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Convex cocompact actions of relatively hyperbolic groups

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We consider discrete groups in $\mathrm{PGL}_d(\mathbb{R})$ acting convex cocompactly on a properly convex domain in real projective space. For such groups, we establish necessary and sufficient conditions for the group to be relatively hyperbolic in terms of the geometry of the convex domain. This answers a question of Danciger, Guéritaud and Kassel and is analogous to a result of Hruska and Kleiner for $\mathrm{CAT}(0)$ spaces.

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1 Introduction

If G is a connected simple Lie group with trivial center and no compact factors and $K \leq G$ is a maximal compact subgroup, then $X = G/K$ has a unique (up to scaling) Riemannian symmetric metric g such that $G = \text{Isom}_0(X, g)$. The metric g is nonpositively curved and X is simply connected; hence, every two points in X are joined by a unique geodesic segment. A subset $\mathcal{C} \subset X$ is called *convex* if, for every $x, y \in \mathcal{C}$, the geodesic joining them is also in \mathcal{C} . Then a discrete group $\Gamma \leq G$ is *convex cocompact* if there exists a nonempty closed convex set $\mathcal{C} \subset X$ such that $\gamma(\mathcal{C}) = \mathcal{C}$ for all $\gamma \in \Gamma$ and Γ acts cocompactly on \mathcal{C} .

When G has real rank one, for instance $G = \text{PSL}_2(\mathbb{R})$, there are an abundance of examples of convex cocompact subgroups in the context of Kleinian groups and hyperbolic geometry. When G has higher real rank, for instance $G = \text{PSL}_3(\mathbb{R})$, there are few examples: Kleiner and Leeb [25] and independently Quint [29] proved that every Zariski dense convex cocompact subgroup is a cocompact lattice.

Danciger, Guéritaud and Kassel [12] have recently introduced a different notion of convex cocompact subgroups in $\text{PGL}_d(\mathbb{R})$ based on the action of the subgroup on the projective space $\mathbb{P}(\mathbb{R}^d)$. Their notion of convex cocompactness requires some preliminary definitions. When $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, the *automorphism group of Ω* is defined to be

$$\text{Aut}(\Omega) := \{g \in \text{PGL}_d(\mathbb{R}) : g\Omega = \Omega\}.$$

For a subgroup $\Lambda \leq \text{Aut}(\Omega)$, the *full orbital limit set of Λ in Ω* is defined to be

$$\mathcal{L}_\Omega(\Lambda) := \bigcup_{p \in \Omega} (\overline{\Lambda \cdot p} \setminus \Lambda \cdot p).$$

Next, let $\mathcal{C}_\Omega(\Lambda)$ denote the convex hull of $\mathcal{L}_\Omega(\Lambda)$ in Ω .

Definition 1.1 (Danciger, Guéritaud and Kassel [12, Definition 1.10]) Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain. An infinite discrete subgroup $\Lambda \leq \text{Aut}(\Omega)$ is called *convex cocompact* if $\mathcal{C}_\Omega(\Lambda)$ is nonempty and Λ acts cocompactly on $\mathcal{C}_\Omega(\Lambda)$.

When Λ is word hyperbolic, there is a close connection between this class of discrete groups in $\text{PGL}_d(\mathbb{R})$ and Anosov representations; see [12] for details and Danciger, Guéritaud and Kassel [13] and Zimmer [35] for related results. Further, by adapting an argument of Benoist [3], Danciger, Guéritaud and Kassel proved a characterization of hyperbolicity in terms of the geometry of $\mathcal{C}_\Omega(\Lambda)$. To state their result we need two definitions.

Definition 1.2 A subset $S \subset \mathbb{P}(\mathbb{R}^d)$ is a *simplex* if there exist $g \in \text{PGL}_d(\mathbb{R})$ and $1 \leq k \leq d$ such that

$$gS = \{[x_1 : \cdots : x_k : 0 : \cdots : 0] \in \mathbb{P}(\mathbb{R}^d) : x_1 > 0, \dots, x_k > 0\}.$$

In this case we define the dimension of S to be $\dim(S) = k - 1$ (notice that S is homeomorphic to \mathbb{R}^{k-1}) and say that the k points

$$g^{-1}[1 : 0 : \cdots : 0], g^{-1}[0 : 1 : 0 : \cdots : 0], \dots, g^{-1}[0 : \cdots : 0 : 1 : 0 : \cdots : 0] \in \partial S$$

are the vertices of S .

Definition 1.3 Suppose $A \subset B \subset \mathbb{P}(\mathbb{R}^d)$. Then A is *properly embedded in B* if the inclusion map $A \hookrightarrow B$ is a proper map (relative to the subspace topology).

Finally, given a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ let H_Ω denote the Hilbert metric on Ω (see Section 3.3 for the definition).

Theorem 1.4 (Danciger, Guéritaud and Kassel [12, Theorem 1.15]) *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $\Lambda \leq \text{Aut}(\Omega)$ is convex cocompact. Then the following are equivalent:*

- (1) $\mathcal{C}_\Omega(\Lambda)$ contains no properly embedded simplices with dimension at least two.
- (2) $(\mathcal{C}_\Omega(\Lambda), H_\Omega)$ is Gromov hyperbolic.
- (3) Λ is word hyperbolic.

Remark 1.5 In the special case when Λ acts cocompactly on Ω , Theorem 1.4 is due to Benoist [3].

The case when Λ is not word hyperbolic is less understood and Danciger, Guéritaud and Kassel asked the following:

Question 1.6 (Danciger, Guéritaud and Kassel [12, Question A.2]) Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $\Lambda \leq \text{Aut}(\Omega)$ is convex cocompact. Under what conditions is Λ relatively hyperbolic with respect to a collection of virtually abelian subgroups?

We provide an answer to this question in terms of the geometry of the family of all maximal properly embedded simplices (note a properly embedded simplex is called *maximal* if it is not contained in a larger properly embedded simplex; they are not necessarily simplices of codimension one).

Our approach is motivated by previous work of Hruska and Kleiner [20] for CAT(0) spaces (see Section 1.2 for details). In some ways the Hilbert metric on a properly convex domain behaves like a CAT(0) metric; see the discussion of Marquis [27]. However, an old result of Kelly and Strauss [24] says that a Hilbert geometry (Ω, H_Ω) is CAT(0) if and only if it is isometric to real hyperbolic $(d-1)$ -space (in which case Ω coincides with the interior of the convex hull of an ellipsoid in some affine chart). Thus, one requires different techniques for studying the geometry of properly convex domains and the groups acting on them.

The following theorem is the first main result of this paper:

Theorem 1.7 (see Section 16) Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $\Lambda \leq \text{Aut}(\Omega)$ is convex cocompact and \mathcal{S}_{\max} is the family of all maximal properly embedded simplices in $\mathcal{C}_\Omega(\Lambda)$ of dimension at least two. Then the following are equivalent:

- (1) \mathcal{S}_{\max} is closed and discrete in the local Hausdorff topology induced by H_Ω .
- (2) $(\mathcal{C}_\Omega(\Lambda), H_\Omega)$ is a relatively hyperbolic space with respect to \mathcal{S}_{\max} .
- (3) $(\mathcal{C}_\Omega(\Lambda), H_\Omega)$ is a relatively hyperbolic space with respect to a family of properly embedded simplices in $\mathcal{C}_\Omega(\Lambda)$ of dimension at least two.
- (4) Λ is a relatively hyperbolic group with respect to a collection of virtually abelian subgroups of rank at least two.

Theorem 1.7 can be viewed as a real projective analogue of a result of Hruska and Kleiner [20] for CAT(0) spaces (see Section 1.2 for details). In this analogy, maximal properly embedded simplices correspond to maximal totally geodesic flats in CAT(0) spaces (see Islam and Zimmer [21] and Benoist [3]).

We also establish a number of properties for convex cocompact subgroups satisfying the conditions in [Theorem 1.7](#). Before stating these results, we informally introduce some notation (see [Section 3](#) for precise definitions). Given a properly convex set Ω which is open in its span, let $\text{diam}_\Omega(A)$ and $\mathcal{N}_\Omega(A; r)$ denote the diameter and r -neighborhood of a subset A with respect to the Hilbert metric. Also, given $x \in \overline{\Omega}$, let $F_\Omega(x) \subset \overline{\Omega}$ denote the open face of x in Ω (see [Definition 3.9](#)). Finally, given a properly convex set C , let $\text{rel-int}(C)$ denote the relative interior (see [Definition 3.2](#)) and let $\partial C = \overline{C} \setminus \text{rel-int}(C)$ denote the boundary.

We will prove the following:

Theorem 1.8 (see [Section 16](#)) *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $\Lambda \leq \text{Aut}(\Omega)$ is convex cocompact and \mathcal{S}_{\max} is the family of all maximal properly embedded simplices in $\mathcal{C}_\Omega(\Lambda)$ of dimension at least two. If \mathcal{S}_{\max} is closed and discrete in the local Hausdorff topology induced by H_Ω , then:*

- (1) Λ has finitely many orbits in \mathcal{S}_{\max} .
- (2) If $S \in \mathcal{S}_{\max}$, then $\text{Stab}_\Lambda(S)$ acts cocompactly on S and contains a finite-index subgroup isomorphic to \mathbb{Z}^k where $k = \dim S$.
- (3) If $A \leq \Lambda$ is an abelian subgroup of rank at least two, then there exists a unique $S \in \mathcal{S}_{\max}$ with $A \leq \text{Stab}_\Lambda(S)$.
- (4) If $S \in \mathcal{S}_{\max}$ and $x \in \partial S$, then $F_\Omega(x) = F_S(x)$.
- (5) If $S_1, S_2 \in \mathcal{S}_{\max}$ are distinct, then $\#(S_1 \cap S_2) \leq 1$ and $\partial S_1 \cap \partial S_2 = \emptyset$.
- (6) For any $r > 0$ there exists $D(r) > 0$ such that, if $S_1, S_2 \in \mathcal{S}_{\max}$ are distinct, then

$$\text{diam}_\Omega(\mathcal{N}_\Omega(S_1; r) \cap \mathcal{N}_\Omega(S_2; r)) \leq D(r).$$

- (7) If $\ell \subset \overline{\mathcal{C}_\Omega(\Lambda)} \cap \partial\Omega$ is a nontrivial line segment, then there exists $S \in \mathcal{S}_{\max}$ with $\ell \subset \partial S$.
- (8) If $x \in \overline{\mathcal{C}_\Omega(\Lambda)} \cap \partial\Omega$ is not a C^1 -smooth point of $\partial\Omega$, then there exists $S \in \mathcal{S}_{\max}$ with $x \in \partial S$.

Remark 1.9 In the special case when $d \leq 4$ and Λ acts cocompactly on $\Omega \subset \mathbb{P}(\mathbb{R}^d)$, [Theorems 1.7](#) and [1.8](#) can be obtained from results of Benoist [\[4\]](#). But, when $d > 4$, [Theorems 1.7](#) and [1.8](#) are new even in the special case when Λ acts cocompactly on Ω .

As alluded to above, convex cocompact subgroups can be seen as a way to extend the theory of Anosov representations to nonhyperbolic groups. Kapovich and Leeb [\[23\]](#) and Zhu [\[34\]](#) have also recently proposed notions of relative Anosov representations

for relatively hyperbolic groups. However, in their definitions, the peripheral subgroups will have unipotent image while convex cocompact subgroups never contain nontrivial unipotent elements. So the groups we consider are very different.

1.1 Naive convex cocompact subgroups

We also establish a variant of [Theorem 1.7](#) for a more general notion of convex cocompact subgroup.

Definition 1.10 Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain. An infinite discrete subgroup $\Lambda \leq \text{Aut}(\Omega)$ is called *naive convex cocompact* if there exists a nonempty closed convex subset $\mathcal{C} \subset \Omega$ such that

- (1) \mathcal{C} is Λ -invariant, that is, $g\mathcal{C} = \mathcal{C}$ for all $g \in \Lambda$, and
- (2) Λ acts cocompactly on \mathcal{C} .

In this case, we say that $(\Omega, \mathcal{C}, \Lambda)$ is a *naive convex cocompact triple*.

It is straightforward to construct examples where $\Lambda \leq \text{Aut}(\Omega)$ is naive convex cocompact but not convex cocompact (see [Section 2.3](#) or [[12](#), Section 3.4]). In these cases, the convex subset \mathcal{C} in [Definition 1.10](#) is a strict subset of $\mathcal{C}_\Omega(\Lambda)$.

For naive convex cocompact subgroups, we also provide a characterization of relative hyperbolicity, but require a technical notion of isolated simplices. In the naive convex cocompact case there exist examples where the group is relatively hyperbolic but the family of maximal properly embedded simplices is not discrete. Instead, maximal properly embedded simplices can occur in parallel families; see [Section 2.3](#). These examples lead to the following definition:

Definition 1.11 Suppose $(\Omega, \mathcal{C}, \Lambda)$ is a naive convex cocompact triple. A family \mathcal{S} of maximal properly embedded simplices in \mathcal{C} of dimension at least two is called

- (1) *isolated* if \mathcal{S} is closed and discrete in the local Hausdorff topology induced by H_Ω ;
- (2) *coarsely complete* if any properly embedded simplex in \mathcal{C} of dimension at least two is contained in a uniformly bounded tubular neighborhood of some maximal properly embedded simplex in \mathcal{S} ;
- (3) Λ -*invariant* if $g \cdot S \in \mathcal{S}$ for all $S \in \mathcal{S}$ and $g \in \Lambda$.

We say that $(\Omega, \mathcal{C}, \Lambda)$ has *coarsely isolated simplices* if there exists an isolated, coarsely complete and Λ -invariant family of maximal properly embedded simplices in \mathcal{C} of dimension at least two.

Remark 1.12 This definition is motivated by Hruska and Kleiner's notion of isolated flats in the first sense for CAT(0) spaces [20, page 1505] (see Section 1.2).

Informally, Definition 1.11 says that a naive convex cocompact triple has coarsely isolated simplices if it is possible to select a closed and discrete family of simplices that contain a representative from each parallel family of maximal properly embedded simplices.

We will prove the following:

Theorem 1.13 (see Section 14) *Suppose $(\Omega, \mathcal{C}, \Lambda)$ is a naive convex cocompact triple. Then the following are equivalent:*

- (1) $(\Omega, \mathcal{C}, \Lambda)$ has coarsely isolated simplices.
- (2) (\mathcal{C}, H_Ω) is a relatively hyperbolic space with respect to a family of properly embedded simplices in \mathcal{C} of dimension at least two.
- (3) Λ is a relatively hyperbolic group with respect to a family of virtually abelian subgroups of rank at least two.

Remark 1.14 Using the basic theory of relatively hyperbolic spaces (see Theorem 4.9), if (\mathcal{C}, H_Ω) is a relatively hyperbolic space with respect to a family of properly embedded simplices in \mathcal{C} , then every simplex in that family is maximal.

There is one subtle aspect of Theorem 1.13: it does *not* say that any isolated, coarsely complete and Λ -invariant family of simplices satisfies (2). In fact, it is possible to construct an example of a naive convex cocompact triple $(\Omega, \mathcal{C}, \Lambda)$ and a family \mathcal{S} of maximal properly embedded simplices where \mathcal{S} is isolated, coarsely complete and Λ -invariant but (\mathcal{C}, H_Ω) is *not* relatively hyperbolic with respect to \mathcal{S} (see Section 2.3 for details). This motivates the following definition:

Definition 1.15 Suppose $(\Omega, \mathcal{C}, \Lambda)$ is a naive convex cocompact triple. A family \mathcal{S} of maximal properly embedded simplices in \mathcal{C} of dimension at least two is called *strongly isolated* if, for every $r > 0$, there exists $D(r) > 0$ such that, if $S_1, S_2 \in \mathcal{S}$ are distinct, then

$$\text{diam}_\Omega(\mathcal{N}_\Omega(S_1; r) \cap \mathcal{N}_\Omega(S_2; r)) \leq D(r).$$

Remark 1.16 This definition is motivated by Hruska and Kleiner’s notion of isolated flats in the second sense for CAT(0) spaces [20, page 1505] (see Section 1.2).

It is fairly easy to show that a family of strongly isolated simplices is also isolated (see Observation 3.21). Although the converse is not always true, we will prove the following theorem:

Theorem 1.17 (see Section 10) *Suppose $(\Omega, \mathcal{C}, \Lambda)$ is a naive convex cocompact triple with coarsely isolated simplices. Then there exists a strongly isolated, coarsely complete and Λ -invariant family of maximal properly embedded simplices in \mathcal{C} of dimension at least two.*

We then prove the following refinement of the Theorem 1.13:

Theorem 1.18 (see Section 13) *Suppose $(\Omega, \mathcal{C}, \Lambda)$ is a naive convex cocompact triple with coarsely isolated simplices. Let S be a strongly isolated, coarsely complete and Λ -invariant family of maximal properly embedded simplices in \mathcal{C} of dimension at least two. Then:*

- (1) (\mathcal{C}, H_Ω) is a relatively hyperbolic space with respect to S .
- (2) Λ has finitely many orbits in S and, if $\{S_1, \dots, S_m\}$ is a set of orbit representatives, then Λ is a relatively hyperbolic group with respect to

$$\{\text{Stab}_\Lambda(S_1), \dots, \text{Stab}_\Lambda(S_m)\}.$$

Further, each $\text{Stab}_\Lambda(S_i)$ is virtually abelian of rank at least two.

Delaying definitions until later in the paper (see Definitions 3.2, 3.9 and 11.1), we will also establish an analogue of Theorem 1.8 for naive convex cocompact subgroups:

Theorem 1.19 (see Section 12) *Suppose $(\Omega, \mathcal{C}, \Lambda)$ is a naive convex cocompact triple with coarsely isolated simplices. Let S be a strongly isolated, coarsely complete and Λ -invariant family of maximal properly embedded simplices in \mathcal{C} of dimension at least two. Then:*

- (1) Λ has finitely many orbits in S .
- (2) If $S \in \mathcal{S}$, then $\text{Stab}_\Lambda(S)$ acts cocompactly on S and contains a finite-index subgroup isomorphic to \mathbb{Z}^k , where $k = \dim S$.

(3) If $A \leq \Lambda$ is an abelian subgroup with rank at least two, then there exists a unique $S \in \mathcal{S}$ with $A \leq \text{Stab}_\Lambda(S)$.

(4) There exists $D > 0$ such that, if $S \in \mathcal{S}$ and $x \in \partial S$, then

$$H_{F_\Omega(x)}^{\text{Haus}}(\bar{C} \cap F_\Omega(x), F_S(x)) \leq D.$$

(5) If $S_1, S_2 \in \mathcal{S}$ are distinct, then $\#(S_1 \cap S_2) \leq 1$ and

$$\left(\bigcup_{x \in \partial S_1} F_\Omega(x) \right) \cap \left(\bigcup_{x \in \partial S_2} F_\Omega(x) \right) = \emptyset.$$

(6) If $a, b, c \in \bar{C} \cap \partial\Omega$ form a half triangle in Ω , then there exists $S \in \mathcal{S}$ where

$$a, b, c \in \bigcup_{x \in \partial S} F_\Omega(x).$$

Remark 1.20 In general, properties (4), (7) and (8) in [Theorem 1.8](#) are not true in the naive convex cocompact case. But, properties (4) and (6) in [Theorem 1.19](#) can be seen as their coarse analogues.

1.2 Motivation from the theory of CAT(0) spaces

The main results of this paper are inspired by previous work of Hruska and Kleiner [\[20\]](#) in the CAT(0) setting. They introduced two notions of isolated flats for CAT(0) spaces and then related these conditions to relative hyperbolicity. In this subsection we recall their definitions and results.

Definition 1.21 (Hruska and Kleiner [\[20\]](#)) Suppose X is a CAT(0)-space and Γ acts geometrically on X (ie the action is properly discontinuous, cocompact and isometric).

(1) (X, Γ) has *isolated flats in the first sense* if there exists a set \mathcal{F} of flats of X such that \mathcal{F} is Γ -invariant, each flat in X is contained in a uniformly bounded tubular neighborhood of some flat in \mathcal{F} , and \mathcal{F} is closed and discrete in the local Hausdorff topology.

(2) (X, Γ) has *isolated flats in the second sense* if there exists a set \mathcal{F} of flats of X such that \mathcal{F} is Γ -invariant, each flat in X is contained in a uniformly bounded tubular neighborhood of some flat in \mathcal{F} , and, for any $r > 0$, there exists $D(r) > 0$ such that, for any two distinct flats $F_1, F_2 \in \mathcal{F}$,

$$\text{diam}_X(\mathcal{N}_X(F_1; r) \cap \mathcal{N}_X(F_2; r)) < D(r).$$

Clearly, if (X, Γ) has isolated flats in the second sense then it also has isolated flats in the first sense. Hruska and Kleiner [\[20\]](#) proved the following:

Theorem 1.22 (Hruska and Kleiner [20]) *Suppose X is a CAT(0)–space and Γ acts geometrically on X . The following are equivalent:*

- (1) (X, Γ) has isolated flats in the first sense.
- (2) (X, Γ) has isolated flats in the second sense.
- (3) X is a relatively hyperbolic space with respect to a family of flats.
- (4) Γ is a relatively hyperbolic group with respect to a collection of virtually abelian subgroups of rank at least two.

Hruska and Kleiner’s work motivated the results in this paper, but the methods of proof are very different.

1.3 Outline of paper

Sections 2–4 are mostly expository. In Section 2, we describe some examples. In Section 3, we set our basic notation and definitions. In Section 4, we recall the definition of relatively hyperbolic metric spaces and some of their basic properties.

The rest of the paper is divided into three parts. In the first part, Sections 5–7, we study properly embedded simplices in general properly convex domains. In the second part of the paper, Sections 8–14, we consider the naive convex cocompact case and prove Theorems 1.13, 1.17, 1.18 and 1.19.

An experienced reader would be able to follow the proof in the naive convex cocompact case by only reading Sections 6 and 8, the statement of Theorem 9.1 then Sections 10–14. In the final part of the paper, Sections 15 and 16, we consider the convex cocompact case and prove Theorems 1.7 and 1.8. In Section 16, we explain how to deduce the convex cocompact case from the naive convex cocompact case. Section 15 proves parts (7) and (8) of Theorem 1.8, which is a refinement of Theorem 1.19.

We now describe some of the proofs in the second part of the paper in the order they are presented.

1.3.1 Outline of the proof of Theorem 1.17 (See Section 10.) When $(\Omega, \mathcal{C}, \Lambda)$ has coarsely isolated simplices, we use the following algorithm to construct $\mathcal{S}_{\text{core}}$, a canonical strongly isolated, coarsely complete and Λ –invariant family of maximal properly embedded simplices of dimension at least two.

First, let \mathcal{S}_{max} denote the family of *all* maximal properly embedded simplices in \mathcal{C} of dimension at least two. In the naive convex cocompact case, this family can have several undesirable properties:

- (a) A maximal simplex could be contained in a tubular neighborhood of a properly embedded simplex with strictly larger dimension.
- (b) \mathcal{S}_{\max} could contain families of parallel maximal simplices (see [Definition 3.17](#)).

In [Section 2.4](#) (respectively [Section 2.3](#)) we construct a simple example where the first (respectively second) problem occurs.

To deal with the first problem, we consider $\widehat{\mathcal{S}}_{\max} \subset \mathcal{S}_{\max}$, the family of all maximal properly embedded simplices that are not contained in a tubular neighborhood of a properly embedded simplex with strictly larger dimension, and show that this subfamily is still coarsely complete.

To deal with the second problem, we select from each family of parallel simplices a canonical “core” simplex. This is accomplished by studying the open boundary faces $F_{\Omega}(x)$ of points $x \in \partial\Omega$.

For every $S \in \widehat{\mathcal{S}}_{\max}$ and vertex v of S , we show that $F_{\Omega}(v) \cap \bar{C}$ is a compact subset of $F_{\Omega}(v)$. Then we exploit the fact that every compact set in a properly convex domain has a well defined “center of mass” (see [Proposition 3.8](#)). Using this, for each simplex $S \in \widehat{\mathcal{S}}_{\max}$ we construct a new simplex $\Phi(S)$ as follows: Let v_1, \dots, v_p be the vertices of S . Then, for $1 \leq j \leq p$, define w_j to be the center of mass of $F_{\Omega}(v_j) \cap \bar{C}$ in $F_{\Omega}(v_j)$. Next define $\Phi(S)$ to be the convex hull of w_1, \dots, w_p in Ω .

