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Algebraic Spivak's theorem and applications

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We prove an analogue of Lowrey and Schürg’s algebraic Spivak’s theorem when working over a base ring A that is either a field or a nice enough discrete valuation ring, and after inverting the residual characteristic exponent e in the coefficients. By this result algebraic bordism groups of quasiprojective derived A -schemes can be generated by classical cycles, leading to vanishing results for low-degree e -inverted bordism classes, as well as to the classification of quasismooth projective A -schemes of low virtual dimension up to e -inverted cobordism. As another application, we prove that e -inverted bordism classes can be extended from an open subset, leading to the proof of homotopy invariance of e -inverted bordism groups for quasiprojective derived A -schemes.

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1. Introduction	351
2. Background	357
3. Presentations of Chern classes and refined projective bundle formulas	372
4. Algebraic Spivak’s theorem and applications	380
References	395

1 Introduction

Algebraic bordism Ω_* is the universal oriented Borel–Moore homology theory in algebraic geometry. It should naturally fit in the more general framework of bivariant algebraic cobordism, together with algebraic cobordism Ω^* , the corresponding ring-valued cohomology theory. These two theories Ω_* and Ω^* contain an abundance of geometric information: they are expected to refine the Chow and the G -theory groups, and the Chow cohomology and K -theory rings, respectively (note that, as of now, there is no well-established Chow cohomology theory), and, furthermore, the latter

groups and rings should be recoverable from the former by a straightforward algebraic operation. Consequently, computing the algebraic (co)bordism groups of schemes should be infeasible in general. However, any formulas proven for (co)bordism classes will automatically hold for the induced classes in other oriented theories, and therefore the role of algebraic (co)bordism is comparable to that of the Grothendieck ring of varieties in the study of Euler characteristics: it serves as the canvas for the universal oriented (co)homology computations, which clarify the geometric essence of formulas of which other theories only see the shadows.

There are, roughly speaking, two approaches to algebraic cobordism, neither of which gives a satisfactory theory as of this moment. The original approach was to consider the theory represented by the motivic spectrum MGL in Morel and Voevodsky's \mathbb{A}^1 -homotopy theory [23]. This approach has several advantages. The associated bivariant theory of Déglise [9] provably satisfies most of the expected properties, such as projective bundle formula, and the existence of homological localization long exact sequences. In fact, it is expected that the Borel–Moore homology theory represented by MGL gives the correct (higher) algebraic bordism groups. These advantages are countered by two main disadvantages, the first of which seems to be a fundamental flaw in the approach: any theory coming from \mathbb{A}^1 -homotopy theory is \mathbb{A}^1 -invariant, but algebraic cobordism should not be \mathbb{A}^1 -invariant (because K -theory is not). The second disadvantage of the approach is its abstractness: it is very hard to give geometric interpretation for the groups, which is in stark contrast to the topological case; see eg Quillen [24].

The second approach, which is also the approach of our paper, is to define (co)bordism directly in geometric terms, namely in terms of cobordism cycles modulo an algebraic geometric analogue of the cobordism relation. This is also the approach that seems to be preeminent in applications, due to its simplicity and geometric nature; see for example the proof of the degree 0 Donaldson–Thomas conjectures by Levine and Pandharipande [21], the computation of K -theoretical degeneracy classes of Hudson, Ikeda, Matsumura and Naruse [14], subsequently generalized to algebraic bordism by Hudson and Matsumura [15], and the relationship between algebraic Morava K -theories and torsion in Chow groups studied by Sechin [25]. All of the above applications employ the algebraic bordism theory of characteristic 0 algebraic schemes, studied by Levine and Morel in their seminal work [20], which was later simplified by the employment of derived algebraic geometry by Lowrey and Schürg [22]. The approach of Lowrey and Schürg also gave the first serious candidate for a geometric algebraic bordism in positive characteristic. This construction was later extended to a bivariant theory by

the author with Yokura [2; 6] using the universal bivariant theory of Yokura [29], and the author studied these theories in greater detail in [5; 3; 4].

Besides its geometric nature, the main advantage of the second approach is that its associated cohomology theory has the expected relationship with K -theory, implying that the theory is not \mathbb{A}^1 -invariant. This gives strong evidence that at least the cohomological part of the theory gives the correct model for algebraic cobordism. However, this approach also has several disadvantages, the most serious being the difficulties with defining higher cobordism groups and proving localization exact sequences, both of which are easily taken care of in the context of motivic homotopy theory. It is conceivable that both of these difficulties could be resolved by a deeper understanding of derived algebraic geometry. The relationship between the two approaches is not well understood: the only known comparison results are the ones obtained in characteristic 0 by Levine [19], identifying Levine–Morel algebraic bordism with part of the motivic homotopy algebraic bordism. Finally, we note that the dichotomy between the two approaches is not perfect, and recent work of Elmanto, Hoyois, Khan, Sosnilo and Yakerson [10] employs both geometric models and motivic homotopy theory.

The purpose of this paper is twofold. The first goal is to further our understanding of the geometric bordism groups in positive and mixed characteristic. The second goal, which uses the improved understanding of algebraic bordism, is to make progress in the following cohomological conjecture:

Conjecture 1.1 *Let A be a local Noetherian ring. Then the natural map*

$$\mathbb{L}^* \rightarrow \Omega^*(\mathrm{Spec}(A))$$

is an isomorphism, where \mathbb{L}^ is the Lazard ring (with cohomological grading).*

The above conjecture is known in characteristic 0; see [20]. Moreover, by standard arguments employing “twisting” of cohomology theories and the Conner–Floyd theorem established in [5], the full conjecture is known for with rational coefficients. We manage to make partial progress on this conjecture in the cases where A is a field or a nice enough discrete valuation ring.

1.1 Summary of results

We denote by Ω^\bullet the base-independent bivariant algebraic cobordism constructed in [4], which has no grading as a bivariant theory. This theory generalizes both algebraic bordism Ω_\bullet and algebraic cobordism Ω^* , the latter of which has a natural grading.

The results of Spivak [27] and Lowrey and Schürg [22] serve as the inspiration for our main result, an *algebraic Spivak's theorem*. These results state that certain cobordism theories, constructed using derived geometry (in the context of differential and algebraic geometry, respectively), coincide with their classical counterparts. Since the corresponding classically defined cobordism theory is the theory of Levine and Morel, which is defined only in characteristic 0, we have no theory to compare derived bordism with in positive and mixed characteristic. Instead, we prove that all derived bordism cycles can be expressed in terms of classical cycles.

Theorem 4.12 *Let A be a field or an excellent Henselian discrete valuation ring with a perfect residue field, and let e be the residual characteristic exponent of A . Then the algebraic cobordism ring $\Omega^*(\mathrm{Spec}(A))[e^{-1}]$ is generated as a $\mathbb{Z}[e^{-1}]$ -algebra by classes of regular projective A -schemes. Moreover, for all quasiprojective derived A -schemes X , $\Omega_\bullet(X)[e^{-1}]$ is generated as an $\Omega^*(\mathrm{Spec}(A))[e^{-1}]$ -module by classes of regular schemes mapping projectively to X .*

In particular, the e -inverted algebraic bordism groups are generated by derived fibre products over A of regular A -schemes. It is possible to use this fact to prove that classical cycles generate these groups as $\mathbb{Z}[e^{-1}]$ -modules, leading to the following corollaries:

Corollary 4.13 *Suppose A is as in Theorem 4.12. Then, for all quasiprojective derived A -schemes X , $\Omega_\bullet(X)[e^{-1}]$ is generated as a $\mathbb{Z}[e^{-1}]$ -module by cycles of the form*

$$[V \rightarrow X]$$

with V a classical complete intersection scheme and the structure morphism $V \rightarrow \mathrm{Spec}(A)$ is either flat or factors through the unique closed point of $\mathrm{Spec}(A)$.

Corollary 4.14 *If k is a field, then the e -inverted bordism groups $\Omega_\bullet(X)[e^{-1}]$, where X is a quasiprojective derived k -scheme, are generated as $\mathbb{Z}[e^{-1}]$ -modules by classes of regular k -varieties mapping projectively to X .*

Note that these results are not strictly speaking generalizations of the results of Lowrey and Schürg: since we do not have a classically defined (geometric) bordism theory against which to compare Ω_\bullet , the best statement we can hope for is this kind of result stating that classical cycles are enough to generate the bordism groups. Notice also how these results bring us closer to proving [Conjecture 1.1](#) after inverting the residual

characteristic: instead of classifying all quasismooth and projective derived A -schemes up to cobordism, we only have to classify the lci projective A -schemes, or even just smooth projective k -varieties if $A = k$ is a perfect field, up to cobordism, and this seems like a much easier problem (and can be done in low dimensions, as we shall see below).

Most of the article is dedicated to proving the above results. The basic idea is to compute the fundamental class of a quasiprojective derived scheme X by resolving the singularities of the deformation to the normal cone of the truncated inclusion $X_{\text{cl}} \hookrightarrow U$, where U is an open subscheme of \mathbb{P}_A^n , and then using the special presentations of Chern classes given by [Lemma 3.5](#). Carrying out this strategy is made difficult by the facts that desingularization by alterations, unlike Hironaka's resolution in characteristic 0, may drastically change the geometry outside the singular locus, and the technique does not provide birational resolutions. To solve the first problem, we need to be able to approximate classes of generically finite morphisms to projective space, which is achieved by [Theorem 3.9](#), and, to solve the second problem, we need to invert the residual characteristic. Note that it is not easy to compute the classes of generically finite morphisms is general without localization exact sequences, which is why we only do the computation when the target is a projective space. Even this case is not simple, and [Theorem 3.9](#) is the main reason we have to restrict our attention to the case where the base ring A is an excellent Henselian discrete valuation ring having a perfect residue field instead of a more general excellent discrete valuation ring. In order to generalize further to, say, the case where A an excellent regular local ring of Krull-dimension ≤ 3 , one would need to generalize also the Bertini-regularity theorems from Ghosh and Krishna [\[13\]](#) to hold in this generality.

[Theorem 4.12](#), [Corollary 4.13](#) and especially [Corollary 4.14](#) considerably simplify the study of algebraic bordism whenever they apply, as derived schemes are much harder to study than classical schemes, let alone smooth varieties over a field. As the first immediate corollary, we obtain the following vanishing result:

Corollary 4.15 *Let X be a quasiprojective A -variety, with A as in [Theorem 4.12](#). Then the groups*

$$\Omega_i^A(X) := \Omega^{-i}(X \rightarrow \text{Spec}(A))$$

vanish for $i < -1$. If $A = k$ is a field, then $\Omega_i^k(X)$ vanish for $i < 0$.

Note that the above result is far from obvious without the algebraic Spivak's theorem, as there exists an abundance of derived schemes having negative virtual dimension.

One can also imagine such vanishing results being very useful, as they allow proving results by induction on degree, rather than by some ad hoc induction scheme. Another easy application is the computation of the cobordism rings in low degrees.

Corollary 4.17 *Let A be as in Theorem 4.12. Then:*

- (1) $\Omega^i(\text{Spec}(A))[e^{-1}]$ vanishes for $i > 0$, and the natural map

$$\mathbb{Z}[e^{-1}] \cong \mathbb{L}^0[e^{-1}] \rightarrow \Omega^0(\text{Spec}(k))[e^{-1}]$$

is an isomorphism; in other words, $\Omega^0(\text{Spec}(A))$ is the free $\mathbb{Z}[e^{-1}]$ -module generated by 1_A ;

- (2) If $A = k$ is a field, then the natural map

$$\mathbb{L}^*[e^{-1}] \rightarrow \Omega^*(\text{Spec}(k))[e^{-1}]$$

is an isomorphism in degrees 0, -1 and -2 ; in other words, $\Omega^1(\text{Spec}(k))$ is the free $\mathbb{Z}[e^{-1}]$ -module generated by $[\mathbb{P}_k^1]$ and $\Omega^2(\text{Spec}(k))$ is the free $\mathbb{Z}[e^{-1}]$ -module generated by $[\mathbb{P}_k^1]^2$ and $[\Sigma_{1,k}]$, where $\Sigma_{1,k}$ is the Hirzebruch surface of degree 1 over k .

We are hopeful that more progress towards the computation of algebraic cobordism of fields can be made in the future.

Another application, which is no longer an immediate corollary of the algebraic Spivak’s theorem, is the following extension result:

Theorem 4.19 *Let A be a field or an excellent Henselian discrete valuation ring with a perfect residue field, and let $j: X \hookrightarrow \bar{X}$ be an open embedding of quasiprojective derived A -schemes. Then the pullback morphism*

$$j^!: \Omega_\bullet(\bar{X})[e^{-1}] \rightarrow \Omega_\bullet(X)[e^{-1}]$$

is surjective.

Let us consider the difficulty in proving such a result: The most straightforward strategy would be to find for each bordism cycle $[Y \rightarrow X]$ a bordism cycle $[\bar{Y} \rightarrow \bar{X}]$ extending it. To achieve this, we would first relatively compactify Y over \bar{X} — see Gaitsgory and Rozenblyum [12, Section 5.2.2] — and then “resolve the singularities” to make the compactification quasismooth. Alas, such resolution results do not exist for derived schemes or in positive characteristic, and the proof of the extension theorem is dedicated

to overcoming this issue. Note also that this result is the right side of the conjectural localization exact sequence (known in characteristic 0 by the work of Levine and Morel), but it seems hard to say anything worthwhile about the kernel of $j^!$.

Combining the extension theorem with the projective bundle formula, we obtain the following homotopy invariance result:

Corollary 4.22 *Let A be as in [Theorem 4.19](#), let X be a quasiprojective derived A -scheme and let $p: E \rightarrow X$ be a vector bundle of rank r on X . Then the pullback map*

$$p^!: \Omega_{\bullet}(X)[e^{-1}] \rightarrow \Omega_{\bullet}(E)[e^{-1}]$$

is an isomorphism.

While this result was expected as the cobordism analogue of \mathbb{A}^1 -invariance of Chow groups and of Grothendieck groups of coherent sheaves, the reader should compare it with the fact that the cobordism rings $\Omega^*(X)$ are usually not \mathbb{A}^1 -invariant.

