise. A consequence of our main theorems is that for $p\le p(n,\epsilon)$, limits will be d_p–complete, and indeed d_p is actually very well behaved. In particular, such limits X will be doubling spaces up to scale 1 with respect to the d_p–distances.

Now that we have the d_p–distance defined and the correct category of spaces to consider it on, namely rectifiable spaces X with a Riemannian structure, let us consider their convergence. As is usual let us begin with the compact case.

Definition 1.3 (d_p–convergence) A sequence $\{(X_i,g_i)\}$ of compact rectifiable Riemannian spaces, in particular a sequence of compact Riemannian manifolds, converges to a compact rectifiable Riemannian space (X,g) in the d_p sense if

$$d_{\mathrm{mGH}}\big((X_i,d_{p,g_i},d\mathrm{vol}_{g_i}),(X,d_{p,g},d\mathrm{vol}_g)\big)\to 0. \tag{1-2}$$

Here d_{mGH} denotes the measured Gromov Hausdorff distance between metric spaces.

Remark 1.4 One could also consider the intrinsic flat distance between the spaces $(X_i,d_{p,g_i},d\mathrm{vol}_{g_i})$ and $(X,d_{p,g},d\mathrm{vol}_g)$. The main observation in this work is the weakening of the usual distance with $p=\infty$ to $p<\infty$. We believe that a counterpart of Theorem 1.7 below should hold with respect to the intrinsic flat distance between the d_p spaces, making use of the key estimates of Theorem 1.11 below.

Remark 1.5 Even if a sequence $\{(M_i, g_i)\}$ of Riemannian manifolds has a (geodesic) Gromov–Hausdorff limit (Y, d), the spaces X and Y need not even be topologically equivalent.

Remark 1.6 Let us briefly mention that pointed convergence for noncompact spaces is defined in a similar spirit as in the Gromov–Hausdorff case, however there is a subtle point due to the behavior of d_p at large distances. See Definition 2.39 for precision.

Throughout the paper we will let $\mathcal{B}_{p,g}(x, r)$ denote the ball of radius r with respect to d_p. That is,

$$\mathcal{B}_{p,g}(x, r) = \{y \in M : d_p(x, y) < r\}. \tag{1-3}$$

1.1 Main ϵ–regularity theorem

Let us now move toward our first main result of the paper. We begin by recalling that the Perelman $\mathcal{W}$–functional, introduced in [43], is defined for a function $f \in C^\infty(M)$ and real number $\tau > 0$ by

$$\mathcal{W}(g, f, \tau) = \frac{1}{(4\pi\tau)^{n/2}} \int_M \{\tau(|\nabla f|^2 + R) + f - n\} e^{-f} \, d\mathrm{vol}_g. \tag{1-4}$$

The Perelman entropy $\mu(g, \tau)$, which can be viewed as the optimal constant in a log-Sobolev inequality at scale $\tau^{1/2}$, is given by

$$\mu(g, \tau) = \inf\left\{\mathcal{W}(g, f, \tau) : \frac{1}{(4\pi\tau)^{n/2}} \int_M e^{-f} \, d\mathrm{vol}_g = 1, \; e^{-f/2} \in W^{1,2}(M)\right\}. \tag{1-5}$$

Finally, Perelman's ν–functional is given by

$$\nu(g, \tau) = \inf\{\mu(g, \tau') : \tau' \in (0, \tau)\},$$

and just guarantees that we are measuring the entropy at all scales below some point. See Section 3.3 for more background. The Perelman entropy $\mu(g, \tau)$ of a complete well-behaved Riemannian manifold (M, g) is nonpositive for all $\tau > 0$. Moreover, if the entropy is equal to zero for some $\tau > 0$, then (M, g) is isometric to Euclidean space. This rigidity statement is the basis of our first main result, which is perturbative in nature.

Theorem 1.7 (ϵ–regularity theorem) *Let (M^n, g) be a complete Riemannian manifold with bounded curvature and fix $\epsilon > 0$ and $p \geq n+1$. There exists a $\delta = \delta(n, \epsilon, p)$ such that if*

$$R \geq -\delta \quad \text{and} \quad \nu(g, 2) \geq -\delta, \tag{1-6}$$

then for all $x \in M$, we have

$$d_{\mathrm{GH}}\big((\mathcal{B}_{p,g}(x,1), d_{p,g}), (\mathcal{B}_{p,g_{\mathrm{euc}}}(0,1), d_{p,g_{\mathrm{euc}}})\big) \le \epsilon, \tag{1-7}$$

and for any $0 < r \le 1$,

$$(1-\epsilon)|\mathcal{B}_{p,g_{\mathrm{euc}}}(0,r)| \le \mathrm{vol}_g(\mathcal{B}_{p,g}(x_0,r)) \le (1+\epsilon)|\mathcal{B}_{p,g_{\mathrm{euc}}}(0,r)|. \tag{1-8}$$

Here $|\cdot|$ denotes the Euclidean volume. In particular, the measure $d\mathrm{vol}_g$ on the metric measure space $(M, d_{p,g}, d\mathrm{vol}_g)$ is a doubling measure for all scales $r \le 1$.

Remark 1.8 The assumption of (nonuniformly) bounded curvature is simply to control degeneration at infinity of M, a local version of these statements would drop this condition.

Remark 1.9 (L^1–Sobolev constant) We may replace the entropy lower bound in Theorems 1.7 by a rigid bound on the L^1–Sobolev constant. Namely, we may replace the assumption $\nu(g,2) \ge -\delta$ in (1-6) with the assumption that for all compactly supported $f : B_g(x,1) \to \mathbb{R}$ with $x \in M$ we have

$$\left(\int_M |f|^{n/(n-1)}\right)^{(n-1)/n} \le (1+\delta)c_n \int_M |\nabla f|, \tag{1-9}$$

where c_n is the sharp Sobolev constant on Euclidean space. However, we avoid focusing on this because, as we will see, metric balls are badly behaved objects, and thus any condition which used a metric ball may be more restrictive than it appears. The μ–entropy intrinsically understands the correct d_p–distance, and thus $\mu(g,1)$ becomes a condition on the unit d_p–scale, as opposed to the $d = d_\infty$ scale.

Remark 1.10 (scaling) For any Riemannian manifold (M, g), the rescaled metric $\tilde{g} = r^{-2} g$ satisfies

$$\mathcal{B}_{p,\tilde{g}}(x_0,\rho) = \mathcal{B}_{g,p}(x_0, \rho r^{1-n/p}), \quad R_{\tilde{g}} = r^{-2} R_g, \quad \nu(g, 2r^2) = \nu(\tilde{g}, 2).$$

If (M, g) is closed or is well behaved at infinity (see eg [54]), then $\lim_{\tau\to 0}\mu(g,\tau) = 0$. In particular, for any such Riemannian manifold (M, g), the hypotheses of Theorem 1.11 hold at some scale.

The proof of Theorem 1.7 depends on the following, which guarantees the existence of $W^{1,p}$ charts on an ϵ–regularity ball. Further, one is able to get that for large but finite p it is possible to control the $W^{1,p}$ energies of limiting functions; this connects to the perspective discussed above of d_p–convergence as a type of convergence of Sobolev spaces.

Theorem 1.11 (L^p–estimates for the metric coefficients) *Let (M^n, g) be a complete Riemannian manifold with bounded curvature. Fix $\epsilon > 0$, $\kappa > 1$ and $p \in [\kappa, \infty)$. There exists $\delta = \delta(n, p, \kappa, \epsilon) > 0$ such that if*

$$R \geq -\delta \quad \text{and} \quad \nu(g, 2) \geq -\delta, \tag{1-10}$$

then for any $x \in M$ there exist an open set $\Omega \subset M$ containing x and a smooth diffeomorphism $\psi : \Omega \to B(0,1) \subset \mathbb{R}^n$ with $\psi(x) = 0$ satisfying

$$\fint_{B(0,1)} |(\psi^{-1})^* g - g_{\mathrm{euc}}|^p \, dy \leq \epsilon \quad \text{and} \quad \fint_{\Omega} |\psi^* g_{\mathrm{euc}} - g|^p \, d\mathrm{vol}_g \leq \epsilon. \tag{1-11}$$

Furthermore, for any $f \in W^{1,p}(B(0,1))$, we have

$$(1-\epsilon)\|\psi^* f\|_{L^{p/\kappa}(\Omega)} \leq \|f\|_{L^p(B(0,1))} \leq (1+\epsilon)\|\psi^* f\|_{L^{\kappa p}(\Omega)}, \tag{1-12}$$

$$(1-\epsilon)\|\nabla\psi^* f\|_{L^{p/\kappa}(\Omega)} \leq \|\nabla f\|_{L^p(B(0,1))} \leq (1+\epsilon)\|\nabla\psi^* f\|_{L^{\kappa p}(\Omega)}. \tag{1-13}$$

The notation $\fint_\Omega u \, d\mathrm{vol}_g$ is used to denote $\mathrm{vol}_g(\Omega)^{-1} \int_\Omega u \, d\mathrm{vol}_g$. In (1-11), the notation $|\cdot|$ indicates the tensor norm with respect to g_{euc} and g respectively.

1.2 Examples and counterexamples

We have been explaining from the beginning what can fail as one converges with sequences of spaces with lower scalar curvature and entropy bounds. In particular, we have discussed how the distance function itself is almost entirely uncontrollable. Let us now make this precise, and in the process see that the d_p–convergence in Theorem 1.7 cannot be replaced with Gromov–Hausdorff convergence or intrinsic flat convergence.

Theorem 1.12 (counterexample to Gromov–Hausdorff convergence) *Fix $n \geq 4$ and $\epsilon > 0$. There exists a sequence of metrics $(\mathbb{T}^n, g_i)$ on the n–dimensional torus with $\delta_i \to 0$ such that*

$$R_{g_i} \geq -\delta_i \quad \text{and} \quad \nu(g_i, 2) \geq -\epsilon, \tag{1-14}$$

and such that $(\mathbb{T}^n, g_i)$ converges in the Gromov–Hausdorff topology to a point, and in the intrinsic flat topology to the zero current as $i \to \infty$. On the other hand, the sequence $(\mathbb{T}^n, g_i)$ converges to a flat torus $(\mathbb{T}^n, g_{\mathrm{flat}})$ in the d_p sense for all finite $p \in [n+1, \infty)$.

The example of Theorem 1.12 is given in Example 9.9. Preserving the lower scalar curvature in the above example is not too challenging, but showing that the entropies are well behaved takes quite a bit more work. Philosophically, this example is similar to situations studied very recently by Allen and Sormani [5], without the lower scalar curvature and entropy requirements; see also [4].

In fact, in Section 9, we construct a variety of other compact and noncompact examples of sequences satisfying (1-14) such that the Gromov–Hausdorff and intrinsic flat limits are not locally Euclidean. For example, we have the following.

Theorem 1.13 *Fix $n \geq 4$. There exist a sequence of metrics $(\mathbb{R}^n, g_i)$ satisfying*

$$R_{g_i} \geq -\frac{1}{i} \quad \text{and} \quad \nu(g_i, 2) \geq -\frac{1}{i}. \tag{1-15}$$

that converge in the d_p–sense to flat Euclidean space for any $p \in [n+1, \infty)$, but whose pointed Gromov–Hausdorff limit is $(\mathbb{R}^n, \ell^\infty)$, ie the taxicab metric on Euclidean space.

The example of Theorem 1.13 is given in Example 9.6. Theorems 1.12 and 1.13 demonstrate that one cannot replace d_p–closeness with Gromov–Hausdorff or intrinsic flat closeness in Theorem 1.7. Furthermore, the following theorem shows that the p for which we establish d_p–convergence in Theorem 1.7 cannot be taken arbitrarily large for fixed δ.

Theorem 1.14 *Fix $n \geq 4$ and $\delta > 0$. There exists a sequence $(\mathbb{R}^n, g_i)$ that satisfies*

$$R_{g_i} \geq -\delta \quad \text{and} \quad \nu(g_i, 2) \geq -\delta, \tag{1-16}$$

and a singular metric g_∞ on $\mathbb{R}^n$ such that $(\mathbb{R}^n, g_i)$ converges in d_p to $(\mathbb{R}^n, g_\infty)$, as a rectifiable Riemannian space, for all $p \in [n+1, p_0)$ for some $p_0 = p_0(\delta)$, but does not d_p–converge to $(\mathbb{R}^n, g_\infty)$ for $p \geq p_0$.

The example of Theorem 1.14 is given in Example 9.5.

1.3 Structure of limit spaces

The next main result of the paper is the following compactness result and structure theorem for limit spaces. We show that under almost nonnegative scalar curvature and entropy, we indeed have a rectifiable Riemannian limit X.

Theorem 1.15 (structure of limit spaces) *Let $\{(M_i, g_i, x_i)\}$ be a sequence of complete pointed Riemannian manifolds with bounded curvature and let $p \geq n+1$. Then there exists $\delta = \delta(n, p) > 0$ such that if*

$$R_{g_i} \geq -\delta \quad \text{and} \quad \nu(g_i, 2) \geq -\delta, \tag{1-17}$$

then there exists a pointed rectifiable Riemannian space (X, g, x), with X topologically but not necessarily metrically a smooth manifold, such that:

(1) *After passing to a subsequence, we have $d_p((M_i, g_i, x_i), (X, g, x)) \to 0$ in the pointed sense of Definition 2.44.*

(2) *The space (X, g, x) is $W^{1,p}$–rectifiably complete and d_p–rectifiably complete, in the sense of Definitions 2.24 and 2.35, respectively.*

The first part of the above theorem just tells us that there exists a rectifiable space X to which the M_i converge. As we have emphasized, it may be that X does not have a well-behaved metric structure and this convergence may not be in the Gromov–Hausdorff or intrinsic flat sense. The second condition in the above theorem touches on some subtle points that we have avoided in the introduction, and essentially tells us that X is a well-behaved rectifiable space which behaves the way one might feel it should in a reasonable scenario. In particular, the gradient of a function is indeed the coordinate gradient that one would compute in rectifiable charts, and the metric d_p generates the topology of X. Note that for $q \gg p$, this may fail for d_q.

1.4 Further results under lower scalar curvature and entropy

Finally, let us conclude by discussing some applications of the results to the underlying structure of spaces with lower scalar curvature and entropy bounds. To begin, we obtain on such spaces an a priori L^q bound for the scalar curvature for $q < 1$:

Theorem 1.16 (L^q scalar curvature estimates) *Let (M^n, g) be a closed Riemannian manifold and let $\epsilon > 0$ and $q \in (0, 1)$ be fixed. There exists $\delta = \delta(n, q, \epsilon) > 0$ such that if*

$$R \geq -\delta \quad \text{and} \quad \nu(g, 2) \geq -\delta, \tag{1-18}$$

then we have

$$⨍_M |R|^q \, d\mathrm{vol}_g \leq \epsilon. \tag{1-19}$$

Motivated by the above we conjecture the following:

Conjecture 1.1 Let (M^n, g) be a closed Riemannian manifold with $R, \nu(g, 2) \geq -A$. Then there exists $B(n, A) > 0$ such that

$$\fint_M |R| \, d\mathrm{vol}_g \leq B. \tag{1-20}$$

In our last main result we prove that Riemannian manifolds satisfying a uniform lower bound on entropy and scalar curvature satisfy a Morrey–Sobolev embedding with a uniform constant.

Theorem 1.17 (L^∞–Sobolev embedding) *Let (M^n, g) be a complete Riemannian manifold with bounded curvature and let $p \geq n+1$ and $q > n$. There exists a $\delta = \delta(n, p, q) > 0$ and $C_{n,q} > 0$ such that if*

$$R \geq -\delta \quad \text{and} \quad \nu(g, 2) \geq -\delta, \tag{1-21}$$

then for all $f \in W^{1,q}(M)$, we have

$$\|f\|_{L^\infty(M)} \leq C_{n,q}(\|\nabla f\|_{L^q(M)} + \|f\|_{L^q(M)}). \tag{1-22}$$

More locally, for all $x_0 \in M$ and $f \in W_0^{1,q}(\mathcal{B}_{p,g}(x_0, 1))$, we have

$$\|f\|_{L^\infty(\mathcal{B}_{p,g}(x_0,1))} \leq C_{n,q}\|\nabla f\|_{L^q(\mathcal{B}_{p,g}(x_0,1))}. \tag{1-23}$$

In terms of the d_p–distance we can upgrade this to a Hölder embedding: there exists $\alpha = \alpha(n, q) \in (0, 1)$ such that

$$|f(x) - f(y)| \leq C_{n,q,p} d_p(x, y)^\alpha \|\nabla f\|_{L^q(\mathcal{B}_{p,g}(x_0,1))} \tag{1-24}$$

for all $x, y \in \mathcal{B}_{p,g}(x_0, 1)$.

Remark 1.18 The examples of Section 9 demonstrate that the Hölder embedding of (1-23) cannot hold with the geodesic distance in place of the d_p–distance.

The following theorem provides a type of stability for a theorem of Schoen and Yau [45] and Gromov and Lawson [28], which states that a metric of nonnegative scalar curvature on a torus must be flat. Stability for this rigidity theorem statement was conjectured by Gromov in [27], with a more concrete formulation of the conjecture given by Sormani in [48]. Progress toward this conjecture has been made in various cases. The first developments were due to Gromov [27], also established by Bamler [10] using Ricci flow, and showed that if a sequence of metrics g_i on a torus that converge in C^0 to a C^2 metric g also have $R_{g_i} \geq -1/i$, then g is the flat metric. Using (regularizing) Ricci flow, Burkhardt-Guim [11] extended this result to limiting metrics that are only C^0, and also proved a generalization of the rigidity result to C^0 metrics with nonnegative scalar curvature in a weak sense. Further progress toward this conjecture, in the form stated by Sormani in [48], has been made in the setting of warped product metrics [3] by Allen, Hernandez-Vazquez, Parise, Payne and Wang, graphical tori [12] by Cabrera Pacheco, Ketterer and Perales, and metrics that are conformal to the flat metric [2] by Allen. We note that the hypotheses in Theorem 1.19 differ from those in the conjecture of Sormani; most notably our assumption of an entropy lower bound takes the place of the lower bound on the minA quantity there, and here stability is with respect to the d_p–distance rather than the intrinsic flat distance.

Theorem 1.19 *Fix $n \geq 2$ and $p \geq n+1$. There exists $\delta = \delta(n,p)$ and $V_0 = V_0(n,p)$ such that the following holds. For any $V \geq V_0$, let (M_i, g_i) be a sequence of compact Riemannian manifolds, diffeomorphic to tori, with $\mathrm{vol}_{g_i}(M_i) \leq V$ and satisfying*

$$\nu(g_i, 2) \geq -\delta \quad \text{and} \quad R_i \geq -\frac{1}{i}. \tag{1-25}$$

Then (M_i, g_i) converges in the d_p sense to a flat torus with $\nu(g,2) \geq -\delta$.

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2 Rectifiable Riemannian spaces and convergence

2.1 Rectifiable Riemannian spaces

We introduce precisely the notion of rectifiable Riemannian spaces, which are the objects which arise as limits in the d_p sense of Riemannian manifolds with uniform lower bounds on scalar curvature and on the entropy.

Let X be a Hausdorff topological space equipped with a Borel measure m on X. We will refer to (X, m) as a topological measure space. We first define the notion of rectifiability of a topological measure space. Since these spaces are not equipped with metrics, the rectifiable structure is provided via an atlas of charts with bi-Lipschitz transition maps.

Definition 2.1 (rectifiable atlas) Let (X,m) be a topological measure space, and consider a collection of charts $\{(U_a,\phi_a)\}_{a\in\mathcal{I}}$, where $U_a\subseteq\mathbb{R}^n$, $\phi_a\colon U_a\to X$ is one-to-one and continuous with continuous inverse on its image, and $\mathcal{I}$ is a countable index set. For each $a,b\in\mathcal{I}$ let us denote $U_{a,b}\equiv U_a\cap\phi_a^{-1}(\phi_b(U_b))\subseteq\mathbb{R}^n$. We say that $\{(U_a,\phi_a)\}_{a\in\mathcal{I}}$ is a rectifiable atlas for (X,m) if:

(1) For each $a,b\in\mathcal{I}$ such that $U_{a,b}$ is nonempty, every point in $U_{a,b}$ has Lebesgue density one.

(2) For each $a,b\in\mathcal{I}$ such that $U_{a,b}$ is nonempty, the transition map
$$\phi_{ba}\equiv\phi_b^{-1}\circ\phi_a\colon U_{a,b}\to U_{b,a}$$
is bi-Lipschitz.

(3) We have
$$m\Bigl(X\setminus\bigcup_{a\in\mathcal{I}}\phi_a(U_a)\Bigr)=0.$$

(4) For each U_a we have that $(\phi_a^{-1})_*m$ is absolutely continuous to the Lebesgue measure.

Given a topological measure space (X,m) equipped with a rectifiable atlas $\{(U_a,\phi_a)\}$, we may define a Riemannian structure on X by defining a (possibly degenerate) Riemannian metric in the charts U_a. Naturally, we must ask that this metric is suitably compatible with the rectifiable atlas and the measure. We call the resulting space a rectifiable Riemannian space.

Definition 2.2 (rectifiable Riemannian space) Let (X,m) be a topological measure space. We say that (X,m) has a rectifiable Riemannian structure if there is a rectifiable atlas $\{(U_a,\phi_a)\}_{a\in\mathcal{I}}$ on (X,m) together with a collection of matrix-valued functions $g_a\colon U_a\to\mathbb{R}^{n\times n}$ such that:

(1) For each $x\in U_a$, $g_a(x)$ is a positive definite symmetric matrix such that
$$\sup_{x\in U_a}\|g_a\|+\|g_a^{-1}\|\le C_a,$$
and g_a is continuous on U_a.

(2) $\{g_a\}_{a\in\mathcal{I}}$ satisfies the compatibility condition: for almost every $x \in U_{a,b}$, ϕ_{ba} is differentiable and $g_a = \phi_{ba}^* g_b$ at x. More precisely, we have

$$(g_a)_{ij}\big|_x = \frac{\partial \phi_{ba}^\alpha}{\partial x^i} \frac{\partial \phi_{ba}^\beta}{\partial x^j} (g_b)_{\alpha\beta}\bigg|_{\phi_{ba}(x)}.$$

(3) The measure $\phi_a^* m$ on U_a is given by

$$\phi_a^* m = \sqrt{\det g_a}\, dx.$$

We say that $\{g_a\}_{a\in\mathcal{I}}$ is the coordinate expression of a rectifiable Riemannian metric g on X and call (X, g) a rectifiable Riemannian space.

Remark 2.3 For condition (1), the continuity assumption is only made for convenience. Indeed, since U_a is not necessarily an open set in $\mathbb{R}^n$, by the Lebesgue differentiation theorem one can always remove an arbitrarily small set in U_a so that the continuity holds under the L^∞ condition.

2.1.1 Examples of rectifiable Riemannian spaces One can imagine a variety of ways in which a rectifiable Riemannian space can degenerate. We will first work our way through some basic examples which explore this. This will give some first intuition on what kind of structure is needed to avoid this. Future sections will explore examples that might arise as limits, which will tell us when these degeneracies can and cannot be avoided.

Example 2.4 Any smooth Riemannian manifold (M, g) is a rectifiable Riemannian space.

With regard to Example 2.4, observe that even for a smooth Riemannian manifold (M, g) a given rectifiable atlas may only cover M up to a set of measure zero:

Example 2.5 Let $X = \mathbb{R}^n$ with g_{euc} the Euclidean metric, and let $m = dx$ be the Lebesgue measure. Consider the rectifiable atlas $\{(U_1, \phi_1), (U_2, \phi_2)\}$ where

$$U_1 = \{(x_1, \dots, x_n) : x_1 > 0\} \quad \text{and} \quad U_2 = \{(x_1, \dots, x_n) : x_1 < 0\}$$

are complementary open half-spaces and ϕ_i is the identity chart restricted to U_i for $i = 1, 2$. Then $(\mathbb{R}^n, g_{\mathrm{euc}})$ is a rectifiable Riemannian space with respect to this rectifiable atlas.

Example 2.6 Any stratified Riemannian manifold X is a rectifiable Riemannian space.

Example 2.7 As a concrete case of Example 2.6, let $X \subset \mathbb{R}^2$ be a countable union of lines $\{\ell_i\}_{i\in\mathbb{N}}$ passing through the origin, and let m be defined by $m|_{\ell_i} = \mathcal{H}^1|_{\ell_i}$. Define $g|_{\ell_i} = g_{\mathbb{R}^2}|_{\ell_i}$ and let $\{(\mathbb{R}\setminus\{0\}, \phi_i)\}_{i\in\mathbb{N}}$ be the rectifiable atlas with $\phi_i : \mathbb{R}\setminus\{0\} \to \ell_i\setminus\{0\}$ defined via the obvious isometric embedding. Then (X, g) is a one-dimensional rectifiable Riemannian space.

Example 2.8 Let (X^n, d, m) be a metric measure space which is also a noncollapsed $\mathrm{RCD}(N, K)$ space, that is, a metric measure space with lower bounds on the Ricci curvature in the generalized sense. Then it follows from [40] that X is a rectifiable Riemannian space.

Due to the flexibility, we can also allow the metric tensor to be mildly singular. Let us consider some examples of this.

Example 2.9 (degenerate metric on $\mathbb{R}^n$) Let $X = \mathbb{R}^n$ and consider the metric defined by $g = \sum_{i=1}^n f_i(x)^2 (dx^i)^2$, where each f_i is a smooth nonnegative function on $\mathbb{R}^n$ such that the set $\Sigma = \bigcup_{i=1}^n \{x : f_i(x) = 0\}$ has Lebesgue measure zero. Further let $m = \sqrt{\det g}\, dx$ be the induced measure. Consider the rectifiable atlas on the topological measure space $(\mathbb{R}^n, m)$ given by $\{(U_a, \phi_a)\}_{a\in\mathbb{N}}$ where

$$U_a = \bigcap_{i=1}^n \{x \in \mathbb{R}^n : a^{-1} \le f_i \le a\}$$

and ϕ_a is the identity chart on $\mathbb{R}^n$ restricted to U_a. Then, with respect to this rectifiable atlas, $(\mathbb{R}^n, g)$ is a rectifiable Riemannian space.

An important feature of Example 2.9 is that, while the geodesic distance gives rise to a metric space structure (X, d), the metric space may not even be topologically equivalent to $\mathbb{R}^n$, as seen in the following example.

Example 2.10 As a special case of Example 2.9, consider the rectifiable Riemannian space $(\mathbb{R}^2, g)$ where $g = dx^2 + |x|^2\, dy^2$. Let d_g be the distance function with respect to g, ie $d_g(x, y) = \inf_\gamma \int_0^1 |\dot\gamma(t)|\, dt$, where the infimum is taken among all curves γ with $\gamma(0) = x$ and $\gamma(1) = y$. Then we see that $d_g(p_1, p_2) = 0$ for all $p_1, p_2 \in \ell$ where $\ell = \{(x, y) : x = 0\}$. In particular, the metric space $(\mathbb{R}^2, d_g)$ collapses ℓ to a point and is not topologically equivalent to $\mathbb{R}^2$. We will examine the d_p–distance for this example in Section 2.2.

In Section 9, we will construct rectifiable Riemannian metrics that are qualitatively similar to Examples 2.9 and 2.10, which arise as limits of smooth Riemannian manifolds with uniform lower bounds on scalar curvature and entropy.

2.1.2 $W^{1,p}$ spaces on rectifiable Riemannian spaces We would like to use the rectifiable structure of a space in order to do analysis. In order to do this, we need to make sense of $W^{1,p}$ functions in our context, which means being able to take gradients of functions and look at their norms. Ideally, we would want to use the rectifiable charts in order to do this in coordinates. Realistically, one has to be quite careful about this. A function might be perfectly differentiable in every coordinate chart, but not really be a $W^{1,p}$ function as its gradient may have a distributional component, as we see in the following example.

Example 2.11 Consider $(\mathbb{R}^n, g_{\text{euc}})$ with the rectifiable atlas $\{(U_1,\phi_1),(U_2,\phi_2)\}$ comprising two open half-spaces as in Example 2.5. The function $f\colon \mathbb{R}^n \to \mathbb{R}$ defined by $f(x)=0$ if $x_1<0$ and $f(x)=1$ if $x_1\ge 0$ clearly does not have gradient in $L^p(\mathbb{R}^n)$, since its distributional gradient is a singular measure supported on $\{x_1=0\}$. However, letting $f_a=\phi_a^* f$ for $a=1,2$, we have $g^{ij}\partial_i f_a \partial_j f_a \equiv 0$ for all $x\in U_a$ for $a=1,2$.

In order to deal with this issue, we will follow a classical approach from metric measure spaces (see for instance [29, Sections 5–7]) to build the Sobolev space theory by considering the behavior of functions along curves. The key is that these ideas adapt themselves very well to this context, as in the end, even when no a priori distance function is available, the notion of an absolutely continuous curve and the behavior of a function along it is available and can be studied.

Let us begin by discussing the notion of an absolutely continuous curve on a rectifiable Riemannian space. In the setting of a smooth manifold or a metric measure space, the theory of Sobolev spaces can be built up by considering the behavior of functions along rectifiable curves. In these settings, a rectifiable curve is defined as one with finite length, where length is defined via approximation by polygonal curves. This definition is independent of parametrization, and every rectifiable curve in a smooth Riemannian manifold or a metric space admits an absolutely continuous (in fact, Lipschitz) parametrization, namely the arc-length parametrization. In practice, it is this absolutely continuous parametrization that is used in the Sobolev space theory.

Moving to the context of rectifiable Riemannian spaces, there are two major factors that must be taken into account when determining the appropriate class of curves along which to study the behavior of functions. First, curves must be appropriately compatible with the rectifiable atlas on the space in order to avoid the difficulty illustrated in Example 2.11. Second, the absence of a distance function prohibits us from speaking about the length of a polygonal curve, and thus of considering rectifiable curves in the sense described above. Instead, we must restrict our attention to absolutely continuous parametrizations of curves. To be more specific about these two considerations, the definition of an absolutely continuous curve in a rectifiable Riemannian space is given in Definition 2.12 below.

Let (X, g) be a rectifiable Riemannian space with rectifiable atlas $\{(U_a, \phi_a)\}_{a \in \mathcal{I}}$, and denote the singular part of X by $X^s = X \setminus \bigcup_a U_a$. This may or may not correspond to topological singularities of the space.

Definition 2.12 (absolutely continuous curves) Let $\gamma : [\alpha, \beta] \to X$ be a continuous curve and define $I_a = \gamma^*(\phi_a(U_a))$ for each $a \in \mathcal{I}$. We say that γ is absolutely continuous if the following properties hold:

(a) $\gamma^*(X^s) \subset [\alpha, \beta]$ is a countable set.

(b) For every $\epsilon > 0$, there exists $\delta > 0$ such that if $\{(s_i, t_i)\}_{i=1}^{\infty}$ is a collection of disjoint intervals in $[\alpha, \beta]$ such that for each i we have $s_i, t_i \in I_{a_i}$ for some $a_i \in \mathcal{I}$ and $\sum_{i=1}^{\infty} |s_i - t_i| < \delta$, then

$$\sum_{i=1}^{\infty} |\gamma_{a_i}(s_i) - \gamma_{a_i}(t_i)|_{g_a(\gamma(s_i))} < \epsilon. \tag{2-1}$$

Some further discussion is in order about Definition 2.12. As we saw in Example 2.11, part (a) of the definition is necessary to guarantee that the behavior of an absolutely continuous curve γ can be entirely reflected in the charts of its rectifiable atlas, since the rectifiable charts may only cover (X, g) up to a set of measure zero. Together with the assumption that γ is continuous, part (a) ensures that there is no contribution to the singular part of the distributional derivative of γ on the set $\gamma^*(X^s)$, and in particular eliminates the issue illustrated in Example 2.11:

Example 2.13 In Example 2.11, the curve $\gamma(t) = (t, 0, \dots, 0)$ is not an absolutely continuous curve in the sense of Definition 2.12 because it violates condition (a).

Part (b) of Definition 2.12 is a replacement of the classical notion of a curve with finite length, as one typically takes the supremum over lengths of polygonal approximations to the curve. As we noted above, in the context of rectifiable Riemannian spaces this notion is unsuitable as one does not have a notion of a distance function with which to measure the length. Instead, we will see that condition (b) guarantees that the curve is absolutely continuous in each chart in a suitably uniform sense.

Note that Definition 2.12 is parametrization dependent, as it requires an absolutely continuous parametrization. This is not restrictive, as on a smooth Riemannian manifold with a classical atlas of charts, every rectifiable curve in the classical sense admits a reparametrization, namely its arc-length parametrization, which is an absolutely continuous curve in the sense of Definition 2.12.

The following lemma provides some basic consequences of the Definition 2.12 that further clarify this notion of absolutely continuous curve and how it fits into the classical notion on smooth spaces.

Lemma 2.14 *Let $\gamma\colon[\alpha,\beta]\to X$ be an absolutely continuous curve in the sense of Definition 2.12 above. Then the following properties hold.*

(1) *For each $a\in\mathcal{I}$, the function $\gamma_a=\phi_a^{-1}\circ\gamma\colon I_a\to U_a$ is differentiable for a.e. $s\in I_a$. Here we again let $I_a=\gamma^*(\phi_a(U_a))\subset[\alpha,\beta]$.*

(2) *For all $\epsilon>0$, there exists $\delta>0$ such that if $S\subset[\alpha,\beta]$ with $|S|<\delta$, then $\int_S|\dot\gamma|_g<\epsilon$.*

(3) *If (X,g) is a smooth Riemannian manifold, then the length of γ is given by $L(\gamma)=\int_\alpha^\beta|\dot\gamma|_g\,dt$.*

Remark 2.15 In Lemma 2.14(2) and (3) and in the sequel, we let $|\dot\gamma|_g\equiv\sqrt{g(\dot\gamma,\dot\gamma)}$, which is well-defined for a.e. $t\in[\alpha,\beta]$ by Lemma 2.14(1) and via the rectifiable atlas $\{(U_a,\phi_a)\}_{a\in\mathcal{I}}$.

Remark 2.16 Having in mind Lemma 2.14(3), we define the length of an absolutely continuous curve $\gamma\colon[\alpha,\beta]\to X$ in a rectifiable Riemannian space by $L(\gamma)=\int_\alpha^\beta|\dot\gamma|_g$. One can check that this notion is independent of Lipschitz reparametrizations.

Proof of Lemma 2.14 We first prove (1). Fix $a\in\mathcal{I}$. Because ϕ_a^{-1} is continuous, we see that $\gamma_a\colon I_a\to U_a$ satisfies the absolute continuity property (2-1) for endpoints $s_i,t_i\in I_a$; thus γ_a extends to a continuous function $\bar\gamma_a\colon\bar I_a\to\bar U_a$. Moreover, the complement $[\alpha,\beta]\setminus\bar I_a$ of $\bar I_a$ is relatively open in $[\alpha,\beta]$ and therefore comprises a countable union of disjoint relatively open intervals (α_j,β_j) in $[\alpha,\beta]$. We may therefore extend $\overline{\gamma}_a$ to a continuous curve $\tilde{\gamma}_a\colon[\alpha,\beta]\to\mathbb{R}^n$ by letting $\tilde{\gamma}_a$ interpolate linearly between $\overline{\gamma}_a(\alpha_j)$ and $\overline{\gamma}_a(\beta_j)$ for each $j\in\mathbb{N}$.

To prove (1), we will show that the curve $\tilde{\gamma}_a\colon[\alpha,\beta]\to\mathbb{R}^n$ is a uniformly continuous curve on $\mathbb{R}^n$. More specifically, we claim that for any $\epsilon>0$, there exists $\delta_0>0$ such that for any disjoint collection of intervals $\{[s_i,t_i]\}_{i=1}^\infty$ such that $\sum_{i=1}^\infty|s_i-t_i|<\delta_0$, we have

$$\sum_{i=1}^{\infty}|\tilde{\gamma}(s_i)-\tilde{\gamma}(t_i)|_{\mathrm{euc}}\le\epsilon. \tag{2-2}$$

It will follow immediately from this absolute continuity of $\tilde{\gamma}$ that $\tilde{\gamma}$ is differentiable for a.e. $t\in[\alpha,\beta]$ and so in particular that γ is differentiable for a.e. $t\in I_a$, proving (1).

Fix $\epsilon>0$, let $\delta_0>0$ be a fixed number to be determined within the proof, and consider a disjoint collection of intervals $\{[s_i,t_i]\}_{i=1}^\infty$ in $[\alpha,\beta]$ such that $\sum_{i=1}^\infty|s_i-t_i|<\delta_0$. Up to refining the collection of intervals (which can only increase (2-2)), we may assume for each $i\in\mathbb{N}$ that s_i,t_i either both lie in $\overline{I}_a$ or both lie in $\overline{[\alpha,\beta]\setminus\overline{I}_a}=\bigcup_{j=1}^\infty[\alpha_j,\beta_j]$. So, subdividing the index set $\mathbb{N}$ by letting

$$\mathcal{J}_0=\{i\in\mathbb{N}:s_i,t_i\in\overline{I}_a\}\quad\text{and}\quad\mathcal{J}_1=\Big\{i\in\mathbb{N}:s_i,t_i\in\bigcup_{j=1}^{\infty}[\alpha_j,\beta_j]\Big\}, \tag{2-3}$$

we will establish (2-2) by showing that

$$\sum_{i\in\mathcal{J}_0}|\tilde{\gamma}(s_i)-\tilde{\gamma}(t_i)|_{\mathrm{euc}}\le\tfrac{1}{2}\epsilon, \tag{2-4}$$

$$\sum_{i\in\mathcal{J}_1}|\tilde{\gamma}(s_i)-\tilde{\gamma}(t_i)|_{\mathrm{euc}}\le\tfrac{1}{2}\epsilon. \tag{2-5}$$

Recall that there exists $C_a>0$ such that

$$C_a^{-1}g_a\le g_{\mathrm{euc}}\le C_a g_a \tag{2-6}$$

in U_a. So, (2-4) follows directly from this fact and (2-1), provided we take $\delta_0\le\delta_1$, where $\delta_1>0$ is a number small enough that (2-1) holds with $\epsilon/4C_a$ in place of ϵ. In order to establish (2-5), we further subdivide the index set $\mathcal{J}_1$ in the following way. Since $\sum_{j=1}^\infty|\beta_j-\alpha_j|\le|\beta-\alpha|$, there exists J_0 such that $\sum_{j=J_0}^\infty|\beta_j-\alpha_j|\le\delta_1$. Up to further refinement of our collection of intervals, we may assume that for each $i\in\mathcal{J}_1$, both endpoints s_i and t_i lie in $\bigcup_{j=1}^{J_0-1}[\alpha_j,\beta_j]$ or both endpoints lie in $\bigcup_{j=J_0}^\infty[\alpha_j,\beta_j]$.

So we let

$$\mathcal{J}_2 = \left\{ i \in \mathbb{N} : s_i, t_i \in \bigcup_{j=1}^{J_0 - 1} [\alpha_j, \beta_j] \right\} \subset \mathcal{J}_1,$$
(2-7)
$$\mathcal{J}_3 = \left\{ i \in \mathbb{N} : s_i, t_i \in \bigcup_{j=J_0}^{\infty} [\alpha_j, \beta_j] \right\} \subset \mathcal{J}_1,$$

so that $\mathcal{J}_2 \cup \mathcal{J}_3 = \mathcal{J}_1$. We establish (2-5) by bounding the sums over $\mathcal{J}_2$ and $\mathcal{J}_3$ separately, starting with $\mathcal{J}_3$. By the triangle inequality and the piecewise linear way that $\tilde{\gamma}$ was defined on $[\alpha, \beta] \setminus \bar{I}_a$, we see that

$$\sum_{i \in \mathcal{J}_3} |\tilde{\gamma}(s_i) - \tilde{\gamma}(t_i)|_{\text{euc}} \le \sum_{j=J_0}^{\infty} |\tilde{\gamma}(\alpha_j) - \tilde{\gamma}(\beta_j)|_{\text{euc}}. \tag{2-8}$$

Then, since $\alpha_j, \beta_j \in \bar{I}_a$, we may apply (2-6) and (2-1) to find that

$$\sum_{j=J_0}^{\infty} |\tilde{\gamma}(\alpha_j) - \tilde{\gamma}(\beta_j)|_{\text{euc}} \le \tfrac{1}{4}\epsilon \tag{2-9}$$

provided $\delta_0 \le \delta_1$, with δ_1 as above. Together, (2-8) and (2-9) show that

$$\sum_{i \in \mathcal{J}_3} |\tilde{\gamma}(s_i) - \tilde{\gamma}(t_i)|_{\text{euc}} \le \tfrac{1}{4}\epsilon. \tag{2-10}$$

Now, to bound the analogous summation over $\mathcal{J}_2$, notice that the linear segment of $\tilde{\gamma}$ defined on $[\alpha_j, \beta_j]$ is absolutely continuous for each $j = 1, \dots, J_0$. Since there are finitely many such intervals, we may find $\delta_2 > 0$ such that if $\delta_0 \le \delta_2$, then

$$\sum_{i \in \mathcal{J}_2} |\tilde{\gamma}(s_i) - \tilde{\gamma}(t_i)|_{\text{euc}} \le \tfrac{1}{4}\epsilon. \tag{2-11}$$

Finally, choose $\delta_0 \le \min\{\delta_1, \delta_2\}$. Then together (2-11) and (2-10) prove (2-5). This shows that $\tilde{\gamma}$ is an absolutely continuous curve on $\mathbb{R}^n$ and thereby establishes (1).

Before moving to the proof of (2), observe that as a consequence of (1), we may define $|\dot{\gamma}_a|_{g_a} = \sqrt{g_a(\dot{\gamma}_a, \dot{\gamma}_a)}$ for a.e. $t \in I_a$, and we have

$$\lim_{s \to t} \frac{|\tilde{\gamma}_a(t) - \tilde{\gamma}_a(s)|_{g_a(t)}}{|s - t|} = |\dot{\gamma}_a(t)|_{g_a(t)} \tag{2-12}$$

for each such t. It is easily checked that this definition is independent of a, and thus we may define $|\dot{\gamma}|_g$ for a.e. $t \in [\alpha, \beta]$.

Now we establish (2). The main idea will be to use (2-12) to relate the integral of $|\dot{\gamma}|_g$ to the absolute continuity assumption (2-1). Fix $\epsilon > 0$ and let $\delta > 0$ be a fixed number, to be specified within the proof. Fix a measurable set $S \subset [\alpha, \beta]$ with $|S| < \delta$, and let $\{S_a\}_{a \in \mathcal{I}}$ be a collection of pairwise disjoint subsets of S with $S_a \subset I_a$ for each $a \in \mathcal{I}$ such that $\left|S \setminus \bigcup_{a \in \mathcal{I}} S_a\right| = 0$, $\tilde{\gamma}_a$ is differentiable for every $t \in S_a$, and every point of S_a is a density point of I_a.

In order to make use of (2-12), we must show that the limit in (2-12) is uniform on a large subset of each S_a. More specifically, we claim that for any $\eta > 0$, there exist $r_\eta > 0$ and a measurable set $S_a^\eta \subset S_a$ with $|S_a^\eta| \geq (1-\eta)|S_a|$ such that if $t \in S_a^\eta$, then

$$|\dot{\gamma}_a(t)|_{g_a(t)} - \eta \leq \frac{|\tilde{\gamma}_a(t) - \tilde{\gamma}_a(s)|_{g_a(t)}}{|t-s|} \leq |\dot{\gamma}_a(t)|_{g_a(t)} + \eta \tag{2-13}$$

for all $s \in S_a$ with $|t-s| < r_\eta$. Indeed, let

$$R(s,t) = \left| \frac{|\tilde{\gamma}_a(t) - \tilde{\gamma}_a(s)|_{g_a(t)}}{|t-s|} - |\dot{\gamma}_a(t)|_{g_a(t)} \right|. \tag{2-14}$$

For each fixed $t \in S_a$, we have $\lim_{s \to t} R(s,t) = 0$. So, if we define the sequence of functions $\rho_k(t) = \sup\{|R(s,t)| : s \in S_a,\ |s-t| < 1/k\}$, we see that $\rho_k(t) \to 0$ pointwise as $k \to \infty$. Applying Egorov's theorem, for any $\eta > 0$, there exists a measurable set S_a^η such that $|S_a \setminus S_a^\eta| \leq \eta |S_a|$ and $\rho_k(t) \to 0$ uniformly on S_a^η. In particular, letting $r_\eta = 1/k$ for k chosen sufficiently large, we see that $|R(s,t)| \leq \eta$ for all $t \in S_a^\eta$. This establishes the claim.

Next, for each $a \in \mathcal{I}$, we aim to estimate the integral of $|\dot{\gamma}|$ over S_a^η. Consider a collection of disjoint intervals $\{I_{a,i} = [s_i, t_i]\}_{i=1}^\infty$ covering S_a^η with endpoints in S_a such that $|I_{a,i}| < r_\eta$ for each i and such that $\sum_i |I_{a,i}| \leq (1+\eta)|S_a^\eta|$. For each i, let $\hat{t}_i \in S_a^\eta \cap I_{a,i}$ be a point such that

$$(1+\eta)|\dot{\gamma}_a(\hat{t}_i)|_{g_a(\hat{t}_i)} \geq \sup\{|\dot{\gamma}_a(t)|_{g_a(t)} : t \in S_a^\eta \cap I_{a,i}\}. \tag{2-15}$$

At least one of the endpoints of $I_{a,i}$ has distance at least $\frac{1}{2}|I_{a,i}|$ from $\hat{t}_i$; without loss of generality suppose it is s_i. So, thanks to (2-13), we find that for each i,

$$\begin{aligned} \int_{S_a^\eta \cap I_{a,i}} |\dot{\gamma}_a(t)|_{g_a(t)}\,dt &\leq (1+\eta)|I_{a,i}|\,|\dot{\gamma}_a(\hat{t}_i)|_{g_a(\hat{t}_i)} \\ &\leq (1+\eta)|I_{a,i}| \frac{|\tilde{\gamma}_a(\hat{t}_i) - \tilde{\gamma}_a(s_i)|_{g_a(\hat{t}_i)}}{|s_i - \hat{t}_i|} + (1+\eta)\eta|I_{a,i}| \\ &\leq 2(1+\eta)|\tilde{\gamma}_a(\hat{t}_i) - \tilde{\gamma}_a(s_i)|_{g_a(\hat{t}_i)} + (1+\eta)\eta|I_{a,i}| \\ &= 2(1+\eta)|\gamma_a(\hat{t}_i) - \gamma_a(s_i)|_{g_a(\hat{t}_i)} + (1+\eta)\eta|I_{a,i}|. \end{aligned} \tag{2-16}$$

The final equality follows from the definition of $\tilde{\gamma}$ and the fact that $\hat{t}_i, s_i \in S_a \subset I_a$. Consequently, summing up over a and i, we find that

$$\begin{aligned}\text{(2-17)}\qquad \sum_{a\in\mathcal{I}}\int_{S_a^\eta}|\dot{\gamma}| &\le \sum_{a\in\mathcal{I}}\sum_{i=1}^{\infty}\int_{S_a^\eta\cap I_{a,i}}|\dot{\gamma}| \\ &\le \sum_{a\in\mathcal{I}}\sum_{i=1}^{\infty}\big(2(1+\eta)|\gamma_a(\hat{t}_i)-\gamma_a(s_i)|+(1+\eta)\eta|I_{a,i}|\big) \\ &\le 2(1+\eta)\epsilon+(1+\eta)^2\eta\delta,\end{aligned}$$

where in the final inequality we have applied (2-1), again using that $\hat{t}_i, s_i \in S_a \subset I_a$.

Finally, we send $\eta \to 0$. The right-hand side of (2-17) tends to 2ϵ, while, making use of the dominated convergence theorem, we see that the left-hand side converges to $\int_S |\dot{\gamma}|$. We therefore see that (2) holds. We omit the proof of (3) since it is standard. □

Following the classical approach in metric measure spaces, we next want to use our absolutely continuous curves to define the notion of a p–weak upper gradient of a function. We will see that most of the Sobolev theory is built up in an identical fashion to the metric measure space setting.