Then $\Phi(S)$ is a properly embedded simplex parallel to S (see [Lemma 3.18](#)). Moreover, if $S_1, S_2 \in \widehat{\mathcal{S}}_{\max}$ are parallel, then $\Phi(S_1) = \Phi(S_2)$. Finally, we define

$$\mathcal{S}_{\text{core}} := \{\Phi(S) : S \in \widehat{\mathcal{S}}_{\max}\}.$$

Showing that this procedure actually produces a strongly isolated, coarsely complete and invariant family requires results from [Sections 8 and 9](#).

In [Section 8](#), we show that any isolated and invariant family of maximal properly embedded simplices satisfies properties (1) and (2) in [Theorem 1.19](#).

Then, in [Section 9](#), we show that, if \mathcal{S}_0 is an isolated, coarsely complete and invariant family of maximal properly embedded simplices of dimension at least two, then there exists a subfamily $\mathcal{S} \subset \mathcal{S}_0$ which satisfies property (4) in [Theorem 1.19](#) while still being isolated, coarsely complete and invariant. In the proof we first construct an explicit subfamily and then argue by contradiction that it must satisfy property (4). The main idea is to use the structure theorem from [Section 7](#) and the action of Λ to construct lots of properly embedded simplices. Then we use these simplices to obtain a contradiction.

This result is a key step in showing that the map Φ is well defined and that $\mathcal{S}_{\text{core}}$ is strongly isolated.

1.3.2 Outline of the proof of Theorem 1.19 (See Section 12.) Properties (1) and (2) are established in Section 8. Properties (3) and (5) are straightforward consequences of the strong isolation property. Property (4) follows from the results in Section 9.

We establish property (6) in Section 11 by combining a Benzécri [6] recentering argument with the strong isolation property. With the notation in Theorem 1.19, let $V := \text{Span}\{a, b, c\}$. By a recentering argument, for any $r > 0$, there exists a neighborhood \mathcal{O} of b such that, if $x \in \mathcal{O} \cap \mathbb{P}(V) \cap \mathcal{C}$, then there exists a simplex $S_x \in \mathcal{S}$ of dimension at least two with

$$B_{\Omega}(x; r) \subset \mathcal{N}_{\Omega}(S_x; D + 1),$$

where D is the constant from the coarsely complete condition. By picking $r > 0$ sufficiently large and using the fact the family \mathcal{S} is strongly isolated, we then show that S_x is independent of x and hence

$$\mathcal{O} \cap \mathbb{P}(V) \cap \mathcal{C} \subset \mathcal{N}_{\Omega}(S; D + 1)$$

for some $S \in \mathcal{S}$. Then it is easy to show that $a, b, c \in \bigcup_{x \in \partial \mathcal{S}} F_{\Omega}(x)$.

This use of the strong isolation property is similar to the proofs of Lemma 3.3.2 and Proposition 3.2.5 in [20].

1.3.3 Outline of the proof of Theorem 1.18 (See Section 13.) Our proof uses Theorem 1.19 and a characterization of relative hyperbolicity due to Sisto [31], which is stated in Theorem 4.15. This characterization involves the existence of a system of projection maps onto the simplices in \mathcal{S} with certain nice metric properties and a technical condition concerning thinness of certain geodesic triangles whose edges infrequently intersect neighborhoods of simplices in \mathcal{S} .

In Section 6, we use supporting hyperplanes to construct natural linear projections from a properly convex domain onto any properly embedded simplex. In the setup of Theorem 1.18, these linear projections end up being coarsely equivalent to the closest-point projection onto simplices of dimension at least two in the Hilbert metric (see Definition 13.5 and Proposition 13.7). The following property of these linear projections plays a key role: if the linear projections of two points onto a simplex are far apart, then the geodesic between those two points spends a significant amount of time in a tubular neighborhood of S (see Proposition 13.11 and Corollary 13.12).

Many of the proofs in [Section 13](#) proceed by contradiction and involve constructing a half triangle in $\partial_1 \mathcal{C}$ using an argument similar to the proof of [Proposition 2.5](#) in [\[3\]](#). Then property (6) in [Theorem 1.19](#) is used.

1.3.4 Outline of the proof of [Theorem 1.13](#) (See [Section 14](#).) The $(1) \implies (3)$ direction is a consequence of [Theorems 1.17](#) and [1.18](#). We show that $(2) \implies (1)$ using the general theory of relatively hyperbolic metric spaces. And we establish $(3) \implies (2)$ by using the following real projective analogue of the flat torus theorem due to Gromoll and Wolf [\[17\]](#) and Lawson and Yau [\[26\]](#):

Theorem 1.23 (Islam and Zimmer [\[21\]](#)) *Suppose that $(\Omega, \mathcal{C}, \Lambda)$ is a naive convex cocompact triple. If $A \leq \Lambda$ is a maximal abelian subgroup of Λ , then there exists a properly embedded simplex $S \subset \mathcal{C}$ such that*

- (1) S is A -invariant,
- (2) A fixes each vertex of S , and
- (3) A acts cocompactly on S .

Moreover, A contains a finite-index subgroup isomorphic to $\mathbb{Z}^{\dim(S)}$.

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2 Examples

2.1 Divisible examples

A properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is called *divisible* if there exists a discrete group $\Lambda \leq \text{Aut}(\Omega)$ which acts cocompactly on Ω . Clearly, in this case, Λ is also a convex cocompact subgroup. Divisible domains have been extensively studied and in this subsection we will recall some examples; for more details see the surveys [\[5; 30; 27\]](#).

An open convex cone $C \subset \mathbb{R}^d$ is *reducible* if there exist a nontrivial decomposition $\mathbb{R}^d = V_1 \oplus V_2$ and convex cones $C_1 \subset V_1$ and $C_2 \subset V_2$ such that $C = C_1 + C_2$. Otherwise C is said to be *irreducible*. The preimage in \mathbb{R}^d of a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is the union of a cone and its negative; when this cone is reducible (respectively irreducible), we say that Ω is *reducible* (respectively *irreducible*).

The Klein–Beltrami model of real hyperbolic d –space is the fundamental example of a convex divisible domain. In particular, if $\mathcal{B} \subset \mathbb{P}(\mathbb{R}^d)$ is the interior of the convex hull of an ellipsoid in some affine chart, then $(\mathcal{B}, H_{\mathcal{B}})$ is isometric to real hyperbolic $(d-1)$ –space $\mathbb{H}_{\mathbb{R}}^{d-1}$ and $\text{Aut}(\mathcal{B})$ coincides with $\text{Isom}(\mathbb{H}_{\mathbb{R}}^{d-1})$. Further, \mathcal{B} is a divisible convex domain because $\text{Aut}(\mathcal{B})$, being a simple Lie group, contains cocompact lattices.

There are many other examples of divisible convex domains; for instance, for every $d \geq 5$, Kapovich [22] has constructed divisible convex domains $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ such that $\text{Aut}(\Omega)$ is discrete, Gromov hyperbolic and not quasi-isometric to any symmetric space.

When $d = 3$, results of Benzécri [6] imply that every irreducible divisible convex domain has word hyperbolic dividing group (see [4, Section 2] for details). In $d = 4$, Benoist established the following dichotomy:

Theorem 2.1 (Benoist [4]) *If $\Omega \subset \mathbb{P}(\mathbb{R}^4)$ is an irreducible properly convex domain and $\Lambda \leq \text{Aut}(\Omega)$ is a discrete group acting cocompactly on Ω , then either*

- (1) Λ is word hyperbolic, or
- (2) Λ is a relatively hyperbolic group with respect to a nonempty collection of virtually abelian subgroups of rank two.

Benoist [4] and Ballas, Danciger and Lee [1] have constructed examples of the second case in [Theorem 2.1](#).

The case when $d > 4$ is fairly mysterious. When $d = 5, 6$ or 7 , Choi, Lee and Marquis [10] have constructed examples where Λ is a relatively hyperbolic group with respect to a collection of virtually abelian subgroups of rank at least two. They also ask whether Benoist’s result is true in any dimension.

Question 2.2 (Choi, Lee and Marquis [10, Remark 1.11]) *Are groups dividing nonsymmetric irreducible properly convex domains always relatively hyperbolic with respect to a (possibly empty) collection of virtually abelian subgroups of rank at least two?*

In the context of the above question, we should mention a recent result of Bobb [7] who proved that, if $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is divisible, then the family of properly embedded simplices in Ω of dimension $(d - 2)$ (ie codimension 1) is closed and discrete in the local Hausdorff topology induced by the Hilbert metric. Thus, if Ω only contains simplices of dimension $(d - 2)$, then [Theorem 1.7](#) implies that Λ is relatively hyperbolic with respect to a collection of virtually abelian subgroups of rank $(d - 2)$.

2.2 Convex cocompact examples

In this subsection we recall a class of examples constructed by Danciger, Guéritaud and Kassel. For details and other examples see [12, Section 12].

Proposition 2.3 (Danciger, Guéritaud and Kassel [12, Section 12.2.2]) *For any $d \geq 4$ there exists a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ with a convex cocompact subgroup $\Lambda \leq \text{Aut}(\Omega)$ such that*

- (1) Λ acts irreducibly on \mathbb{R}^d , and
- (2) Λ is a relatively hyperbolic group with respect to a nonempty collection of virtually abelian subgroups of rank two.

2.3 Naive convex cocompact examples, I

In this subsection we construct examples of

- (a) a naive convex cocompact triple which is not convex cocompact;
- (b) a naive convex cocompact triple where the group is relatively hyperbolic but the family of all maximal properly embedded simplices of dimension at least two is not discrete in the local Hausdorff topology; and
- (c) a naive convex cocompact triple $(\Omega_\star, \mathcal{C}_\star^{(R)}, \Lambda_\star)$ and a family \mathcal{S} of maximal properly embedded simplices of dimension at least two which is isolated, coarsely complete and Λ_\star -invariant but $(\mathcal{C}_\star^{(R)}, H_{\Omega_\star})$ is not relatively hyperbolic with respect to \mathcal{S} .

For the rest of the subsection, we will freely use the notation introduced in [Section 3](#) and make the following assumptions:

Assumptions $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $\Lambda \leq \text{Aut}(\Omega)$ is a discrete group which acts cocompactly on Ω .

Let $\pi : \mathbb{R}^d \rightarrow \mathbb{P}(\mathbb{R}^d)$ be the natural projection. Then $\pi^{-1}(\Omega) = C \cup -C$, where $C \subset \mathbb{R}^d$ is some properly convex cone. Then define

$$\begin{aligned} \Omega_\star &:= \{[(v, w)] : v, w \in C\} \subset \mathbb{P}(\mathbb{R}^{2d}), \\ \mathcal{C}_\star &:= \{[(v, v)] : v \in C\} \subset \mathbb{P}(\mathbb{R}^{2d}), \\ \Lambda_\star &:= \{[g \oplus g] : g \in \text{GL}_d(\mathbb{R}), [g] \in \Lambda\} \subset \text{PGL}_{2d}(\mathbb{R}). \end{aligned}$$

Then, by construction, $(\Omega_\star, \mathcal{C}_\star, \Lambda_\star)$ is a naive convex cocompact triple.

Observation 2.4 $\mathcal{C}_{\Omega_\star}(\Lambda_\star) = \Omega_\star$. In particular, $\Lambda_\star \leq \text{Aut}(\Omega_\star)$ is not a convex cocompact subgroup.

Proof Since Λ acts cocompactly on Ω , it is easy to show that $\mathcal{L}_\Omega(\Lambda) = \partial\Omega$ (see for instance the proof of Lemma 2.2 in [2]). By convexity, to prove that $\mathcal{C}_{\Omega_\star}(\Lambda_\star) = \Omega_\star$ it is enough to show that

$$\overline{\mathcal{C}_{\Omega_\star}(\Lambda_\star)} \supset \{[(v, 0)] : v \in \partial C\} \cup \{[(0, v)] : v \in \partial C\}.$$

Fix $v \in \partial C$. Then $[v] \in \partial\Omega$ and so there exist $p \in \Omega$ and a sequence $g_n \in \Lambda$ such that $\lim_{n \rightarrow \infty} g_n(p) = [v]$. Let $\bar{p} \in C$ be a lift of p and $\bar{g}_n \in \text{GL}_d(\mathbb{R})$ be a lift of g_n . Then, for any $t > 0$,

$$[(tv, v)] = \lim_{n \rightarrow \infty} [\bar{g}_n \oplus \bar{g}_n] \cdot [(t\bar{p}, \bar{p})] \in \mathcal{L}_{\Omega_\star}(\Lambda_\star).$$

Taking limits as $t \rightarrow 0$ and $t \rightarrow \infty$, respectively,

$$\{[(v, 0)], [(0, v)]\} \subset \overline{\mathcal{L}_{\Omega_\star}(\Lambda_\star)} \subset \overline{\mathcal{C}_{\Omega_\star}(\Lambda_\star)}.$$

Since $v \in \partial C$ was arbitrary, the proof is complete. □

We can thicken \mathcal{C}_\star to obtain a one-parameter family of naive convex cocompact triples: for $R \geq 0$ define

$$\mathcal{C}_\star^{(R)} := \{y \in \Omega_\star : H_{\Omega_\star}(y, \mathcal{C}_\star) \leq R\}.$$

Observation 2.5 For any $R \geq 0$, $(\Omega_\star, \mathcal{C}_\star^{(R)}, \Lambda_\star)$ is a naive convex cocompact triple.

Proof Note that $\mathcal{C}_\star^{(R)}$ is the closed R -neighborhood of the convex set \mathcal{C}_\star in the Hilbert metric H_{Ω_\star} . So $\mathcal{C}_\star^{(R)}$ is a closed convex set containing \mathcal{C}_\star ; see for instance [11, Corollary 1.10]. The observation then follows since Λ_\star acts cocompactly on \mathcal{C}_\star . □

For $x \in \overline{\Omega}$, define $\bar{x} \in \overline{C}$ to be the unique lift of x with $\|\bar{x}\|_2 = 1$. Then define $x_\star := [(\bar{x}, \bar{x})] \in \overline{\Omega}_\star$.

By definition we have the following description of the faces of Ω_\star :

Observation 2.6 *If $x \in \overline{\Omega}$, then*

$$F_{\Omega_\star}(x_\star) = \{[(v, w)] : v, w \in \overline{C} \text{ and } [v], [w] \in F_\Omega(x)\}.$$

In particular, if x is an extreme point of Ω , then

$$F_{\Omega_\star}(x_\star) = \{[(s\bar{x}, \bar{x})] : s \in (0, +\infty)\} = \{[(\bar{x}, t\bar{x})] : t \in (0, +\infty)\}.$$

Observations 2.5 and 2.6 and the definition of the Hilbert metric imply the following:

Observation 2.7 *If $x \in \partial\Omega$ is an extreme point and $R > 0$, then*

$$\overline{\mathcal{C}_\star^{(R)}} \cap F_{\Omega_\star}(x_\star)$$

is a compact interval containing x_\star with nonempty interior.

We now specialize our assumptions.

Additional assumption Λ is relatively hyperbolic group with respect to a nonempty collection of virtually abelian subgroups of rank at least two.

Let \mathcal{S}_{\max} denote the set of all maximal properly embedded simplices in Ω of dimension at least two. Then, by [Theorem 1.7](#),

- (1) \mathcal{S}_{\max} is closed and discrete in local Hausdorff topology induced by H_Ω ,
- (2) (Ω, H_Ω) is relatively hyperbolic with respect to \mathcal{S}_{\max} , and
- (3) if $S \in \mathcal{S}_{\max}$, then each vertex of S is an extreme point of Ω .

For each $S \in \mathcal{S}_{\max}$, define

$$S_\star := \{x_\star : x \in S\} \subset \mathcal{C}_\star.$$

Then S_\star is a maximal properly embedded simplex in \mathcal{C}_\star .

Observation 2.8 *For any $R > 0$, the metric space $(\mathcal{C}_\star^{(R)}, H_{\Omega_\star})$ is relatively hyperbolic with respect to $\mathcal{S}_\star = \{S_\star : S \in \mathcal{S}_{\max}\}$.*

Proof sketch Consider the map $F : \Omega \rightarrow \Omega_\star$ defined by $F(x) = x_\star$. Then F induces a quasi-isometry $(\Omega, H_\Omega) \rightarrow (\mathcal{C}_\star^{(R)}, H_{\Omega_\star})$ and so the observation follows from the general theory of relatively hyperbolic spaces (see [Theorem 4.8](#)). \square

Next we show that the family of all maximal properly embedded simplices in $\mathcal{C}_\star^{(R)}$ of dimension at least two is not discrete in the local Hausdorff topology. By construction, if $S \subset \Omega$ is a properly embedded simplex of dimension at least two (ie $S \in \mathcal{S}_{\max}$), then S_\star is a maximal properly embedded simplex in Ω_\star of the same dimension. Let v_1, \dots, v_p be the vertices of S . Then, by [Lemma 3.18](#),

$$\mathbb{P}(\text{Span}\{w_1, \dots, w_p\}) \cap \Omega_\star$$

is a maximal properly embedded simplex in Ω_\star for any choice of

$$w_j \in \overline{\mathcal{C}_\star^{(R)}} \cap F_{\Omega_\star}(v_{j,\star}), \quad j = 1, \dots, p.$$

This construction combined with [Observation 2.7](#) yields the following:

Observation 2.9 *For any $R > 0$, the family of maximal properly embedded simplices in $\mathcal{C}_\star^{(R)}$ of dimension at least two is not discrete. In particular, $\mathcal{C}_\star^{(R)}$ contains parallel properly embedded simplices (see [Definition 3.17](#)).*

We also can construct the following:

Observation 2.10 *For any $R > 0$, there exists a family $\mathcal{S}_\diamond^{(R)}$ of maximal properly embedded simplices in $\mathcal{C}_\star^{(R)}$ of dimension at least two where*

- (1) $\mathcal{S}_\diamond^{(R)}$ is isolated, coarsely complete and Λ_\star -invariant;
- (2) $(\mathcal{C}_\star^{(R)}, H_{\Omega_\star})$ is not a relatively hyperbolic space with respect to $\mathcal{S}_\diamond^{(R)}$.

Proof For each extreme point $x \in \partial\Omega$, define $x^+, x^- \in F_{\Omega_\star}(x_\star)$ to be the points such that

$$[x^+, x^-] = \overline{\mathcal{C}_\star^{(R)}} \cap F_{\Omega_\star}(x_\star).$$

Notice that $x^+ \neq x^-$ by [Observation 2.7](#).

Given $S \in \mathcal{S}_{\max}$, fix a labeling v_1, \dots, v_p of its vertices. Then, for $\sigma = (\sigma_1, \dots, \sigma_p) \in \{+, -\}^p$, define

$$S_\sigma := \mathbb{P}(\text{Span}\{v_1^{\sigma_1}, \dots, v_p^{\sigma_p}\}) \cap \Omega_\star.$$

Then, by [Lemma 3.18](#), S_σ is a properly embedded simplex in Ω_\star .

By construction, the set

$$\mathcal{S}_\diamond^{(R)} := \{S_\sigma : S \in \mathcal{S}_{\max}, \sigma \in \{+, -\}^{\dim S + 1}\}$$

is isolated, coarsely complete and Λ_\star -invariant. However, if $S \in \mathcal{S}_{\max}$ and $\sigma, \tau \in \{+, -\}^{\dim S + 1}$ are distinct, then $S_\sigma \neq S_\tau$ and

$$H_\Omega^{\text{Haus}}(S_\sigma, S_\tau) < +\infty$$

by Lemma 3.18. So, by Theorem 4.6, $(\mathcal{C}_\star^{(R)}, H_{\Omega_\star})$ is not a relatively hyperbolic space with respect to $\mathcal{S}_\diamond^{(R)}$. □

2.4 Naive convex cocompact examples, II

In this section we construct a naive convex cocompact triple $(\Omega, \mathcal{C}, \Lambda)$ where \mathcal{C} contains a maximal properly embedded simplex which is contained in a bounded neighborhood of a properly embedded simplex with strictly larger dimension.

Let $C := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 < x_3^2\}$ and $\mathcal{B} := \{[v] : v \in C\}$. Then $(\mathcal{B}, H_{\mathcal{B}})$ is the Klein–Beltrami model of real hyperbolic 2-space. In particular, if we fix $x, y \in \partial \mathcal{B}$ distinct, there exists $h \in \text{Aut}(\mathcal{B})$ which translates along the line segment $(x, y) \subset \mathcal{B}$. Let $\bar{x}, \bar{y} \in \partial C$ be lifts of x and y , respectively, and let $\bar{h} \in \text{Aut}(C)$ be a lift of h .

Next define

$$\Omega := \{[(v, w)] : v, w \in C\} \subset \mathbb{P}(\mathbb{R}^6)$$

and let

$$x_1 := [(\bar{x}, 0)], \quad x_2 := [(0, \bar{x})], \quad y_1 := [(\bar{y}, 0)], \quad y_2 := [(0, \bar{y})].$$

Then x_1, x_2, y_1 and y_2 are the vertices of a properly embedded simplex $S \subset \Omega$. Further, if

$$\Lambda := \langle [\bar{h} \oplus \text{id}], [\text{id} \oplus \bar{h}], [(2 \text{id}) \oplus (\frac{1}{2} \text{id})] \rangle \leq \text{Aut}(\Omega),$$

then (Ω, S, Λ) is a naive convex cocompact triple.

Fix $R > 0$. Then $\mathcal{C} := \{p \in \Omega : H_\Omega(p, S) \leq R\}$ is a closed convex subset of Ω ; see for instance [11, Corollary 1.10]. So $(\Omega, \mathcal{C}, \Lambda)$ is also a naive convex cocompact triple. Now

$$F := \{[(v, 0)] : v \in C\}$$

is an open boundary face of Ω and $\bar{C} \cap F$ has nonempty interior in F . So there exists

$$z \in \bar{C} \cap F \setminus (x_1, y_1).$$

Then z, x_2 and y_2 are the vertices of a properly embedded simplex S' in \mathcal{C} . Further, S' is maximal and S' is contained in a bounded neighborhood of S .

3 Some notation and definitions

In this section we set some notation that we will use for the rest of the paper.

3.1 Basic notation in projective geometry

If $V \subset \mathbb{R}^d$ is a nonzero linear subspace, we will let $\mathbb{P}(V) \subset \mathbb{P}(\mathbb{R}^d)$ denote its projectivization. In most other cases, we will use $[o]$ to denote the projective equivalence class of an object o , for instance:

- (1) If $v \in \mathbb{R}^d \setminus \{0\}$, then $[v]$ denotes the image of v in $\mathbb{P}(\mathbb{R}^d)$,
- (2) If $\phi \in \text{GL}_d(\mathbb{R})$, then $[\phi]$ denotes the image of ϕ in $\text{PGL}_d(\mathbb{R})$.
- (3) If $T \in \text{End}(\mathbb{R}^d) \setminus \{0\}$, then $[T]$ denotes the image of T in $\mathbb{P}(\text{End}(\mathbb{R}^d))$.

We also identify $\mathbb{P}(\mathbb{R}^d) = \text{Gr}_1(\mathbb{R}^d)$; so, for instance, if $x \in \mathbb{P}(\mathbb{R}^d)$ and $V \subset \mathbb{R}^d$ is a linear subspace, then $x \in \mathbb{P}(V)$ if and only if $x \subset V$.

Given a subset X of \mathbb{R}^d or $\mathbb{P}(\mathbb{R}^d)$, we will let $\text{Span } X \subset \mathbb{R}^d$ denote the smallest linear subspace containing X .

Next, for a subset $X \subset \mathbb{P}(\mathbb{R}^d)$ we define the *automorphism group of X* to be

$$\text{Aut}(X) := \{g \in \text{PGL}_d(\mathbb{R}) : gX = X\}.$$

Further, given a group $G \leq \text{PGL}_d(\mathbb{R})$ the *stabilizer of X in G* is

$$\text{Stab}_G(X) := \{g \in G : gX = X\} = G \cap \text{Aut}(X).$$

3.2 Convexity

- Definition 3.1**
- (1) A subset $C \subset \mathbb{P}(\mathbb{R}^d)$ is *convex* if there exists an affine chart \mathbb{A} of $\mathbb{P}(\mathbb{R}^d)$ where $C \subset \mathbb{A}$ is a convex subset.
 - (2) A subset $C \subset \mathbb{P}(\mathbb{R}^d)$ is *properly convex* if there exists an affine chart \mathbb{A} of $\mathbb{P}(\mathbb{R}^d)$ where $C \subset \mathbb{A}$ is a bounded convex subset.
 - (3) An open subset of $\mathbb{P}(\mathbb{R}^d)$ that is properly convex is called a *properly convex domain* in $\mathbb{P}(\mathbb{R}^d)$.