Conventions

All derived schemes are assumed to be Noetherian and of finite Krull dimension. Derived fibre product is denoted by $X \times_{\mathbb{Z}}^R Y$, and truncation is denoted by X_{cl} . A map of derived schemes is called *projective* if it is proper and admits a relatively ample line bundle. The *Krull dimension* of a derived scheme means the Krull dimension of its truncation. A flat lci morphism is called *syntomic*. An *snc scheme* is a scheme that can be globally expressed as an snc divisor inside a regular scheme. We will write

$$[r] := \{1, \dots, r\}.$$

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2 Background

In this section we recall the necessary background material.

2.1 Quasismooth morphisms and derived complete intersection schemes

In this section we review the background of various notions of derived complete intersections that we are going to use in the work. Let us recall that a closed embedding $Z \hookrightarrow X$ of derived schemes is called a *derived regular embedding (of virtual codimension r)* if it is locally on X of the form

$$\mathrm{Spec}(A// (a_1, \dots, a_r)) \hookrightarrow \mathrm{Spec}(A),$$

where $//$ denotes the derived quotient; see [18, 2.3.1]. In other words, Z is locally given as the derived vanishing locus of r functions on X . Let us begin with the following definition:

Definition 2.1 A finite-type morphism $f: X \rightarrow Y$ of Noetherian derived schemes is called *quasismooth* if the relative cotangent complex $\mathbb{L}_{X/Y}$ has Tor-dimension ≤ 1 (it follows that $\mathbb{L}_{X/Y}$ is perfect and f is of finite presentation; see eg [4, Proposition 2.23]). If $\mathbb{L}_{X/Y}$ has constant virtual rank d on X , then we say that f is quasismooth of *relative virtual dimension d* .

Let us then recall the basic properties of quasismooth morphisms.

- Proposition 2.2** (1) *Quasismooth morphisms are stable under composition and derived base change. Relative virtual dimension is additive under composition, and is preserved in derived pullbacks.*
- (2) *A morphism of classical schemes is quasismooth if and only if it is lci.*
- (3) *A closed embedding is quasismooth if and only if it is a derived regular embedding.*

Proof The first claim is obvious, the second is classical and the third is [18, Proposition 2.3.8]. \square

We are also going to need the following absolute version of quasismoothness:

Definition 2.3 A Noetherian derived scheme X is called a *derived complete intersection scheme* if only finitely many of the homotopy sheaves $\pi_i(\mathcal{O}_X)$ are nontrivial, and if for all points $x \in X$ the cotangent complex $\mathbb{L}_{\kappa(x)/X}$ has Tor-dimension ≤ 2 , where $\kappa(x)$ is the residue field of X_{cl} at x .

Derived complete intersection schemes admit the following alternative characterization:

Proposition 2.4 *Let X be a Noetherian derived scheme. Then the following are equivalent:*

- (1) X is a derived complete intersection scheme.
- (2) The cotangent complex $\mathbb{L}_{X/\mathbb{Z}}$ has Tor-dimension ≤ 1 .
- (3) For all morphisms $X \rightarrow Y$ with Y a regular scheme, the relative cotangent complex $\mathbb{L}_{X/Y}$ has Tor-dimension ≤ 1 .
- (4) There exists a morphism $X \rightarrow Y$ with Y regular such that the relative cotangent complex $\mathbb{L}_{X/Y}$ has Tor-dimension ≤ 1 .

Proof The equivalence of (1), (2) and (3) is [4, Proposition 2.28], but the proof shows also that they are all equivalent to (4). □

Example 2.5 The following types derived schemes are derived complete intersections:

- (1) Classical complete intersection schemes (in particular, regular schemes).
- (2) If X is a derived complete intersection and $f: Y \rightarrow X$ is a morphism such that $\mathbb{L}_{Y/X}$ has Tor-dimension ≤ 1 (eg if f is quasismooth), then Y is a derived complete intersection scheme.

2.2 Derived blow-ups

One of the main technical tools we are going to need in this article is the construction of derived blow-ups and derived deformation to normal cone from [18]. Let us recall the definitions and the results we are going to use:

Definition 2.6 Let $Z \hookrightarrow X$ be a derived regular embedding. Then, for any X -scheme S , a *virtual Cartier divisor* on S lying over Z is the datum of a commutative diagram

$$\begin{array}{ccc}
 D & \xrightarrow{i_D} & S \\
 \downarrow g & & \downarrow \\
 Z & \hookrightarrow & X
 \end{array}$$

such that:

- (1) i_D is a derived regular embedding of virtual codimension 1 (ie a virtual Cartier divisor).

- (2) The truncation is a Cartesian square.
- (3) The canonical morphism

$$g^* \mathcal{N}_{Z/X}^\vee \rightarrow \mathcal{N}_{D/S}^\vee$$

induces a surjection on π_0 .

It is then possible to define derived blow-ups via its functor of points.

Definition 2.7 Let $Z \hookrightarrow X$ be a derived regular embedding. Then the *derived blow-up* $\mathrm{Bl}_Z(X)$ is the X -scheme representing virtual Cartier divisors lying over Z . In other words, given an X -scheme S , the space of X -morphisms

$$S \rightarrow \mathrm{Bl}_Z(X)$$

is naturally identified with the maximal sub- ∞ -groupoid of the ∞ -category of virtual Cartier divisors of S that lie over Z .

Theorem 2.8 Let $i : Z \hookrightarrow X$ be a derived regular embedding with X Noetherian. Then:

- (1) The derived blow-up $\mathrm{Bl}_Z(X)$ exists as a derived scheme and is unique up to contractible space of choices.
- (2) The structure morphism $\pi : \mathrm{Bl}_Z(X) \rightarrow X$ is projective, quasismooth, and induces an equivalence

$$\mathrm{Bl}_Z(X) - \mathcal{E} \rightarrow X - Z,$$

where \mathcal{E} is the universal virtual Cartier divisor on $\mathrm{Bl}_Z(X)$ lying over Z (also called the **exceptional divisor**).

- (3) The derived blow-up $\mathrm{Bl}_Z(X) \rightarrow X$ is stable under derived base change.
- (4) The exceptional divisor \mathcal{E} is naturally identified with $\mathbb{P}_Z(\mathcal{N}_{Z/X})$.
- (5) If $Z \xrightarrow{i} X \xrightarrow{j} Y$ is a sequence of quasismooth closed embeddings, then there exists a natural derived regular embedding $\tilde{j} : \mathrm{Bl}_Z(X) \hookrightarrow \mathrm{Bl}_Z(Y)$, called the **strict transform**.
- (6) Given derived regular embeddings $i : Z \hookrightarrow X$ and $j : Y \hookrightarrow X$, the strict transforms \tilde{i} and \tilde{j} do not meet in $\mathrm{Bl}_{Z \cap Y}(X)$.

(7) If Z and X are classical schemes (so that $Z \hookrightarrow X$ is lci), there exists a natural equivalence

$$\mathrm{Bl}_Z(X) \simeq \mathrm{Bl}_Z^{\mathrm{cl}}(X),$$

where the right-hand side is the classical blow-up.

Proof The statements (1), (3), (4), (5) and (7) are directly from [18, Theorem 4.1.5]. The second claim is essentially from loc. cit., but the authors only prove that π is proper; projectivity of π follows from the fact that the line bundle $\mathcal{O}(-\mathcal{E})$ is π -ample. For a proof of (6), see for example [2, Lemma 4.5]. \square

Another result we are going to need is the following:

Proposition 2.9 *Let $Z \hookrightarrow X$ be a derived regular embedding. Then there exists a natural closed embedding $\mathrm{Bl}_{Z_{\mathrm{cl}}}^{\mathrm{cl}}(X_{\mathrm{cl}}) \hookrightarrow \mathrm{Bl}_Z(X)$ and a derived Cartesian square*

$$\begin{array}{ccc} \mathcal{E}_{\mathrm{cl}} & \hookrightarrow & \mathrm{Bl}_{Z_{\mathrm{cl}}}^{\mathrm{cl}}(X_{\mathrm{cl}}) \\ \downarrow & & \downarrow \\ \mathcal{E} & \hookrightarrow & \mathrm{Bl}_Z(X) \end{array}$$

where $\mathcal{E}_{\mathrm{cl}}$ is the classical exceptional divisor.

Proof Indeed, the closed embedding $i_{Z/X}$ from [3, Appendix B.4] has this property; see Lemma B.15 and Theorem B.16. \square

2.3 Algebraic cobordism

In this section, we will recall the base-independent algebraic cobordism of finite Krull-dimensional Noetherian derived schemes that admit an ample line bundle [4]. There exists a well-behaved extension of this theory to those (finite-dimensional, Noetherian) derived schemes that admit an ample family of line bundles, but, since we will not need this generality, we choose to restrict to this less complicated theory. An even simpler choice would have been to utilize the bivariant algebraic A -cobordism theories [3] for a fixed Noetherian ring A of finite Krull dimension. Although these theories have the advantage of being graded, this is outweighed by the clarity provided by working with a single base-independent theory rather than multiple base-dependent ones, which in any case can be recovered from the former. There is also the added benefit of (absolute)

algebraic bordism being naturally part of the more general theory: it is the bivariant group associated to the morphism $X \rightarrow \text{Spec}(\mathbb{Z})$.

Throughout this section we will denote by \mathcal{C} the ∞ -category of finite-dimensional Noetherian derived schemes admitting an ample line bundle.

2.3.1 Definition of algebraic (co)bordism Let us start with the definition of universal precobordism.

Definition 2.10 Let $X \rightarrow Y$ be a morphism in \mathcal{C} . Then the universal precobordism group $\underline{\Omega}^\bullet(X \rightarrow Y)$ is the group completion of the abelian monoid on cycles

$$[V \xrightarrow{f} X],$$

where f is projective, $\mathbb{L}_{V/Y}$ has Tor-dimension ≤ 1 and the monoid operation is given by taking disjoint union. These cycles are subjected to the *derived double point relations*: given a projective morphism $W \rightarrow \mathbb{P}^1 \times X$ such that $\mathbb{L}_{W/\mathbb{P}^1 \times X}$ has Tor-dimension ≤ 1 , and virtual Cartier divisors D_1 and D_2 on W such that their sum is the fibre W_∞ of $W \rightarrow \mathbb{P}^1$ over ∞ ,

$$[W_0 \rightarrow X] = [D_1 \rightarrow X] + [D_2 \rightarrow X] - [\mathbb{P}_{D_1 \times_W^R D_2}(\mathcal{O}(D_1) \oplus \mathcal{O}) \rightarrow X] \in \underline{\Omega}^\bullet(X),$$

where W_0 is the fibre of $W \rightarrow \mathbb{P}^1$ over 0. To see how the bivariant operations are defined, see for example [4, Definition 2.38].

Definition 2.11 If $X \rightarrow Y$ is a finite-type morphism in \mathcal{C} , then $\underline{\Omega}^\bullet(X \rightarrow Y)$ (and its quotient discussed below) has a naturally defined grading. Indeed, if

$$[V \rightarrow X] \in \underline{\Omega}^\bullet(X \rightarrow Y),$$

then $V \rightarrow Y$ is of finite type, and the condition on Tor-dimension implies that it is quasismooth. The *degree* of $[V \rightarrow X]$ is $-d$, where d is the relative virtual dimension of $V \rightarrow Y$. The bivariant group of $X \rightarrow Y$, equipped with this grading, is denoted by $\Omega^*(X \rightarrow Y)$.

Definition 2.12 (bivariant fundamental classes) Given a morphism $X \rightarrow Y$ in \mathcal{C} such that $\mathbb{L}_{X/Y}$ has Tor-dimension ≤ 1 , we have the *bivariant fundamental class*

$$1_{X/Y} := [X \rightarrow X] \in \underline{\Omega}^\bullet(X \rightarrow Y).$$

These classes give rise to a stable orientation on the bivariant theory $\underline{\Omega}^\bullet$. If $X \rightarrow X$ is the identity morphism, we will often use the shorthand notation $1_X := 1_{X/X}$, and, if either X or Y is a spectrum of a ring, we will often drop “Spec” from the notation.

It was shown in [4, Theorem 3.15] that there exists a formal group law

$$F(x, y) = \sum_{i, j \geq 0} a_{ij} x^i y^j \in \underline{\Omega}^\bullet(\text{Spec}(\mathbb{Z}))[[x, y]]$$

such that, for every $X \in \mathcal{C}$ and for all line bundles \mathcal{L}_1 and \mathcal{L}_2 on X , the equality

$$c_1(\mathcal{L}_1 \otimes \mathcal{L}_2) = F(c_1(\mathcal{L}_1), c_1(\mathcal{L}_2)) \in \underline{\Omega}^\bullet(X)$$

holds, where $c_1(\mathcal{L})$ is the *first Chern class* (also called the *Euler class*) of \mathcal{L} , ie the cycle represented by $[Z_s \hookrightarrow X]$, where s is any global section of \mathcal{L} and Z_s is the derived vanishing locus of s .

Definition 2.13 Consider the stack $[\mathbb{A}^1/\mathbb{G}_m] \times [\mathbb{A}^1/\mathbb{G}_m]$, which classifies the data

$$(\mathcal{L}_1, s_1, \mathcal{L}_2, s_2),$$

where \mathcal{L}_i are line bundles and s_i are global sections of \mathcal{L}_i . Let

$$(\mathcal{L}_1^u, s_1^u, \mathcal{L}_2^u, s_2^u)$$

be the universal such data classified by the identity morphism, and let us denote by \mathcal{V} , \mathcal{V}_i and \mathcal{V}_{12} the derived vanishing loci of $s_1^u s_2^u$, s_i^u and (s_1^u, s_2^u) inside $[\mathbb{A}^1/\mathbb{G}_m] \times [\mathbb{A}^1/\mathbb{G}_m]$. Notice that there exists a commutative square

$$\begin{array}{ccc} \mathcal{V}_{12} & \hookrightarrow & \mathcal{V}_2 \\ \downarrow & \searrow \iota^{12} & \downarrow \iota^2 \\ \mathcal{V}_1 & \hookrightarrow & \mathcal{V} \end{array}$$

and none of the maps ι^1 , ι^2 or ι^{12} is a derived regular embedding. The embeddings $\mathcal{V}_{12} \hookrightarrow \mathcal{V}_i$ are derived regular.

The stack \mathcal{V} is a gadget allowing us to detect and decompose derived schemes that look like a sum of two virtual Cartier divisors, and it is a fundamental tool in our definition of algebraic cobordism.