To begin, we need a notion of a collection of curves that have p–measure zero, an idea first introduced by Ahlfors and Beurling in [1] and further developed by Fuglede in [23] in the Euclidean and Riemannian settings. To this end, let $\mathfrak{M}$ denote the collection of all absolutely continuous curves on (X, g). For $1 \le p < \infty$, we say that a family of curves $\Gamma \subset \mathfrak{M}$ has $\mathrm{Mod}_p(\Gamma) = 0$ if there exists a nonnegative Borel measurable function $f \in L^p(X)$ such that $\int_\gamma f = +\infty$ for every $\gamma \in \Gamma$. Here and in the sequel, we use the notation $\int_\gamma f$ to mean

$$\text{(2-18)}\qquad \int_\gamma f := \int_\alpha^\beta f(\gamma(t))|\dot{\gamma}|_g\, dt.$$

A property is said to hold for p–a.e absolutely continuous curve if it holds for every curve in $\mathfrak{M} \setminus \Gamma$ where $\mathrm{Mod}_p(\Gamma) = 0$. For the corresponding definition of families of curves with $\mathrm{Mod}_p(\Gamma) = 0$ in the metric measure space context; see [29, Definition 5.1] and the equivalent formulation of the definition given in [29, Theorem 5.5].

It follows directly from the definition that for any nonnegative Borel measurable function $f \in L^p(X)$, then $\int_\gamma f < \infty$ for p–a.e. absolutely continuous curve. In a similar vein, the following lemma shows that convergent sequences in $L^p(X)$ converge along p–a.e. absolutely continuous curve.

Lemma 2.17 (cf Theorem 5.7 of [29]) *Fix $1 \le p < \infty$. Let $u_k\colon X \to \mathbb{R} \cup \{\pm\infty\}$ be a sequence of Borel measurable functions converging in $L^p(X)$ to a Borel measurable function $u\colon X \to \mathbb{R} \cup \{\pm\infty\}$. Up to a subsequence,*

$$\int_a^b |u_k - u||\dot{\gamma}|_g \, dt \to 0 \tag{2-19}$$

as $k \to \infty$ for all $\gamma \in \mathfrak{M} \setminus \Gamma$, where $\mathrm{Mod}_p(\Gamma) = 0$.

Lemma 2.17 was shown in the context of metric measure spaces in [29, Theorem 5.7]. In our setting, the proof carries over without modification.

Having in hand the notions of absolutely continuous curves and families Γ of absolutely continuous curves with $\mathrm{Mod}_p(\Gamma) = 0$, we are now in a position to define upper gradients and p–weak upper gradients of functions $u\colon X \to \mathbb{R}$. The notion of weak upper gradient was first introduced by Heinonen and Koskela in [34], and the definition we give here is analogous to [29, Definition 6.1].

Definition 2.18 (upper gradients and p–weak upper gradients) Let $u\colon X \to \mathbb{R}$ and $G\colon X \to [0, \infty]$ be Borel measurable functions. We say that G is an upper gradient for u if

$$|u(\gamma(a)) - u(\gamma(b))| \le \int_\gamma G \tag{2-20}$$

for every absolutely continuous curve $\gamma\colon [a, b] \to X$. For $1 \le p < \infty$, we say that G is a p–weak upper gradient for u if the upper gradient condition (2-20) holds for p–a.e. absolutely continuous curve $\gamma\colon [a, b] \to X$.

The following example shows that this is a natural notion of gradient.

Example 2.19 Consider Euclidean space as a rectifiable Riemannian space with the rectifiable atlas comprising only the identity chart. For a smooth function $u\colon \mathbb{R}^n \to \mathbb{R}$, the classical gradient $|\nabla u|$ is an upper gradient for u.

Furthermore, we note that the potential issues highlighted by Example 2.11 are eliminated with respect to this definition.

Example 2.20 Consider the rectifiable Riemannian space and the function f defined in Example 2.11. We see clearly that $G = 0$ is not a p–weak upper gradient for f, since the upper gradient condition (2-20) fails for any curve that crosses the hyperplane $\{x_1 = 0\}$. In fact, considering the family Γ of absolutely continuous curves of the form $\gamma(t) = (0, x') + te_1$ for $t \in (-\epsilon, \epsilon)$, we easily see that f has no p–weak upper gradient in $L^p(X)$.

Now, let $\widetilde{W}^{1,p}(X)$ be the collection of all Borel measurable functions $u \colon X \to \mathbb{R}$ such that u is L^p integrable and u has a p–weak upper gradient $G \in L^p(X)$. The Sobolev space $W^{1,p}(X)$ on a rectifiable Riemannian space is defined in the following way, following the definition first introduced by Shanmugalingam in [46] in the context of metric measure spaces and presented in Definition 7.1 of [29]. We note that a closely related definition of Sobolev spaces on a metric measure space was given by Cheeger in [16].

Definition 2.21 For any $u \in \widetilde{W}^{1,p}(X)$, we define

(2-21) $$\|u\|_{W^{1,p}(X)} = \|u\|_{L^p(X)} + \inf_G \|G\|_{L^p(X)},$$

where the infimum is taken over all p–weak upper gradients G of u. We define the space $W^{1,p}(X) = \widetilde{W}^{1,p}(X)/\sim$, where $u \sim v$ for $u, v \in \widetilde{W}^{1,p}(X)$ if $\|u - v\|_{W^{1,p}(X)} = 0$.

Remark 2.22 One subtlety of Definition 2.21, which is also present in the analogous metric measure space setting, is that it is possible to modify a function $u \in \widetilde{W}^{1,p}(X)$ on a set of m measure zero to obtain a function $\tilde{u}$ that is not in $\widetilde{W}^{1,p}(X)$. For instance, on Euclidean space, consider the functions $u \equiv 0$ in $W^{1,p}(\mathbb{R}^n)$ and $\tilde{u} = \chi_E$, where E is the set of all rational points in $\mathbb{R}^n$. Then $\tilde{u}$ has no p–weak upper gradient in L^p. This subtlety explains why some of the statements in Proposition 2.23 below require the choice of a certain representative for an L^p function. However, if $u, \tilde{u} \in \widetilde{W}^{1,p}(X)$ and $u = \tilde{u}$ m–a.e., then u and $\tilde{u}$ define the same element in $W^{1,p}(X)$.

From this point, we can establish a number of basic properties of the space $W^{1,p}(X)$ showing that this space possesses many of the important features of Sobolev spaces in smooth settings, which we collect in the following proposition. The analogous properties are established in the metric measure space setting in [29, Section 7]. In fact, the proofs there can be carried over almost verbatim, with only the modification being the distinction between the use of absolutely continuous curves in our setting as opposed to rectifiable curves in the setting of metric measure spaces. For this reason, we omit the proofs and instead point the reader to the corresponding statements in [29, Section 7]. Properties (1)–(3) and (5) were originally proven by Shanmugalingam [46], and (4) was established by Cheeger in [16] for $p > 1$ and Hajłasz [29] for $p = 1$.

Proposition 2.23 (basic properties of the Sobolev space $W^{1,p}(X)$) *Let (X,g) be a rectifiable Riemannian space and fix $1<p<\infty$. Then the following properties hold*:

(1) ***Closedness*** (cf [29, Lemma 7.8]) *Suppose $\{u_i\}_{i=1}^\infty, \{G_i\}_{i=1}^\infty$ are sequences in $L^p(X)$ such that u_i and G_i converge weakly to $u\in L^p(X)$ and $G\in L^p(X)$, respectively. If G_i is a p–weak upper gradient of u_i for each $i\in\mathbb{N}$, then there is a representative of u in L^p such that G is a p–weak upper gradient of u.*

(2) ***Lower semicontinuity*** (cf [29, Corollary 7.10]) *Let $u_i\in W^{1,p}$ be a bounded sequence converging weakly in $L^p(X)$ to u. Then there is a representative of u such that $u\in W^{1,p}(X)$ and*

$$\|u\|_{W^{1,p}(X)} \le \liminf_{i\to\infty} \|u_i\|_{W^{1,p}(X)}. \tag{2-22}$$

(3) ***Banach space*** (cf [29, Theorem 7.12]) *The space $W^{1,p}(X)$ is a Banach space.*

(4) ***Minimal p–weak upper gradient*** (cf [29, Theorem 7.16]) *There exists a minimal p–weak upper gradient $G_u\in L^p(X)$ in the sense that $G_u\le G$ m–a.e. for every p–weak upper gradient $G\in L^p(X)$.*

(5) ***Smooth spaces*** (cf [29, Theorem 7.13, Corollary 7.15]) *Suppose (X,g) is a smooth Riemannian manifold, then $W^{1,p}(X)$ coincides with the standard Sobolev space of X. Moreover, the norm of gradient vector $|\nabla u|_g$ is the least p–weak upper gradient for $u\in W^{1,p}(X)$.*

Thanks to Proposition 2.23, for any $u\in W^{1,p}(X)$ we may write

$$\|u\|_{W^{1,p}(X)} = \|u\|_{L^p(X)} + \|G_u\|_{L^p(X)}, \tag{2-23}$$

where G_u is the least p–weak upper gradient of u.

Without imposing any additional structure, the space $W^{1,p}$ on a rectifiable Riemannian space may be trivial. Moreover, as we saw in Example 2.11, the usual coordinate expression for the norm of the gradient may not be meaningful. For this reason, we introduce the notion of rectifiable Riemannian spaces that are $W^{1,p}$*–rectifiably complete*, that is, spaces for which the space $W^{1,p}$ is sufficiently large and the minimal p–weak upper gradient coincides with the derivative in charts almost everywhere.

Definition 2.24 ($W^{1,p}$–rectifiable completeness) Fix $p>n$. We say that (X,g) is $W^{1,p}$–rectifiably complete if the following hold:

(a) $W^{1,p}(X)$ is dense in $L^p(X)$.

(b) For all $u \in W^{1,p}(X)$ and $a \in I$, the function $u_a = \phi_a^* u \colon U_a \to \mathbb{R}$ is differentiable a.e. and

$$G_u(\phi_a(x)) = |\nabla u|_g \equiv \sqrt{g_a^{-1}(\partial u_a(x), \partial u_a(x))}$$

for $\phi_a^* m$–a.e. $x \in U_a$. Here, ∂u_a denotes the Euclidean gradient of u_a.

Example 2.25 A smooth Riemannian manifold is $W^{1,p}$–rectifiably complete for any $p \in (n, \infty)$.

Example 2.26 It is easy to check that the rectifiable Riemannian space of Example 2.7 is $W^{1,p}$–rectifiably complete for any $p \in (n, \infty)$.

Example 2.27 For $\alpha > 0$, consider the rectifiable Riemannian space $(\mathbb{R}^2, g_\alpha)$, where $g_\alpha = dx^2 + |x|^{2\alpha}\, dy^2$. This is a generalization of Example 2.10 and a special case of Example 2.9. Fix $p > 2$. There exists $\alpha = \alpha(p) \in (0, 1/2p)$ such that $(\mathbb{R}^2, g_\alpha)$ is rectifiably complete; the proof of this fact is a special case of the proof of Proposition 8.5 in Section 8.2.

Example 2.28 Thanks to Theorem 1.15, the d_p–limits of sequences of smooth Riemannian manifolds satisfying uniform lower bounds on scalar curvature and entropy are $W^{1,p}$–rectifiably complete for suitably chosen p. See Section 8 for further discussion and the proof of this fact.

2.2 The d_p–distance

In view of Example 2.9, we see that the geodesic distance does not reflect the underlying structure of a rectifiable Riemannian space. This is mainly due to the degeneracy of the metric. More seriously, we will see in Section 9 that these types of examples can arise as limits of manifolds with lower scalar curvature and entropy bounds. At its heart, this occurs because the distance function requires $W^{1,\infty}$–control, which will be too much to ask for. We introduce and discuss the notion of the d_p–distance here, which depends only on $W^{1,p}$–control of our space. In the context of lower scalar curvature and entropy bounds, this will be obtainable for at least some $n < p < \infty$.

Definition 2.29 (d_p–distance) Let (X, g) be a rectifiable Riemannian space. Given $p \in (1, \infty)$ and $x, y \in X$, the d_p–distance $d_{p,g,X}(x, y)$ between x and y is defined as

$$d_{p,g,X}(x, y) = \sup\left\{ |f(x) - f(y)| : \int_X |\nabla f|^p \, dm \le 1,\ f \in W^{1,p}_{\mathrm{loc}}(X) \cap C^0_{\mathrm{loc}}(X) \right\}.$$

When there is no ambiguity, we will frequently write d_p or $d_{p,g}$ or $d_{p,X}$ in place of $d_{p,g,X}$. On Euclidean space, we will often use the short-hand $d_{p,\mathrm{euc}} = d_{p,g_{\mathrm{euc}},\mathbb{R}^n}$. The definition of d_p makes sense for any $p \in (1,\infty)$, but is only interesting for $p > n$. For instance, on Euclidean space, $d_p(x,y) = +\infty$ for all $x \neq y$ whenever $p \leq n$. Indeed, for $p \in (1,n)$, consider any function f with $f(x) \neq 0$ that vanishes outside of $B\big(x, \frac{1}{2}|x-y|\big)$ and has $\|\nabla f\|_{L^p(\mathbb{R}^n)} \leq 1$, and consider the maximizing sequence given by $f_\lambda(z) = \lambda^{1-n/p} f(z/\lambda)$. For the borderline case $p = n$, take a maximizing sequence of smooth functions approximating $f(z) = c_n \log\log(1 + 1/|z-x|)$, where c_n is chosen so that $\|\nabla f\|_{L^n(\mathbb{R}^n)} = 1$.

2.2.1 Examples We consider two examples of the d_p–distance on some rectifiable Riemannian spaces. To begin with, we study the behavior of the d_p–distance on Euclidean space.

Example 2.30 (the d_p–distance on Euclidean space) On Euclidean space, for $p > n$ we directly compute that $d_p(x,y) = S|x-y|^{1-n/p}$, where

$$S = S_{n,p} = \sup\left\{ |f(x) - f(0)| : x \in B(0,1),\ \int_{\mathbb{R}^n} |\nabla f|^p \, dx \leq 1 \right\} \tag{2-24}$$

is a normalizing constant. Note that $S_{n,p} \to 1$ as $p \to \infty$. Thus,

$$\mathcal{B}_{p,g_{\mathrm{euc}}}(0, S r^{1-n/p}) = B(0,r) \tag{2-25}$$

for any $r > 0$. By taking the test function that is equal to $|x|\omega_n^{-n/p}$ in $B(0,1)$ and is the constant $\omega_n^{-n/p}$ on $\mathbb{R}^n \setminus B(0,1)$, we see that $S = S_{n,p} \geq \omega^{-n/p}$, and thus $\mathcal{B}_{p,g_{\mathrm{euc}}}(0,r) \subseteq B(0, \omega_n^{n/(p-n)} r^{p/(p-n)})$. In particular, if $p > n$ then

$$\mathcal{B}_{p,g_{\mathrm{euc}}}(0,r) \subseteq B(0, C_n r^{p/(p-n)}) \tag{2-26}$$

for all $r > 0$, where C_n depends only on the dimension.

Example 2.31 (hyperbolic space) Given $n \geq 2$ and $n < p < \infty$, hyperbolic space $(\mathbb{H}^n, g_{\mathrm{hyp}})$ has finite bounded diameter with respect to d_p. More specifically, there is a constant $C = C(n,p)$ such that $d_p(x,y) \leq C$ for all $x, y \in \mathbb{H}^n$. Indeed, this follows from the Morrey–Sobolev inequality on hyperbolic space established in [42] (see also [41] for the two-dimensional case), which states that there exists $C = C(n,p) > 0$ (which is, in fact, explicit and sharp) such that for any $f \in W^{1,p}(\mathbb{H}^n)$, we have

$$\sup_{x \in \mathbb{H}^n} |f(x)| \leq C \|\nabla f\|_{L^p(\mathbb{H}^n)}. \tag{2-27}$$

In the definition of d_p, we do not require f to be globally integrable. However, the proof of (2-27) is based on the Polya–Szegö principle and the symmetric decreasing rearrangement of f, so it is easy to check that the same proof implies that, for any $R>0$ and $f\in W^{1,p}(B_{g_{\mathrm{hyp}}}(R))$,

$$\sup_{x,y\in B_{g_{\mathrm{hyp}}}(R)}|f(x)-f(y)|\le C\|\nabla f\|_{L^p(B_{g_{\mathrm{hyp}}}(R))}, \tag{2-28}$$

where C is the same constant as in (2-27) and in particular is independent of R. Thus, passing $R\to\infty$, we arrive at the inequality

$$\sup_{x,y\in\mathbb{H}^n}|f(x)-f(y)|\le C\|\nabla f\|_{L^p(\mathbb{H}^n)}. \tag{2-29}$$

Consequently, $d_p(x,y)\le C$ for all $x,y\in\mathbb{H}^n$.

On any smooth closed Riemannian manifold (M,g), $d_{p,g}$ defines a distance metric on X as long as $p>n$. On the other hand, d_p may only define a pseudometric if the metric is degenerate, as we see in the following example.

Example 2.32 Fix $\alpha>0$ and consider the rectifiable Riemannian space $(\mathbb{R}^2,g_\alpha)$, where $g_\alpha=dx^2+|x|^{2\alpha}dy^2$. This is a generalization of Example 2.10 and a special case of Example 2.9. For p such that $\alpha p\ge 1$, we have $d_{p,g_\alpha}(x,y)=0$ for all $x,y\in\{0\}\times\mathbb{R}$.

Remark 2.33 The definition of the d_p–distance makes sense more generally for any space equipped with a $W^{1,p}$ structure. For instance, one may define the d_p–distance on a metric measure space (X,d,m). For reasonable metric measure spaces, for instance those which are doubling and have a Poincaré inequality, one can show $d_p(x,y)\to d(x,y)$ as $p\to\infty$ for any pair of points $x,y\in X$. To see this, for each $x,y\in X$, $p>n$ and $\epsilon>0$ we choose f_p so that $f_p(x)=0$,

$$d_p(x,y)\le|f_p(y)-f_p(x)|+\epsilon\quad\text{and}\quad\int_X|\nabla f_p|^p\,dm\le 1. \tag{2-30}$$

Then the Sobolev embedding theorem for such spaces [30] shows that $\{f_p\}_{p>0}$ is locally uniformly Hölder continuous and is bounded in $W^{1,q}$ on each bounded open subset for any $q>0$. Hence, there is a sequence $p_i\to+\infty$ and an $f\in\bigcap_{p>0}W^{1,p}_{\mathrm{loc}}(X)$ such that f_{p_i} converges to f weakly in $W^{1,q}_{\mathrm{loc}}(X)$ for all $q>0$ and locally uniformly in $C^0_{\mathrm{loc}}(X)$. In particular, we have $|\nabla f|\le 1$ and

$$\limsup_{i\to+\infty}d_{p_i}(x,y)\le|f(y)-f(x)|+\epsilon\le d(x,y)+\epsilon, \tag{2-31}$$

where we have used the characterization of the geodesic distance on such spaces: $d(x,y)=\sup\{|f(x)-f(y)|:|\nabla f|\le 1\}$; see Sections 5 and 6 of [16]. Thus, we have $\limsup_{p\to+\infty} d_p(x,y) \le d(x,y)$. For the opposite direction, if X is of finite diameter, X is of finite measure. By taking $f_p(z) = m(X)^{-1/p} d(x,z)$ for $z \in X$, $\|\nabla f_p\|_{L^p} = 1$ and hence f_p is an admissible function. By letting $p \to +\infty$, we have the reverse inequality, $\limsup_{p\to+\infty} d_p(x,y) \ge d(x,y)$. If the diameter is infinite, by the volume-doubling properties, X is of polynomial volume growth. By modifying the test function with a cutoff $f_p(z) = c_p\phi(d(z,y)/p)\cdot\max\{d(z,y), 2d(x,y)\}$ where c_p is chosen so that $\|\nabla f_p\|_{L^p} = 1$ and $c_p \to 1$ as $p \to +\infty$ thanks to the volume growth, a similar argument will give us the reverse inequality as well.

The proof above can be easily extended to the measured Gromov–Hausdorff convergence version for RCD spaces. More precisely, fix a measured Gromov–Hausdorff convergent sequence of compact $\mathrm{RCD}(K,N)$ spaces with $N < +\infty$:

$$(X^i, d^i, m^i) \to (X^\infty, d^\infty, m^\infty). \tag{2-32}$$

By the above argument, we may write $d^i_\infty = d^i$. The same argument together with the convergence [7; 8; 25] for p–Cheeger energies with respect to (2-32) shows that whenever $p_i \to p_\infty$, $x_i, y_i \to x_\infty, y_\infty$ for $x_i, y_i \in X_i$ and $x_\infty, y_\infty \in X_\infty$, we have $d^i_{p_i}(x_i,y_i) \to d^\infty_{p_\infty}(x_\infty,y_\infty)$. In fact, by using a contradiction argument, one can in addition show that the convergence is uniform. Namely, for any $\epsilon > 0$ and $\mathrm{RCD}(K,N)$ space (X,d,m) with $\mathrm{diam}(X) = 1$ and $0 < v < m(X) < V < +\infty$, there is a $p_0(K,N,v,V)$ such that for all $p > p_0$ and $x, y \in X$, we have $|d_p(x,y) - d(x,y)| < \epsilon$.

2.2.2 Properties of the d_p–distance

Let us discuss some basic facts about the d_p–distance.

Remark 2.34 (scaling) Given a $W^{1,p}$–complete rectifiable Riemannian space (X,g), let $\tilde g = \rho^{-2} g$. Then for any $x, y \in M$, we have

$$d_{p,\tilde g}(x,y) = \rho^{n/p-1} d_{p,g}(x,y). \tag{2-33}$$

Next, in Example 2.32, we saw an example of a degenerate metric on a rectifiable Riemannian space for which d_p only defined a pseudometric. It would therefore be sensible to formalize knowing when this does and does not happen:

Definition 2.35 (d_p–rectifiable completeness) Given a rectifiable Riemannian space (X,g), we say that (X,g) is d_p–rectifiably complete if d_p defines a metric on X and the topology induced by d_p coincides with the topology of X.

By Sobolev embedding, smooth compact manifolds are indeed d_p–complete for $p > n$. Morally speaking, that is to say that the d_p metric geometry coincides with the original geometry on compact manifolds. The following shows that two Riemannian manifolds which are d_p–isometric are in fact isometric as Riemannian manifolds.

Proposition 2.36 *Fix $n \geq 2$ and $p > n$. Let (M, g) and (N, h) be compact Riemannian n–manifolds and suppose that $(M, d_{p,g})$ and $(N, d_{p,h})$ are isometric as metric spaces. Then (M, g) and (N, h) are isometric as Riemannian manifolds.*

The proof relies on multiple steps of approximation and is a corollary of the proof of Theorem 1.7; we will postpone the proof to Section 7.

Remark 2.37 From the definition of d_p, we see that for all $x \in X$ we obtain the local Sobolev inequality

$$\sup_{y \in \mathcal{B}_p(x,R)} |f(x) - f(y)| \leq R\|\nabla f\|_{L^p(X)}. \tag{2-34}$$

In particular, if (X, g) is a compact rectifiable Riemannian space that is d_p–rectifiably complete, then it satisfies the Sobolev embedding $W^{1,p}(X) \hookrightarrow L^\infty(X)$.

Remark 2.38 (d_p as the $(W^{1,p})^*$–norm) Fix $p > n$ and let (X, g) be a rectifiable Riemannian space satisfying the Sobolev embedding $W^{1,p}(X) \hookrightarrow L^\infty(X)$. For instance, one can consider any compact d_p–rectifiably complete (X, g) by Remark 2.37. Then for any $x \in X$, the Dirac delta δ_x is an element of the dual space $(W^{1,p}(X))^*$, and

$$d_p(x, y) \equiv \|\delta_x - \delta_y\|_{(W^{1,p}(X))^*}. \tag{2-35}$$

In particular, reinterpreting the usual distance function $d(x, y) = \|\delta_x - \delta_y\|_{(W^{1,\infty}(X))^*}$, we see that the d_p–distance function has been obtained by weakening the function space norm we use to measure the distance between the distributions δ_x and δ_y.

2.2.3 d_p–convergence Our primary interest in the d_p–distance is to give rise to a notion of convergence which captures some Sobolev control. We begin with d_p–convergence of compact sequences.

Definition 2.39 (d_p–convergence) Let (X, g) and (Y, h) be compact d_p–complete rectifiable Riemannian spaces. Given $\epsilon > 0$, we say that

$$d_p((X, g), (Y, h)) \leq \epsilon \tag{2-36}$$

if there exist collections of points $\{x_i\}_{i=1}^N \subset X$ and $\{y_i\}_{i=1}^N \subset Y$ such that each collection is ϵ–dense with respect to d_p and

$$|d_{p,g,X}(x_i, x_j) - d_{p,h,Y}(y_i, y_j)| \le \epsilon, \tag{2-37}$$

and further

$$1-\epsilon \le \frac{\mathrm{vol}_g(\mathcal{B}_p(x_i, r))}{\mathrm{vol}_h(\mathcal{B}_p(y_i, r))} \le 1+\epsilon \tag{2-38}$$

for all $r \in [\epsilon, 1]$.

Remark 2.40 It is more standard to replace (2-38) with something like

$$1-\epsilon \le \frac{\int_{\mathcal{B}_p(x_i,r)} 1 - r^{-1} d_p(x_i, z)}{\int_{\mathcal{B}_p(y_i,r)} 1 - r^{-1} d_p(y_i, z)} \le 1+\epsilon, \tag{2-39}$$

which is strictly weaker and does not rule out the possibility that the measures are concentrating. In our context we can work with the stronger condition, so we leave it as in (2-38).

In other words, two compact spaces are ϵ close in the d_p sense if their d_p metric spaces are ϵ–Gromov–Hausdorff close and the volumes of balls above scale ϵ are close.

Remark 2.41 In (2-38), we require volumes of balls to be ϵ–close up to scale 1. Up to scaling, we may replace 1 with any other fixed number.

Recall that a sequence of pointed proper metric spaces (X_i, d_i, x_i) is said to converge to a pointed proper metric space (X, d, x) in the pointed Gromov–Hausdorff topology if $(\bar{B}_{d_i}(x_i, R), d_i) \to (\bar{B}_d(x, R), d)$ in the Gromov–Hausdorff topology for every $R > 0$. In view of Example 2.31, we clearly cannot adopt a direct analogue of this definition when defining pointed d_p–convergence. Indeed, we have seen that the hyperbolic space equipped with the d_p metric is *not* a proper metric space for $p > n$, since sufficiently large balls have noncompact closure. For this reason, we cannot define pointed d_p–convergence by asking for d_p–convergence on d_p–balls of increasingly large radius. Instead, we make use of d_p–completeness to construct an exhaustion that plays the role of balls of large radius.

More specifically, let (X, g, x) be a d_p–complete pointed rectifiable Riemannian space. By d_p–completeness, for any $y \in X$ there is some radius $r \le 1$ sufficiently small that $\mathcal{B}_p(y, 4r) \Subset X$ has compact closure. Roughly speaking, we define $\mathrm{Cov}(x, N)$ to be the set of points that are linked to x by a sequence of N precompact balls of radius at most 1. More concretely, define $\mathrm{Cov}(x,N)$ be to the collection of points y such that there is a sequence $\{(z_i,r_i)\}_{i=1}^N$ satisfying

(1) $r_i \le 1$,

(2) $x, y \in \bigcup_{i=1}^N \mathcal{B}_p(z_i, r_i)$,

(3) $\mathcal{B}_p(z_i,r_i)\cap\mathcal{B}_p(z_{i+1},r_{i+1}) \neq \varnothing$ for all $i = 1,\dots,N-1$,

(4) $\mathcal{B}_p(z_i, 4r_i)$ is precompact.

Note that $\mathrm{Cov}(x,N)$ is an open set, and that by the triangle inequality, we always have the containment $\mathrm{Cov}(x,N) \subseteq \mathcal{B}_p(x,2N)$. To get an intuitive idea for how the sets $\mathrm{Cov}(x,N)$ behave, if we define the analogue of $\mathrm{Cov}(x,N)$ with respect to the geodesic distance instead of the d_p–distance, then on any Riemannian manifold (or more generally, on any proper length space), this set is simply a geodesic ball of radius $2N$.

The main advantage of working with the sets $\mathrm{Cov}(x,N)$ instead of p–balls of increasing radius is highlighted in the following two initial lemmas, which show the sense in which $\{\mathrm{Cov}(x,N)\}_{N\in\mathbb{N}}$ provides an exhaustion of X. The first lemma shows that any $y \in X$ is contained in $\mathrm{Cov}(x,N)$ for some $N \in \mathbb{N}$.

Lemma 2.42 *Let (X,g,x) be a d_p–complete rectifiable Riemannian space. For any compact connected set $\Omega \subset X$ containing x, there exists an $N \in \mathbb{N}$ such that $\Omega \subset \mathrm{Cov}(x,N)$.*

Proof By compactness of Ω, we can find $1 > r > 0$ sufficiently small that for all $z \in \Omega$, $\mathcal{B}_p(z,4r) \Subset X$. Then by compactness of Ω, we can find N such that Ω is covered by $\{\mathcal{B}_p(z_i,r)\}_{i=1}^N$ for $z_i \in \Omega$. Hence $\mathrm{Cov}(x,N)$ contains Ω. □

The second lemma is more subtle, and shows that $\mathrm{Cov}(x,N)$ has compact closure for any $N \in \mathbb{N}$.

Lemma 2.43 *Let (X,g,x) be a d_p–complete rectifiable Riemannian space. For any $N \in \mathbb{N}$ the set $\mathrm{Cov}(x,N)$ has compact closure.*

Proof We argue by induction. For $N = 1$, let $r_0 \le 1$ be the supremum over radii $r \le 1$ such that for some $y \in X$ we have $x \in \mathcal{B}_p(y,r)$ and $\mathcal{B}_p(y,4r) \Subset X$. So we may find some $y_0 \in X$ and $r_{y_0} \in \left(\frac{9}{10}r_0, r_0\right)$ such that $x \in \mathcal{B}_p(y_0, r_{y_0})$ and $\mathcal{B}_p(y_0, 4r_{y_0}) \Subset X$. We claim that

$$\mathrm{Cov}(x,1) \subset \mathcal{B}_p(y_0, 4r_{y_0}), \tag{2-40}$$

the latter of which has compact closure by assumption. Indeed, for any $z \in \mathrm{Cov}(x,1)$, we have $z \in \mathcal{B}_p(y,r)$ for some $y \in X$ and for some $r \le r_0$ with $\mathcal{B}_p(y,4r) \Subset X$. Then by repeatedly applying the triangle inequality,

$$z \in \mathcal{B}_p(y,r_0) \subset \mathcal{B}_p(x,2r_0) \subset \mathcal{B}_p(y_0,3r_0) \subset \mathcal{B}_p(y_0,4r_{y_0}). \tag{2-41}$$

This establishes (2-40).

Now, suppose that we have shown that $\mathrm{Cov}(x,N)$ is precompact. We claim that $\mathrm{Cov}(x,N+1) \Subset X$. It suffices to show that for all sequences in $\mathrm{Cov}(x,N+1)$, there is a subsequence that converges with respect to the underlying topology on X.

Let x_i be a sequence in $\mathrm{Cov}(x,N+1)$. For each i, we can find $\{(z_{i,a},r_{i,a})\}_{a=1}^{N+1}$ such that

(1) $r_{i,a} \le 1$ for all $a \le N+1$,
(2) $x, x_i \in \bigcup_{a=1}^{N+1} \mathcal{B}_p(z_{i,a},r_{i,a})$,
(3) $\mathcal{B}_p(z_{i,a},r_{i,a}) \cap \mathcal{B}_p(z_{i,a+1},r_{i,a+1}) \neq \varnothing$ for $a = 1,\dots,N$,
(4) $\mathcal{B}_p(z_{i,a},4r_{i,a}) \Subset X$.

We may assume $x_i \in \mathcal{B}_p(z_{i,N+1},r_{i,N+1})$ and $x \in \bigcup_{a=1}^{N} \mathcal{B}_p(z_{i,a},r_{i,a})$ for all i large enough. Otherwise, $x_i \in \mathrm{Cov}(x,N)$, which we already know to have compact closure and thus x_i has a convergent subsequence with respect to the underlying topology. Let $z_i \in \mathcal{B}_p(z_{i,N+1},r_{i,N+1}) \cap \mathcal{B}_p(z_{i,N},r_{i,N}) \subseteq \mathrm{Cov}(x,N)$. By compactness, we can assume $z_i \to z_\infty \in \overline{\mathrm{Cov}(x,N)}$, which is compactly contained in X. Therefore for all i and k,

$$d_p(x_i,z_k) \le d_p(x_i,z_i) + d_p(z_i,z_k) \le 2r_{i,N+1} + d_p(z_i,z_k). \tag{2-42}$$

By compactness, we may assume $r_{i,N+1} \to r_\infty \in [0,1]$ as $i \to \infty$.

Case 1 If $r_\infty > 0$, then we can find K such that for all $i \ge K$,

$$d_p(z_i,z_K) < \tfrac{1}{2} r_{K,N+1} \quad \text{and} \quad r_{i,N+1} < \tfrac{3}{2} r_{K,N+1}. \tag{2-43}$$

Therefore for $i \ge K$, we have $x_i \in \mathcal{B}_p(z_K, 3r_{K,N+1}) \subset \mathcal{B}_p(z_{K,N+1},4r_{K,N+1})$, which is compactly contained in X.

Case 2 If $r_\infty = 0$, then

$$d_p(x_i,z_\infty) \le d_p(x_i,z_i) + d_p(z_i,z_\infty) \le 2r_{i,N+1} + d_p(z_i,z_\infty) \to 0. \tag{2-44}$$

By d_p–completeness, there is an $r_{z_\infty} > 0$ such that $\mathcal{B}_p(z_\infty, r_{z_\infty}) \Subset X$. By (2-44), $x_i \in \mathcal{B}_p(z_\infty, r_{z_\infty})$ for i sufficiently large. This completes the proof. □

With these two lemmas in hand, we can now define pointed d_p–convergence.

Definition 2.44 Let (X_i, g_i, x_i) and (X, g, x) be d_p–complete pointed rectifiable Riemannian spaces. We say that

$$(X_i, g_i, x_i) \to (X, g, x) \tag{2-45}$$

in the pointed d_p sense if the following holds. For all $N \in \mathbb{N}$, there exists $N' \geq N$ and compact sets $\Omega \subset X$ and $\Omega_i \subset X_i$ such that:

(1) $\mathrm{Cov}(x, N) \subset \Omega \subset \mathrm{Cov}(x, N')$.

(2) $\mathrm{Cov}_i(x_i, N) \subset \Omega_i \subset \mathrm{Cov}_i(x_i, N')$ for i sufficiently large.

(3) For all $\epsilon > 0$, there exists an N such that for all $i > N$, the following holds: There exist $\{x_j^i\}_{j=1}^N \subset \Omega_i$ and $\{y_j\}_{j=1}^N \subset \Omega$ such that each collection is ϵ–dense with respect to d_{p,Ω_i} and $d_{p,\Omega}$, respectively, and for all $r \in [\epsilon, 1]$ satisfy

$$|d_{p,\Omega_i}(x_k^i, x_l^i) - d_{p,\Omega}(y_k, y_l)| \leq \epsilon, \tag{2-46}$$

$$1 - \epsilon \leq \frac{\mathrm{vol}_g(\mathcal{B}_p(x_k^i, r))}{\mathrm{vol}_h(\mathcal{B}_p(y_k, r))} \leq 1 + \epsilon. \tag{2-47}$$

Remark 2.45 In part (3) of Definition 2.44 above, the d_p–convergence on the compact sets Ω_i and Ω corresponds to the relative d_p–distances d_{p,g_i,Ω_i} and $d_{p,g,\Omega}$. This is necessary as d_p is not a local object.

3 Further preliminaries

In this section, we introduce further preliminaries that will be needed in the paper.

3.1 Ricci flows

Let us cover some of the basics of the Ricci flow in this subsection. A Ricci flow $(M, g(t))_{t \in (0,T)}$ is a family of smooth metrics $g(t)$ on a smooth manifold M^n satisfying the evolution equation

$$\partial_t g(t) = -2\,\mathrm{Ric}_{g(t)}\,. \tag{3-1}$$

If (M^n, g) is a complete Riemannian manifold with bounded curvature,

$$\sup_{x \in M} |\mathrm{Rm}_g|(x) < +\infty, \tag{3-2}$$

then Shi in [47] established the short-time existence of the Ricci flow for such a metric, following the existence theory on closed Riemannian manifolds due to Hamilton [31] and the trick by DeTurck [21].

A sequence $\{(M_i, g_i(t), x_i)_{t\in(0,T)}\}$ of complete pointed solutions of the Ricci flow satisfying

$$|\mathrm{Rm}_{g(t)}| \le \frac{C}{t} \quad \text{and} \quad \mathrm{inj}(M, g(t)) \ge c_0 t^{1/2} \tag{3-3}$$

is compact in the C^∞ Cheeger–Gromov topology. That is, up to a subsequence, $\{(M_i, g_i(t), x_i)_{t\in(0,T)}\}$ converges smoothly to a pointed complete solution of the Ricci flow $(M_\infty, g_\infty(t), x_\infty)_{t\in(0,T)}$, which also satisfies (3-3). This compactness theorem was originally observed in Hamilton [32]; see also [19, Theorem 6.35].

Along the Ricci flow, the scalar curvature $R = R_{g(t)}(x)$ evolves by

$$\partial_t R = \Delta R + 2|\mathrm{Ric}|^2. \tag{3-4}$$

The equation is coupled to the Ricci flow in the sense that $\Delta = \Delta_{g(t)}$. Because it is a supersolution of the heat equation, lower bounds on the scalar curvature are preserved under the Ricci flow. In other words, if $R_{g(0)} \ge -\delta$ for all $x \in M$, then

$$R_{g(t)} \ge -\delta \tag{3-5}$$

for all $x \in M$ and $t \in (0, T)$. This monotonicity provides one-sided control on the expansion of volumes under the Ricci flow, since the volume form evolves by

$$\partial_t \, d\mathrm{vol}_{g(t)} = -R_{g(t)} \, d\mathrm{vol}_{g(t)}. \tag{3-6}$$

As such, a flow satisfying (3-5) has $d\mathrm{vol}_{g(t)} \le \exp\{\delta(t-s)\} \, d\mathrm{vol}_{g(s)}$ for all $s \le t$. So, provided $\delta \le \frac{1}{2}$, a Taylor expansion shows that for all $0 \le s \le t \le \min\{1, T\}$,

$$d\mathrm{vol}_{g(t)} \le \{1 + 2\delta(t-s)\} \, d\mathrm{vol}_{g(s)}. \tag{3-7}$$

3.2 Heat flows coupled to Ricci flow

Given a complete bounded curvature Ricci flow $(M, g(t))$, with $t \in [0, T)$, we consider the heat operator $\partial_t - \Delta$ coupled to Ricci flow, along with its formal adjoint the conjugate heat operator $-\partial_t - \Delta + R$. The Cauchy problem for the conjugate heat equation is well-posed backward in time. The two operators are conjugate in the sense that if u and v are smooth functions, with suitable decay at infinity if M is noncompact, then

$$\begin{aligned} \int_M uv \, d\mathrm{vol}_{g(T)} - \int_M uv \, d\mathrm{vol}_{g(0)} \\ = \int_0^T \int_M \{v(\partial_t - \Delta)u - u(-\partial_t - \Delta + R)v\} \, d\mathrm{vol}_{g(t)} \, dt. \end{aligned} \tag{3-8}$$

For $0 \le s < t \le T$, we let $K(x,t;y,s)$ denote the heat kernel with singularity at (y,s), ie the solution to

$$\begin{cases} (\partial_t - \Delta_x)K(x,t;y,s) = 0 & \text{for } t \in (s,t), \\ \lim_{t \searrow s} K(\,\cdot\,,t;y,s) = \delta_y. \end{cases} \tag{3-9}$$

As a function of (y,s), $K(x,t;y,s)$ is the kernel for the conjugate heat equation with singularity at (x,t), that is,

$$\begin{cases} (\partial_s + \Delta_y - R)K(x,t;y,s) = 0 & \text{for } s \in (0,t), \\ \lim_{s \nearrow t} K(x,t;\,\cdot\,,s) = \delta_x. \end{cases} \tag{3-10}$$

Suppose φ satisfies $(\partial_t + \Delta - R)\varphi = 0$ on $M \times [0,T]$. Then

$$\int_M \varphi(x,t)\,d\mathrm{vol}_{g(t)}(x) = \int_M \varphi(x,T)\,d\mathrm{vol}_{g(T)}(x) \tag{3-11}$$

for all $t \in [0,T]$. In particular, for any $s \in [0,t)$ we have

$$\int_M K(x,t;y,s)\,d\mathrm{vol}_{g(s)}(y) = 1. \tag{3-12}$$

If (3-5) holds, then (3-7) and (3-9) imply that for $s \le t \le \min\{1,T\}$,

$$\int_M K(x,t;y,s)\,d\mathrm{vol}_{g(t)}(x) \le 1 + 2\delta(t-s). \tag{3-13}$$

From the evolution (3-4), we have the following representation formula for the scalar curvature in terms of the heat kernel:

$$\begin{aligned} R_{g(t)}(x) = \int_M & K(x,t;y,0)R_{g(0)}(y)\,d\mathrm{vol}_{g(0)}(y) \\ &+ 2\int_0^t \int_M K(x,t;y,s)|\mathrm{Ric}_{g(s)}(y)|^2\,d\mathrm{vol}_{g(s)}(y)\,ds. \end{aligned} \tag{3-14}$$

We refer the reader to [17, Chapter 26.1] for these basic properties on kernels for heat-type equations coupled to Ricci flow.

3.3 The $\mathcal{W}$–functional and Perelman entropy

Let $(M,g(t))_{t\in(0,T]}$ be a complete Ricci flow with bounded curvature. The $\mathcal{W}$–functional defined in (1-4) is monotone along the Ricci flow in the following sense. Set $\tau(t) = T - t$ and let

$$u(x,t) = (4\pi\tau)^{-n/2}e^{-f(x,t)} \tag{3-15}$$

be a solution of the conjugate heat equation along the flow on $[0,T]$. Provided u decays suitably at infinity (eg if $u(T)$ is compactly supported) if M is noncompact, then

$$\mathcal{W}(g(s), f(s), \tau(s)) \le \mathcal{W}(g(t), f(t), \tau(t)) \tag{3-16}$$

for $s \le t$. This monotonicity was shown in [43] for closed manifolds and in [13, Theorem 7.1(i)–(ii)] for complete manifolds with bounded curvature. By taking a compactly supported minimizing sequence for $\mu(g(t), T)$, we see from (3-16) that the Perelman entropy $\mu(\tau)$ defined in (1-5) is also monotone along the Ricci flow in the sense that

$$\mu(g(s), \tau(s)) \le \mu(g(t), \tau(t)) \tag{3-17}$$

for $s \le t$. Correspondingly, if $(M, g(t))_{t\in(0,2)}$ is a Ricci flow and if $\nu(g(0), 2) \ge -\delta$, then $\nu(g(t), 1) \ge -\delta$ for all $t \in (0, 1]$.

Suppose the infimum in $\mu(g(t), \tau(t))$ is achieved. This is the case, for instance, if M is closed; see [54] for necessary and sufficient conditions in the complete noncompact case with bounded geometry. Then the entropy $\mu(g(t), \tau(t))$ is constant in t only if the Ricci flow is a gradient shrinking soliton that becomes singular at time T. This means that, if f is a function achieving the infimum in $\mu(g(t), \tau(t))$ and $\varphi(t)$ is the diffeomorphism generated by $\nabla f(t)$, then

$$g(t) = (T-t)\varphi(t)^* g(0). \tag{3-18}$$

On Euclidean space $(\mathbb{R}^n, g_{\mathrm{euc}})$, the entropy $\mu(g_{\mathrm{euc}}, \tau)$ is equal to zero for all $\tau > 0$, and the infimum in $\mu(g_{\mathrm{euc}}, \tau)$ is achieved by $f(x) = |x|^2/4\tau$. The following lemma asserts that Euclidean space is the only complete Riemannian manifold with bounded curvature with $\mu(g, \tau) = 0$; cf [43, Section 3.1].

Lemma 3.1 *Let $(M, g(t))_{t\in(0,T)}$ be a complete Ricci flow with bounded curvature. Then $\mu(g(0), \tau) \le 0$ for all $\tau \in (0, T)$, with equality if and only if the flow is isometric to Euclidean space. Furthermore, fix $t \in (0, T)$ and $x \in M$. Set $\tau(s) = t - s$, and let $f(y, s)$ be defined by*

$$K(x, t; y, s) = (4\pi\tau)^{-n/2} e^{-f(y,s)}. \tag{3-19}$$

If

$$\mathcal{W}(g(s), f(s), \tau(s)) = 0, \tag{3-20}$$

then the flow is isometric to the constant Euclidean flow.

Proof Fix $t \in (0, T)$ and $x \in M$. Set $\tau(s) = t - s$, and let $f(y, s)$ be defined by (3-19). Then $\lim_{s \to t} \mathcal{W}(g(s), f(s), \tau(s)) = 0$. Hence, by monotonicity and the definition of $\mu(g, \overline{\tau})$ as an infimum,

$$\mu(g, t) \le \mathcal{W}(g(0), f(0), t) \le 0. \tag{3-21}$$

This concludes the proof of the first claim. Furthermore, suppose that $\mu(g(0), t) = 0$ for some $t \in (0, T)$. By (3-21), f achieves the infimum in $\mu(g(0), t)$, where again we let f be defined by (3-19). So, $g(t)$ is a gradient-shrinking soliton given by (3-18). In particular,

$$|\mathrm{Rm}_{g(s)}| = \frac{1}{t-s} |\mathrm{Rm}_{g(0)}|. \tag{3-22}$$

The flow exists and has bounded curvature for $s \in (0, T)$, with $T > t$. Thus (3-22) implies that $|\mathrm{Rm}_{g(0)}| = 0$. This (in particular, that $|\mathrm{Ric}_{g(0)}| = 0$) together with (3-18) implies that $(M, g(t))$ is a metric cone, cf [51]. However, a flat manifold which is also a metric cone can only be the Euclidean space. We conclude $(M, g) = (\mathbb{R}^n, g_{\mathrm{euc}})$. □

Finally, let us recall Perelman's no-local-collapsing theorem, which ensures that small balls are noncollapsed along the flow if the entropy is bounded below. More specifically, fix $x \in M$ and $t \in (0, T)$. Then for $r \in (0, t^{1/2})$, if $R_{g(t)} \le r^{-2}$ on $B_{g(t)}(x, r)$, then

$$\mathrm{vol}_{g(t)}(B_{g(t)}(x, r)) \ge \kappa r^n, \tag{3-23}$$

where κ depends only on n and $\nu(g(0), 2T)$.

3.4 Uniform existence time and scale-invariant estimates

A crucial point throughout the paper is that, under the lower bounds on the entropy assumed in our main theorems, the Ricci flow starting from (M, g) exists for a uniform time and enjoys small scale-invariant estimates on the curvature tensor. The following theorem establishes this fact, and essentially follows from an epsilon regularity theorem of Hein and Naber in [33]. For the sake of completeness, and because the assumptions made here are slightly different than the ones there, we include the proof.

Theorem 3.2 *Fix $n \ge 2$ and $\lambda > 0$. There exists $\delta = \delta(n, \lambda) > 0$ such that the following holds. Let (M, g) be a complete Riemannian n–manifold with bounded curvature satisfying*

$$\nu(g, 2) \ge -\delta. \tag{3-24}$$

Then the Ricci flow $(M, g(t))$ *with* $g(0) = g$ *exists for* $t \in (0, 1]$. *Furthermore, for all* $x \in M$ *and* $t \in (0, 1]$, *we have the scale-invariant estimates*

$$|\mathrm{Rm}_{g(t)}| \le \frac{\lambda}{t}. \tag{3-25}$$

Remark 3.3 Thanks to Shi's derivative estimates, (3-25) implies that for all $k \in \mathbb{N}$, we have

$$|\nabla^k \mathrm{Rm}| \le \frac{C(n,k,\lambda)}{t^{1+k/2}}. \tag{3-26}$$

The key point of the proof of Theorem 3.2 is the continuity property contained in the following lemma.

Lemma 3.4 *Fix* $n \ge 2$ *and* $C > 0$. *Let* $\{(M_i, g_i(t), x_i)_{t\in(-1,1]}\}$ *be a sequence of complete Ricci flows such that each time slice has bounded curvature and*

$$|\mathrm{Rm}| \le C \quad \text{on } M \times (-1, 0]. \tag{3-27}$$

Suppose that

$$\nu(g_i(-1), 2) \ge -\delta_i, \tag{3-28}$$

where $\delta_i \to 0$. *Then, up to a subsequence,* $\{(M_i, g_i(t), x_i)_{t\in(-1,0]}\}$ *converges smoothly to the constant Euclidean flow* $(\mathbb{R}^n, g_{\mathrm{euc}}, 0^n)_{t\in(-1,0]}$.