Notice that, if $C \subset \mathbb{P}(\mathbb{R}^d)$ is convex, then C is a convex subset of every affine chart that contains it.

A *line segment* in $\mathbb{P}(\mathbb{R}^d)$ is a connected subset of a projective line. Given two points $x, y \in \mathbb{P}(\mathbb{R}^d)$, there is no canonical line segment with endpoints x and y , but we will use the following convention: if $C \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex set and $x, y \in \overline{C}$,

then when the context is clear we will let $[x, y]$ denote the closed line segment joining x to y which is contained in \bar{C} . In this case, we will also let $(x, y) = [x, y] \setminus \{x, y\}$, $[x, y) = [x, y] \setminus \{y\}$ and $(x, y] = [x, y] \setminus \{x\}$.

Along similar lines, given a properly convex subset $C \subset \mathbb{P}(\mathbb{R}^d)$ and a subset $X \subset \bar{C}$, we will let

$$\text{ConvHull}_C(X)$$

denote the smallest convex subset of \bar{C} which contains X . For instance, with our notation, $[x, y] = \text{ConvHull}_C(\{x, y\})$ when $x, y \in \bar{C}$.

We also make the following topological definitions:

Definition 3.2 Suppose $C \subset \mathbb{P}(\mathbb{R}^d)$ is a convex set. The *relative interior* of C , denoted by $\text{rel-int}(C)$, is the interior of C in $\mathbb{P}(\text{Span } C)$. In the case that $C = \text{rel-int}(C)$, we will say that C is *open in its span*. The *boundary* of C is $\partial C := \bar{C} \setminus \text{rel-int}(C)$, the *ideal boundary* of C is

$$\partial_i C := \partial C \setminus C,$$

and the *nonideal boundary* of C is

$$\partial_n C := \partial C \cap C.$$

Finally, we define $\dim C$ to be the dimension of $\text{rel-int}(C)$ (notice that $\text{rel-int}(C)$ is homeomorphic to $\mathbb{R}^{\dim C}$).

Recall that a subset $A \subset B \subset \mathbb{P}(\mathbb{R}^d)$ is properly embedded if the inclusion map $A \hookrightarrow B$ is proper. With the notation in [Definition 3.2](#), we have the following characterization of properly embedded subsets:

Observation 3.3 Suppose $C \subset \mathbb{P}(\mathbb{R}^d)$ is a convex set. A convex subset $S \subset C$ is properly embedded if and only if $\partial_i S \subset \partial_i C$.

We also recall the definition of supporting hyperplanes.

Definition 3.4 Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain.

- (1) A projective hyperplane $H \subset \mathbb{P}(\mathbb{R}^d)$ is a *supporting hyperplane* of Ω if $H \cap \Omega = \emptyset$ and $H \cap \partial\Omega \neq \emptyset$.
- (2) A boundary point $x \in \partial\Omega$ is a C^1 -*smooth point* of $\partial\Omega$ if there exists a unique supporting hyperplane containing x .

3.3 The Hilbert metric

For distinct points $x, y \in \mathbb{P}(\mathbb{R}^d)$, let \overline{xy} be the projective line containing them. Suppose $C \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex set which is open in its span. If $x, y \in C$ are distinct, let a and b be the two points in $\overline{xy} \cap \partial C$ ordered a, x, y, b along \overline{xy} . Then define the Hilbert distance between x and y to be

$$H_C(x, y) = \frac{1}{2} \log[a, x, y, b],$$

where

$$[a, x, y, b] = \frac{|x - b||y - a|}{|x - a||y - b|}$$

is the cross ratio. We also define $H_C(x, x) = 0$ for all $x \in C$. Using the invariance of the cross ratio under projective maps and the convexity of C , it is possible to establish the following (see for instance [9, Section 28]):

Proposition 3.5 *Suppose $C \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex set which is open in its span. Then H_C is a complete $\text{Aut}(C)$ -invariant metric on C which generates the standard topology on C . Moreover, if $p, q \in C$, then there exists a geodesic joining p and q whose image is the line segment $[p, q]$.*

As a corollary to Proposition 3.5, we observe the following:

Corollary 3.6 *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain. Then $\text{Aut}(\Omega)$ acts properly on Ω .*

We will frequently use the following notation:

Definition 3.7 *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain.*

- (1) For $x \in \Omega$ and $r > 0$, define

$$B_\Omega(x; r) = \{y \in \Omega : H_\Omega(x, y) < r\}.$$

- (2) For a subset $A \subset \Omega$ and $r > 0$, define

$$\mathcal{N}_\Omega(A; r) = \bigcup_{x \in A} B_\Omega(x; r).$$

- (3) For a subset $A \subset \Omega$, define

$$\text{diam}_\Omega(A) = \sup\{H_\Omega(x, y) : x, y \in A\}.$$

3.4 The center of mass of a compact subset

It is possible to define a “center of mass” for a compact set in a properly convex domain. Let \mathcal{K}_d denote the set of all pairs (Ω, K) where $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $K \subset \Omega$ is a compact subset.

Proposition 3.8 *There exists a function $\mathcal{K}_d \rightarrow \mathbb{P}(\mathbb{R}^d)$,*

$$(\Omega, K) \mapsto \text{CoM}_\Omega(K),$$

such that

- (1) $\text{CoM}_\Omega(K) \in \text{ConvHull}_\Omega(K)$,
- (2) $\text{CoM}_\Omega(K) = \text{CoM}_\Omega(\text{ConvHull}_\Omega(K))$, and
- (3) if $g \in \text{PGL}_d(\mathbb{R})$, then $g\text{CoM}_\Omega(K) = \text{CoM}_{g\Omega}(gK)$

for every $(\Omega, K) \in \mathcal{K}_d$.

There are several constructions of such a center of mass; see for instance [27, Lemma 4.2] or [21, Proposition 4.5]. The approach in [21] is based on an argument of Frankel [16, Section 12] in several complex variables.

3.5 The faces of a convex domain

In this section we define the faces of a convex set and then describe some of their properties.

Definition 3.9 Given a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ and $x \in \overline{\Omega}$, let $F_\Omega(x)$ denote the *open face* of x ; that is,

$$F_\Omega(x) = \{x\} \cup \{y \in \overline{\Omega} : \text{there is an open line segment in } \overline{\Omega} \text{ containing } x \text{ and } y\}.$$

Remark 3.10 Notice that $F_\Omega(x) = \Omega$ when $x \in \Omega$. Further, a properly convex set C that is open in its span is a properly convex domain in $\mathbb{P}(\text{Span } C)$. Thus, the above definition (and the results of this subsection) apply to any properly convex set open in its span.

We also introduce the following notation:

Definition 3.11 Given a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ and a subset $X \subset \overline{\Omega}$, define

$$F_\Omega(X) := \bigcup_{x \in X} F_\Omega(x).$$

The next observation is a simple consequence of convexity.

Observation 3.12 Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain.

- (1) $F_\Omega(x)$ is open in its span.
- (2) $y \in F_\Omega(x)$ if and only if $x \in F_\Omega(y)$ if and only if $F_\Omega(x) = F_\Omega(y)$.
- (3) If $y \in \partial F_\Omega(x)$, then $F_\Omega(y) \subset \partial F_\Omega(x)$.
- (4) If $x, y \in \overline{\Omega}$, $z \in (x, y)$, $p \in F_\Omega(x)$ and $q \in F_\Omega(y)$, then

$$(p, q) \subset F_\Omega(z).$$

In particular, $(p, q) \subset \Omega$ if and only if $(x, y) \subset \Omega$.

The next two results relate the faces to the Hilbert metric.

Proposition 3.13 Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, x_n is a sequence in Ω and $\lim_{n \rightarrow \infty} x_n = x \in \overline{\Omega}$. If y_n is another sequence in Ω , $\lim_{n \rightarrow \infty} y_n = y \in \overline{\Omega}$ and

$$\liminf_{n \rightarrow \infty} H_\Omega(x_n, y_n) < +\infty,$$

then $y \in F_\Omega(x)$ and

$$H_{F_\Omega(x)}(x, y) \leq \liminf_{n \rightarrow \infty} H_\Omega(x_n, y_n).$$

Proof This is a straightforward consequence of the definition of the Hilbert metric. \square

Proposition 3.14 Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain. Assume $p_1, p_2, q_1, q_2 \in \overline{\Omega}$, $F_\Omega(p_1) = F_\Omega(p_2)$ and $F_\Omega(q_1) = F_\Omega(q_2)$. If $(p_1, q_1) \cap \Omega \neq \emptyset$, then

$$H_\Omega^{\text{Haus}}((p_1, q_1), (p_2, q_2)) \leq \max\{H_{F_\Omega(p_1)}(p_1, p_2), H_{F_\Omega(q_1)}(q_1, q_2)\}.$$

Remark 3.15 Observation 3.12 implies that $(p_1, q_1), (p_2, q_2) \subset \Omega$.

Proof This is a straightforward consequence of the fact that a R -neighborhood of a closed convex set in the Hilbert metric is convex (see for instance [11, Corollary 1.10]). For details see [21, Proposition 5.3]. \square

The final result of this subsection relates the faces to the behavior of automorphisms.

Proposition 3.16 [21, Proposition 5.6] *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $p_0 \in \Omega$ and $g_n \in \text{Aut}(\Omega)$ is a sequence such that*

- (1) $\lim_{n \rightarrow \infty} g_n(p_0) = x \in \partial\Omega$,
- (2) $\lim_{n \rightarrow \infty} g_n^{-1}(p_0) = y \in \partial\Omega$, and
- (3) g_n converges in $\mathbb{P}(\text{End}(\mathbb{R}^d))$ to $T \in \mathbb{P}(\text{End}(\mathbb{R}^d))$.

Then $\text{image}(T) \subset \text{Span}\{F_\Omega(x)\}$, $\mathbb{P}(\ker T) \cap \Omega = \emptyset$ and $y \in \mathbb{P}(\ker T)$.

3.6 Parallel properly embedded simplices

Definition 3.17 *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain. Two properly embedded simplices $S_1, S_2 \subset \Omega$ are called *parallel* if $\dim S_1 = \dim S_2 \geq 1$ and there is a labeling v_1, \dots, v_p of the vertices of S_1 and a labeling w_1, \dots, w_p of the vertices of S_2 such that $F_\Omega(v_k) = F_\Omega(w_k)$ for all $1 \leq k \leq p$.*

The following lemma allows us to “wiggle” the vertices of a properly embedded simplex and obtain a new parallel properly embedded simplex.

Lemma 3.18 *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $S \subset \Omega$ is a properly embedded simplex with vertices v_1, \dots, v_p . If $w_j \in F_\Omega(v_j)$ for $1 \leq j \leq p$, then*

$$S' := \Omega \cap \mathbb{P}(\text{Span}\{w_1, \dots, w_p\})$$

is a properly embedded simplex with vertices w_1, \dots, w_p . Moreover,

$$H_\Omega^{\text{Haus}}(S, S') \leq \max_{1 \leq j \leq p} H_{F_\Omega(v_j)}(v_j, w_j).$$

Proof For $1 \leq q \leq p$ and $1 \leq i_1 < i_2 < \dots < i_q \leq p$, define

$$S(i_1, \dots, i_q) := \text{rel-int}(\text{ConvHull}_\Omega\{v_{i_1}, \dots, v_{i_q}\}),$$

$$S'(i_1, \dots, i_q) := \text{rel-int}(\text{ConvHull}_\Omega\{w_{i_1}, \dots, w_{i_q}\}).$$

We claim that, for each $1 \leq q \leq p$ and $1 \leq i_1 < i_2 < \dots < i_q \leq p$, there exists a face $F(i_1, \dots, i_q)$ of Ω such that

$$S(i_1, \dots, i_q) \cup S'(i_1, \dots, i_q) \subset F(i_1, \dots, i_q).$$

When $q = 1$, this is by hypothesis. Then the claim follows from induction on q and **Observation 3.12**(4).

Then $S'(i_1, \dots, i_q) \subset \partial\Omega$ if and only if $S(i_1, \dots, i_q) \subset \partial\Omega$. Hence, S' is a properly embedded simplex.

The “moreover” part follows from a similar induction argument and **Proposition 3.14**. \square

3.7 The Hausdorff distance and local Hausdorff topology

When (X, d) is a metric space, the *Hausdorff distance* between two subsets $A, B \subset X$ is defined by

$$d^{\text{Haus}}(A, B) = \max\left\{\sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b)\right\}.$$

When (X, d) is a complete metric space, d^{Haus} is a complete metric on the set of nonempty compact subsets of X .

The local Hausdorff topology is a natural topology on the set of closed sets in X . For a closed set C_0 , a basepoint $x_0 \in X$ and $r_0, \epsilon_0 > 0$, define the set $U(C_0, x_0, r_0, \epsilon_0)$ to consist of all closed subsets $C \subset X$ where

$$d^{\text{Haus}}(C_0 \cap B_X(x_0; r_0), C \cap B_X(x_0; r_0)) < \epsilon_0,$$

where

$$B_X(x_0; r_0) = \{x \in X : d(x_0, x) < r_0\}.$$

The *local Hausdorff topology* on the set of nonempty closed subsets of X is the topology generated by the sets $U(\cdot, \cdot, \cdot, \cdot)$.

3.8 The local Hausdorff topology on slices of a properly convex domain

Fix a distance $d_{\mathbb{P}}$ on $\mathbb{P}(\mathbb{R}^d)$ induced by a Riemannian metric. Then the following observation is a consequence of convexity and the definition of the Hilbert metric:

Observation 3.19 *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain. If $V_n \in \text{Gr}_p(\mathbb{R}^d)$ is a sequence of p -dimensional linear subspaces, $V_n \rightarrow V$ in $\text{Gr}_p(\mathbb{R}^d)$ and $\mathbb{P}(V) \cap \Omega \neq \emptyset$, then*

- (1) $\overline{\mathbb{P}(V_n) \cap \Omega}$ converges to $\overline{\mathbb{P}(V) \cap \Omega}$ in the Hausdorff topology induced by $d_{\mathbb{P}}$,
- (2) $\mathbb{P}(V_n) \cap \Omega$ converges to $\mathbb{P}(V) \cap \Omega$ in the local Hausdorff topology induced by H_{Ω} .

Proof To prove (1), notice that the set of nonempty compact subsets in $\mathbb{P}(\mathbb{R}^d)$ endowed with $d_{\mathbb{P}}^{\text{Haus}}$ is a compact metric space. Hence, to show that $\overline{\mathbb{P}(V_n) \cap \Omega}$ converges to $\overline{\mathbb{P}(V) \cap \Omega}$, it suffices to show that every convergent subsequence of $\overline{\mathbb{P}(V_n) \cap \Omega}$ converges to $\overline{\mathbb{P}(V) \cap \Omega}$. So suppose that $\overline{\mathbb{P}(V_{n_j}) \cap \Omega}$ converges to some compact set C . Since Ω is open, it is clear that $\overline{\mathbb{P}(V) \cap \Omega} \subset C$. Since every point in $\partial\Omega$ has a supporting hyperplane, it is clear that $C \subset \overline{\mathbb{P}(V) \cap \Omega}$. Thus, $\overline{\mathbb{P}(V_{n_j}) \cap \Omega}$ converges to $\overline{\mathbb{P}(V) \cap \Omega}$.

The proof of (2) is similar. □

This observation has the following consequence:

Observation 3.20 *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain. Then the set of properly embedded simplices in Ω of dimension at least two is closed in the local Hausdorff topology.*

Proof Suppose $S_n \subset \Omega$ is a sequence of properly embedded simplices of dimension at least two and S_n converges to S in the local Hausdorff topology. We claim that S is also a properly embedded simplex. By passing to a subsequence, we can suppose that $\dim S_n = p - 1 \geq 2$ for all n .

Let $V_n = \text{Span } S_n$, then $S_n = \mathbb{P}(V_n) \cap \Omega$. By passing to a subsequence, we can suppose that

$$V := \lim_{n \rightarrow \infty} V_n$$

exists in $\text{Gr}_p(\mathbb{R}^d)$. Then **Observation 3.19(2)** implies that $S = \mathbb{P}(V) \cap \Omega$. Next let $v_1^{(n)}, \dots, v_p^{(n)}$ be the vertices of S_n . Then

$$(1) \quad S_n = \text{rel-int}(\text{ConvHull}_\Omega\{v_1^{(n)}, \dots, v_p^{(n)}\}).$$

By passing to a subsequence, we can suppose that $v_1^{(n)}, \dots, v_p^{(n)}$ converge to v_1, \dots, v_p . Then **Observation 3.19(1)** and (1) imply that

$$S = \mathbb{P}(V) \cap \Omega = \text{rel-int}(\text{ConvHull}_\Omega\{v_1, \dots, v_p\}).$$

Since $\dim \mathbb{P}(V) \cap \Omega = p - 1$, the lines v_1, \dots, v_p must be linearly independent. Hence, S is a properly embedded simplex. □

As a consequence of **Observation 3.20** we will show the following:

Observation 3.21 *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and \mathcal{S} is a family of maximal properly embedded simplices of dimension at least two. If \mathcal{S} is strongly isolated, then it is also isolated.*

Proof Suppose $S_n \in \mathcal{S}$ is a sequence and S_n converges to S in the local Hausdorff topology induced by H_Ω . Then S is a properly embedded simplex by **Observation 3.20**. Since S is unbounded in (Ω, H_Ω) , for any $r > 0$,

$$\lim_{n \rightarrow \infty} \text{diam}_\Omega(\mathcal{N}_\Omega(S_n; r) \cap \mathcal{N}_\Omega(S_{n+1}; r)) = \infty.$$

Then, since \mathcal{S} is strongly isolated, there exists $N \geq 0$ such that $S_n = S_m$ for all $n, m \geq N$. Then $S = S_N$. So S is closed and discrete in the local Hausdorff topology. □

4 Background on relatively hyperbolic metric spaces

In this section we recall the definition of relatively hyperbolic metric spaces and then state a useful characterization due to Sisto.

Definition 4.1 Suppose ω is a nonprincipal ultrafilter, (X, d) is a metric space, (x_n) is a sequence of points in X and (λ_n) is a sequence of positive numbers with $\lim_{\omega} \lambda_n = \infty$. The *asymptotic cone of X with respect to (x_n) and (λ_n)* , denoted by $C_{\omega}(X, x_n, \lambda_n)$, is the ultralimit $\lim_{\omega}(X, \lambda_n^{-1} d, x_n)$.

For more background on asymptotic cones, see [14].

Definition 4.2 (Druţu and Sapir [15, Definition 2.1]) Let (X, d) be a complete geodesic metric space and let \mathcal{S} be a collection of closed geodesic subsets (called *pieces*). Suppose that the following two properties are satisfied:

- (1) Every two different pieces have at most one common point.
- (2) Every simple geodesic triangle (a simple loop composed of three geodesics) in X is contained in one piece.

Then we say that the metric space (X, d) is *tree-graded with respect to \mathcal{S}* .

Definition 4.3 A complete geodesic metric space (X, d) is *relatively hyperbolic with respect to a collection of subsets \mathcal{S}* if all its asymptotic cones, with respect to a fixed nonprincipal ultrafilter, are tree-graded with respect to the collection of ultralimits of the elements of \mathcal{S} .

Definition 4.4 A finitely generated group G is *relatively hyperbolic with respect to a family of subgroups $\{H_1, \dots, H_k\}$* if the Cayley graph of G with respect to some (and hence any) finite set of generators is relatively hyperbolic with respect to the collection of left cosets $\{gH_i : g \in G, i = 1, \dots, k\}$.

Remark 4.5 This is one among several equivalent definitions of relatively hyperbolic groups. We direct the interested reader to [15] for more details.

We now recall some basic properties of relatively hyperbolic metric spaces. Given a metric space (X, d) , a subset $A \subset X$ and $r > 0$, define

$$\mathcal{N}_X(A; r) = \{x \in X : d(x, a) < r \text{ for some } a \in A\}.$$

Theorem 4.6 (Druţu and Sapir [15, Theorem 4.1]) *Suppose (X, d) is relatively hyperbolic with respect to \mathcal{S} . For any $r > 0$ there exists $Q(r) > 0$ such that, if $S_1, S_2 \in \mathcal{S}$ are distinct, then*

$$\text{diam}_X(\mathcal{N}_X(S_1; r) \cap \mathcal{N}_X(S_2; r)) \leq Q(r).$$

The next two results involve quasi-isometric embeddings.

Definition 4.7 Suppose (X, d_X) and (Y, d_Y) are complete geodesic metric spaces. A map $f: X \rightarrow Y$ is a (A, B) -quasi-isometric embedding if

$$\frac{1}{A} d_X(x_1, x_2) - B \leq d_Y(f(x_1), f(x_2)) \leq A d_X(x_1, x_2) + B$$

for all $x_1, x_2 \in X$. If, in addition, there exists a quasi-isometry $g: Y \rightarrow X$ and $R > 0$ such that

$$d_X(x, (g \circ f)(x)) \leq R$$

for all $x \in X$ and

$$d_Y(y, (f \circ g)(y)) \leq R$$

for all $y \in Y$, then f is a (A, B) -quasi-isometry.

Being relatively hyperbolic is invariant under quasi-isometries.

Theorem 4.8 (Druţu and Sapir [15, Theorem 5.1]) *Suppose (X, d_X) and (Y, d_Y) are complete geodesic metric spaces and $f: X \rightarrow Y$ is a quasi-isometry. Then (X, d_X) is relatively hyperbolic with respect to \mathcal{S} if and only if (Y, d_Y) is relatively hyperbolic with respect to $f(\mathcal{S})$.*

Being relatively hyperbolic also constrains the possible quasi-isometric embeddings of \mathbb{R}^k when $k \geq 2$.

Theorem 4.9 (Druţu and Sapir [15, Corollary 5.8]) *Suppose (X, d_X) is relatively hyperbolic with respect to \mathcal{S} . Then, for any $A \geq 1$ and $B \geq 0$ there exists $M = M(A, B)$ such that, if $k \geq 2$ and $f: \mathbb{R}^k \rightarrow X$ is an (A, B) -quasi-isometric embedding, then there exists some $S \in \mathcal{S}$ such that*

$$f(\mathbb{R}^k) \subset \mathcal{N}_X(S; M).$$

We end this section by describing a useful characterization of relative hyperbolicity due to Sisto.

Definition 4.10 Let (X, d) be a complete geodesic metric space and \mathcal{S} a collection of subsets of X . A family of maps $\Pi = \{\pi_S : X \rightarrow S\}_{S \in \mathcal{S}}$ is an *almost-projection system* for \mathcal{S} if there exists $C > 0$ such that, for all $S \in \mathcal{S}$,

- (1) if $x \in X$ and $p \in S$, then

$$d(x, p) \geq d(x, \pi_S(x)) + d(\pi_S(x), p) - C;$$

- (2) $\text{diam}_X \pi_S(S') \leq C$ for all $S, S' \in \mathcal{S}$ distinct; and
- (3) if $x \in X$ and $d(x, S) = R$, then $\text{diam}_X \pi_S(B(x; R)) \leq C$.

In a relatively hyperbolic metric space, almost projection systems appear naturally.

Theorem 4.11 (Sisto [31, Theorem 2.14]) *Suppose (X, d) is relatively hyperbolic with respect to a collection \mathcal{S} . For each $S \in \mathcal{S}$ and $x \in X$, let $\pi_S(x)$ be any point in S satisfying*

$$d(x, \pi_S(x)) \leq d(x, S) + 1,$$

then $\Pi = \{\pi_S : X \rightarrow S\}_{S \in \mathcal{S}}$ is an almost-projection system for \mathcal{S} .

To obtain a characterization of relatively hyperbolicity in terms of the existence of almost-projection systems, one needs an additional property.

Definition 4.12 Let (X, d) be a complete geodesic metric space. A collection of geodesics \mathcal{G} is a *geodesic path system* if:

- (1) For every distinct $x_1, x_2 \in X$ there exists a geodesic in \mathcal{G} whose endpoints are x_1 and x_2 .
- (2) If $\alpha \in \mathcal{G}$, then every subpath of α is also in \mathcal{G} .

Definition 4.13 Let (X, d) be a complete geodesic metric space and \mathcal{S} a collection of subsets of X .