Definition 2.14 Let $X \rightarrow Y$ be a morphism in \mathcal{C} . Then the *bivariant algebraic cobordism* group $\underline{\Omega}^\bullet(X \rightarrow Y)$ is obtained by imposing the following *decomposition relation* on $\underline{\Omega}^\bullet(X \rightarrow Y)$: Given a projective morphism $f: V \rightarrow X$ such that $\mathbb{L}_{V/Y}$ has Tor-dimension ≤ 1 , and a morphism $V \rightarrow \mathcal{V}$, denote by V_i and V_{12} the derived pullbacks $V \times_{\mathcal{V}} \mathcal{V}_i$ and $V \times_{\mathcal{V}} \mathcal{V}_{12}$ and by f^i and f^{12} the induced projective morphisms

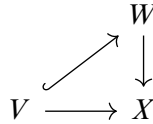
$V_i \rightarrow X$ and $V_{12} \rightarrow X$, respectively. If $\mathbb{L}_{V_1/Y}$ and $\mathbb{L}_{V_2/Y}$ have Tor-dimension ≤ 1 , then

$$(1) \quad [V \rightarrow X] = [V_1 \rightarrow X] + [V_2 \rightarrow X] + f_*^{12} \left(\sum_{i,j \geq 1} a_{ij} c_1(\mathcal{L}_1)^{i-1} \cdot c_1(\mathcal{L}_2)^{j-1} \cdot 1_{V_{12}/Y} \right)$$

in $\Omega^\bullet(X \rightarrow Y)$, where \mathcal{L}_i are the pullbacks of the universal line bundles \mathcal{L}_i^u and $1_{V_{12}/Y}$ is the bivariant fundamental class

$$[V_{12} \rightarrow V_{12}] \in \underline{\Omega}^\bullet(V_{12} \rightarrow Y).$$

Remark 2.15 The decomposition relation (1) holds already in $\underline{\Omega}^\bullet(X \rightarrow Y)$ if V is the sum of virtual Cartier divisors on a derived scheme W admitting a projective map $W \rightarrow X$ such that the triangle



commutes and $\mathbb{L}_{W/Y}$ has Tor-dimension ≤ 1 . It is not known whether or not such a W can be found in general.

Let us then recall in more detail the homology and cohomology theories associated to Ω^\bullet , starting with the homology theory.

Definition 2.16 (algebraic bordism) Given $X \in \mathcal{C}$, we define its *algebraic bordism* group as

$$\Omega_\bullet(X) := \Omega^\bullet(X \rightarrow \text{Spec}(\mathbb{Z})).$$

Let $f: X \rightarrow Y$ be a morphism in \mathcal{C} . If f is projective, then there exists a *pushforward morphism*

$$f_*: \Omega_\bullet(X) \rightarrow \Omega_\bullet(Y)$$

given by the formula

$$[V \xrightarrow{g} X] \mapsto [V \xrightarrow{f \circ g} Y],$$

and, if \mathbb{L}_f has Tor-dimension ≤ 1 , there exists a *Gysin pullback morphism*

$$f^!: \Omega_\bullet(Y) \rightarrow \Omega_\bullet(X)$$

given by the formula

$$[W \rightarrow Y] \mapsto [W \times_Y^R X \rightarrow X].$$

Pushforwards and Gysin pullbacks are functorial in the obvious sense.

The following result is an immediate consequence of [Proposition 2.4](#) and the bivariant formalism:

Proposition 2.17 *Let $Y \in \mathcal{C}$ be a regular scheme. Then, for all $X \rightarrow Y$ in \mathcal{C} , the morphism*

$$-\bullet 1_{Y/\mathbb{Z}}: \Omega_{\bullet}^Y(X) := \Omega^{\bullet}(X \rightarrow Y) \rightarrow \Omega_{\bullet}(X)$$

is an isomorphism, and these maps commute with pushforwards and Gysin pullbacks. □

Let us then recall the cohomology theory.

Definition 2.18 (algebraic cobordism) Given $X \in \mathcal{C}$, we define its *algebraic cobordism ring* as

$$\Omega^*(X) := \Omega^*(X \rightarrow X),$$

where the ring structure is given by the formula

$$[V \rightarrow X] \bullet [W \rightarrow X] = [V \times_X^R W \rightarrow X] \in \Omega^*(X),$$

and the grading is defined as in [Definition 2.11](#). Suppose then that $f: X \rightarrow Y$ is a morphism in \mathcal{C} . Then there exists a *pullback morphism*

$$f^*: \Omega^*(Y) \rightarrow \Omega^*(X)$$

given by the formula

$$[W \rightarrow Y] \mapsto [W \times_Y^R X \rightarrow X],$$

and, if f is projective and quasismooth, there exists a *Gysin pushforward morphism*

$$f_!: \Omega^*(X) \rightarrow \Omega^*(Y)$$

given by the formula

$$[V \xrightarrow{g} X] \mapsto [V \xrightarrow{f \circ g} Y].$$

Note that f^* is multiplicative and preserves the grading while $f_!$ does not have to do either. Both pullbacks and Gysin pushforwards are functorial in the obvious sense.

These theories have the following formal properties:

Proposition 2.19 (projection formula) *The group $\Omega_{\bullet}(X)$ is an $\Omega^*(X)$ -module, with the action given by*

$$[V \rightarrow X] \bullet [W \rightarrow X] = [V \times_X^R W \rightarrow X].$$

Moreover, if $f: X \rightarrow Y$ is a projective morphism, then, for all $\alpha \in \Omega^(Y)$ and all $\beta \in \Omega_{\bullet}(X)$, the equality*

$$f_*(f^*(\alpha) \bullet \beta) = \alpha \bullet f_*(\beta) \in \Omega_{\bullet}(Y)$$

holds. If f is projective and quasismooth, then, for all $\alpha \in \Omega^(Y)$ and all $\gamma \in \Omega^*(X)$, the equality*

$$f_!(f^*(\alpha) \bullet \gamma) = \alpha \bullet f_!(\gamma) \in \Omega^*(Y)$$

holds.

Proof This is an immediate consequence of bivariant formalism, but can also be easily checked on the level of cycles. □

Proposition 2.20 (push-pull formula) *Suppose that*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow g' & & \downarrow g \\ X & \xrightarrow{f} & Y \end{array}$$

is a derived Cartesian square in \mathcal{C} . Then:

(1) *If \mathbb{L}_f has Tor-dimension ≤ 1 and g is projective,*

$$f^! \circ g_* = g'_* \circ f'^! : \Omega_{\bullet}(Y') \rightarrow \Omega_{\bullet}(X).$$

(2) *If g is projective and quasismooth,*

$$f^* \circ g_! = g'_! \circ f'^* : \Omega^*(Y') \rightarrow \Omega^*(X).$$

Proof This is an immediate consequence of bivariant formalism, but can also be easily checked on the level of cycles. □

Proposition 2.21 (naturality of the duality map) *For every $X \in \mathcal{C}$ derived complete intersection, there exists a natural duality morphism*

$$-\bullet 1_{X/\mathbb{Z}} : \Omega^*(X) \rightarrow \Omega_{\bullet}(X).$$

Moreover:

(1) If $f: X \rightarrow Y$ is a projective quasismooth morphism, then the square

$$\begin{array}{ccc} \Omega^*(X) & \xrightarrow{-\bullet 1_{X/\mathbb{Z}}} & \Omega_\bullet(X) \\ \downarrow f! & & \downarrow f_* \\ \Omega^*(Y) & \xrightarrow{-\bullet 1_{Y/\mathbb{Z}}} & \Omega_\bullet(Y) \end{array}$$

commutes.

(2) If $f: X \rightarrow Y$ is such that $\mathbb{L}_{X/Y}$ has Tor-dimension ≤ 1 , then the square

$$\begin{array}{ccc} \Omega^*(Y) & \xrightarrow{-\bullet 1_{Y/\mathbb{Z}}} & \Omega_\bullet(Y) \\ \downarrow f^* & & \downarrow f^! \\ \Omega^*(X) & \xrightarrow{-\bullet 1_{X/\mathbb{Z}}} & \Omega_\bullet(X) \end{array}$$

commutes.

Proof This is an immediate consequence of bivariant formalism, but can also be easily checked on the level of cycles. □

2.3.2 Basic properties Let us then recall the basic properties of algebraic cobordism. Note that the formal group law acting on the Chern classes of Ω^* induces a map

$$\mathbb{L}^* \rightarrow \Omega^*(X)$$

for all $X \in \mathcal{C}$, which is compatible with pullbacks. We will have to use some of the basic properties of this morphism later in the article, which we collect below.

Proposition 2.22 *Let $X \in \mathcal{C}$. Then the image of $\mathbb{L}^* \rightarrow \Omega^*(X)$ is generated by derived schemes smooth over X .*

Proof By the same argument as in [20], one shows that the image of $\mathbb{L}^* \rightarrow \Omega^*(X)$ is generated by towers of projective bundles over X . □

Proposition 2.23 *Let $X \in \mathcal{C}$. Then the natural map*

$$\mathbb{L}^* \rightarrow \Omega^*(X)$$

is an injection.

Proof Since the morphism is compatible with pullbacks, we can pull back to the generic point of an irreducible component of X to reduce to the case where $X \simeq \text{Spec}(k)$ is the spectrum of a field k . Moreover, following the arguments of [20, Section 4.1.9],

which are based on the ideas of Quillen in [24] and depend only on the formal properties of Chern classes, we can construct the *total Landweber–Novikov operator*

$$\Omega^* \rightarrow \Omega^*[b_1, b_2, \dots]^{(t)},$$

and, combining this with the degree morphism

$$\text{deg}: \Omega^0(\text{Spec}(k)) \rightarrow K^0(\text{Spec}(k)) \cong \mathbb{Z},$$

we obtain a natural morphism

$$\Omega^*(\text{Spec}(k)) \rightarrow \mathbb{Z}[b_1, b_2, \dots].$$

As in [20, Lemma 4.3.1], one proves that the composition $\mathbb{L}^* \rightarrow \mathbb{Z}[b_1, b_2, \dots]$ is an injection, proving the claim. □

Next we have to recall that the decomposition relations used in the construction of Ω^\bullet imply the so-called snc relations of Lowrey and Schürg, which follow as a special case of the following result:

Proposition 2.24 [4, Lemma 3.5] *Let $X \in \mathcal{C}$ be a derived complete intersection scheme and let*

$$D \simeq n_1 D_1 + \dots + n_r D_r$$

be a virtual Cartier divisors on X with $n_i > 0$. Let us denote for every $I \subset [r]$ by ι^I the canonical inclusion of the derived intersection

$$D_I := \bigcap_{i \in I} D_i \hookrightarrow D.$$

Then

$$1_{D/\mathbb{Z}} = \sum_{I \subset [r]} \iota_*^I (F_I^{n_1, \dots, n_r} (c_1(\mathcal{O}(D_1)), \dots, c_1(\mathcal{O}(D_r)))) \bullet 1_{D_I/\mathbb{Z}} \in \Omega_\bullet(D)$$

for universal homogeneous power series

$$F_I^{n_1, \dots, n_r} (x_1, \dots, x_r) \in \mathbb{L}^* \llbracket x_1, \dots, x_r \rrbracket$$

of degree $1 - |I|$. □

We also record the following dimension formula, which can be used to calculate the Krull dimension of V in a nice enough cycle $[V \rightarrow X] \in \Omega^{-d}(X)$:

Lemma 2.25 *Let A be a discrete valuation ring with residue field κ , let V and X be integral and projective A -schemes and let $V \rightarrow X$ be an lci A -morphism of relative virtual dimension d . Then*

$$\dim(V) = \dim(X) + d.$$

Proof Note that the analogous claim is known over fields, and therefore we obtain the result in the case that X is not flat over A (and hence X and V are projective κ -schemes). To prove the general case, we will use the dimension formula (see eg [7, Tag 02JU]) to conclude that, if X is flat over A , then

$$\dim(X) = \dim(X_\eta) + 1 = \dim(X_\kappa) + 1,$$

where X_η and X_κ are the general and the special fibres of $X \rightarrow \text{Spec}(A)$ respectively.

If both V and X are flat over A , then

$$\begin{array}{ccc} V_\kappa & \longrightarrow & V \\ \downarrow & & \downarrow \\ X_\kappa & \longrightarrow & X \end{array}$$

is derived Cartesian and therefore $\dim_v(V/X) = \dim_v(V_\kappa/X_\kappa)$, where \dim_v stands for the relative virtual dimension. It follows that

$$\dim(X) = \dim(X_\kappa) + 1 = \dim(V_\kappa) + d + 1 = \dim(V) + d,$$

and we are done in this case.

If X is flat over A but V is not, then $V \rightarrow X$ factors as

$$V \rightarrow X_\kappa \hookrightarrow X$$

and $\dim_v(V/X_\kappa) = \dim_v(V/X) + 1$. Therefore,

$$\dim(V) = \dim(X_\kappa) + d + 1 = \dim(X) + d,$$

proving the claim in the last remaining case. □

Finally, we record the following formulas, which are going to be useful when classifying low-dimensional varieties up to cobordism:

Lemma 2.26 *Let $X \in \mathcal{C}$ and let E be a vector bundle of rank r on X . Then*

$$[\mathbb{P}_X^{r-1} \rightarrow X] - [\mathbb{P}(E) - X]$$

is an \mathbb{L} -linear combination of elements of positive degree in $\Omega^{1-r}(X)$.

Proof By twisting E if necessary, we may assume that there exists a short exact sequence

$$0 \rightarrow E \rightarrow \mathcal{O}^{\oplus N} \rightarrow F \rightarrow 0$$

of vector bundles. Let π be the morphism $\mathbb{P}_X^{N-1} \rightarrow X$; it follows that

$$[\mathbb{P}(E) \rightarrow X] = \pi_!(c_{N-r}(F(1))) \in \Omega^{1-r}(X)$$

is an \mathbb{L} -linear combination of products of Chern classes of F , proving the claim. \square

Lemma 2.27 *Let $Z \hookrightarrow X$ be a derived regular embedding in \mathcal{C} . Then*

$$1_X - [\text{Bl}_Z(X) \rightarrow X]$$

is an \mathbb{L} -linear combination of elements of positive degree in $\Omega^0(X)$.