Proof Perelman's no-local-collapsing (3-23) together with the curvature bounds (3-27) provide a lower bound on the injectivity radius for each time-slice; see for instance [44, Chapter 10, Lemma 51]. So, by Hamilton's compactness theorem, the flows converge smoothly to a smooth limiting Ricci flow $(M_\infty, g_\infty(t), x_\infty)_{t\in[0,1]}$ satisfying (3-27).

For each $i \in \mathbb{N}$, let $f_i(y,s) = f_i(x_i, 1; y, s)$ be the function defined by

$$K_i(x_i, 1; y, s) = (4\pi(1-s))^{-n/2} \exp\left\{\frac{-f_i(y,s)}{4(1-s)}\right\}. \tag{3-29}$$

Here K_i is the heat kernel on (M_i, g_i, x_i) defined in (3-9). From (3-28), we see that

$$-\delta_i \le \mathcal{W}(g_i(s), f_i(s), 1-s) \le 0. \tag{3-30}$$

The uniform curvature bounds (3-27) ensure that the heat kernels are uniformly Gaussian. That is, setting $\tau = 1-s$ for $s \in [0,1]$, we have

$$\frac{c}{\tau^{n/2}} \exp\left\{\frac{-d_{g_i(1)}(x_i, y)^2}{4\tau}\right\} \le K_i(x_i, 1; y, s) \le \frac{C}{\tau^{n/2}} \exp\left\{\frac{-d_{g_i(1)}(x_i, y)^2}{4\tau}\right\}. \tag{3-31}$$

Since $K_i(x_i, 1; y, s)$ is also a solution of the conjugate heat equation (3-8), we determine that for $s \in (0, 1)$, the sequence $\{f_i\}$ converges smoothly on compact sets to a function $f_\infty\colon M \times (0, 1) \to \mathbb{R}$ that satisfies the minimization constraint in the definition of $\mu(g(0), \tau)$. Thus from (3-30), we have

$$\mathcal{W}(g_\infty(s), f_\infty(s), \tau(s)) = 0. \tag{3-32}$$

We conclude by applying Lemma 3.1. □

Now, Theorem 3.2 follows from Lemma 3.4 by a standard contradiction argument, which we show below. Before giving the proof, it will be convenient to introduce the following notation. Let $(M, g(t))_{t\in(0,T)}$ be a smooth Ricci flow such that each time-slice is complete with bounded curvature. Given $x \in M$ and $t \in (0, T)$, we define the regularity scale $r_{|\mathrm{Rm}|}(x, t)$ to be

$$r_{|\mathrm{Rm}|}(x, t) = \sup\Bigl\{r > 0 : \sup_{P(x,t,r)} |\mathrm{Rm}| \le r^{-2}\Bigr\}. \tag{3-33}$$

Here $P(x, t, r) = B_{g(t)}(x, r) \times (t - r^2, t]$ is a parabolic cylinder.

Proof of Theorem 3.2 In order to prove the theorem, it suffices to prove the following claim.

Claim *Fix $n \in \mathbb{N}$ and $\lambda > 0$. There exists $\delta = \delta(n, \lambda)$ such that if $(M, g(t))_{t\in(0,T]}$ satisfying $\nu(g(0), 2T) \ge -\delta$, then*

$$r_{|\mathrm{Rm}|}(x, T)^2 \ge \frac{T}{\lambda}. \tag{3-34}$$

Before proving the claim, let us see how it implies the theorem. Let $T_{\max} > 0$ be the maximal existence time for the Ricci flow with $g(0) = g$, and let $T_0 = \min\{1, T_{\max}\}$. For any $x \in M$ and $t \in (0, T_0)$, we apply the claim to find that (3-25) holds at (x, t). Finally, recall that if $T_{\max} < +\infty$, then $\sup_M |\mathrm{Rm}| \to +\infty$ as $t \to T_{\max}$. So, we conclude that $T_{\max} > T_0$, and thus $T_0 = 1$. This concludes the proof of the theorem.

Let us now prove the claim. Up to rescaling the flow and translating in time, it is equivalent to showing that if $(M, g(t))_{t\in(-1,0]}$ satisfies $\nu(g(-1), 2) \ge -\delta$, then $r_{|\mathrm{Rm}|}(x, 0)^2 \ge 1/\lambda$. Suppose for the sake of contradiction that the claim fails. Then we may find a sequence of flows $\{(M_i, g_i(t))_{t\in[-1,0]}\}$ with $\nu(g_i(-1), 2) \ge -\delta_i$ for a sequence $\delta_i \to 0$, but $\inf\{r_{|\mathrm{Rm}|}(x, 0)^2 : x \in M_i\} < 1/\lambda$. Let $x_i \in M_i$ be chosen so that

$$r_{|\mathrm{Rm}|}(x_i, 0)^2 \le 2 \inf\{r_{|\mathrm{Rm}|}(x, 0)^2 : x \in M_i\} \tag{3-35}$$

and set $\rho_i^2 = \frac{1}{2}\lambda r_{|\mathrm{Rm}|}(x_i,0)^2 < 1$. Then the rescaled flow $\tilde{g}_i(t) = \rho_i^{-2} g_i(\rho_i^2 t)$ is defined on $[-\rho_i^{-2}, 0]$ and satisfies $\nu(\tilde{g}_i(-\rho_i^{-2}), 2\rho_i^{-2}) \ge -\delta_i$. In particular, thanks to the monotonicity of the Perelman entropy, (3-28) holds with $\tilde{g}_i$ replacing g_i. Furthermore, with respect to the rescaled metric,

$$r_{|\mathrm{Rm}|}(x_i,0)^2 = \frac{2}{\lambda} \tag{3-36}$$

and $r_{|\mathrm{Rm}|}(y,0)^2 \ge 1/\lambda$ for all $y \in M_i$. This latter fact implies, in particular, that $|\mathrm{Rm}_{\tilde{g}_i(t)}| \le \lambda$ for all $(y,t) \in M_i \times [-1,0]$. Applying Lemma 3.4, we see that the flows converge smoothly to the constant Euclidean flow. On the other hand, the regularity scale (3-36) passes to the limit, and we reach a contradiction. □

3.5 Applications of Theorem 3.2

Finally, we have two direct applications of Theorem 3.2. See also [52] for applications of pseudolocality for a localized entropy. These applications could in fact be understood without most of the structure of Theorem 1.15. The first is a finiteness theorem and is closely related to the pseudolocality finiteness in [35] and the ϵ–regularity of [33].

Theorem 3.5 (finiteness theorem) *Fix $n \ge 2$. There exists $\delta_0 = \delta_0(n) > 0$ such that for any $\delta \le \delta_0$, and for any positive C_0, τ_0, V, the space $\mathcal{M}_{\delta,C_0,\tau_0,V}$ of compact Riemannian manifolds satisfying*

$$R \ge -C_0, \quad \nu(g,\tau_0) \ge -\delta \quad \text{and} \quad \mathrm{vol}_g(M) \le V$$

contains finitely many diffeomorphism types.

Proof The proof is analogous to [35, Theorem 37.1], which was proved by Perelman [43, Remark 10.5] using Perelman's pseudolocality. In our case, we replace the use of Perelman's pseudolocality by Theorem 3.2 under assumptions on entropy and the almost-monotonicity of the volume (3-7). □

We also show that a compact Riemannian manifold with entropy and scalar curvature lower bounds admits a metric of nonnegative scalar curvature.

Theorem 3.6 *Fix $n \ge 2$. There exists $\delta_0 = \delta_0(n) > 0$ such that for any positive τ_0 and V, there exists $\epsilon = \epsilon(\delta_0, \tau_0, V)$ such that if (M,g) is a compact manifold with*

$$R \ge -\epsilon, \quad \nu(g,\tau_0) \ge -\delta_0, \quad \mathrm{vol}_g(M) \le V. \tag{3-37}$$

Then M admits a metric of nonnegative scalar curvature.

Proof Suppose by way of contradiction that the claim is false, so we may find a sequence of compact Riemannian manifolds (M_i, g_i) such that $\nu(g_i, \tau_0) \geq -\delta_0$, $\mathrm{vol}_{g_i}(M_i) \leq V$, $R_{g_i} \geq -1/i$ and M_i does not admit a metric of nonnegative scalar curvature. Up to rescaling each metric, we may assume without loss of generality that $\tau_0 = 1$. Applying Theorem 3.2, we obtain a sequence of Ricci flows $(M_i, g_i(t))_{t\in(0,1]}$ such that the sequence $(M_i, g_i(1))$ has uniformly bounded geometry, $g_i(1)$–diameter uniformly bounded above, and satisfies $R_{g_i(1)} \geq -1/i$. So, up to a subsequence, $(M_i, g_i(1))$ converges in the Cheeger–Gromov sense to a compact Riemannian manifold (M, g) with $R_g \geq 0$ and $\mathrm{vol}_g(M) \leq V$. In particular, for i sufficiently large, M_i is diffeomorphic to M. This contradicts the assumption that each M_i does not admit a metric of nonnegative scalar curvature. □

3.6 Basic Ricci flow estimates

In the final subsection of this preliminaries section, we give two further basic estimates that will be needed in the paper.

First, the following lemma is a consequence of the proof of Theorem 3.2, which shows that the $g(1)$ ball of radius 16 is smoothly close to the Euclidean ball of radius 16, provided that the entropy is chosen to be sufficiently small; see also [52, Theorem 1.2]. This observation will allow us to compare the metrics $g = g(0)$ and g_{euc} by way of comparing $g(0)$ and $g(1)$, and will be used repeatedly throughout the paper.

Lemma 3.7 *Given any fixed $\epsilon > 0$, we may choose $\delta > 0$ and $\lambda > 0$ sufficiently small that if (3-24) and (3-25) hold, then for any $x_0 \in M$ and $t \in (0,1]$, we may find a diffeomorphism $\phi\colon B_{g(t)}(x_0, 16t^{1/2}) \to \Omega \subset \mathbb{R}^n$, with inverse $\psi = \phi^{-1}$ such that $\phi(x_0) = 0$ and*

$$(1-\epsilon) g_{\mathrm{euc}} \leq \psi^* g(t) \leq (1+\epsilon) g_{\mathrm{euc}} \tag{3-38}$$

for all $x \in \Omega$. In particular,

$$(1-\epsilon)\omega_n r^n \leq \mathrm{vol}_{g(t)}(B_{g(t)}(x, r)) \leq (1+\epsilon)\omega_n r^n \tag{3-39}$$

for any $r \in (0, 16t^{1/2})$.

The second fact contained in this section is the following elementary lemma using the scale invariant estimates (3-25) to bound the evolution of the metric from one dyadic scale to the next. We will make use of this lemma in Section 5.

Lemma 3.8 *Fix $n \geq 2$ and $\beta \in \left(0, \frac{1}{4}\right)$. There exists a $\lambda = \lambda(n, \beta) > 0$ such that the following holds. Let $(M, g(t))_{t\in(0,1]}$ be a Ricci flow satisfying (3-25). For any $x \in M$ and $0 < s_1 \leq s_2 \leq 1$, we have*

$$\left(\frac{s_1}{s_2}\right)^{\beta} g(s_1) \leq g(s_2) \leq \left(\frac{s_1}{s_2}\right)^{-\beta} g(s_1). \tag{3-40}$$

Consequently, for any $r > 0$ we have

$$B_{g(s_1)}(x, r s_1^{1/2}) \subseteq B_{g(s_2)}(x, r s_2^{1/2}) \quad \text{for all } s_1 \leq s_2. \tag{3-41}$$

Furthermore, there is a universal constant λ_0 such that if $\lambda \leq \lambda_0$, then for $x \in B_{g(1)}(p, 2)$, we have

$$B_{g(t)}(x, 4t^{1/2}) \subseteq B_{g(1)}(p, 4) \quad \text{for all } t \leq \frac{1}{2^5}. \tag{3-42}$$

Proof We first show (3-40). Fix $v \in TM$ and consider the function $g(t)(v, v)$. By the scale-invariant bounds (3-25), we have for all $t > 0$ that

$$-\frac{C_n\lambda}{t} g(v, v) \leq \partial_t g(v, v) = -2\,\mathrm{Ric}(v, v) \leq \frac{C_n\lambda}{t} g(v, v) \tag{3-43}$$

for some dimensional constant C_n. This can be seen by taking normal coordinates at each point so that Ric is diagonal while g is a identity matrix. Hence, by integrating the function $\log g(v, v)$, we see that for any $0 < s \leq t \leq 1$ we have

$$\left(\frac{s}{t}\right)^{C_n\lambda} g(s) \leq g(t) \leq \left(\frac{t}{s}\right)^{C_n\lambda} g(s). \tag{3-44}$$

This showed (3-40) by choosing λ sufficiently small.

From (3-40) we directly deduce that for all $r > 0$ and $s_1 \leq s_2$, we have

$$B_{g(s_1)}(x, r s_1^{1/2}) \subseteq B_{g(s_2)}\left(x, r s_1^{1/2} \left(\frac{s_2}{s_1}\right)^{\beta}\right). \tag{3-45}$$

Then (3-41) follows from (3-45) because $\beta < \frac{1}{2}$. To see (3-42), we take $r = 4$ and $s_2 = 1$ in (3-45) to find

$$B_{g(t)}(x, 4t^{1/2}) \subseteq B_{g(1)}(x, 4t^{1/2-\beta}) \subseteq B_{g(1)}(x, 2), \tag{3-46}$$

where the second containment holds for all $t \leq 1/2^5$ because $\beta < \frac{1}{4}$. The triangle inequality then ensures that (3-42) holds. □

4 Integral estimates for Ricci and scalar curvature

This section has two main goals. The first is to prove an integral estimate for the Ricci curvature under the hypotheses of Theorem 1.7: this key estimate is used in Section 5 to prove the decomposition theorem, Theorem 5.1. The second goal is to prove the integral bounds of scalar curvature in Theorem 1.16, whose proof goes along similar lines to that of Theorem 4.1.

Theorem 4.1 (integral Ricci estimate) *Fix $n \geq 2$, $\epsilon > 0$ and $\theta \in \left[0, \frac{1}{2}\right)$. There exists a $\delta = \delta(n, \theta, \epsilon) > 0$ such that the following holds. Suppose $(M, g(t))_{t \in (0,1]}$ is a Ricci flow satisfying*

$$R_{g(0)} \geq -\delta, \tag{4-1}$$

$$\nu(g(0), 2) \geq -\delta. \tag{4-2}$$

Then for any $x \in M$ and any $t \in (0, 1]$,

$$\int_0^t \left(\frac{s}{t}\right)^{-\theta} \fint_{B_{g(t)}(x, 4t^{1/2})} |\mathrm{Ric}_{g(s)}| \, d\mathrm{vol}_{g(s)} \, ds \leq \epsilon^2. \tag{4-3}$$

This section is organized in the following way. In Section 4.1, we prove an almost-Gaussian lower bound for the conjugate heat kernel and for a cutoff function evolving by the conjugate heat equation. A major tool is a heat kernel estimate due to Zhang; see Proposition 4.5 below. In Sections 4.2–4.3, we establish Theorems 4.1 and 1.16, respectively, by integrating the evolution equation for the scalar curvature (3-4) against suitably chosen functions to which we apply the estimates of Section 4.1.

By Theorem 3.2, the hypotheses of Theorem 4.1 imply that the scale-invariant curvature bounds

$$|\mathrm{Rm}_{g(t)}| \leq \frac{\lambda}{t} \tag{4-4}$$

hold for all $x \in M$ and $t \in (0, 1]$, with λ as small as desired by choosing δ sufficiently small.

4.1 Heat kernel lower bounds and evolving cutoff function

In Proposition 4.2 below, we establish lower bounds on the heat kernel, and in Proposition 4.4 we prove lower bounds for a cutoff function that evolves by the conjugate heat equation. Both lower bounds become degenerate for small times. Nonetheless, the degeneration occurs in a sufficiently controlled way for our application in the proofs of Theorems 4.1 and 1.16.

Proposition 4.2 *Fix $n \geq 2$ and $\lambda > 0$. There exist $\delta = \delta(n, \lambda) > 0$ and $C = C(n) > 0$ such that the following holds. Suppose that $(M, g(t))_{t\in[0,1]}$ is a complete Ricci flow with bounded curvature satisfying (4-1), (4-2) and (4-4). Then for any $0 < s < t < 1$, letting $\tau = t - s$, we have*

$$K(x,t;y,s) \geq \left(\frac{s}{t}\right)^{\lambda} C\tau^{-n/2} \exp\left\{\frac{-4d_{g(t)}(x,y)^2}{\tau}\right\}. \tag{4-5}$$

Remark 4.3 The hypothesis (4-1) is not actually necessary in Proposition 4.2, but for convenience we assume it, to allow for a simpler proof; in particular, we can directly call upon Zhang's heat kernel lower bound Proposition 4.5 below.

Before proving Proposition 4.2, let us state its main consequence. Fix $t \in (0, 1]$ and $r^2 \geq t$. Consider a smooth function $\varphi\colon M \times \{t\} \to \mathbb{R}$ such that

$$\varphi(y) = \begin{cases} 1 & \text{in } B_{g(t)}(x_0, 8r), \\ 0 & \text{in } M \setminus B_{g(t)}(x_0, 16r). \end{cases} \tag{4-6}$$

For $s \in [0, t)$, let $\varphi(y, s)$ be the solution of the conjugate heat equation with terminal data given by $\varphi(y, t)$:

$$\begin{cases} (\partial_s + \Delta - R)\varphi(y,s) = 0 & \text{in } M \times [0,t), \\ \varphi(y,t) = \varphi(y). \end{cases} \tag{4-7}$$

Proposition 4.4 below shows that $\varphi(y, s)$ behaves sufficiently like a cutoff function for all $s \in (0, 1]$ that we can derive useful estimates.

Proposition 4.4 *Fix $n \geq 2$. There exist $\delta = \delta(n) > 0$ and $C = C(n) > 0$ such that the following holds. Suppose that $(M, g(t))_{t\in[0,1]}$ is a Ricci flow satisfying (4-1), (4-2) and (4-4) for $\lambda \leq 1$. Then we have*

$$\varphi(y,s) \geq C\left(\frac{s}{t}\right)^{\lambda} \tag{4-8}$$

for all $(y, s) \in B_{g(t)}(x_0, 4r) \times (0, t)$.

We now proceed to the proof of Proposition 4.2, which follows from the following lower heat kernel bound, due to Zhang in [53].

Proposition 4.5 (Zhang) *Fix $n \geq 3$ and let $(M, g(t))_{t\in[0,1]}$ be a Ricci flow satisfying (4-1) and (4-2). Then for any $0 \leq s < t \leq 1$, we have*

$$K(x,t;y,s) \geq \frac{c}{\tau^{n/2}} \exp\left\{\frac{-4}{\tau} d_{g(t)}(x,y)^2 - \frac{1}{\sqrt{\tau}} \int_0^{\tau} \sqrt{s'} R(y, t-s')\, ds' - 2\delta\right\}. \tag{4-9}$$

Here $c = c(n)$ and we let $\tau = t - s$.

Proof of Proposition 4.2 Provided we choose $\delta \le 1$, the estimate (4-9) implies that

$$K(x,t;y,s) \ge c\tau^{-n/2}\exp\left\{\frac{-4d_{g(t)}(x,y)^2}{\tau}\right\} \times F(y,s), \tag{4-10}$$

where c is a dimensional constant and

$$F(y,s) = \exp\left\{-\frac{1}{\sqrt{\tau}}\int_0^\tau \sqrt{s'}\,R(y,t-s')\,ds'\right\}. \tag{4-11}$$

We claim that

$$\log F(s,y) \ge \log\left(\frac{s}{t}\right)^\lambda. \tag{4-12}$$

Exponentiating (4-12) will conclude the proof of the proposition. To this end, we change variables and then bound the scalar curvature using the scale-invariant curvature bound (4-4), finding that

$$\begin{aligned} -\log F(s,y) &= \frac{1}{\tau^{1/2}}\int_0^\tau \sqrt{s'}\,R(y,t-s')\,ds' \\ &\le \frac{\lambda}{\tau^{1/2}}\int_0^\tau \sqrt{s'}(t-s')^{-1}\,ds' \\ &\le \lambda\int_0^\tau (t-s')^{-1}\,ds' = \lambda\int_s^t \rho^{-1}\,d\rho = \lambda\log\frac{t}{s}. \end{aligned} \tag{4-13}$$

Negating this expression establishes (4-12) and thus concludes the proof. □

We now prove Proposition 4.4.

Proof of Proposition 4.4 Expressing the solution with respect to the conjugate heat kernel, we have

$$\varphi(y,s) = \int_M \varphi(x)K(x,t;y,s)\,d\mathrm{vol}_{g(t)}(x). \tag{4-14}$$

Fix any $y \in B_{g(t)}(p,4r)$. Having chosen $r^2 \ge t$, we note that $B_{g(t)}(y,t^{1/2}) \subset B_{g(t)}(p,8r)$, and in particular $\varphi(x,t) = 1$ in this set. Using this observation, followed by Proposition 4.2, we find that

$$\begin{aligned} \left(\frac{s}{t}\right)^{-\lambda}\varphi(y,s) &\ge \left(\frac{s}{t}\right)^{-\lambda}\int_{B_{g(t)}(y,t^{1/2})} K(x,t;y,s)\,d\mathrm{vol}_{g(t)}(x) \\ &\ge \int_{B_{g(t)}(y,t^{1/2})} \frac{C}{\tau^{n/2}}\exp\left\{\frac{-4d_{g(t)}(x,y)^2}{\tau}\right\}d\mathrm{vol}_{g(t)}(x). \end{aligned} \tag{4-15}$$

Since $\tau = t - s \leq t$, we see from Lemma 3.7 that the right-hand side is bounded below by a universal constant, namely, by

$$\int_{B(0,\tau^{1/2})} \tau^{-n/2} \exp\left\{\frac{-4|x|^2}{\tau}\right\} dx = \int_{B(0,1)} \exp\{-4|x|^2\}\, dx, \tag{4-16}$$

so long as $\lambda \leq \lambda_0$. This completes the proof. □

4.2 Proof of Theorem 4.1

Before proving Theorem 4.1, let us make the following observation.

Lemma 4.6 *Fix $n \geq 2$, $\delta > 0$ and $\lambda > 0$. Let $(M, g(t))_{t\in[0,1]}$ be a Ricci flow satisfying (4-1) and (4-4). For any $t \in (0,1]$, let $\varphi\colon M \times \{t\} \to \mathbb{R}$ be a nonnegative smooth function, and if M is noncompact then assume φ has compact support. Let $\varphi(y,s)$ be the evolution of φ by the conjugate heat equation for $s \in (0,t)$. Then*

$$2\int_0^t \int_M |\mathrm{Ric}_{g(s)}(y)|^2 \varphi(y,s)\, d\mathrm{vol}_{g(s)}(y)\, ds \leq \left(\frac{\lambda}{t} + \delta\right) \int_M \varphi(y,t)\, d\mathrm{vol}_{g(t)}(y). \tag{4-17}$$

Proof We multiply $\varphi(y,s)$ by the evolution equation for the scalar curvature (3-4) and integrate in space and time to obtain the following. After an integration by parts, we find that

$$\begin{aligned} 2\int_0^t \int_M &|\mathrm{Ric}_{g(s)}(y)|^2 \varphi(y,s)\, d\mathrm{vol}_{g(s)}(y)\, ds \\ &= \int_0^t \int_M (\partial_s - \Delta) R_{g(s)}(y) \varphi(y,s)\, d\mathrm{vol}_{g(s)}(y)\, ds \\ &= \int_0^t \int_M R_{g(s)}(y)\, (\partial_s + \Delta - R_{g(s)}) \varphi(y,s)\, d\mathrm{vol}_{g(s)}(y)\, ds \\ &\qquad + \int_M R_{g(t)} \varphi(y,t)\, d\mathrm{vol}_{g(t)}(y) - \int_M R_{g(0)} \varphi(y,0)\, \mathrm{vol}_{g(0)}(y) \\ &= \int_M R_{g(t)} \varphi(y,t)\, d\mathrm{vol}_{g(t)}(y) - \int_M R_{g(0)} \varphi(y,0)\, \mathrm{vol}_{g(0)}(y). \end{aligned} \tag{4-18}$$

This integration by parts is justified because, for each fixed time-slice, φ and $|\nabla \varphi|$ decay exponentially with respect to $d_{g(t)}(x,\cdot)$; see [17, Chapter 26.1]. We wish to bound the right-hand side of the last equation on (4-18) from above. By the maximum principle the function $\varphi(y,s)$ is nonnegative for all y, s. Hence, making use first of the lower bound on scalar curvature (4-1) and then of the conservation of the L^1–norm under the conjugate heat equation (3-11), we have

$$-\int_M R\varphi \mathrm{vol}_{g(0)} \le \delta \int_M \varphi \mathrm{vol}_{g(0)} = \delta \int_M \varphi \mathrm{vol}_{g(t)}. \tag{4-19}$$

Pairing this with (4-18) and applying the scale-invariant curvature estimates (4-4) to bound the scalar curvature in the t–time slice, we find that

$$\begin{aligned} 2\int_0^t \int_M |\mathrm{Ric}_{g(s)}(y)|^2 \varphi(y,s)\, d\mathrm{vol}_{g(s)}(y)\, ds &\le \int_M (R_{g(t)}(y)+\delta)\varphi(y,t)\, d\mathrm{vol}_{g(t)}(y) \\ &\le \Big(\frac{\lambda}{t}+\delta\Big)\int_M \varphi(y,t)\, d\mathrm{vol}_{g(t)}(y). \end{aligned} \tag{4-20}$$

This conclude the proof of the lemma. □

Finally, we prove Theorem 4.1.

Proof of Theorem 4.1 Up to rescaling the flow, we may assume that $t=1$. Together Lemma 4.6 and Proposition 4.4 (with $r=t^{1/2}=1$) imply that

$$\begin{aligned} \int_0^1 s^\lambda \int_{B_{g(1)}(x,4)} |\mathrm{Ric}_{g(s)}|^2\, d\mathrm{vol}_{g(s)}\, ds &\le C(\lambda+\delta)\,\mathrm{vol}_{g(1)}(B_{g(1)}(x,16)) \\ &\le C(\lambda+\delta), \end{aligned} \tag{4-21}$$

where the second inequality comes from (3-39). Further, by (3-7) and (3-39), we have

$$\inf_{0<s<1} \mathrm{vol}_{g(s)}(B_{g(1)}(x,4r)) \ge (1-2\delta)\,\mathrm{vol}_{g(1)}(B_{g(1)}(x,4r)) \ge c. \tag{4-22}$$

Hence,

$$\int_0^1 s^\lambda \fint_{B_{g(1)}(x,4)} |\mathrm{Ric}_{g(s)}|^2\, d\mathrm{vol}_{g(s)}\, ds \le C(\lambda+\delta). \tag{4-23}$$

Now, fix $\lambda>0$ sufficiently small that $\lambda \le \frac{1}{2}-\theta$. In this way, if we set $\theta_0 := \theta + \frac{1}{2}\lambda$, we ensure that

$$1-2\theta_0 \ge \tfrac{1}{2}-\theta. \tag{4-24}$$

Choose δ sufficiently small that (4-4) holds for this choice of λ. Then by Hölder's inequality, for any $\Omega \subseteq M$ we have

$$\begin{aligned}(4\text{-}25)\quad \int_0^1 s^{-\theta} \fint_\Omega |\mathrm{Ric}_{g(s)}| \, d\mathrm{vol}_{g(s)} \, ds &\leq \left(\int_0^1 s^{-2\theta_0} \, ds\right)^{1/2} \left(\int_0^1 s^\lambda \fint_\Omega |\mathrm{Ric}|^2 \, d\mathrm{vol}_{g(s)} \, ds\right)^{1/2} \\ &= (1-2\theta_0)^{-1/2} \left(\int_0^1 s^\lambda \fint_\Omega |\mathrm{Ric}|^2 \, d\mathrm{vol}_{g(s)} \, ds\right)^{1/2}.\end{aligned}$$

The constant $(1-2\theta_0)^{-1/2}$ is bounded above by $\left(\frac{1}{2}-\theta\right)^{-1/2}$ thanks to (4-24). By choosing λ and δ such that $C(\lambda+\delta)^{1/2} \leq \epsilon^2$, together with (4-25) and (4-23) this concludes the proof. □

4.3 Integral bounds for the scalar curvature

We now prove Theorem 1.16, which we restate below as Theorem 4.7 below. The proof is similar to that of Theorem 4.1.

Theorem 4.7 (L^q scalar curvature estimates) *Fix $n \geq 2$, $q \in (0,1)$ and $\epsilon > 0$. There exists a $\delta = \delta(n,q,\epsilon) > 0$ such that the following holds. Let (M,g) be a closed Riemannian n–manifold such that*

$$(4\text{-}26)\qquad R \geq -\delta \quad \text{and} \quad \nu(g,2) \geq -\delta.$$

Then we have

$$(4\text{-}27)\qquad \fint_M |R|^q \, d\mathrm{vol}_g \leq \epsilon.$$

Proof of Theorem 1.16 Let R_+ and R_- denote the positive and negative parts of R, respectively. Since $\fint R_-^q \, d\mathrm{vol}_g \leq \delta^q$, we choose $\delta^q \leq \frac{1}{2}\epsilon$ and it suffices to show that $\fint R_+^q \, d\mathrm{vol}_g \leq \frac{1}{2}\epsilon$. By Theorem 3.2, for any fixed $\lambda > 0$, we may choose δ small enough that the Ricci flow $(M, g(t))$ with $g(0) = 0$ exists for $t \in (0,1]$ and enjoys the scale-invariant curvature bounds (4-4) for all $x \in M$ and $t \in (0,1]$. Consider the nonnegative function

$$(4\text{-}28)\qquad f(x,t) = R_{g(t)}(x) + 2\delta.$$

Note that $f \geq R_+$, so it suffices to show that $⨍_M f^q \, d\mathrm{vol}_{g(0)} \leq \frac{1}{2}\epsilon$. For any $q \in (0,1]$, we see that f^q is a supersolution of the heat equation coupled to Ricci flow. Indeed, noting that $q(q-1) < 1$ and recalling (3-4), we compute that

$$(\partial_t - \Delta) f^q = q f^{q-1}(\partial_t - \Delta) f - q(q-1) f^{q-2} |\nabla f|^2 \geq q f^{q-1}(\partial_t - \Delta) R \geq 0. \tag{4-29}$$

So, applying (3-8) with $u = f^q$ and $v = 1$, we find

$$\begin{aligned} \int_M f^q \, d\mathrm{vol}_{g(0)} &= \int_M f^q \, d\mathrm{vol}_{g(1)} - \int_0^1 \int_M \{(\partial_t - \Delta) f^q - R f^q\} \, d\mathrm{vol}_{g(t)} \, dt \\ &\leq \int_M f^q \, d\mathrm{vol}_{g(1)} + \int_0^1 \int_M R f^q \, d\mathrm{vol}_{g(t)} \, dt. \end{aligned} \tag{4-30}$$

We bound each of the terms on the right-hand side of (4-30) separately. For the first term, using the scale-invariant curvature bounds (4-4) and (3-7), we see that

$$\int_M f^q \, d\mathrm{vol}_{g(1)} \leq (\lambda + 2\delta)^q \mathrm{vol}_{g(1)}(M) \leq 2(\lambda + 2\delta)^q \mathrm{vol}_{g(0)}(M). \tag{4-31}$$

As for the second term on the right-hand side of (4-30), we note that

$$R f^q \leq f^{q+1} \leq 2^{q+1}(|R|^{q+1} + (2\delta)^{q+1}).$$

So, again making use of (3-7), we find

$$\int_0^1 \int_M R f^q \, d\mathrm{vol}_{g(t)} \, dt \leq C \delta^q \mathrm{vol}_{g(0)}(M) + C \int_0^1 \int_M |R|^{q+1} \, d\mathrm{vol}_{g(t)} \, dt, \tag{4-32}$$

where C is a constant depending on q. We bound the second term on the right-hand side of (4-32) using the same argument as in the proof of Theorem 4.1. More specifically, let $\varphi \colon M \times (0,1) \to \mathbb{R}$ be the solution to the conjugate heat equation with terminal data $\varphi(x,1) = 1$ on $M \times \{1\}$. By (the proof of) Proposition 4.4, we see that $\varphi(y,s) \geq c s^\lambda$ for all $y \in M$ and $s \in (0,1]$, where $c = c(n)$. Thus, applying Lemma 4.6 to this choice of φ, we find that

$$\begin{aligned} \int_0^1 t^\lambda \int_M |R|^2 \, d\mathrm{vol}_{g(t)} \, dt &\leq C \int_0^1 \int_M |R|^2 \varphi(y,t) \, d\mathrm{vol}_{g(t)} \, dt \\ &\leq C(\lambda + \delta) \, \mathrm{vol}_{g(1)}(M) \\ &\leq C(\lambda + \delta) \, \mathrm{vol}_{g(0)}(M). \end{aligned} \tag{4-33}$$

We choose δ sufficiently small that $\lambda < (1-q)/(1+q)$, and set $\theta = \lambda(1+q)/(1-q) < 1$. Hölder's inequality with $p = 2/(1+q)$ and $p' = 2/(1-q)$, together with (4-33), allows us to deduce that

$$\begin{aligned} \text{(4-34)} \quad & \int_0^1 \int_M |R|^{q+1} \, d\mathrm{vol}_{g(t)} \, dt \\ & \le \left(2 \, \mathrm{vol}_{g(0)}(M) \int_0^1 s^{-\theta} \, ds \right)^{(1-q)/2} \left(\int_0^1 s^{\lambda} \int_M |R|^2 \, d\mathrm{vol}_{g(s)} \, ds \right)^{(1+q)/2} \\ & \le C(q)(\lambda + \delta)^{(1+q)/2} \mathrm{vol}_{g(0)}(M). \end{aligned}$$

Pairing this estimate with (4-30), (4-31) and (4-32), we find that

$$\text{(4-35)} \qquad \fint_M f^q \, d\mathrm{vol}_{g(0)} \le C(\delta + \lambda)^q + C\delta^q + C(\lambda + \delta)^{(1+q)/2},$$

where C depends on q and n. Choose δ sufficiently small that the right-hand side of (4-35) is bounded above by $\frac{1}{2}\epsilon$. Recalling that $f \ge R_+$, this concludes the proof. □

5 Decomposition theorem

The main goal of this section is to establish the decomposition theorem, Theorem 5.1 below. The end purpose of this decomposition is to allow us to gain $W^{1,p}$–control on our initial manifold for large but finite $p < \infty$. Thus, Theorem 5.1 will be an essential tool used to prove Theorem 1.11. The integral estimate for Ricci curvature established in Theorem 4.1 is the key estimate in the proof.

Before stating the decomposition theorem precisely, let us give an informal description of its contents. Given a complete Ricci flow satisfying $-\delta$ lower bounds on the scalar curvature and the entropy, each ball $B_{g(1)}(x_0, 2)$ can be decomposed as a countable union of "good sets" $\mathcal{G}^k$ and a "bad set" $\mathcal{A}$. The bad set has measure zero and on the k^{th} good set, the metrics $g(0)$ and $g(1)$ are equivalent up to an error of size $(1+\epsilon)^k$. Furthermore, the volumes of the $\mathcal{G}^k$ decay geometrically, and the complement of the first k good sets satisfies a geometrically decaying content bound.

In fact, if we restrict the time to compare $g(0)$ and $g(t)$ for t small, then we can obtain the same kind of decomposition with smaller error.

Theorem 5.1 (decomposition theorem) *For each $\epsilon > 0$ there exists a $\delta = \delta(n, \epsilon) > 0$ such that the following holds. Let $(M, g(t))_{t \in (0,1]}$ be a complete Ricci flow with bounded curvature satisfying*

$$\text{(5-1)} \qquad R_{g(0)} \ge -\delta,$$

$$\text{(5-2)} \qquad \nu(g(0), 2) \ge -\delta.$$

Fix $x_0 \in M$ and $\eta \le \epsilon$. There exists a $\hat{t} = \hat{t}(n,\eta) \in (0,1]$ such that every ball $B_{g(1)}(x_0,2)$ can be decomposed into good sets $\mathcal{G}^k$ and a bad set $\mathcal{A}$ as

$$B_{g(1)}(x_0,2) = \bigcup_{k=1}^{\infty} \mathcal{G}^k \cup \mathcal{A}, \tag{5-3}$$

where the sets have the following properties:

(1) $\operatorname{vol}_{g(0)}(\mathcal{A}) = 0$.

(2) *For all $x \in \mathcal{G}^k$ and for all $s,t \in (0,\hat{t}]$, the metrics satisfy*

$$(1-\eta)(1-\epsilon)^{k-1} g(s) \le g(t) \le (1-\eta)(1+\epsilon)^{k-1} g(s). \tag{5-4}$$

(3) *For each $k \ge 2$, we have* $\operatorname{vol}_{g(0)}(\mathcal{G}^k) \le (1+\epsilon)^k \eta \epsilon^{k-2}$.

(4) *For each $k \in \mathbb{N}$, let $\mathcal{A}^k = B_{g(1)}(x_0,2) \setminus \bigcup_{\ell=1}^k \mathcal{G}^\ell$ be the complement of the first k good sets. There is a countable collection $\mathcal{C}^k$ and a mapping $y \mapsto t_y$ for $y \in \mathcal{C}^k$ such that*

$$\mathcal{A}^k \subseteq \bigcup_{y \in \mathcal{C}^k} B_{g(t_y)}(y, 12 t_y^{1/2}), \tag{5-5}$$

with $\sum_{y \in \mathcal{C}^k} t_y^{n/2} \le \eta \epsilon^{k-1}$.

When $\eta = \epsilon$, then we may take $\hat{t} = 1$.

Remark 5.2 The effect of $\eta > 0$ in the above theorem is that, for small t, one can force the bad sets comparing $g(0)$ to $g(t)$ to have decreasingly small volume. In this way one gets that $g(t)$ is converging to $g(0)$ in various norms.

Throughout this section we use the notation

$$\underline{B}_t(x) := B_{g(t)}(x, 4t^{1/2}) \tag{5-6}$$

to denote the scale-invariant balls of radius 4 and we let

$$\underline{B} := \underline{B}_1(x_0). \tag{5-7}$$

In this notation, Theorem 4.1 states that for any $\theta \in \left(0, \tfrac{1}{2}\right)$ and $\epsilon > 0$, we may find $\delta = \delta(n,\theta,\epsilon) > 0$ such that, under the hypotheses of Theorem 5.1, we have

$$\int_0^t \left(\frac{s}{t}\right)^{-\theta} \fint_{\underline{B}_t(x)} |\mathrm{Ric}_{g(s)}| \, d\mathrm{vol}_{g(s)} \, ds \le \epsilon^2. \tag{5-8}$$

5.1 Preliminary results

As in the previous sections we have by Theorem 3.2 that for any $\lambda > 0$, we may choose δ sufficiently small in Theorem 5.1 that

$$|\mathrm{Rm}_{g(t)}| \le \frac{\lambda}{t} \tag{5-9}$$

holds for all $x \in M$ and $t \in (0,1]$. The norm of the Ricci curvature $|\mathrm{Ric}_{g(t)}|$ evolves along the Ricci flow by

$$(\partial_t - \Delta)|\mathrm{Ric}_{g(t)}| \le c_n |\mathrm{Rm}_{g(t)}|\,|\mathrm{Ric}_{g(t)}|; \tag{5-10}$$

see [18, Lemma 6.38]. For a Ricci flow satisfying (5-9), the evolution (5-10) becomes

$$\left(\partial_t - \Delta - \frac{c_n\lambda}{t}\right)|\mathrm{Ric}_{g(t)}| \le 0. \tag{5-11}$$

That is to say, when t is uniformly bounded away from zero, the norm of the Ricci curvature evolves as a subsolution of a heat-type equation with smooth bounded potential. Note that (5-9) provides uniform lower bounds for the Ricci tensor when t is bounded away from zero, a necessary ingredient for establishing parabolic regularity estimates. So, after rescaling the metric, a standard Moser iteration argument (along with a trick of Li and Schoen [36] to pass from the L^2–norm to the L^1–norm) leads to the following pointwise estimates for the norm of the Ricci curvature; see [17, Theorem 25.2] for a proof.

Proposition 5.3 *Fix $n \ge 2$. There exist constants $C = C(n)$ and $\lambda_0(n)$ such that if $(M, g(t))_{t\in(0,1]}$ is a Ricci flow satisfying (5-9) with $\lambda \le \lambda_0(n)$, then for any $t \in (0,1]$ we have*

$$|\mathrm{Ric}_{g(s)}(y)| \le C \fint_{t/4}^{t} \fint_{\underline{B}_t(x)} |\mathrm{Ric}_{g(s)}|\, d\mathrm{vol}_s(y)\, ds \tag{5-12}$$

for all $(y,s) \in B_{g(t)}(x, 3t^{1/2}) \times \left(\frac{1}{2}t, t\right)$.

In the proof of Theorem 5.1, we will need the following Vitali-type lemma. The difference from a usual Vitali cover is that the balls are not taken with respect to a fixed metric, but rather the covering comprises geodesic balls with respect to different time slices $g(t)$ along a Ricci flow. At various points in the proof, we will call upon the elementary containments of balls established in Lemma 3.8.

Lemma 5.4 (Vitali-type lemma) *Given $n \geq 2$, there exists a $\lambda_0(n)$ such that the following holds. Let $(M, g(t))_{t\in(0,1]}$ be a Ricci flow satisfying (5-9) with $\lambda \leq \lambda_0(n)$. For any $x_0 \in M$ and $t_0 \in (0,1]$, consider a set $\mathcal{A} \subseteq B_{g(t_0)}(x_0, 2t_0^{1/2})$ and a mapping $y \mapsto t_y \in \left(0, \frac{1}{200}t_0\right]$ defined for all $y \in \mathcal{A}$. There exists a countable collection $\mathcal{C} \subseteq \mathcal{A}$ such that*

(1) *the balls $\underline{B}_{t_y}(y)$ are pairwise disjoint for all $y \in \mathcal{C}$,*

(2) *the collection $\{B_{g(36t_y)}(x, 12t_y^{1/2})\}_{y\in\mathcal{C}}$ is a covering of $\mathcal{A}$,*

(3) *for each $y \in \mathcal{C}$, $\underline{B}_{36t_y}(y) \subseteq \underline{B}_{t_0}(x_0)$ and $\underline{B}_{t_y}(y) \subseteq \underline{B}_{t_0}(x_0)$.*

Definition 5.5 We call a pair $(\mathcal{C}, y \mapsto t_y)$ satisfying (1)–(3) a *covering pair* of $\mathcal{A}$ in $\underline{B}_{t_0}(x_0)$.

Proof of Lemma 5.4 Up to rescaling the flow, we may assume that $t_0 = 1$. The inductive construction of the cover is similar to a standard Vitali covering argument. For each $k \in \mathbb{N}$, let

$$F_k = \{\underline{B}_{t_y}(y) : y \in \mathcal{A}, t_y \in (2^{-k-1}, 2^{-k}]\}. \tag{5-13}$$

Set $H_0 = F_0$ and let G_0 be a maximal disjoint subcollection of H_0. Now, suppose we have defined $G_0, \dots G_{k-1}$. Then let

$$H_k = \{B \in F_k : B \cap B' = \varnothing \text{ for all } B' \in G_0 \cup \dots \cup G_{k-1}\}, \tag{5-14}$$

and take G_k to be a maximal disjoint subcollection of H_k. Note that G_k contains finitely many balls. We define the countable set $\mathcal{C} \subseteq \mathcal{A}$ by

$$\mathcal{C} = \bigcup_{k=1}^{\infty} \{y \in \mathcal{A} : \underline{B}_{t_y}(y) \in G_k\}. \tag{5-15}$$

Let us verify that the three properties claimed in the lemma are valid. Lemma 5.4(1) holds by construction, and both parts of Lemma 5.4(3) follow directly from (3-42) in Lemma 3.8 thanks to the assumption that $t_y \leq \frac{1}{200}$.

To establish Lemma 5.4(2), fix any $x \in \mathcal{A}$ and choose $k \in \mathbb{N}$ so that $t_x \in (2^{-k-1}, 2^{-k}]$. Then either $\underline{B}_{t_x}(x) \in H_k$ or not. In the first case, since G_k is a maximal set, we know that $\underline{B}_{t_x}(x)$ intersects some $\underline{B}_{t_y}(y) \in G_k$ (where possibly $x = y$). In this case $\frac{1}{2}t_y \leq t_x \leq 2t_y$. So, if we take $z \in \underline{B}_{t_x}(x) \cap \underline{B}_{t_y}(y)$, the triangle inequality and (3-40) imply that

$$d_{g(t_y)}(x,y) \leq d_{g(t_y)}(y,z) + d_{g(t_y)}(x,z) \leq 4t_y^{1/2} + 4^{\lambda+1}t_x^{1/2} \leq 10t_y^{1/2}. \tag{5-16}$$

The final inequality holds provided we have taken λ sufficiently small. In the second case, when $\underline{B}_{t_x}(x) \notin H_k$, we see that $\underline{B}_{t_x}(x)$ must intersect some $\underline{B}_{t_y}(y) \in H_\ell$ with $\ell \in \{1, \dots, k-1\}$. Then, since $t_x \le t_y$, we find by (3-41) that $\underline{B}_{t_x}(x) \subseteq B_{g(t_y)}(x, 5t_y^{1/2})$. In particular, $B_{g(t_y)}(x, 5t_y^{1/2})$ and $\underline{B}_{t_y}(y)$ intersect nontrivially, and thus by the triangle inequality $x \in B_{g(t_y)}(y, 10t_y^{1/2})$.

So, in both the first and second cases, we have $x \in B_{g(t_y)}(y, 10t_y^{1/2})$ for some $y \in \mathcal{C}$. In order to complete the proof of (2), we apply (3-40) to find that

$$B_{g(t_y)}(y, 10t_y^{1/2}) \subseteq B_{g(36t_y)}(y, 2(36t_y)^{1/2}), \tag{5-17}$$

where the final containment holds provided we choose λ small enough. This completes the proof of the lemma. □

5.2 Good and bad sets on an arbitrary ball

Throughout this section, we fix $\epsilon \in (0,1)$ and $\theta \in \left(0, \frac{1}{2}\right)$ and assume that $(M, g(t))_{t\in[0,1]}$ is a Ricci flow satisfying (5-1) and (5-2) with δ chosen according to Theorem 4.1.

For a ball $\underline{B}$, we define the *stopping time* $t(x)$ for each $x \in B_{g(1)}(x_0, 2)$ by

$$t(x) = \inf\left\{t' \le \tfrac{1}{200} : \fint_{t/4}^{t} \fint_{\underline{B}_t(x)} |\mathrm{Ric}_{g(s)}| \, d\mathrm{vol}_{g(s)} \, ds \le t^{\theta/2-1}\epsilon \text{ for all } t \in \left[t', \tfrac{1}{200}\right]\right\}. \tag{5-18}$$

Observe that, provided we take $\epsilon < 200^{-2}$, applying (5-8) with $t = 200^{-1}$ ensures that

$$\fint_{1/800}^{1/200} \fint_{\underline{B}_{1/200}(x)} |\mathrm{Ric}_{g(t)}| \, d\mathrm{vol}_{g(t)} \, dt \le 200^{\theta/2-1}\epsilon, \tag{5-19}$$

so the stopping condition holds at $t' = \frac{1}{200}$.

For a ball $\underline{B}_{t_0}(x_0)$ with $t_0 < 1$, we define the stopping time $t(x)$ by

$$t(x) = t_0 \, \tilde{t}(x) \tag{5-20}$$

for each $x \in B_{g(t_0)}(x_0, 2t_0^{1/2})$, where $\tilde{t}(x)$ is the stopping time defined in (5-18) applied to the rescaled flow $\tilde{g}(t) = t_0^{-1} g(t_0 t)$.

The good and bad sets on $\underline{B}_{t_0}(x_0)$, respectively, are defined by

$$\begin{aligned} \mathcal{G}(\underline{B}_{t_0}(x_0)) &= \{x \in B_{g(t_0)}(x_0, 2t_0^{1/2}) : t(x) = 0\}, \\ \mathcal{A}(\underline{B}_{t_0}(x_0)) &= \{x \in B_{g(t_0)}(x_0, 2t_0^{1/2}) : t(x) > 0\}. \end{aligned} \tag{5-21}$$

In the following proposition, we establish estimates on the good and bad sets that will be iteratively applied to establish Theorem 5.1. When convenient, we adopt the shorthand $t_x = t(x)$.