- (1) A geodesic triangle \mathcal{T} in X is *\mathcal{S} -almost-transverse with constants κ and Δ* if

$$\text{diam}_X(\mathcal{N}_X(S; \kappa) \cap \gamma) \leq \Delta$$

for every $S \in \mathcal{S}$ and edge γ of \mathcal{T} .

- (2) The collection \mathcal{S} is *asymptotically transverse-free* if there exist λ and σ such that, for each $\Delta \geq 1$ and $\kappa \geq \sigma$, if \mathcal{T} is a geodesic triangle in X which is \mathcal{S} -almost-transverse with constants κ and Δ , then \mathcal{T} is $(\lambda\Delta)$ -thin.

- (3) The collection S is asymptotically transverse-free relative to a geodesic path system \mathcal{G} if there exist λ and σ such that, for each $\Delta \geq 1$ and $\kappa \geq \sigma$, if \mathcal{T} is a geodesic triangle in X whose sides are in \mathcal{G} and is S -almost-transverse with constants κ and Δ , then \mathcal{T} is $(\lambda\Delta)$ -thin.

Observation 4.14 *Suppose \mathcal{T} is geodesic triangle that is S -almost-transverse with constants κ and Δ . If $\kappa' < \kappa$ and $\Delta' > \Delta$, then \mathcal{T} is S -almost-transverse with constants κ' and Δ' .*

Theorem 4.15 (Sisto [31, Theorem 2.14]) *Let (X, d) be a complete geodesic metric space and S a collection of subsets of X . Then the following are equivalent:*

- (1) X is relatively hyperbolic with respect to S .
- (2) S is asymptotically transverse-free and there exists an almost-projection system for S .
- (3) S is asymptotically transverse-free relative to a geodesic path system and there exists an almost-projection system for S ,

To be precise, Sisto [31, Theorem 2.14] only explicitly proves that (1) and (2) are equivalent; however, his proof can be adapted to show the equivalence of (2) and (3). We will explain how in the [appendix](#).

Part I General remarks about properly embedded simplices

5 Basic properties of simplices

In this section we explain some properties of simplices that we will use throughout the paper. We begin by considering the standard open simplex in $\mathbb{P}(\mathbb{R}^d)$.

Example 5.1 Let

$$S = \{[x_1 : \dots : x_d] \in \mathbb{P}(\mathbb{R}^d) : x_1 > 0, \dots, x_d > 0\}.$$

Then S is a $(d-1)$ -dimensional simplex. Let $G \leq \text{GL}_d(\mathbb{R})$ denote the group generated by the group of diagonal matrices with positive entries and the group of permutation matrices. Then

$$\text{Aut}(S) = \{[g] \in \text{PGL}_d(\mathbb{R}) : g \in G\}.$$

The Hilbert metric on S can be explicitly computed as

$$H_S([x_1 : \cdots : x_d], [y_1 : \cdots : y_d]) = \max_{1 \leq i, j \leq d} \frac{1}{2} \left| \log \frac{x_i y_j}{y_i x_j} \right|.$$

For details, see [28, Proposition 1.7], [18] or [32].

We also observe the following:

Observation 5.2 *Suppose $S \subset \mathbb{P}(\mathbb{R}^d)$ is a simplex and $H \leq \text{Aut}(S)$ acts cocompactly on S . Then:*

- (1) *If $H_0 \leq H$ is the subgroup of elements that fix the vertices of S , then H_0 also acts cocompactly on S .*
- (2) *If $F \subset \partial S$ is a face of S , then $\text{Stab}_H(F)$ acts cocompactly on F .*

Proof Notice that H_0 is a finite-index subgroup of H and so (1) follows.

By changing coordinates, we can assume that

$$S = \{[x_1 : \cdots : x_k : 0 : \cdots : 0] : x_1 > 0, \dots, x_k > 0\},$$

$$F = \{[x_1 : \cdots : x_l : 0 : \cdots : 0] : x_1 > 0, \dots, x_l > 0\},$$

where $l < k$.

Consider the onto map $\pi : S \rightarrow F$ which projects to the first l coordinates. Then $h \circ \pi = \pi \circ h$ for all $h \in H_0$. By (1) there exists a compact set $K \subset S$ with $H_0 \cdot K = S$. Then $\pi(K) \subset F$ is compact and

$$H_0 \cdot \pi(K) = \pi(H_0 \cdot K) = \pi(S) = F.$$

So H_0 acts cocompactly on F . Since $H_0 \leq \text{Stab}_H(F)$, this proves (2). □

Observation 5.3 *If $S \subset \mathbb{P}(\mathbb{R}^d)$ is a simplex, then (S, H_S) is quasi-isometric to real Euclidean $(\dim S)$ -space.*

Proof By replacing \mathbb{R}^d with $\text{Span } S$ and picking suitable coordinates we can assume that

$$S = \{[x_1 : \cdots : x_{k+1}] \in \mathbb{P}(\mathbb{R}^{k+1}) : x_1 > 0, \dots, x_{k+1} > 0\}.$$

Next consider the map $\Phi : S \rightarrow \mathbb{R}^k$ defined by

$$\Phi([x_1 : \cdots : x_{k+1}]) = \left(\log \frac{x_2}{x_1}, \dots, \log \frac{x_{k+1}}{x_1} \right)$$

and define a distance d on \mathbb{R}^k by

$$d(v, w) = \frac{1}{2} \max \left\{ \max_{1 \leq i \leq k} |v_i - w_i|, \max_{1 \leq i, j \leq k} |(v_i - v_j) - (w_i - w_j)| \right\}.$$

Since d is induced by a norm, (\mathbb{R}^k, d) is quasi-isometric to real Euclidean k -space. Further,

$$d(\Phi([x_1 : \cdots : x_{k+1}]), \Phi([y_1 : \cdots : y_{k+1}])) = \max_{1 \leq i, j \leq k+1} \frac{1}{2} \left| \log \frac{x_i y_j}{y_i x_j} \right|$$

and so [Example 5.1](#) implies that Φ induces an isometry $(S, H_S) \rightarrow (\mathbb{R}^k, d)$. Hence, (S, H_S) is quasi-isometric to real Euclidean k -space. \square

We will frequently use the following observation about the faces of properly embedded simplices:

Observation 5.4 *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $S \subset \Omega$ is a properly embedded simplex. If $x \in \partial S$, then:*

- (1) $F_S(x)$ is properly embedded in $F_\Omega(x)$.
- (2) $F_S(x) = \bar{S} \cap F_\Omega(x)$.

Proof By definition $F_S(x) \subset F_\Omega(x)$. So, to show that $F_S(x)$ is properly embedded in $F_\Omega(x)$, it is enough to show that

$$\partial F_S(x) \subset \partial F_\Omega(x).$$

Suppose not. Then, since $\overline{F_S(x)} \subset \overline{F_\Omega(x)}$, there exists some $y \in F_\Omega(x) \cap \partial F_S(x)$. Let v_1, \dots, v_p denote the vertices of S . Then, after relabeling, there exist $0 < l < k < p$ such that

$$\begin{aligned} \overline{F_S(x)} &= \partial S \cap \mathbb{P}(\text{Span}\{v_1, \dots, v_k\}) \\ \overline{F_S(y)} &= \partial S \cap \mathbb{P}(\text{Span}\{v_1, \dots, v_l\}). \end{aligned}$$

Pick $z \in \partial S$ such that

$$\overline{F_S(z)} = \partial S \cap \mathbb{P}(\text{Span}\{v_{k+1}, \dots, v_p\}).$$

Then $(z, x) \subset \Omega$, but

$$[z, y] \subset \bar{S} \cap \mathbb{P}(\text{Span}\{v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_p\}) \subset \partial S \subset \partial \Omega.$$

But, since $y \in F_\Omega(x)$, this contradicts [Observation 3.12\(4\)](#). Hence, $\partial F_S(x) \subset \partial F_\Omega(x)$ and so $F_S(x)$ is properly embedded in $F_\Omega(x)$.

Next we show that $F_S(x) = \bar{S} \cap F_\Omega(x)$. By definition $F_S(x) \subset \bar{S} \cap F_\Omega(x)$. To establish the other inclusion, fix $u \in \bar{S} \cap F_\Omega(x)$ and then fix $w \in (x, u)$. Then, since $F_\Omega(x) = F_\Omega(w)$, we have

$$x, u \in F_\Omega(w) \cap \overline{F_S(w)}.$$

But, by part (1),

$$\emptyset = F_\Omega(w) \cap \partial F_S(w).$$

So $x, u \in F_S(w)$. So $F_S(x) = F_S(w) = F_S(u)$ and, in particular, $u \in F_S(x)$. Since $u \in \bar{S} \cap F_\Omega(x)$ was arbitrary, we see that $F_S(x) \supset \bar{S} \cap F_\Omega(x)$. □

6 Linear projections onto simplices

In this section we construct certain linear maps associated to a properly embedded simplex in a properly convex domain.

Definition 6.1 Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $S \subset \Omega$ is a properly embedded simplex with $\dim S = (q - 1) \geq 1$. A set of codimension one linear subspaces $\mathcal{H} := \{H_1, \dots, H_q\}$ is *S-supporting* when:

- (1) Each $\mathbb{P}(H_j)$ is a supporting hyperplane of Ω ,
- (2) If $F_1, \dots, F_q \subset \partial S$ are the boundary faces of maximal dimension, then (up to relabeling) $F_j \subset \mathbb{P}(H_j)$ for all $1 \leq j \leq q$.

Proposition 6.2 Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $S \subset \Omega$ is a properly embedded simplex and \mathcal{H} is a set of *S-supporting* hyperplanes. Then

$$\text{Span } S \oplus \left(\bigcap_{H \in \mathcal{H}} H \right) = \mathbb{R}^d \quad \text{and} \quad \Omega \cap \mathbb{P} \left(\bigcap_{H \in \mathcal{H}} H \right) = \emptyset.$$

Proof Suppose $\mathcal{H} := \{H_1, \dots, H_q\}$, $F_1, \dots, F_q \subset \partial S$ are the boundary faces of maximal dimension and v_1, \dots, v_q are the vertices of S labeled so that $F_j \subset \mathbb{P}(H_j)$ and $v_j \notin \bar{F}_j$. Let $\bar{v}_1, \dots, \bar{v}_q \in \mathbb{R}^d \setminus \{0\}$ be lifts of v_1, \dots, v_q , respectively.

First notice that

$$\Omega \cap \mathbb{P} \left(\bigcap_{H \in \mathcal{H}} H \right) = \emptyset$$

since $\mathbb{P}(H_j) \cap \Omega = \emptyset$ for every j .

Since $S \subset \mathbb{P}(v_j + H_j)$ and $S \cap \mathbb{P}(H_j) = \emptyset$, we must have $v_j \notin \mathbb{P}(H_j)$ and hence

$$(2) \quad v_j \oplus H_j = \mathbb{R}^d$$

for every j . Further,

$$(3) \quad v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_q \in \bar{F}_j \subset \mathbb{P}(H_j)$$

for each j .

Define $W := \bigcap_{H \in \mathcal{H}} H$. We claim that

$$\text{Span } S \oplus W = \mathbb{R}^d.$$

Since

$$\dim W + \dim \text{Span } S \geq (d - q) + q = d,$$

it suffices to show that

$$\text{Span } S \cap W = \{0\}.$$

If not, we can find $\alpha_1, \dots, \alpha_q \in \mathbb{R}$ such that

$$0 \neq \sum_{j=1}^q \alpha_j \bar{v}_j \in W.$$

By relabeling we can assume that $\alpha_1 \neq 0$. Then, by (3),

$$v_1 \subset \text{Span}\{v_2, \dots, v_q\} + W \subset H_1,$$

which contradicts (2). So

$$\text{Span } S \oplus W = \mathbb{R}^d. \quad \square$$

Using Proposition 6.2, we define the following linear projection:

Definition 6.3 Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $S \subset \Omega$ is a properly embedded simplex and \mathcal{H} is a set of S -supporting hyperplanes. Define $L_{S, \mathcal{H}} \in \text{End}(\mathbb{R}^d)$ to be the linear projection

$$\text{Span } S \oplus \left(\bigcap_{H \in \mathcal{H}} H \right) \rightarrow \text{Span } S.$$

We call $L_{S, \mathcal{H}}$ the linear projection of Ω onto S relative to \mathcal{H} .

Calling $L_{S, \mathcal{H}}$ the linear projection of Ω onto S is motivated by the following observation:

Observation 6.4 $L_{S, \mathcal{H}}(\Omega) = S.$

Proof By Proposition 6.2, $\mathbb{P}(\ker L_{S, \mathcal{H}}) \cap \Omega = \emptyset$, so $L_{S, \mathcal{H}}$ is well defined on Ω . The set $L_{S, \mathcal{H}}(\Omega) \subset \mathbb{P}(\text{Span } S)$ is connected and contains $S = L_{S, \mathcal{H}}(S)$. Further,

$$L_{S, \mathcal{H}}^{-1}(\partial S) = \bigcup_{j=1}^q L_{S, \mathcal{H}}^{-1}(\bar{F}_j) \subset \bigcup_{j=1}^q \mathbb{P}(H_j)$$

and so $\Omega \cap L_{S, \mathcal{H}}^{-1}(\partial S) = \emptyset$. Thus, $L_{S, \mathcal{H}}(\Omega) = S.$ □

We now derive some basic properties of these projection maps. First, recall, from Definition 3.11, that

$$F_{\Omega}(X) = \bigcup_{x \in X} F_{\Omega}(x)$$

when Ω is a properly convex domain and $X \subset \bar{\Omega}$.

Proposition 6.5 *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $S \subset \Omega$ is a properly embedded simplex and \mathcal{H} is a set of S -supporting hyperplanes. Then:*

- (1) *If $x \in \partial\Omega \cap \mathbb{P}(\bigcap_{H \in \mathcal{H}} H)$ and $y \in \partial S$, then $[x, y] \subset \partial\Omega$.*
- (2) $\mathbb{P}(\bigcap_{H \in \mathcal{H}} H) \cap F_{\Omega}(\partial S) = \emptyset.$

Proof Suppose $x \in \partial\Omega \cap \mathbb{P}(\bigcap_{H \in \mathcal{H}} H)$ and $y \in \partial S$. Then there exists a boundary face $F \subset \partial S$ of maximal dimension such that $y \in \bar{F}$. Then there exists some $H \in \mathcal{H}$ such that $F \subset \mathbb{P}(H)$. Then $[x, y] \subset \mathbb{P}(H)$ and so $[x, y] \cap \Omega = \emptyset$. Thus, $[x, y] \subset \partial\Omega$.

Next, suppose for a contradiction that

$$x \in \mathbb{P}\left(\bigcap_{H \in \mathcal{H}} H\right) \cap F_{\Omega}(\partial S).$$

Then there exists $y \in \partial S$ with $x \in F_{\Omega}(y)$. Pick $y' \in \partial S$ such that $(y, y') \subset S$. Then, by Observation 3.12(4), we also have $(x, y') \subset \Omega$. But this contradicts part (1). □

Proposition 6.6 *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $S \subset \Omega$ is a properly embedded simplex and \mathcal{H} is a set of S -supporting hyperplanes. If $x \in F_{\Omega}(\partial S)$, then $L_{S, \mathcal{H}}(x) \in F_{\Omega}(x) \cap \bar{S}$.*

Remark 6.7 Notice that $L_{S,\mathcal{H}}$ is defined and continuous on

$$\mathbb{P}(\mathbb{R}^d) \setminus \mathbb{P}(\ker L_{S,\mathcal{H}}) = \mathbb{P}(\mathbb{R}^d) \setminus \mathbb{P}\left(\bigcap_{H \in \mathcal{H}} H\right)$$

and so the previous proposition implies that $L_{S,\mathcal{H}}(x)$ is well defined.

Proof Fix $y \in \partial S$ such that $x \in F_\Omega(y)$. Then there exists an open line segment $\ell \subset F_\Omega(y)$ with $x, y \in \ell$. Then $L_{S,\mathcal{H}}(y) = y$ and $L_{S,\mathcal{H}}(\ell)$ is either an open line segment or a single point. So either $L_{S,\mathcal{H}}(x) = y$ or $L_{S,\mathcal{H}}(x)$ and y are contained in an open line segment in $\bar{S} \subset \bar{\Omega}$. So $L_{S,\mathcal{H}}(x) \in F_\Omega(y) = F_\Omega(x)$. \square

For a general properly embedded simplex, there could be many different sets of supporting hyperplanes, but the next result shows that the corresponding linear projections form a compact set.

Definition 6.8 Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $S \subset \Omega$ is a properly embedded simplex. Define

$$\mathcal{L}_S := \{L_{S,\mathcal{H}} : \mathcal{H} \text{ is a set of } S\text{-supporting hyperplanes}\} \subset \text{End}(\mathbb{R}^d).$$

Proposition 6.9 Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $S \subset \Omega$ is a properly embedded simplex. Then \mathcal{L}_S is a compact subset of $\text{End}(\mathbb{R}^d)$.

Proof Suppose that $F_1, \dots, F_q \subset \partial S$ are the boundary faces of S of maximal dimension. Fix a sequence L_{S,\mathcal{H}_n} of projections. Then

$$\mathcal{H}_n = \{H_{n,1}, \dots, H_{n,q}\},$$

where $F_j \subset \mathbb{P}(H_{n,j})$. Since $\text{Gr}_{d-1}(\mathbb{R}^d)$ is compact, we can find $n_k \rightarrow \infty$ such that

$$H_j := \lim_{k \rightarrow \infty} H_{n_k,j}$$

exists in $\text{Gr}_{d-1}(\mathbb{R}^d)$ for every $1 \leq j \leq q$. Then $F_j \subset \mathbb{P}(H_j)$ and $\mathbb{P}(H_j) \cap \Omega = \emptyset$ for every $1 \leq j \leq q$. So $\mathcal{H} = \{H_1, \dots, H_q\}$ is a set of S -supporting hyperplanes. Further, by definition,

$$L_{S,\mathcal{H}} = \lim_{k \rightarrow \infty} L_{S,\mathcal{H}_{n_k}}$$

in $\text{End}(\mathbb{R}^d)$. Since L_{S,\mathcal{H}_n} was an arbitrary sequence, \mathcal{L}_S is compact. \square

7 Opposite faces of periodic properly embedded simplices

In this section we establish a structure theorem for periodic properly embedded simplices.

Definition 7.1 Given a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ and a properly embedded simplex $S \subset \Omega$, we say that S is *periodic* if $\text{Stab}_{\text{Aut}(\Omega)}(S)$ acts cocompactly on S .

Definition 7.2 Suppose $S \subset \mathbb{P}(\mathbb{R}^d)$ is a simplex. Two faces $F_1, F_2 \subset \partial S$ are *opposite* when

- (1) $\bar{F}_1 \cap \bar{F}_2 = \emptyset$, and
- (2) $\text{Span } S = \text{Span}(F_1 \cup F_2)$.

Two points $x_1, x_2 \in \partial S$ are *opposite* if their faces $F_S(x_1)$ and $F_S(x_2)$ are opposite.

Observation 7.3 If $S \subset \mathbb{P}(\mathbb{R}^d)$ is a simplex, then two faces $F_1, F_2 \subset \partial S$ are opposite if and only if there exists a labeling v_1, \dots, v_p of the vertices of S such that

$$F_1 = \text{rel-int}(\text{ConvHull}_S\{v_1, \dots, v_q\}),$$

$$F_2 = \text{rel-int}(\text{ConvHull}_S\{v_{q+1}, \dots, v_p\})$$

for some $1 \leq q \leq p - 1$.

Observation 7.4 If $S \subset \mathbb{P}(\mathbb{R}^d)$ is a simplex and $x_1, x_2 \in \partial S$ are opposite, then $(x_1, x_2) \subset S$.

Theorem 7.5 Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, S is a properly embedded simplex in Ω and $\text{Stab}_{\text{Aut}(\Omega)}(S)$ acts cocompactly on S . Assume $x_1, x_2 \in \partial S$ are opposite points. If

$$V_1 = \text{Span } F_\Omega(x_1), \quad V_2 = \text{Span } F_\Omega(x_2) \quad \text{and} \quad V = V_1 + V_2,$$

then $V_1 \cap V_2 = \{0\}$ and

$$\Omega \cap \mathbb{P}(V) = \text{rel-int}(\text{ConvHull}_\Omega(F_\Omega(x_1) \cup F_\Omega(x_2))).$$

- Remark 7.6**
- (1) For the last equality, notice that “ \supset ” is a consequence of convexity.
 - (2) Since we are assuming that opposite points exist, we must have $\dim S \geq 1$.

Before proving the theorem we state and prove two corollaries.

Corollary 7.7 *With the hypotheses of Theorem 7.5:*

- (1) *If $y_1 \in F_\Omega(x_1)$ and $y_2 \in F_\Omega(x_2)$, then $(y_1, y_2) \subset \Omega$.*
- (2) *If $y_1 \in \partial F_\Omega(x_1)$ and $y_2 \in \overline{F_\Omega(x_2)}$, then $[y_1, y_2] \subset \partial\Omega$.*
- (3) *If $y_1 \in \overline{F_\Omega(x_1)}$ and $y_2 \in \partial F_\Omega(x_2)$, then $[y_1, y_2] \subset \partial\Omega$*

Proof This follows immediately from Theorem 7.5 and the fact that $\partial(\Omega \cap \mathbb{P}(V)) \subset \partial\Omega$. □

Corollary 7.8 *With the hypotheses of Theorem 7.5, if $S_1 \subset F_\Omega(x_1)$ and $S_2 \subset F_\Omega(x_2)$ are properly embedded simplices, then*

$$S' := \text{rel-int}(\text{ConvHull}_\Omega(S_1 \cup S_2))$$

is a properly embedded simplex of Ω with

$$\dim S' = \dim S_1 + \dim S_2 + 1.$$

Proof Let v_1, \dots, v_p be the extreme points of S_1 and let w_1, \dots, w_q be the extreme points of S_2 . Since $V_1 \cap V_2 = \{0\}$, the lines $v_1, \dots, v_p, w_1, \dots, w_q$ are linearly independent in \mathbb{R}^d . Then Corollary 7.7 implies that S' is a properly embedded simplex with vertices $v_1, \dots, v_p, w_1, \dots, w_q$ and so

$$\dim S' = p + q - 1 = (p - 1) + (q - 1) + 1 = \dim S_1 + \dim S_2 + 1. \quad \square$$

The rest of the section is devoted to the proof of Theorem 7.5. So fix $S \subset \Omega \subset \mathbb{P}(\mathbb{R}^d)$ and $x_1, x_2 \in \partial S$ as in the statement of the theorem.

First, $\text{Stab}_{\text{Aut}(\Omega)}(S)$ permutes the faces of S and S has finitely many faces, so there exists a finite-index subgroup $G \leq \text{Stab}_{\text{Aut}(\Omega)}(S)$ which stabilizes each face of S . Then $G(V_1) = V_1$, $G(V_2) = V_2$, $G|_{V_1} \leq \text{Aut}(F_\Omega(x_1))$ and $G|_{V_2} \leq \text{Aut}(F_\Omega(x_2))$. Further, G acts cocompactly on S since G has finite index in $\text{Stab}_{\text{Aut}(\Omega)}(S)$.

Lemma 7.9 *Fix $p_0 \in (x_1, x_2)$. There exist $y_1 \in F_S(x_1)$, $y_2 \in F_S(x_2)$ and a sequence $a_n \in G$ such that*

- (1) $y_1 = \lim_{n \rightarrow \infty} a_n p_0$,
- (2) $y_1 = \lim_{n \rightarrow \infty} a_n x_1$, and
- (3) $y_2 = \lim_{n \rightarrow \infty} a_n x_2$.

Proof Let v_1, \dots, v_p be the extreme points of S labeled so that

$$F_S(x_1) = \text{rel-int}(\text{ConvHull}_\Omega\{v_1, \dots, v_q\}),$$

$$F_S(x_2) = \text{rel-int}(\text{ConvHull}_\Omega\{v_{q+1}, \dots, v_p\}),$$

where $q = 1 + \dim F_S(x_1)$.