Proof Considering the algebraic cobordism given by the blow-up of $\mathbb{P}^1 \times X$ at $\infty \times Z$, we obtain the formula

$$1_X - [\text{Bl}_Z(X) \rightarrow X] = [\mathbb{P}_Z(\mathcal{N}_{Z/X} \oplus \mathcal{O}) \rightarrow X] - [\mathbb{P}_{\mathbb{P}_Z(\mathcal{N}_{Z/X})}(\mathcal{O}(1) \oplus \mathcal{O}) \rightarrow X]$$

in $\Omega^0(X)$. The claim follows from applying [Lemma 2.26](#) to $[\mathbb{P}_Z(\mathcal{N}_{Z/X} \oplus \mathcal{O}) \rightarrow Z]$ and $[\mathbb{P}_{\mathbb{P}_Z(\mathcal{N}_{Z/X})}(\mathcal{O}(1) \oplus \mathcal{O}) \rightarrow Z]$ and then pushing forward. \square

2.4 Desingularization by alterations

In this section we are going to recall desingularization results, which will play an important role later in this article. Desingularization by alterations was first introduced by de Jong in [\[16\]](#), after which it (and several improvements of the original theorem) have found numerous applications in the study of algebraic geometry in positive and mixed characteristic. Our main reference will be the fairly recent article [\[28\]](#) by Temkin.

We start by recalling alterations.

Definition 2.28 A map $\pi : X \rightarrow Y$ of integral Noetherian schemes is called an *alteration* if it is proper, dominant and generically finite. If \mathcal{P} is a subset of the prime numbers, then π is called a \mathcal{P} -alteration if its degree is a \mathcal{P} -number, ie all of its prime divisors lie in \mathcal{P} . If $|\mathcal{P}| \leq 1$, then we will also use the term *e-alteration*, where

$$e = \begin{cases} p \in \mathcal{P} & \text{if } |\mathcal{P}| = 1, \\ 1 & \text{if } \mathcal{P} = \emptyset, \end{cases}$$

is the characteristic exponent of \mathcal{P} .

Given an integral scheme X , we are going to denote by $\text{char}(X)$ the set of nonzero residual characteristic of X . The main result of Temkin is that, under certain assumptions, X admits a $\text{char}(X)$ -alteration $X' \rightarrow X$ from a regular scheme X . Before giving the main result, we recall the following terminology:

Definition 2.29 A Noetherian ring A is *quasiexcellent* if, for all primes $\mathfrak{p} \subset A$, the completion morphism $A_{\mathfrak{p}} \rightarrow \widehat{A}_{\mathfrak{p}}$ is flat (automatic) and has geometrically regular fibres, and, for all finite-type A -algebras B , the regular locus of B (ie the set of primes $\mathfrak{q} \subset B$ such that $B_{\mathfrak{q}}$ is a regular local ring) is an open subset of $\text{Spec}(B)$. A quasiexcellent ring A is called *excellent* if it is also universally catenary (eg A is a regular local ring). A Noetherian scheme X is *(quasi)excellent* if it admits an open affine cover by spectra of (quasi)excellent rings.

The following result is a special case of the main result of [28] (combined with [8]):

Theorem 2.30 *Let X be an integral Noetherian scheme admitting a finite-type morphism to a quasiexcellent scheme Y of dimension at most 3 and let $Z \hookrightarrow X$ be a closed subscheme. Then there exists a projective $\text{char}(X)$ -alteration*

$$\pi : X' \rightarrow X$$

with X' regular and $\pi^{-1}(Z)$ a strict normal crossing divisor. □

We will only use this result in the case where Y is the spectrum of a field or an excellent discrete valuation ring.

2.5 Bertini theorems

Bertini theorems, saying that ample enough line bundles have global sections whose (derived) vanishing locus has good properties, are going to play an important role in the arguments of this paper. We start with the following easy observation, which shows the existence of enough “classical sections” in great generality:

Lemma 2.31 *Let X be a quasiprojective scheme over a Noetherian ring A and let \mathcal{L} be an ample line bundle. Then, for all $n \gg 0$, the line bundle $\mathcal{L}^{\otimes n}$ has a global section s which is not a zerodivisor on X . In particular, the derived vanishing locus of s coincides with the classical vanishing locus of s .*

Proof This is classical, but see for example [11, Theorem 5.1] for a reference. □

When A is a discrete valuation ring, we can say a lot more using the recent Bertini-regularity result over discrete valuation rings by Ghosh and Krishna.

Theorem 2.32 [13, Theorem 9.6] *Let A be a discrete valuation ring, let X be a regular quasiprojective A -scheme, and let \mathcal{L} be a very ample line bundle on X . Then, for all $n \gg 0$, the line bundle $\mathcal{L}^{\otimes n}$ has a global section s whose vanishing locus Z_s is regular and of codimension 1. Moreover, if X is flat over A , then we can find such a global section s with Z_s flat over A .*

Proof We may assume without loss of generality that X is connected. Note that the authors assume the structure morphism $X \rightarrow \text{Spec}(A)$ to be surjective. However, if this is not the case, then X is quasiprojective over either the fraction field of A or the residue field of A , and the claim follows from the Bertini-regularity theorems over fields. Moreover, the authors assume the Krull dimension of X to be at least 2, but this is only because the surjectivity of $X \rightarrow \text{Spec}(A)$ implies that otherwise X is going to be affine and semilocal, and hence all line bundles are going to be trivial, rendering the claim trivial. To prove the last claim, we note that, if X is flat over A , then Z_s is not flat over A if and only if it has a component defined over the residue field of A . But, looking at the proof of [13, Theorem 9.6], it is clear that this does not happen, so we are done. \square

3 Presentations of Chern classes and refined projective bundle formulas

The purpose of this section is to prove that Chern classes of vector bundles and line bundles often admit presentations in terms of nice cobordism cycles, and to use this to prove several refined versions of the projective bundle formula, which will play an important role in Section 4. We note the unfortunate expositional fact that some results of Section 3.2 for discrete valuation rings use the results of Section 4 for fields, so the logic of Sections 3 and 4 proceeds really as follows:

Section 3 for fields \implies Section 4 for fields
 \implies Section 3 for discrete valuation rings
 \implies Section 4 for discrete valuation rings.

We hope that the reader does not get confused because of this nonlinear narrative.

3.1 Presentations of Chern classes

The purpose of this section is to record several presentability results of Chern classes that are going to be useful in the proofs of the refined projective bundle formulas as

well as later in the article. We start with the following observation, which is useful when we know that large enough powers of a line bundle admit nice sections:

Lemma 3.1 *Let X be a finite-dimensional divisorial Noetherian derived scheme and let \mathcal{L} be a line bundle. Then, given coprime integers p and q and integers $a, b \in \mathbb{Z}$ such that $ap + bq = 1$,*

$$c_1(\mathcal{L}) = (ac_1(\mathcal{L}^{\otimes p}) + bc_1(\mathcal{L}^{\otimes q})) \bullet \left(1_X + \sum_{i=1}^{\infty} b_i c_1(\mathcal{L})^i \right) \in \Omega^1(X),$$

where $b_i := b_i(p, q, a, b) \in \mathbb{L}^{-i}$.

Proof Indeed, from the formal group law it follows that

$$ac_1(\mathcal{L}^{\otimes p}) + bc_1(\mathcal{L}^{\otimes q}) = c_1(\mathcal{L}) \bullet \left(1_X + \sum_{i=1}^{\infty} a_i c_1(\mathcal{L})^i \right),$$

where $a_i \in \mathbb{L}^{-i}$. The claim follows from the fact that $1_X + \sum_{i=1}^{\infty} a_i c_1(\mathcal{L})^i$ is invertible. □

The next result will enable us to find nice presentations of Chern classes of all vector bundles by arguing inductively on the rank.

Lemma 3.2 *Let X be a finite-dimensional Noetherian derived scheme having an ample line bundle, and let E be a vector bundle on X . Then, denoting by*

$$0 \rightarrow \mathcal{O}(-1) \rightarrow E \rightarrow Q \rightarrow 0$$

the tautological exact sequence of vector bundles on $\mathbb{P}(E)$, the class

$$\frac{1_{\mathbb{P}(E)} + c_1(\mathcal{O}(-1)) \bullet [\mathbb{P}_{\mathbb{P}(E)}(\mathcal{O}(-1) \oplus \mathcal{O}) \rightarrow \mathbb{P}(E)]}{1 - c_r(E) \bullet [\mathbb{P}_{\mathbb{P}(E)}(E \oplus \mathcal{O}) \rightarrow \mathbb{P}(E)]} \bullet c_{r-1}(Q) \in \Omega^{r-1}(\mathbb{P}(E))$$

pushes forward to $1_X \in \Omega^0(X)$.

Proof Indeed, this is just [4, Lemma 3.28]. □

Our first two results show that Chern classes of vector bundles on classical schemes can be often presented by classical cycles. These results are not needed later in the article, but they might be useful for other purposes, which is why we record them here. The uninterested reader may skip ahead to [Lemma 3.5](#).

We begin with the case of line bundles.

Lemma 3.3 *Let X be a quasiprojective scheme over a Noetherian ring A and let \mathcal{L} be a line bundle. Then $c_1(\mathcal{L}) \in \Omega^1(X)$ is equivalent to an \mathbb{L} -linear combination of cycles of the form $[Z \hookrightarrow X]$, where $Z \hookrightarrow X$ is a classical regular embedding.*

Proof We start by proving the claim for ample line bundles by arguing inductively on the Krull dimension of X , the base case of an empty scheme being obvious. By Lemma 2.31, we can find coprime integers p and q and effective Cartier divisors $i_1 : Z_1 \hookrightarrow X$ and $i_2 : Z_2 \hookrightarrow X$ in the linear systems of $\mathcal{L}^{\otimes p}$ and $\mathcal{L}^{\otimes q}$, respectively. By Lemma 3.1 and the projection formula it follows that

$$\begin{aligned} c_1(\mathcal{L}) &= (a[Z_1 \hookrightarrow X] + b[Z_2 \hookrightarrow X]) \bullet \left(1_X + \sum_{i=1}^{\infty} b_i c_1(\mathcal{L})^i \right) \\ &= a[Z_1 \hookrightarrow X] + b[Z_2 \hookrightarrow X] + a i_1! \left(\sum_{i=1}^{\infty} b_i c_1(\mathcal{L}|_{Z_1})^i \right) + b i_2! \left(\sum_{i=1}^{\infty} b_i c_1(\mathcal{L}|_{Z_2})^i \right) \end{aligned}$$

with $a, b \in \mathbb{Z}$ and $b_i \in \mathbb{L}$, so we have proven the claim for \mathcal{L} ample by the inductive assumption.

Suppose then that \mathcal{L} is an arbitrary line bundle on X . By the quasiprojectivity of X we can find ample line bundles \mathcal{L}_1 and \mathcal{L}_2 such that $\mathcal{L} \cong \mathcal{L}_1 \otimes \mathcal{L}_2^\vee$, and therefore it follows from the formal group law that

$$c_1(\mathcal{L}) = \sum_{i,j} b_{ij} c_1(\mathcal{L}_1)^i \bullet c_1(\mathcal{L}_2)^j$$

for some $b_{ij} \in \mathbb{L}$. Hence the claim follows from the ample case using the projection formula and the fact that ample line bundles are stable under pullbacks along immersions. □

It is then not very hard to deal with arbitrary vector bundles.

Proposition 3.4 *Let X be a quasiprojective scheme over a Noetherian ring A and let E be a vector bundle on X . Then $c_i(E) \in \Omega^i(X)$ is equivalent to an integral combination of cycles of the form $[V \rightarrow X]$ with $V \rightarrow X$ lci.*

Proof We will proceed by induction on the rank r of the vector bundle, the base case $r = 1$ following from Lemma 3.3. Suppose that E is a rank r vector bundle on X and

that the claim is known for all quasiprojective A -schemes and for all vector bundles of rank at most r . Then, by [Lemma 3.2](#) and the projection formula,

$$c_i(E) = \pi_* \left(\frac{1_{\mathbb{P}(E)} + c_1(\mathcal{O}(-1)) \bullet [\mathbb{P}_{\mathbb{P}(E)}(\mathcal{O}(-1) \oplus \mathcal{O}) \rightarrow \mathbb{P}(E)]}{1 - c_r(E) \bullet [\mathbb{P}_{\mathbb{P}(E)}(E \oplus \mathcal{O}) \rightarrow \mathbb{P}(E)]} \bullet c_{r-1}(Q) \bullet c_i(E) \right) \in \Omega^*(X),$$

where π is the natural map $\mathbb{P}(E) \rightarrow X$. Moreover, the element

$$\frac{1_{\mathbb{P}(E)} + c_1(\mathcal{O}(-1)) \bullet [\mathbb{P}_{\mathbb{P}(E)}(\mathcal{O}(-1) \oplus \mathcal{O}) \rightarrow \mathbb{P}(E)]}{1 - c_r(E) \bullet [\mathbb{P}_{\mathbb{P}(E)}(E \oplus \mathcal{O}) \rightarrow \mathbb{P}(E)]} \bullet c_{r-1}(Q) \bullet c_i(E)$$

can be expressed in the desired form by the inductive assumption and the Whitney sum formula, so the claim follows by pushing forward. \square

Next we show that we can prove stronger results when working over discrete valuation rings. Again, we start with the case of line bundles.

Lemma 3.5 *Let X be a regular quasiprojective scheme over a discrete valuation ring (or a field) A , and let \mathcal{L} be a line bundle on X . Then $c_1(\mathcal{L}) \in \Omega^1(X)$ is equivalent to an \mathbb{L} -linear combination of cycles of the form $[Z \hookrightarrow X]$, where Z is a regular scheme. If X is flat over A , then we can moreover assume the Z to be flat over A .*

Proof Let us first assume that \mathcal{L} is very ample. We will proceed by induction on the Krull dimension of X , the base case of an empty scheme being obvious. By [Theorem 2.32](#) we can find coprime integers p and q and regular divisors $i_1: Z_1 \hookrightarrow X$ and $i_2: Z_2 \hookrightarrow X$ in the linear systems of $\mathcal{L}^{\otimes p}$ and $\mathcal{L}^{\otimes q}$, respectively. By [Lemma 3.1](#) and the projection formula it follows that

$$\begin{aligned} c_1(\mathcal{L}) &= (a[Z_1 \hookrightarrow X] + b[Z_2 \hookrightarrow X]) \bullet \left(1_X + \sum_{i=1}^{\infty} b_i c_1(\mathcal{L})^i \right) \\ &= a[Z_1 \hookrightarrow X] + b[Z_2 \hookrightarrow X] + ai_1! \left(\sum_{i=1}^{\infty} b_i c_1(\mathcal{L}|_{Z_1})^i \right) \\ &\quad + bi_2! \left(\sum_{i=1}^{\infty} b_i c_1(\mathcal{L}|_{Z_2})^i \right) \end{aligned}$$

with $a, b \in \mathbb{Z}$ and $b_i \in \mathbb{L}$, and the claim follows from the inductive assumption. Note that, if X was flat over A , then we could have chosen Z_i to be flat over A too, proving the stronger claim as well.