Proposition 5.6 *Fix $n \geq 2$, $\epsilon \in (0,1)$ and $\theta \in \left(0, \frac{1}{2}\right)$. There exists a $\delta = \delta(n,\epsilon,\theta) > 0$ such that the following holds. Let $(M, g(t))_{t\in[0,1]}$ be a Ricci flow satisfying (5-1) and (5-2). Fix $\underline{B}_{t_0}(x_0)$.*

(1) *For any $x \in \mathcal{G}(\underline{B}_{t_0}(x_0))$ and for any $s, s' \in [0, t_0]$, the metrics $g(s)$ and $g(s')$ at x satisfy*

$$(1-\epsilon)g(s) \leq g(s') \leq (1+\epsilon)g(s). \tag{5-22}$$

(2) *Suppose $x \in \mathcal{A}(\underline{B}_{t_0}(x_0))$ and fix $s_0 \in [t_x, t_0]$. Then for all $s, s' \in [s_0, t_0]$ and $y \in B_{g(s_0)}(x, 2s_0^{1/2})$, the metrics $g(s)$ and $g(s')$ at y satisfy*

$$(1-\epsilon)g(s) \leq g(s') \leq (1+\epsilon)g(s). \tag{5-23}$$

(3) *There is a countable collection $\mathcal{C} = \mathcal{C}(\underline{B}_{t_0}(x_0)) \subseteq \mathcal{A}(\underline{B}_{t_0}(x_0))$ such that $(\mathcal{C}, t_y)$ is a covering pair for $\mathcal{A}(\underline{B}_{t_0}(x_0))$ in the sense of Definition 5.5, where $t_y = t(y)$ is the stopping time, and*

$$\sum_{y\in\mathcal{C}} t_y^{(n-\theta)/2} \leq \epsilon\, t_0^{(n-\theta)/2}. \tag{5-24}$$

Proof Up to rescaling the flow, we may assume without loss of generality that $t_0 = 1$. We choose δ sufficiently small that (5-8) holds.

Observe that (1) follows immediately from (2), since the estimate (5-22) is a particular case of (5-23) with $s_0 = 0$. Let us prove (2). Thanks to (3-40), it suffices to establish (5-23) for $s, s' \in \left[s_0, \frac{1}{200}\right]$. Together the pointwise estimates of Proposition 5.3 and the definition of the stopping time imply that

$$|\mathrm{Ric}_{g(t)}| \leq C\epsilon t^{\theta/2-1} \quad \text{on } B_{g(t)}(x, 3t^{1/2}) \times \{t\} \tag{5-25}$$

for all $t \in \left[s_0, \frac{1}{200}\right]$. So, calling upon (3-41), we find that

$$|\mathrm{Ric}_{g(t)}| \leq C\epsilon t^{\theta/2-1} \quad \text{on } B_{g(s_0)}(x, 2s_0^{1/2}) \times \left[s_0, \tfrac{1}{200}\right]. \tag{5-26}$$

Now, for any $y \in B_{g(s_0)}(x, s_0^{1/2})$, we integrate (5-26) from s to s' precisely as in the proof of Lemma 3.8 to find

$$(1-C\epsilon)g(s) \leq g(s') \leq (1+C\epsilon)g(s'), \tag{5-27}$$

where $C = C(n)$. Up to further decreasing δ so that (5-27) holds with $\epsilon' = \epsilon/C$ in place of ϵ, this establishes (5-23) and hence (1) and (2).

We now prove (3). We apply the Vitali-type Lemma 5.4, taking $\mathcal{A} = \mathcal{A}(\underline{B})$ with the mapping $y \mapsto t_y$ given by the stopping time $t_y = t(y)$. Lemma 5.4 ensures the existence of a covering pair $(\mathcal{C}, t_y)$ for $\mathcal{A}(\underline{B})$ in $\underline{B}$ with $\mathcal{C} \subseteq \mathcal{A}(\underline{B})$. We prove the content bound (5-24) in the following way. For any $y \in \mathcal{C}$, the definition of the stopping time t_y guarantees that

$$\int_{t_y/4}^{t_y} \fint_{\underline{B}_{t_y}(y)} |\mathrm{Ric}_{g(s)}| \, d\mathrm{vol}_{g(s)} \, ds = \epsilon t_y^{\theta/2}. \tag{5-28}$$

By rearranging terms in (5-28) and calling upon the volume lower bound in (3-39), we find that

$$\begin{aligned} t_y^{(n-\theta)/2} &\le \frac{C t_y^{-\theta}}{\epsilon} \int_{t_y/4}^{t_y} \int_{\underline{B}_{t_y}(y)} |\mathrm{Ric}_{g(s)}| \, d\mathrm{vol}_{g(s)} \, ds \\ &\le \frac{C}{\epsilon} \int_{t_y/4}^{t_y} s^{-\theta} \int_{\underline{B}_{t_y}(y)} |\mathrm{Ric}_{g(s)}| \, d\mathrm{vol}_{g(s)} \, ds. \end{aligned} \tag{5-29}$$

Now, by Lemma 5.4(1) and (3), respectively, the balls $\{\underline{B}_{t_y}(y)\}_{y \in \mathcal{C}}$ are pairwise disjoint and contained in $\underline{B}$. So, we sum (5-29) over all $y \in \mathcal{C}$ and apply (5-8) to discover that

$$\sum_{y \in \mathcal{C}} t_y^{(n-\theta)/2} \le \frac{C}{\epsilon} \int_0^1 s^{-\theta} \int_{\underline{B}} |\mathrm{Ric}_{g(s)}| \, d\mathrm{vol}_{g(s)} \, ds \le C\epsilon. \tag{5-30}$$

Again, up to further decreasing δ so that (5-30) holds for $\epsilon' = \epsilon / C$, this concludes the proof of (3) and thus the proposition. □

5.3 The k^{th} good and bad sets and the proof of Theorem 5.1 with $\eta = \epsilon$

In this section, we apply Proposition 5.6 inductively in order to define k^{th} good and bad sets and establish Theorem 5.1 in the case when $\eta = \epsilon$, and thus $\hat{t} = 1$. Separating the proofs when $\eta < \epsilon$ and $\eta = \epsilon$ is convenient as it allows us to apply Corollary 5.7 below to establish estimates needed for the case $\eta < \epsilon$.

Proof of Theorem 5.1 when $\eta = \epsilon$ Let δ be chosen according to Proposition 5.6.

Step 1 We inductively define sets $\mathcal{G}^k \subseteq \tilde{\mathcal{A}}^{k-1}$, $\tilde{\mathcal{A}}^k \subseteq \tilde{\mathcal{A}}^{k-1}$ and $\mathcal{C}^k \subseteq \tilde{\mathcal{A}}^k$ for each $k \in \mathbb{N}$ satisfying the following properties:

(1) For all $x \in \mathcal{G}^k$ and $s, s' \in [0, 1]$, we have

$$(1-\epsilon)^k g(s') \le g(s) \le (1+\epsilon)^k g(s'). \tag{5-31}$$

(2) For each $x \in \mathcal{C}^k$ we have a mapping $y \mapsto t_y \in (0, 200^{-k})$ such that $(\mathcal{C}^k, t_y)$ is a covering pair for $\widetilde{\mathcal{A}}^k$ in $\underline{B}$ as in Definition 5.5, and such that

$$\sum_{y \in \mathcal{C}^k} t_y^{n/2} \le \epsilon^k. \tag{5-32}$$

(3) Furthermore, if $y \in \mathcal{C}^k$, then

$$(1-\epsilon)^k g(s') \le g(s) \le (1+\epsilon)^k g(s') \tag{5-33}$$

for all $x \in B_{g(t_y)}(y, 2t_y^{1/2})$ and for all $s, s' \in [t_y, 1]$.

In the claim above and in its proof, we suppress in the notation the dependence of t_y on k for $y \in \mathcal{C}^k$. Let $\widetilde{\mathcal{A}}^0 = \underline{B}$, and for $k = 1$, we set

$$\mathcal{G}^1 = \mathcal{G}(\underline{B}) \quad \text{and} \quad \widetilde{\mathcal{A}}^1 = \mathcal{A}(\underline{B}), \tag{5-34}$$

as defined in (5-21). Let $(\mathcal{C}^1, t_y)$ be the covering pair provided by Proposition 5.6. Then properties (1)–(3) above for $k = 1$ follow directly from Proposition 5.6.

Now, suppose that we have defined the sets $\mathcal{G}^k \subseteq \widetilde{\mathcal{A}}^{k-1}$, $\widetilde{\mathcal{A}}^k \subseteq \widetilde{\mathcal{A}}^{k-1}$ and $\mathcal{C}^k \subseteq \widetilde{\mathcal{A}}^k$ satisfying properties (1)–(3). We define $\mathcal{G}^{k+1}$ by

$$\mathcal{G}^{k+1} = \widetilde{\mathcal{A}}^k \cap \bigcup_{y \in \mathcal{C}^k} \mathcal{G}(\underline{B}_{t_y}(y)). \tag{5-35}$$

If $x \in \mathcal{G}^{k+1}$, then $x \in \mathcal{G}(\underline{B}_{t_y}(y))$ for some $y \in \mathcal{C}^k$. So, Proposition 5.6(1) applied to $\mathcal{G}(\underline{B}_{t_y}(y))$ implies that for all $s, s' \in [0, t_y]$, we have

$$(1-\epsilon) g(s') \le g(s) \le (1+\epsilon) g(s') \tag{5-36}$$

at x. The inductive hypothesis ensures that (5-33) holds at x. Together, (5-36) and (5-33) imply that

$$(1-\epsilon)^k g(s') \le g(s) \le (1+\epsilon)^k g(s') \tag{5-37}$$

for all $s, s' \in [0, 1]$. Therefore, (5-31) holds with $k+1$ replacing k for each $x \in \mathcal{G}^{k+1}$.

Similarly, define

$$\widetilde{\mathcal{A}}^{k+1} = \widetilde{\mathcal{A}}^k \cap \bigcup_{y \in \mathcal{C}^k} \mathcal{A}(\underline{B}_{t_y}(y)). \tag{5-38}$$

For each $y \in \mathcal{C}^k$, Proposition 5.6(3) ensures the existence of a covering pair $(\mathcal{C}_y, z \mapsto t_z)$ for $\mathcal{A}(\underline{B}_{t_y}(y))$ in $\underline{B}_{t_y}(y)$ with $t_z \in \big(0, \frac{1}{200} t_y\big) \subseteq (0, 200^{-(k+1)})$. We set

$$\mathcal{C}^{k+1} = \bigcup_{y \in \mathcal{C}^k} \mathcal{C}_y. \tag{5-39}$$

Together Proposition 5.6(3) and the inductive hypothesis (2) ensure that $(\mathcal{C}^{k+1}, z \mapsto t_z)$ is a covering pair for $\widetilde{\mathcal{A}}^{k+1}$ in $\underline{B}$. Further, Proposition 5.6(3) and (5-32) show that

$$\sum_{z \in \mathcal{C}^{k+1}} t_z^{n/2} \le \epsilon^{k+1}, \tag{5-40}$$

so (5-32) holds for $\mathcal{C}^{k+1}$ with $k+1$ replacing k. Thus property (2) holds for $k+1$. Finally, Proposition 5.6(2) along with (5-33) ensures that

$$(1-\epsilon)^{k+1} g(s') \le g(s) \le (1+\epsilon)^{k+1} g(s') \tag{5-41}$$

for all $x \in B_{g(t_z)}(z, 2t_z^{1/2})$ and for all $s, s' \in [t_z, 1]$. Thus property (3) holds for $k+1$. This concludes the proof of the claim.

Step 2 We now finish the proof of Theorem 5.1. Theorem 5.1(2) follows directly from (5-31) and the definition of the sets $\mathcal{G}^k$. Noting that

$$\mathcal{A}^k := B_{g(1)}(x_0, 2) \setminus \bigcup_{\ell=1}^{k} \mathcal{G}^\ell \subset \widetilde{\mathcal{A}}^k, \tag{5-42}$$

Theorem 5.1(4) follows from property (2) above. Next, as $\mathcal{G}^k \subseteq \widetilde{\mathcal{A}}^{k-1}$ by construction, we have

$$\mathrm{vol}_{g(1)}(\mathcal{G}^k) \le \mathrm{vol}_{g(1)}(\widetilde{\mathcal{A}}^{k-1}) \le \sum_{y \in \mathcal{C}^{k-1}} \mathrm{vol}_{g(1)}(B_{g(36t_y)}(y, 2(36t_y)^{1/2}). \tag{5-43}$$

We apply (3-7) followed by (3-39) to find that

$$\begin{aligned} \mathrm{vol}_{g(1)}(B_{g(36t_y)}(x, 2(36t_y)^{1/2}) &\le (1+2\delta)\, \mathrm{vol}_{g(36t_y)}(B_{g(36t_y)}(x, 2(36t_y)^{1/2}) \\ &\le C t_y^{n/2}, \end{aligned} \tag{5-44}$$

where C is a dimensional constant. Thus, by (5-43), (5-44) and (5-32), we see that

$$\mathrm{vol}_{g(1)}(\mathcal{G}^k) \le C \sum_{y \in \mathcal{C}^{k-1}} t_y^{1/2} \le C\epsilon^{k-1}. \tag{5-45}$$

Pairing this estimate with Theorem 5.1(2), this establishes Theorem 5.1(3). Finally, observe that the same argument shows that $\mathrm{vol}_{g(200^{-k})}(\widetilde{\mathcal{A}}^k) \le C\epsilon^k$. So, defining $\mathcal{A} = \bigcap_{k=1}^{\infty} \widetilde{\mathcal{A}}^k$, we see that

$$\mathrm{vol}_{g(0)}(\mathcal{A}) = \lim_{k\to\infty} \mathrm{vol}_{g(200^{-k})}(\mathcal{A}) \le \lim_{k\to\infty} \mathrm{vol}_{g(200^{-k})}(\widetilde{\mathcal{A}}^k) = 0. \tag{5-46}$$

This proves Theorem 5.1(1) and thus concludes the proof of the theorem. □

An immediate consequence of Theorem 5.1 with $\hat{t} = 1$ is the following comparison of volumes of balls.

Corollary 5.7 *Fix $n \geq 2$ and $\epsilon > 0$. There exists a $\delta = \delta(n, \epsilon) > 0$ such that the following holds. Let $(M, g(t))_{t \in (0,1]}$ be a Ricci flow satisfying (5-1) and (5-2). Fix $x_0 \in M$ and $t_0 \in (0, 1]$. For all $s, t \in (0, t_0]$, we have*

$$1 - \epsilon \leq \frac{\operatorname{vol}_{g(t)}(B_{g(t_0)}(x_0, 2t_0^{1/2}))}{\operatorname{vol}_{g(s)}(B_{g(t_0)}(x_0, 2t_0^{1/2}))} \leq 1 + \epsilon. \tag{5-47}$$

Proof Up to parabolic rescaling, we may assume that $t_0 = 1$. Thanks to (3-7), it suffices to show that $\operatorname{vol}_{g(0)}(B_{g(1)}(x_0, 2)) \leq (1 + \epsilon) \operatorname{vol}_{g(1)}(B_{g(1)}(x_0, 2))$ provided δ is taken to be sufficiently small depending on ϵ. Let δ also be taken sufficiently small according to Theorem 5.1. Applying Theorem 5.1 with $\eta = \epsilon$ and thus $\hat{t} = 1$, we find that

$$\begin{aligned} \operatorname{vol}_{g(0)}(B_{g(1)}(x_0, 2)) &= \sum_{k=1}^{\infty} \operatorname{vol}_{g(0)}(\mathcal{G}^k) \leq \operatorname{vol}_{g(0)}(\mathcal{G}^k) + \sum_{k=2}^{\infty} (1+\epsilon)^k \epsilon^{k-1} \\ &\leq (1+\epsilon) \operatorname{vol}_{g(1)}(x_0, 2) + C\epsilon \leq (1+C\epsilon) \operatorname{vol}_{g(1)}(x_0, 2). \end{aligned} \tag{5-48}$$

The final inequality follows from Lemma 3.7. By further decreasing δ, we may replace ϵ by ϵ/C to complete the proof. □

5.4 Improved Ricci estimate and good and bad sets on the initial ball

When $\eta < \epsilon$, we will need to apply a refined form of Proposition 5.6. To this end, we first show that an improved integral estimate for the Ricci curvature holds outside a set of small content. As usual, we use the notation $\underline{B}_t(x) = B_{g(t)}(x, 4t^{1/2})$ as defined in (5-6).

Lemma 5.8 (improved Ricci integral estimate) *Fix $n \geq 2$, $\epsilon > 0$ and $\theta \in \left(0, \frac{1}{2}\right)$. There exists a $\delta = \delta(n, \epsilon, \theta) > 0$ such that the following holds. Let $(M, g(t))_{t \in [0,1]}$ be a Ricci flow satisfying (5-1) and (5-2). Then for any $x_0 \in M$ and $\eta \leq \epsilon$, there exist $\hat{t} = \hat{t}(n, \epsilon, \theta, \eta) \in (0, 1]$ and an exceptional set $E \subset B_{g(1)}(x_0, 2)$ such that the following properties hold.*

(1) *For all $x \in B_{g(1)}(x_0, 2) \setminus E$, we have the improved integral Ricci curvature estimate*

$$\int_0^{\hat{t}} \left(\frac{s}{\hat{t}}\right)^{-\theta/2} \fint_{\underline{B}_{\hat{t}}(x)} |\operatorname{Ric}_{g(s)}(y)| \, d\operatorname{vol}_{g(s)}(y) \, ds \leq \eta^2. \tag{5-49}$$

(2) *There is a finite set $\{y_\ell\}_{\ell=1}^N \subset B_{g(1)}(x_0, 2)$ such that*

$$E \subset \bigcup_{\ell=1}^{N} B_{g(\hat{t})}(y_\ell, 2\hat{t}^{1/2}) \quad \text{with } N \leq \eta \hat{t}^{-n/2}. \tag{5-50}$$

Proof Fix any $\epsilon \in \left(0, \frac{1}{2}\right)$. Let $\delta = \delta(n, \epsilon, \theta)$ be chosen sufficiently small that we may apply Theorem 4.1 with $t = \epsilon = 1$ and Lemmas 3.8 and 3.7 with λ to be determined in the course of the proof. Throughout the proof, we use the shorthand $B = B_{g(1)}(x_0, 2)$.

Step 1 We first claim that

$$\int_B \int_0^t \left(\frac{s}{t}\right)^{-\theta/2} \fint_{\underline{B}_t(x)} |\mathrm{Ric}_{g(s)}(y)| \, d\mathrm{vol}_{g(s)}(y) \, ds \, d\mathrm{vol}_{g(0)}(x) \le 2t^\theta \tag{5-51}$$

for any $t \le \hat{t}$, provided $\hat{t}$ is taken small enough.

Step 1a Recall from Corollary 5.7 and Lemma 3.7 that

$$\mathrm{vol}_{g(s)}(B_{g(1)}(x_0, 4)) \le (1+\epsilon) \, \mathrm{vol}_{g(1)}(B_{g(1)}(x_0, 4)) \le (1+\epsilon) 4^n \omega_n. \tag{5-52}$$

This together with Theorem 4.1 (taking $t = \epsilon = 1$) implies that for any $t \le 1$ we have

$$\begin{aligned}\int_0^t s^{-\theta/2} \int_{\underline{B}_1(x_0)} &|\mathrm{Ric}_{g(s)}(y)| \, d\mathrm{vol}_{g(s)}(y) \\ &\le t^{\theta/2} \int_0^t s^{-\theta} \int_{\underline{B}_1(x_0)} |\mathrm{Ric}_{g(s)}(y)| \, d\mathrm{vol}_{g(s)}(y) \le (1+\epsilon) 4^n \omega_n t^{\theta/2}.\end{aligned} \tag{5-53}$$

Step 1b A basic maximal function argument shows that

$$\begin{aligned}\int_B \fint_{\underline{B}_t(x)} &|\mathrm{Ric}_{g(s)}| \, d\mathrm{vol}_{g(s)}(y) \, d\mathrm{vol}_{g(0)}(x) \\ &\le (1+2\epsilon) \int_{B_{g(1)}(x,4)} |\mathrm{Ric}_{g(s)}(y)| \, d\mathrm{vol}_{g(t)}(y).\end{aligned} \tag{5-54}$$

To see this, take $\hat{t}$ small enough such that $4\hat{t}^{1/2} \le 1$. By the proof of Corollary 5.7 and Lemma 3.7, we have

$$(1-\epsilon)\omega_n (4t)^{n/2} \le \mathrm{vol}_{g(s)}(\underline{B}_t(x)) \le (1+\epsilon)\omega_n (4t)^{n/2} \tag{5-55}$$

for any $s \in [0, t]$. Furthermore, by Lemma 3.8, for any $x \in B$, we have $\underline{B}_t(x) \subset \underline{B}_1(x_0)$. With these observations in hand, we see that for fixed $s \in (0, t)$,

$$\begin{aligned}\int_B \fint_{\underline{B}_t(x)} &|\mathrm{Ric}_{g(s)}| \, d\mathrm{vol}_{g(s)}(y) \, d\mathrm{vol}_{g(0)}(x) \\ &\le \frac{1+\epsilon}{\omega_n t^{n/2}} \int_B \int_{\underline{B}_1(x_0)} \chi_{\underline{B}_t(x)}(y) |\mathrm{Ric}_{g(s)}(y)| \, d\mathrm{vol}_{g(s)}(y) \, d\mathrm{vol}_{g(0)}(x) \\ &\le \frac{1+\epsilon}{\omega_n t^{n/2}} \int_{B_{g(1)}(x,2)} \left(\int_B \chi_{\underline{B}_t(y)}(x) \, d\mathrm{vol}_{g(0)}(x) \right) |\mathrm{Ric}_{g(s)}|(y) \, d\mathrm{vol}_{g(s)}(y) \\ &\le (1+2\epsilon) \int_{B_{g(1)}(x,2)} |\mathrm{Ric}_{g(s)}(y)| \, d\mathrm{vol}_{g(t)}(y).\end{aligned} \tag{5-56}$$

Here $\chi_{\underline{B}_t(x)}$ is the indicator function of $\underline{B}_t(x)$. This establishes (5-54).

Step 1c Multiplying (5-54) by $s^{-\theta/2}$ and integrating with respect to s and then combining the subsequent estimate with (5-53), we establish that

$$\int_0^t s^{-\theta/2} \fint_{\underline{B}(t_\eta)} |\mathrm{Ric}_{g(s)}(y)| \, d\mathrm{vol}_{g(s)}(y) \, ds \le t^{\theta/2} \tag{5-57}$$

holds for $t \le \hat{t}$. Multiplying both sides of (5-57) by $t^{\theta/2}$, we obtain (5-51).

Step 2 Now, let

$$E = \left\{ x \in B : \int_0^{\hat{t}} \left(\frac{s}{\hat{t}}\right)^{-\theta/2} \fint_{\underline{B}_t(x)} |\mathrm{Ric}_{g(s)}(y)| \, d\mathrm{vol}_{g(s)}(y) \, ds \ge \hat{t}^{\theta/2} \right\}. \tag{5-58}$$

By definition, (5-49) holds on $B \setminus E$; it remains to show the content bound (5-50). Let

$$B_{g(\hat{t})}(E, \hat{t}^{1/2}) = \bigcup_{x \in E} B_{g(\hat{t})}(x, \hat{t}^{1/2}). \tag{5-59}$$

We see that every $x \in B_{g(\hat{t})}(E, t^{1/2})$ satisfies

$$\int_0^{\hat{t}} \left(\frac{s}{\hat{t}}\right)^{-\theta/2} \fint_{B_{g(\hat{t})}(x, 8\hat{t}^{1/2})} |\mathrm{Ric}_{g(s)}(y)| \, d\mathrm{vol}_{g(s)}(y) \, ds \ge c_n \hat{t}^{\theta/2}. \tag{5-60}$$

Thus, by applying Chebyshev's inequality to (5-51), we find that

$$\mathrm{vol}_{g(0)}(B_{g(\hat{t})}(E, \hat{t}^{1/2})) \le C_n \hat{t}^{\theta/2}. \tag{5-61}$$

Take $\{y_\ell\}_{\ell=1}^N$ to be a maximal $\frac{1}{4}\hat{t}^{1/2}$–dense subset of E with respect to $g(\hat{t})$. We conclude that (5-50) holds by taking $\hat{t}$ sufficiently small depending on η. □

We now proceed as in Section 5.2 to decompose a ball $\underline{B}_t$ satisfying the improved Ricci integral estimate (5-49). For a ball $\underline{B}$, we define the refined stopping time $t(x)$ for each $x \in B_{g(1)}(x_0, 2)$ by

$$t(x) = \inf\left\{ t' \le \tfrac{1}{200} : \fint_{t/4}^{t} \fint_{\underline{B}_t(x)} |\mathrm{Ric}_{g(s)}| \, d\mathrm{vol}_{g(s)} \, ds \le t^{\theta/4-1} \hat{t}^{\theta/4} \text{ for all } t \in \left[t', \tfrac{1}{200}\right] \right\}. \tag{5-62}$$

As in Section 5.2, provided we take $t_0 < 200^{-2}$, if a ball $\underline{B}$ satisfies (5-49), then

$$\fint_{1/800}^{\frac{1}{200}} \fint_{\underline{B}_{1/200}(x)} |\mathrm{Ric}_{g(t)}| \, d\mathrm{vol}_{g(t)} \, dt \le 200^{\theta/4-1} \hat{t}^{\theta/4}, \tag{5-63}$$

so the stopping condition holds at $t' = \frac{1}{200}$.

For a ball $\underline{B}_t(x_0)$ with $t<1$, we define the stopping time $t(x)$ by

$$t(x)=t\,\tilde{t}(x) \tag{5-64}$$

for each $x\in B_{g(t)}(x_0,2t^{1/2})$, where $\tilde{t}(x)$ is the stopping time defined in (5-62) applied to the rescaled flow $\tilde{g}(s)=t^{-1}g(ts)$. The refined good and bad sets on $\underline{B}_t(x_0)$, respectively, are defined by

$$\begin{aligned}\hat{\mathcal{G}}(\underline{B}_t(x_0))&=\{x\in B_{g(t)}(x_0,2t^{1/2}):t(x)=0\},\\ \hat{\mathcal{A}}(\underline{B}_t(x_0))&=\{x\in B_{g(t)}(x_0,2t^{1/2}):t(x)>0\}.\end{aligned} \tag{5-65}$$

In the following proposition, we establish estimates on the good and bad sets. This is exactly Proposition 5.6 with $t_0^{\theta/4}$ in place of ϵ and $\frac{1}{2}\theta$ in place of θ; the proof is identical and thus omitted. Again, when convenient, we adopt the shorthand $t_x=t(x)$.

Proposition 5.9 *Fix $n\geq 2$, $\theta\in\left(0,\frac{1}{2}\right)$, $\eta>0$, $\hat{t}\in(0,1]$ and $x_0\in M$. Suppose that the improved Ricci estimate (5-49) holds on $\underline{B}_{\hat{t}}(x_0)$. Then the following properties hold:*

(1) *For any $x\in\mathcal{G}(\underline{B}_{\hat{t}}(x_0))$ and for any $s,s'\in[0,\hat{t}]$, the metrics $g(s)$ and $g(s')$ at x satisfy*

$$(1-\eta)g(s)\leq g(s')\leq(1+\eta)g(s). \tag{5-66}$$

(2) *Suppose we have $x\in\mathcal{A}(\underline{B}_{\hat{t}}(x_0))$, and fix $s_0\in[t_x,\hat{t}]$. Then for all $s,s'\in[s_0,\hat{t}]$ and $y\in B_{g(s_0)}(x,2s_0^{1/2})$, the metrics $g(s)$ and $g(s')$ at y satisfy*

$$(1-\eta)g(s)\leq g(s')\leq(1+\eta)g(s). \tag{5-67}$$

(3) *There is a countable collection $\mathcal{C}=\mathcal{C}(\underline{B}_{\hat{t}}(x_0))\subseteq\mathcal{A}(\underline{B}_{\hat{t}}(x_0))$ such that $(\mathcal{C},t_y)$ is a covering pair for $\mathcal{A}(\underline{B}_{\hat{t}}(x_0))$ in the sense of Definition 5.5, where $t_y=t(y)$ is the stopping time and*

$$\sum_{y\in\mathcal{C}}t_y^{(n/2-\theta/4)}\leq\hat{t}^{n/2}. \tag{5-68}$$

In particular,

$$\sum_{y\in\mathcal{C}}t_y^{n/2}\leq\eta\hat{t}^{n/2}. \tag{5-69}$$

Now we prove Theorem 5.1 in the case when $\eta<\epsilon$. The proof by induction is completely analogous to the proof when $\eta=\epsilon$, with the only modification coming from the first step of the iteration.

Proof of Theorem 5.1 when $\eta < \epsilon$ Let δ be chosen according to Lemma 5.8. We inductively define sets $\mathcal{G}^k \subseteq \widetilde{\mathcal{A}}^{k-1}$, $\widetilde{\mathcal{A}}^k \subseteq \widetilde{\mathcal{A}}^{k-1}$ and $\mathcal{C}^k \subseteq \widetilde{\mathcal{A}}^k$ for each $k \in \mathbb{N}$ satisfying the following properties:

(1) For all $x \in \mathcal{G}^k$, we have

$$(1-\eta)(1-\epsilon)^{k-1} g(s') \le g(s) \le (1+\eta)(1+\epsilon)^{k-1} g(s') \tag{5-70}$$

for all $s, s' \in [0, \hat{t}]$.

(2) For each $x \in \mathcal{C}^k$ we have a mapping $y \mapsto t_y \in (0, 200^{-k})$ such that $(\mathcal{C}^k, t_y)$ is a covering pair for $\widetilde{\mathcal{A}}^k$ in $\underline{B}$ as in Definition 5.5 and such that

$$\sum_{y \in \mathcal{C}^k} t_y^{n/2} \le \hat{t}^{n/2} \eta \epsilon^{k-2}. \tag{5-71}$$

(3) Furthermore, if $y \in \mathcal{C}^k$, then

$$(1-\eta)(1-\epsilon)^{k-1} g(s') \le g(s) \le (1+\eta)(1+\epsilon)^{k-1} g(s') \tag{5-72}$$

for all $x \in B_{g(t_y)}(y, 2t_y^{1/2})$ and for all $s, s' \in [t_y, \hat{t}]$.

As before, we suppress in the notation the dependence of t_y on k for $y \in \mathcal{C}^k$. The proof of the inductive step is identical to that in the case when $\hat{t} = 1$, so we need only to establish the base case. Let $\widetilde{\mathcal{A}}^0 = \underline{B}$.

Let E be chosen according to Lemma 5.8; by Lemma 5.8(2) we see that

$$E \subset \bigcup_{\ell=1}^{N} \underline{B}_{\hat{t}}(x_\ell), \quad \text{with } N \le \eta \hat{t}^{-n/2}. \tag{5-73}$$

Next, consider a maximal $\frac{1}{4}\hat{t}^{1/2}$–dense set $\{x_i\}_{i=1}^{N'}$ in $\underline{B} \setminus E$ with respect to $g(\hat{t})$. In this way,

$$\underline{B} \setminus E \subseteq \bigcup_{i=1}^{N'} B_{g(\hat{t})}(x_i, \hat{t}^{1/2}), \tag{5-74}$$

and thanks to Corollary 5.7, we have that

$$N' \le C_n \hat{t}^{-n/2}. \tag{5-75}$$

Since (5-49) holds for each $i = 1, \dots, N'$, we apply Proposition 5.9 to decompose $B_{g(\hat{t})}(x_i, 4\hat{t}^{1/2})$ for each $i = 1, \dots, N'$. Now, we set

$$\mathcal{G}^1 = \bigcup_{i=1}^{N'} \widehat{\mathcal{G}}(\underline{B}_{\hat{t}(x_i)}) \quad \text{and} \quad \widetilde{\mathcal{A}}^1 = E \cup \bigcup_{\ell=1}^{N'} \widehat{\mathcal{A}}(\underline{B}_{\hat{t}(x_\ell)}) \tag{5-76}$$

as defined in (5-65). For each x_ℓ for $\ell = 1, \dots, N$, we define $t_{x_\ell} = \hat{t}^{1/2}$, and set

$$\mathcal{C}^1 = \bigcup_{\ell=1}^{N} \{x_\ell\} \cup \bigcup_{\ell=1}^{N'} \hat{\mathcal{C}}(\underline{B}_{\hat{t}(x_\ell)}). \tag{5-77}$$

Then properties (1)–(3) follow directly from Proposition 5.6 along with (5-73) and (5-75), since

$$\sum_{\mathcal{C}^1} t_y^{n/2} = N\hat{t}^{n/2} + \eta N'\hat{t}^{n/2} \le C_n \eta. \tag{5-78}$$

We have thus inductively defined the sets $\mathcal{G}^k$, $\widetilde{\mathcal{A}}^k$ and $\mathcal{C}^k$. The remainder of the proof is identical to the proof in the case when $\eta = \epsilon$, shown in the previous subsection. This completes the proof of Theorem 5.1. □

6 L^P bounds for the metric coefficients

In this section, we prove Theorem 1.11, which we restate below as Theorem 6.1 for the convenience of the reader.

Theorem 6.1 (Theorem 1.11 restated) *Fix $n \ge 2$, $P \in [1, \infty)$ and $\epsilon > 0$. There exists $\delta = \delta(n, P, \epsilon) > 0$ such that the following holds. Let (M, g) be a complete Riemannian n–manifold with bounded curvature satisfying*

$$R \ge -\delta \quad \text{and} \quad \nu(g, 2) \ge -\delta. \tag{6-1}$$

Then for any $x_0 \in M$, there is an open set $\Omega \subset M$ containing x_0 and a smooth diffeomorphism $\psi : \Omega \to B(0,1) \subset \mathbb{R}^n$ with $\psi(x_0) = 0$ satisfying

$$⨍_{B(0,1)} |(\psi^{-1})^* g - g_{\mathrm{euc}}|^P \, dy \le \epsilon \quad \text{and} \quad ⨍_{\Omega} |\psi^* g_{\mathrm{euc}} - g|^P \, d\mathrm{vol}_g \le \epsilon. \tag{6-2}$$

Furthermore, for any $\kappa > 1$ and $q_0 \in [\kappa, \infty)$ we may choose δ additionally small depending on q_0 and κ such that for any $f \in W^{1,q}(B(0,1))$ with $q \in [\kappa, q_0]$, we have

$$(1-\epsilon)\|\psi^* f\|_{L^{q/\kappa}(\Omega)} \le \|f\|_{L^q(B(0,1))} \le (1+\epsilon)\|\psi^* f\|_{L^{\kappa q}(\Omega)}, \tag{6-3}$$

$$(1-\epsilon)\|\nabla \psi^* f\|_{L^{q/\kappa}(\Omega)} \le \|\nabla f\|_{L^q(B(0,1))} \le (1+\epsilon)\|\nabla \psi^* f\|_{L^{\kappa q}(\Omega)}. \tag{6-4}$$

Remark 6.2 The estimates in (6-2) are equivalent to

$$⨍_{\Omega} \|d\psi\|_\infty^P \, d\mathrm{vol}_g \le 1 + \epsilon \quad \text{and} \quad ⨍_{B(0,1)} \|d\psi^{-1}\|_\infty^P \, dy \le 1 + \epsilon. \tag{6-5}$$

Here, given a map $u\colon (M,g)\to(N,h)$, the notation $\|du\|_\infty(x)$ indicates the operator norm of the linear map $du_x\colon (T_xM,g)\to(T_{u(x)}M,h)$. Given $B\colon TM\times TM\to\mathbb{R}$ a bilinear form, we let $\|B\|_\infty=\sup\|B(v,\cdot)\|_\infty/\|v\|_g$.

Remark 6.3 It will be apparent in the proof that, given any number $R\ge 1$, by choosing $\delta>0$ additionally small depending on R, we may obtain the conclusion of Theorem 6.1 with $B(0,R)$ replacing $B(0,1)$.

Proof Let us begin with some initial observations. Without loss of generality we may assume that $\epsilon\le\epsilon(n,P)$, where $\epsilon(n,P)$ will be determined in the proof. By Theorem 3.2, the Ricci flow $g(t)$ with $g(0)=g$ exists for $t\in[0,1]$ and $|\mathrm{Rm}_{g(t)}|\le\lambda/t$ for all $x\in M$ and $t\in(0,1]$. Here λ may be taken as small as needed by choosing δ sufficiently small. We let

$$\Omega=B_{g(1)}(x_0,1). \tag{6-6}$$

By Lemma 3.7, we obtain a smooth diffeomorphism $\psi\colon B_{g(1)}(x_0,2)\to U\subset\mathbb{R}^n$, with inverse $\phi=\psi^{-1}$ such that $\psi(x_0)=0$, $\psi(\Omega)=B(0,1)$ and

$$\left(1-\tfrac{1}{2}\epsilon\right)g_{\mathrm{euc}}\le\phi^*g(1)\le\left(1+\tfrac{1}{2}\epsilon\right)g_{\mathrm{euc}} \tag{6-7}$$

for all $x\in U$. This holds as long as λ (and hence δ) has been chosen to be sufficiently small depending on ϵ and n.

Now, let

$$B_{g(1)}(x_0,2)=\bigcup_{k=1}^{\infty}\mathcal{G}^k\cup\mathcal{A} \tag{6-8}$$

be the decomposition provided by the decomposition theorem, Theorem 5.1, with $\epsilon=\eta$. From (6-7) and Theorem 5.1(2), for every $x\in\phi^*\mathcal{G}^k$ we have

$$(1-\epsilon)^{k+1}g_{\mathrm{euc}}\le\phi^*g\le(1+\epsilon)^{k+1}g_{\mathrm{euc}}. \tag{6-9}$$

Therefore, we have

$$|\phi^*g-g_{\mathrm{euc}}|_{g_{\mathrm{euc}}}\le(1+\epsilon)^{k+1}-1\le\epsilon(k+1)(1+\epsilon)^k \tag{6-10}$$

for all $x\in\phi^*\mathcal{G}^k$, and likewise

$$|g-\psi^*g_{\mathrm{euc}}|_g\le\epsilon(k+1)(1+\epsilon)^k \tag{6-11}$$

for all $x \in \mathcal{G}^k$. Furthermore, $\mathrm{vol}_g(\mathcal{A}) = \mathrm{vol}_{\mathrm{euc}}(\phi^*\mathcal{A}) = 0$ and for all $k \geq 2$, we have

$$\mathrm{vol}_g(\mathcal{G}^k) \leq (1+\epsilon)^k \epsilon^{k-1}, \tag{6-12}$$

$$\mathrm{vol}_{\mathrm{euc}}(\phi^*\mathcal{G}^k) \leq (1+\epsilon)^k \epsilon^{k-1}. \tag{6-13}$$

We begin by proving the first estimate in (6-2). Set

$$\mu(r) = \mathrm{vol}_{\mathrm{euc}}(\{y \in B(0,1) : |\phi^* g - g_{\mathrm{euc}}|_{g_{\mathrm{euc}}} \geq r\}) \tag{6-14}$$

and $r_k = \epsilon(k+1)(1+\epsilon)^k$. Note that $\mu(0) \leq \omega_n$, while (6-10) and (6-13) ensure that

$$\begin{aligned}\mu(r_k) &\leq \mathrm{vol}_{\mathrm{euc}}\Bigl(B(0,1) \setminus \bigcup_{\ell=1}^{k} \phi^*\mathcal{G}^\ell\Bigr) \\ &\leq \sum_{\ell=k+1}^{\infty} \mathrm{vol}_{\mathrm{euc}}(\phi^*\mathcal{G}^\ell) \\ &\leq \sum_{\ell=k+1}^{\infty} (1+\epsilon)^\ell \epsilon^{\ell-1} \leq C\epsilon^k,\end{aligned} \tag{6-15}$$

where C is a universal constant. We apply the layer cake formula to find that

$$\begin{aligned}\int_{B(0,1)} |\phi^* g - g_{\mathrm{euc}}|^P \, dy &= P \int_0^\infty r^{P-1} \mu(r) \, dr \\ &\leq \mu(0) P \int_0^{r_0} r^{P-1} \, dr + P \sum_{k=0}^{\infty} \mu(r_k) \int_{r_k}^{r_{k+1}} r^{P-1} \, dr \\ &\leq \epsilon\mu(0) + \sum_{k=0}^{\infty} \mu(r_k) r_{k+1}^P \\ &\leq \epsilon\omega_n + C\epsilon^P \sum_{k=0}^{\infty} \epsilon^k (k+2)^P (1+\epsilon)^{P(k+1)} \leq C\epsilon\omega_n\end{aligned} \tag{6-16}$$

for some $C = C(n, P)$ provided that ϵ is small enough, depending on P. Further decreasing δ, we may replace ϵ by ϵ/C. Dividing through by ω_n, we establish the first estimate in (6-2). The second estimate is entirely analogous, with the only additional point to note being that $\mathrm{vol}_g(\Omega) \leq (1+\epsilon)\omega_n$ thanks to (6-10) and (6-12).

We now show (6-3). Let $\sigma = \sigma(\epsilon, \kappa, q)$ be chosen later in the proof and let $\delta = \delta(n, \sigma)$ be sufficiently small to apply the decomposition theorem, Theorem 5.1, with σ in place of ϵ. Fix any $f \in W^{1,q}(B(0,1))$. From (6-10) and $\mathrm{vol}_{\mathrm{euc}}(\phi^*\mathcal{A}) = 0$, we find that

$$\|f\|^q_{L^q(B(0,1))} \le \sum_{k=1}^{\infty} \int_{\phi^*\mathcal{G}^k\cap B(0,1)} |f|^q \, dx \le \sum_{k=1}^{\infty} (1+\sigma)^{nk/2+1} \int_{\mathcal{G}^k\cap\Omega} |\psi^* f|^q \, d\mathrm{vol}_g. \tag{6-17}$$

Next, for each k, we apply Hölder's inequality with κ and $\kappa' = \kappa/(\kappa-1)$ and apply (6-12) to find that

$$\|f\|^q_{L^q(B(0,1))} \le \sum_{k=1}^{\infty} (1+\sigma)^{nk/2+1} \mathrm{vol}_g(\mathcal{G}^k)^{1/\kappa'} \left(\int_{\mathcal{G}^k} |\psi^* f|^{\kappa q} \, d\mathrm{vol}_g \right)^{1/\kappa} \le \|\psi^* f\|^q_{L^{\kappa q}(\Omega)} \sum_{k=1}^{\infty} (1+\sigma)^{(n+1/\kappa')k} \sigma^{k/\kappa'}. \tag{6-18}$$

Now, by choosing $\sigma = \sigma(\epsilon,\kappa)$ sufficiently small, the sum on the right-hand side is bounded above by $1+\epsilon$, and thus by $(1+\epsilon)^q$ for any $q \ge 1$; after taking the $(1/q)^{\text{th}}$ power of both sides, we have shown that

$$\|f\|_{L^q(B(0,1))} \le (1+\epsilon)\|\psi^* f\|_{L^{\kappa q}(\Omega)}. \tag{6-19}$$

The proof of the other inequality in (6-3) is completely analogous.

The proof of (6-4) is similar. From (6-10), we have

$$\begin{aligned} \int_{B(0,1)} |\nabla_{\mathrm{euc}} f|^q_{\mathrm{euc}} \, dx &\le \sum_{k=1}^{\infty} \int_{\phi^*\mathcal{G}^k\cap B(0,1)} |\nabla_{\mathrm{euc}} f|^q_{\mathrm{euc}} \, dx \\ &\le \sum_{k=1}^{\infty} (1+\sigma)^{(q+n/2)k+1} \int_{\mathcal{G}^k} |\nabla_g \psi^* f|^q \, d\mathrm{vol}_g \\ &\le \left(\int_{\Omega} |\nabla_g \psi^* f|^{\kappa q} \, d\mathrm{vol}_g \right)^{1/\kappa} \sum_{k=1}^{\infty} (1+\sigma)^{(q+n/2)k+1} \mathrm{vol}_g(\mathcal{G}^k)^{1/\kappa'} \\ &\le \left(\int_{\Omega} |\nabla_g \psi^* f|^{\kappa q} \, d\mathrm{vol}_g \right)^{1/\kappa} \sum_{k=1}^{\infty} (1+\sigma)^{(q+n)k} \sigma^{k/\kappa'} \\ &\le (1+\epsilon)\|\nabla_g \psi^* f\|^q_{L^{\kappa q}(\Omega)}, \end{aligned} \tag{6-20}$$

where again the final inequality holds provided that σ is sufficiently small depending on q_0, ϵ and κ. The other inequality in (6-4) is analogous. This completes the proof. □

As a consequence of the proof of Theorem 6.1, we obtain the following estimate for the Ricci flow. In short, it tells us that, at in least in L^P, the metrics $g(t)$ converge as $t \to 0$.

Corollary 6.4 *Fix $n \geq 2$, $P \in [1,\infty)$ and $\epsilon > 0$. There exists a $\delta = \delta(n,P,\epsilon) > 0$ such that the following holds. Let (M,g) be a complete Riemannian n–manifold with bounded curvature satisfying*

$$R \geq -\delta \quad \text{and} \quad \nu(g,2) \geq -\delta. \tag{6-21}$$

For all $\eta \in (0,\epsilon)$, there exists $\hat{t}_\eta = \hat{t}_\eta(n,P,\epsilon,\eta)$ such that for any $x_0 \in M$, $s,t \in [0,\hat{t}_\eta]$,

$$\fint_{B_{g(1)}(x_0,1)} |g(s)-g(t)|^P_{g(t_\eta)} \, d\mathrm{vol}_{g(t_\eta)} \leq \eta. \tag{6-22}$$

Moreover, for any $\kappa > 1$, we may choose δ additionally depending on κ such that for all $s,t \in [0,\hat{t}_\eta]$,

$$\fint_{B_{g(1)}(x_0,1)} |g(s)-g(t)|^{P/\kappa}_{g(1)} \, d\mathrm{vol}_{g(1)} \leq \eta. \tag{6-23}$$

Remark 6.5 Upgrading this L^P coefficient convergence to d_p–convergence is morally, though not explicitly stated, in Section 8. This is in fact significantly harder, as one needs to understand how the analysis on $g(s)$ varies, not just the metric coefficients.

Proof The proof of (6-22) is identical to the proof of (6-2), but now we apply the decomposition theorem, Theorem 5.1, with η in place of ϵ. Then the proof of (6-23) follows precisely by repeating the proof of (6-4). □

7 Epsilon regularity

In this section, we prove the epsilon regularity theorem, Theorem 1.7, and the uniform L^∞–Sobolev embedding, Theorem 1.17. The main tool is Theorem 6.1, established in the previous section.

7.1 Preliminary lemmas

Let us first establish two preliminary lemmas that will be needed in the proof of Theorem 1.7. The first lemma allows us to localize the d_p–distance in Euclidean space.

We use the notation $d_{q,B(0,R)}$ to denote $d_{q,g_{\mathrm{euc}},B(0,R)}$, that is,

$$d_{q,B(0,R)}(x,y) = \sup\left\{ |f(x)-f(y)| : \int_{B(0,R)} |\nabla f|^q \, dx \le 1 \right\}. \tag{7-1}$$

Our first lemma tells us that on Euclidean space we may localize the d_p–distance.