Let $W := \text{Span } S$. Since G fixes each extreme point and acts cocompactly on S , we can find a sequence $a_n \in G$ with

$$a_n|_W = \begin{bmatrix} \lambda_{n,1} & & \\ & \ddots & \\ & & \lambda_{n,p} \end{bmatrix}$$

relative to the basis v_1, \dots, v_p of W and

$$\lim_{n \rightarrow \infty} \frac{\lambda_{n,i}}{\lambda_{n,j}} = \begin{cases} c_{i,j} \in (0, \infty) & \text{if } 1 \leq i, j \leq q, \\ \infty & \text{if } 1 \leq i \leq q < j \leq p, \\ c_{i,j} \in (0, \infty) & \text{if } q < i, j \leq p. \end{cases}$$

Then define

$$y_1 := [\text{diag}(c_{1,1}, \dots, c_{q,1}, 0, \dots, 0)] \cdot x_1 = \lim_{n \rightarrow \infty} a_n x_1 \in F_S(x_1),$$

$$y_2 := [\text{diag}(0, \dots, 0, c_{q+1,q+1}, \dots, c_{p,q+1})] \cdot x_2 = \lim_{n \rightarrow \infty} a_n x_2 \in F_S(x_2).$$

Then $\lim_{n \rightarrow \infty} a_n p_0 = y_1$. □

Next let $a_{1,n} = a_n|_{V_1} \in \text{Aut}(F_\Omega(x_1))$ and $a_{2,n} = a_n|_{V_2} \in \text{Aut}(F_\Omega(x_2))$. Since $\lim_{n \rightarrow \infty} a_{1,n}(x_1) = y_1 \in F_S(x_1)$ and $\text{Aut}(F_\Omega(x_1))$ acts properly on $F_\Omega(x_1)$, we see that

$$\{a_{1,n} : n \geq 0\} \subset \text{Aut}(F_\Omega(x_1))$$

is relatively compact. The same argument implies that

$$\{a_{2,n} : n \geq 0\} \subset \text{Aut}(F_\Omega(x_2))$$

is relatively compact. So, by passing to a subsequence, we can suppose that $a_1 := \lim_{n \rightarrow \infty} a_{1,n}$ exists in $\text{Aut}(F_\Omega(x_1))$ and $a_2 := \lim_{n \rightarrow \infty} a_{2,n}$ exists in $\text{Aut}(F_\Omega(x_1))$.

Lemma 7.10 $V_1 \cap V_2 = \{0\}$.

Proof Suppose not. Then fix a decomposition

$$V = (V_1 \cap V_2) \oplus W_1 \oplus W_2$$

with $V_1 = (V_1 \cap V_2) \oplus W_1$ and $V_2 = (V_1 \cap V_2) \oplus W_2$. Then, relative to this decomposition,

$$a_n|_V = \begin{bmatrix} A_{1,1}^{(n)} & A_{1,2}^{(n)} & A_{1,3}^{(n)} \\ 0 & A_{2,2}^{(n)} & 0 \\ 0 & 0 & A_{3,3}^{(n)} \end{bmatrix}$$

for some $A_{1,1}^{(n)} \in \text{GL}(V_1 \cap V_2)$, $A_{1,2}^{(n)} \in \text{Lin}(W_1, V_1 \cap V_2)$, $A_{1,3}^{(n)} \in \text{Lin}(W_2, V_1 \cap V_2)$, $A_{2,2}^{(n)} \in \text{GL}(W_1)$ and $A_{3,3}^{(n)} \in \text{GL}(W_2)$.

Then

$$a_{1,n} = \begin{bmatrix} A_{1,1}^{(n)} & A_{1,2}^{(n)} \\ 0 & A_{2,2}^{(n)} \end{bmatrix} \quad \text{and} \quad a_{2,n} = \begin{bmatrix} A_{1,1}^{(n)} & A_{1,3}^{(n)} \\ 0 & A_{3,3}^{(n)} \end{bmatrix}$$

relative to the decompositions $V_1 = (V_1 \cap V_2) \oplus W_1$ and $V_2 = (V_1 \cap V_2) \oplus W_2$, respectively.

Since $a_{1,n}$ converges to a_1 in $\text{PGL}(V_1)$, there exists a sequence $t_{1,n} \in \mathbb{R}$ such that

$$A_1 := \lim_{n \rightarrow \infty} t_{1,n} \begin{pmatrix} A_{1,1}^{(n)} & A_{1,2}^{(n)} \\ 0 & A_{2,2}^{(n)} \end{pmatrix} \in \text{GL}(V_1)$$

and $a_1 = [A_1]$. For the same reasons, there exists a sequence $t_{2,n} \in \mathbb{R}$ such that

$$A_2 := \lim_{n \rightarrow \infty} t_{2,n} \begin{pmatrix} A_{1,1}^{(n)} & A_{1,3}^{(n)} \\ 0 & A_{3,3}^{(n)} \end{pmatrix} \in \text{GL}(V_2)$$

and $a_2 = [A_2]$.

Since $t_{1,n}A_{1,1}^{(n)}$ and $t_{2,n}A_{1,1}^{(n)}$ both converge in $\text{GL}(V_1 \cap V_2)$, we must have

$$0 \neq c := \lim_{n \rightarrow \infty} \frac{t_{2,n}}{t_{1,n}}.$$

But then

$$t_{1,n} \begin{pmatrix} A_{1,1}^{(n)} & A_{1,2}^{(n)} & A_{1,3}^{(n)} \\ 0 & A_{2,2}^{(n)} & 0 \\ 0 & 0 & A_{3,3}^{(n)} \end{pmatrix}$$

converges in $\text{GL}(V)$, which implies that $a_n|_V$ converges in $\text{PGL}(V)$. But, by construction, $a_n|_V$ diverges in $\text{PGL}_d(\mathbb{R})$. So we have a contradiction and hence $V_1 \cap V_2 = \{0\}$. □

Next let $\pi_1: V \rightarrow V_1$ and $\pi_2: V \rightarrow V_2$ be the projections relative to the decomposition $V = V_1 \oplus V_2$. To show that

$$\Omega \cap \mathbb{P}(V) = \text{rel-int}(\text{ConvHull}_\Omega(F_\Omega(x_1) \cup F_\Omega(x_2))),$$

it is enough to show that $\pi_1(\Omega \cap \mathbb{P}(V)) = F_\Omega(x_1)$ and $\pi_2(\Omega \cap \mathbb{P}(V)) = F_\Omega(x_2)$.

Lemma 7.11 $\pi_1(\Omega \cap \mathbb{P}(V)) = F_\Omega(x_1).$

Proof Since $F_\Omega(x_1) \subset \overline{\Omega \cap \mathbb{P}(V)}$, we clearly have

$$F_\Omega(x_1) \subset \pi_1(\Omega \cap \mathbb{P}(V)).$$

Relative to the decomposition $V = V_1 \oplus V_2$, we have

$$a_n|_V = \begin{bmatrix} B_{1,n} & 0 \\ 0 & B_{2,n} \end{bmatrix},$$

where $B_{1,n} \in \text{GL}(V_1)$, $[B_{1,n}] = a_{1,n}$, $B_{2,n} \in \text{GL}(V_2)$ and $[B_{2,n}] = a_{2,n}$. Since $a_{1,n}$ converges to a_1 in $\text{PGL}(V_1)$, there exists a sequence $s_{1,n} \in \mathbb{R}$ such that

$$B_1 := \lim_{n \rightarrow \infty} s_{1,n} B_{1,n} \in \text{GL}(V_1)$$

and $a_1 = [B_1]$. Similarly, there exists a sequence $s_{2,n} \in \mathbb{R}$ such that

$$B_2 := \lim_{n \rightarrow \infty} s_{2,n} B_{2,n} \in \text{GL}(V_2)$$

and $a_2 = [B_2]$.

Since $\lim_{n \rightarrow \infty} a_n p_0 = y_1 \in F_\Omega(x_1)$, we must have

$$\lim_{n \rightarrow \infty} \frac{s_{1,n}}{s_{2,n}} = 0.$$

Then $a_n|_V$ converges to $[T] \in \mathbb{P}(\text{End}(V))$, where

$$T = \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix} \in \text{End}(V).$$

Notice that $a_1^{-1} \circ [T] = [\pi_1]$.

Fix $p \in \Omega \cap \mathbb{P}(V)$. We claim that $T(p) \in F_\Omega(x_1)$. Since $\ker T = V_2$ and $\mathbb{P}(V_2) \cap \Omega = \emptyset$, we have $p \notin \mathbb{P}(\ker T)$. So

$$T(p) = \lim_{n \rightarrow \infty} a_n(p).$$

Then, since $\lim_{n \rightarrow \infty} a_n(p_0) = y_1 \in F_\Omega(x_1)$, [Proposition 3.13](#) implies that $T(p) \in F_\Omega(x_1)$. Since $p \in \Omega \cap \mathbb{P}(V)$ was arbitrary,

$$T(\Omega \cap \mathbb{P}(V)) \subset F_\Omega(x_1).$$

Then

$$\pi_1(\Omega \cap \mathbb{P}(V)) = (a_1^{-1} \circ [T])(\Omega \cap \mathbb{P}(V)) \subset a_1^{-1}(F_\Omega(x_1)) = F_\Omega(x_1),$$

since $a_1 \in \text{Aut}(F_\Omega(x_1))$. □

Applying the same argument to a_n^{-1} shows that

$$\pi_2(\Omega \cap \mathbb{P}(V)) = F_\Omega(x_2),$$

which completes the proof.

Part II The naive convex cocompact case

8 Invariant and isolated sets of simplices are periodic

In this section we show that any isolated and invariant family of properly embedded simplices of dimension at least two satisfies properties (1) and (2) of [Theorem 1.19](#).

Proposition 8.1 *Suppose $(\Omega, \mathcal{C}, \Lambda)$ is a naive convex cocompact triple and \mathcal{S} is an isolated and Λ -invariant family of maximal properly embedded simplices in \mathcal{C} of dimension at least two. Then:*

- (1) \mathcal{S} is a locally finite collection; that is, for any compact set $K \subset \Omega$, the set

$$\{S \in \mathcal{S} : S \cap K \neq \emptyset\}$$

is finite.

- (2) Λ has finitely many orbits in \mathcal{S} .
- (3) If $S \in \mathcal{S}$, then $\text{Stab}_\Lambda(S)$ acts cocompactly on S and contains a finite-index subgroup isomorphic to \mathbb{Z}^k , where $k = \dim S$.

Parts (1) and (2) are simple consequences of the definition. The proof of the first assertion in part (3) is nearly identical to the proof of the analogous result in the CAT(0) setting; see Wise [\[33, Proposition 4.0.4\]](#), Hruksa [\[19, Theorem 3.7\]](#) or Hruska and Kleiner [\[20, Section 3.1\]](#).

Proof Since \mathcal{S} is closed and discrete in the local Hausdorff topology, part (1) is true. To prove part (2), fix a compact set $K \subset \mathcal{C}$ such that $\Lambda \cdot K = \mathcal{C}$. Then each Λ -orbit of \mathcal{S} intersects K and, by part (1), there are only finitely many such intersections. Hence, there are only finitely many Λ -orbits in \mathcal{S} .

To prove part (3), fix $S \in \mathcal{S}$. Let

$$X := \{g \in \Lambda : S \cap gK \neq \emptyset\}.$$

Then $S = \bigcup_{g \in X} (S \cap gK)$. Since $(g^{-1}S) \cap K \neq \emptyset$ when $g \in X$, part (1) implies that the set

$$\{g^{-1}S : g \in X\}$$

is finite. Since

$$g^{-1}S = h^{-1}S \iff gh^{-1} \in \text{Stab}_\Lambda(S) \iff \text{Stab}_\Lambda(S)g = \text{Stab}_\Lambda(S)h,$$

there exist $g_1, \dots, g_m \in X$ such that

$$\bigcup_{g \in X} \text{Stab}_\Lambda(S)g = \bigcup_{j=1}^m \text{Stab}_\Lambda(S)g_j.$$

Then the set $\widehat{K} := \bigcup_{j=1}^m S \cap g_j K$ is compact and

$$\text{Stab}_\Lambda(S) \cdot \widehat{K} = \bigcup_{g \in X} S \cap gK = S.$$

So $\text{Stab}_\Lambda(S)$ acts cocompactly on S .

Finally, we show that $\text{Stab}_\Lambda(S)$ contains a finite-index subgroup isomorphic to \mathbb{Z}^k , where $k = \dim S$. First, let $G \leq \text{Stab}_\Lambda(S)$ denote the subgroup of elements which fix every vertex of S . Then G has finite index in $\text{Stab}_\Lambda(S)$. Next let $V := \text{Span } S$ and consider the homomorphism

$$\varphi : G \rightarrow \text{Aut}(S) \leq \text{PGL}(V), \quad \varphi(g) = g|_V.$$

Fix $v_1, \dots, v_{k+1} \in V$ such that $[v_1], \dots, [v_{k+1}]$ are the vertices of S . Then, relative to the basis v_1, \dots, v_{k+1} ,

$$\varphi(G) \leq \mathcal{D} := \{[\text{diag}(a_1, \dots, a_{k+1})] : a_1, \dots, a_{k+1} > 0\} \cong (\mathbb{R}^k, +).$$

Further, $\varphi(G)$ is a lattice of \mathcal{D} since G acts properly and cocompactly on S . So $\varphi(G)$ is isomorphic to \mathbb{Z}^k .

Every element of $\ker \varphi$ acts trivially on S . Then, since $\text{Aut}(\Omega)$ acts properly on Ω , the group $\ker \varphi \leq G$ is finite. By Selberg’s lemma, there exists a torsion-free finite-index subgroup $\Lambda_0 \leq \Lambda$. Then

$$H := \Lambda_0 \cap G \leq \text{Stab}_\Lambda(S)$$

has finite index. Further, $\ker \varphi \cap H = \{1\}$ since H is torsion-free. So H is isomorphic to $\varphi(H)$. Finally, $\varphi(H) \leq \varphi(G) \cong \mathbb{Z}^k$ has finite index so $\varphi(H)$ is also isomorphic to \mathbb{Z}^k . □

9 Faces of properly embedded simplices

In this section we show that any isolated, coarsely complete and invariant family of properly embedded simplices can be refined to a family which also satisfies property (4) of [Theorem 1.19](#).

Theorem 9.1 *Suppose $(\Omega, \mathcal{C}, \Lambda)$ is a naive convex cocompact triple with coarsely isolated simplices. Let S_0 be an isolated, coarsely complete and Λ -invariant family of maximal properly embedded simplices in \mathcal{C} of dimension at least two. Then there exists an isolated, coarsely complete and Λ -invariant subfamily $S \subset S_0$ with the following additional property:*

(\star) *There exists $D_1 > 0$ such that, if $S \in S$ and $x \in \partial S$, then*

$$H_{F_{\Omega}(x)}^{\text{Haus}}(\bar{\mathcal{C}} \cap F_{\Omega}(x), F_S(x)) \leq D_1.$$

Remark 9.2 By [Observation 5.4](#), if $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $S \subset \Omega$ is a properly embedded simplex and $x \in \partial S$, then

$$\bar{S} \cap F_{\Omega}(x) = \overline{F_S(x)} \cap F_{\Omega}(x) = F_S(x).$$

However, if $\mathcal{C} \subset \Omega$ is a general convex subset and $x \in \partial_i \mathcal{C}$, then

$$\bar{\mathcal{C}} \cap F_{\Omega}(x) \supset \overline{F_{\mathcal{C}}(x)} \cap F_{\Omega}(x) \supset F_{\mathcal{C}}(x)$$

and both inclusions can be strict.

The rest of the section is devoted to the proof of this theorem. We will need the following observation about simplices:

Lemma 9.3 *Suppose $S \subset \mathbb{P}(\mathbb{R}^d)$ is a simplex and $(a, b) \subset S$ is a properly embedded line. If $p \in S$, then there exist $a' \in F_S(a)$ and $b' \in F_S(b)$ such that $p \in (a', b')$.*

Proof Let $X \subset S$ be the set of points $p \in S$ where there exist $a' \in F_S(a)$ and $b' \in F_S(b)$ such that $p \in (a', b')$. By hypothesis, X is nonempty. Next let $G \leq \text{Aut}(S)$ denote the group of automorphisms that fix all vertices. Then G acts transitively on S (see [Example 5.1](#)). Further, $G \cdot F_S(a) = F_S(a)$ and $G \cdot F_S(b) = F_S(b)$. So $G \cdot X = X$. Then $X = S$ since G acts transitively on S . □

We start by making an initial reduction.

Lemma 9.4 Suppose $(\Omega, \mathcal{C}, \Lambda)$ is a naive convex cocompact triple with coarsely isolated simplices. Let \mathcal{S}_0 be an isolated, coarsely complete and Λ -invariant family of maximal properly embedded simplices in \mathcal{C} of dimension at least two. Then there exists an isolated, coarsely complete and Λ -invariant subfamily $\mathcal{S} \subset \mathcal{S}_0$ with the following additional property:

(\star) If $S_1, S_2 \in \mathcal{S}$ and $\dim S_1 < \dim S_2$, then

$$\infty = \sup_{p \in S_1} H_\Omega(p, S_2).$$

Proof Since \mathcal{S}_0 is coarsely complete there exists $D_0 > 0$ such that, if $S \subset \mathcal{C}$ is a properly embedded simplex of dimension at least two, then there exists some $S' \in \mathcal{S}_0$ with

$$S \subset \mathcal{N}_\Omega(S'; D_0).$$

Let $X \subset \mathcal{S}_0$ be the set of simplices $S \in \mathcal{S}_0$ where there exists some $S' \in \mathcal{S}_0$ with $\dim S < \dim S'$ and

$$\sup_{p \in S} H_\Omega(p, S') < +\infty.$$

Notice that X is a Λ -invariant subset of \mathcal{S}_0 . Next, for each $S \in X$, define

$$m(S) := \inf_{\substack{S' \in \mathcal{S}_0 \\ \dim S < \dim S'}} \sup_{p \in S} H_\Omega(p, S').$$

Then $m(S)$ is finite and Λ -invariant. Further, [Proposition 8.1](#) implies that there are only finitely many Λ -orbits in X . So

$$M := \sup_{S \in X} m(S) = \max_{S \in X} m(S) < +\infty.$$

We claim that $\mathcal{S} := \mathcal{S}_0 \setminus X$ satisfies the conclusion of the proposition. By construction, if $S_1, S_2 \in \mathcal{S}$ and $\dim S_1 < \dim S_2$, then

$$\infty = \sup_{p \in S_1} H_\Omega(p, S_2).$$

Further, since X is Λ -invariant, the set \mathcal{S} is Λ -invariant. Also, since $\mathcal{S} \subset \mathcal{S}_0$, the set \mathcal{S} is closed and discrete in the local Hausdorff topology. The following claim proves that \mathcal{S} is coarsely complete:

Claim If $S \subset \mathcal{C}$ is a properly embedded simplex of dimension at least two, then there exists some $S' \in \mathcal{S}$ with

$$S \subset \mathcal{N}_\Omega(S'; D),$$

where $D := D_0 + (d - 3)M$.

Fix a properly embedded simplex $S \subset \mathcal{C}$ of dimension at least two. Then there exists some $S_1 \in \mathcal{S}_0$ such that

$$S \subset \mathcal{N}_\Omega(S_1; D_0).$$

If $S_1 \in \mathcal{S}$, we are done. Otherwise there exist $k \in \{2, \dots, d-2\}$, $S_2, S_3, \dots, S_{k-1} \in X$ and $S_k \in \mathcal{S}$ such that

- (1) $S_j \subset \mathcal{N}_\Omega(S_{j+1}; M)$ for $j = 1, \dots, k-1$, and
- (2) $\dim S_j < \dim S_{j+1}$ for $j = 1, \dots, k-1$.

Then

$$S \subset \mathcal{N}_\Omega(S_k; D_0 + (k-1)M) \subset \mathcal{N}_\Omega(S_k; D).$$

Since S was an arbitrary properly embedded simplex of dimension at least two in \mathcal{C} , this completes the proof of the claim and the proposition. □

For the rest of the section fix $\Omega, \mathcal{C}, \Lambda$ and \mathcal{S}_0 satisfying the hypotheses of [Theorem 9.1](#). Fix $S \subset \mathcal{S}_0$ satisfying [Lemma 9.4](#). Since \mathcal{S} is coarsely complete, there exists $D_0 > 0$ such that, if $S \subset \mathcal{C}$ is a properly embedded simplex of dimension at least two, then there exists some $S' \in \mathcal{S}$ such that

$$S \subset \mathcal{N}_\Omega(S'; D_0).$$

Since each simplex has finitely many faces and there are only finitely many distinct orbits of properly embedded simplices in \mathcal{S} (see [Proposition 8.1](#)), it is enough to show that, if $S \in \mathcal{S}$ and $x \in \partial S$, then

$$(4) \quad H_{F_\Omega(x)}^{\text{Haus}}(\bar{\mathcal{C}} \cap F_\Omega(x), F_S(x)) < +\infty.$$

Suppose for a contradiction that (4) fails for some choice of $S \in \mathcal{S}$ and $x \in \partial S$. We can choose S and x so that

$$(5) \quad (\dim F_\Omega(x), \dim F_\Omega(x) - \dim F_S(x))$$

is minimal in lexicographical order among all examples which fail to satisfy (4).

9.1 The vertex case

In this subsection we show that x is not a vertex of S .

Lemma 9.5 *x is not a vertex of S .*

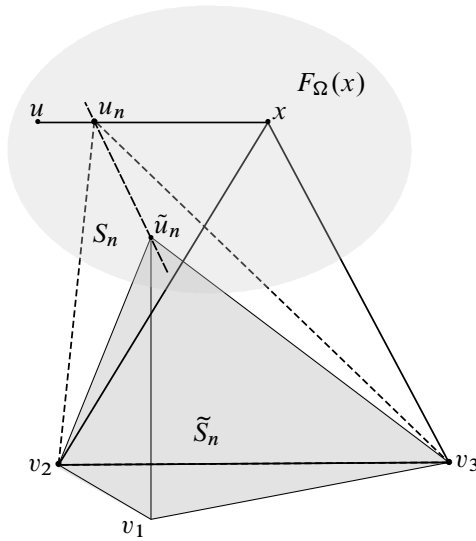


Figure 1: Diagram illustrating the proof of Lemma 9.5.

Proof Suppose for a contradiction that x is a vertex of S . Then let v_2, \dots, v_p be the other vertices of S .

We will first show that $\bar{C} \cap F_\Omega(x)$ contains a properly embedded line and then use this line to obtain a contradiction.

Claim $\bar{C} \cap F_\Omega(x)$ contains a properly embedded line.

By assumption,

$$\infty = H_{F_\Omega(x)}^{\text{Haus}}(\bar{C} \cap F_\Omega(x), F_S(x)) = H_{F_\Omega(x)}^{\text{Haus}}(\bar{C} \cap F_\Omega(x), \{x\}).$$

So there exists $u \in \bar{C} \cap \partial F_\Omega(x)$. Fix a sequence

$$u_n \in [x, u) \subset F_\Omega(x)$$

such that $\lim_{n \rightarrow \infty} u_n = u$. Lemma 3.18 implies that u_n, v_2, \dots, v_p are the vertices of a properly embedded simplex $S_n \subset \Omega$ of dimension $\dim S_n = \dim S \geq 2$. Then, for each n , there exists $\tilde{S}_n \in \mathcal{S}$ such that

$$S_n \subset \mathcal{N}_\Omega(\tilde{S}_n; D_0).$$

Then, by Proposition 3.13, for each n there exists $\tilde{u}_n \in \partial \tilde{S}_n$ such that

$$H_{F_\Omega(x)}(\tilde{u}_n, u_n) \leq D_0.$$

If \tilde{u}_n is not a vertex of \tilde{S}_n , then [Observation 5.4](#) implies that $F_{\tilde{S}_n}(\tilde{u}_n)$ is properly embedded in $F_\Omega(x)$ and the claim is established (see [Figure 1](#)). So we may assume that \tilde{u}_n is a vertex of \tilde{S}_n .