In general, we can find very ample line bundles \mathcal{L}_1 and \mathcal{L}_2 such that $\mathcal{L} \cong \mathcal{L}_1 \otimes \mathcal{L}_2^\vee$, and therefore, by the formal group law,

$$c_1(\mathcal{L}) = \sum_{i,j \geq 0} b_{ij} c_1(\mathcal{L}_1)^i \bullet c_1(\mathcal{L}_2)^j \in \Omega^*(X)$$

for some $b_{ij} \in \mathbb{L}$. As very ample line bundles are stable under restrictions to closed subschemes, the general case follows from the very ample case. □

Finally, we deal with general vector bundles.

Proposition 3.6 *Let X be a regular quasiprojective scheme over a discrete valuation ring (or a field) A , and let E a vector bundle on X . Then $c_i(E) \in \Omega^i(X)$ is equivalent to an integral combination of cycles of the form $[V \rightarrow X]$ with V regular. If X is flat over A , then we can moreover assume the V to be flat over A .*

Proof This is identical to the proof of Proposition 3.4, using Lemma 3.5 instead of Lemma 3.3 for the base case. □

3.2 Refined projective bundle formula

The purpose of this section is to prove refined versions of projective bundle formula that are going to be useful later in the article. We start with the easier one. Note that the first part of the following result is not needed later in the article:

Proposition 3.7 *Let X be a quasiprojective derived scheme over a finite-dimensional Noetherian ring A , and let π be the projection $\mathbb{P}^n \times X \rightarrow X$. Then a cobordism class*

$$[V \rightarrow \mathbb{P}^n \times X] \in \Omega^d(\mathbb{P}^n \times X)$$

is equivalent to

$$\sum_{i=0}^n c_1(\mathcal{O}(1))^i \bullet \pi^*(\alpha_i)$$

and:

- (1) *If V is a complete intersection scheme, then $\alpha_i \in \Omega^{d-i}(X)$ are equivalent to integral combinations of complete intersection schemes mapping projectively to X .*
- (2) *If A is a discrete valuation ring (or a field) and V is regular, then $\alpha_i \in \Omega^{d-i}(X)$ are integral combinations of regular schemes mapping projectively to X .*

Proof By the projective bundle formula there exist unique $\alpha_i \in \Omega^*(X)$ such that

$$[V \rightarrow \mathbb{P}^n \times X] = \sum_{i=0}^n c_1(\mathcal{O}(1))^i \bullet \pi^*(\alpha_i) \in \Omega_\bullet(\mathbb{P}^n \times X)$$

holds; our task is to find the desired presentation for α_i . But this is easy: clearly

$$\pi_*(c_1(\mathcal{O}(1))^i \bullet [V \rightarrow \mathbb{P}^n \times X]) = \alpha_{n-i} + [\mathbb{P}^1]\alpha_{n-i-1} + \dots + [\mathbb{P}^{n-i}]\alpha_0,$$

where $[\mathbb{P}^i]$ is the class of \mathbb{P}^i in \mathbb{L}^{-i} , so the first claim follows from [Proposition 3.4](#) and the second claim follows from [Proposition 3.6](#). □

Remark 3.8 There is a more general version of [Proposition 3.7](#) that holds for all projective bundles and not just the trivial ones. However, the proof is much more complicated and, since we will not need the more general result, we have chosen only to prove a special case.

The following result is one of the crucial results needed in the proof of the algebraic Spivak's theorem as it allows us to approximate the class of a generically finite morphisms to a projective space:

Theorem 3.9 *Let A be a Henselian discrete valuation ring with a perfect residue field (or let A be an arbitrary field), and let $f: V \rightarrow \mathbb{P}_A^n$ be a projective morphism from an integral regular scheme V of relative virtual dimension 0. If f is generically finite of degree $d \geq 0$, then*

$$[V \rightarrow \mathbb{P}_A^n] = d + \sum_{i=1}^n c_1(\mathcal{O}(1))^i \bullet \pi^*(\alpha_i) \in \Omega^0(\mathbb{P}_A^n)[e^{-1}]$$

where the α_i are integral combinations of regular projective A -schemes and e is the residual characteristic exponent of A . If A is a field, then the equality holds without inverting e .

Proof Suppose first that f is dominant, which also implies that V is flat over A . Using [Proposition 3.7](#) and [Lemma 2.25](#) it follows that

$$[V \rightarrow \mathbb{P}_A^n] = \sum_{i=0}^n c_1(\mathcal{O}(1))^i \bullet \pi^*(\alpha_i)$$

with

$$\alpha_0 = \sum_i n_i [\text{Spec}(B_i) \rightarrow \text{Spec}(A)] \in \Omega^0(\text{Spec}(A)),$$

where B_i are finite, flat, integral and regular A -algebras, and $n_i \in \mathbb{Z}$. Using the specialization morphism of cohomology theories $\Omega^* \rightarrow K^0$ one checks that

$$\sum_i n_i \deg(B_i/A) = d,$$

so the claim follows from [Lemma 3.10](#) below.

Suppose then that f is not dominant. If V is flat over A (eg if A is a field), then the above proof shows what we want. Otherwise V is a regular κ -variety, and, by [Proposition 3.7](#) and [Lemma 2.25](#),

$$\alpha_0 = \sum_i n_i [C_i \rightarrow \text{Spec}(A)] \in \Omega^0(\text{Spec}(A))$$

with C_i smooth curves over κ and $n_i \in \mathbb{Z}$. Using the computation of $\Omega^{-1}(\text{Spec}(\kappa))[e^{-1}]$ from [Corollary 4.17](#) (whose proof only needs this result for fields), we see that

$$\alpha_0 = b[\text{Spec}(\kappa) \hookrightarrow \text{Spec}(A)] = 0 \in \Omega^0(\text{Spec}(\kappa))[e^{-1}],$$

where $b \in \mathbb{L}^{-1}[e^{-1}]$. □

We needed the following computation of cobordism classes of finite regular algebras in the above proof:

Lemma 3.10 *Let A be a Henselian discrete valuation ring with a perfect residue field (or let A be an arbitrary field) and let B be a finite, flat, integral and regular A -algebra of degree d . Then*

$$[\text{Spec}(B) \rightarrow \text{Spec}(A)] = d \in \Omega^0(\text{Spec}(A)).$$

Proof Note that if A is a field, then B is a finite field extension of A and the claim follows trivially from [Lemma 3.11](#), as any field extension factors as a composition of primitive field extensions.

Let us then deal with the case where A is a Henselian discrete valuation ring with a perfect residue field κ . By [\[7, Tag 04GG\]](#), B is a Henselian discrete valuation ring; let us denote its residue field by l . We will write $n := [l : \kappa]$. Let $\alpha \in l$ be a primitive element over κ , let \tilde{f} be its minimal polynomial and let $f \in A[x]$ be a monic lift of \tilde{f} . It is clear that $A' := A[x]/(f)$ is a Henselian discrete valuation ring of degree n over A . Moreover, as B is Henselian and α is a simple root of \tilde{f} , we can lift α to a root $\tilde{\alpha}$

of f in B , giving rise to a morphism $\psi: A' \rightarrow B$ by sending x to $\tilde{\alpha}$. We have factored $A \rightarrow B$ as

$$A \xrightarrow{\phi} A' \xrightarrow{\psi} B$$

with ϕ primitive by construction and ψ primitive by [26, Chapter I, Proposition 18], so the claim follows from Lemma 3.11 below. \square

The above lemma needed the following result in its proof:

Lemma 3.11 *Let A be a Noetherian ring of finite Krull dimension, and let $\psi: A \rightarrow B$ be a finite syntomic morphism of degree d . If ψ factors as a composition of primitive finite syntomic morphisms, then*

$$[\text{Spec}(B) \rightarrow \text{Spec}(A)] = d \in \Omega^0(\text{Spec}(A)).$$

Proof It is enough to consider the case where B itself is primitive over A . Hence,

$$B \cong A[x]/(f)$$

for some $f = x^d + a_{d-1}x^{d-1} + \dots + a_0 \in A[x]$, and therefore $\text{Spec}(B)$ is the derived vanishing locus of

$$F(x, y) = x^d + a_{d-1}x^{d-1}y + \dots + a_0y^d \in \Gamma(\mathbb{P}_A^1; \mathcal{O}(d)),$$

and hence

$$[\text{Spec}(B) \rightarrow \text{Spec}(A)] = \pi_!(c_1(\mathcal{O}(d))),$$

where π is the projection $\mathbb{P}_A^1 \rightarrow \text{Spec}(A)$. The claim then follows from noticing that $c_1(\mathcal{O}(d)) = dc_1(\mathcal{O}(1))$ and $\pi_!(c_1(\mathcal{O}(1))) = 1$. \square

We are going to use Theorem 3.9 in Section 4 in the form of the following corollary:

Corollary 3.12 *Let A be a Henselian discrete valuation ring with a perfect residue field κ (or let A be an arbitrary field), and let $f: D \rightarrow \mathbb{P}_A^n$ be a projective morphism of relative virtual dimension 0 from an snc scheme D . If f is generically finite of degree $d \geq 0$, then*

$$[D \rightarrow \mathbb{P}_A^n] = d \cdot 1_{\mathbb{P}_A^n/\mathbb{Z}} + \sum_{i=1}^n \pi^*(\alpha_i) \cdot c_1(\mathcal{O}(1))^i \cdot 1_{\mathbb{P}_A^n/\mathbb{Z}} \in \Omega_\bullet(\mathbb{P}_A^n)[e^{-1}]$$

with $\alpha_i \in \Omega^{-i}(X)$ integral combinations of regular projective A -schemes. If A is a field, then the equality holds without inverting e .

Proof Let X be a regular scheme in which D is an snc divisor, let D_1, \dots, D_r be the prime components of D and let $n_1, \dots, n_r > 0$ be such that

$$D = n_1 D_1 + \dots + n_r D_r$$

as effective Cartier divisors on X . Then the snc relations and Poincaré duality imply that

$$[D \rightarrow \mathbb{P}_A^n] = \sum_{I \subset [r]} f_1^I (F_I^{n_1, \dots, n_r} (c_1(\mathcal{O}(D_1)), \dots, c_1(\mathcal{O}(D_r)))) \in \Omega^0(\mathbb{P}_A^n),$$

where f^I is the canonical morphism $D_I := \bigcap_{i \in I} D_i \rightarrow \mathbb{P}_A^n$. Applying Proposition 3.6 to the elements

$$F_I^{n_1, \dots, n_r} (c_1(\mathcal{O}(D_1)), \dots, c_1(\mathcal{O}(D_r))) \in \Omega^{1-|I|}(D_I),$$

we conclude that

$$[D \rightarrow \mathbb{P}_A^n] = n_1 [D_1 \rightarrow \mathbb{P}_A^n] + \dots + n_r [D_r \rightarrow \mathbb{P}_A^n] + \beta \in \Omega^0(\mathbb{P}_A^n),$$

where β is an integral combination of cycles of the form $[V \rightarrow \mathbb{P}_A^n] \in \Omega^0(\mathbb{P}_A^n)$, where V is regular and the morphism $V \rightarrow \mathbb{P}_A^n$ is nondominant. Combining this with the fact that

$$d = n_1 \deg(D_1/\mathbb{P}_A^n) + \dots + n_r \deg(D_r/\mathbb{P}_A^n)$$

with $\deg(D_i/\mathbb{P}_A^n) = 0$ whenever $D_i \rightarrow \mathbb{P}_A^n$ is nondominant, the claim follows from Theorem 3.9 and right multiplication by $1_{\mathbb{P}_A^n/\mathbb{Z}}$. □

4 Algebraic Spivak’s theorem and applications

The purpose of this section is to prove the algebraic Spivak’s theorem (Theorem 4.12) and apply it to prove further properties of algebraic bordism. We start by fixing notation in Section 4.1, after which we prove Spivak’s theorem in Section 4.2. Section 4.3 is dedicated to proving the extension theorem (Theorem 4.19).

Throughout the section A will be either a field or an excellent Henselian discrete valuation ring with a perfect residue field κ . We will denote by

$$\Omega_*^A(X) := \Omega^{-*}(X \rightarrow \text{Spec}(A))$$

the A -bordism groups of quasiprojective derived A -schemes X , where the grading is given by the relative virtual dimension over A . As A is regular, we have the natural isomorphism

$$-\bullet 1_{A/\mathbb{Z}} : \Omega_*^A(X) \xrightarrow{\cong} \Omega_\bullet(X)$$

but note that the right-hand side does not have a natural grading. The A -bordism groups $\Omega_*^A(X)$ have a canonical $\Omega^*(\text{Spec}(A))$ -module structure given by the bivariant product, which also has an explicit formula

$$[V \rightarrow \text{Spec}(A)].[W \rightarrow X] = \pi_X^*([V \rightarrow \text{Spec}(A)]) \bullet [W \rightarrow X] = [V \times_{\text{Spec}(A)}^R W \rightarrow X],$$

where π_X is the structure morphism $X \rightarrow \text{Spec}(A)$. Pushforwards and Gysin pullbacks are maps of $\Omega^*(\text{Spec}(A))$ -modules.

4.1 Deformation diagrams

Suppose that X is a connected quasismooth and quasiprojective derived A -scheme, and let $i : X \hookrightarrow U$ be a closed embedding with U an open subscheme of \mathbb{P}_A^n for some $n \geq 0$. We will denote by

$$\bar{i}_{\text{cl}} : \bar{X}_{\text{cl}} \hookrightarrow \mathbb{P}_A^n$$

the scheme-theoretic closure of the truncation X_{cl} of X in U . The purpose of this section is to record and study the basic properties of three deformation diagrams playing an important role in Sections 4.2 and 4.3. Let us start with the derived deformation diagram.