Lemma 7.1 *For all $\epsilon > 0$ and $p \ge n+1$ there exists $R = R(n,\epsilon,p)$ such that for all $q \ge p - \frac{1}{2}$ and for all $x \in \mathcal{B}_{q,g_{\mathrm{euc}}}(0,1)$, we have*

$$|d_{q,B(0,R)}(x,0) - d_{q,g_{\mathrm{euc}}}(x,0)| \le \epsilon. \tag{7-2}$$

Proof One inequality in (7-2) follows immediately from the definition: for any $x \in B(0,R)$, we have

$$d_{q,g_{\mathrm{euc}}}(x,0) \le d_{q,B(0,R)}(x,0). \tag{7-3}$$

Next, we show that for any x such that $d_{q,B(0,R)}(x,0) \le 1+\epsilon$, we have

$$d_{q,B(0,R)}(x,0) \le d_{q,g_{\mathrm{euc}}}(x,0) + \epsilon. \tag{7-4}$$

Note that this immediately implies that (7-4) holds for any $x \in \mathcal{B}_{q,g_{\mathrm{euc}}}(0,1)$ and thus together with (7-3) will complete the proof. Let f be a function such that

$$\int_{B(0,R)} |\nabla f|^q \, dx \le 1, \quad f(0) = 0 \quad \text{and} \quad d_{q,B(0,R)}(x,0) < f(x) + \epsilon. \tag{7-5}$$

We will modify f to produce an admissible test function for $d_{p,g_{\mathrm{euc}}}(x,0)$. Without loss of generality, we may replace f with $\min\{|f|,2\}$, which also satisfies (7-5). Now, consider a cutoff function ψ such that $0 \le \psi \le 1$ and

$$\psi = \begin{cases} 1 & \text{in } B(0,2), \\ 0 & \text{in } \mathbb{R}^n \setminus B(0,R) \end{cases} \qquad \text{and} \qquad |\nabla \psi| \le CR^{-1} \tag{7-6}$$

for a dimensional constant C. By choosing R sufficiently large depending on n, ϵ and p, we have $\|\nabla\psi\|_{L^q(\mathbb{R}^n)} \le CR^{n/q-1} \le \frac{1}{2}\epsilon$ for any $q \ge p - \frac{1}{2}$. So, letting $\hat{f} = (1-\epsilon) f \psi$, we have

$$\hat{f}(0) = 0, \quad \hat{f}(x) \ge (1-2\epsilon) d_{q,B(0,R)}(x,0) \tag{7-7}$$

and

$$\begin{aligned} \|\nabla \hat{f}\|_{L^q(\mathbb{R}^n)} &\le (1-\epsilon)\|\psi \nabla f\|_{L^q(\mathbb{R}^n)} + (1-\epsilon)\|f \nabla \psi\|_{L^q(\mathbb{R}^n)} \\ &\le (1-\epsilon)\|\nabla f\|_{L^q(B(0,R))} + 2(1-\epsilon)\|\nabla\psi\|_{L^q(\mathbb{R}^n)} \\ &\le 1 - \epsilon + \epsilon = 1. \end{aligned} \tag{7-8}$$

In particular, $\hat{f}$ is an admissible test function for $d_{q,g_{\mathrm{euc}}}(x,0)$ and so

$$d_{q,g_{\mathrm{euc}}}(x,0) \geq (1-2\epsilon) d_{q,B(0,R)}(x,0). \tag{7-9}$$

After replacing ϵ with $\frac{1}{2}\epsilon$, this completes the proof of (7-4) and thus of the lemma. □

Remark 7.2 In the course of proving of Lemma 7.1, we have established the following statement under the same hypotheses. For any $x \in \mathbb{R}^n$ such that $d_{q,B(0,R)}(x,0) \leq 2$, we may find a function $\hat{f} \in W_0^{1,q}(B(0,R))$ such that $\|\nabla f\|_{L^q(\mathbb{R}^n)} \leq 1$, $\hat{f}(0) = 0$ and $\hat{f}(x) \geq (1-2\epsilon) d_{q,B(0,R)}(x,0)$.

The next lemma shows the continuity of the $d_{p,U}$–distance with respect to the Euclidean distance, and will be used in the proof of Lemma 7.4 below.

Lemma 7.3 *Fix $p > n$, let $U \subset \mathbb{R}^n$ be a bounded open set with smooth boundary and $V \Subset U$ be precompact. For any sequences $\{x_i\}, \{y_i\} \subset V$ with $x_i, y_i \to x_\infty, y_\infty \in \overline{V}$, we have $d_{p,U}(x_i, y_i) \to d_{p,U}(x_\infty, y_\infty)$.*

Proof Fix $\epsilon > 0$ and let $f \in W^{1,p}(U)$ be a continuous function such that $f(y_\infty) = 0$, $f(x_\infty) \geq d_{p,U}(x_\infty, y_\infty) - \epsilon$ and $\int_U |\nabla f|^p \, dx \leq 1$. Taking f as a test function in the definition of $d_{p,U}(x_i, y_i)$ for each i, we find that

$$\begin{aligned} \liminf_{i\to\infty} d_{p,U}(x_i, y_i) \geq \lim_{i\to\infty} |f(x_i) - f(y_i)| &= |f(x_\infty) - f(y_\infty)| \\ &\geq d_{p,U}(x_\infty, y_\infty) - \epsilon. \end{aligned} \tag{7-10}$$

Letting $\epsilon \to 0$, we see that $\liminf_{i\to\infty} d_{p,U}(x_i, y_i) \geq d_{p,U}(x_\infty, y_\infty)$.

On the other hand, fix $\epsilon > 0$ and for each i let $f_i \in W^{1,p}(U)$ be such that $f_i(y_i) = 0$, $f_i(x_i) \geq d_{p,U}(x_i, y_i) - \epsilon$ and $\int_U |\nabla f_i|^p \, dx \leq 1$. Up to a subsequence, $f_i \rightharpoonup f_\infty$ in $W^{1,p}(U)$ with $\int_U |\nabla f_\infty|^p \, dx \leq 1$, and by the Morrey–Sobolev inequality, $\{f_i\}$ is locally uniformly Hölder continuous and so, up to a further subsequence, $f_i \to f_\infty$ locally uniformly. In particular, f_∞ is continuous and we use the triangle inequality to see that $f_i(x_i) \to f_\infty(x_\infty)$ and $f_i(y_i) \to f_\infty(x_\infty)$ as $i \to \infty$. So, f_∞ is an admissible test function in the definition of $d_{p,U}(x_\infty, y_\infty)$ and

$$\begin{aligned} d_{p,U}(x_\infty, y_\infty) \geq |f_\infty(x_\infty) - f_\infty(y_\infty)| &= \lim_{i\to\infty} |f_i(x_i) - f_i(y_i)| \\ &\geq \lim_{i\to\infty} d_{p,U}(x_i, y_i) - \epsilon. \end{aligned} \tag{7-11}$$

Letting $\epsilon \to 0$, we see that $\lim_{i\to\infty} d_{p,U}(x_i, y_i) \leq d_{p,U}(x_\infty, y_\infty)$ along this subsequence. So, we see that $\lim_{i\to\infty} d_{p,U}(x_i, y_i) = d_{p,U}(x_\infty, y_\infty)$ along this subsequence;

since any subsequence has a further subsequence to which the argument can be applied, we see that the convergence holds for the full sequence. This concludes the proof. □

The next lemma also deals with the d_p–distance on Euclidean space, and establishes uniform continuity of $d_{q,U}$ with respect to q. Again, the notation $d_{q,U}$ is used here to denote $d_{q,g_{\mathrm{euc}},U}$ for a set $U \subset \mathbb{R}^n$.

Lemma 7.4 *Fix $p > n$ and let $U \subset \mathbb{R}^n$ be a bounded open set with smooth boundary. Then $d_{q,U}(x,y)$ converges to $d_{p,U}(x,y)$ uniformly on a precompact set V with $\overline{V} \Subset U$ as $q \to p$.*

Proof We first consider the case $q \to p^-$. By definition, for $q < p$, we have

(7-12) $$d_{p,U}(x,y) \le |U|^{1/q-1/p} d_{q,U}(x,y).$$

In particular, for any $\epsilon > 0$ we have $d_{p,U}(x,y) \le (1+\epsilon) d_{q,U}(x,y)$ for $q < p$ sufficiently close to p. Suppose by way of contradiction that there are $\epsilon_0 > 0$, $x_i, y_i \in V$ and $p_i \to p^-$ such that for all $i \gg 1$,

(7-13) $$d_{p,U}(x_i,y_i) + 2\epsilon_0 \le d_{p_i,U}(x_i,y_i).$$

For each i, let $f_i \in W^{1,p}(U) \cap C^0_{\mathrm{loc}}(U)$ be such that $f_i(y_i) = 0$, $\int_U |\nabla f_i|^{p_i} \le 1$ and

(7-14) $$d_{p_i,U}(x_i,y_i) < |f_i(x_i) - f_i(y_i)| + \epsilon_0.$$

After passing to a subsequence, $x_i, y_i \to x_\infty, y_\infty \in \overline{V}$ by compactness. By Lemma 7.3, $d_{p,U}(x_i,y_i) \to d_{p,U}(x_\infty,y_\infty)$ as $i \to \infty$. For any fixed $q < p$, for i sufficiently large we may apply Hölder's inequality to find

(7-15) $$\int_U |\nabla f_i|^q \, dx \le |U|^{1-q/p_i}.$$

In particular, we see that $\|\nabla f_i\|_{L^q(U)}$ is bounded uniformly in i. Moreover, by the Morrey–Sobolev inequality, $\{f_i\}$ is uniformly Hölder continuous. We therefore see that $f_i \rightharpoonup f_\infty$ in $W^{1,q}(U)$ with $\int_U |\nabla f_\infty|^q \, dx \le 1$ for all $q < p$, and $f_i \to f_\infty$ uniformly. Letting q tend to p, we see that $\int_U |\nabla f_\infty|^p \le 1$, and so f_∞ is an admissible test function for $d_{p,U}(x_\infty,y_\infty)$. Thus, passing to the limit and using (7-13) and (7-14), we reach a contradiction, since

(7-16) $$\begin{aligned} d_{p,U}(x_\infty,y_\infty) + \epsilon_0 \le \lim_{i \to +\infty} |f_i(x_i) - f_i(y_i)| &= |f_\infty(x_\infty) - f_\infty(y_\infty)| \\ &\le d_{p,U}(x_\infty,y_\infty). \end{aligned}$$

Now we consider the case when $q \to p^+$. Again by (7-12), we see that for any $\epsilon > 0$, we have $d_{p,U}(x,y) \geq (1-\epsilon) d_{q,U}(x,y)$ for $q > p$ sufficiently close to p. Suppose by way of contradiction that there exist $p_i > p$, $\epsilon_0 > 0$, $p_i \to p^+$ and $x_i, y_i \in V$ such that for all i,

$$d_{p_i,U}(x_i,y_i) + \epsilon_0 < d_{p,U}(x_i,y_i). \tag{7-17}$$

We may assume $x_i, y_i \to x_\infty, y_\infty \in \overline{V}$. Let σ be a small constant to be determined later. By smooth approximation, let $f \in C^1(\overline{U})$ be such that $\int_U |\nabla f|^p \leq 1 + \sigma\epsilon_0$ and

$$d_{p,U}(x_\infty, y_\infty) \leq |f(x_\infty) - f(y_\infty)| + \sigma\epsilon_0. \tag{7-18}$$

We let $\Lambda = \sup_{\overline{U}} |\nabla f|$. For i sufficiently large, we have

$$\left(\int_U |\nabla f|^{p_i}\, dx\right)^{1/p_i} \leq \Lambda^{(p_i - p)/p_i} \left(\int_U |\nabla f|^p\right)^{1/p_i} \leq (1 + \sigma\epsilon_0). \tag{7-19}$$

So $f/(1+\sigma\epsilon_0)$ is an admissible test function for $d_{p_i,U}(x_i,y_i)$. We thus see that

$$|f(x_i) - f(y_i)| \leq (1+\sigma\epsilon_0) d_{p_i,U}(x_i,y_i). \tag{7-20}$$

So, by (7-17), (7-18) and (7-20), respectively, we find

$$\begin{aligned} \limsup_{i\to\infty} d_{p_i,U}(x_i,y_i) + \epsilon_0 &\leq d_{p,U}(x_\infty,y_\infty) \\ &\leq \lim_{i\to\infty} |f(x_i) - f(y_i)| + \sigma\epsilon_0 \\ &\leq (1+\sigma\epsilon_0) \liminf_{i\to\infty} d_{p_i,U}(x_i,y_i) + \sigma\epsilon_0. \end{aligned} \tag{7-21}$$

We reach a contradiction by choosing σ small enough. □

7.2 Proof of Theorem 1.7

In this section, we prove Theorem 1.7, which we restate below as Theorem 7.5 for the convenience of the reader.

Theorem 7.5 (Theorem 1.7 restated) *Fix $n \geq 2$. For any $\epsilon > 0$ and $p \geq n+1$, there exists a $\delta = \delta(n,\epsilon,p)$ such that the following holds. Let (M^n, g) be a complete Riemannian manifold with bounded curvature satisfying*

$$R \geq -\delta \quad \textit{and} \quad \nu(g,2) \geq -\delta. \tag{7-22}$$

Then for all $x \in M$,

$$d_{\mathrm{GH}}\big((\mathcal{B}_{p,g}(x,1), d_{p,g}), (\mathcal{B}_{p,g_{\mathrm{euc}}}(0,1), d_{p,g_{\mathrm{euc}}})\big) < \epsilon. \tag{7-23}$$

Moreover,

$$(1-\epsilon)|\mathcal{B}_{p,g_{\mathrm{euc}}}(0,r)| \le \mathrm{vol}_g(\mathcal{B}_{p,g}(x_0,r)) \le (1+\epsilon)|\mathcal{B}_{p,g_{\mathrm{euc}}}(0,r)| \tag{7-24}$$

for all $0<r<1$, where $|\cdot|$ denotes the Euclidean volume. In particular, the measure $d\mathrm{vol}_g$ on the metric measure space $(M,d_{p,g},d\mathrm{vol}_g)$ is a doubling measure for all scales $r\le 1$.

Proof of Theorem 7.5 Let $\epsilon'>0$ be a fixed number, to be specified later in the proof. To begin, notice that we may choose $R=R\big(n,\frac{1}{2}p,\epsilon'\big)\ge 2$ according to Lemma 7.1, and $\kappa=\kappa(n,p,\epsilon',R)>1$ sufficiently close to 1 according to Lemma 7.4 so that for any $z,y\in\mathcal{B}_{p,g_{\mathrm{euc}}}(0,2)$, we have

$$\begin{aligned} |d_{p/\kappa,B(0,R)}(z,y)-d_{p,g_{\mathrm{euc}}}(z,y)| &< \epsilon', \\ |d_{\kappa p,B(0,R)}(z,y)-d_{p,g_{\mathrm{euc}}}(z,y)| &< \epsilon'. \end{aligned} \tag{7-25}$$

Up to possibly increasing R depending on p and n, we have $\mathcal{B}_{p,g_{\mathrm{euc}}}(0,2)\subset B\big(0,\frac{1}{2}R\big)$ by (2-26).

Now, choose $\delta=\delta(n,p,\epsilon',\kappa,R)=\delta(n,p,\epsilon')$ sufficiently small that we may apply Theorem 6.1 with $\epsilon=\epsilon'$ and with R as above (see Remark 6.3), obtaining a diffeomorphism $\psi\colon\Omega'\to B(0,R)$ satisfying the properties of Theorem 6.1. We claim that for any $z,y\in\mathcal{B}_{p,g_{\mathrm{euc}}}(0,2)$ we have

$$|d_{p,g}(\psi^{-1}(z),\psi^{-1}(y))-d_{p,g_{\mathrm{euc}}}(z,y)|<\epsilon, \tag{7-26}$$

so, in particular, the diffeomorphism ψ^{-1} is an ϵ–Gromov–Hausdorff approximation between $(\mathcal{B}_{p,g_{\mathrm{euc}}}(0,2),d_{p,\mathrm{euc}})$ and $(\Omega,d_{p,g})$, where $\Omega=\psi^{-1}(\mathcal{B}_{p,g_{\mathrm{euc}}}(0,2))\subset M$.

Fix $z,y\in\mathcal{B}_{p,g_{\mathrm{euc}}}(0,2)$ and set $y_0=\psi^{-1}(y)$, $z_0=\psi^{-1}(z)\in\Omega$ for brevity of notation. Thanks to (7-25), in order to prove (7-26) it suffices to show that

$$d_{p,g}(z_0,y_0)\le(1+3\epsilon')\,d_{\kappa p,B(0,R)}(z,y), \tag{7-27}$$

$$d_{p,g}(z_0,y_0)\ge(1-3\epsilon')d_{p/\kappa,B(0,R)}(z,y). \tag{7-28}$$

We begin by showing (7-27). Let $f\in W^{1,p}(M)\cap C^0(M)$ be a function such that $\int_M|\nabla f|^p\,d\mathrm{vol}_g\le 1$ and

$$d_{p,g}(z_0,y_0)\le(1+\epsilon')|f(z_0)-f(y_0)|. \tag{7-29}$$

Thanks to Theorem 6.1, we find that

$$\begin{aligned} \|\nabla\psi^*f\|_{L^{\kappa p}(B(0,R))} &\le (1+\epsilon')\|\nabla f\|_{L^p(\Omega)} \\ &\le (1+\epsilon')\|\nabla f\|_{L^p(M)}\le 1+\epsilon'. \end{aligned} \tag{7-30}$$

So $h = \psi^* f/(1+\epsilon')$ is an admissible test function for $d_{\kappa p,B(0,R)}(y,z)$, and thus $d_{\kappa p,B(0,R)}(y,z) \ge |h(z)-h(y)|$. Furthermore,

$$|h(z)-h(y)| \ge (1+\epsilon')^{-1}|f(z_0)-f(y_0)| \ge (1+\epsilon')^{-2} d_{p,g}(z_0,y_0). \tag{7-31}$$

This establishes (7-27).

The proof of (7-28) is similar. Fix a function $h \in W^{1,p/\kappa}(B(0,R)) \cap C^0(B(0,R))$ such that $\int_{B(0,R)} |\nabla h|^{p/\kappa}\,dx \le 1$ and

$$d_{p/\kappa,B(0,R)}(z_0,y_0) < (1+\epsilon')|h(z_0)-h(y_0)|. \tag{7-32}$$

By Remark 7.2, we may assume h to be vanishing outside $B(0,R)$ and hence $f = \psi^* h$ can be extended to a function on M. By Theorem 6.1, we have

$$\|\nabla f\|_{L^p(M)} \le (1+\epsilon')\|\nabla h\|_{L^{p/\kappa}(B(0,R))} \le 1+\epsilon'. \tag{7-33}$$

Hence, $f/(1+\epsilon')$ is an admissible test function for $d_{p,g}(y,z)$, and so

$$(1+\epsilon')d_{p,g}(y,z) \ge |f(y)-f(z)| = |h(y_0)-h(z_0)|. \tag{7-34}$$

This establishes (7-28), and hence (7-26). From (7-26), it follows immediately that

$$\psi^{-1}(\mathcal{B}_{p,g_{\mathrm{euc}}}(0,1-8\epsilon')) \subseteq \mathcal{B}_{p,g}(x,1) \subseteq \psi^{-1}(\mathcal{B}_{p,g_{\mathrm{euc}}}(0,1+8\epsilon')), \tag{7-35}$$

and hence (7-23) holds taking $\epsilon' = \frac{1}{8}\epsilon$. Finally, (7-24) is now an immediate consequence of (7-35), Theorem 5.1 and a rescaling argument. This completes the proof of Theorem 7.5. $\square$

Using the same approximation strategy as in obtaining (7-26), we now give the proof of the d_p version of the Myers–Steenrod theorem, namely Proposition 2.36.

Proof of Proposition 2.36 Since the manifolds are d_p–complete this immediately tells us that we have a homeomorphism $\phi\colon M \to N$ such that $d_{p,g}(x,y) = d_{p,h}(\phi(x),\phi(y))$ for all $x, y \in M$. By the Myers–Steenrod theorem, it suffices to show that d_g and d_h are locally isometric. We first show that $d_{p,g}$ is locally a function of d_g.

Claim 1 *For any $x \in M$, we have*

$$\lim_{y\to x} \frac{d_{p,g}(x,y)}{d_g(x,y)^{1-n/p}} = S, \tag{7-36}$$

where S is the Euclidean d_p–distance from the origin to a point on the standard unit sphere; see (2-24).

Proof of Claim 1 Suppose that, on the contrary, we can find a sequence $\{y_i\} \subset M$ with $d_i = d_g(x, y_i) \to 0$ and $\epsilon_0 > 0$ such that

$$\left| \frac{d_{p,g}(x, y_i)}{d_i^{1-n/p}} - S \right| \geq \epsilon_0 > 0. \tag{7-37}$$

Consider the rescaled metric $g_i = d_i^{-2} g$ so that $d_{g_i}(x, y_i) = 1$ for all $i \in \mathbb{N}$ and $d_{p,g_i}(x, y_i) = d_i^{n/p-1} d_{p,g}(x, y_i)$. Clearly, we have $(M, g_i, x) \to (\mathbb{R}^n, g_{\mathrm{euc}}, 0)$ in the C^∞–Cheeger–Gromov sense. By the smooth convergence, the argument used in obtaining (7-26) shows that $d_{p,g_i}(x, y_i) \to S$, which contradicts (7-37). □

In particular, from the claim and the d_p–isometry, we conclude that for any $x \in M$,

$$\lim_{y \to x} \frac{d_h(\phi(x), \phi(y))}{d_g(x, y)} = 1. \tag{7-38}$$

By repeating the argument on ϕ^{-1}, we conclude that ϕ and ϕ^{-1} are both 1–Lipschitz. In particular, this implies that ϕ is a isometry with respect to geodesic distance. Now, the classical Myers–Steenrod theorem implies that ϕ is differentiable at x and satisfies $(\phi^* h)(x) = g(x)$ for all $x \in M$. This completes the proof. □

7.3 Proof of Theorem 1.17

Now we prove the L^∞–Sobolev inequality on manifolds with small entropy and R_-. We restate Theorem 1.17 below as Theorem 7.6.

Theorem 7.6 (L^∞–Sobolev embedding) *Fix $p, q_0 \geq n+1$. Then there exists a $\delta = \delta(n, p, q_0) > 0$ such that the following holds. Let (M^n, g) be a complete Riemannian manifold with bounded curvature satisfying*

$$R \geq -\delta \quad \text{and} \quad \nu(g, 2) \geq -\delta. \tag{7-39}$$

Then for all $x_0 \in M$ and $q \in (n, q_0)$, there exists a $C_{n,q} = C(n, q) > 0$ such that for all $f \in W_0^{1,q}(\mathcal{B}_{p,g}(x_0, 1))$, we have

$$\|f\|_{L^\infty(\mathcal{B}_{p,g}(x_0,1))} \leq C_{n,q} \|\nabla f\|_{L^q(\mathcal{B}_{p,g}(x_0,1))}. \tag{7-40}$$

For all $f \in W^{1,q}(\mathcal{B}_{p,g}(x_0, 1))$ and $x, y \in \mathcal{B}_{p,g}(x_0, 1)$, we have

$$|f(x) - f(y)| \leq C_{n,q} \|\nabla f\|_{L^q(B_{p,g}(x_0,1))}. \tag{7-41}$$

For all $f \in W^{1,q}(M)$, *we have*

$$\|f\|_{L^\infty(M)} \le C_{n,q}(\|\nabla f\|_{L^q(M)} + \|f\|_{L^q(M)}). \tag{7-42}$$

In terms of the d_p*–distance we can upgrade* (7-41) *to a Hölder embedding: there exists an* $\alpha = \alpha(n,q) \in (0,1)$ *such that*

$$|f(x) - f(y)| \le C_{n,q,p} d_p(x,y)^\alpha \|\nabla f\|_{L^q(\mathcal{B}_{p,g}(x_0,1))} \tag{7-43}$$

for all $x, y \in \mathcal{B}_{p,g}(x_0,1)$.

Proof of Theorem 7.6 Fix any $\epsilon \le \frac{1}{4}$, and fix $\delta = \delta(n,p,\epsilon)$ according to Theorem 7.5. As in the proof of Theorem 7.5, we obtain a diffeomorphism $\psi : \mathcal{B}_{p,g}(x_0, 5+\epsilon) \to \tilde{\Omega}$, where $\mathcal{B}_{p,\mathrm{euc}}(0,5) \subset \tilde{\Omega} \subset \mathcal{B}_{p,\mathrm{euc}}(0,5+2\epsilon)$, satisfying the properties of Theorem 6.1. Recall from (2-26) that there exists an $R = R(n)$ such that $\mathcal{B}_{p,\mathrm{euc}}(0,6) \subset B(0,R)$ for all $p \ge n+1$.

Let $f \in W_0^{1,q}(\mathcal{B}_{p,g}(x_0,1))$ and extend f by zero to be defined in all of $\tilde{\Omega}$. Let $h = \psi_* f$ and then naturally extend h by zero to be defined on $B(0,R)$. By (6-4) of Theorem 6.1 and Remark 6.3, for any $q \in (\kappa n, q_0)$ we have

$$\|\nabla h\|_{L^{q/\kappa}(B(0,R))} \le (1+\epsilon)\|\nabla f\|_{L^q(\mathcal{B}_{p,g}(x_0,1))}. \tag{7-44}$$

So, applying the Morrey–Sobolev embedding on $B(0,R)$ followed by (7-44), we have

$$\begin{aligned}\|f\|_{L^\infty(\mathcal{B}_{p,g}(x_0,1))} = \|h\|_{L^\infty(B(0,R))} &\le \tilde{C}_{n,q}\|\nabla h\|_{L^{q/\kappa}(B(0,R))} \\ &\le C_{n,q}\|\nabla f\|_{L^q(\mathcal{B}_{p,g}(x_0,1))}.\end{aligned} \tag{7-45}$$

This completes the proof of (7-40).

To prove (7-41) consider any function $f \in W^{1,q}(\mathcal{B}_{p,g}(x_0,1))$ and let $h = \psi_* f$ be the function defined on $\tilde{\Omega}' = \psi(\mathcal{B}_{p,g}(x_0,1))$. By the Morrey–Sobolev inequality on Euclidean space, for all $x, y \in \mathcal{B}_{p,g}(x_0,1)$ we have

$$\begin{aligned}|f(x) - f(y)| = |h(\psi(x)) - h(\psi(y))| &\le C_{n,q}\|\nabla h\|_{L^{q/\kappa}(\tilde{\Omega}')} \\ &\le C_{n,q}\|\nabla f\|_{L^q(\mathcal{B}_{p,g}(x_0,1))}.\end{aligned} \tag{7-46}$$

In fact, we may apply the Hölder Morrey–Sobolev embedding on $\tilde{\Omega}'$ in (7-46) above to see that

$$\begin{aligned}|f(x) - f(y)| &= |h(\psi(x)) - h(\psi(y))| \\ &\le C_{n,q}|\psi(x) - \psi(y)|^{1-n\kappa/q}\|\nabla h\|_{L^{q/\kappa}(\tilde{\Omega}')} \\ &\le C_{n,q}|\psi(x) - \psi(y)|^{1-n\kappa/q}\|\nabla f\|_{L^q(\mathcal{B}_{p,g}(x_0,1))}.\end{aligned} \tag{7-47}$$

Hence it suffices to show that $|\psi(x)-\psi(y)| \le C d_{p,g}(x,y)$ for some $C(n,p)>0$ for $x, y \in \mathcal{B}_{p,g}(x_0,1)$. Since $\tilde{\Omega} \supset \mathcal{B}_{p,\mathrm{euc}}(5)$, and by the proof of Theorem 7.5, we get that $\psi(x), \psi(y) \in \mathcal{B}_{p,\mathrm{euc}}(2) = B(0,(2S^{-1})^{p/(p-n)})$, where $S=S(n,p)$ is given by (2-25). Consider the test function $\varphi(z) = \min\{|\psi(x)-\psi(z)|, 2(2S^{-1})^{p/(p-n)}\}$, which is compactly supported on $\tilde{\Omega}$. By (6-4) of Theorem 6.1 and Remark 6.3, we see that $C\varphi$ is an admissible function for $d_{p,g}(x,y)$ for some $C(n,p)>0$. The claim follows.

To prove (7-42), let $f \in W^{1,q}(M)$. Fix $x_0 \in M$ and let ψ be the diffeomorphism obtained above such that $\psi(x_0)=0$. Let $h=\psi_* f$ be the pushforward of f, which is defined on $\tilde{\Omega}$. Let φ be a cutoff function such that $\varphi \equiv 1$ on $\mathcal{B}_{p,\mathrm{euc}}\big(0,\frac{1}{2}\big)$, φ vanishes outside $\mathcal{B}_{p,\mathrm{euc}}(0,1)$ and $|\partial\varphi| \le C_n$; recall (2-25). Then by the Morrey–Sobolev inequality on Euclidean space followed by Theorem 6.1, we have

$$
\begin{aligned}
|f(x_0)| &\le \|h\varphi\|_{L^\infty(\mathcal{B}_{p,\mathrm{euc}}(0,1))} \\
&\le C_{n,q}\|\nabla(h\varphi)\|_{L^{q/\kappa}(\mathcal{B}_{p,\mathrm{euc}}(0,1))} \\
&\le C_{n,q}\big(\|\nabla h\|_{L^{q/\kappa}(\mathcal{B}_{p,\mathrm{euc}}(0,1))} + \|h\|_{L^{q/\kappa}(\mathcal{B}_{p,\mathrm{euc}}(0,1))}\big) \\
&\le C_{n,q}(\|\nabla f\|_{L^q(M)} + \|f\|_{L^q(M)}).
\end{aligned}
\tag{7-48}
$$

Since x_0 is arbitrarily chosen, this completes the proof. □

8 Global convergence theorem

In this section we prove Theorem 1.15, which we restate below as Theorem 8.1.

Theorem 8.1 (Theorem 1.15, restated) *For $p \ge n+1$, there exists a $\delta=\delta(n,p)>0$ such that if $\{(M_i,g_i,x_i)\}$ is a sequence of complete pointed Riemannian manifolds with bounded curvature such that*

$$
R_{g_i} \ge -\delta \quad \text{and} \quad \nu(g_i,2) \ge -\delta,
\tag{8-1}
$$

then there exists a pointed rectifiable Riemannian space $(M_\infty,g_\infty,x_\infty)$, with M_∞ topologically a smooth manifold, such that the following holds:

1. *After passing to a subsequence, we have $d_p((M_i,g_i,x_i),(M_\infty,g_\infty,x_\infty)) \to 0$ in the sense of Definition 2.39.*
2. *The space $(X_\infty,g_\infty,x_\infty)$ is $W^{1,p}$–rectifiably complete and d_p–rectifiably complete in the sense of Definitions 2.24 and 2.35, respectively.*

This section is organized in the following way. In Section 8.1, we construct the rectifiable Riemannian space that will ultimately be shown to be the pointed d_p limit in Theorem 8.1. Then, in Section 8.2, we show that this limit is $W^{1,p}$–rectifiably complete. In Section 8.3, we establish the pointed d_p convergence of the sequence, and finally in Section 8.4 we put these pieces together to conclude the proof of Theorem 8.1.

8.1 Constructing the limit space

We first obtain the rectifiable Riemannian space $(M_\infty, g_\infty, x_\infty)$ that will ultimately be the pointed d_p limit, and establish the integral convergence of the metric tensors to this limit. For notational convenience, all convergence means subsequence convergence.

Proposition 8.2 *Fix $P \ge n+1$. There exists $\delta = \delta(n, P) > 0$ such that the following holds. Suppose $\{(M_i^n, g_i, x_i)\}$ is a sequence of complete pointed Riemannian manifolds with bounded curvature satisfying*

$$R_{g_i} \ge -\delta \quad \text{and} \quad \nu(g_i, 2) \ge -\delta. \tag{8-2}$$

Then there exists a pointed rectifiable Riemannian space $(M_\infty, g_\infty, x_\infty)$, with M_∞ topologically a smooth manifold, such that up to a subsequence, the following holds. For any compact subset $\Omega \Subset M_\infty$, we can find subsets $\Omega_i \Subset M_i$ and diffeomorphisms $\psi_{i,\Omega}\colon \Omega \to \Omega_i$ such that $\psi_{i,\Omega}(x_\infty) = x_i$ and

$$\|\psi_{i,\Omega}^* g_i - g\|_{L^q(\Omega)} \to 0 \tag{8-3}$$

for any $q \in [1, P]$.

Proof We proceed in several steps. The first three steps involve constructing the pointed rectifiable Riemannian space $(M_\infty, g_\infty, x_\infty)$ that will later be shown to be the d_p limit of (M_i, g_i, x_i), while in the fourth step we establish the convergence of the metric tensors (8-3).

Step 1 (constructing the smooth pointed topological manifold (M_∞, x_∞)) Fix $\lambda > 0$ to be specified later in the proof. By Theorem 3.2 and Perelman's no-local-collapsing theorem (3-23), if $\delta = \delta(n, \lambda)$ is taken sufficiently small, then there is a sequence of complete Ricci flow solutions $\{(M_i, g_i(t))\}_{t\in[0,1]}$ such that $g_i(0) = g_i$ and

$$\begin{cases} |\mathrm{Rm}(g_i(t))| \le \lambda t^{-1}, \\ \mathrm{inj}(g_i(t)) \ge c_n \sqrt{t}. \end{cases} \tag{8-4}$$

By Hamilton's compactness theorem [32], after passing to a subsequence, we get that $(M_i, g_i(t), x_i) \to (M_\infty, g_\infty(t), x_\infty)$ in the pointed C^∞–Cheeger–Gromov sense so that $g_\infty(t)$ is a solution to the Ricci flow on $M_\infty \times (0,1]$ also satisfying (8-4).

Step 2 (constructing the rectifiable Riemannian metric g_∞ and measure m_∞) We now construct a rectifiable Riemannian metric as an L^P limit of $g_\infty(t)$ as t tends to zero.

From the previous step, for any precompact set $\Omega \Subset M_\infty$ containing x_0, we can find a sequence of maps $\psi_{i,\Omega}\colon \Omega \to M_i$, each a diffeomorphism onto its image, such that $\psi_{i,\Omega}(x_\infty) = x_i$ and $\psi_i^* g_i(t) \to g_\infty(t)$ smoothly on any compact subsets of $M_\infty \times (0,1]$. For notational convenience, we will use g_i and $g_i(t)$ to denote $\psi_{i,\Omega}^* g_i$ and $\psi_{i,\Omega}^* g_i(t)$, respectively. By way of a covering argument, it suffices to consider the case $\Omega = B_{g_\infty(1)}(x_\infty, 1)$.

Fix any $\eta > 0$. By (6-23) of Corollary 6.4 (taking $\epsilon = \frac{1}{10}$, $\kappa = 2$ and $2P$ in place of P), we may find $t_\eta = t_\eta(n, P, \eta) \in (0,1)$ such that for all $s, t \in (0, t_\eta)$, we have

$$\int_\Omega |g_i(s) - g_i(t)|^P_{g_i(1)} \, d\mathrm{vol}_{g_i(1)} \le \eta. \tag{8-5}$$

So, passing to the smooth limit as $i \to \infty$, we find that

$$\int_\Omega |g_\infty(s) - g_\infty(t)|^P_{g_\infty(1)} \, d\mathrm{vol}_{g_\infty(1)} \le \eta \tag{8-6}$$

for all $s, t \in (0, t_\eta)$. Then we see that $g_\infty(t)$ is a Cauchy sequence and hence $g_\infty = \lim_{t\to 0^+} g_\infty(t)$ exists in $L^P(\Omega, g_\infty(1))$. Next, by letting $\Omega_j \Subset M_\infty$ be an exhaustion of M_∞, we define g_∞ on the whole M_∞. With g_∞ in hand, we define the measure m_∞ on M_∞ by setting m_∞ to be the "induced measure" of g_∞. More precisely, consider the function $f(x) = (\det g_\infty / \det g_\infty(1))^{1/2}$ on M_∞. Thanks to the L^P_{loc}–convergence of $g_\infty(t)$ to g_∞ as $t \to 0$, f is coordinate-free and is in $L^P_{\mathrm{loc}}(M_\infty)$ for $P > 1$. Now we define $m_\infty = f \, d\mathrm{vol}_{g_\infty(1)}$, which is a well-defined Radon measure on M_∞. This completes the construction.

Step 3 (verifying that $(M_\infty, g_\infty, x_\infty)$ is a rectifiable Riemannian space) Now we claim that $(M_\infty, g_\infty, x_\infty)$ is a rectifiable Riemannian space. To construct a rectifiable atlas for $(M_\infty, g_\infty, x_\infty)$, let $\{x_j\} \subset M_\infty$ be a collection of points such that $\{B_{g_\infty(1)}(x_j, \frac{1}{2})\}$ covers M_∞ and $\{B_{g_\infty(1)}(x_j, \frac{1}{4})\}$ are pairwise disjoint. By taking $\lambda > 0$ sufficiently small in Step 1, we may apply Lemma 3.7 and (8-5) to obtain charts $\phi_j : U_j \to B_{g_\infty(1)}(x_j, 2)$, where $U_j \subset \mathbb{R}^n$ is such that

$$\int_{U_j} |\phi_j^* g_\infty - g_{\mathrm{euc}}|^P_{g_{\mathrm{euc}}} \, dx \le C. \tag{8-7}$$

Let $U_{a,j} = \{x \in U_j : |\phi_j^* g_\infty - g_{\mathrm{euc}}|_{g_{\mathrm{euc}}} \le a\}$ and let $\phi_{a,i}\colon U_{a,j} \to M_\infty$ be defined by $\phi_j|_{U_{a,j}}$. We easily check from (8-7) that $\{(U_{a,j}, \phi_{a,j})\}_{a,i\in\mathbb{N}}$ is a rectifiable atlas for $(M_\infty, x_\infty, m_\infty)$ and that g_∞ is a rectifiable Riemannian metric with respect to this rectifiable atlas, by the construction of g_∞ and m_∞.

Step 4 (convergence) Finally, the L^P–convergence (8-3) of $g_i \to g_\infty$ follows from the L^P–convergence in (8-5) and a diagonal subsequence argument. This completes the proof of the proposition. □

Remark 8.3 From the proofs of Proposition 8.2 and Theorem 6.1, we may immediately deduce the following properties of the limit space $(M_\infty, g_\infty, x_\infty)$ constructed in Proposition 8.2. Given any $x \in M_\infty$, we may apply Lemma 3.7 to obtain a diffeomorphism $\phi\colon B(0,2) \to \Omega' \subset M_\infty$ such that $\phi(0) = x$ and $\Omega := \phi(B(0,1))$ satisfies $B_{g_\infty(1)}(x, 1-\epsilon) \subset \Omega \subset B_{g_\infty(1)}(x, 1-\epsilon)$. In a slight abuse of notation, we identify $\phi^* g$ and $\phi^* g_k$ with g and g_k, respectively. We have the following estimates:

$$\text{(8-8)}\qquad \begin{aligned} \int_B |g_\infty - g_k|^P_{\mathrm{euc}} \to 0, &\qquad \int_B |g_\infty^{-1} - g_k^{-1}|^P_{\mathrm{euc}} \to 0,\\ \int_B |g_\infty g_k^{-1} - \mathrm{Id}|^P_{\mathrm{euc}} \to 0, &\qquad \int_B |g_k g^{-1} - \mathrm{Id}|^P_{\mathrm{euc}} \to 0. \end{aligned}$$

Here the measure of integration can be taken to be dx, $d\mathrm{vol}_g$ or $d\mathrm{vol}_{g_k}$. Furthermore, for any fixed $p \ge n+1$ and for $\kappa^2 = (p+n)/2n > 1$, we may choose δ in Proposition 8.2 additionally depending on p and κ so that

$$\text{(8-9)}\quad (1-\epsilon)\|f\|_{L^{p/\kappa}(B(0,1),g_{\mathrm{euc}})} \le \|f\|_{L^p(B(0,1),g_\infty)} \le (1+\epsilon)\|f\|_{L^{\kappa p}(B(0,1),g_{\mathrm{euc}})}.$$

Similarly, we may replace g_{euc} with g_k above.

8.2 $W^{1,p}$–rectifiable completeness of the limit space

In this section, we prove that the limiting rectifiable Riemannian space $(M_\infty, g_\infty, x_\infty)$ obtained in Proposition 8.2 is $W^{1,p}$–rectifiably complete as in Definition 2.24. Given $1 < p < \infty$ fixed, we let $W^{1,p}(M_\infty, g_\infty)$ denote the Sobolev space as defined in Section 2.1.2. For any function $u \in W^{1,p}(M_\infty, g_\infty)$, we let $G_{M_\infty, g_\infty, u}$ denote the least p–weak upper gradient of u, whose existence is guaranteed by Proposition 2.23. We will show that the function $u_a = \phi_a^* u$ is differentiable a.e. in U_a, and thus we may let $|\nabla_{g_\infty} u|\colon X \to \mathbb{R}$ be the function defined in charts by

$$\text{(8-10)}\qquad |\nabla_{g_\infty} u|(\phi_a(x)) = \bigl(g_\infty^{ij}\partial_i u_a(x)\partial_j u_a(x)\bigr)^{1/2}$$

for a.e. $x \in U_a$.

Proposition 8.4 *Fix $p \geq n+1$. By choosing $P = P(p,n)$ sufficiently large and thus $\delta = \delta(n,p)$ sufficiently small in Proposition 8.2, the limiting rectifiable Riemannian space $(M_\infty, g_\infty, x_\infty)$ constructed in Proposition 8.2 is $W^{1,p}$–rectifiably complete in the sense of Definition 2.24. That is:*

(a) *$W^{1,p}(M_\infty, g_\infty)$ is dense in $L^p(M_\infty, g_\infty)$.*

(b) *For any $u \in W^{1,p}(M_\infty, g_\infty)$, the function $u_a = \phi_a^* u$ is weakly differentiable in U_a, and thus the function $|\nabla_{g_\infty} u|$ in (8-10) is defined for m–a.e. $x \in M_\infty$. Moreover, the least weak upper gradient satisfies $G_{M_\infty, g_\infty, u} = |\nabla_{g_\infty} u|$ for m–a.e. $x \in M_\infty$.*

More precisely, in Proposition 8.4(b), we mean the following when we say that u_a is weakly differentiable in U_a (recall that U_a may not be an open set). From Proposition 8.2 and Step 4 of its proof, in which we construct the rectifiable atlas for (M_∞, g_∞), we see that we have a classical atlas of charts $\{\widehat{\phi}_j\}$ for the smooth manifold M_∞, and these charts agree with the charts $\{\phi_a\}$ of the rectifiable atlas where they are both defined. The function $\widehat{\phi}_j^* u$, defined on an open ball in Euclidean space and agreeing with $u_a = \phi_a^* u$ on the intersection of their domains, has weak partial derivatives in L^p in the sense of Sobolev spaces, defined for instance in [22, Chapter 4].

We will first establish a localized version of Proposition 8.4(b) in Section 8.2.1 below (Proposition 8.5). Using this proposition, we will then prove Proposition 8.4 in Section 8.2.2.

8.2.1 The local estimate In this section, we will prove the following local version of Proposition 8.4(b). In order to alleviate notation, we let $g = g_\infty$. Fix $x \in M_\infty$ and let $\phi\colon B(0,2) \to \Omega'$ with $\phi(0) = x$ be as in Remark 8.3. As in Remark 8.3, we slightly abuse notation by identifying $\phi^* g$ and $\phi^* g_k$ with g and g_k, respectively. We set $B = B(0,1)$ and denote by $W^{1,p}(B, g)$ the $W^{1,p}$ space as defined in Section 2.1.2 with respect to the metric g on the space B. We let $G_{B,g,u}$ denote the least p–weak upper gradient of u in $W^{1,p}(B, g)$ and $|\dot\gamma|_g = g(\dot\gamma, \dot\gamma)^{1/2}$. We will use the analogous notation for g_k and g_{euc}.

Proposition 8.5 *Fix $p \geq n+1$. By choosing $P = P(n,p)$ sufficiently large and thus $\delta = \delta(n,p)$ sufficiently small in Proposition 8.2, any $u \in W^{1,p}(B, g)$ is differentiable m–a.e. and we have $G_{B,g,u} = |\nabla_g u|$ for m–a.e. $x \in B$.*

In preparation for the proof of Proposition 8.5, we first prove some preliminary lemmas. Let $\mathfrak{M}_g$ and $\mathfrak{M}_{\mathrm{euc}}$ denote the collection of all absolutely continuous curves with respect to g and g_{euc}, respectively, in B. Note that $\mathfrak{M}_{\mathrm{euc}}$ is also the collection of g_k absolutely continuous curves in B for every k.

Lemma 8.6 *Fix P as in Proposition 8.2. For any $1 \le p \le \frac{1}{2}P$, we have*

$$\mathrm{Mod}_{g,p}(\mathfrak{M}_g \setminus \mathfrak{M}_{\mathrm{euc}}) = 0 \quad \textit{and} \quad \mathrm{Mod}_{g_{\mathrm{euc}},p}(\mathfrak{M}_{\mathrm{euc}} \setminus \mathfrak{M}_g) = 0.$$

Proof Consider the family $\Gamma \subset \mathfrak{M}_g$ of curves $\gamma \colon [a,b] \to B(0,1)$ such that

$$\int_a^b |g^{-1}(\gamma)|_{\mathrm{euc}}^{1/2} |\dot{\gamma}|_g \, dt = +\infty.$$

Because $|g^{-1}|_{\mathrm{euc}}^{1/2} \in L^{P/2}(B,g)$ by (8-8), it follows from the definition of Mod_p that $\mathrm{Mod}_{g,P/2}(\Gamma) = 0$. Notice that $\mathfrak{M}_g \setminus \mathfrak{M}_{\mathrm{euc}} \subset \Gamma$. Indeed, for any $\gamma \in \mathfrak{M}_g \setminus \mathfrak{M}_{\mathrm{euc}}$, we apply the Cauchy–Schwarz inequality to find that

$$\int_a^b |g^{-1}(\gamma)|_{\mathrm{euc}}^{1/2} |\dot{\gamma}|_g \, dt \ge \int_a^b |\dot{\gamma}(t)|_{\mathrm{euc}} \, dt = +\infty,$$

and so $\gamma \in \Gamma$. It follows that $\mathrm{Mod}_{g,p}(\mathfrak{M}_g \setminus \mathfrak{M}_{\mathrm{euc}}) = 0$. The other claim is proven analogously. □

In view of Lemma 8.6, we will henceforth let $\mathfrak{M} = \mathfrak{M}_g \cap \mathfrak{M}_{\mathrm{euc}}$ and will restrict our attention to curves in $\mathfrak{M}$.

Lemma 8.7 *Fix $p \ge n+1$ and let $\kappa^2 = (p+n)/2n > 1$. By choosing $P = P(n,p)$ sufficiently large and thus $\delta = \delta(n,p)$ sufficiently small in equation (9-3), for any family of curves $\Gamma \subset \mathfrak{M}$ with $\mathrm{Mod}_{g,p}(\Gamma) = 0$ we have $\mathrm{Mod}_{g_k,p/\kappa}(\Gamma) = \mathrm{Mod}_{g_{\mathrm{euc}},p/\kappa}(\Gamma) = 0$.*

Proof Let $\Gamma \subset \mathfrak{M}$ be a family of curves with $\mathrm{Mod}_{g,p}(\Gamma) = 0$. By definition, there exists an $F \ge 0$ such that $F \in L^p(B,g)$ and $\int_a^b F(\gamma)|\dot{\gamma}|_g \, dt = +\infty$ for all $g \in \Gamma$. For each k, consider the function $F_k = F|g_k g^{-1}|_{\mathrm{euc}}^{1/2}$. By Remark 8.3, we have $F_k \in L^{p/\kappa}(B, g_k)$ provided P is chosen sufficiently large depending on p and n. Then

$$\int_a^b F_k(\gamma)|\dot{\gamma}|_{g_k} \, dt \ge \int_a^b F(\gamma)|\dot{\gamma}|_g \, dt = +\infty,$$

and so $\mathrm{Mod}_{g_k,p/\kappa}(\Gamma) = 0$. The same proof holds for g_{euc} letting $F_{\mathrm{euc}} = F|g^{-1}|_{\mathrm{euc}}^{1/2}$. □

Lemma 8.8 *Fix $p \geq n+1$ and let $\kappa^2 = (p+n)/2n > 1$. By choosing $P = P(n,p)$ sufficiently large and thus $\delta = \delta(n,p)$ sufficiently small in equation (9-3), we have $W^{1,\kappa p}(B, g_{\mathrm{euc}}) \subset W^{1,p}(B,g) \subset W^{1,p/\kappa}(B, g_{\mathrm{euc}})$.*

Proof We prove the second inclusion; the proof of the first inclusion is analogous. Fix $u \in W^{1,p}(B,g)$, so by definition we have

$$(8\text{-}11)\quad |u(\gamma(a)) - u(\gamma(b))| \leq \int_a^b G_{B,g,u}(\gamma)|\dot{\gamma}|_g \, dt \leq \int_a^b G_{B,g,u}(\gamma)|g|_{\mathrm{euc}}^{1/2}|\dot{\gamma}|_{g_{\mathrm{euc}}} \, dt$$

for all $\gamma \in \mathfrak{M} \setminus \Gamma$, where $\Gamma \subset \mathfrak{M}$ is a family of curves such that $\mathrm{Mod}_{g,p}(\Gamma) = 0$. Letting $H = G_{B,g,u}|g|_{\mathrm{euc}}^{1/2}$, we directly see that H satisfies the weak upper gradient condition for u with respect to g_{euc}. Moreover, by Lemma 8.6, we see that $\mathrm{Mod}_{g_{\mathrm{euc}},p/\kappa}(\Gamma) = 0$. This implies that H is a p/κ–weak upper gradient for u with respect to g_{euc}. Moreover, we deduce from Remark 8.3 that $H \in L^{p/\kappa}(B, g_{\mathrm{euc}})$, and so $u \in W^{1,p/\kappa}(B, g_{\mathrm{euc}})$. □

In Lemma 8.8, we have used the Newtonian space definition of $W^{1,p/\kappa}(B, g_{\mathrm{euc}})$. It is known (see [29, Theorem 7.13]) that this space agrees with the typical definitions of Sobolev spaces on Euclidean space.