Next pick $g_n \in \Lambda$ and a compact set $K \subset \Omega$ such that $g_n \tilde{S}_n \cap K \neq \emptyset$ for all $n \geq 0$. Then, by [Proposition 8.1](#), the set

$$\{g_n \tilde{S}_n : n \geq 0\}$$

is finite. So, by passing to a subsequence, we can suppose that

$$\tilde{S} := g_n \tilde{S}_n = g_m \tilde{S}_m$$

for all $n, m \geq 0$. Since \tilde{u}_n is a vertex of \tilde{S}_n , $g_n \tilde{u}_n$ is a vertex of \tilde{S} for all $n \geq 0$. Since \tilde{S} has finitely many vertices, by passing to a subsequence we can suppose that

$$\tilde{u} := g_n \tilde{u}_n = g_m \tilde{u}_m$$

for all $n, m \geq 0$. Then, for all $n \geq 0$, let

$$F := F_\Omega(\tilde{u}) = g_n F_\Omega(x).$$

After passing to a subsequence, we can assume that $g_n(u, x)$ converges to $(u_\infty, x_\infty) \subset \bar{F}$. We claim that (u_∞, x_∞) is properly embedded in F . By construction, $g_n u \in \partial F$ for all n , so $u_\infty \in \partial F$. Since

$$\lim_{n \rightarrow \infty} H_F(\tilde{u}, g_n u_n) = \lim_{n \rightarrow \infty} H_{F_\Omega(x)}(\tilde{u}_n, u_n) \leq D_0,$$

we can pass to another subsequence such that $\lim_{n \rightarrow \infty} g_n u_n$ exists in F . Since $g_n u_n \in g_n(u, x)$, this implies that $(u_\infty, x_\infty) \subset F$. Further,

$$\lim_{n \rightarrow \infty} H_F(g_n u_n, g_n x) = \lim_{n \rightarrow \infty} H_{F_\Omega(x)}(u_n, x) = \infty.$$

Hence, $x_\infty \in \partial F$ and so (u_∞, x_∞) is a properly embedded line in F . Then $F_\Omega(x) = g_1^{-1} F$ also contains a properly embedded line. This completes the proof of the claim.

Now suppose that $(a, b) \subset \bar{C} \cap F_\Omega(x)$ is a properly embedded line. Then [Corollary 7.8](#) implies that a, b, v_2, \dots, v_p are the vertices of a properly embedded simplex S' in Ω of dimension p . Then there exists $\tilde{S} \in \mathcal{S}$ such that

$$S' \subset \mathcal{N}_\Omega(\tilde{S}; D_0).$$

Then [Observation 5.3](#) implies that $\dim \tilde{S} \geq p > \dim S$.

Now fix $x' \in (a, b)$. Since $x' \in F_\Omega(x)$, Lemma 3.18 implies that x', v_2, \dots, v_p are the vertices of a properly embedded simplex S'' and

$$H_\Omega^{\text{Haus}}(S, S'') \leq H_{F_\Omega(x)}(x, x').$$

Since $S'' \subset S'$, we then have

$$S \subset \mathcal{N}_\Omega(\tilde{S}; D_0 + H_{F_\Omega(x)}(x, x')).$$

Finally, since $\dim S < \dim \tilde{S}$, we have a contradiction with the condition in Lemma 9.4. □

9.2 Using the fact that simplices are coarsely isolated

By Lemma 9.5 and Observation 5.4, $F_S(x)$ is a properly embedded simplex in $F_\Omega(x)$ with dimension at least one. In particular, $\partial F_S(x) \neq \emptyset$. Next, recall from Definition 3.11 that

$$F_\Omega(\partial F_S(x)) = \bigcup_{y \in \partial F_S(x)} F_\Omega(y).$$

In this subsection we will prove the following:

Proposition 9.6 $\bar{C} \cap F_\Omega(\partial F_S(x))$ is a connected component of $\bar{C} \cap \partial F_\Omega(x)$.

The rest of this subsection is devoted to the proof of Proposition 9.6.

Lemma 9.7 $\bar{C} \cap F_\Omega(\partial F_S(x))$ is closed in $\bar{C} \cap \partial F_\Omega(x)$.

Proof Suppose $v_n \in \bar{C} \cap F_\Omega(\partial F_S(x))$ converges to $v_\infty \in \bar{C} \cap \partial F_\Omega(x)$. Since $F_S(x)$ has finitely many faces, by passing to a subsequence we can suppose that there exists $y \in \partial F_S(x)$ such that $v_n \in F_\Omega(y)$ for all n .

Since $F_\Omega(y) \subset \partial F_\Omega(x)$, our minimality assumption — see (5) — implies that

$$R := H_{F_\Omega(y)}^{\text{Haus}}(F_S(y), \bar{C} \cap F_\Omega(y)) < +\infty.$$

So, for every n , there exists $v'_n \in F_S(y) \subset \partial F_S(x)$ such that $H_{F_\Omega(y)}(v_n, v'_n) \leq R$. Passing to a subsequence, we can suppose that $v'_\infty := \lim_{n \rightarrow \infty} v'_n \in \partial F_S(x)$ exists. Then, by Proposition 3.13,

$$v_\infty \in F_\Omega(v'_\infty) \subset F_\Omega(\partial F_S(x)). \quad \square$$

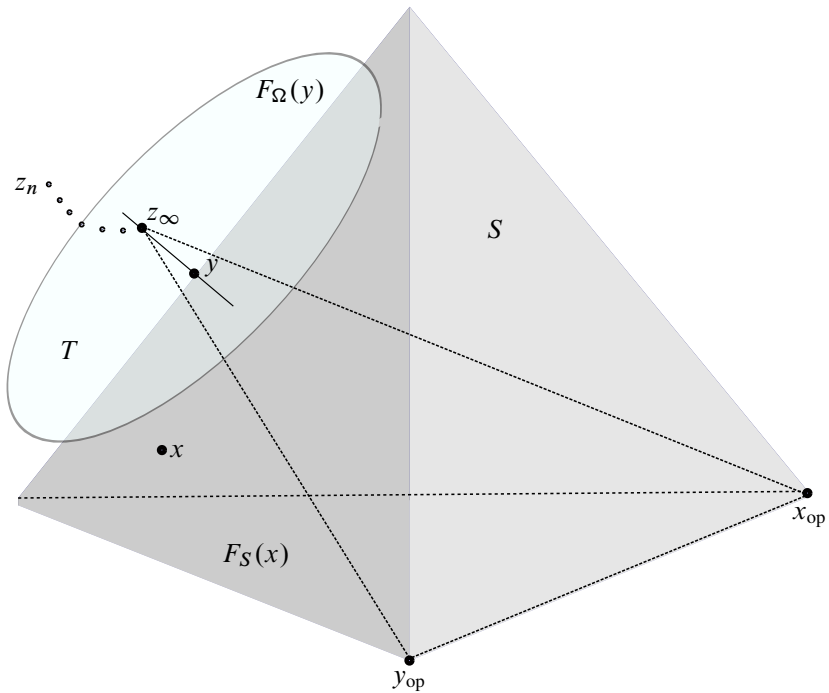


Figure 2: Diagram illustrating the proof by contradiction that $\bar{\mathcal{C}} \cap F_\Omega(\partial F_S(x))$ is open in $\bar{\mathcal{C}} \cap \partial F_\Omega(x)$. The sequence of points $z_n \notin F_\Omega(\partial F_S(x))$ but converge to $z_\infty \in F_\Omega(y)$ where $y \in \partial F_S(x)$.

Proving that $\bar{\mathcal{C}} \cap F_\Omega(\partial F_S(x))$ is open in $\bar{\mathcal{C}} \cap \partial F_\Omega(x)$ is much more involved (see Figure 2). Assume for a contradiction that this is false. Then there exist $y \in \partial F_S(x)$, $z_\infty \in F_\Omega(y)$ and a sequence

$$z_n \in \bar{\mathcal{C}} \cap \partial F_\Omega(x) \setminus F_\Omega(\partial F_S(x))$$

such that $\lim_{n \rightarrow \infty} z_n = z_\infty$.

Then let $y_{op} \in \partial F_S(x)$ be a point opposite to y in $F_S(x)$ and let $x_{op} \in \partial S$ be a point opposite to x in S . For each n , define

$$T_n := \text{rel-int}(\text{ConvHull}_\Omega\{z_n, y_{op}, x_{op}\}).$$

Then define

$$T := \text{rel-int}(\text{ConvHull}_\Omega\{z_\infty, y_{op}, x_{op}\}).$$

Lemma 9.8 *T is a properly embedded simplex in C.*

Proof By construction,

$$\text{rel-int}(\text{ConvHull}_\Omega\{y, y_{\text{op}}, x_{\text{op}}\})$$

is a properly embedded simplex in S and hence in Ω (see [Corollary 7.7](#)). Then T is also a properly embedded simplex in Ω by [Lemma 3.18](#). \square

Lemma 9.9 For n sufficiently large, T_n is a properly embedded simplex in \mathcal{C} .

Proof By construction, $[y_{\text{op}}, x_{\text{op}}] \subset \partial S \subset \partial\Omega$ and

$$[z_n, y_{\text{op}}] \subset \overline{F_\Omega(x)} \subset \partial\Omega.$$

Since $z_n \in \partial F_\Omega(x)$, [Corollary 7.7](#) implies that $[z_n, x_{\text{op}}] \subset \partial\Omega$. Finally, T_n converges to T and so, for n large enough, T_n intersects Ω . Thus, for n sufficiently large, T_n is a properly embedded simplex in Ω . \square

Then, by passing to a subsequence, we can suppose that $T_n \subset \mathcal{C}$ is a properly embedded simplex of dimension at least two for all n . Since S is coarsely complete, for each n there exists $\tilde{S}_n \in \mathcal{S}$ such that

$$T_n \subset \mathcal{N}_\Omega(\tilde{S}_n; D_0).$$

Since the sequence T_n converges to T , there exists some compact subset $K \subset \Omega$ such that $\tilde{S}_n \cap K \neq \emptyset$ for all n . Thus, by [Proposition 8.1](#) and passing to a subsequence, we can suppose that

$$\tilde{S} := \tilde{S}_n$$

for all $n \geq 0$. Then

$$T \cup \bigcup_{n \geq 0} T_n \subset \mathcal{N}_\Omega(\tilde{S}; D_0).$$

By [Proposition 3.13](#), there exist $\tilde{x}, \tilde{y}_{\text{op}}, \tilde{y}, \tilde{z}_n, \tilde{x}_{\text{op}} \in \partial\tilde{S}$ such that

$$\tilde{x} \in F_\Omega(x), \tilde{y}_{\text{op}} \in F_\Omega(y_{\text{op}}), \tilde{y} \in F_\Omega(y) = F_\Omega(z_\infty), \tilde{z}_n \in F_\Omega(z_n), \tilde{x}_{\text{op}} \in F_\Omega(x_{\text{op}}).$$

Lemma 9.10 $H_{F_\Omega(x)}^{\text{Haus}}(F_S(x), F_{\tilde{S}}(\tilde{x})) < +\infty$.

Proof Since

$$F_\Omega(y_{\text{op}}) = F_\Omega(\tilde{y}_{\text{op}}) \subset \partial F_\Omega(x),$$

our minimality assumption — see (5) — implies that

$$R_1 := H_{F_\Omega(y_{\text{op}})}^{\text{Haus}}(F_S(y_{\text{op}}), F_{\tilde{S}}(\tilde{y}_{\text{op}})) \leq H_{F_\Omega(y_{\text{op}})}^{\text{Haus}}(F_S(y_{\text{op}}), \bar{C} \cap F_\Omega(y_{\text{op}})) + H_{F_\Omega(\tilde{y}_{\text{op}})}^{\text{Haus}}(\bar{C} \cap F_\Omega(\tilde{y}_{\text{op}}), F_{\tilde{S}}(\tilde{y}_{\text{op}})) < +\infty.$$

Likewise,

$$R_2 := H_{F_\Omega(y)}^{\text{Haus}}(F_S(y), F_{\tilde{S}}(\tilde{y})) < +\infty.$$

By Lemma 9.3, for every $p \in F_S(x)$, there exist $p_1 \in F_S(y)$ and $p_2 \in F_S(y_{\text{op}})$ such that $p \in (p_1, p_2)$. Then there exist $\tilde{p}_1 \in F_{\tilde{S}}(\tilde{y})$ and $\tilde{p}_2 \in F_{\tilde{S}}(\tilde{y}_{\text{op}})$ such that

$$H_{F_\Omega(y)}(p_1, \tilde{p}_1), H_{F_\Omega(y_{\text{op}})}(p_2, \tilde{p}_2) \leq \max\{R_1, R_2\}.$$

Then, by Proposition 3.14,

$$H_{F_\Omega(x)}(p, F_{\tilde{S}}(\tilde{x})) \leq H_{F_S(x)}(p, (\tilde{p}_1, \tilde{p}_2)) \leq H_{F_S(x)}^{\text{Haus}}((p_1, p_2), (\tilde{p}_1, \tilde{p}_2)) \leq \max\{H_{F_\Omega(y)}(p_1, \tilde{p}_1), H_{F_\Omega(y_{\text{op}})}(p_2, \tilde{p}_2)\} \leq \max\{R_1, R_2\}.$$

The same argument shows that, if $q \in F_{\tilde{S}}(\tilde{x})$, then

$$H_{F_\Omega(x)}(q, F_S(x)) \leq \max\{R_1, R_2\}.$$

So

$$H_{F_\Omega(x)}^{\text{Haus}}(F_S(x), F_{\tilde{S}}(\tilde{x})) \leq \max\{R_1, R_2\}. \quad \square$$

Since $\tilde{z}_n \in \partial F_{\tilde{S}}(\tilde{x})$, Lemma 9.10 and Proposition 3.13 imply that there exists $a_n \in \partial F_S(x)$ with $a_n \in F_\Omega(\tilde{z}_n) = F_\Omega(z_n)$. So $z_n \in F_\Omega(a_n) \subset F_\Omega(\partial F_S(x))$, which contradicts our assumption that

$$z_n \in \bar{C} \cap \partial F_\Omega(x) \setminus F_\Omega(\partial F_S(x)).$$

Hence, $\bar{C} \cap F_\Omega(\partial F_S(x))$ is open in $\bar{C} \cap \partial F_\Omega(x)$.

9.3 Using the group action

In this subsection we use the action of $\text{Stab}_\Lambda(F_S(x))$ to upgrade Proposition 9.6. We begin with the following observation:

Observation 9.11 $\text{Stab}_\Lambda(F_S(x))$ acts cocompactly on $F_S(x)$ and $\text{Stab}_\Lambda(F_S(x)) \leq \text{Stab}_\Lambda(F_\Omega(x))$.

Proof The first assertion follows from [Proposition 8.1](#) and [Observation 5.2](#).

For the second, if $g \in \text{Stab}_\Delta(F_S(x))$, then

$$F_\Omega(x) \cap gF_\Omega(x) \supset F_S(x) \cap gF_S(x) = F_S(x) \neq \emptyset.$$

Hence, $gF_\Omega(x) = F_\Omega(x)$. □

Proposition 9.12 $\bar{C} \cap F_\Omega(\partial F_S(x)) = \bar{C} \cap \partial F_\Omega(x)$.

Proof By [Observation 5.4](#), $\bar{C} \cap F_\Omega(\partial F_S(x)) \subset \bar{C} \cap \partial F_\Omega(x)$. For the other inclusion, fix $z \in \bar{C} \cap \partial F_\Omega(x)$.

Case 1 (there exists $y \in \partial F_S(x)$ such that $[y, z] \subset \partial F_\Omega(x)$) Then y and z are in the same connected component of $\bar{C} \cap \partial F_\Omega(x)$. So [Proposition 9.6](#) implies that

$$z \in \bar{C} \cap F_\Omega(\partial F_S(x)).$$

Case 2 ($(y, z) \subset F_\Omega(x)$ for every $y \in \partial F_S(x)$) Using [Observation 9.11](#), there exists an unbounded sequence $g_n \in \text{Stab}_\Delta(F_S(x))$. By passing to a subsequence, we can suppose that $y_1 := \lim_{n \rightarrow \infty} g_n(x)$ and $y_2 := \lim_{n \rightarrow \infty} g_n^{-1}(x)$ both exist. Then $y_1, y_2 \in \partial F_S(x)$. [Observation 9.11](#) also implies that

$$\{g_n z : n \geq 0\} \subset \bar{C} \cap \partial F_\Omega(x).$$

We claim that this set intersects $F_\Omega(\partial F_S(x))$.

Let $V := \text{Span } F_\Omega(x)$. By passing to a subsequence, we can assume that $g_n|_V$ converges in $\mathbb{P}(\text{End}(V))$ to some $T \in \mathbb{P}(\text{End}(V))$. Then

$$T(w) = \lim_{n \rightarrow \infty} g_n(w)$$

for all $w \in \mathbb{P}(V) \setminus \mathbb{P}(\ker T)$. By [Proposition 3.16](#),

$$\text{image}(T) \subset \text{Span } F_{F_\Omega(x)}(y_1) = \text{Span } F_\Omega(y_1),$$

$y_2 \in \mathbb{P}(\ker T)$ and $\mathbb{P}(\ker T) \cap F_\Omega(x) = \emptyset$.

We claim that $z \notin \mathbb{P}(\ker T)$. Otherwise, $[y_2, z] \subset \mathbb{P}(\ker T)$ and so $[y_2, z] \subset \partial F_\Omega(x)$. This contradicts our assumptions for Case 2. Then

$$T(z) = \lim_{n \rightarrow \infty} g_n(z) \in \bar{C} \cap \mathbb{P}(\text{Span } F_\Omega(y_1)) = \bar{C} \cap \overline{F_\Omega(y_1)} \subset \bar{C} \cap \overline{F_\Omega(\partial F_S(x))}.$$

Then [Proposition 9.6](#) implies that

$$g_n z \in \bar{C} \cap F_\Omega(\partial F_S(x))$$

for n sufficiently large. Then

$$z \in g_n^{-1}(\bar{C} \cap F_\Omega(\partial F_S(x))) = \bar{C} \cap F_\Omega(\partial F_S(x))$$

and the proof is complete in this case. □

9.4 Finishing the proof of Theorem 9.1

Since $H_{F_\Omega(x)}^{\text{Haus}}(\bar{C} \cap F_\Omega(x), F_S(x)) = \infty$, for every $n \geq 1$, there exists $w_n \in \bar{C} \cap F_\Omega(x)$ with

$$H_{F_\Omega(x)}(w_n, F_S(x)) \geq n.$$

Then, for each n , pick $x_n \in F_S(x)$ such that

$$H_{F_\Omega(x)}(w_n, x_n) = H_{F_\Omega(x)}(w_n, F_S(x)).$$

Using [Observation 9.11](#), translating by elements in $\text{Stab}_\Lambda(F_S(x))$ and passing to a subsequence, we can assume that

$$x_\infty := \lim_{n \rightarrow \infty} x_n \text{ exists and } x_\infty \in F_S(x).$$

By passing to a further subsequence, we can also assume that

$$w_\infty := \lim_{n \rightarrow \infty} w_n \in \bar{C} \cap \overline{F_\Omega(x)}$$

exists. In fact, $w_\infty \in \bar{C} \cap \partial F_\Omega(x)$ since

$$\lim_{n \rightarrow \infty} H_{F_\Omega(x)}(w_n, x_\infty) \geq \lim_{n \rightarrow \infty} (H_{F_\Omega(x)}(w_n, x_n) - H_{F_\Omega(x)}(x_n, x_\infty)) = \infty.$$

So [Proposition 9.12](#) implies that $w_\infty \in F_\Omega(y)$ for some $y \in \partial F_S(x)$.

Next fix $p \in [x_\infty, w_\infty) \subset F_\Omega(x)$. Then, by [Proposition 3.14](#),

$$\begin{aligned} H_{F_\Omega(x)}(p, F_S(x)) &\leq H_{F_\Omega(x)}(p, [x_\infty, y)) \leq H_{F_\Omega(x)}^{\text{Haus}}([x_\infty, w_\infty), [x_\infty, y)) \\ &\leq H_{F_\Omega(y)}(w_\infty, y). \end{aligned}$$

Then fix a sequence $p_n \in [x_n, w_n]$ with $\lim_{n \rightarrow \infty} p_n = p$. Then

$$\begin{aligned} H_{F_\Omega(x)}(p_n, x_n) &= H_{F_\Omega(x)}(w_n, x_n) - H_{F_\Omega(x)}(w_n, p_n) \\ &= H_{F_\Omega(x)}(w_n, F_S(x)) - H_{F_\Omega(x)}(w_n, p_n) \\ &\leq H_{F_\Omega(x)}(w_n, p_n) + H_{F_\Omega(x)}(p_n, F_S(x)) - H_{F_\Omega(x)}(w_n, p_n) \\ &= H_{F_\Omega(x)}(p_n, F_S(x)). \end{aligned}$$

Taking $n \rightarrow \infty$ yields

$$H_{F_{\Omega}(x)}(p, x_{\infty}) \leq H_{F_{\Omega}(x)}(p, F_S(x)) \leq H_{F_{\Omega}(y)}(w_{\infty}, y).$$

Since $p \in [x_{\infty}, w_{\infty})$ is arbitrary, we have

$$\infty = \lim_{\substack{q \in [x_{\infty}, w_{\infty}) \\ q \rightarrow w_{\infty}}} H_{F_{\Omega}(x)}(q, x_{\infty}) \leq H_{F_{\Omega}(y)}(w_{\infty}, y) < \infty$$

and we have a contradiction. This finishes the proof of [Theorem 9.1](#).

10 Proof of [Theorem 1.17](#)

For the rest of the section let $(\Omega, \mathcal{C}, \Lambda)$ be a naive convex cocompact triple with coarsely isolated simplices. We will describe a procedure for producing a family of strongly isolated, coarsely complete and Λ -invariant maximal properly embedded simplices in \mathcal{C} of dimension at least two.

Let \mathcal{S}_{\max} denote the family of all maximal properly embedded simplices in \mathcal{C} of dimension at least two. Then let $X \subset \mathcal{S}_{\max}$ be the set of simplices $S \in \mathcal{S}_{\max}$ where there exists some $S' \in \mathcal{S}_{\max}$ with $\dim S < \dim S'$ and

$$\sup_{p \in S} H_{\Omega}(p, S') < +\infty.$$

Next let $\widehat{\mathcal{S}}_{\max} := \mathcal{S}_{\max} \setminus X$. That is, $\widehat{\mathcal{S}}_{\max}$ consists of the maximal properly embedded simplices of dimension at least two that are not contained in a tubular neighborhood of a properly embedded simplex with strictly larger dimension.

For each simplex $S \in \widehat{\mathcal{S}}_{\max}$ we construct a new simplex $\Phi(S)$ as follows. Let v_1, \dots, v_p be the vertices of S . Then define (see [Remark 10.2](#) to see why this is well defined)

$$w_j := \text{CoM}_{F_{\Omega}(v_j)}(\bar{\mathcal{C}} \cap F_{\Omega}(v_j)) \quad \text{for } 1 \leq j \leq p$$

and

$$\Phi(S) := \Omega \cap \mathbb{P}(\text{Span}\{w_1, \dots, w_p\}).$$

Finally, define

$$\mathcal{S}_{\text{core}} := \{\Phi(S) : S \in \widehat{\mathcal{S}}_{\max}\}.$$

[Theorem 1.17](#) will be a consequence of the following theorem:

Theorem 10.1 *$\mathcal{S}_{\text{core}}$ is a well-defined, strongly isolated, coarsely complete and Λ -invariant family of maximal properly embedded simplices in \mathcal{C} of dimension at least two.*

Remark 10.2 (1) To show that $\mathcal{S}_{\text{core}}$ is well defined we need to show that $\bar{\mathcal{C}} \cap F_{\Omega}(v)$ is a compact subset of $F_{\Omega}(v)$ for every simplex $S \in \hat{\mathcal{S}}_{\text{max}}$ and vertex v of S .

(2) The map Φ selects from each family of parallel simplices a canonical “core” simplex, thus motivating the notation $\mathcal{S}_{\text{core}}$.

The rest of this section is devoted to the proof of [Theorem 10.1](#). Let \mathcal{S}_0 be an isolated, coarsely complete and Λ -invariant family of maximal properly embedded simplices in \mathcal{C} of dimension at least two. By [Theorem 9.1](#) and passing to a subfamily of \mathcal{S}_0 , we can assume that there exists $R > 0$ such that, if $S \in \mathcal{S}_0$ and $x \in \partial S$, then

$$(6) \quad H_{F_{\Omega}(x)}^{\text{Haus}}(\bar{\mathcal{C}} \cap F_{\Omega}(x), F_S(x)) = H_{F_{\Omega}(x)}^{\text{Haus}}(\bar{\mathcal{C}} \cap F_{\Omega}(x), \bar{S} \cap F_{\Omega}(x)) \leq R.$$

By [Lemma 9.4](#) and passing to another subfamily of \mathcal{S}_0 , we can also assume that, if $S_1, S_2 \in \mathcal{S}_0$ and $\dim S_1 < \dim S_2$, then

$$(7) \quad \infty = \sup_{p \in S_1} H_{\Omega}(p, S_2).$$

The next two lemmas show that \mathcal{S}_0 and $\hat{\mathcal{S}}_{\text{max}}$ are “coarsely the same”.