Construction 4.1 (derived deformation diagram of i) Denoting the derived blow-up of $\infty \times X \hookrightarrow \mathbb{P}^1 \times U$ by $M(X/U)$ gives rise to a derived Cartesian diagram

$$\begin{array}{ccccc} X & \xleftarrow{i^\infty} & \mathbb{P}(\mathcal{N}_{X/U} \oplus \mathcal{O}) + \text{Bl}_X(U) & \longrightarrow & \infty \\ \downarrow j'^\infty & & \downarrow j^\infty & & \downarrow \\ \mathbb{P}^1 \times X & \xleftarrow{\hat{i}} & M(X/U) & \longrightarrow & \mathbb{P}^1 \\ \uparrow j^0 & & \uparrow j^0 & & \uparrow \\ X & \xleftarrow{i} & U & \longrightarrow & 0 \end{array}$$

where $+$ denotes the sum of virtual Cartier divisors. Let us denote the induced morphisms $\mathbb{P}(\mathcal{N}_{X/U} \oplus \mathcal{O}) \hookrightarrow M(X/U)$ and $\text{Bl}_X(U) \hookrightarrow M(X/U)$ by j_1^∞ and j_2^∞ , respectively.

Since $\mathbb{P}^1 \times X$ does not meet $\text{Bl}_X(U)$ by Theorem 2.8, we have the following result:

Lemma 4.2 *Let everything be as above. Then*

$$i^! \circ j^0! = s^! \circ j_1^{\infty!} : \Omega_*^A(M(X/U)) \rightarrow \Omega_*^A(X),$$

where s is the zero section $X \hookrightarrow \mathbb{P}(\mathcal{N}_{X/U} \oplus \mathcal{O})$.

Proof This follows immediately from the fact that the squares

$$\begin{array}{ccc}
 X \xleftarrow{s} \mathbb{P}(\mathcal{N}_{X/U} \oplus \mathcal{O}) & & X \xleftarrow{i} U \\
 \downarrow j'^{\infty} & \text{and} & \downarrow j'^0 \\
 \mathbb{P}^1 \times X \xleftarrow{\hat{i}} M(X/U) & & \mathbb{P}^1 \times X \xleftarrow{\hat{i}} M(X/U) \\
 \downarrow j_1^{\infty} & & \downarrow j^0
 \end{array}$$

commute up to homotopy. □

We will combine the above result with the following standard observation in [Section 4.2](#):

Lemma 4.3 *Let X be a quasiprojective derived A scheme, let $\rho: \mathbb{P}(E \oplus \mathcal{O}) \rightarrow X$ be a projective bundle over X with $r = \text{rank}(E)$, and let $s: X \hookrightarrow \mathbb{P}(E \oplus \mathcal{O})$ be the zero section. Then*

$$s^1(-) = \rho_*(c_r(E(1)) \bullet -): \Omega_*^A(\mathbb{P}(E \oplus \mathcal{O})) \rightarrow \Omega_{*-r}^A(X).$$

Proof This follows immediately from the fact that s is the derived vanishing locus of a global section of $E(1)$. □

We will also need the following classical deformation diagrams:

Construction 4.4 (classical deformation diagram of \bar{i}_{cl}) Denoting the classical blow-up of $\infty \times \bar{X}_{\text{cl}} \hookrightarrow \mathbb{P}^1 \times \mathbb{P}_A^n$ by $M^{\text{cl}}(\bar{X}_{\text{cl}}/\mathbb{P}_A^n)$ gives us the diagram

$$\begin{array}{ccccc}
 \bar{X}_{\text{cl}} & \xleftarrow{\bar{i}_{\text{cl}}^{\infty}} & \mathcal{E} + \text{Bl}_{\bar{X}_{\text{cl}}}^{\text{cl}}(\mathbb{P}_A^n) & \longrightarrow & \infty \\
 \downarrow \bar{j}_{\text{cl}}'^{\infty} & & \downarrow \bar{j}_{\text{cl}}^{\infty} & & \downarrow \\
 \mathbb{P}^1 \times \bar{X}_{\text{cl}} & \xleftarrow{\hat{i}_{\text{cl}}} & M^{\text{cl}}(\bar{X}_{\text{cl}}/\mathbb{P}_A^n) & \longrightarrow & \mathbb{P}^1 \\
 \uparrow \bar{j}_{\text{cl}}'^0 & & \uparrow \bar{j}_{\text{cl}}^0 & & \uparrow \\
 \bar{X}_{\text{cl}} & \xleftarrow{\bar{i}_{\text{cl}}} & \mathbb{P}_A^n & \longrightarrow & 0
 \end{array}$$

where \mathcal{E} denotes the exceptional divisor of the blow-up. We will denote by $\bar{j}_{\text{cl},1}^{\infty}$ the induced morphism $\mathcal{E} \hookrightarrow M^{\text{cl}}(\bar{X}_{\text{cl}}/\mathbb{P}_A^n)$.

Construction 4.5 (classical deformation diagram of i_{cl}) Restricting the deformation diagram of [Construction 4.4](#) to the open subscheme $U \subset \mathbb{P}_A^n$ gives rise to the diagram

$$\begin{array}{ccccc}
 X_{\text{cl}} & \xleftarrow{i_{\text{cl}}^{\infty}} & \mathcal{E}^{\circ} + \text{Bl}_{X_{\text{cl}}}^{\text{cl}}(U) & \longrightarrow & \infty \\
 \downarrow j_{\text{cl}}'^{\infty} & & \downarrow j_{\text{cl}}^{\infty} & & \downarrow \\
 \mathbb{P}^1 \times X_{\text{cl}} & \xleftarrow{\hat{i}_{\text{cl}}^{\circ}} & M^{\text{cl}}(X_{\text{cl}}/U) & \longrightarrow & \mathbb{P}^1 \\
 \uparrow j_{\text{cl}}'^0 & & \uparrow j_{\text{cl}}^0 & & \uparrow \\
 X_{\text{cl}} & \xleftarrow{i_{\text{cl}}} & U & \longrightarrow & 0
 \end{array}$$

The induced morphism $\mathcal{E}^{\circ} \hookrightarrow M^{\text{cl}}(X_{\text{cl}}/U)$ is denoted by $j_{\text{cl},1}^{\infty}$.

As a special case of [Proposition 2.9](#), we obtain the following result:

Lemma 4.6 *The square*

$$\begin{array}{ccc} \mathcal{E}^\circ & \xleftarrow{j_{\text{cl},1}^\infty} & M^{\text{cl}}(X_{\text{cl}}/U) \\ \downarrow \iota' & & \downarrow \iota \\ \mathbb{P}(\mathcal{N}_{X/U} \oplus \mathcal{O}) & \xleftarrow{j_1^\infty} & M(X/U) \end{array}$$

is derived Cartesian. □

4.2 Proof of the algebraic Spivak's theorem

In this section we prove the algebraic Spivak's theorem. For simplicity we will call elements of the form

$$[V \rightarrow X] \in \Omega_*^A(X) \quad \text{and} \quad [V \rightarrow \text{Spec}(A)] \in \Omega^*(\text{Spec}(A))$$

with V a regular scheme *regular cycles*. The algebraic Spivak's theorem will easily follow from the following result:

Lemma 4.7 *Let X be a connected quasismooth and quasiprojective derived A -scheme of virtual A -dimension d . Then the fundamental class*

$$1_{X/A} \in \Omega_d^A(X)[e^{-1}]$$

is equivalent to a $\mathbb{Z}[e^{-1}]$ -linear combination of elements of the form

$$(\alpha_1 \bullet \cdots \bullet \alpha_s) \cdot \beta$$

with $\alpha_i \in \Omega^*(\text{Spec}(A))$ and $\beta \in \Omega_*^A(X)$ integral combinations of regular cycles.

Throughout this section, we will use the notation of [Section 4.1](#). Let us start with the following construction:

Construction 4.8 Choose a desingularization by an e -alteration

$$\pi: W \rightarrow M^{\text{cl}}(\bar{X}_{\text{cl}}/\mathbb{P}_A^n)$$

with $\mathcal{E}' := \pi^{-1}(\mathcal{E})$ and $W_0 := \pi^{-1}(\mathbb{P}_A^n)$ snc divisors (here \mathbb{P}_A^n is the copy of the projective space over 0), and denote by

$$\pi^\circ: W^\circ \rightarrow M^{\text{cl}}(X_{\text{cl}}/U),$$

\mathcal{E}'° and W_0° the pullbacks of π , \mathcal{E}' and W'_0 along $U \subset \mathbb{P}^n_A$. Note that, since

$$\begin{array}{ccc} W_0^\circ & \hookrightarrow & W^\circ \\ \downarrow & & \downarrow \\ U & \hookrightarrow & M(X/U) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{E}'^\circ & \hookrightarrow & W^\circ \\ \downarrow & & \downarrow \\ \mathbb{P}(\mathcal{N}_{X/U} \oplus \mathcal{O}) & \hookrightarrow & M(X/U) \end{array}$$

are derived Cartesian, it follows from Lemma 4.2 that

$$(2) \quad i^!([W_0^\circ \rightarrow U]) = s^!([\mathcal{E}'^\circ \rightarrow \mathbb{P}(\mathcal{N}_{X/U} \oplus \mathcal{O})])$$

in $\Omega_*^A(X)$.

We then have the following lemmas:

Lemma 4.9 *We have that*

$$[W_0^\circ \rightarrow U] = e^m \cdot 1_{U/A} + \sum_{i=1}^n \alpha_i \cdot (c_1(\mathcal{O}_U(1)))^i \cdot 1_{U/A} \in \Omega_*^A(U),$$

where m is a nonnegative integer, $\mathcal{O}_U(1)$ is the restriction of $\mathcal{O}(1)$ to U , and the $\alpha_i \in \Omega^{-i}(\text{Spec}(A))$ are integral combinations of regular cycles.

Proof By construction, $\pi: W \rightarrow M^{\text{cl}}(\bar{X}_{\text{cl}}/\mathbb{P}^n_A)$ is surjective and generically finite of degree e^m for some $m \in \mathbb{Z}$ nonnegative (and therefore also of relative virtual dimension 0 over the regular locus of $M^{\text{cl}}(\bar{X}_{\text{cl}}/\mathbb{P}^n_A)$). As $M^{\text{cl}}(\bar{X}_{\text{cl}}/\mathbb{P}^n_A)$ is regular away from the fibre over ∞ , as the inclusion $\mathbb{P}^n_A \hookrightarrow M^{\text{cl}}(\bar{X}_{\text{cl}}/\mathbb{P}^n_A)$ of the fibre over 0 is a prime divisor, and as a surjective morphism from an integral scheme to the spectrum of a discrete valuation ring is necessarily flat, it follows that also the morphism $W_0 \rightarrow \mathbb{P}^n_A$ is generically finite and of degree e^m . As W_0 is an snc scheme, the claim follows from Corollary 3.12 and restricting along the inclusion $U \subset \mathbb{P}^n_A$. \square

Moreover, the other term of (2) is of the correct form by the following result:

Lemma 4.10 *We have that*

$$s^!([\mathcal{E}'^\circ \rightarrow \mathbb{P}(\mathcal{N}_{X/U} \oplus \mathcal{O})]) \in \Omega_*^A(X)$$

is an integral combination of regular cycles.

Proof By Lemma 4.3,

$$s^!([\mathcal{E}'^\circ \rightarrow \mathbb{P}(\mathcal{N}_{X/U} \oplus \mathcal{O})]) = \rho_* (c_r(\mathcal{N}_{X/U}(1)) \cdot [\mathcal{E}'^\circ \rightarrow \mathbb{P}(\mathcal{N}_{X/U} \oplus \mathcal{O})]) \in \Omega_*^A(X),$$

where ρ is the canonical projection $\mathbb{P}(\mathcal{N}_{X/U} \oplus \mathcal{O}) \rightarrow X$. Moreover, denoting by $\mathcal{E}'_1, \dots, \mathcal{E}'_r$ the prime components of \mathcal{E}'_o , letting $n_1, \dots, n_r > 0$ be such that

$$\mathcal{E}'_o = n_1 \mathcal{E}'_1 + \dots + n_r \mathcal{E}'_r$$

as effective Cartier divisors on W^o , and denoting for each $I \subset [r]$ the natural inclusion $\mathcal{E}'_I := \bigcap_{i \in I} \mathcal{E}'_i \hookrightarrow \mathcal{E}'_o$, the snc relations imply that

$$\begin{aligned} c_r(\mathcal{N}_{X/U}(1)) \bullet 1_{\mathcal{E}'_o/A} &= c_r(\mathcal{N}_{X/U}(1)) \bullet \sum_{I \subset [r]} \iota_*^I(F_I^{n_1, \dots, n_r}(c_1(\mathcal{O}(D_1)), \dots, c_1(\mathcal{O}(D_r)))) \bullet 1_{\mathcal{E}'_I/A} \\ &= \sum_{I \subset [r]} \iota_*^I(c_r(\mathcal{N}_{X/U}(1)) \bullet F_I^{n_1, \dots, n_r}(c_1(\mathcal{O}(D_1)), \dots, c_1(\mathcal{O}(D_r)))) \bullet 1_{\mathcal{E}'_I/A} \end{aligned}$$

in $\Omega_*^A(\mathcal{E}'_o)$, so the claim follows from Proposition 3.6 and pushing forward to X . \square

We are now ready to prove the main lemma needed for the proof of the algebraic Spivak's theorem.

Proof of Lemma 4.7 We will proceed by induction on the Krull dimension of X , the base case of an empty scheme being trivial. Suppose that X has Krull dimension d and the result has been proven for derived quasiprojective and quasismooth derived A schemes of Krull dimension at most $d - 1$. Denoting by π_X the structure morphism $X \rightarrow \text{Spec}(A)$ and by $\mathcal{O}_X(1)$ the restriction of $\mathcal{O}(1)$ to X , and combining (2) with Lemmas 4.9 and 4.10, we see that

$$1_{X/A} = \frac{\beta - \sum_{i=1}^n \alpha_i \cdot (c_1(\mathcal{O}_X(1)))^i \bullet 1_{X/A}}{e^m} \in \Omega_*^A(X)[e^{-1}],$$

where $\beta \in \Omega_*^A(X)[e^{-1}]$ and $\alpha_i \in \Omega^{-i}(\text{Spec}(A))$ are integral combinations of regular cycles. By Lemma 4.11 the classes

$$c_1(\mathcal{O}_X(1))^i \bullet 1_{X/A}$$

for $i > 0$ are equivalent to an \mathbb{L} -linear combination of pushforwards of fundamental classes of derived schemes of Krull dimension at most $d - 1$, and therefore these classes can be dealt with inductively. We can therefore express $1_{X/A}$ in the desired form, proving the claim. \square

We used the following lemma in the above proof:

Lemma 4.11 *Let X be a quasiprojective derived A -scheme of Krull dimension d , and let \mathcal{L} be a line bundle on X . Then*

$$c_1(\mathcal{L})^i = \sum_{j=1}^N a_j [V_j \hookrightarrow X] \in \Omega^i(X),$$

where $a_j \in \mathbb{L}$ and V_j are derived schemes of Krull dimension at most $d - 1$.