Lemma 8.9 *Fix $1 \leq q \leq p < \infty$. Let $u \in W^{1,p}(B,g)$ and let G be a q–weak upper gradient of u. Then G is a p–weak upper gradient of u.*

Proof As usual, let $G_{B,g,u}$ denote the least p–weak upper gradient of u. We will show that $G \geq G_{B,g,u}$ for m–a.e. $x \in B$, which implies directly that G is a p–weak upper gradient. To this end, we employ a trick from [29, Lemma 6.3]. Since G is a q–weak upper gradient for u, we know that the weak upper gradient condition is satisfied for G for all $\gamma \in \mathfrak{M} \setminus \Gamma$, where $\mathrm{Mod}_{g,q}(\Gamma) = 0$. By definition, there exists a function $F \in L^q(B,g)$ such that $\int_a^b F(\gamma)|\dot{\gamma}|_g \, dt = +\infty$ for all $g \in \Gamma$. Therefore, the function $G_k = G + F/k$ is an upper gradient of u, and in particular, a p–weak upper gradient for u. Therefore, $G_k \geq G_{B,g,u}$ for m–a.e. $x \in B$. Now, since F is finite m–a.e., we have $G_k \to G$ and thus $G \geq G_{B,g,u}$ for m–a.e. $x \in B$. □

We are now ready to prove Proposition 8.5.

Proof of Proposition 8.5 We consider two cases: first, the case when u is a smooth function in B, and then the general case when $u \in W^{1,p}(B,g)$. In each case, we proceed in two steps: first showing that $|\nabla_g u|$ is a p–weak upper gradient of u with respect to g, and then showing that it is the least p–weak upper gradient of u with respect to g.

Case 1 $u \in C^\infty(B, g_{\mathrm{euc}})$.

Step 1 *$|\nabla_g u|$ is a p–weak upper gradient for u with respect to g.*

The main estimate toward showing Step 1 is the following. Up to passing to a subsequence, we have

$$\int_a^b |\nabla_{g_k} u|(\gamma)|\dot\gamma|_{g_k}\,dt \to \int_a^b |\nabla_g u|(\gamma)|\dot\gamma|_g\,dt \tag{8-12}$$

for every curve $\gamma\colon [a,b] \to B$ in $\mathfrak{M} \setminus \Gamma$, where Γ is a family of curves such that $\mathrm{Mod}_{g,p}(\Gamma) = 0$. Once we establish this fact, it will easily follow that $|\nabla_g u|$ is a p–weak upper gradient for u with respect to g. Indeed, since g_k is a smooth metric for each k, we know that $|\nabla_{g_k} u|$ is the least upper gradient for u with respect to g_k. So, for any $\gamma \in \mathfrak{M} \setminus \Gamma$, we have

$$|u(\gamma(a)) - u(\gamma(b))| \le \int_a^b |\nabla_{g_k} u|(\gamma)|\dot\gamma|_{g_k}\,dt \to \int_a^b |\nabla_g u|(\gamma)|\dot\gamma|_g\,dt \tag{8-13}$$

as $k \to \infty$. Since $\mathrm{Mod}_{g,p}(\Gamma) = 0$, it follows that $|\nabla_g u|$ is a p–weak upper gradient for u with respect to g.

We employ Lemma 2.17 in order to prove (8-12). More specifically, for any curve $\gamma \in \mathfrak{M}$, we have

$$\left| \int_a^b |\nabla_{g_k} u|(\gamma)|\dot\gamma|_{g_k}\,dt - \int_a^b |\nabla_g u|(\gamma)|\dot\gamma|_g\,dt \right| \le \mathrm{I} + \mathrm{II},$$

where

$$\begin{aligned} \mathrm{I} &= \left| \int_a^b (|\nabla_{g_k} u| - |\nabla_g u|)(\gamma)|\dot\gamma|_{g_k}\,dt \right|, \\ \mathrm{II} &= \left| \int_a^b |\nabla_g u|(\gamma)(|\dot\gamma|_{g_k} - |\dot\gamma|_g)\,dt \right|. \end{aligned} \tag{8-14}$$

We claim that terms I and II tend to zero as $k \to \infty$ for all $\gamma \in \mathfrak{M} \setminus \Gamma$, where $\mathrm{Mod}_{g,p}(\Gamma) = 0$. Indeed, consider the sequence of functions

$$F_k = \big| |\nabla_{g_k} u| - |\nabla_g u| \big| \, |g_k g^{-1}|_{\mathrm{euc}}^{1/2},$$

and notice that $\mathrm{I} \le \int_a^b F_k(\gamma)|\dot\gamma|_g\,dt$. Thanks to (8-8), we see that $F_k \to 0$ in $L^p(B, g)$, provided P is large enough depending on p. Then Lemma 2.17 implies that, after passing to a subsequence, $\mathrm{I} \to 0$ for all $\gamma \in \mathfrak{M} \setminus \Gamma_{\mathrm{I}}$, where $\Gamma_{\mathrm{I}} \subset \mathfrak{M}$ is a family of curves with $\mathrm{Mod}_{g,p}(\Gamma_{\mathrm{I}}) = 0$.

We argue in a similar fashion for term II. Consider the sequences

$$\widetilde{F}_k = |\nabla_g u|(|g_k g^{-1}|_{\text{euc}}^{1/2} - 1) \quad \text{and} \quad \widehat{F}_k = |\nabla_g u|(1 - |g g_k^{-1}|_{\text{euc}}^{-1/2}).$$

Since $|\nabla_g u| \geq 0$, we have

$$\int_a^b |\nabla_g u|(\gamma)(|\dot\gamma|_{g_k} - |\dot\gamma|_g)\,dt \leq \int_a^b \widetilde{F}_k(\gamma)|\dot\gamma|_g\,dt,$$
$$\int_a^b |\nabla_g u|(\gamma)(|\dot\gamma|_g - |\dot\gamma|_{g_k})\,dt \leq \int_a^b \widehat{F}_k(\gamma)|\dot\gamma|_g\,dt.$$

Furthermore, thanks to (8-8), we see that $\widetilde{F}_k, \widehat{F}_k \to 0$ in $L^p(B,g)$ provided that P is chosen sufficiently large depending on p. So, by Lemma 2.17, we see that, after passing to a further subsequence, we have $\mathrm{II} \to 0$ for all $\gamma \in \mathfrak{M} \setminus \Gamma_{\mathrm{II}}$ where $\mathrm{Mod}_{g,p}(\Gamma_{\mathrm{II}}) = 0$. Letting $\Gamma = \Gamma_{\mathrm{I}} \cup \Gamma_{\mathrm{II}}$, we conclude that (8-12) holds. This completes Step 1.

Step 2 *$|\nabla_g u|$ is the least p–weak upper gradient of u with respect to $W^{1,p}(B,g)$.*

Let $H \in L^p(B,g)$ be any p–weak upper gradient of u with respect to g. By definition, we have

$$|u(\gamma(a)) - u(\gamma(b))| \leq \int_a^b H(\gamma)|\dot\gamma|_g\,dt \leq \int_a^b H|g\,g_k^{-1}|_{\text{euc}}^{1/2}|\dot\gamma|_{g_k}\,dt \tag{8-15}$$

for all $\gamma \in \mathfrak{M} \setminus \Gamma$, where Γ is a family of curves such that $\mathrm{Mod}_{g,p}(\Gamma) = 0$. In particular, letting $H_k = H|g\,g_k^{-1}|_{\text{euc}}^{1/2}$, we see immediately that H_k satisfies the upper gradient condition for every $\gamma \in \mathfrak{M} \setminus \Gamma$. By Lemma 8.6, we see that $\mathrm{Mod}_{g_k,p/\kappa}(\Gamma) = 0$, and thus H_k is a p/κ–weak upper gradient for u with respect to g_k. Moreover, it follows from Remark 8.3 that $H_k \in L^{p/\kappa}(B,g) \cap L^{p/\kappa}(B,g_k)$ and that $H_k \to H$ in $L^{p/\kappa}(B,g)$. In particular, $H_k \to H$ for m–a.e. $x \in B$.

Similarly, since $u \in C^\infty(B, g_{\text{euc}})$, we deduce from (8-8) that $|\nabla_{g_k} u| \to |\nabla_g u|$ in $L^p(B,g)$, and therefore m–a.e. Since $|\nabla_{g_k} u|$ is the least q–weak upper gradient for u with respect to g_k for any q, we have that $|\nabla_{g_k} u| \leq H_k$ m–a.e. Passing $k \to \infty$, it follows that $|\nabla_g u| \leq H$, and so $|\nabla_g u|$ is the least p–weak upper gradient for u with respect to g.

Case 2 $u \in W^{1,p}(B,g)$.

Step 1 *$|\nabla_g u|$ is a p–weak upper gradient of u in $W^{1,p}(B,g)$.*

By Lemma 8.8, we know that $u \in W^{1,p/\kappa}(B, g_{\text{euc}})$, and thus we may find a sequence $u_i \in C^\infty(B, g_{\text{euc}})$ such that $u_i \to u$ in $W^{1,p/\kappa}(B, g_{\text{euc}})$. So, using Remark 8.3, we see that $|\nabla_g u_i| \to |\nabla_g u|$ in $L^q(B,g)$ for $q = p/\kappa^2$. Therefore, applying Lemma 2.17 once again, after passing to a subsequence, we have

$$\int_a^b |\nabla_g u_i|(\gamma)|\dot\gamma|_g \, dt \to \int_a^b |\nabla_g u|(\gamma)|\dot\gamma|_g \, dt \tag{8-16}$$

as $i\to\infty$ for all $\gamma \in \mathfrak{M}\setminus\Gamma_0$, where $\mathrm{Mod}_{g,q}(\Gamma_0)=0$. Moreover, applying Case 1 to each smooth function u_i, we know that $|\nabla_g u_i|$ is a q–weak upper gradient for u_i and thus

$$|u_i(\gamma(a)) - u_i(\gamma(b))| \le \int_a^b |\nabla_g u_i|(\gamma)|\dot\gamma|_g \, dt \tag{8-17}$$

for all $\gamma \in \mathfrak{M}\setminus\Gamma_i$, where Γ_i is a family of curves such that $\mathrm{Mod}_{g,q}(\Gamma_i)=0$. Now, since $p/\kappa > n$, we have that $u_i \to u$ uniformly by the Morrey–Sobolev inequality on Euclidean space. So, letting $\Gamma = \Gamma_0 \cup \bigcup \Gamma_i$, we see that $\mathrm{Mod}_{g,q}(\Gamma) = 0$, and letting $i\to\infty$ in (8-17), we find that $|u(\gamma(a)) - u(\gamma(b))| \le \int_a^b |\nabla_g u|(\gamma)|\dot\gamma|_g \, dt$ for all $\gamma\in\mathfrak{M}\setminus\Gamma$. This proves that $|\nabla_g u|$ is a q–weak upper gradient of u with respect to g. Applying Lemma 8.9, we see that $|\nabla_g u|$ is also a p–weak upper gradient with respect to g, completing Step 1.

Step 2 *$|\nabla_g u|$ is the least p–weak upper gradient of u in $W^{1,p}(B,g)$.*

Let $v_i \equiv u_i - u \in W^{1,p}(B,g)$. As noted in the previous step, $v_i \to 0$ in $W^{1,p/\kappa}(B, g_{\mathrm{euc}})$, so in particular $|\nabla_g v_i| \to 0$ m–a.e. Moreover, applying the previous step, $|\nabla_g v_i|$ is a p–weak upper gradient of v_i in $W^{1,p}(B,g)$.

Now, consider any p–weak upper gradient $H \in L^p(B,g)$ for u with respect to g and let $H_i = H + |\nabla_g v_i|$, which converges pointwise m–a.e. to H. Applying the triangle inequality, we see that H_i is a p–weak upper gradient for u_i. Moreover, from Case 1, we know that $|\nabla_g u_i|$ is the least p–weak upper gradient for u_i with respect to g, and hence

$$H_i \ge |\nabla_g u_i| \tag{8-18}$$

for m–a.e. $x \in B(0,1)$. Since $|\nabla_g u_i| \to |\nabla_g u|$ in $L^q(B, g_{\mathrm{euc}})$, after passing to further subsequences, the sequence also converges pointwise a.e, and thus m–a.e. Thus sending $i\to\infty$ in (8-18), we see that $|\nabla_g u|$ is the least p–weak upper gradient for u with respect to g. This concludes the proof of Step 2 and thus of the proposition. □

8.2.2 Proof of Proposition 8.4

We are now ready to prove Proposition 8.4.

We first prove part (b). Fix $u \in W^{1,p}(M_\infty, g_\infty)$. By Proposition 8.5, we know that the pullback of u is weakly differentiable in charts and that $|\nabla_{g_\infty} u|$ is defined for a.e. $x \in M_\infty$. To show that $|\nabla_{g_\infty}| = G_{M_\infty,g_\infty,u}$, we proceed in two steps: first showing that $|\nabla_{g_\infty} u|$ is a p–weak upper gradient of u, and then showing that it is the least p–weak upper gradient.

We claim that $|\nabla_{g_\infty} u|$ is a p–weak upper gradient of u with respect to $W^{1,p}(M_\infty, g_\infty)$. To this end, let $\{x_j\} \subset M_\infty$ be a collection of points such that $\{B_{g_\infty(1)}(x_j, \frac{1}{2})\}$ covers M_∞ and the $B_{g_\infty(1)}(x_j, \frac{1}{4})$ are pairwise disjoint. For each j, let $\phi_j : B(0,2) \to \Omega_j \subset B_{g_\infty(1)}(x_j, \frac{1}{2})$ be as in Remark 8.3.

Consider any absolutely continuous curve $\gamma : [a,b] \to M_\infty$. The continuous image under γ of the compact set $[a,b]$ is compact, and thus the image of γ in M_∞ intersects finitely many of the Ω_j. Take a finite partition $a = a_0 < a_1 < \cdots < a_N = b$ of the interval $[a,b]$ such that the image of $[a_i, a_{i+1}]$ is contained entirely in Ω_j for one j. Then, applying the triangle inequality and Proposition 8.5, we see that

$$\text{(8-19)}\quad |u(\gamma(a)) - u(\gamma(b))| \le \sum_{i=1}^{N} |u(\gamma(a_{i-1})) - u(\gamma(a_i))| \le \sum_{i=1}^{N} \int_{a_{i-1}}^{a_i} |\nabla_{g_\infty} u| |\dot\gamma|_{g_\infty}\, dt = \int_a^b |\nabla_{g_\infty} u| |\dot\gamma|_{g_\infty}\, dt.$$

It follows that $|\nabla_{g_\infty} u|$ is a p–weak upper gradient for u with respect to $W^{1,p}(M_\infty, g_\infty)$.

Now we show that $|\nabla_{g_\infty} u|$ is the least p–weak upper gradient of u with respect to $W^{1,p}(M_\infty, g_\infty)$, thus proving (b). Notice that, for each j, the restriction of $G_{M_\infty,g_\infty,u}$ to Ω_j is a p–weak upper gradient for u with respect to $W^{1,p}(\Omega_j, g_\infty)$. So, since we know from Proposition 8.5 that $|\nabla_{g_\infty} u|$ is the least p–weak upper gradient of u with respect to $W^{1,p}(\Omega_j, g_\infty)$, it follows that $|\nabla_{g_\infty} u| \le G_{M_\infty,g_\infty,u}$ for m–a.e. $x \in \Omega_j$. Since the collection $\{\Omega_j\}$ covers M_∞, we thus see that $|\nabla_{g_\infty} u| \le G_{M_\infty,g_\infty,u}$ for m–a.e. $x \in M_\infty$. This completes the proof of (b).

Finally, we prove (a). Fix any $v \in L^p(M_\infty, g_\infty)$ and $\epsilon > 0$. We wish to show that there exists $u \in W^{1,p}(M_\infty, g_\infty)$ such that $\|v - u\|_{L^p(M_\infty,g_\infty)} \le \epsilon$. It is apparent that bounded and compactly supported functions are dense in $L^p(M_\infty, g_\infty)$, and thus we may assume without loss of generality that v is bounded and compactly supported. Let $\{x_j\}_{j=1}^N \subset M_\infty$ be a finite collection such that the support of v is contained in $\bigcup_{j=1}^N \Omega_j$, where the Ω_j are defined as in the previous step. Let $\{\psi_j\}$ be a partition of unity subordinate to $\{\Omega_j\}_{j=1}^N$ such that $\phi_j^* \psi_j$ is a smooth function in $B(0,1)$. Since u is bounded, for each j we have $\phi_j^* u \in L^{\kappa p}(B(0,1), g_{\mathrm{euc}})$. So we may find a smooth function $\tilde{v}_j$ such that $\|\phi_j^* u - \tilde{v}_j\|_{L^{\kappa p}(B,g_{\mathrm{euc}})} \le \epsilon/2N$. Thus, by Remark 8.3, we have $\|u - v_j\|_{L^p(\Omega_j, g_\infty)} = \|\phi_j^* u - \tilde{v}_j\|_{L^p(B,g)} \le \epsilon/N$. Here we let $v_j = \phi_* \tilde{v}_j$. Let $v = \sum_{j=1}^N \psi_j v_j$. Thanks to Lemma 8.8 and part (b) above, we see that $v \in W^{1,p}(M_\infty, g_\infty)$. Finally,

$$\|u - v\|_{L^p(M_\infty, g_\infty)} \le \sum_{j=1}^N \|u - v_j\|_{L^p(\Omega_j, g_\infty)} \le \epsilon. \tag{8-20}$$

This completes the proof of (a) and thus of the proposition. □

8.3 Convergence in d_p

In this section, we establish the following proposition, which is Theorem 8.1(1).

Proposition 8.10 *Fix $p \ge n+1$. We may choose $P = P(n,p)$ sufficiently large and thus $\delta = \delta(n,p)$ sufficiently small in Proposition 8.2 such that if (M_i, g_i, x_i) and $(M_\infty, g_\infty, x_\infty)$ are as in Proposition 8.2, then, after passing to a subsequence,*

$$d_p((M_i, g_i, x_i), (M_\infty, g_\infty, x_\infty)) \to 0. \tag{8-21}$$

In order to prove Proposition 8.10, we will first show the following local version.

Proposition 8.11 *Fix $p \ge n+1$. We may choose $P = P(n,p)$ sufficiently large and thus $\delta = \delta(n,p)$ sufficiently small in Proposition 8.2 so that if (M_i, g_i, x_i) and $(M_\infty, g_\infty, x_\infty)$ are as in Proposition 8.2, the following holds. For any compact set $\Omega \Subset M_\infty$, after passing to a subsequence, we can find $\Omega_i \Subset M_i$ such that*

$$d_p\big((\Omega_i, g_i), (\Omega, g_\infty)\big) \to 0 \tag{8-22}$$

as $i \to +\infty$, in the sense of Definition 2.39.

Before proving the propositions, we establish two lemmas that will be needed in the proofs. First, we show that the supremum in the definition of $d_{p,g,\Omega}(x,y)$ is achieved.

Lemma 8.12 *For $p \ge n+1$, there exists $\delta(n,p) > 0$ such that the following holds. Let (M^n, g) be a complete Riemannian manifold with bounded curvature such that*

$$R_g \ge -\delta \quad \text{and} \quad \nu(g,2) \ge -\delta.$$

Then for any bounded subset $\Omega \Subset M$ and $x, y \in \Omega$, there exists a function $f \in W^{1,p}(\Omega)$ such that $d_{p,g,\Omega}(x,y) = |f(x) - f(y)|$ and $\int_\Omega |\nabla f|^p \, d\mathrm{vol}_g = 1$.

Proof Consider a maximizing sequence $f_i \in W^{1,p}(\Omega)$ with $\int_\Omega |\nabla f_i|^p \, d\mathrm{vol}_g \le 1$, $f_i(x) = 0$ and $f_i(y) \to d_{p,g,\Omega}(x,y)$. Choosing $\delta = \delta(n,p)$ according to Theorem 7.6, we may apply Theorem 7.6 to see that $|f_i| \le C = C(n,p,g,\Omega)$ on Ω. In particular, the sequence f_i is uniformly bounded in $W^{1,p}(\Omega)$. Hence, after passing to a subsequence, f_i converges weakly in $W^{1,p}(\Omega)$ to a function $f \in W^{1,p}(\Omega)$ with $\int_\Omega |\nabla f|^p \, d\mathrm{vol}_g \le 1$. Moreover, by applying Theorem 7.6 to rescalings of the metric, we see that each f_i is continuous with a modulus of continuity that is uniform in i. So, by the Arzelà–Ascoli theorem, f_i converges uniformly to f. In particular, $f(x) = 0$ and $f(y) = d_{p,g,\Omega}(x,y)$. Finally, note that $\int_\Omega |\nabla f|^p \, d\mathrm{vol}_g = 1$, otherwise a multiple κf for $\kappa > 1$ would be an admissible test function for $d_{p,g,\Omega}(x,y)$. This completes the proof. □

Next, we use Gehring's lemma and the doubling property of the d_p metrics to show that a function $f \in W^{1,p}(\Omega)$ achieving the supremum in $d_{p,g,\Omega}(x,y)$ enjoys higher integrability properties.

Lemma 8.13 *Fix $p \ge n+1$. There exist constants $\delta(n,p) > 0$, $\kappa(n,p) > 1$ and $C_0(n,p) > 0$ such that the following holds. Let (M,g) be a complete Riemannian manifold with bounded curvature and*

$$R_g \ge -\delta \quad \text{and} \quad \nu(g,2) \ge -\delta.$$

Fix $\Omega \Subset M$ and $x, y \in \Omega$, and let $f \in W^{1,p}(\Omega)$ be a function achieving the supremum in $d_{p,g,\Omega}(x,y)$. Then for all $\mathcal{B}_{p,g}(z,4r) \subset \Omega$ such that $r < \frac{1}{10}\min\{d_{p,g}(x,y), 1\}$, we have

$$\left(\fint_{\mathcal{B}_{p,g}(z,r)} |\nabla f|^{p\kappa} \, d\mathrm{vol}_g \right)^{1/p\kappa} \le C_0 \left(\fint_{\mathcal{B}_{p,g}(z,4r)} |\nabla f|^p \, d\mathrm{vol}_g \right)^{1/p}. \tag{8-23}$$

Proof Let $x, y \in \Omega \Subset M$ and $f \in W^{1,p}(\Omega)$ be a function such that $\int_\Omega |\nabla f|^p \, d\mathrm{vol}_g = 1$ and $d_{p,g,\Omega}(x,y) = |f(x) - f(y)|$. We may assume without loss of generality that $f(x) = 0$. Fix $r < \frac{1}{10}\min\{d_{p,g}(x,y), 1\}$ and $z \in M$ such that $\mathcal{B}_{p,g}(z,4r) \subset \Omega$.

Step 1 Fix any $q \in (n,p)$. We claim that there exists $C_{n,p,q} > 0$ such that

$$\left(\fint_{B_{p,g}(z,r)} |\nabla f|^p \, d\mathrm{vol}_g \right)^{1/p} \le C_{n,p,q} \left(\fint_{B_{p,g}(z,4r)} |\nabla f|^q \, d\mathrm{vol}_g \right)^{1/q}. \tag{8-24}$$

To see this, notice that by the definition of $d_{p,g,\Omega}$, the function f satisfies

$$\int_\Omega |\nabla f|^p \, d\mathrm{vol}_g = \inf\left\{ \int_\Omega |\nabla h|^p \, d\mathrm{vol}_g : h \in W^{1,p}(\Omega),\ |h(x) - h(y)| = d_{p,g,\Omega}(x,y) \right\}. \tag{8-25}$$

So, computing the Euler–Lagrange equation associated to (8-25), we see that for any $h \in W^{1,p}(\Omega)$ satisfying $h(x) = h(y)$, we have

$$\int_\Omega |\nabla f|^{p-2} \langle \nabla f, \nabla h \rangle \, d\mathrm{vol}_g = 0. \tag{8-26}$$

We first consider the case when $x \in \mathcal{B}_{p,g}(z, 2r)$ and $y \notin \mathcal{B}_{p,g}(z, 2r)$. Let ϕ be a cutoff function on M such that $\phi \equiv 1$ on $\mathcal{B}_{p,g}(z, r)$ and vanishes outside $\mathcal{B}_{p,g}(z, 2r)$. We will make the construction more precise below. Since $f(x) = 0$, the function $h = f\phi^p$ satisfies $h(x) = h(y) = 0$. Therefore, choosing $h = f\phi^p$ as a test function in (8-26) and applying Young's inequality, we find that

$$\begin{aligned} \int_\Omega |\nabla f|^p \phi^p \, d\mathrm{vol}_g &\le p \int_\Omega |\nabla f|^{p-1} \phi^{p-1} \, |\nabla \phi| f \, d\mathrm{vol}_g \\ &\le \epsilon \int_\Omega |\nabla f|^p \phi^p \, d\mathrm{vol}_g + \frac{p-1}{\epsilon} \|f\|^p_{L^\infty(\mathcal{B}_{p,g}(z,2r))} \int_\Omega |\nabla \phi|^p \, d\mathrm{vol}_g. \end{aligned} \tag{8-27}$$

Consequently, absorbing the first term on the right-hand side of (8-27) and using the volume estimate (7-24), we find that

$$\left(⨍_{\mathcal{B}_{p,g}(z,r)} |\nabla f|^p \, d\mathrm{vol}_g \right)^{1/p} \le C \|f\|_{L^\infty(\mathcal{B}_{p,g}(z,2r))} \left(⨍_{\mathcal{B}_{p,g}(z,2r)} |\nabla \phi|^p \, d\mathrm{vol}_g \right)^{1/p}, \tag{8-28}$$

where $C = C(n, p)$. By applying the Sobolev inequality of Theorem 7.6 and the volume estimate (7-24) to the rescaled metric $\tilde{g} = \lambda^2 g$, where $4r\lambda^{1-n/p} = 1$ so that $\mathcal{B}_{p,\tilde{g}}(z, 1) = \mathcal{B}_{p,g}(z, 4r)$, we find that

$$\|f\|_{L^\infty(\mathcal{B}_{p,g}(z,2r))} \le C_{n,p,q} r^{p/(p-n)} \left(⨍_{\mathcal{B}_{p,g}(z,4r)} |\nabla f|^q \, d\mathrm{vol}_g \right)^{1/q}. \tag{8-29}$$

On the other hand, let us now construct a good cutoff function ϕ. Begin by constructing a cutoff function Φ on Euclidean space such that $\Phi = 1$ on $\mathcal{B}_{p,\mathrm{euc}}\big(0, \frac{2}{3}\big)$, Φ vanishes outside $\mathcal{B}_{p,\mathrm{euc}}\big(0, \frac{4}{5}\big)$ and $|\partial \Phi|^{10p} \le C_{n,p} \Phi^{10p-1}$. Let ϕ be its pullback along the diffeomorphism obtained from $\tilde{g}$. A similar argument as in Theorem 7.6 using Theorem 5.1 shows that

$$\left(⨍_{\mathcal{B}_{p,g}(z,2r)} |\nabla \phi|^p \, d\mathrm{vol}_g \right)^{1/p} \le C_{n,p} r^{-p/(p-n)}. \tag{8-30}$$

By combining (8-28)–(8-30), we conclude that (8-24) holds in this case.

Next, consider the case when $y \in \mathcal{B}_{p,g}(z,2r)$, and so $x \notin \mathcal{B}_{p,g}(z,2r)$. Applying the same argument to the function $\tilde{f} = d_{p,g,\Omega}(x,y) - f$, we deduce the same inequality (8-24) because $|\nabla f| = |\nabla \tilde{f}|$. Finally, if $x, y \notin \mathcal{B}_{p,g}(z,2r)$, we consider $\tilde{f} = f - \bar{f}$, where $\bar{f} \in \mathbb{R}$ so that $\int_{\mathcal{B}_{p,g}(z,2r)} \tilde{f}\, d\mathrm{vol}_g = 0$. Hence, we still have (8-29) and thus the proof above can be carried over without any change. We thus have (8-24) in this case as well.

Step 2 Since $(M, d_{p,g}, d\mathrm{vol}_g)$ is a metric measure space and the measure $d\mathrm{vol}_g$ is a doubling measure with respect to $d_{p,g}$ for scales $r \le 1$, by choosing $q = \frac{1}{2}(n+p)$ we may apply the form of Gehring's lemma in [39, Theorem 3.1] to see that there are a $\tilde{p} > p$ and a $C_0 > 1$ depending only on n and p such that for all $\mathcal{B}_{p,g}(z,4r) \subset \Omega$ where $r < \frac{1}{10}\min\{d_{p,g}(x,y), 1\}$, we have

$$\left(\fint_{\mathcal{B}_{p,g}(z,r)} |\nabla f|^{\tilde{p}}\, d\mathrm{vol} \right)^{1/\tilde{p}} \le C_0 \left(\fint_{\mathcal{B}_{p,g}(z,4r)} |\nabla f|^{p}\, d\mathrm{vol} \right)^{1/p}. \tag{8-31}$$

Note that the reverse Hölder inequality assumption on [39, Theorem 3.1] is stated on balls of the same radius. It is clear from the proof that (8-24) suffices; see also the classical Gehring lemma [24] on Euclidean space. Letting $\kappa = \tilde{p}/p$ completes the proof. □

Remark 8.14 Assume that $P = P(n,p)$ is taken sufficiently large and thus $\delta = \delta(n,p)$ is taken sufficiently small in Proposition 8.2, so that we may apply Proposition 8.4 to the limit space $(M_\infty, g_\infty, x_\infty)$ constructed in Proposition 8.2. Since $(M_\infty, g_\infty, x_\infty)$ is $W^{1,p}$–rectifiably complete, we see from the proof of Proposition 8.2 that the ϵ–regularity theorem, Theorem 7.5, and Sobolev inequalities of Theorem 7.6 pass to the limit $(M_\infty, g_\infty, x_\infty)$. In particular, the proofs and conclusions of Lemmas 8.12 and 8.13 also hold for $(M_\infty, g_\infty, x_\infty)$.

We are now ready to prove Proposition 8.11.

Proof of Proposition 8.11 Assume that $P = P(n,p)$ in Proposition 8.2 is taken large enough to apply Proposition 8.4 and thus Remark 8.14. Fix $\Omega \Subset M_\infty$ and choose $\bar{\Omega} \subsetneq \Omega \subsetneq \tilde{\Omega} \Subset M_\infty$. From the Cheeger–Gromov convergence established as in Step 1 of the proof of Proposition 8.2, we may assume that the g_i are all defined on $\tilde{\Omega}$ via the diffeomorphisms $\psi_i : \tilde{\Omega} \to \tilde{\Omega}_i \subset M_i$.

Step 1 We claim that $d_{p,g_i,\Omega}(x,y) \to d_{p,g_\infty,\Omega}(x,y)$ for every $x, y \in \Omega$.

Fix $x, y \in \Omega$. By Lemma 8.12 and Remark 8.14, we may find $f \in W^{1,p}(\widetilde{\Omega}, g_\infty)$ such that $d_{p,g_\infty,\widetilde{\Omega}}(x,y) = |f(x)-f(y)|$ and $\int_{\widetilde{\Omega}} |\nabla^{g_\infty} f|^p \, d\mathrm{vol}_{g_\infty} = 1$. Furthermore, applying Lemma 8.13 (see Remark 8.14) to a covering of $\widetilde{\Omega}$ by g_∞ p–balls, we find that

$$\left(\int_\Omega |\nabla^{g_\infty} f|^{\kappa p} \, d\mathrm{vol}_{g_\infty}\right)^{1/\kappa p} \le C_1, \tag{8-32}$$

where C_1 depends on n, p, Ω, $\widetilde{\Omega}$ and $d_{p,g_\infty}(x,y)$, and where $\kappa = \kappa(n,p) > 1$ is the constant obtained in Lemma 8.13. Here we have used the fact that the topologies induced by $d_{g_\infty(1)}$ and d_{p,g_∞} are equivalent. This can be seen by taking a multiple of $d_{g_\infty(1)}$ as a test function for d_{p,g_∞} together with (8-7) and the Morrey–Sobolev inequality.

Fix $\epsilon > 0$. We aim to show that $f/(1+\epsilon)$ is an admissible test function for $d_{p,g_i,\Omega}(x,y)$ for i sufficiently large. By Proposition 8.2 and the $W^{1,p}$–rectifiable completeness of the limit space, we see that $|\nabla^{g_i} f|^p = (1+\mathcal{E}_{1,i})|\nabla^{g_\infty} f|^p$ and $d\mathrm{vol}_{g_i} = (1+\mathcal{E}_{2,i})\, d\mathrm{vol}_{g_\infty}$, where $\mathcal{E}_{1,i}, \mathcal{E}_{2,i} \colon \Omega \to \mathbb{R}$ are errors such that $\mathcal{E}_{1,i}, \mathcal{E}_{2,i} \to 0$ in $L^q(\Omega, g_\infty)$ for any $q \le P/(n+p)$. Therefore, provided we choose $P = P(n,p)$ large enough that the Hölder conjugate κ' of κ is less than $P/2(n+p)$, we use Hölder's inequality and (8-32) to see that

$$\begin{aligned}
\left(\int_\Omega |\nabla^{g_i} f|^p \, d\mathrm{vol}_{g_i}\right)^{1/p} \\
&= \left(\int_\Omega |\nabla^{g_\infty} f|^p (1 + \mathcal{E}_{1,i} + \mathcal{E}_{2,i} + \mathcal{E}_{1,i}\mathcal{E}_{2,i}) \, d\mathrm{vol}_{g_\infty}\right)^{1/p} \\
&\le 1 + \left(\int_\Omega |\nabla^{g_\infty} f|^p (\mathcal{E}_{1,i} + \mathcal{E}_{2,i} + \mathcal{E}_{1,i}\mathcal{E}_{2,i}) \, d\mathrm{vol}_{g_\infty}\right)^{1/p} \\
&\le 1 + \|\nabla^{g_\infty} f\|_{L^{\kappa p}(\Omega)} \|\mathcal{E}_{1,i} + \mathcal{E}_{2,i} + \mathcal{E}_{1,i}\mathcal{E}_{2,i}\|_{L^{\kappa'}(\Omega)} \\
&\le 1 + \epsilon,
\end{aligned} \tag{8-33}$$

where the final inequality holds for i sufficiently large and makes use of (8-32). Therefore, $f/(1+\epsilon)$ is an admissible test function for $d_{p,g_i,\Omega}(x,y)$ and consequently

$$d_{p,g_\infty,\widetilde{\Omega}}(x,y) \le (1+\epsilon) d_{p,g_i,\Omega}(x,y) \tag{8-34}$$

for i sufficiently large. Letting $i \to +\infty$ and then $\epsilon \to 0$, we find that

$$d_{p,g_\infty,\widetilde{\Omega}}(x,y) \le \liminf_{i\to+\infty} d_{p,g_i,\Omega}(x,y). \tag{8-35}$$

Then an argument analogous to the proof of Lemma 7.4 shows that $d_{p,g,\Omega}$ is continuous with respect to the domain and hence we may let $\widetilde{\Omega}$ tend to Ω to conclude that

$$d_{p,g_\infty,\Omega}(x,y) \le \liminf_{i\to+\infty} d_{p,g_i,\Omega}(x,y). \tag{8-36}$$

We now apply the analogous argument to $x, y \in \bar{\Omega}$ with the roles of g_∞ and g_i swapped, making use of the crucial fact that the upper bound in Lemma 8.13 and thus (8-32) are uniform in i. We find that for any $\epsilon > 0$, $d_{p,g_i,\Omega}(x, y) \le (1+\epsilon) d_{p,g_\infty,\bar{\Omega}}(x, y)$ for i sufficiently large and hence

$$\limsup_{i \to +\infty} d_{p,g_i,\Omega}(x, y) \le d_{p,g_\infty,\Omega}(x, y). \tag{8-37}$$

This completes the proof of Step 1.

Step 2 We claim that

$$d_{\mathrm{GH}}((\Omega, d_{p,g_i,\Omega}), (\Omega, d_{p,g_\infty,\Omega})) \to 0 \tag{8-38}$$

and that the volumes of p–balls converge, thereby establishing the proposition. Indeed, fix any $\epsilon > 0$. Letting $g_i(t)$ and $g_\infty(t)$ be the Ricci flows as in the proof of Proposition 8.2. From the smooth convergence of $g_i(1)$ to $g_\infty(1)$, we see that there exists an $N \in \mathbb{N}$ such that Ω can be covered by $\{B_{g_i(1)}(z_j, \epsilon)\}_{j=1}^N$ for i sufficiently large. For each i and $z, w \in \Omega$, let $f \in W^{1,p}(\Omega)$ be a maximizer of $d_{p,g_i,\Omega}(z, w)$, whose existence is guaranteed by Lemma 8.12. Since $g_i(1)$ has uniformly bounded geometry, we apply the Morrey–Sobolev inequality and estimate (1-13) of Theorem 6.1 to f to see that

$$d_{p,g_i,\Omega}(z, w) \le C_0(n, p, \Omega) d_{g_i(1)}(z, w)^{1-n/p} \tag{8-39}$$

for all $z, w \in \Omega$ and i sufficiently large.

In particular, Ω can be covered by $\{\mathcal{B}_{i,\Omega}(z_j, C_0\epsilon^{1-n/p})\}_{j=1}^N$, where $\mathcal{B}_{i,\Omega}$ is the ball with respect to $d_{p,g_i,\Omega}$ and N is independent of $i \to +\infty$. Together with the convergence $d_{p,g_i,\Omega}(z_j, z_k) \to d_{p,g_\infty,\Omega}(z_j, z_k)$ for each pair of z_j, z_k, this proves the Gromov–Hausdorff convergence. The volume convergence in Definition 2.39 follows from this Gromov–Hausdorff convergence together with the L^P–convergence of the metric coefficients on Proposition 8.2. This completes the proof of the proposition. □

Finally, we use Proposition 8.11 to establish Proposition 8.10.

Proof of Proposition 8.10 Let $P(n, p)$ and thus $\delta(n, p)$ be as in Proposition 8.11. Note that the largest radii less than or equal to 1 such that $\mathcal{B}_{p,g_\infty}(y, 4r) \Subset M_\infty$ for $y \in M_\infty$, and $\mathcal{B}_{p,g_i}(y, 4r) \Subset M_i$ for $y \in M_i$, respectively, are both equal to 1 thanks to the ϵ–regularity theorem (Theorem 1.7) and Remark 8.14. Moreover, again by

Theorem 1.7 and Remark 8.14 and (2-25), there exists an $r < R$ depending on n and p such that for any $y \in M_\infty$ and $y_i \in M_i$, we have

$$B_{g_\infty(1)}(y, r) \subset \mathcal{B}_{p,g_\infty}(y, 1) \subset B_{g_\infty(1)}(y, R),$$
$$B_{g_i(1)}(y_i, r) \subset \mathcal{B}_{p,g_i}(y_i, 1) \subset B_{g_i(1)}(y_i, R), \tag{8-40}$$

where $g_\infty(t)$ and $g_i(t)$ are the Ricci flows as in the proof of Proposition 8.2. Hence, from the second containments in (8-40), we see that for any $N \in \mathbb{N}$,

$$\mathrm{Cov}_{g_\infty}(x_\infty, N) \subset B_{g_\infty(1)}(x_\infty, 2NR),$$
$$\mathrm{Cov}_{g_i}(x_i, N) \subset B_{g_i(1)}(x_i, 2NR). \tag{8-41}$$

So, for any $N \in \mathbb{N}$, choose $\Omega \subset M_\infty$ to be a compact set such that

$$B_{g_\infty(1)}(x_\infty, 2NR + 1) \subset \Omega \subset B_{g_\infty(1)}(x_\infty, 2NR + 2). \tag{8-42}$$

By the Cheeger–Gromov convergence used in Proposition 8.2 to obtain the map $\psi : \Omega \to M_i$ and the set $\Omega_i := \psi(\Omega) \subset M_i$, we see that

$$B_{g_i(1)}(x_i, 2NR) \subset \Omega_i \subset B_{g_i(1)}(x_i, 2NR + 3) \tag{8-43}$$

for i sufficiently large. Therefore, combining (8-41), (8-42) and (8-43), we find that

$$\mathrm{Cov}_{g_\infty}(x_\infty, N) \subset \Omega \quad \text{and} \quad \mathrm{Cov}_{g_i}(x_i, N) \subset \Omega_i. \tag{8-44}$$

Now, recall that the metrics $g_\infty(1)$ and $g_i(1)$ satisfy a uniform curvature bound. Therefore, by volume comparison, there exists an $N' \in \mathbb{N}$ depending only on N, n and p such that

$$B_{g_\infty(1)}(x_\infty, 2NR + 2) \subset \bigcup_{a=1}^{N'} B_{g_\infty(1)}(y_a, r),$$
$$B_{g_i(1)}(x_i, 2NR + 3) \subset \bigcup_{a=1}^{N'} B_{g_i(1)}(y_{a,i}, r) \tag{8-45}$$

for some $\{y_a\}_{a=1}^{N'} \subset M_\infty$ and $\{y_{a,i}\}_{a=1}^{N'} \subset M_i$, where r and R are as in (8-40) and depend only on n and p. So, applying the first containment of (8-40) to each ball above, we find that

$$B_{g_\infty(1)}(x_\infty, 2NR + 2) \subset \mathrm{Cov}_{g_\infty}(x_\infty, N'),$$
$$B_{g_i(1)}(x_i, 2NR + 3) \subset \mathrm{Cov}_{g_i}(x_i, N'). \tag{8-46}$$

Together with (8-42) and (8-43), this shows that

$$\Omega \subset \mathrm{Cov}_{g_\infty}(x_\infty, N') \quad \text{and} \quad \Omega_i \subset \mathrm{Cov}_{g_i}(x_i, N'). \tag{8-47}$$

Now, having obtained the appropriate sets Ω and Ω_i for each $N \in \mathbb{N}$, we may apply Proposition 8.11 to complete the proof. □

8.4 Proof of Theorem 8.1

We finally prove Theorem 8.1.

Fix $p \geq n+1$. Choose $P = P(n,p)$ sufficiently large according to Propositions 8.5 and 8.10. Now let $\delta = \delta(n,P) = \delta(n,p)$ sufficiently small to apply Proposition 8.2.

By Proposition 8.2, we obtain the space $(M_\infty, g_\infty, x_\infty)$, and applying Proposition 8.10 implies the pointed d_p–convergence of Theorem 1.15(1). Proposition 8.5 yields the first claim in (2), that is, that the limit space $(M_\infty, g_\infty, x_\infty)$ is $W^{1,p}$–rectifiably complete. Finally, we show that $(M_\infty, g_\infty, x_\infty)$ is d_p–rectifiably complete, that is, that the topology generated by d_{p,g_∞} agrees with the topology of M_∞. Indeed, this follows from the observation that Propositions 8.2 and 8.5 imply that the topology generated by d_{p,M_∞,g_∞} agrees with the topology generated by $d_{p,M_\infty,g_\infty(1)}$, which in turn agrees with the topology of a smooth manifold M_∞. □

Remark 8.15 Let $(M_\infty, g_\infty, x_\infty)$ be the limit rectifiable Riemannian space obtained in Theorem 8.1. We have shown that for any suitable compact set $\Omega \subset M_\infty$, we have convergence along the sequence of the relative d_p–distances on M_i to $d_{p,g_\infty,\Omega}$. It is worth noting that, for any $x, y \in M_\infty$ and for any exhaustion $\{\Omega_a\}$ of M_∞ by compact sets, we have

$$\lim_{a\to\infty} d_{p,g_\infty,\Omega_a}(x,y) = d_{p,g_\infty,M}(x,y). \tag{8-48}$$

To see this, we first see directly from the definition that the relative d_p–distance is monotone decreasing with respect to set inclusion, so the limit on the left-hand side of (8-48) exists and

$$\lim_{a\to\infty} d_{p,g_\infty,\Omega_a}(x,y) \geq d_{p,g_\infty,M}(x,y). \tag{8-49}$$

On the other hand, for each $a \in \mathbb{N}$ consider a function $f_a \in W^{1,p}(\Omega_a)$ achieving the supremum in $d_{p,g_\infty,\Omega_a}(x,y)$; recall Lemma 8.12 and Remark 8.14. Then there exists an $f \in W^{1,p}(M_\infty)$ such that on every compact set $\Omega \subset M_\infty$, $f_a \to f$ weakly in $W^{1,p}(\Omega)$ and uniformly. Thus, $\int_{M_\infty} |\nabla f|^p \, d\mathrm{vol}_g \leq 1$ and so f is an admissible test function for $d_{p,g_\infty,M}(x,y)$. Moreover,

$$|f(x) - f(y)| = \lim_{a\to\infty} |f_a(x) - f_a(y)| = \lim_{a\to\infty} d_{p,g_\infty,\Omega_a}(x,y). \tag{8-50}$$

We therefore establish the opposite inequality in (8-48) and conclude.

8.5 Proof of Theorem 1.19

We now prove Theorem 1.19, which provides a form of stability for rigidity of the flat metric as the only metric on a torus with nonnegative scalar curvature.

Fix $\delta = \delta(n, p)$ according to Theorem 1.15. Any compact Riemannian manifold with $\nu(g, 2) \geq -\delta$ has volume bounded below by a constant $C = C(\delta)$, so choose $V_0 > C$ so that the hypotheses of the theorem are not vacuous.

Consider a sequence of tori (M_i, g_i) with

$$\nu(g_i, 2) \geq -\delta, \quad \mathrm{vol}_{g_i}(M_i) \leq V_0 \quad \text{and} \quad R_{g_i} \geq -\frac{1}{i}.$$

By Theorem 1.15, up to a subsequence, (M_i, g_i) converges in the d_p sense to a rectifiable Riemannian space (M_∞, g_∞), which is constructed in Proposition 8.2. Moreover, M_∞ is diffeomorphic to a torus.

By the proof of Proposition 8.2, the Ricci flows $(M_i, g_i(t))_{t\in[0,1]}$ exist and, at each time slice have uniformly bounded geometry,

$$\mathrm{vol}_{g_i(t)}(M_i) \leq 2V_0 \quad \text{and} \quad R_{g_i(t)} \geq -\frac{1}{i}.$$

By Hamilton compactness [32], after passing to a subsequence, we have convergence $(M_i, g_i(t)) \to (M_\infty, g_\infty(t))$ in the pointed C^∞–Cheeger–Gromov sense so that $g_\infty(t)$ is a solution to the Ricci flow on $M_\infty \times (0, 1]$ with $R_{g_\infty(t)} \geq 0$ for all $t \in (0, 1]$. So, by Schoen and Yau [45] and Gromov and Lawson [28], we see that $g_\infty(t)$ is a flat metric on the torus for each t. Since $(M_\infty, g_\infty(t))_{t\in(0,1]}$ is a Ricci flow, it follows that each $g_\infty(t)$ is the same flat metric.

Furthermore, from the proof of Proposition 8.2, we know that the metric coefficients of $g_\infty(t)$ converge in L^p to g_∞, the limiting rectifiable Riemannian metric. It follows that g_∞ is the flat metric on M_∞. □

9 Examples

In this section, we will construct various examples of sequences of complete Riemannian manifolds (M_i, g_i) with bounded curvature that satisfy the almost nonnegative entropy and scalar curvature assumptions of our main theorems. In each example, the d_p limits of our spaces will be either Euclidean space or a flat torus, and these limits do not agree with their Gromov–Hausdorff and intrinsic flat limits.

9.1 The basic building block: a two-parameter family of metrics

We begin by constructing a two-parameter family of metrics on $\mathbb{R}^{n+1}$ for $n \geq 3$, which serves as the basic building block for constructing all of our examples. Let h denote the standard metric on $\mathbb{S}^{n-1}$. We define the two-parameter family of metrics $g_{\delta,\epsilon}$ on $M = \mathbb{R}_+ \times \mathbb{S}^{n-1} \times \mathbb{R}$ by

(9-1) $$g_{\delta,\epsilon} = dr^2 + f_{\delta,\epsilon}^2(r)h + \varphi_{\delta,\epsilon}^2(r)\,dx^2.$$

The warping factor f will be used to identify $\mathbb{R}^+ \times \mathbb{S}^{n-1}$ topologically with $\mathbb{R}^n$; however, geometrically this will be done in a way to add a large amount of positive curvature to the space. The warping factor φ will be constructed so that it will *slowly* degenerate as $r \to 0$. If this degeneration is sufficiently slow we will see that we can preserve the positive scalar curvature, and, much more challenging, the lower entropy as well. If $\varphi(0) = 0$, then this would imply that the line $\{0^n\} \times \mathbb{R}$ has a fully degenerate metric g along it, in particular $d_g((0^n, s), (0^n, t)) = 0$ for any two points along the line $\{0^n\} \times \mathbb{R}$. The parameters $\epsilon, \delta > 0$ are built so that we may approach such a degenerate limit smoothly and in different ways, depending on our end goal.