Lemma 10.3 $\mathcal{S}_0 \subset \hat{\mathcal{S}}_{\text{max}}.$

Proof Fix $S_0 \in \mathcal{S}_0$ and suppose $S_1 \in \mathcal{S}_{\text{max}}$ with $\dim S_1 > \dim S_0$. Since \mathcal{S}_0 is coarsely complete there exists $S_2 \in \mathcal{S}_0$ and $r > 0$ such that

$$S_1 \subset \mathcal{N}_{\Omega}(S_2; r).$$

Then [Observation 5.3](#) implies that $\dim S_1 \leq \dim S_2$. So, by (7),

$$\sup_{p \in S_0} H_{\Omega}(p, S_1) \geq -r + \sup_{p \in S_0} H_{\Omega}(p, S_2) = \infty.$$

Since $S_1 \in \mathcal{S}_{\text{max}}$ was an arbitrary simplex with $\dim S_1 > \dim S_0$, we see that $\mathcal{S}_0 \in \mathcal{S}_{\text{max}} \setminus X = \hat{\mathcal{S}}_{\text{max}}.$ □

Lemma 10.4 *If $S \in \hat{\mathcal{S}}_{\text{max}}$ has vertices v_1, \dots, v_p , then there exists $S_0 \in \mathcal{S}_0$ with $\dim S_0 = \dim S$ and a labeling w_1, \dots, w_p of the vertices of S_0 such that*

$$F_{\Omega}(v_j) = F_{\Omega}(w_j)$$

for all $1 \leq j \leq p$. Moreover,

$$H_{\Omega}^{\text{Haus}}(S, S_0) \leq R.$$

Proof Since \mathcal{S}_0 is coarsely complete, there exists $S_0 \in \mathcal{S}_0$ and $r > 0$ such that

$$S \subset \mathcal{N}_\Omega(S_0; r).$$

Then [Observation 5.3](#) implies that $\dim S \leq \dim S_0$. Then, since $S \in \widehat{\mathcal{S}}_{\max}$, we must have $\dim S = \dim S_0$.

By [Proposition 3.13](#), there exist $w_1, \dots, w_p \in \partial S_0$ such that

$$F_\Omega(v_j) = F_\Omega(w_j)$$

for all $1 \leq j \leq p$. Then, by [Lemma 3.18](#),

$$S'_0 := \mathbb{P}(\text{Span}\{w_1, \dots, w_p\}) \cap \Omega \subset S_0$$

is a properly embedded simplex with vertices w_1, \dots, w_p . Then, since

$$\dim S'_0 = \dim S = \dim S_0,$$

we must have $S_0 = S'_0$. This proves the first assertion in the lemma.

Now the “moreover” part is a consequence of [\(6\)](#) and [Lemma 3.18](#). □

Lemma 10.5 *If $S \in \widehat{\mathcal{S}}_{\max}$, then $\Phi(S)$ is a well-defined maximal properly embedded simplex in \mathcal{C} and*

$$(8) \quad H_\Omega^{\text{Haus}}(S, \Phi(S)) \leq 2R.$$

Moreover,

$$\mathcal{S}_{\text{core}} = \{\Phi(S) : S \in \mathcal{S}_0\}.$$

Proof Fix $S \in \widehat{\mathcal{S}}_{\max}$ and let v_1, \dots, v_p be the vertices of S . By [Lemma 10.4](#), there exist $S_0 \in \mathcal{S}_0$ and a labeling w_1, \dots, w_p of the vertices of S_0 such that $F_\Omega(w_j) = F_\Omega(v_j)$ for all $1 \leq j \leq p$. Then, by [\(6\)](#),

$$\begin{aligned} H_{F_\Omega(v_j)}^{\text{Haus}}(\bar{\mathcal{C}} \cap F_\Omega(v_j), \{v_j\}) &\leq H_{F_\Omega(w_j)}^{\text{Haus}}(\bar{\mathcal{C}} \cap F_\Omega(w_j), \{w_j\}) + H_{F_\Omega(v_j)}(w_j, v_j) \\ &\leq 2H_{F_\Omega(w_j)}^{\text{Haus}}(\bar{\mathcal{C}} \cap F_\Omega(w_j), \{w_j\}) \leq 2R. \end{aligned}$$

So $\bar{\mathcal{C}} \cap F_\Omega(v_j)$ is a compact subset of $F_\Omega(v_j)$. Hence,

$$\text{CoM}_{F_\Omega(v_j)}(\bar{\mathcal{C}} \cap F_\Omega(v_j))$$

is well defined. Thus, $\Phi(S)$ is well defined.

Then, Lemma 3.18 implies that $\Phi(S)$ is a properly embedded simplex and

$$H_{\Omega}^{\text{Haus}}(S, \Phi(S)) \leq 2R.$$

Further $\Phi(S)$ is maximal since $S \in \widehat{\mathcal{S}}_{\max}$.

Finally, since $F_{\Omega}(w_j) = F_{\Omega}(v_j)$ for all $1 \leq j \leq p$, we have $\Phi(S) = \Phi(S_0)$. Since $S \in \widehat{\mathcal{S}}_{\max}$ was arbitrary, this implies the “moreover” part of the lemma. \square

Lemma 10.6 *If $S_1, S_2 \in \mathcal{S}_{\text{core}}$ and $H_{\Omega}^{\text{Haus}}(S_1, S_2) < \infty$, then $S_1 = S_2$.*

Proof Arguing as in the proof of Lemma 10.4, there exist a labeling v_1, \dots, v_p of the vertices of S_1 and a labeling w_1, \dots, w_p of the vertices of S_2 such that

$$F_{\Omega}(v_j) = F_{\Omega}(w_j)$$

for all $1 \leq j \leq p$. Then, by the definition of Φ ,

$$w_j = \text{CoM}_{F_{\Omega}(w_j)}(\bar{C} \cap F_{\Omega}(w_j)) = \text{CoM}_{F_{\Omega}(v_j)}(\bar{C} \cap F_{\Omega}(v_j)) = v_j$$

for all $1 \leq j \leq p$. \square

Lemma 10.7 *$\mathcal{S}_{\text{core}}$ is coarsely complete and Λ -invariant.*

Proof By construction, $\mathcal{S}_{\text{core}}$ is Λ -invariant.

Since S_0 is coarsely complete, there exists $D_0 > 0$ such that, if S is a properly embedded simplex in \mathcal{C} of dimension at least two, then there exists $S' \in S_0$ such that

$$S \subset \mathcal{N}_{\Omega}(S'; D_0).$$

Then (8) implies that

$$S \subset \mathcal{N}_{\Omega}(\Phi(S'); D_0 + 2R).$$

So $\mathcal{S}_{\text{core}}$ is coarsely complete. \square

We complete the proof of the theorem by establishing the following lemma:

Lemma 10.8 *$\mathcal{S}_{\text{core}}$ is strongly isolated: for any $r > 0$ there exists $D_2(r) > 0$ such that, if $S_1, S_2 \in \mathcal{S}_{\text{core}}$ are distinct, then*

$$\text{diam}_{\Omega}(\mathcal{N}_{\Omega}(S_1; r) \cap \mathcal{N}_{\Omega}(S_2; r)) \leq D_2(r).$$

Proof Fix $r > 0$. Suppose for a contradiction that such a $D_2(r) > 0$ does not exist. Then, by [Lemma 10.5](#), for every $n \geq 0$, there exist $S_{1,n}, S_{2,n} \in \mathcal{S}_0$ such that $\Phi(S_{1,n}) \neq \Phi(S_{2,n})$ and

$$\text{diam}_\Omega(\mathcal{N}_\Omega(\Phi(S_{1,n}); r) \cap \mathcal{N}_\Omega(\Phi(S_{2,n}); r)) > n.$$

Then, by [\(8\)](#),

$$\text{diam}_\Omega(\mathcal{N}_\Omega(S_{1,n}; r_0) \cap \mathcal{N}_\Omega(S_{2,n}; r_0)) > n,$$

where $r_0 := r + 2R$.

Pick

$$a_n, b_n \in \mathcal{N}_\Omega(S_{1,n}; r_0) \cap \mathcal{N}_\Omega(S_{2,n}; r_0)$$

with $H_\Omega(a_n, b_n) \geq n$. Let $m_n \in [a_n, b_n]$ be such that

$$(9) \quad H_\Omega(a_n, m_n) \geq \frac{1}{2}n \quad \text{and} \quad H_\Omega(b_n, m_n) \geq \frac{1}{2}n.$$

For each n , we can find $\gamma_n \in \Lambda$ such that

$$\{\gamma_n m_n : n \geq 0\}$$

is relatively compact in Ω . So, by passing to a subsequence, we can suppose that $m := \lim_{n \rightarrow \infty} \gamma_n m_n$ exists in \mathcal{C} . Passing to another subsequence, we can assume that $a := \lim_{n \rightarrow \infty} \gamma_n a_n$ and $b := \lim_{n \rightarrow \infty} \gamma_n b_n$ exist in $\bar{\mathcal{C}}$. Then [\(9\)](#) implies that $a, b \in \partial_i \mathcal{C}$ and so (a, b) is a properly embedded line in \mathcal{C} . Finally, using [Proposition 8.1](#) and passing to another subsequence, we can suppose that

$$S_1 := \gamma_n S_{n,1} = \gamma_m S_{m,1} \quad \text{and} \quad S_2 := \gamma_n S_{n,2} = \gamma_m S_{m,2}$$

for all $n, m \geq 0$. Then, by construction, $S_1, S_2 \in \mathcal{S}_0$ and $\Phi(S_1) \neq \Phi(S_2)$.

Notice that

$$(a, b) \subset \mathcal{N}_\Omega(S_1; r_0 + 1) \cap \mathcal{N}_\Omega(S_2; r_0 + 1).$$

[Proposition 3.13](#) then implies that \bar{S}_1 and \bar{S}_2 both intersect $F_\Omega(a)$. Then [\(6\)](#) implies that

$$\begin{aligned} & H_{F_\Omega(a)}^{\text{Haus}}(\bar{S}_1 \cap F_\Omega(a), \bar{S}_2 \cap F_\Omega(a)) \\ & \leq H_{F_\Omega(a)}^{\text{Haus}}(\bar{S}_1 \cap F_\Omega(a), \bar{\mathcal{C}} \cap F_\Omega(a)) + H_{F_\Omega(a)}^{\text{Haus}}(\bar{\mathcal{C}} \cap F_\Omega(a), \bar{S}_2 \cap F_\Omega(a)) \\ & \leq 2R. \end{aligned}$$

The same reasoning shows that

$$H_{F_\Omega(b)}^{\text{Haus}}(\bar{S}_1 \cap F_\Omega(b), \bar{S}_2 \cap F_\Omega(b)) \leq 2R.$$

Now we claim that

$$H_{\Omega}^{\text{Haus}}(S_1, S_2) \leq 2R.$$

By symmetry, it is enough to fix $p \in S_1$ and show that

$$H_{\Omega}(p, S_2) \leq 2R.$$

Fix $a' \in \bar{S}_1 \cap F_{\Omega}(a)$ and $b' \in \bar{S}_1 \cap F_{\Omega}(b)$. Since $(a, b) \subset \Omega$, [Observation 3.12\(4\)](#) implies that $(a', b') \subset \Omega$.

Then, by [Lemma 9.3](#), there exist $a_1 \in F_{S_1}(a') \subset \bar{S}_1 \cap F_{\Omega}(a)$ and $b_1 \in F_{S_1}(b') \subset \bar{S}_2 \cap F_{\Omega}(b)$ such that $p \in (a_1, b_1)$. Then there exist $a_2 \in \bar{S}_2 \cap F_{\Omega}(a)$ and $b_2 \in \bar{S}_2 \cap F_{\Omega}(b)$ with

$$\max\{H_{F_{\Omega}(a)}(a_1, a_2), H_{F_{\Omega}(b)}(b_1, b_2)\} \leq 2R.$$

Then, by [Proposition 3.14](#),

$$\begin{aligned} H_{\Omega}(p, S_2) &\leq H_{\Omega}(p, (a_2, b_2)) \leq H_{\Omega}^{\text{Haus}}((a_1, b_1), (a_2, b_2)) \\ &\leq \max\{H_{F_{\Omega}(a)}(a_1, a_2), H_{F_{\Omega}(b)}(b_1, b_2)\} \leq 2R. \end{aligned}$$

So

$$H_{\Omega}^{\text{Haus}}(S_1, S_2) \leq 2R.$$

By [Lemma 10.5](#), $H_{\Omega}^{\text{Haus}}(\Phi(S_1), \Phi(S_2)) \leq 6R$. Then, by [Lemma 10.6](#), $\Phi(S_1) = \Phi(S_2)$ and we have a contradiction.

Thus, there exists $D_2(r) > 0$ such that, if $S_1, S_2 \in \mathcal{S}_{\text{core}}$ are distinct, then

$$\text{diam}_{\Omega}(\mathcal{N}_{\Omega}(S_1; r) \cap \mathcal{N}_{\Omega}(S_2; r)) \leq D_2(r). \quad \square$$

11 Half triangles in the ideal boundary

In this section we verify property (6) of [Theorem 1.19](#).

Definition 11.1 Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain. A list of three points a, b, c form a *half triangle in Ω* if $[a, b], [b, c] \subset \partial\Omega$, $(a, c) \subset \Omega$, and $a \neq c$.

Theorem 11.2 Suppose that $(\Omega, \mathcal{C}, \Lambda)$ is a naive convex cocompact triple with coarsely isolated simplices. Let \mathcal{S} be a strongly isolated, coarsely complete and Λ -invariant family of maximal properly embedded simplices in \mathcal{C} of dimension at least two. If $a, b, c \in \partial_i \mathcal{C}$ form a half triangle, then there exists $S \in \mathcal{S}$ such that $a, b, c \in F_{\Omega}(\partial S)$.

As a corollary we observe that simplices in \mathcal{S} cannot have “half triangles sticking out”.

Corollary 11.3 *Suppose that $(\Omega, \mathcal{C}, \Lambda)$ is a naive convex cocompact triple with coarsely isolated simplices. Let \mathcal{S} be a strongly isolated, coarsely complete and Λ -invariant family of maximal properly embedded simplices in \mathcal{C} of dimension at least two. If $S \in \mathcal{S}$, $a, c \in \partial S$, $b \in \partial_i \mathcal{C}$ and a, b, c form a half triangle in Ω , then $b \in F_\Omega(\partial S)$.*

Proof of Corollary 11.3 assuming Theorem 11.2 By Theorem 11.2, there exists $S' \in \mathcal{S}$ such that $a, b, c \in F_\Omega(\partial S')$. So there exist $a', b', c' \in \partial S'$ such that $a \in F_\Omega(a')$, $b \in F_\Omega(b')$ and $c \in F_\Omega(c')$. Define

$$M := \max\{H_{F_\Omega(a)}(a, a'), H_{F_\Omega(c)}(c, c')\}.$$

By Proposition 3.14,

$$H_\Omega^{\text{Haus}}((a, c), (a', c')) \leq M$$

and so

$$(a, c) \subset S \cap \mathcal{N}_\Omega(S'; M).$$

Then

$$\infty = \text{diam}_\Omega(\mathcal{N}_\Omega(S; M) \cap \mathcal{N}_\Omega(S'; M)).$$

Since \mathcal{S} is strongly isolated, $S = S'$. So $b' \in \partial S' = \partial S$ and hence $b \in F_\Omega(b') \subset F_\Omega(\partial S)$. □

We begin the proof of Theorem 11.2 with a lemma.

Lemma 11.4 *Suppose $(\Omega, \mathcal{C}, \Lambda)$ is a naive convex cocompact triple. Assume $a, b, c \in \partial_i \mathcal{C}$ form a half triangle and $V = \text{Span}\{a, b, c\}$. For any $r > 0$ and $\epsilon > 0$, there exists a neighborhood U of b in $\mathbb{P}(V)$ such that, if $x \in U \cap \mathcal{C}$, then there exists a properly embedded simplex $S = S(x) \subset \mathcal{C}$ of dimension at least two such that*

$$(10) \quad B_\Omega(x; r) \cap \mathbb{P}(V) \subset \mathcal{N}_\Omega(S; \epsilon).$$

Proof Fix $r > 0$ and $\epsilon > 0$. Suppose for a contradiction that such a neighborhood U does not exist. Then we can find a sequence $p_n \in \mathcal{C} \cap \mathbb{P}(V)$ such that $\lim_{n \rightarrow \infty} p_n = b$ and each p_n does not satisfy (10) for any properly embedded simplex $S \subset \mathcal{C}$ of dimension at least two.

After passing to a subsequence, we can find $\gamma_n \in \Lambda$ such that $\gamma_n p_n \rightarrow p \in \mathcal{C}$. Passing to a further subsequence, we can suppose that $\gamma_n a \rightarrow a_\infty$, $\gamma_n b \rightarrow b_\infty$ and $\gamma_n c \rightarrow c_\infty$. Then $[a_\infty, b_\infty], [b_\infty, c_\infty] \subset \partial_i \mathcal{C}$ and, by the definition of the Hilbert metric,

$$\infty = \lim_{n \rightarrow \infty} H_\Omega(p_n, (a, c)) = \lim_{n \rightarrow \infty} H_\Omega(\gamma_n p_n, (\gamma_n a, \gamma_n c)).$$

So $[a_\infty, c_\infty] \subset \partial_i \mathcal{C}$. Thus, a_∞, b_∞ and c_∞ are the vertices of a properly embedded simplex $S \subset \mathcal{C}$. However, for n sufficiently large, we have

$$B_\Omega(\gamma_n p_n; r) \cap \gamma_n \mathbb{P}(V) \subset \mathcal{N}_\Omega(S; \epsilon)$$

and so

$$B_\Omega(p_n; r) \cap \mathbb{P}(V) \subset \mathcal{N}_\Omega(\gamma_n^{-1} S; \epsilon).$$

Hence, we have a contradiction. □

Proof of Theorem 11.2 By Theorem 9.1, there exists an isolated, coarsely complete and Λ -invariant subfamily $S' \subset S$ where

$$(11) \quad D_0 := \sup_{S \in S'} \sup_{x \in \partial S} H_{F_\Omega(x)}^{\text{Haus}}(\bar{\mathcal{C}} \cap F_\Omega(x), F_S(x)) < +\infty.$$

Since S' is coarsely complete, there exists $D_1 > 0$ such that, if $S \subset \mathcal{C}$ is a properly embedded simplex of dimension at least two, then there exists $S' \in S'$ with

$$S \subset \mathcal{N}_\Omega(S'; D_1).$$

As S is strongly isolated, so is S' . Thus, there exists $D_2 > 0$ such that, if $S_1, S_2 \in S'$ and

$$\text{diam}_\Omega(\mathcal{N}_\Omega(S_1; 1 + D_1) \cap \mathcal{N}_\Omega(S_2; 1 + D_1)) \geq D_2,$$

then $S_1 = S_2$.

Define $V := \text{Span}\{a, b, c\}$. By Lemma 11.4, there exists a neighborhood U of b in $\mathbb{P}(V)$ such that, if $x \in U \cap \mathcal{C}$, then there exists a properly embedded simplex $S = S(x)$ in \mathcal{C} of dimension at least two with

$$B_\Omega(x; D_2) \cap \mathbb{P}(V) \subset \mathcal{N}_\Omega(S; 1).$$

Then, for each $x \in U \cap \mathcal{C}$, there exists some $S_x \in S'$ such that

$$B_\Omega(x; D_2) \cap \mathbb{P}(V) \subset \mathcal{N}_\Omega(S_x; 1 + D_1).$$

By shrinking U we can also assume that $U \cap \mathcal{C}$ is convex.

We claim that $S_x = S_y$ for all $x, y \in U \cap \mathcal{C}$. Since $U \cap \mathcal{C}$ is convex, it is enough to verify this when $H_\Omega(x, y) \leq \frac{1}{2}D_2$. In that case,

$$B_\Omega(y; \frac{1}{2}D_2) \cap \mathbb{P}(V) \subset B_\Omega(x; D_2) \cap \mathbb{P}(V) \subset \mathcal{N}_\Omega(S_x; 1 + D_1)$$

and so

$$B_\Omega(y; \frac{1}{2}D_2) \cap \mathbb{P}(V) \subset \mathcal{N}_\Omega(S_x; 1 + D_1) \cap \mathcal{N}_\Omega(S_y; 1 + D_1).$$

Since

$$\text{diam}_\Omega(\mathcal{N}_\Omega(S_x; 1 + D_1) \cap \mathcal{N}_\Omega(S_y; 1 + D_1)) \geq \text{diam}_\Omega(B_\Omega(y; \frac{1}{2}D_2) \cap \mathbb{P}(V)) = D_2,$$

we then have $S_x = S_y$.

Next let $S = S_x$ for some (and hence any) $x \in U \cap \mathcal{C}$. Then

$$U \cap \mathcal{C} \subset \mathcal{N}_\Omega(S; 1 + D_1).$$

Fix some $a_1 \in (a, b) \cap U$ and $c_1 \in (b, c) \cap U$. Then, by [Proposition 3.13](#), there exist $a'_1, b', c'_1 \in \partial S$ such that $a'_1 \in F_\Omega(a_1)$, $b' \in F_\Omega(b)$ and $c'_1 \in F_\Omega(c_1)$. So $b \in F_\Omega(b') \subset F_\Omega(\partial S)$.

We now show that $a \in F_\Omega(\partial S)$. We can find a sequence

$$q_n \in \partial_i \mathcal{C} \cap [a_1, a) \subset \partial_i \mathcal{C} \cap F_\Omega(a'_1)$$

such that $\lim_{n \rightarrow \infty} q_n = a$. Then, by [\(11\)](#), there exists $q'_n \in F_S(a'_1)$ with

$$H_\Omega(q_n, q'_n) \leq D_0.$$

Then, passing to a subsequence, $a' := \lim_{n \rightarrow \infty} q'_n$ exists in $\overline{F_S(a'_1)}$ and, by [Proposition 3.13](#), $a \in F_\Omega(a')$. Thus, $a \in F_\Omega(\partial S)$.

The same argument shows that $c \in F_\Omega(\partial S)$. □

12 Proof of Theorem 1.19

Suppose $(\Omega, \mathcal{C}, \Lambda)$ is a naive convex cocompact triple with coarsely isolated simplices and \mathcal{S} is a strongly isolated, coarsely complete and Λ -invariant family of maximal properly embedded simplices in \mathcal{C} of dimension at least two.

(1) and (2) This is [Proposition 8.1](#).

(3) Since \mathcal{S} is coarsely complete there exists $D_0 > 0$ such that, if S is a properly embedded simplex in \mathcal{C} of dimension at least two, then there exists $S' \in \mathcal{S}$ with

$$S \subset \mathcal{N}_\Omega(S'; D_0).$$

Applying [Theorem 1.23](#) to a maximal abelian subgroup which contains A shows that there exists a properly embedded simplex $S_0 \subset \mathcal{C}$ with $A \leq \text{Stab}_\Lambda(S_0)$. Since \mathcal{S} is strongly isolated, there exists a unique $S \in \mathcal{S}$ with

$$S_0 \subset \mathcal{N}_\Omega(S; D_0).$$

So, by uniqueness, $A \leq \text{Stab}_\Lambda(S)$.

(4) By [Theorem 9.1](#), there exists a coarsely complete subfamily $\mathcal{S}' \subset \mathcal{S}$ and a constant $D_1 > 0$ such that, if $S \in \mathcal{S}'$ and $x \in \partial S$, then

$$H_{F_\Omega(x)}^{\text{Haus}}(\bar{\mathcal{C}} \cap F_\Omega(x), F_S(x)) \leq D_1.$$

We claim that $\mathcal{S}' = \mathcal{S}$. Suppose that $S \in \mathcal{S}$. Since \mathcal{S}' is coarsely complete, there exist $S' \in \mathcal{S}'$ and $D'_0 > 0$ such that

$$S \subset \mathcal{N}_\Omega(S'; D'_0).$$

But $S' \in \mathcal{S}' \subset \mathcal{S}$ and

$$\text{diam}_\Omega(\mathcal{N}_\Omega(S'; D'_0) \cap \mathcal{N}_\Omega(S; D'_0)) \geq \text{diam}_\Omega(S) = \infty,$$

so $S = S' \in \mathcal{S}'$. Since $S \in \mathcal{S}$ was arbitrary, we see that $\mathcal{S}' = \mathcal{S}$.