Proof It is enough to prove the claim for $i = 1$: if

$$c_1(\mathcal{L}) = \sum_{j=1}^N a_j [V_j \hookrightarrow X] \in \Omega^1(X)$$

is the desired expression, then

$$c_1(\mathcal{L})^i = \sum_{j=1}^N a_j [V'_j \hookrightarrow X] \in \Omega^i(X),$$

where V'_j is the derived vanishing locus of the zero section of $\mathcal{L}|_{V_j}^{\oplus i-1}$ on V_j . Suppose first that \mathcal{L} is a very ample line bundle. Then, for all $n \gg 0$, the line bundle $\mathcal{L}^{\otimes n}$ has a global section s whose vanishing locus does not contain any of the irreducible components of X_{cl} . In particular, we can find coprime integers p and q and global sections $s_p \in \Gamma(X; \mathcal{L}^{\otimes p})$ and $s_q \in \Gamma(X; \mathcal{L}^{\otimes q})$ whose derived vanishing loci $[Z_1 \hookrightarrow X]$ and $[Z_2 \hookrightarrow X]$ have Krull dimension at most $d - 1$, and therefore Lemma 3.1 implies that

$$c_1(\mathcal{L}) = \left(1_X + \sum_{i=1}^{\infty} b_i c_1(\mathcal{L})^i \right) \cdot (a[Z_1 \hookrightarrow X] + b[Z_2 \hookrightarrow X]) \in \Omega^1(X)$$

for a and b integers and $b_i \in \mathbb{L}$. Hence the claim follows for \mathcal{L} very ample. As a general line bundle \mathcal{L} is equivalent to $\mathcal{L}_1 \otimes \mathcal{L}_2^\vee$ with \mathcal{L}_1 and \mathcal{L}_2 very ample, the general case follows using the formal group law. □

It is now easy to prove the algebraic Spivak’s theorem.

Theorem 4.12 (algebraic Spivak’s theorem) *Let A be a field or an excellent Henselian discrete valuation ring with a perfect residue field κ , and let e be the residual characteristic exponent of A . Then the algebraic cobordism ring $\Omega^*(\text{Spec}(A))[e^{-1}]$ is generated as a $\mathbb{Z}[e^{-1}]$ -algebra by regular projective A -schemes. Moreover, for all quasiprojective derived A -schemes X , $\Omega_\bullet(X)[e^{-1}]$ is generated as an $\Omega^*(\text{Spec}(A))[e^{-1}]$ -module by classes of regular schemes mapping projectively to X .*

Proof Suppose X is a quasiprojective derived A -scheme. Then $\Omega_{\bullet}(X)[e^{-1}]$ is by definition generated as a $\mathbb{Z}[e^{-1}]$ -module by cycles of the form

$$[V \xrightarrow{f} X] = f_*(1_{V/\mathbb{Z}})$$

with f projective and V a derived complete intersection scheme, and the claim follows from Lemma 4.7 under the natural isomorphism $-\bullet 1_{A/\mathbb{Z}}: \Omega_{*}^A(X) \xrightarrow{\cong} \Omega_{\bullet}(X)$. \square

This result has several immediate corollaries. Let us start with the following:

Corollary 4.13 *Suppose A is as in Theorem 4.12. Then, for all quasiprojective derived A -schemes X , $\Omega_{\bullet}(X)[e^{-1}]$ is generated as a $\mathbb{Z}[e^{-1}]$ -module by cycles of the form*

$$[V \rightarrow X]$$

with V a classical complete intersection scheme, and the structure morphism $V \rightarrow \text{Spec}(A)$ is either flat or factors through the unique closed point of $\text{Spec}(A)$.

Proof Let X be a quasiprojective derived A -scheme. Then, by Theorem 4.12, the algebraic bordism group $\Omega_{\bullet}(X)$ is generated by $\mathbb{Z}[e^{-1}]$ -linear combinations of elements of the form

$$\begin{aligned} &([V_1 \rightarrow \text{Spec}(A)] \cdots [V_1 \rightarrow \text{Spec}(A)]) \cdot [W \rightarrow X] \\ &= [V_1 \times_{\text{Spec}(A)}^R \cdots \times_{\text{Spec}(A)}^R V_r \times_{\text{Spec}(A)}^R W \rightarrow X], \end{aligned}$$

where V_i and W are regular. If A is a field, then the derived fibre product is classical, proving the claim for fields.

We are left with proving the first claim for discrete valuation rings. We first observe that since V_i and W are integral, each of them is either flat over A or factors through the residue field κ of A . If all or all but one of the V_1, \dots, V_r and W are flat over A , the derived fibre product is classical, so we are left with the case where at least two of them factor through κ . Without loss of generality, we can assume them to be $V = V_1$ and W . But then one computes that the derived fibre product $V \times_{\text{Spec}(A)}^R W$ is the derived vanishing locus of the zero section of the trivial line bundle in $V \times_{\text{Spec}(\kappa)}^R W \simeq V \times_{\text{Spec}(\kappa)} W$ and the claim follows from the fact that

$$[V \times_{\text{Spec}(A)}^R W \rightarrow X] = f_*(c_1(\mathcal{O}) \bullet 1_{V \times_{\text{Spec}(\kappa)} W/\mathbb{Z}}) = 0 \in \Omega_{\bullet}(X),$$

where f is the induced morphism $V \times_{\text{Spec}(\kappa)} W \rightarrow X$. \square

Corollary 4.14 *If k is a field, then the e -inverted bordism groups $\Omega_{\bullet}(X)[e^{-1}]$, where X is a quasiprojective derived k -scheme, are generated as $\mathbb{Z}[e^{-1}]$ -modules by classes of regular k -varieties mapping projectively to X .*

Proof If k is perfect, then this is an immediate corollary of [Theorem 4.12](#) because a quasiprojective derived k -scheme is regular if and only if it is smooth over k , and smoothness is stable under derived base change.

Suppose k is not perfect and let k^{perf} denote its perfection. Then, by the previous case, for every bordism class $[Y \rightarrow X]$, the base change $[Y_{k^{\text{perf}}} \rightarrow X_{k^{\text{perf}}}]$ is equivalent to a linear combination of regular k^{perf} -varieties mapping projectively to $X_{k^{\text{perf}}}$. Chasing the coefficients, we see that the same is true for $[Y_{k'} \rightarrow X_{k'}]$, where k' is a finite field extension of k of degree e^m . Therefore,

$$[Y \rightarrow X] = e^{-m}[Y_{k'} \rightarrow X_{k'}]$$

is equivalent to a linear combination of regular k -varieties. □

An immediate corollary of the above is the following vanishing result:

Corollary 4.15 *Let X be a quasiprojective A -variety. Then:*

- (1) *If $A = k$ is a field, $\Omega_i^k(X)[e^{-1}] = 0$ for all $i < 0$.*
- (2) *If A is a discrete valuation ring, $\Omega_i^A(X)[e^{-1}] = 0$ for all $i < -1$.*

Proof This follows from [Corollary 4.13](#) because lci A -schemes have relative virtual dimension at least 0 when A is a field, and at least -1 when A is a discrete valuation ring. □

Note that the above bounds are strict as the fundamental class $1_{\kappa/A}$ of $\text{Spec}(\kappa)$, where κ is the residue field of a discrete valuation ring A , lies in degree -1 . The above vanishing result also has the following amusing corollary, which seems to be difficult to prove directly. Similar results can be proven using virtual fundamental classes in intersection theory (or graded K -theory) combined with Grothendieck–Riemann–Roch formulas; see eg [\[17\]](#).

Corollary 4.16 *Let k be a field and let X be a quasismooth projective derived k -scheme with negative virtual dimension. Then*

$$\sum_{i \in \mathbb{Z}} (-1)^i \dim_k(H^i(X; \mathcal{O}_X)) = 0 \in \mathbb{Z}.$$

Proof Indeed, the left-hand side is the image of

$$[X \rightarrow \text{Spec}(k)] \in \Omega_*(\text{Spec}(k))[e^{-1}]$$

under the natural morphism

$$\Omega_*(\text{Spec}(k))[e^{-1}] \rightarrow K^0(\text{Spec}(k))[e^{-1}] \cong \mathbb{Z}[e^{-1}],$$

so the claim follows from [Corollary 4.15](#). □

Let us then turn to the structure of the algebraic cobordism ring of A . Using birational geometry of varieties of dimension ≤ 2 , we are able to make modest progress towards [Conjecture 1.1](#).

Corollary 4.17 *Let A be as in [Theorem 4.12](#). Then:*

- (1) $\Omega^i(\text{Spec}(A))[e^{-1}]$ vanishes for $i > 0$, and the natural map

$$\mathbb{Z}[e^{-1}] \cong \mathbb{L}^0[e^{-1}] \rightarrow \Omega^0(\text{Spec}(k))[e^{-1}]$$

is an isomorphism; in other words, $\Omega^0(\text{Spec}(A))$ is the free $\mathbb{Z}[e^{-1}]$ -module generated by 1_A .

- (2) If $A = k$ is a field, then the natural map

$$\mathbb{L}^*[e^{-1}] \rightarrow \Omega^*(\text{Spec}(k))[e^{-1}]$$

is an isomorphism in degrees $0, -1$ and -2 ; in other words, $\Omega^1(\text{Spec}(k))$ is the free $\mathbb{Z}[e^{-1}]$ -module generated by $[\mathbb{P}_k^1]$ and $\Omega^2(\text{Spec}(k))$ is the free $\mathbb{Z}[e^{-1}]$ -module generated by $[\mathbb{P}_k^1]^2$ and $[\Sigma_{1,k}]$, where $\Sigma_{1,k}$ is the Hirzebruch surface of degree 1 over k .

Proof (1) Let us first assume that $A = k$ is a field. The vanishing of $\Omega^i(\text{Spec}(A))[e^{-1}]$ for $i > 0$ follows from [Corollary 4.15](#), and by [Theorem 4.12](#) $\Omega^0(\text{Spec}(A))[e^{-1}]$ is generated as a $\mathbb{Z}[e^{-1}]$ -algebra by classes of field extensions of k . It then follows from [Lemma 3.10](#) that the natural morphism

$$\mathbb{Z}[e^{-1}] \cong \mathbb{L}^0[e^{-1}] \rightarrow \Omega^0(\text{Spec}(k))[e^{-1}]$$

is surjective, and hence an isomorphism by [Proposition 2.23](#).

Suppose then that A is an excellent Henselian discrete valuation ring with a perfect residue field κ . Then the vanishing of $\Omega^i(\text{Spec}(A))[e^{-1}]$ for $i > 1$ follows from [Corollary 4.15](#). Moreover, by [Corollary 4.13](#), the Gysin pushforward map

$$\mathbb{Z}[e^{-1}] \cong \Omega^0(\text{Spec}(\kappa))[e^{-1}] \rightarrow \Omega^1(\text{Spec}(A))[e^{-1}]$$

is a surjection because syntomic A -schemes cannot have negative relative virtual dimension. As $[\text{Spec}(\kappa) \hookrightarrow \text{Spec}(A)] = 0$, it follows that $\Omega^1(\text{Spec}(A))[e^{-1}]$ vanishes.

We are left to compute $\Omega^0(\text{Spec}(A))[e^{-1}]$. It follows from the previous paragraph (and Lemma 2.25) that it is generated as a $\mathbb{Z}[e^{-1}]$ -algebra by classes of the form $[V \rightarrow \text{Spec}(A)]$, where V is a 1-dimensional regular and integral projective A -scheme. If V is flat over A , then $[V \rightarrow \text{Spec}(A)]$ lies in the image of the natural morphism $\mathbb{L}^0[e^{-1}] \rightarrow \Omega^0(\text{Spec}(A))[e^{-1}]$ by Lemma 3.10. On the other hand, if V is not flat over A , then it is a smooth curve over κ . Using the second part of this result,

$$[V \rightarrow \text{Spec}(\kappa)] = b \in \Omega^{-1}(\text{Spec}(\kappa))[e^{-1}],$$

where b comes from $\mathbb{L}^{-1}[e^{-1}]$, and therefore

$$[V \rightarrow \text{Spec}(A)] = b[\text{Spec}(\kappa) \hookrightarrow \text{Spec}(A)] = 0 \in \Omega^0(\text{Spec}(A))[e^{-1}].$$

It follows that the natural map

$$\mathbb{Z}[e^{-1}] \cong \mathbb{L}^0[e^{-1}] \rightarrow \Omega^0(\text{Spec}(A))[e^{-1}]$$

is surjective, and hence an isomorphism by Proposition 2.23.

(2) The proof splits into three cases.

(a) (k is infinite and perfect) We follow the argument of Levine and Morel from [20, Section 4.3]. Let X be a smooth k -variety of dimension $n \leq 2$, and suppose that the claim is known up to dimension $n - 1$. It is well known that X admits a finite birational morphism $X \rightarrow X'$, where $X' \hookrightarrow \mathbb{P}_k^{n+1}$ is a hypersurface; let us denote the degree of X' by d . By the embedded resolution of surfaces (see [1, Section 0]), we may find a morphism

$$\pi: Y \rightarrow \mathbb{P}_k^{n+1}$$

which is a composition of finitely many blow-ups whose centres are smooth subvarieties of dimension $\leq n - 1$ of \mathbb{P}_k^{n+1} such that

$$\pi^{-1}(X') = \tilde{X} + n_1 \mathcal{E}_1 + \dots + n_r \mathcal{E}_r$$

as divisors, where the strict transform \tilde{X} of X' is smooth, \mathcal{E}_i are the exceptional divisors (which are projective bundles over smooth varieties of dimension $\leq n - 1$) and $n_i > 0$. As X is the normalization of X' , we obtain a canonical morphism

$$\rho: \tilde{X} \rightarrow X.$$

Notice that, if $n = 1$, then ρ is an isomorphism, and if $n = 2$, then ρ is a composition of blow-ups at closed points (see eg [7, Tag 0C5RJ]). It follows from Lemma 2.27 that $[X]$ lies in the image of $\mathbb{L}^*[e^{-1}]$ if and only if $[\tilde{X}]$ does.