The functions $f_{\delta,\epsilon}$ and $\varphi_{\delta,\epsilon}$ are now precisely defined in the following way.

Definition of $\varphi_{\delta,\epsilon}$ For $\epsilon > 0$, let $\phi_\epsilon : \mathbb{R}_+ \to \mathbb{R}_+$ be a smooth function such that

(9-2) $$\phi_\epsilon(r) = \begin{cases} \epsilon & \text{for } r \leq \frac{1}{2}\epsilon, \\ \psi_1 & \text{for } \frac{1}{2}\epsilon \leq r \leq 2\epsilon, \\ r & \text{for } 2\epsilon \leq r \leq \frac{1}{2}, \\ \psi_2 & \text{for } \frac{1}{2} \leq r \leq 2, \\ 1 & \text{for } r \geq 2, \end{cases}$$

where $\psi_i(r)$ are smooth nondecreasing functions with $\psi_2'' \leq 0$,

(9-3) $$|\psi_1^{(k)}| \leq 8\epsilon^{-k+1} \quad \text{and} \quad |\psi_2^{(k)}| \leq 4^k \quad \text{for } k = 1, 2.$$

We then let $\varphi_{\delta,\epsilon} : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by

(9-4) $$\varphi_{\delta,\epsilon}(r) = \phi_\epsilon(r)^\delta.$$

Observe that $\varphi_{\delta,\epsilon}$ satisfies the properties

(9-5) $$\left|\frac{\varphi_{\delta,\epsilon}'(r)}{\varphi_{\delta,\epsilon}(r)}\right| \leq \frac{50\delta}{r}, \quad \left|\frac{\varphi_{\delta,\epsilon}''(r)}{\varphi_{\delta,\epsilon}(r)}\right| \leq \frac{50\delta}{r^2}, \quad \left|\frac{\varphi_{\delta,\epsilon}^{(k)}(r)}{\varphi_{\delta,\epsilon}(r)}\right| \leq \frac{C_k\delta}{r^k}.$$

Remark 9.1 As we have defined the two-parameter family of metrics, when $\epsilon = 0$ the corresponding metric $g_{\delta,\epsilon}$ vanishes at $r = 0$. In fact, we can modify the construction so that $g_{\delta,\epsilon}|_{r=0}$ agrees with any prescribed singular metric $l(x)\,dx^2$ along $r = 0$. More precisely, given fixed $\delta, \epsilon > 0$ and a smooth function $l\colon \mathbb{R} \to [0, +\infty)$, we could replace $\varphi_{\delta,\epsilon}$ by the function $\varphi_{\delta,\epsilon}(r,x) = [1 - \phi_\epsilon(r)^\delta] l(x) + \phi_\epsilon(r)^\delta$ so that $\varphi_{\delta,\epsilon}(r_0, x_0) \to 1$ when $r_0 > 0$ and $\varphi_{\delta,\epsilon}(0, x_0) \to l(x_0)$ pointwise as $\delta \to 0$.

Definition of $f_{\delta,\epsilon}$ Let $\zeta\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a smooth nonincreasing cutoff function such that $\zeta(x) \equiv 1$ on $\left[0, \frac{1}{2}\right]$, vanishes on $[1, \infty)$ and satisfies $|\zeta'|^2 + |\zeta''| \le 100$.

Define $\widetilde{f}_{\delta,\epsilon}$ to be the solution of the ODE

$$\begin{cases} \widetilde{f}'_{\delta,\epsilon} = \left[1 - 10^4 n\delta\left(1 - \zeta\left(\frac{r}{100\epsilon}\right)\right)\right], \\ \widetilde{f}_{\delta,\epsilon}(0) = 0. \end{cases} \tag{9-6}$$

In this way, the corresponding metric $dr^2 + f_{\delta,\epsilon}^2 h$ coincides with the Euclidean metric on $\mathbb{R}^n$ for r sufficiently small. Finally, we define

$$f_{\delta,\epsilon}(r) = \zeta\left(\frac{1}{4}r\right)\widetilde{f}_{\delta,\epsilon}(r) + \left(1 - \zeta\left(\frac{1}{4}r\right)\right)r, \tag{9-7}$$

so that $f_{\delta,\epsilon}$ is equal to the solution to the ODE for $r \le 2$, the function r for $r \ge 4$, and interpolates smoothly in between.

Crucially, this two-parameter family of metrics satisfies a lower bound on entropy and scalar curvature that is uniform for all ϵ and δ sufficiently small. Geometrically, what is happening is that the warping factor is changing so slowly that even though the actual metric geometry may be behaving very poorly, in some weaker sense (d_p sense! though we will not directly appeal to this) the geometry looks very Euclidean at all points and scales. This sense of closeness to Euclidean space will be good enough to force the small lower entropy bound on the example.

Theorem 9.2 *Fix $n \ge 3$, $\eta > 0$ and $L > 0$. There exist $\epsilon_0 > 0$ and $\delta_0 > 0$, depending on n, η and L, such that the following holds. For all $\epsilon \le \epsilon_0$ and $\delta \le \delta_0$, the metric $g_{\delta,\epsilon}$ defined in (9-1) satisfies*

$$R_{g_{\delta,\epsilon}} \ge -\eta, \tag{9-8}$$

$$\nu(g_{\delta,\epsilon}, L) \ge -\eta. \tag{9-9}$$

The scalar curvature lower bound and entropy lower bound of Theorem 9.2 will be established in Sections 9.3 and 9.4, respectively.

Remark 9.3 The metrics $g_{\delta,\epsilon}$ are defined on an $(n+1)$–dimensional space, so fixing $n \geq 3$ means that our examples are of dimension 4 or higher.

Let us again discuss the examples geometrically, this time with more of a focus on how each parameter behaves in the construction. One can think of the metric $g_{\delta,\epsilon}$ defined in (9-1) in the following way. The portion $dr^2 + f_{\delta,\epsilon}^2(r)h$ of the metric $g_{\delta,\epsilon}$ agrees with the Euclidean metric on $\mathbb{R}^n$ far from $0 \in \mathbb{R}^n$, while in a neighborhood of $0 \in \mathbb{R}^n$, it is a smoothed-out cone metric on $\mathbb{R}^n$ with cone angle proportional to δ. The parameter ϵ governs the scale at which this cone metric is smoothed out. This component can roughly be thought of as Euclidean $\mathbb{R}^n$, although taking the smoothed cone in place of $\mathbb{R}^n$ provides a crucial positive scalar curvature contribution in order to guarantee that (9-8) holds as long as $n \geq 3$. The component $\varphi_{\delta,\epsilon}^2(r)\,dx^2$ adds a fiber at each point on $(\mathbb{R}^n, dr^2 + f_{\delta,\epsilon}^2(r)h)$. Away from $0 \in \mathbb{R}^n$, these fibers are Euclidean, but for r small, the fibers become increasingly degenerate.

If we choose $\epsilon = \delta$ and let $\epsilon \to 0$, then the metric tensors converge smoothly to the Euclidean metric $g_\infty = \lim_{\epsilon\to 0} g_{\epsilon,\epsilon} = \sum_{i=1}^{n+1}(dx^i)^2$. However, if ϵ is relatively small compared to δ, then the limiting metric will be Euclidean away from a ray $\ell = \{x : x_i = 0 \text{ for } i = 1, \dots, n\}$ in $\mathbb{R}^{n+1}$ and ℓ will be collapsed to a point along the sequence. For instance, if we choose $\delta = (-\log\epsilon)^{-1/2}$ and let $\epsilon \to 0$, then in a pointwise sense, $\lim_{\epsilon\to 0} g_{\delta(\epsilon),\epsilon} = \sum_{i=1}^{n}(dx^i)^2 + (1-\chi_{r=0})(dx^{n+1})^2$. In both of these two examples, the constructed sequence converges to the Euclidean metric in L^p_{loc} for all $p > 1$, while in the latter case the Gromov–Hausdorff limit is very different; see Example 9.4 below. This will correspond to our general d_p ϵ–regularity theorem when the entropy and scalar curvature have lower bound converging to 0.

9.2 Examples constructed from the main building block

In the following, we will make use of the metrics $g_{\delta,\epsilon}$ with $\delta = (-\log\epsilon)^{-1/2}$ to produce sequences of metrics whose d_p limits and Gromov–Hausdorff limits are entirely different. First, we go into greater depth concerning the basic metric $g_{\delta,\epsilon}$ with $\delta = (-\log\epsilon)^{-1/2}$. We will take $n = 3$ since this is the borderline case. The case of $n \geq 3$ can be constructed similarly.

Example 9.4 (collapsing along a line in Euclidean space) Let $n \geq 3$. By choosing $\delta = (-\log\epsilon)^{-1/2}$ in (9-1), we obtain a sequence of metrics which degenerate along a ray in $\mathbb{R}^{n+1}$ and remain the flat metric away from it. In the Gromov–Hausdorff limit, the ray collapses to a point. On the other hand, from construction, it is easy to see that g_ϵ converges to the Euclidean metric in $L^p_{\mathrm{loc}}(\mathbb{R}^{n+1})$ for all $p>1$. By Theorem 1.15 (and the proof of Proposition 8.2), the pointed d_p limit is the flat Euclidean space. In particular, notice that Theorem 1.7 implies that $\mathrm{vol}_{g_\epsilon}\bigl(\mathcal{B}_{p,g_\epsilon}(0,1)\bigr) \to \mathrm{vol}_{g_{\mathrm{euc}}}(\mathcal{B}_{p,\mathrm{euc}}(0,1))$ as $\epsilon \to 0$, while the volumes of metric balls are tending to infinity:

$$\mathrm{vol}_{g_\epsilon}(B_{g_\epsilon}(0,1)) \to +\infty. \tag{9-10}$$

Indeed, for ϵ sufficiently small, $B_{g_\epsilon}(0,1)$ contains the Euclidean strip

$$\left\{(x,y)\in\mathbb{R}^n\times\mathbb{R} : \tfrac{1}{4}\le |x|\le \tfrac{1}{2},\ |y|\le \tfrac{1}{2\epsilon}\right\}.$$

Since g_ϵ converges smoothly uniformly to the Euclidean metric away from $|x|=0$, we see that (9-10) holds.

We see from the above example that the metric degeneration which causes the metric collapse occurs along a line in $\mathbb{R}^4$. More generally, we conjecture that this metric collapsing can occur only along codimension-3 subsets along converging sequences.

Example 9.5 (d_p–convergence does not hold for all p) In contrast to Example 9.4, in this example we only pass $\epsilon\to 0$ but fix $\delta>0$ small in the construction of (9-1). The corresponding sequence of metrics converges pointwise, and in L^p_{loc} for p less than some $p_1(\delta)$, to $g_\infty = g_{\mathrm{cone}} + r^\delta\,dx^2$, which degenerates at $r=0$. By Theorem 1.15 (and the proof of Proposition 8.2), the sequence converges in the pointed d_p sense to $(\mathbb{R}^{n+1}, g_\infty, 0^n)$ for $p\in[n+2, p_0(\delta)]$. However, this d_p convergence to $(\mathbb{R}^{n+1}, g_\infty, 0^n)$ does not hold for all $p\in[n+2,\infty)$. Indeed, for p sufficiently large, the metric space $(\mathbb{R}^{n+1}, d_{p,g_\infty}, 0^n)$ is topologically distinct from the underlying topology on $\mathbb{R}^{n+1}$, and in particular is not d_p–complete.

This illustrates that δ must be taken to depend on p in our ϵ–regularity theorems, and if we only assume a lower bound on the entropy lower bound and scalar curvature along the sequence, then the limiting rectifiable Riemannian metric g_∞ may have an inverse that is only bounded in $L^{p_0}_{\mathrm{loc}}$ for some $p_0(\delta)>1$ but not all $p>1$.

Example 9.6 (collapsing lines in Euclidean space) In this example, we use the building block of Example 9.4 to construct a sequence of metrics on $\mathbb{R}^{n+1}$ for $n\ge 3$ whose Gromov–Hausdorff limit is the taxicab metric, while the d_p limit is the flat metric on $\mathbb{R}^{n+1}$. The basic idea of the construction is to cut off the building block of Example 9.4 to obtain a degenerating metric on a tubular neighborhood of a line in Euclidean space, and to glue this metric into tubular neighborhoods of an increasing dense collection of lines in $\mathbb{R}^{n+1}$.

Let us now go into the details of this construction. Let $n \geq 3$. First, we obtain a collection of disjoint strips (ie tubular neighborhoods around lines) $\{\mathcal{S}_{i,j}(r_0)\}$ for $i \in \mathbb{N}$ and $j \in \{1, \ldots, n+1\}$ in the following way. Define the projection $\pi_1 : \mathbb{R}^{n+1} \to \mathbb{R}^n$ by $\pi_1(x) = (x_2, \ldots, x_{n+1})$, and let π_j be defined analogously for each $j = 1, \ldots, n+1$. Next, for each $j = 1, \ldots, n+1$ and $(k_1, \ldots, k_n) \in \mathbb{Z}^n$, we consider the collection of points $\{z_{k_1,\ldots,k_n,j}\} \subset \mathbb{R}^n = \pi_j(\mathbb{R}^{n+1})$ with coordinates given by

$$z_{k_1,\ldots,k_n,j} = ((100n + 10j)r_0 k_1, \ldots, (100n + 10j)r_0 k_n). \tag{9-11}$$

Up to reindexing the countable set of $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ by $i \in \mathbb{N}$, we let $\{z_{i,j}\} = \{z_{k_1,\ldots,k_n,j}\}$. Now, let $B_{\mathbb{R}^n}(z, r)$ denote the Euclidean ball in $\mathbb{R}^n$ and define the strip $S_{i,j}(r_0)$ of radius r_0 around the line $\{\pi_j(x) = z_{i,j}\}$ by

$$\mathcal{S}_{i,j}(r_0) = \pi_j^{-1}(B_{\mathbb{R}^n}(z_{i,j}, r_0)). \tag{9-12}$$

It is easy to check that the collection of strips $\{\mathcal{S}_{i,j}(r_0)\}$ are $200nr_0$ dense, in the sense that for any $x \in \mathbb{R}^{n+1}$, there exists $\mathcal{S}_{i,j}$ such that $\mathrm{dist}_{g_{\mathrm{euc}}}(x, S_{i,j}) \leq 200nr_0$, and that these strips are pairwise disjoint.

Now, with the collection $\{\mathcal{S}_{i,j}(r_0)\}$ of disjoint strips in hand, and for any $r_0 > 0$ fixed, we use (9-1) to define a metric g_{r_0} on each strip in the following way. Up to a rigid motion of $\mathbb{R}^{n+1}$, it suffices to define g_{r_0} on the strip $\mathcal{S} = \pi_{n+1}^{-1}(B_{\mathbb{R}^n}(0, r_0))$. Let $\delta = (-\log \epsilon)^{-1/2}$, and take ϵ depending on r_0 to be sufficiently small that $R_{g_{\delta,\epsilon}} \geq -r_0^3$ and $\nu(g_{\delta,\epsilon}, 2r_0^{-2}) \geq -r_0$ by Theorem 9.2. Then consider the rescaled metric

$$g_{r_0} = dr^2 + \left(\tfrac{1}{5}r_0\right)^2 f_\epsilon^2\left(\frac{5r}{r_0}\right) h + \varphi_\epsilon^2\left(\frac{5r}{r_0}\right) dx_{n+1}^2, \tag{9-13}$$

where $\epsilon = \epsilon(r_0)$, which satisfies $R \geq -r_0$ and $\nu(g_{r_0}, 2) \geq -r_0$. Note that after this rescaling, the metric g_{r_0} agrees with the Euclidean metric outside of the strip $B_{g_{\mathrm{euc}}}(\ell, r_0)$. Finally, by restricting this metric to the set $|z| < r_0$, we define the metric g_{r_0} on the strip $\mathcal{S}$.

Finally, let $\tilde{g}_{r_0}$ be the metric on $\mathbb{R}^{n+1}$ be defined by

$$\tilde{g}_{r_0} = \begin{cases} g_{\mathrm{euc}} & \text{if } x \in \mathbb{R}^{n+1} \setminus \bigcup \mathcal{S}_{i,j}(r_0), \\ g_{r_0} & \text{if } x \in \mathcal{S}_{i,j}(r_0). \end{cases} \tag{9-14}$$

Direct computation shows that $\tilde{g}_{r_0}$ converges to $g_{\mathbb{R}^{n+1}}$ in $L^p_{\mathrm{loc}}(\mathbb{R}^{n+1})$ for all $p \geq 1$. In particular, this implies that $\mathrm{vol}_{\tilde{g}_{r_0}}(\Omega) \to \mathrm{vol}_{\mathrm{euc}}(\Omega)$ for any compact set $\Omega \subset \mathbb{R}^{n+1}$. By Theorem 1.15 (and the proof of Proposition 8.2), the sequence converges to $(\mathbb{R}^n, g_{\mathrm{euc}}, 0^n)$ in the pointed d_p sense for all $p \in [n+1, \infty)$. However, in the pointed Gromov–Hausdorff topology, this sequence converges to the taxicab metric.

To roughly explain this, consider the metrics $(\mathbb{R}^{n+1}, \hat{g}_\epsilon) \equiv (\mathbb{R}^{n+1}, \epsilon^{-2}\tilde{g}_{r_0})$. Here, $\epsilon = \epsilon(r)$ is the parameter chosen above. Clearly the metrics $(\mathbb{R}^{n+1}, \hat{g}_\epsilon)$ are isometric to $(\mathbb{R}^{n+1}, \tilde{g}_{r_0})$ by a Euclidean dilation. Let $\ell_{ij} \equiv \pi_j^{-1}(z_{ij})$ denote the lines we have glued around. Then, roughly, we have on each such line ℓ_{ij} that $\hat{g}_\epsilon = 1$ in the direction of this line, and that $\hat{g}_\epsilon \approx \epsilon^{-2}$ in all other directions and at all other points. We also have, in coordinates, that these lines are $o(\epsilon)$–dense. Clearly, a path of minimal length from x to y is now one which stays on these lines as long as possible, and moving from one line to another now causes an error which is approximately $\epsilon^{-1}o(\epsilon) = o(1)$. In particular, we see that $d_{\hat{g}_\epsilon}(x,y) = \sum |x_i - y_i| + o(1)$. Further, as $\epsilon \to 0$ we see a minimal path is any path which is always moving in coordinate directions (specifically, along our increasing dense collection of lines ℓ_{ij}). Hence, $(\mathbb{R}^n, \hat{g}_\epsilon)$ is converging to the taxi-cab metric.

Next, we construct some examples in the compact setting.

Example 9.7 (collapsing circle in torus) In this example, we construct a sequence of metrics $\{g_i\}_{i\in\mathbb{N}}$ on the torus $\mathbb{T}^{n+1}$ for $n \geq 3$ so that each g_i coincides with the flat metric away from a shrinking tubular neighborhood of a fixed $S^1 \subset \mathbb{T}^{n+1}$. The sequence g_i becomes degenerate along this S^1, and in the Gromov–Hausdorff limit, the S^1 collapses to a point. In particular, the metric space arising as the Gromov–Hausdorff limit is not topologically a torus. The d_p limit will be the flat torus for any $p \geq n+2$.

We begin the construction. Fix $n \geq 3$ and $r_0 > 0$. Let g_{r_0} be the metric on Euclidean space $\mathbb{R}^{n+1}$ defined in (9-13) in the previous example, which we recall agrees with the Euclidean metric outside the strip $B_{g_{\mathrm{euc}}}(\ell, r_0)$ and is translation invariant in the x_{n+1} direction. Now, consider the torus $\mathbb{T}^{n+1} = \mathbb{R}^{n+1}/\mathbb{Z}^{n+1}$, equipped with the metric $\tilde{g}_{r_0}$ given by g_{r_0} descending to the quotient.

We now let $r_0 \to 0$. For every r_0, the smooth Riemannian manifold $(\mathbb{T}^{n+1}, \tilde{g}_{r_0})$ satisfies $R_{\tilde{g}_{r_0}} \geq -r_0$. Moreover, for any $\delta > 0$, there exists $\tau_0 = \tau_0(\delta)$ such that $\nu(g_{\mathrm{flat}}, \tau_0) \geq -\frac{1}{2}\delta$. So, arguing as in the proof of Proposition 9.12 below, we find that for $r_0 \leq \bar{r}_0(\delta)$, we have $\nu(\tilde{g}_{r_0}, \tau_0) \geq -\delta$. We directly see that the metrics $\tilde{g}_{r_0}$ converge in L^p for every p to the flat metric on $\mathbb{T}^{n+1}$. Applying the proof of Theorem 1.7, we see that $(\mathbb{T}^{n+1}, \tilde{g}_{r_0})$ converges to $(\mathbb{T}^{n+1}, g_{\mathrm{flat}})$ in the d_p sense for all $p \in [n+2, \infty)$. On the other hand, in the Gromov–Hausdorff topology, the S^1 factor corresponding to the projection of the degenerating line ℓ collapses to a point in the limit. In particular, the metric space arising in the Gromov–Hausdorff limit is not topologically a torus.

By replicating the construction of the degeneracy, we can construct examples so that the sequence of metrics on the torus $\mathbb{T}^{n+1}$ converges to a metric space Y^k with $k < n+1$ or even $k = 0$ (a point) in the Gromov–Hausdorff topology.

Example 9.8 (collapsing of $\mathbb{T}^{n+1}$ to $\mathbb{T}^n$) In the previous example, Example 9.7, we constructed a sequence of metrics on $\mathbb{T}^{n+1}$ which degenerate along a single S^1 inside. In this example, we will modify the construction to obtain a sequence of metrics g_i on the torus that degenerate along increasingly dense sequences of parallel copies of S^1 and remain flat away from them. In the Gromov–Hausdorff topology, the $(\mathbb{T}^{n+1}, g_i)$ collapses to the n–dimensional flat torus $\mathbb{T}^n$. In particular, the Gromov–Hausdorff limit is one dimension lower than the dimension of each manifold in the sequence. On the other hand, the limit with respect to d_p is the flat torus $\mathbb{T}^{n+1}$ for $p \geq n+2$.

More precisely, we again fix an $(n+1)$–dimensional flat torus for $n \geq 3$, identified with $[0,1]^{n+1}/\sim$ and with coordinates $(x_1, \dots, x_{n+1})$. Now consider a maximal $100r_0$ dense set $\{z_i\}$ in $[0,1]^{n+1}/\sim$.

Let $S_i^1 = \{(x_1, \dots, x_n) = z_i\} \subset \mathbb{T}^{n+1}$, and let $\mathcal{S}_i$ be the strip around S_i^1 of radius r_0 as in the previous example. We let

$$\widetilde{g}_{r_0} = \begin{cases} g_{\text{euc}} & \text{for } x \in \mathbb{T}^{n+1} \setminus \bigcup \mathcal{S}_i(r_0), \\ g_{r_0} & \text{for } x \in \mathcal{S}_i. \end{cases} \tag{9-15}$$

Here, g_{r_0} is the same metric on a strip around S^1 as defined in Example 9.7 above. For every r_0, the smooth Riemannian manifold $(\mathbb{T}^{n+1}, \widetilde{g}_{r_0})$ satisfies $R_{\widetilde{g}_{r_0}} \geq -r_0$, and for every $\delta > 0$, there exist $\tau_0 = \tau_0(\delta)$ and $\overline{r}_0 = \overline{r}_0(\delta)$ such that for $r_0 \leq \overline{r}_0$, we have $\nu(\widetilde{g}_{r_0}, \tau_0) \geq -\delta$. In the Gromov–Hausdorff topology, $(\mathbb{T}^{n+1}, \widetilde{g}_{r_0})$ converges to the n–dimensional flat torus $\mathbb{T}^n$ with the usual distance as $r_0 \to 0$. On the other hand, $(\mathbb{T}^{n+1}, \widetilde{g}_{r_0})$ converges to the $(n+1)$–dimensional flat torus with respect to d_p–convergence for each $p \geq n+2$ by (the proof of) Theorem 1.7.

Example 9.9 (collapsing $\mathbb{T}^{n+1}$ to a point) In this example, we further modify the construction in Examples 9.7 and 9.8 to produce a sequence of metrics on $\mathbb{T}^{n+1}$ such that the sequence collapses to a point in the Gromov–Hausdorff and intrinsic flat topologies. Once again, the d_p limit will still be the flat torus $\mathbb{T}^{n+1}$ for $p \geq n+2$. The basic idea of the construction is to choose an increasingly dense collection of strips around copies of $S^1 \subset \mathbb{T}^{n+1}$ with all different orientations, in a similar fashion to Example 9.6, and then to paste the degenerating metrics of Examples 9.7 and 9.8 into each of these strips.

More specifically, we begin with the sequence of metrics $\tilde{g}_{r_0}$ on $\mathbb{R}^{n+1}$ constructed in Example 9.6. Without loss of generality, we may assume that r_0 is always chosen so that for each $j = 1, \dots, n+1$, we have that $1/(100n + 10j)r_0$ is an integer. With this assumption, the metrics $\tilde{g}_{r_0}$ are invariant under the $\mathbb{Z}^{n+1}$ action on $\mathbb{R}^{n+1}$, so we may consider the quotient $\mathbb{T}^{n+1} = \mathbb{R}^{n+1}/\mathbb{Z}^{n+1}$ equipped with the metric descending from $\tilde{g}_{r_0}$ under the quotient, which we again denote by $\tilde{g}_{r_0}$.

The smooth Riemannian manifolds $(\mathbb{T}^{n+1}, \tilde{g}_{r_0})$ satisfy $R_{\tilde{r}_0} \geq -r_0$, and for any $\delta > 0$, there exist $\tau_0 = \tau_0(\delta)$ and $\bar{r}_0 = \bar{r}_0(\delta)$ such that for $r_0 \leq \bar{r}_0$, we have $\nu(\tilde{g}_{r_0}, \tau_0) \geq -\delta$, provided $\epsilon(r_0)$ is taken to be sufficiently small depending on r_0. Sending $r_0 \to 0$, the metrics converge in the Gromov–Hausdorff topology to a point. To see this, we claim that for any $x, y \in (\mathbb{T}^{n+1}, \tilde{g}_{r_0})$,

$$\mathrm{dist}_{\tilde{g}_{r_0}}(x, y) \leq 10 \times 200n^2 r_0 + n\epsilon(r_0). \tag{9-16}$$

Indeed, let S^1_{ij} denote the S^1 factor in $\mathbb{T}^{n+1}$ that is the projection of the line $\pi_j^{-1}(z_{ij}) \in \mathbb{R}^{n+1}$ in the construction of Example 9.6. Then we have $\mathrm{dist}_{\tilde{g}_{r_0}}\big(y, \bigcup S^1_{ij}\big) \leq 200nr_0$ and $\mathrm{dist}_{\tilde{g}_{r_0}}\big(x, \bigcup S^1_{ij}\big) \leq 200nr_0$ because the collection $\{S^1_{ij}\}$ is $200nr_0$–dense. Furthermore, for any two points $\tilde{x}, \tilde{y} \in \bigcup S^1_{ij}$, we have $\mathrm{dist}_{\tilde{g}_{r_0}}(\tilde{x}, \tilde{y}) \leq n\epsilon(r_0) + \times 200n^2 r_0$. Furthermore, by [49, Corollary 3.21], $(\mathbb{T}^{n+1}, \tilde{g}_{r_0})$ converges to the zero current in the intrinsic flat sense. However, we directly see that the metric tensors converge to the flat metric on $\mathbb{T}^{n+1}$ in L^p for all $p < \infty$. Using this fact and appealing to the proof of Theorem 1.7, we see that $(\mathbb{T}^{n+1}, \tilde{g}_{r_0})$ converges to the flat $(n+1)$–dimensional torus in the d_p sense for each $p \in [n+2, \infty)$.

9.3 Scalar curvature of the metrics $g_{\delta,\epsilon}$

In this subsection, we show that the negative part of the scalar curvature of $g_{\delta,\epsilon}$ can be made arbitrarily small if δ, ϵ are small enough. More precisely, we have the following.

Proposition 9.10 *For any $n \geq 3$ and $\eta > 0$, there exist $\epsilon_0 > 0$ and $\delta_0 > 0$ depending on η such that for all $\epsilon \leq \epsilon_0$ and $\delta \leq \delta_0$, the metric $g_{\delta,\epsilon}$ defined in (9-1) satisfies $R_{g_{\delta,\epsilon}} \geq -\eta$.*

To begin with, we need the following expression for the scalar curvature of a metric g taking the general form of (9-1).

Lemma 9.11 *Let $M = \mathbb{R}_+ \times \mathbb{S}^{n-1} \times \mathbb{R}$ and let h denote the standard metric on $\mathbb{S}^{n-1}$. For any metric g on M taking the form*

$$g = dr^2 + f^2(r)h + \varphi^2(x,r)\,dx^2, \tag{9-17}$$

the scalar curvature R_g of g is given by

$$R_g = \frac{n-1}{f^2}[2-(f^2)''] + \frac{(n-4)(n-1)}{f^2}[1-(f')^2] - \frac{2\varphi''}{\varphi} - \frac{2(n-1)\varphi' f'}{\varphi f}. \tag{9-18}$$

Here, the prime denotes a derivative with respect to r.

Proof Let $i, j, k, \dots$ be the coordinates on $\mathbb{S}^{n-1}$. As noted in the statement of the lemma, for a function F we will use F' to denote $\partial_r F$. We let F_x denote $\partial_x F$. We first compute the Christoffel symbols that will be needed in our computation. First, we have

$$\Gamma^B_{rA} = \tfrac{1}{2} g^{BC}(\partial_r g_{AC} + \partial_A g_{rC} - \partial_C g_{rA}) = \tfrac{1}{2} g^{BC}\partial_r g_{AC} = \begin{cases} \dfrac{f'}{f}\delta^i_j & \text{if } A = j, B = i, \\ \dfrac{\varphi'}{\varphi} & \text{if } A = x, B = x, \\ 0 & \text{otherwise.} \end{cases} \tag{9-19}$$

Next, note that

$$\Gamma^k_{ij} = \tfrac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) = \tfrac{1}{2} h^{kl}(\partial_i h_{jl} + \partial_j h_{il} - \partial_l h_{ij}) = \widetilde{\Gamma}^k_{ij}, \tag{9-20}$$

where $\widetilde{\Gamma}^k_{ij}$ denotes the Christoffel symbol for the standard metric h. Next, we compute

$$\begin{aligned} \Gamma^r_{ij} &= -\tfrac{1}{2} g^{rr}\partial_r g_{ij} = -ff'h_{ij}, \\ \Gamma^r_{xx} &= -\tfrac{1}{2}\partial_r \varphi^2 = -\varphi\varphi', \\ \Gamma^x_{xx} &= \tfrac{1}{2} g^{xx}(\partial_x g_{xx} + \partial_x g_{xx} - \partial_x g_{xx}) = \frac{\varphi_x}{\varphi}. \end{aligned} \tag{9-21}$$

The remaining Christoffel symbols vanish: $\Gamma^x_{ij} = \Gamma^\bullet_{ix} = \Gamma^i_{xx} = 0$.

With these Christoffel symbols in hand, we compute the Ricci curvatures R_{rr}, R_{ij} and R_{xx}. We have

$$\begin{aligned} R_{rr} &= \partial_A \Gamma^A_{rr} - \partial_r \Gamma^A_{rA} + \Gamma^A_{AB}\Gamma^B_{rr} - \Gamma^B_{rA}\Gamma^A_{Br} \\ &= 0 - (\partial_r \Gamma^i_{ri} + \partial_r \Gamma^x_{rx}) + 0 - (\Gamma^j_{ri}\Gamma^j_{ri} + \Gamma^x_{rx}\Gamma^x_{rx}) \\ &= -\sum_{i=1}^{n-1} \partial_r\left(\frac{f'}{f}\right) - \partial_r\left(\frac{\varphi'}{\varphi}\right) - \sum_{i=1}^{n-1}\left(\frac{f'}{f}\right)^2 - \frac{(\varphi')^2}{\varphi^2} = -(n-1)\frac{f''}{f} - \frac{\varphi''}{\varphi}. \end{aligned} \tag{9-22}$$

For R_{ij}, we have

$$
\begin{aligned}
R_{ij} &= \partial_A \Gamma^A_{ij} - \partial_j \Gamma^A_{iA} + \Gamma^A_{AB}\Gamma^B_{ij} - \Gamma^B_{jA}\Gamma^A_{Bi} \\
&= \widetilde{R}_{ij} + (\partial_r \Gamma^r_{ij} + \partial_x \Gamma^x_{ij}) - (\partial_j \Gamma^r_{ir} + \partial_j \Gamma^x_{ix}) \\
&\qquad + (\Gamma^r_{rk}\Gamma^k_{ij} + \Gamma^x_{xk}\Gamma^k_{ij} + \Gamma^A_{Ar}\Gamma^r_{ij} + \Gamma^A_{Ax}\Gamma^x_{ij}) \\
&\qquad\qquad - (\Gamma^r_{jk}\Gamma^k_{ri} + \Gamma^x_{jk}\Gamma^k_{xi} + \Gamma^B_{jr}\Gamma^r_{iB} + \Gamma^B_{jx}\Gamma^x_{iB}) \\
&= \widetilde{R}_{ij} + \partial_r \Gamma^r_{ij} + \Gamma^k_{ij}(\Gamma^r_{rk} + \Gamma^x_{xk}) + \Gamma^r_{ij}(\Gamma^k_{kr} + \Gamma^x_{xr}) - (\Gamma^r_{jk}\Gamma^k_{ri} + \Gamma^k_{jr}\Gamma^r_{ik}) \\
&= \widetilde{R}_{ij} - (f')^2 h_{ij} - f f'' h_{i\bar{j}} + (-f f' h_{ij})\left[(n-1)\frac{f'}{f} + \frac{\varphi'}{\varphi}\right] + 2(f')^2 h_{ij} \\
&= \widetilde{R}_{ij} - h_{ij}\left(f f'' + (n-2)(f')^2 + \frac{\varphi' f f'}{\varphi}\right).
\end{aligned}
\tag{9-23}
$$

Here, $\widetilde{R}_{ij}$ denotes the Ricci curvature of h. Finally, for R_{xx}, we have

$$
\begin{aligned}
R_{xx} &= \partial_A \Gamma^A_{xx} - \partial_x \Gamma^A_{xA} + \Gamma^A_{AB}\Gamma^B_{xx} - \Gamma^B_{xA}\Gamma^A_{Bx} \\
&= (\partial_r \Gamma^r_{xx} + \partial_x \Gamma^x_{xx}) - (\partial_x \Gamma^x_{xx}) + (\Gamma^r_{xx}\Gamma^A_{Ar} + \Gamma^x_{xx}\Gamma^A_{Ax}) \\
&\qquad - (\Gamma^B_{xr}\Gamma^r_{Bx} + \Gamma^B_{xx}\Gamma^x_{Bx}) \\
&= (-\varphi\varphi'' - (\varphi')^2) + \Gamma^r_{xx}(\Gamma^i_{ir} + \Gamma^x_{rx}) + \Gamma^x_{xx}\Gamma^x_{xx} \\
&\qquad - (\Gamma^x_{xr}\Gamma^r_{xx} + \Gamma^r_{xx}\Gamma^x_{rx} + \Gamma^x_{xx}\Gamma^x_{xx}) \\
&= (-\varphi\varphi'' - (\varphi')^2) + \Gamma^r_{xx}\Gamma^i_{ir} - \Gamma^x_{xr}\Gamma^r_{xx} \\
&= (-\varphi\varphi'' - (\varphi')^2) + (n-1)(-\varphi\varphi')\frac{f'}{f} + (\varphi')^2 \\
&= -\varphi\varphi'' - (n-1)\frac{\varphi\varphi' f'}{f}.
\end{aligned}
\tag{9-24}
$$

Therefore, using the fact that $\widetilde{R}_{ij} = (n-2)h_{ij}$, we have

$$
\begin{aligned}
R &= g^{rr} R_{rr} + g^{ij} R_{ij} + g^{xx} R_{xx} \\
&= -(n-1)\frac{f''}{f} - \frac{\varphi''}{\varphi} \\
&\qquad + \frac{1}{f^2} h^{ij}\left(\widetilde{R}_{ij} - h_{ij}\left(f f'' + (n-2)(f')^2 + \frac{\varphi' f f'}{\varphi}\right)\right) \\
&\qquad\qquad - \frac{1}{\varphi^2}\left(\varphi\varphi'' + (n-1)\frac{\varphi\varphi' f'}{f}\right) \\
&= -(n-1)\frac{f''}{f} - \frac{2\varphi''}{\varphi} - \frac{(n-1)\varphi' f'}{\varphi f} \\
&\qquad + (n-1)\left[\frac{n-2}{f^2} - \frac{f''}{f} - (n-2)\frac{(f')^2}{f^2} - \frac{\varphi' f'}{\varphi f}\right]
\end{aligned}
\tag{9-25}
$$

$$
\begin{aligned}
&= -\frac{2(n-1)f''}{f} - \frac{2\varphi''}{\varphi} + \frac{(n-2)(n-1)}{f^2} \\
&\qquad\qquad - \frac{(n-2)(n-1)(f')^2}{f^2} - \frac{2(n-1)\varphi' f'}{\varphi f} \\
&= -\frac{2(n-1)f''}{f} + \frac{(n-2)(n-1)}{f^2}[1-(f')^2] - \frac{2(n-1)\varphi' f'}{\varphi f} - \frac{2\varphi''}{\varphi}.
\end{aligned}
$$

Finally, to arrive at (9-18), we note that $2f''/f = (f^2)''/f^2 - 2(f')^2/f^2$, and so

$$
\begin{aligned}
-\frac{2(n-1)f''}{f} &= -\frac{(n-1)(f^2)''}{f^2} + \frac{2(n-1)(f')^2}{f^2} \\
&= \frac{n-1}{f^2}[2-(f^2)''] - \frac{2(n-1)}{f^2}[1-(f')^2].
\end{aligned} \tag{9-26}
$$

Rearranging this expression, we arrive at (9-18), completing the proof. □

With Lemma 9.11 in hand, we are now ready to prove Proposition 9.10.

Proof of Proposition 9.10 For notational convenience, we will omit the indices δ and ϵ, letting g, f and φ denote $g_{\delta,\epsilon}$, $f_{\delta,\epsilon}$, and $\varphi_{\delta,\epsilon}$, respectively. We will assume that $\epsilon \le \epsilon_0$ and $\delta \le \delta_0$, where $\epsilon_0, \delta_0 < \frac{1}{4}$ will be fixed within the proof. By Lemma 9.11, the scalar curvature of g takes the form

$$
\begin{aligned}
R &= \frac{n-1}{f^2}[2-(f^2)''] + \frac{(n-4)(n-1)}{f^2}[1-(f')^2] - \frac{2\varphi''}{\varphi} - \frac{2(n-1)\varphi' f'}{\varphi f} \\
&= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.
\end{aligned} \tag{9-27}
$$

We estimate the scalar curvature from below in three different intervals of r in the cases that follow. First, recall from the definition (9-7) of f that

$$
f(r) = \begin{cases} \tilde{f}(r) & \text{for } r \le 2, \\ r & \text{for } r \ge 4, \end{cases} \tag{9-28}
$$

where $\tilde{f}$ is the solution of the ODE (9-6). We first collect some useful estimates for $\tilde{f}$. For notational convenience, we write $\sigma = 10^4 n\delta(1-\zeta)$ and $\sigma_0 = 10^4 n\delta$ in the definition of (9-6) so that $\tilde{f}' = 1-\sigma$ where σ increases from 0 to $\sigma_0 = 10^4 n\delta$. We will assume that δ_0 is small enough that $\sigma_0 < \frac{1}{4}$. Since $\tilde{f}(0) = 0$, by integrating, we find that

$$
\tfrac{3}{4} r < (1-\sigma_0) r \le \tilde{f}(r) \le r \quad \text{and} \quad 1-\sigma_0 \le \tilde{f}'(r) \le 1. \tag{9-29}
$$

Case 1 $\left(r \le \frac{1}{2}\epsilon\right)$ In this case, $\varphi(r) \equiv \epsilon$, so terms III and IV in (9-27) vanish, and $f = \tilde{f}$ thanks to (9-28). We first consider the case when $n \ge 4$, since II is clearly nonnegative by (9-6). By (9-6) and (9-29), if δ_0 is sufficiently small that $\sigma < 1$, then

$$\begin{aligned} \mathrm{I}+\mathrm{II} &\geq \frac{n-1}{f^2}(2-(f^2)'') = \frac{2(n-1)}{f^2}(1-ff''-(f')^2) \\ &\geq \frac{2(n-1)(2-\sigma)\sigma}{f^2} > \frac{2(n-1)\sigma}{f^2}. \end{aligned} \tag{9-30}$$

Here we have used $\sigma' \geq 0$ and hence $R > 0$ when $n \geq 4$. It remains to consider $n = 3$. In this case, we have

$$\begin{aligned} \mathrm{I}+\mathrm{II} &= \frac{2}{f^2}(1-(f^2)''+(f')^2) = \frac{2}{f^2}(1-2ff''-(f')^2) \\ &\geq \frac{2}{f^2}(1-(1-\sigma)^2) \geq \frac{2\sigma}{f^2}. \end{aligned} \tag{9-31}$$

Hence we also have $R > 0$ when $n = 3$ as long as δ_0 is small enough.

Case 2 $\left(r \in \left[\frac{1}{2}\epsilon, 2\right]\right)$ In this case, we still have $f = \tilde{f}$ by (9-28) and $\sigma \equiv 10^4 n\delta$ in this range. Therefore by (9-29) and the computation in Case 1, for $n \geq 3$ and sufficiently small δ_0,

$$\mathrm{I}+\mathrm{II} \geq \frac{\sigma}{f^2} \geq \frac{10^4 n\delta}{r^2}. \tag{9-32}$$

On the other hand, by (9-5), we see that

$$\mathrm{III} = -\frac{2\varphi''}{\varphi} \geq -\frac{100\delta}{r^2}. \tag{9-33}$$

Similarly, using (9-5) and (9-29) we find that

$$\mathrm{IV} = -\frac{2(n-1)f'\varphi'}{f\varphi} \geq -\frac{1200n\delta}{r^2}. \tag{9-34}$$

Hence, $R = \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV} > 0$.

Case 3 $(r > 2)$ In this case, $\varphi \equiv 1$, and so terms III and IV in (9-27) vanish. Furthermore, we directly see that $\mathrm{I} = \mathrm{II} = 0$ when $r \geq 4$, since $f(r) = r$ there. So it remains to show that $\mathrm{I}+\mathrm{II} \geq -\eta$ in the case when $r \in (2,4)$. Note that for r in this interval, we have from (9-29) that

$$f(r) = \zeta\left(\tfrac{1}{4}r\right)\tilde{f}(r) + \left(1-\zeta\left(\tfrac{1}{4}r\right)\right)r \geq \tfrac{1}{2}r \geq 1. \tag{9-35}$$

Therefore, it remains to estimate $|2-(f^2)''|$ and $|1-(f')^2|$ for $r \in (2,4)$. By rewriting $2-(f^2)'' = (r^2-f^2)''$ and $1-(f')^2 = (r')^2-(f')^2$, it suffices to estimate $(r-f)''$.

By the construction of ζ and (9-29) for $r \in (2,4)$,

$$(9\text{-}36)\qquad \begin{aligned}|(f-r)''| &\le \left|\zeta''\left(\tfrac{1}{4}r\right)(\tilde f - r)\right| + \left|\zeta'\left(\tfrac{1}{4}r\right)(\tilde f - r)'\right| + \left|\zeta\left(\tfrac{1}{4}r\right)(\tilde f - r)''\right| \\ &\le 10|\tilde f - r| + 10|\tilde f' - 1| \le 100\sigma_0.\end{aligned}$$

Combining with (9-29), we conclude that $\| f - r\|_{C^2((2,4))} \le C_n\delta_0$. Hence for $r \in (2,4)$,

$$(9\text{-}37)\qquad \begin{aligned}R &= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} \\ &\ge -(n-1)|(r^2 - f^2)''| - (n-4)(n-1)|1-(f')^2| \ge -C_n\delta_0.\end{aligned}$$

Therefore, if δ_0 is sufficiently small, then the right-hand side will be larger than $-\eta$. This completes the proof. □

9.4 Entropy lower bound for the metrics $g_{\delta,\epsilon}$

In this section, we will show that the entropy of the metric $g_{\delta,\epsilon}$ on $\mathbb{R}^{n+1}$ can be made arbitrarily small by taking ϵ and δ to be sufficiently small for $n \ge 3$.

Proposition 9.12 *For any $n \ge 3$ and $\eta, L > 0$, there exists an $\epsilon_0 > 0$ such that for any $\epsilon, \delta \le \epsilon_0$, the metric $g_{\delta,\epsilon}$ defined in (9-1) satisfies*

$$(9\text{-}38)\qquad \nu(g_{\delta,\epsilon}, L) \ge -\eta.$$

Before we give the proof of Proposition 9.12, we start with some basic notation and preliminaries. Throughout this section, it will be convenient to rescale the metric by τ^{-1} so that we only need to estimate $\mu(g,\tau) = \mu(\tau^{-1}g, 1)$. Given a minimizer f_τ of $\mu(\tau^{-1}g, 1)$, we define the associated function

$$u_\tau = (4\pi)^{-n/4}\exp\left\{-\tfrac{1}{2}f_\tau\right\},$$

which satisfies $\int_M u_\tau^2\, d\mathrm{vol}_{\tau^{-1}g} = 1$. In a slight abuse of terminology, we will refer to this function u_τ as a minimizer of $\mu(\tau^{-1}g, 1)$. A minimizer u_τ of $\mu(\tau^{-1}g, 1)$ is a positive smooth function satisfying the Euler–Lagrange equation

$$(9\text{-}39)\qquad -4\Delta u_\tau + Ru_\tau - 2u_\tau \ln u_\tau - \left(\tfrac{1}{2}(n+1)\log(4\pi) + (n+1) + \mu(\tau^{-1}g, 1)\right)u_\tau = 0,$$

where the Laplacian and scalar curvature are with respect to $\tau^{-1}g$. On Euclidean space $\mathbb{R}^{n+1}$, it is well-known that the minimizers u of $\mu(g_{\mathrm{euc}}, 1)$ are uniquely given by the Gaussian functions

$$(9\text{-}40)\qquad u^2 = (4\pi)^{-(n+1)/2}\exp\left\{-\tfrac{1}{4}|x-y|^2\right\},$$

where $|\cdot|$ is the Euclidean metric and y is a fixed point on $\mathbb{R}^{n+1}$. Indeed, this is precisely the log-Sobolev inequality on Euclidean space. The following lemma shows that these functions are in fact the only bounded $W^{1,2}$ subsolutions of (9-39) on Euclidean space.

Lemma 9.13 (characterization of solutions on Euclidean space) *Fix $\mu \le 0$ and let $u \in W^{1,2}(\mathbb{R}^{n+1})$ be a bounded solution to*

$$4\Delta u + u \log u^2 + \big(\tfrac{1}{2}(n+1)\log(4\pi) + (n+1) + \mu\big)u \ge 0 \tag{9-41}$$

in $\mathbb{R}^{n+1}$ with

$$\int_{\mathbb{R}^n} u^2\, dx \le 1. \tag{9-42}$$

Then $\mu = 0$ and u takes the form (9-40) for some $y \in \mathbb{R}^{n+1}$.

Proof By [54, Lemma 2.3], a bounded solution u of (9-41) with $\|u\|_{L^2(\mathbb{R}^{n+1})} \le 1$ has Gaussian decay in the sense that for any $y \in \mathbb{R}^{n+1}$, there exist positive numbers r_0, a and A depending on y such that

$$u(y) \le A \exp\{-a|x-y|^2\} \quad \text{when } |x-y| \ge r_0.$$

In particular, multiplying the equation (9-41) by u and integrating over $\mathbb{R}^{n+1}$, we are justified in integrating by parts to find

$$\begin{aligned} 0 &\ge \int_{\mathbb{R}^{n+1}} -4u\Delta u - u^2 \log u^2 - \big((n+1) + \tfrac{1}{2}(n+1)\log(4\pi) + \mu\big)u^2\, dx \\ &= \int_{\mathbb{R}^{n+1}} 4|\nabla u|^2 - u^2 \log u^2 - \big((n+1) + \tfrac{1}{2}(n+1)\log(4\pi) + \mu\big)u^2\, dx. \end{aligned} \tag{9-43}$$

Now, set $w = u/\|u\|_{L^2(\mathbb{R}^{n+1})}$ so that $\|w\|_{L^2(\mathbb{R}^{n+1})} = 1$. Divide (9-43) by $\|u\|^2_{L^2(\mathbb{R}^{n+1})}$ to see that

$$\begin{aligned} 0 &\ge \int_{\mathbb{R}^{n+1}} 4|\nabla w|^2 - w^2 \log w^2 - w^2 \log \|u\|^2_{L^2(\mathbb{R}^{n+1})} \\ &\qquad\qquad - \big((n+1) + \tfrac{1}{2}(n+1)\log(4\pi) + \mu\big)w^2\, dx \\ &= \int_{\mathbb{R}^{n+1}} 4|\nabla w|^2 - w^2 \log w^2 - \big((n+1) + \tfrac{1}{2}(n+1)\log(4\pi)\big)w^2\, dx \\ &\qquad\qquad - \log \|u\|^2_{L^2(\mathbb{R}^{n+1})} - \mu \\ &\ge -\log \|u\|^2_{L^2(\mathbb{R}^{n+1})} - \mu, \end{aligned} \tag{9-44}$$

where the final inequality is obtained by applying the Euclidean log-Sobolev inequality to w. By (9-42) and $\mu \le 0$, we conclude that $\mu = 0$ and $\|u\|_{L^2(\mathbb{R}^{n+1})} = 1$. Then from (9-43), we see that u is a minimizer of the Euclidean log-Sobolev inequality and so u^2 is a Gaussian as in (9-42). This concludes the proof. □

The following lemma ensures the existence of the minimizer of the entropy for $(M, g_{\delta,\epsilon})$ with exponential decay at infinity.