Note that it follows from [Lemma 2.27](#) and [Corollary 4.14](#) that

$$c_1(\mathcal{O}(d)) \bullet [Y \rightarrow \mathbb{P}_k^{n+1}] = [X' \rightarrow \mathbb{P}_k^{n+1}] + \sum_{i=1}^m a_i [V_i \rightarrow \mathbb{P}_k^{d+1}],$$

where $a_i \in \mathbb{L}^*$ and V_i are smooth varieties of dimension $\leq n - 1$, and, as $[X']$ lies in the image of $\mathbb{L}^*[e^{-1}]$ (this follows from the formal group law and the fact that $[\mathbb{P}_k^i]$ are in the image of \mathbb{L}^*), it follows that $[\pi^{-1}(X')]$ lies in the image of $\mathbb{L}^*[e^{-1}]$. On the other hand, by [Lemma 2.26](#), $[\mathcal{E}_i]$ lie in the image of $\mathbb{L}^*[e^{-1}]$, and therefore, computing the class of $[\pi^{-1}(X') \hookrightarrow Y]$ using the formal group law, we see that also $[\tilde{X}]$ lies in the image of $\mathbb{L}^*[e^{-1}]$. But, as we already noted, this implies that $[X]$ lies in the image of $\mathbb{L}^*[e^{-1}]$, so we are done.

(b) (k is not perfect) Let X be a regular k -variety of dimension $n \leq 2$, and let k^{perf} be the perfection of k . It follows from (a) that $[X_{k^{\text{perf}}}]$ lies in the image of $\mathbb{L}^*[e^{-1}]$. Moreover, by chasing coefficients we see that this is true for some finite intermediate field extension

$$k \subset k' \subset k^{\text{perf}},$$

and therefore

$$[X] = e^{-m} [X_{k'}]$$

lies in the image of $\mathbb{L}^*[e^{-1}]$, where $e^m = [k' : k]$.

(c) (k is finite) The problem with the strategy of (a) in this case is that a smooth projective k -variety V might not admit a finite birational projection to a hypersurface. But this is easy to fix: clearly every such a variety admits such a projection for all large enough finite extensions of k , so in particular we can find extensions k' and k'' of coprime degrees p and q such that $X_{k'}$ and $X_{k''}$ admit the desired projections over k' and k'' , respectively. Therefore, we can run the argument of part (a) for $X_{k'}$ and $X_{k''}$, and therefore

$$\begin{aligned} a[X_{k'} \rightarrow \text{Spec}(k)] + b[X_{k''} \rightarrow \text{Spec}(k)] &= a[X_{k'} \rightarrow \text{Spec}(k)] + b[X_{k''} \rightarrow \text{Spec}(k)] \\ &= (ap + bq)[X \rightarrow \text{Spec}(k)] \\ &= [X \rightarrow \text{Spec}(k)] \end{aligned}$$

lies in the image of $\mathbb{L}^*[e^{-1}] \rightarrow \Omega^*(\text{Spec}(k))[e^{-1}]$, where $a, b \in \mathbb{Z}$ are such that $ap + bq = 1$. □

Let us also record the following result, concerning the generators of the cobordism ring of a discrete valuation ring:

Proposition 4.18 *Let A be an excellent Henselian discrete valuation ring with a perfect residue field κ . Then:*

- (1) $\Omega^{-1}(\text{Spec}(A))[e^{-1}]$ is generated as a $\mathbb{Z}[e^{-1}]$ -module by regular and flat projective A -schemes.
- (2) If κ has characteristic 0, then $\Omega^*(\text{Spec}(A))$ is generated as an abelian group by the classes of syntomic projective A -schemes.

Proof In both cases the proof is essentially the same. By [Corollary 4.13](#), we see that $\Omega^i(\text{Spec}(A))[e^{-1}]$ is generated by cycles of the form $[V \rightarrow \text{Spec}(A)]$ with V a complete intersection scheme that is either flat over A or factors through $\text{Spec}(\kappa)$, and we have to show that nonflat cycles vanish at least in certain degrees. Note that, if V is not flat, then

$$[V \rightarrow \text{Spec}(\kappa)] \in \Omega^{-d}(\text{Spec}(\kappa))[e^{-1}]$$

lies in the image of the natural morphism $\mathbb{L}^{-d}[e^{-1}] \rightarrow \Omega^{-d}(\text{Spec}(\kappa))[e^{-1}]$ if d is at most 2 or if κ has characteristic 0 [[20](#), [Theorem 4.3.7](#)], and therefore

$$[V \rightarrow \text{Spec}(A)] = b[\text{Spec}(\kappa) \hookrightarrow \text{Spec}(A)] = 0 \in \Omega^{1-d}(A)$$

for some $b \in \mathbb{L}^{-d}[e^{-1}]$. Hence $\Omega^{-i}(\text{Spec}(A))$ is generated by classes of syntomic projective A -schemes if i is at most 1 or if κ has characteristic 0. Moreover, as $\mathbb{Z}[e^{-1}] \cong \Omega^0(\text{Spec}(A))[e^{-1}]$, it follows from [Theorem 4.12](#) that $\Omega^{-1}(\text{Spec}(A))[e^{-1}]$ is generated as a $\mathbb{Z}[e^{-1}]$ -module by regular cycles, proving the first claim. \square

4.3 Proof of the extension theorem

The goal of this section is to prove the following result:

Theorem 4.19 (extension theorem) *Let A be a field or an excellent Henselian discrete valuation ring with a perfect residue field κ , and let $j: X \hookrightarrow \bar{X}$ be an open embedding of quasiprojective derived A -schemes. Then the pullback morphism*

$$j^!: \Omega_*^A(\bar{X})[e^{-1}] \rightarrow \Omega_*^A(X)[e^{-1}]$$

is surjective.

As $j^!$ is $\Omega^*(\text{Spec}(A))[e^{-1}]$ -linear, it is enough by [Theorem 4.12](#) to show that an element

$$[V \rightarrow X] \in \Omega_d^A(X)$$

with V a regular scheme lies in the image of $j^!$. Letting \bar{V} be the scheme-theoretic closure of V in some locally closed embedding

$$V \hookrightarrow \mathbb{P}^n \times X_{\text{cl}} \subset \mathbb{P}^n \times \bar{X}_{\text{cl}}$$

which clearly maps projectively to \bar{X} , it is enough to extend the fundamental class of V to an element of $\Omega_*^A(\bar{V})[e^{-1}]$. As all quasiprojective A -schemes admit a projective compactification, [Theorem 4.19](#) follows from an easy inductive argument combined with the following lemma and [Corollary 4.15](#).

Lemma 4.20 *Let \bar{X} be a projective A -scheme and let $j: X \hookrightarrow \bar{X}$ be an open embedding with X a regular scheme of virtual A -dimension d . If the Gysin pullback morphism*

$$j^!: \Omega_i^A(\bar{X})[e^{-1}] \rightarrow \Omega_i^A(X)[e^{-1}]$$

is surjective for $i \leq d - 1$, then there exists a class $\alpha \in \Omega_d^A(\bar{X})[e^{-1}]$ with $j^!(\alpha) = 1_{X/A}$.

Proof Note that we can find a (derived) Cartesian square

$$\begin{array}{ccc} X & \xhookrightarrow{j} & \bar{X} \\ \downarrow i & & \downarrow \bar{i} \\ U & \hookrightarrow & \mathbb{P}^n_A \end{array}$$

where the vertical arrows are closed embeddings and the horizontal arrows are open embeddings. Since i is a (derived) regular embedding, we can apply [Construction 4.8](#), and conclude that, in the notation of [Section 4.1](#) and [Construction 4.8](#), the equality

$$i^!([W_0^\circ \rightarrow U]) = s^!([\mathcal{E}'^\circ \rightarrow \mathbb{P}(\mathcal{N}_{X/U} \oplus \mathcal{O})]) \in \Omega_*^A(X)$$

holds. By [Lemma 4.9](#),

$$1_{X/A} = \frac{s^!([\mathcal{E}'^\circ \rightarrow \mathbb{P}(\mathcal{N}_{X/U} \oplus \mathcal{O})]) - \sum_{i=1}^n \alpha_i \cdot (c_1(\mathcal{O}_X(1)))^i \cdot 1_{X/A}}{e^m} \in \Omega_*^A(X)[e^{-1}],$$

where $\mathcal{O}_X(1)$ is the restriction of $\mathcal{O}(1)$ on X , and $\alpha_i \in \Omega^{-i}(\text{Spec}(A))$. As

$$c_1(\mathcal{O}_X(1))^i \cdot 1_{X/A} \in \Omega_{d-i}^A(X)[e^{-1}],$$

we have reduced the problem to showing that

$$s^!([\mathcal{E}'^\circ \rightarrow \mathbb{P}(\mathcal{N}_{X/U} \oplus \mathcal{O})])$$

extends. By [Lemma 4.3](#),

$$s^!([\mathcal{E}'^\circ \rightarrow \mathbb{P}(\mathcal{N}_{X/U} \oplus \mathcal{O})]) = \rho_*(c_r(\mathcal{N}_{X/U}(1)) \cdot [\mathcal{E}'^\circ \rightarrow \mathbb{P}(\mathcal{N}_{X/U} \oplus \mathcal{O})]) \in \Omega_*^A(X),$$

where ρ is the projection $\mathbb{P}(\mathcal{N}_{X/U} \oplus \mathcal{O}) \rightarrow X$, and we can find the desired extension by applying Lemma 4.21 to the open immersion $\mathcal{E}^{\circ} \hookrightarrow \mathcal{E}'$ and pushing forward to \bar{X} . \square

We needed the following lemma in the above proof:

Lemma 4.21 *Let $j: D^{\circ} \hookrightarrow D$ be an open immersion of snc schemes admitting an ample line bundle, and let E be a vector bundle on D° . Then, for all $i \geq 0$, there exists a class $\alpha \in \Omega_{\bullet}(D)$ such that*

$$j^!(\alpha) = c_i(E) \bullet 1_{D^{\circ}/\mathbb{Z}} \in \Omega_{\bullet}(D^{\circ}).$$

Proof Let D_1, \dots, D_r be the prime components of D , and denote for each $I \subset [r]$ by ι^I the closed embedding

$$D_I := \bigcap_{i \in I} D_i \hookrightarrow D.$$

By the snc relations,

$$1_{D/\mathbb{Z}} = \sum_{I \subset [r]} \iota_*^I(\alpha_I) \in \Omega_{\bullet}(D),$$

and, moreover, if \mathfrak{J} contains all I such that D_I meets the image of j , then

$$1_{D^{\circ}/\mathbb{Z}} = j^! \left(\sum_{I \in \mathfrak{J}} \iota_*^I(\alpha_I) \right) \in \Omega_{\bullet}(D^{\circ}).$$

For each $I \in \mathfrak{J}$ choose a coherent sheaf \mathcal{F}_I on D_I extending $E|_{j^{-1}(D_I)}$. As D_I is regular, \mathcal{F}_I is a perfect complex, and therefore has well-defined Chern classes. By the naturality of Chern classes, it follows that

$$c_i(E) \bullet 1_{D^{\circ}/\mathbb{Z}} = j^! \left(\sum_{I \in \mathfrak{J}} \iota_*^I(c_i(\mathcal{F}_I) \bullet \alpha_I) \right) \in \Omega_{\bullet}(D^{\circ}),$$

which is exactly what we wanted. \square

Combining the extension theorem with the projective bundle formula, we obtain the following \mathbb{A}^1 -invariance statement:

Corollary 4.22 (homotopy invariance) *Let A be as in Theorem 4.19, let X be a quasiprojective derived A -scheme and let $p: E \rightarrow X$ be a vector bundle of rank r on X . Then the pullback map*

$$p^!: \Omega_*^A(X)[e^{-1}] \rightarrow \Omega_{*+r}^A(E)[e^{-1}]$$

is an isomorphism.

Proof As p admits a section, $p^!$ is at least an injection. On the other hand, by the projective bundle formula, the morphism

$$\bigoplus_{i=0}^r \Omega_{*-r+i}^A(X)[e^{-1}] \rightarrow \Omega_*^A(\mathbb{P}(E \oplus \mathcal{O}))[e^{-1}],$$

where the i^{th} morphism is defined as

$$c_1(\mathcal{O}(1))^i \bullet \bar{p}^!(-),$$

is an isomorphism, where \bar{p} is the natural projection $\mathbb{P}(E \oplus \mathcal{O}) \rightarrow X$. As $\mathcal{O}(1)$ has a global section with derived vanishing locus $\mathbb{P}(E)$ whose complement in $\mathbb{P}(E \oplus \mathcal{O})$ is the open immersion $j: E \hookrightarrow \mathbb{P}(E \oplus \mathcal{O})$, it follows that all classes of the form $c_1(\mathcal{O}(1))^i \bullet \bar{p}^!(\alpha)$ vanish when pulled back along j . As $j^!$ is surjective by the extension theorem, we can conclude that $p^!$ is a surjection, which finishes the proof. \square

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2023

- Isometry groups with radical, and aspherical Riemannian manifolds with large symmetry, I 1
OLIVER BAUES and YOSHINOBU KAMISHIMA
- Symplectic resolutions of character varieties 51
GWYN BELLAMY and TRAVIS SCHEDLER
- Odd primary analogs of real orientations 87
JEREMY HAHN, ANDREW SENGER and DYLAN WILSON
- Examples of non-Kähler Calabi–Yau 3–folds with arbitrarily large b_2 131
KENJI HASHIMOTO and TARO SANO
- Rotational symmetry of ancient solutions to the Ricci flow in higher dimensions 153
SIMON BRENDLE and KEATON NAFF
- d_p –convergence and ϵ –regularity theorems for entropy and scalar curvature lower bounds 227
MAN-CHUN LEE, AARON NABER and ROBIN NEUMAYER
- Algebraic Spivak’s theorem and applications 351
TONI ANNALA
- Collapsing Calabi–Yau fibrations and uniform diameter bounds 397
YANG LI