Lemma 9.14 (existence and estimates for extremals) *Fix $\epsilon, \delta, L > 0$ and $\tau \in (0, L)$ and let $g_{\delta,\epsilon}$ be the metric defined in (9-1). A minimizer u_τ of the entropy $\mu(\tau^{-1} g_{\delta,\epsilon}, 1)$ exists. Furthermore there exist constants $a, A > 0$ and a point $y \in M$ depending on δ, ϵ and τ such that u_τ satisfies*

$$u_\tau(x) \le A \exp\{-a\, d^2(x, y)\}. \tag{9-45}$$

Here $d(\cdot,\cdot)$ denotes the geodesic distance with respect to $\tau^{-1} g_{\delta,\epsilon}$.

Proof The existence of minimizers follows from minor modifications of the existence proof of [54, Theorem 1.1(a)]; we therefore only outline these modifications. As in the proof of [54, Theorem 1.1(a)], we let v_k denote a minimizer of the entropy restricted to the ball $B_g(0, k)$ (here we let $g = \tau^{-1} g_{\delta,\epsilon}$) and let x_k be a point where v_k achieves its maximum. Viewing g as a metric on $\mathbb{R}^n \times \mathbb{R}$ with coordinates $(z, y) \in \mathbb{R}^n \times \mathbb{R}$, the metric is translation invariant with respect to the y component. Hence we may assume without loss of generality that $x_k = (z_k, 0)$ for all k. There are then two cases to consider: either the sequence $\{z_k\}$ is bounded in $\mathbb{R}^n$, or it is not. In the case that the sequence is bounded, the existence of a minimizer follows just as in [54, Theorem 1.1(a)]. The case that $\{z_k\}$ is unbounded is even simpler than the corresponding case in [54]: since $\mu(\tau^{-1} g_{\delta,\epsilon}, 1) < 0$, then arguing as in [54, Theorem 1.1(a)] we obtain a bounded $W^{1,2}$ solution to (9-41) with $\mu < 0$ on $\mathbb{R}^{n+1}$, a contradiction to Lemma 9.13. Finally, the Gaussian decay is established in [54, Lemma 2.3]. □

As usual, let $g_{\delta,\epsilon}$ be the metric defined in (9-1). In the next lemma, we show that the rescaled metric $\tau^{-1} g_{\delta,\epsilon}$ is uniformly close to the Euclidean metric in any compact subset centered at $\bar{x}$, after appropriate change of coordinate. To do this, we define an explicit diffeomorphism centered at $\bar{x}$ that takes into account how and in which direction the metric is degenerating.

In the application, we will take $\bar{x}$ to be a point such that u_τ, the minimizer of $\mu(\tau^{-1} g_{\delta,\epsilon}, 1)$, achieves its maximum at $\bar{x}$. Thanks to the symmetry of $g_{\delta,\epsilon}$, we can and will assume without loss of generality that $\bar{y} = 0$ and that $\bar{z} = \bar{r}\,\bar{e}$ for a fixed unit vector $\bar{e} \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$. For this reason, in what follows we will always assume $\bar{x}$ is as above. Before giving the precise statement, we would like to introduce some notation. Consider $\delta, \epsilon > 0$ and $\tau > 0$ fixed. Let $\ell \in \mathbb{R}^n \times \mathbb{R}$ be the line defined by

$$\ell = \{(0^n, y) : y \in \mathbb{R}\}. \tag{9-46}$$

For $\bar{x} = (\bar{r}\bar{e}, 0) \in M = \mathbb{R}^n \times \mathbb{R}$, where $\bar{e} \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$, we define an associated diffeomorphism $\Phi_{\tau,\bar{r},\delta,\epsilon} \colon \mathbb{R}^n \times \mathbb{R} \to M$ by

$$\Phi_{\tau,\bar{r},\delta,\epsilon}(x) = \left(\tau^{1/2} z + \bar{r}\bar{e}, \frac{\tau^{1/2}}{\varphi_{\delta,\epsilon}(\bar{r} + \tau^{1/2})} y\right). \tag{9-47}$$

Notice that $\Phi_{\tau,\bar{r},\delta,\epsilon}(0) = \bar{x}$. In principle, there are two rescalings performed by $\Phi_{\tau,\bar{r},\delta,\epsilon}$. Firstly, we rescale the coordinates centered at $\bar{x}$ by τ to compensate for the scaling of the metric by τ^{-1}. Secondly, we additionally rescale the y coordinate to account for the degeneracy of $\varphi_{\delta,\epsilon}$. Under the diffeomorphism, the image of ℓ where the metric degenerates will play a key role. Let $\bar{\rho} = \tau^{-1/2}\bar{r}$ and denote the pullback of ℓ as

$$\tilde{\ell} := \Phi^*_{\tau,\bar{r},\delta,\epsilon}(\ell) = \{(-\overline{\rho e}, y) : y \in \mathbb{R}\}. \tag{9-48}$$

In particular, if $\bar{\rho}$ tends to infinity along the sequence, then the degeneracy $\tilde{\ell}$ is pushed off to infinity and becomes invisible in the limit. On the other hand, if $\bar{\rho}$ stays bounded, we will show that the singularity of the sequence of rescaled pullback metric is still mild and close to the Euclidean metric away from $\tilde{\ell}$.

The next lemma shows that the pullback of $\tau^{-1} g_{\delta,\epsilon}$ under $\Phi_{\tau,\bar{r},\delta,\epsilon}$ converges to the Euclidean metric away from $\tilde{\ell}$ as $\delta, \epsilon \to 0$.

Lemma 9.15 (good charts) *Fix $n \geq 3$ and $L > 0$, and consider sequences $\epsilon_i, \delta_i \to 0$, $\tau_i \in (0, L]$ and $\bar{r}_i \in (0, \infty)$. Let $\bar{\rho}_i = \tau_i^{-1/2}\bar{r}_i$ and assume that $\bar{\rho}_i \to \bar{\rho}_\infty \in [0, \infty]$. Define $\tilde{\ell}_\infty = \{(-\bar{\rho}_\infty\bar{e}, y) : y \in \mathbb{R}\}$ if $\bar{\rho}_\infty < \infty$ and $\tilde{\ell}_\infty = \varnothing$ if $\bar{\rho} = \infty$.*

Then the metrics $g_i := \Phi^*_{\tau_i,\bar{r}_i,\epsilon_i}(\tau_i^{-1} g_{\delta_i,\epsilon_i})$ converge to the Euclidean metric $g_{\mathbb{R}^{n+1}}$ in $C^\infty_{\mathrm{loc}}(\mathbb{R}^{n+1} \setminus \tilde{\ell}_\infty)$. Furthermore, in the case when $\bar{\rho}_\infty < \infty$, we have

$$\min\left\{\tfrac{1}{2}, \left(\frac{|z + \bar{\rho}_i\bar{e}_i|}{1 + \bar{\rho}_i}\right)^{\delta_i}\right\} g_{\mathrm{euc}} \leq g_i(z, y) \leq g_{\mathrm{euc}}. \tag{9-49}$$

Here $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$.

Proof For convenience, we omit the index i and let $\varphi = \varphi_{\delta,\epsilon}$. Let $g = \Phi^*_{\tau,\bar{r},\delta,\epsilon}(\tau^{-1} g_{\delta,\epsilon})$. Then at a point $(z, y) \in \mathbb{R}^n \times \mathbb{R}$, we have

$$g|_{(z,y)} = g_{\mathrm{cone},f,\tau}(z + \overline{\rho e}) + \tilde{\varphi}^2(|z + \overline{\rho e}|)\, dy^2, \tag{9-50}$$

where $g_{\mathrm{cone},f,\tau}$ denotes the pullback of the cone metric on the $\mathbb{R}^n$ component and we set

$$\tilde{\varphi}(s) = \frac{\varphi(\tau^{1/2} s)}{\varphi(\tau^{1/2}(1 + \bar{\rho}))}. \tag{9-51}$$

It is clear that the cone metrics $g_{\mathrm{cone},f,\tau}$ converge smoothly to the standard Euclidean metric on $\mathbb{R}^n$ on any compact set as $i \to +\infty$. Therefore, to show the desired smooth convergence of the metrics away from $\tilde{\ell}_\infty$, it remains to show that

$$\tilde{\varphi}(|\cdot + \overline{\rho}\overline{e}|) \to 1 \tag{9-52}$$

smoothly on compact subsets of $\mathbb{R}^n \setminus \{-\overline{\rho}_\infty \overline{e}\}$. To this end, note that $\tilde{\varphi}$ satisfies $|\tilde{\varphi}'(s)/\tilde{\varphi}(s)| \le 50\delta/s$ by (9-5), and that $\tilde{\varphi}(1+\overline{\rho}) = 1$ by definition. In this way, we have

$$\begin{aligned} |\log \tilde{\varphi}(|z+\overline{\rho}\overline{e}|)| &= \left| \int_{1+\overline{\rho}}^{|z+\overline{\rho}\overline{e}|} \frac{\tilde{\varphi}'(s)}{\tilde{\varphi}(s)}\, ds \right| \\ &\le \left| \int_{1+\overline{\rho}}^{|z+\overline{\rho}\overline{e}|} \frac{50\delta}{s}\, ds \right| = 50\delta \left| \log\left(\frac{|z+\overline{\rho}\overline{e}|}{1+\overline{\rho}} \right) \right|. \end{aligned} \tag{9-53}$$

To establish (9-52), and hence the desired convergence, we consider two cases.

Case 1 ($\overline{\rho}_\infty = \infty$) When $\overline{\rho} = \infty$, it is clear that $|\cdot + \overline{\rho}_i \overline{e}|/(1+\overline{\rho}_i) \to 1$ uniformly on compact subsets as $i \to \infty$, and hence $\tilde{\varphi}(|\cdot + \overline{\rho}_i \overline{e}|) \to 1$ uniformly on compact subsets by (9-53). The higher-order convergence follows analogously by using (9-5), and thus we establish (9-52) in this case.

Case 2 ($\overline{\rho}_\infty < \infty$) Fix $\gamma \in (0,1]$. For any $z \in \mathbb{R}^n$ in the annular region defined by

$$\gamma(1+\overline{\rho}_\infty) \le |z + \overline{\rho}_\infty \overline{e}| \le \gamma^{-1}(1+\overline{\rho}_\infty), \tag{9-54}$$

we see from (9-53) that $|\log \tilde{\varphi}(|z+\overline{\rho}\overline{e}|)| \le 2\delta |\log \gamma|$ for i sufficiently large. Exponentiating both sides, we discover that

$$\gamma^{2\delta} \le \tilde{\varphi}(|z+\overline{\rho}\overline{e}|) \le \gamma^{-2\delta}. \tag{9-55}$$

So, as ϵ and δ tend to zero, we see that $\tilde{\varphi}(|z+\overline{\rho}\overline{e}|)$ converges uniformly to 1 for all z in this set. The convergence of the higher derivatives of φ follows in the same way thanks to (9-5), and we see that (9-52) holds in this case as well.

Finally, we show (9-49). The upper bound is immediate from the construction of the metrics. To establish the lower bound in (9-49), we note that, by construction,

$$g_{\mathrm{cone},f,\tau} \ge \tfrac{1}{2} g_{\mathbb{R}^n} \tag{9-56}$$

Next, notice that for any z such that $|z+\overline{\rho}\overline{e}| \ge 1+\overline{\rho}$, we have $\tilde{\varphi}(|z+\overline{\rho}\overline{e}|) \ge 1$ because φ is a monotone increasing function. On the other hand, for any z with $0 < |z+\rho\overline{\epsilon}| \le 1+\overline{\rho}$, we exponentiate the left- and right-hand sides of (9-53) to find that

$$\tilde{\varphi}(|z+\overline{\rho}\overline{e}|) \ge \left(\frac{|z+\overline{\rho}\overline{e}|}{1+\overline{\rho}} \right)^{\delta}. \tag{9-57}$$

Together, (9-56) and (9-57) establish (9-49). This concludes the proof of the lemma. □

In the proof of Proposition 9.12, we would like to show that the entropy $\mu(\tau^{-1}g_{\delta,\epsilon},1)$ converges to 0 for any $\tau\in(0,L]$ as $\epsilon,\delta\to 0$ by analyzing the limit of the corresponding sequence of minimizers. A key point will be to show that the limit is nontrivial, for which we need the following uniform mean value inequality.

Lemma 9.16 (mean value inequality) *Fix $n\ge 3$ and $L>0$, and consider sequences $\delta_i,\epsilon_i\to 0$, $\tau_i\in(0,L]$ and $\overline{r}_i\in(0,\infty)$. Let u_i be a minimizer of $\mu(\tau_i^{-1}g_{\delta_i,\epsilon_i},1)$, let $\Phi_{\tau_i,\overline{r}_i,\delta_i,\epsilon_i}$ be the diffeomorphism defined in (9-47), and let $v_i=\Phi^*_{\tau_i,\overline{r}_i,\delta_i,\epsilon_i}u_i$.*

Suppose that $\overline{\rho}_i=\overline{r}_i/\tau_i^{1/2}\to\overline{\rho}_\infty$ for some $\overline{\rho}_\infty\in[0,+\infty)$. Then there are $N=N(n,\overline{\rho}_\infty)\in\mathbb{N}$ and $C(n)>0$ such that if $i>N$, then for all $x\in\mathbb{R}^{n+1}$ we have

$$\|v_i\|_{L^\infty(B_{\mathrm{euc}}(x,1/4))}\le C(n)\left(\int_{B_{\mathrm{euc}}(x,1)} v_i^2\,d\mathrm{vol}_{g_i}\right)^{1/2} \tag{9-58}$$

for i sufficiently large. Here, the balls are taken with respect to the Euclidean metric on $\mathbb{R}^{n+1}$. In particular, $\|v_i\|_{L^\infty(\mathbb{R}^{n+1})}\le C(n)$.

If $\overline{\rho}_i\to+\infty$, then the L^∞–estimate holds in the following sense: for all $\Omega\Subset\mathbb{R}^{n+1}$, there is an $N=N(\Omega)\in\mathbb{N}$ such that if $i>N$, $\|v_i\|_{L^\infty(\Omega)}\le C(n)$.

Proof We let $g_i=\Phi^*_{\tau_i,\overline{r}_i,\delta_i,\epsilon_i}\tau_i^{-1}g_{\delta_i,\epsilon_i}$. For notational convenience, we will omit the index i when no confusion can arise. Moreover, each ball is taken with respect to the Euclidean metric.

If $\overline{\rho}_i\to+\infty$, then by Lemma 9.15, $g_i\to g_{\mathrm{euc}}$ in $C^k_{\mathrm{loc}}(\mathbb{R}^{n+1})$ for all $k\in\mathbb{N}$, and the result follows from standard Moser iteration; see for example [37] or [54]. It therefore suffices to consider the case where $\overline{\rho}_\infty<\infty$. In this case, we modify the Moser iteration argument to account for the mild singularity of g near ℓ.

Keeping in mind the lower bound for the metric (9-49) established in Lemma 9.15, we define the function $\Lambda(x)=\max\{2,\left(|z+\overline{\rho}\overline{e}|/(1+\overline{\rho})\right)^{-\delta}\}$ (where $x=(z,y)\in\mathbb{R}^n\times\mathbb{R}$) so that $g_{\mathrm{euc}}\le\Lambda(x)g$. We will make repeated use of the upper bound in (9-49), which implies that $d\mathrm{vol}_g\le dx$ and $|\partial u|\le|\nabla_g u|$ for any function u, where $|\partial u|$ denotes the Euclidean norm of the Euclidean gradient.

As a first step, fix any $p>1$ and $\Omega\Subset\mathbb{R}^{n+1}$. Provided $\delta<\delta_0(n,p)$, we note that $\|v\|_{L^{2/p}(\Omega,g_{\mathrm{euc}})}$ is uniformly bounded for any i. Indeed, apply Hölder's inequality and use the fact that $\int_{\mathbb{R}^n} v^2 \, d\mathrm{vol}_g = 1$ to see that

$$\text{(9-59)} \qquad \int_\Omega v^{2/p}\,dx \le \int_\Omega \Lambda^{(n+1)/2} v^{2/p}\,d\mathrm{vol}_g \le \left(\int_\Omega v^2\,d\mathrm{vol}_g\right)^{1/p}\left(\int_\Omega \Lambda^{(n+1)p'/2}\,dx\right)^{1/p'} \le C,$$

where $C = C(\mathrm{diam}(\Omega), p, n)$, provided that δ is small enough so that $\Lambda^{(n+1)p'/2}$ is integrable. Here p' denotes the Hölder conjugate of p.

Next, let $\bar{x} = (-\bar{\rho}\bar{e}, \bar{y})$ for some $\bar{y} \in \mathbb{R}$. Let $1 \ge r_1 > r_0 > \frac{1}{4}$ and let ϕ be a Lipschitz cutoff on $\mathbb{R}^{n+1}$ defined by

$$\text{(9-60)} \qquad \phi(x) = \begin{cases} 1 & \text{on } B(\bar{x}, r_0), \\ \dfrac{r_1 - |x-\bar{x}|}{r_1 - r_0} & \text{on } B(\bar{x}, r_1), \\ 0 & \text{outside } B(\bar{x}, r_1). \end{cases}$$

We will henceforth use B_r to denote $B(\bar{x}, r)$ for simplicity. We now establish an energy estimate. On one hand, since v satisfies (9-39), we multiply the equation by $v^{p-1}\phi^2$ and integrate to find that for $p > 1$,

$$\text{(9-61)} \qquad \begin{aligned} &\int_{B_{r_1}} \phi^2 v^{p-1}(-4\Delta v)\,d\mathrm{vol}_g \\ &\quad = \int_{B_{r_1}} \phi^2 v^p(-R + 2\log v + ((n+1) + \mu + \log(4\pi)^{(n+1)/2})\,d\mathrm{vol}_g \\ &\quad \le \int_{B_{r_1}} C_n \phi^2 v^p + 2\phi^2 v^p \log v\,d\mathrm{vol}_g. \end{aligned}$$

In the final line, we have used the fact that $R \ge -1$ thanks to Proposition 9.10, provided we choose ϵ and δ sufficiently small (thus i sufficiently large), and that $\mu = \mu(g_i, 1) \le 0$. Here the connection is with respect to g. On the other hand, we integrate by parts and apply the Cauchy–Schwarz inequality to find

$$\text{(9-62)} \qquad \begin{aligned} &\int_{B_{r_1}} \phi^2 v^{p-1}(-4\Delta v)\,d\mathrm{vol}_g \\ &\quad = \int_{B_{r_1}} 4\nabla v \cdot \nabla(\phi^2 v^{p-1})\,d\mathrm{vol}_g \\ &\quad \ge \int_{B_{r_1}} 4(p-1)\phi^2 v^{p-2}|\nabla v|^2 - 8v^{p-1}\phi|\nabla v|\,|\nabla\phi|\,d\mathrm{vol}_g \\ &\quad \ge \int_{B_{r_1}} 2(p-1)\phi^2 v^{p-2}|\nabla v|^2 - \frac{8}{p-1} v^p |\nabla\phi|^2\,d\mathrm{vol}_g. \end{aligned}$$

Combining (9-61) and (9-62), we see that

$$2(p-1)\int_{B_{r_1}} \phi^2 v^{p-2}|\nabla v|^2 \, d\mathrm{vol}_g \le \int_{B_{r_1}} C_n\phi^2 v^p + 2\phi^2 v^p \log v + \frac{8}{p-1} v^p |\nabla\phi|^2 \, d\mathrm{vol}_g. \tag{9-63}$$

Now, let us use the shorthand $\bar{\mu} = (n+2)/(n+1) > 1$ and $\nu = (n+2)/(n+3) < 1$. Noticing that $(2\nu)^* = 2(n+2)\nu/(n+2-2\nu) = 2\bar{\mu}$, we apply the Euclidean Sobolev inequality, followed by (9-49) and then Hölder's inequality to see that

$$\begin{aligned} \left(\int_{B_{r_1}} |\phi v^{p/2}|^{2\bar{\mu}} \, dx\right)^{1/2\bar{\mu}} &\le C_n\left(\int_{B_{r_1}} |\partial(\phi v^{p/2})|^{2\nu} \, dx\right)^{1/2\nu} \\ &\le C_n\left(\int_{B_{r_1}} \Lambda^{(n+1)/2}|\nabla(\phi v^{p/2})|^{2\nu} \, d\mathrm{vol}_g\right)^{1/2\nu} \\ &\le C_n\left(\int_{B_{r_1}} \Lambda^{(n+2)(n+1)/2} \, dx\right)^{1/2(n+1)}\left(\int_{B_r} |\nabla(\phi v^{p/2})|^2 \, d\mathrm{vol}_g\right)^{1/2} \\ &\le C_n(1+\bar{\rho}^2)^{c_n\delta}\left(\int_{B_r} |\nabla(\phi v^{p/2})|^2 \, d\mathrm{vol}_g\right)^{1/2}. \end{aligned} \tag{9-64}$$

Using the same trick of interchanging g and g_{euc}, (9-63) and Hölder's inequality imply

$$\begin{aligned} \int_{B_{r_1}} |\nabla(\phi v^{p/2})|^2 \, d\mathrm{vol}_g &\le \int_{B_{r_1}} p^2\phi^2 v^{p-2}|\nabla v|^2 + 2v^p|\nabla\phi|^2 \, d\mathrm{vol}_g \\ &\le \int_{B_{r_1}} C_n p\phi^2 v^p + C_n p v^p|\nabla\phi|^2 + C_n p\phi^2 v^p \log v \, d\mathrm{vol}_g \\ &\le \int_{B_{r_1}} C_n p\phi^2 v^p + C_n\Lambda v^p|\partial\phi|^2 + C_n p\phi^2 v^p \log v \, dx \\ &\le \frac{C_n p(1+\bar{\rho}^2)^{c_n\delta}}{(r_1-r_0)^2}\left(\int_{B_{r_1}} v^{p\lambda} \, dx\right)^{1/\lambda}. \end{aligned} \tag{9-65}$$

Here we let $\lambda = 2(n+2)/(2n+3) \in (1, \bar{\mu})$, and we have assumed that p is bounded uniformly away from 1 so that $p/(p-1)$ is bounded above by a universal constant. We have used the fact that $\int_{\mathbb{R}^{n+1}} v^2 \, d\mathrm{vol}_g = 1$ to control the term arising from $\log v$.

By combining this with (9-64) and replacing p by $p\lambda$, we conclude that for all $p > (2n+4)/(2n+3)$,

$$\|v\|_{L^{p\sigma}(B_{r_0}, g_{\mathrm{euc}})} \le \left[\frac{C_n p(1+\bar{\rho}^2)^{c_n\delta}}{(r_1-r_0)^2}\right]^{1/p} \|v\|_{L^p(B_{r_1}, g_{\mathrm{euc}})} \tag{9-66}$$

where $\sigma = (2n+3)/(2n+2) > 1$. Let $R_0 = 1$ and $R_k = 1 - \sum_{i=1}^k 2^{-i-1}$ so that $\lim_{k\to+\infty} R_k = \frac{1}{2}$. Applying (9-66) inductively with $r_1 = R_k$, $r_0 = R_{k+1}$ and $p = p_0\sigma^k$ for $p_0 \in ((2n+4)/(2n+3), 2)$, we have that if δ is small depending only on n, p_0 and $\bar{\rho}_\infty$, then

$$\begin{aligned} \log \|v\|_{L^{p_0\sigma^{k+1}}(B_{R_{k+1}}, g_{\mathrm{euc}})} &\le \log \|v\|_{L^{p_0}(B_1, g_{\mathrm{euc}})} + \sum_{i=1}^k \frac{C_n}{\sigma^i} \log\bigl(C_n(1+\bar{\rho}^2)^{c_n\delta}\sigma^i 2^i\bigr) \\ &\le C(n) + \log \|v\|_{L^{p_0}(B_1, g_{\mathrm{euc}})}. \end{aligned} \tag{9-67}$$

Here we have used the fact that $\bar{\rho}_i \to \bar{\rho}_\infty$. By letting $k \to +\infty$, we conclude that

$$\|v_i\|_{L^\infty(B_{1/2}(\bar{x}))} \le C(n)\left(\int_{B_1(\bar{x})} v_i^{p_0}\, dx\right)^{1/p_0} \tag{9-68}$$

for some $p_0 \in ((2n+4)/(2n+3), 2)$. Together with (9-59), we have the upper bound of v_i on $B_{1/2}(\bar{x})$. The result follows by choosing $p_0 < 2$ and applying Hölder's inequality once more using the integrability of Λ when we replace the volume form dx with $d\mathrm{vol}_g$. For $\bar{x} = (\bar{z}, \bar{y})$ where $|\bar{z} + \bar{\rho}_i \bar{e}| \ge \frac{1}{2}$, we can repeat the argument but apply the iteration on a small ball $B_{1/4}(\bar{x})$ so that g is uniformly equivalent to g_{euc}. ◻

Now we are ready to prove Proposition 9.12.

Proof of Proposition 9.12 Let $\eta > 0$ and for a complete Riemannian metric g, define

$$\tau_0(g) = \sup\{s > 0 : \mu(g,\tau) > -\eta \text{ for all } \tau \in (0,s)\}. \tag{9-69}$$

Since $g_{\delta,\epsilon}$ has bounded curvature and is not isometric to Euclidean space, we have $\mu(g_{\delta,\epsilon},\tau) < 0$; see Lemma 3.1 or the proof of [17, Lemma 17.19], replacing Perelman's differential Harnack estimate with its noncompact generalization established in [13].

Claim 1 *For $\delta, \epsilon \in (0,1)$ fixed, we have $\tau_0(g_{\delta,\epsilon}) > 0$.*

Proof of claim Since it is standard and similar to the proof in the compact case, see [17, Proposition 17.20], we sketch the proof only.

It suffices to show that $\lim_{\tau\to 0^+}\mu(g_{\delta,\epsilon},\tau)=0$. Since the constants ϵ and δ are fixed, we will omit the subscripts δ and ϵ for notational convenience. Supposing the claim is not true, we can find a sequence of $\tau_i\to 0^+$ such that $\lim_{i\to+\infty}\mu(g,\tau_i)<-\eta$ for some $\eta>0$. Consider the rescaled metric $g_i=\tau_i^{-1}g$, so that $\mu(g_i,1)<-\eta$ for i sufficiently large. By Lemma 9.14, we can find a sequence of minimizers u_i of $\mu(g_i,1)$ such that

$$-4\Delta u_i+R_iu_i-2u_i\log u_i-\big(\tfrac{1}{2}(n+1)\log(4\pi)+(n+1)+\mu(\tau_i^{-1}g,1)\big)u_i=0. \tag{9-70}$$

Since u_i has exponential decay at infinity by Lemma 9.14, we can find $p_i\in\mathbb{R}^{n+1}$ such that $u_i(p_i)=\max u_i$. Moreover, $u_i(p_i)\ge C(n,\eta)$ by applying the maximum principle to (9-70). On the other hand, since g is a smooth metric, it is easy to show that $(\mathbb{R}^{n+1},g_i,p_i)$ converges to $(\mathbb{R}^{n+1},g_{\mathrm{euc}},0)$ as $i\to+\infty$ in the C^∞–Cheeger–Gromov sense. Hence u_i will converge to $u_\infty>0$ (modulo diffeomorphism) in $C^\infty_{\mathrm{loc}}(\mathbb{R}^{n+1})$, and u_∞ satisfies $\int_{\mathbb{R}^{n+1}}u_\infty^2\,dx\le 1$ and solves the equation

$$4\Delta u_\infty+2u_\infty\log u_\infty+\big(\tfrac{1}{2}(n+1)\log(4\pi)+(n+1)+\mu_\infty\big)u_\infty=0, \tag{9-71}$$

where $\mu_\infty=\lim_{i\to\infty}\mu(\tau_i^{-1}g,1)\le-\eta$. Moreover, a standard Moser iteration argument shows that the u_i are uniformly bounded and hence u_∞ is bounded as well. This can be proved using the argument in the work of [37] or [54], or by modifying the proof of Lemma 9.16. Therefore, using the equation we see that $u_\infty\in W^{1,2}(\mathbb{R}^{n+1})$ and hence $\mu_\infty=0$ by Lemma 9.13, which contradicts $\mu(\tau_i^{-1}g,1)<-\eta$ for i sufficiently large. □

We now prove that there is an ϵ_0 small enough that for all $\epsilon,\delta<\epsilon_0$, we have $\tau_0(g_{\delta,\epsilon})\ge L$. The proof is similar to that of Claim 1 above, but additional care must be taken with respect to the convergence of the metrics and their corresponding minimizers. Suppose the conclusion is not true, and that we can find a sequence of $\delta_i,\epsilon_i\to 0^+$ such that $\tau_i=\tau_0(g_{\delta_i,\epsilon_i})<L$ for all i. By definition, we have $\mu_i=\mu(g_{\delta_i,\epsilon_i},\tau_i)=\mu(\tau_i^{-1}g_{\delta_i,\epsilon_i},1)=-\eta$. By Lemma 9.14, we can find a minimizer u_i of $\mu(\tau_i^{-1}g_{\delta_i,\epsilon_i},1)$ which attains its maximum at some $\bar{x}_i=(\bar{z}_i,\bar{y}_i)\in\mathbb{R}^n\times\mathbb{R}$. We may assume without loss of generality that $\bar{y}_i=0$ and $\bar{z}_i=\bar{r}_i\bar{e}$ for some fixed $\bar{e}\in\mathbb{S}^{n-1}\subset\mathbb{R}^n$ by the symmetry of g_{δ_i,ϵ_i}. We again let $\bar{\rho}_i=\tau_i^{1/2}/\bar{r}_i$. Up to a subsequence, which we will not relabel, we have $\bar{\rho}_i\to\bar{\rho}_\infty\in[0,\infty]$.

Let $\Phi_{\tau_i,\bar{r}_i,\delta_i,\epsilon_i}$ be the diffeomorphism defined in (9-47) using $\bar{x}_i$ above and let $g_i=\Phi_{\tau_i,\bar{r}_i,\delta_i,\epsilon_i}(\tau_i^{-1}g_{\delta_i,\epsilon_i})$. If $\bar{\rho}_\infty=+\infty$, then by Lemma 9.15, we have convergence $(\mathbb{R}^{n+1},\tau_i^{-1}g_{\delta_i,\epsilon_i},\bar{x}_i)\to(\mathbb{R}^{n+1},g_{\mathrm{euc}},0)$ in the C^∞–Cheeger–Gromov sense. Then the proof of Claim 1 carries over and we reach a contradiction to the assumption that $\mu_i = -\eta$. It therefore suffices to consider the case when $\bar{\rho}_\infty < +\infty$.

As in Lemma 9.15, we let $\tilde{\ell}_\infty = \{(-\bar{\rho}_\infty \bar{e}, y) : y \in \mathbb{R}\}$. By Lemma 9.15, g_i converges to g_{euc} in $C^k_{\mathrm{loc}}(\mathbb{R}^{n+1} \setminus \tilde{\ell}_\infty)$. Together with the L^∞ estimate of $v_i = \Phi^*_{\tau_i, \bar{r}_i, \delta_i, \epsilon_i} u_i$ from Lemma 9.16, we have $v_i \to v_\infty$ in $C^\infty_{\mathrm{loc}}(\mathbb{R}^{n+1} \setminus \tilde{\ell}_\infty)$ and $L^p_{\mathrm{loc}}(\mathbb{R}^{n+1})$ for $p > 0$, by the dominated convergence theorem. Therefore, $\int_{\mathbb{R}^{n+1}} v_\infty^2 \, dx \le 1$ by Fatou's lemma and v_∞ solves

$$4\Delta v_\infty + 2v_\infty \log v_\infty + \big(\tfrac{1}{2}(n+1)\log(4\pi) + (n+1) - \eta\big) v_\infty = 0 \tag{9-72}$$

on $\mathbb{R}^{n+1} \setminus \tilde{\ell}_\infty$. It remains to establish the following claim.

Claim 2 *The limit v_∞ is nontrivial and $v_\infty \in W^{1,2}(\mathbb{R}^{n+1})$.*

Proof of claim We first show that v_∞ is nontrivial. By Lemma 9.16, we have

$$v_i(0) \le C \left(\int_{B_1(0)} v_i^2 \, d\mathrm{vol}_{g_i} \right)^{1/2} \tag{9-73}$$

for some universal constant C independent of i. On the other hand, by the decay rate of v_i from Lemma 9.14, we may apply the maximum principle to the Euler–Lagrange equation at its maximum point, which by our selection of $\bar{x}_i$ is the origin, to show that at $x = 0$,

$$2v_i \log v_i \ge R_{g_i} v_i - \big(\tfrac{1}{2}(n+1)\log(4\pi) + (n+1) - \eta\big) v_i, \tag{9-74}$$

and hence

$$\int_{B_1(0)} v_i^2 \, d\mathrm{vol}_{g_i} \ge c(n, \eta). \tag{9-75}$$

This shows that $v_\infty \ne 0$ as $v_i \to v_\infty$ in $L^p_{\mathrm{loc}}(\mathbb{R}^{n+1})$ and $g_i \to g_{\mathbb{R}^{n+1}}$ in $L^p_{\mathrm{loc}}(\mathbb{R}^{n+1})$ by construction for all $p > 0$.

It remains to show that $v_\infty \in W^{1,2}(\mathbb{R}^{n+1})$. Since $\int_{\mathbb{R}^{n+1}} v_\infty^2 \, dx \le 1$, it remains to consider $\|\partial v_\infty\|_{L^2(\mathbb{R}^{n+1})}$. We first point out that for each g_i, $\|\nabla v_i\|_{L^2(\mathbb{R}^{n+1}, g_i)}$ is uniformly bounded. This can be seen by integrating the Euler–Lagrange equation with a cutoff function ϕ with respect to g_i,

$$0 = \int_{\mathbb{R}^{n+1}} \phi v_i \Big(-4\Delta_{g_i} v_i + R_{g_i} v_i - \big(\tfrac{1}{2}(n+1)\log(4\pi) + (n+1) - \eta\big) v_i - 2v_i \log v_i \Big) \, d\mathrm{vol}_{g_i}. \tag{9-76}$$

Since the scalar curvature of g_i is uniformly bounded from below and v_i is uniformly bounded by Lemma 9.16 for i sufficiently large, we have for a suitable cutoff function ϕ,

$$\begin{aligned}(9\text{-}77)\quad C(n,\eta)\int_{\mathbb{R}^{n+1}} v_i^2\, d\mathrm{vol}_{g_i} &\geq \int_{\mathbb{R}^{n+1}} \phi v_i(-\Delta_{g_i} v_i)\, d\mathrm{vol}_{g_i} + 8\int_{\mathbb{R}^{n+1}} v_i^2 \frac{|\nabla\phi|^2}{\phi}\, d\mathrm{vol}_{g_i} \\ &= \int_{\mathbb{R}^{n+1}} \nabla v_i\cdot\nabla(\phi v_i)\, d\mathrm{vol}_{g_i} + 8\int_{\mathbb{R}^{n+1}} v_i^2 \frac{|\nabla\phi|^2}{\phi}\, d\mathrm{vol}_{g_i} \\ &\geq \frac{1}{2}\int_{\mathbb{R}^{n+1}} \phi|\nabla v_i|^2\, d\mathrm{vol}_{g_i}.\end{aligned}$$

By letting $\phi\to 1$, this gives the uniform boundedness of $\|\nabla v_i\|_{L^2(\mathbb{R}^{n+1},g_i)}$. Now let $B_{\mathbb{R}^n}(-\bar{\rho}_\infty\bar{e},r)$ be the ball of radius r in $\mathbb{R}^n$ and $p>1$. By the metric equivalence from Lemma 9.15, $\Lambda^{-1}g_{\mathrm{euc}}\leq g_i\leq g_{\mathrm{euc}}$ for some $\Lambda(z,y)\leq\max\{|z+\bar{\rho}_\infty\bar{e}|^{-\delta},2\}$,

$$\begin{aligned}(9\text{-}78)\quad \int_{B_{\mathbb{R}^n}(-\bar{\rho}_\infty\bar{e},r)\times[-r,r]} |\partial v_i|^{2/p}\, dx &\leq \int_{B_{\mathbb{R}^n}(-\bar{\rho}_\infty\bar{e},r)\times[-r,r]} \Lambda^{n/2}|\nabla v_i|^{2/p}\, d\mathrm{vol}_{g_i} \\ &\leq \left(\int_{\mathbb{R}^{n+1}} |\nabla v_i|^2\, d\mathrm{vol}_{g_i}\right)^{1/p}\left(\int_{B_{\mathbb{R}^n}(-\bar{\rho}_\infty\bar{e},r)\times[-r,r]} \Lambda^{np^*/2}\, dx\right)^{1/p^*} \\ &\leq C^{1/p}\left[\frac{Cr^{n+1}}{1-c_n\delta_i p^*}\right]^{1/p^*}.\end{aligned}$$

Letting $i\to+\infty$ followed by $p\to 1$ and $r\to+\infty$, we have $v_\infty\in W^{1,2}(\mathbb{R}^{n+1})$. ◻

By the claim and the proof of Lemma 9.13, we deduce that

$$\lim_{i\to+\infty}\mu(\tau_i^{-1}g_{\delta_i,\epsilon_i},1)=\mu_\infty=0,$$

which contradicts the fact that $\mu(\tau_i^{-1}g_{\delta_i,\epsilon_i},1)=-\eta$. This completes the proof. ◻

References

[1] **L Ahlfors**, **A Beurling**, *Conformal invariants and function-theoretic null-sets*, Acta Math. 83 (1950) 101–129 MR Zbl

[2] **B Allen**, *Almost non-negative scalar curvature on Riemannian manifolds conformal to tori*, J. Geom. Anal. 31 (2021) 11190–11213 MR Zbl

[3] **B Allen**, **L Hernandez-Vazquez**, **D Parise**, **A Payne**, **S Wang**, *Warped tori with almost non-negative scalar curvature*, Geom. Dedicata 200 (2019) 153–171 MR Zbl

[4] **B Allen**, **C Sormani**, *Contrasting various notions of convergence in geometric analysis*, Pacific J. Math. 303 (2019) 1–46 MR Zbl

[5] **B Allen**, **C Sormani**, *Relating notions of convergence in geometric analysis*, Nonlinear Anal. 200 (2020) art. id. 111993 MR Zbl

[6] **L Ambrosio**, **N Gigli**, **G Savaré**, *Heat flow and calculus on metric measure spaces with Ricci curvature bounded below—the compact case*, from "Analysis and numerics of partial differential equations" (F Brezzi, P Colli Franzone, U Gianazza, G Gilardi, editors), Springer INdAM Ser. 4, Springer (2013) 63–115 MR Zbl

[7] **L Ambrosio**, **S Honda**, *New stability results for sequences of metric measure spaces with uniform Ricci bounds from below*, from "Measure theory in non-smooth spaces" (N Gigli, editor), De Gruyter Open, Warsaw (2017) 1–51 MR Zbl

[8] **L Ambrosio**, **S Honda**, *Local spectral convergence in* $\mathrm{RCD}^*(K, N)$ *spaces*, Nonlinear Anal. 177 (2018) 1–23 MR Zbl

[9] **D Bakry**, **M Émery**, *Hypercontractivité de semi-groupes de diffusion*, C. R. Acad. Sci. Paris Sér. I Math. 299 (1984) 775–778 MR Zbl

[10] **R H Bamler**, *A Ricci flow proof of a result by Gromov on lower bounds for scalar curvature*, Math. Res. Lett. 23 (2016) 325–337 MR Zbl

[11] **P Burkhardt-Guim**, *Pointwise lower scalar curvature bounds for* C^0 *metrics via regularizing Ricci flow*, Geom. Funct. Anal. 29 (2019) 1703–1772 MR Zbl

[12] **A J Cabrera Pacheco**, **C Ketterer**, **R Perales**, *Stability of graphical tori with almost nonnegative scalar curvature*, Calc. Var. Partial Differential Equations 59 (2020) art. id. 134 MR Zbl

[13] **A Chau**, **L-F Tam**, **C Yu**, *Pseudolocality for the Ricci flow and applications*, Canad. J. Math. 63 (2011) 55–85 MR Zbl

[14] **J Cheeger**, *Comparison and finiteness theorems for Riemannian manifolds*, PhD thesis, Princeton University (1967) MR Zbl Available at `https://www.proquest.com/docview/302249825`

[15] **J Cheeger**, *Finiteness theorems for Riemannian manifolds*, Amer. J. Math. 92 (1970) 61–74 MR Zbl

[16] **J Cheeger**, *Differentiability of Lipschitz functions on metric measure spaces*, Geom. Funct. Anal. 9 (1999) 428–517 MR Zbl

[17] **B Chow**, **S-C Chu**, **D Glickenstein**, **C Guenther**, **J Isenberg**, **T Ivey**, **D Knopf**, **P Lu**, **F Luo**, **L Ni**, *The Ricci flow: techniques and applications, III: Geometric-analytic aspects*, Mathematical Surveys and Monographs 163, Amer. Math. Soc., Providence, RI (2010) MR Zbl

[18] **B Chow**, **D Knopf**, *The Ricci flow: an introduction*, Mathematical Surveys and Monographs 110, Amer. Math. Soc., Providence, RI (2004) MR Zbl

[19] **B Chow**, **P Lu**, **L Ni**, *Hamilton's Ricci flow*, Graduate Studies in Math. 77, Amer. Math. Soc., Providence, RI (2006) MR Zbl

[20] **G De Cecco**, **G Palmieri**, *Integral distance on a Lipschitz Riemannian manifold*, Math. Z. 207 (1991) 223–243 MR Zbl

[21] **D M DeTurck**, *Deforming metrics in the direction of their Ricci tensors*, J. Differential Geom. 18 (1983) 157–162 MR Zbl

[22] **L C Evans**, **R F Gariepy**, *Measure theory and fine properties of functions*, CRC Press, Boca Raton, FL (1992) MR Zbl

[23] **B Fuglede**, *Extremal length and functional completion*, Acta Math. 98 (1957) 171–219 MR Zbl

[24] **F W Gehring**, *The L^p–integrability of the partial derivatives of a quasiconformal mapping*, Acta Math. 130 (1973) 265–277 MR Zbl

[25] **N Gigli**, **A Mondino**, **G Savaré**, *Convergence of pointed non-compact metric measure spaces and stability of Ricci curvature bounds and heat flows*, Proc. Lond. Math. Soc. 111 (2015) 1071–1129 MR Zbl

[26] **M Gromov**, *Metric structures for Riemannian and non-Riemannian spaces*, Birkhäuser (2007) MR Zbl

[27] **M Gromov**, *Dirac and Plateau billiards in domains with corners*, Cent. Eur. J. Math. 12 (2014) 1109–1156 MR Zbl

[28] **M Gromov**, **H B Lawson, Jr**, *Spin and scalar curvature in the presence of a fundamental group, I*, Ann. of Math. 111 (1980) 209–230 MR Zbl

[29] **P Hajłasz**, *Sobolev spaces on metric-measure spaces*, from "Heat kernels and analysis on manifolds, graphs, and metric spaces" (P Auscher, T Coulhon, A Grigor'yan, editors), Contemp. Math. 338, Amer. Math. Soc., Providence, RI (2003) 173–218 MR Zbl

[30] **P Hajłasz**, **P Koskela**, *Sobolev met Poincaré*, Mem. Amer. Math. Soc. 688, Amer. Math. Soc., Providence, RI (2000) MR Zbl

[31] **R S Hamilton**, *Three-manifolds with positive Ricci curvature*, J. Differential Geometry 17 (1982) 255–306 MR Zbl

[32] **R S Hamilton**, *A compactness property for solutions of the Ricci flow*, Amer. J. Math. 117 (1995) 545–572 MR Zbl

[33] **H-J Hein**, **A Naber**, *New logarithmic Sobolev inequalities and an ϵ–regularity theorem for the Ricci flow*, Comm. Pure Appl. Math. 67 (2014) 1543–1561 MR Zbl

[34] **J Heinonen**, **P Koskela**, *Quasiconformal maps in metric spaces with controlled geometry*, Acta Math. 181 (1998) 1–61 MR Zbl

[35] **B Kleiner**, **J Lott**, *Notes on Perelman's papers*, Geom. Topol. 12 (2008) 2587–2855 MR Zbl

[36] **P Li**, **R Schoen**, *L^p and mean value properties of subharmonic functions on Riemannian manifolds*, Acta Math. 153 (1984) 279–301 MR Zbl

[37] **Y Li**, *Ricci flow on asymptotically Euclidean manifolds*, Geom. Topol. 22 (2018) 1837–1891 MR Zbl

[38] **J Lott**, **C Villani**, *Ricci curvature for metric-measure spaces via optimal transport*, Ann. of Math. 169 (2009) 903–991 MR Zbl

[39] **O E Maasalo**, *The Gehring lemma in metric spaces*, preprint (2007) arXiv 0704.3916

[40] **A Mondino**, **A Naber**, *Structure theory of metric measure spaces with lower Ricci curvature bounds*, J. Eur. Math. Soc. 21 (2019) 1809–1854 MR Zbl

[41] **F Mugelli**, **G Talenti**, *Sobolev inequalities in 2-D hyperbolic space: a borderline case*, J. Inequal. Appl. 2 (1998) 195–228 MR Zbl

[42] **V H Nguyen**, *The sharp Poincaré–Sobolev type inequalities in the hyperbolic spaces $\mathbb{H}^n$*, J. Math. Anal. Appl. 462 (2018) 1570–1584 MR Zbl

[43] **G Perelman**, *The entropy formula for the Ricci flow and its geometric applications*, preprint (2002) Zbl arXiv math/0211159

[44] **P Petersen**, *Riemannian geometry*, Graduate Texts in Math. 171, Springer (1998) MR Zbl

[45] **R Schoen**, **S T Yau**, *Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds with nonnegative scalar curvature*, Ann. of Math. 110 (1979) 127–142 MR Zbl

[46] **N Shanmugalingam**, *Newtonian spaces: an extension of Sobolev spaces to metric measure spaces*, Rev. Mat. Iberoamericana 16 (2000) 243–279 MR Zbl

[47] **W-X Shi**, *Deforming the metric on complete Riemannian manifolds*, J. Differential Geom. 30 (1989) 223–301 MR Zbl

[48] **C Sormani**, *Scalar curvature and intrinsic flat convergence*, from "Measure theory in non-smooth spaces" (N Gigli, editor), De Gruyter Open, Warsaw (2017) 288–338 MR Zbl

[49] **C Sormani**, **S Wenger**, *The intrinsic flat distance between Riemannian manifolds and other integral current spaces*, J. Differential Geom. 87 (2011) 117–199 MR Zbl

[50] **K-T Sturm**, *Generalized Ricci bounds and convergence of metric measure spaces*, C. R. Math. Acad. Sci. Paris 340 (2005) 235–238 MR Zbl

[51] **B Wang**, *The local entropy along Ricci flow, A: The no-local-collapsing theorems*, Camb. J. Math. 6 (2018) 267–346 MR Zbl

[52] **B Wang**, *The local entropy along Ricci flow*, *B*: *The pseudo-locality theorems*, preprint (2020) arXiv 2010.09981

[53] **Q S Zhang**, *Bounds on volume growth of geodesic balls under Ricci flow*, Math. Res. Lett. 19 (2012) 245–253 MR Zbl

[54] **Q S Zhang**, *Extremal of log Sobolev inequality and W entropy on noncompact manifolds*, J. Funct. Anal. 263 (2012) 2051–2101 MR Zbl

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