(5) Suppose $S_1, S_2 \in \mathcal{S}$ and $\#(S_1 \cap S_2) > 1$. Then $S_1 \cap S_2$ contains a properly embedded line and hence

$$\text{diam}_\Omega(\mathcal{N}(S_1; r) \cap \mathcal{N}(S_2; r)) = \infty$$

for any $r > 0$. Thus, $S_1 = S_2$ since \mathcal{S} is strongly isolated.

Suppose $S_1, S_2 \in \mathcal{S}$ and $F_\Omega(\partial S_1) \cap F_\Omega(\partial S_2) \neq \emptyset$. Then there exist $s_1 \in \partial S_1$ and $s_2 \in \partial S_2$ with $F_\Omega(s_1) = F_\Omega(s_2)$. Fix $p_1 \in S_1$ and $p_2 \in S_2$. Then, by [Proposition 3.14](#),

$$H_\Omega^{\text{Haus}}([p_1, s_1], [p_2, s_2]) \leq \max\{H_\Omega(p_1, p_2), H_{F_\Omega(s_1)}(s_1, s_2)\}.$$

So, for any $r > \max\{H_\Omega(p_1, p_2), H_{F_\Omega(s_1)}(s_1, s_2)\}$,

$$\text{diam}_\Omega(\mathcal{N}_\Omega(S_1; r) \cap \mathcal{N}_\Omega(S_2; r)) = \infty.$$

Thus, $S_1 = S_2$ since \mathcal{S} is strongly isolated.

(6) This is [Theorem 11.2](#). □

13 Proof of Theorem 1.18

In this section we prove [Theorem 1.18](#), which we recall here.

Theorem 1.18 *Suppose $(\Omega, \mathcal{C}, \Lambda)$ is a naive convex cocompact triple with coarsely isolated simplices. Let \mathcal{S} be a strongly isolated, coarsely complete and Λ -invariant family of maximal properly embedded simplices in \mathcal{C} of dimension at least two. Then:*

- (1) (\mathcal{C}, H_Ω) is a relatively hyperbolic space with respect to \mathcal{S} .
- (2) Λ has finitely many orbits in \mathcal{S} and, if $\{S_1, \dots, S_m\}$ is a set of orbit representatives, then Λ is a relatively hyperbolic group with respect to

$$\{\text{Stab}_\Lambda(S_1), \dots, \text{Stab}_\Lambda(S_m)\}.$$

Further, each $\text{Stab}_\Lambda(S_i)$ is virtually abelian of rank at least two.

Notation 13.1 For the rest of the section, fix a naive convex cocompact triple $(\Omega, \mathcal{C}, \Lambda)$ with coarsely isolated simplices. Then fix a strongly isolated, coarsely complete and Λ -invariant family \mathcal{S} of maximal properly embedded simplices in \mathcal{C} of dimension at least two. By [Proposition 8.1](#), Λ has finitely many orbits in \mathcal{S} and, for each $S \in \mathcal{S}$, the group $\text{Stab}_\Lambda(S)$ is virtually abelian of rank at least two. Finally, fix orbit representatives S_1, \dots, S_m of the Λ action on \mathcal{S} .

By [Proposition 8.1](#) again, if $S \in \mathcal{S}$, then $\text{Stab}_\Lambda(S)$ acts cocompactly on S . Thus, by [Theorem 4.8](#), (\mathcal{C}, H_Ω) is relatively hyperbolic with respect to \mathcal{S} if and only if Λ is relatively hyperbolic relative to $\{\text{Stab}_\Lambda(S_1), \dots, \text{Stab}_\Lambda(S_m)\}$.

So it is enough to prove that (\mathcal{C}, H_Ω) is relatively hyperbolic with respect to \mathcal{S} . To accomplish this we will use Sisto's characterization of relative hyperbolicity stated in [Theorem 4.15](#).

Recall from [Definition 6.8](#) that, for a properly embedded simplex S , \mathcal{L}_S is the family of linear projections onto S . For each $S \in \mathcal{S}$, choose a set of S -supporting hyperplanes \mathcal{H}_S to form a collection of linear projections

$$\Pi_S := \{L_{S, \mathcal{H}_S} : S \in \mathcal{S}\}.$$

Next fix the geodesic path system on (\mathcal{C}, H_Ω) defined by

$$\mathcal{G} := \{[x, y] : x, y \in \mathcal{C}\}.$$

By [Theorem 4.15](#), it is enough to show that Π_S is an almost-projection system and \mathcal{S} is asymptotically transverse-free relative to \mathcal{G} .

Remark 13.2 In general, $\#\mathcal{L}_S > 1$, so there is some choice involved in the construction of Π_S . However, by [Proposition 13.7](#),

$$\sup_{S \in \mathcal{S}} \sup_{L_1, L_2 \in \mathcal{L}_S} \sup_{x \in \mathcal{C}} H_\Omega(L_1(x), L_2(x)) < +\infty.$$

So Π_S will be an almost-projection system, independent of the choices involved in its construction.

13.1 Π_S is an almost-projection system

Theorem 13.3 Π_S is an almost-projection system for S on the complete geodesic metric space (\mathcal{C}, H_Ω) .

The proof of [Theorem 13.3](#) will require a series of preliminary results. We first prove a continuity lemma for linear projections that will be used repeatedly in this section.

Lemma 13.4 If $S \in \mathcal{S}$, then the map

$$(L, x) \in \mathcal{L}_S \times \bar{\mathcal{C}} \rightarrow L(x) \in \bar{S}$$

is continuous.

Proof We first show that $\mathbb{P}(\ker L) \cap \bar{\mathcal{C}} = \emptyset$ for all $L \in \mathcal{L}_S$. Suppose for a contradiction that $L \in \mathcal{L}_S$ and

$$x \in \mathbb{P}(\ker L) \cap \bar{\mathcal{C}}.$$

[Proposition 6.2](#) implies that $x \in \partial_i \mathcal{C}$. Then [Proposition 6.5](#) implies that $[y, x] \subset \partial_i \mathcal{C}$ for every $y \in \partial S$. Next fix $y_1, y_2 \in \partial S$ such that $(y_1, y_2) \subset S$. Then y_1, x, y_2 form a half triangle. So $x \in F_\Omega(\partial S)$ by [Corollary 11.3](#). But [Proposition 6.5](#) implies that

$$F_\Omega(\partial S) \cap \mathbb{P}(\ker L) = \emptyset.$$

So we have a contradiction. Thus, $\mathbb{P}(\ker L) \cap \bar{\mathcal{C}} = \emptyset$ for all $L \in \mathcal{L}_S$.

Now suppose that $\lim_{n \rightarrow \infty} (L_n, x_n) = (L, x)$ in $\mathcal{L}_S \times \bar{\mathcal{C}}$. Let $\tilde{x}_n, \tilde{x} \in \mathbb{R}^d$ denote lifts of x_n and x , respectively, such that $\lim_{n \rightarrow \infty} \tilde{x}_n = \tilde{x}$. Then

$$L(\tilde{x}) = \lim_{n \rightarrow \infty} L_n(\tilde{x}_n) \in \mathbb{R}^d.$$

Since $\mathbb{P}(\ker L) \cap \bar{\mathcal{C}} = \emptyset$, we have $L(\tilde{x}) \neq 0$. So

$$L(x) = [L(\tilde{x})] = \lim_{n \rightarrow \infty} [L_n(\tilde{x}_n)] = \lim_{n \rightarrow \infty} L_n(x_n). \quad \square$$

Next we introduce the “closest points” projection onto a properly embedded simplex.

Definition 13.5 If $S \subset \Omega$ is a properly embedded simplex and $p \in \Omega$, the *closest points projection of p onto S* is the set

$$\pi_S(p) := S \cap \{q \in \Omega : H_\Omega(p, q) \leq H_\Omega(p, S)\}.$$

Observation 13.6 Suppose $S \subset \Omega$ is a properly embedded simplex. Then:

- (1) If $p \in \Omega$, then $\pi_S(p)$ is compact and convex.
- (2) If $g \in \text{Aut}(\Omega)$, then $g \circ \pi_S = \pi_{gS} \circ g$.

Proof Part (2) is obvious and part (1) follows from the fact that metric balls in the Hilbert metric are convex. □

Now, we establish the coarse equivalence between the two projections.

Proposition 13.7 There exists $\delta_1 \geq 0$ such that, if $S \in \mathcal{S}$, \mathcal{H} is a set of S -supporting hyperplanes and $x \in \mathcal{C}$, then

$$\max_{p \in \pi_S(x)} H_\Omega(L_{S, \mathcal{H}}(x), p) \leq \delta_1.$$

Proof Since \mathcal{S} has finitely many Λ orbits (see [Proposition 8.1](#)), it is enough to prove the result for some fixed $S \in \mathcal{S}$.

Suppose the proposition is false. Then, for every $n \geq 0$, there exist $x_n \in \mathcal{C}$, a set of S -supporting hyperplanes \mathcal{H}_n and $p_n \in \pi_S(x_n)$ such that

$$H_\Omega(p_n, L_{S, \mathcal{H}_n}(x_n)) \geq n.$$

Let m_n be the midpoint of the projective line segment $[p_n, L_{S, \mathcal{H}_n}(x_n)]$ in the Hilbert distance. Since $\text{Stab}_\Lambda(S)$ acts cocompactly on S (see [Proposition 8.1](#)), by translating by elements of $\text{Stab}_\Lambda(S)$ and passing to a subsequence, we can assume that $m := \lim_{n \rightarrow \infty} m_n$ exists in S . Passing to a further subsequence and using [Proposition 6.9](#), we can assume that there exist $x, p, x' \in \partial_i \mathcal{C}$ and $L_{S, \mathcal{H}} \in \mathcal{L}_S$ where $x := \lim_{n \rightarrow \infty} x_n$, $p := \lim_{n \rightarrow \infty} p_n$, $x' := \lim_{n \rightarrow \infty} L_{S, \mathcal{H}_n}(x_n)$ and $L_{S, \mathcal{H}} := \lim_{n \rightarrow \infty} L_{S, \mathcal{H}_n}$. By [Lemma 13.4](#),

$$L_{S, \mathcal{H}}(x) = \lim_{n \rightarrow \infty} L_{S, \mathcal{H}_n}(x_n) = x'.$$

We first show that $[x', x] \subset \partial_i \mathcal{C}$. Observe that $L_{\mathcal{S}, \mathcal{H}}(v) = x'$ for all $v \in [x', x]$ since $L_{\mathcal{S}, \mathcal{H}}$ is linear and $L_{\mathcal{S}, \mathcal{H}}(x') = x' = L_{\mathcal{S}, \mathcal{H}}(x)$. But $L_{\mathcal{S}, \mathcal{H}}(\Omega) = \mathcal{S}$, implying $[x', x] \cap \Omega = \emptyset$. Hence,

$$[x', x] \subset \partial_i \mathcal{C}.$$

Next we show that $[p, x] \subset \partial_i \mathcal{C}$. Suppose not; then $(p, x) \subset \mathcal{C}$. Choose any $v \in (p, x) \cap \mathcal{C}$ and a sequence $v_n \in [p_n, x_n]$ such that $v = \lim_{n \rightarrow \infty} v_n$. Since $p \in \partial_i \mathcal{C}$ and $v \in \mathcal{C}$,

$$\lim_{n \rightarrow \infty} H_{\Omega}(v_n, p_n) = \infty.$$

Fix any $v_S \in \mathcal{S}$. Then, choosing n large enough such that $H_{\Omega}(v_n, p_n) \geq 2 + H_{\Omega}(v, v_S)$ and $H_{\Omega}(v, v_n) \leq 1$, we have

$$\begin{aligned} H_{\Omega}(x_n, v_S) &\leq H_{\Omega}(x_n, v_n) + H_{\Omega}(v_n, v) + H_{\Omega}(v, v_S) \\ &= H_{\Omega}(x_n, p_n) - H_{\Omega}(p_n, v_n) + H_{\Omega}(v_n, v) + H_{\Omega}(v, v_S) \\ &\leq H_{\Omega}(x_n, p_n) - 1, \end{aligned}$$

which is a contradiction since $p_n \in \pi_{\mathcal{S}}(x_n)$. Hence, $[p, x] \subset \partial_i \mathcal{C}$.

Thus, $[p, x] \cup [x, x'] \subset \partial_i \mathcal{C}$ and, by construction, $m \in (p, x) \subset \mathcal{C}$. Thus, the three points x, x', p form a half triangle. Then, by Corollary 11.3, $x \in F_{\Omega}(\partial \mathcal{S})$. So, by Proposition 6.6, $x' = L_{\mathcal{S}, \mathcal{H}}(x) \in F_{\Omega}(x)$. Since $[p, x] \subset \partial_i \mathcal{C}$, Observation 3.12(4) implies that $(p, x') \subset \partial_i \mathcal{C}$. This is a contradiction since

$$m \in (p, x') \cap \mathcal{C} \neq \emptyset. \quad \square$$

The next step is to prove δ -thinness of some special triangles built using linear projections (see Proposition 13.9). The following lemma provides a criterion for δ -thinness of triangles in Hilbert geometry.

Lemma 13.8 *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $x, y, z \in \Omega$. If*

$$[x, y] \subset \mathcal{N}_{\Omega}([x, z] \cup [z, y]; R),$$

then the geodesic triangle

$$[x, y] \cup [y, z] \cup [z, x]$$

is $(2R)$ -thin.

Proof The sets

$$I_x = [x, y] \cap \mathcal{N}_{\Omega}([x, z]; R) \quad \text{and} \quad I_y = [x, y] \cap \mathcal{N}_{\Omega}([y, z]; R)$$

are nonempty and relatively open in $[x, y]$. Since $[x, y] = I_x \cup I_y$ and $[x, y]$ is connected, there exists $c \in I_x \cap I_y$. Then there exist $c_x \in [x, z]$ and $c_y \in [y, z]$ such that $H_\Omega(c, c_x) < R$ and $H_\Omega(c, c_y) < R$. Then

$$H_\Omega^{\text{Haus}}([x, c_x], [x, c]) \leq H_\Omega(x, x) + H_\Omega(c_x, c) < R$$

and

$$H_\Omega^{\text{Haus}}([c_x, z], [c_y, z]) \leq H_\Omega(c_x, c_y) + H_\Omega(z, z) < 2R.$$

So

$$[x, z] \subset \mathcal{N}_\Omega([x, y] \cup [y, z]; 2R).$$

A similar argument shows that

$$[y, z] \subset \mathcal{N}_\Omega([z, x] \cup [x, y]; 2R).$$

So the geodesic triangle is $(2R)$ -thin. □

Proposition 13.9 *There exists $\delta_2 \geq 0$ such that, if $x \in \mathcal{C}$, $S \in \mathcal{S}$, $z \in S$ and \mathcal{H} is a set of S -supporting hyperplanes, then the geodesic triangle*

$$[x, z] \cup [z, L_{S, \mathcal{H}}(x)] \cup [L_{S, \mathcal{H}}(x), x]$$

is δ_2 -thin.

Proof Since \mathcal{S} has finitely many Λ orbits (see Proposition 8.1), it is enough to prove the result for some fixed $S \in \mathcal{S}$. By Lemma 13.8, it is enough to show that there exists $\delta_2 \geq 0$ such that

$$[L_{S, \mathcal{H}}(x), z] \subset \mathcal{N}_{\delta_2/2}([z, x] \cup [x, L_{S, \mathcal{H}}(x)])$$

for all $x \in \mathcal{C}$, $z \in S$ and \mathcal{H} a set of S -supporting hyperplanes.

Suppose such a δ_2 does not exist. Then, for every $n \geq 0$, there exist $z_n \in S$, a set of S -supporting hyperplanes \mathcal{H}_n , $p_n := L_{S, \mathcal{H}_n}(x_n)$ and $u_n \in [z_n, p_n]$ such that

$$H_\Omega(u_n, [z_n, x_n] \cup [x_n, p_n]) \geq n.$$

Since $\text{Stab}_\Lambda(S)$ acts cocompactly on S , by translating by elements of $\text{Stab}_\Lambda(S)$ and passing to a subsequence, we can assume that $u := \lim_{n \rightarrow \infty} u_n$ exists and $u \in S$. Passing to a further subsequence and using Proposition 6.9, we can assume there exist $x, z, p \in \bar{\mathcal{C}}$ and $L_{S, \mathcal{H}} \in \mathcal{L}_S$ where $x := \lim_{n \rightarrow \infty} x_n$, $z := \lim_{n \rightarrow \infty} z_n$, $p := \lim_{n \rightarrow \infty} p_n$ and $L_{S, \mathcal{H}} := \lim_{n \rightarrow \infty} L_{S, \mathcal{H}_n}$. Since

$$\begin{aligned} \lim_{n \rightarrow \infty} H_\Omega(u, [x_n, z_n] \cup [x_n, p_n]) &\geq \lim_{n \rightarrow \infty} (H_\Omega(u_n, [x_n, z_n] \cup [x_n, p_n]) - H_\Omega(u, u_n)) \\ &= \infty, \end{aligned}$$

we have

$$[x, z] \cup [x, p] \subset \partial_i \mathcal{C}.$$

By construction, $u \in (p, z) \subset \mathcal{C}$. Thus, p, x, z form a half triangle. Then, by [Corollary 11.3](#), $x \in F_\Omega(\partial S)$. [Lemma 13.4](#) and [Proposition 6.6](#) then imply

$$p = \lim_{n \rightarrow \infty} p_n = \lim_{n \rightarrow \infty} L_{S, \mathcal{H}_n}(x_n) = L_{S, \mathcal{H}}(x) \in F_\Omega(x).$$

Then, since $[x, z] \subset \partial_i \mathcal{C}$, [Observation 3.12\(4\)](#) implies that $[p, z] \subset \partial_i \mathcal{C}$. This is a contradiction, since

$$u \in (p, z) \cap \mathcal{C} \neq \emptyset. \quad \square$$

Proposition 13.10 *Set $\delta_3 := \delta_1 + 3\delta_2$. If $x \in \mathcal{C}$, $S \in \mathcal{S}$, \mathcal{H} is a set of S -supporting hyperplanes and $z \in S$, then $H_\Omega(L_{S, \mathcal{H}}(x), [x, z]) \leq \delta_3$.*

Proof By [Proposition 13.9](#), the geodesic triangle

$$[x, z] \cup [z, L_{S, \mathcal{H}}(x)] \cup [L_{S, \mathcal{H}}(x), x]$$

is δ_2 -thin. Thus, there exist $y \in [L_{S, \mathcal{H}}(x), z]$, $y_1 \in [x, L_{S, \mathcal{H}}(x)]$ and $y_2 \in [x, z]$ such that $H_\Omega(y, y_1) \leq \delta_2$ and $H_\Omega(y, y_2) \leq \delta_2$.

We claim that

$$H_\Omega(L_{S, \mathcal{H}}(x), y_1) \leq \delta_1 + \delta_2.$$

Choose any $p \in \pi_S(x)$. Since $[L_{S, \mathcal{H}}(x), z] \subset S$,

$$H_\Omega(x, p) = H_\Omega(x, S) \leq H_\Omega(x, y).$$

Then, using [Proposition 13.7](#),

$$H_\Omega(x, L_{S, \mathcal{H}}(x)) \leq H_\Omega(x, p) + H_\Omega(p, L_{S, \mathcal{H}}(x)) \leq H_\Omega(x, y) + \delta_1.$$

Then,

$$\begin{aligned} H_\Omega(L_{S, \mathcal{H}}(x), y_1) &= H_\Omega(L_{S, \mathcal{H}}(x), x) - H_\Omega(y_1, x) \\ &\leq H_\Omega(x, y) + \delta_1 - H_\Omega(y_1, x) \\ &\leq H_\Omega(y, y_1) + \delta_1 \leq \delta_2 + \delta_1. \end{aligned}$$

Hence,

$$\begin{aligned} H_\Omega(L_{S, \mathcal{H}}(x), [x, z]) &\leq H_\Omega(L_{S, \mathcal{H}}(x), y_2) \\ &\leq H_\Omega(L_{S, \mathcal{H}}(x), y_1) + H_\Omega(y_1, y) + H_\Omega(y, y_2) \\ &\leq \delta_1 + 3\delta_2 = \delta_3. \end{aligned} \quad \square$$

Our next goal is to prove that, if the distance between the linear projections of two points onto a simplex $S \in \mathcal{S}$ is large, then the geodesic between the two points spends a significant amount of time in a tubular neighborhood of S . This is accomplished in [Corollary 13.12](#) using the next result.

Proposition 13.11 *There exists $\delta_4 \geq 0$ such that, if $S \in \mathcal{S}$, \mathcal{H} is a set of S -supporting hyperplanes, $x, y \in \mathcal{C}$ and $H_\Omega(L_{S,\mathcal{H}}(x), L_{S,\mathcal{H}}(y)) \geq \delta_4$, then*

$$H_\Omega(L_{S,\mathcal{H}}(x), [x, y]) \leq \delta_4 \quad \text{and} \quad H_\Omega(L_{S,\mathcal{H}}(y), [x, y]) \leq \delta_4.$$

Proof Observe that the linear projections are Λ -equivariant; that is,

$$L_{gS, g\mathcal{H}} \circ g = g \circ L_{S,\mathcal{H}}$$

for any $g \in \Lambda$, $S \in \mathcal{S}$ and \mathcal{H} a set of S -supporting hyperplanes. Moreover, by [Proposition 8.1](#), there are only finitely many Λ -orbits in \mathcal{S} . Thus, it is enough to prove this proposition for a fixed $S \in \mathcal{S}$.

Suppose the proposition is false. Then, for every $n \geq 0$, there exist $x_n, y_n \in \mathcal{C}$ and a set of S -supporting hyperplanes \mathcal{H}_n with

$$H_\Omega(L_{S,\mathcal{H}_n}(x_n), L_{S,\mathcal{H}_n}(y_n)) \geq n \quad \text{and} \quad H_\Omega(L_{S,\mathcal{H}_n}(x_n), [x_n, y_n]) \geq n.$$

Let $a_n := L_{S,\mathcal{H}_n}(x_n)$ and $b_n := L_{S,\mathcal{H}_n}(y_n)$. Then pick $c_n \in [a_n, b_n]$ such that

$$(12) \quad H_\Omega(c_n, a_n) = \frac{1}{2}n.$$

Then

$$(13) \quad H_\Omega(c_n, b_n) \geq H_\Omega(a_n, b_n) - H_\Omega(c_n, a_n) \geq \frac{1}{2}n$$

and

$$(14) \quad H_\Omega(c_n, [x_n, y_n]) \geq H_\Omega(a_n, [x_n, y_n]) - H_\Omega(c_n, a_n) \geq \frac{1}{2}n.$$

Since $\text{Stab}_\Lambda(S)$ acts cocompactly on S (see [Proposition 8.1](#)), by translating by elements of $\text{Stab}_\Lambda(S)$ and passing to a subsequence, we may assume that $c := \lim_{n \rightarrow \infty} c_n$ exists and $c \in S$. After taking a further subsequence, we can assume that the following limits exist in $\bar{\mathcal{C}}$: $a := \lim_{n \rightarrow \infty} a_n$, $b := \lim_{n \rightarrow \infty} b_n$, $x := \lim_{n \rightarrow \infty} x_n$ and $y := \lim_{n \rightarrow \infty} y_n$.

We now observe that $a, b, x, y \in \partial_i \mathcal{C}$. Equations (12) and (13) imply that $a, b \in \partial_i \mathcal{C}$. Equation (14) implies that $[x, y] \subset \partial_i \mathcal{C}$.

We claim that $x \in F_\Omega(a)$ and $y \in F_\Omega(b)$. Since $c_n \in S$, by Proposition 13.10, there exists $a'_n \in [x_n, c_n]$ such that $H_\Omega(a_n, a'_n) \leq \delta_3$. Up to passing to a subsequence, we can assume that $a' := \lim_{n \rightarrow \infty} a'_n$ exists in \bar{C} . Observe that $a' \in \partial_1 C$ since

$$\lim_{n \rightarrow \infty} H_\Omega(a'_n, c) \geq \lim_{n \rightarrow \infty} (H_\Omega(a_n, c_n) - H_\Omega(c_n, c) - H_\Omega(a_n, a'_n)) = \infty.$$

Since $a'_n \in [x_n, c_n]$,

$$a' \in \partial_1 C \cap [x, c] = \{x\}.$$

Thus, $\lim_{n \rightarrow \infty} a'_n = x$. Since $\lim_{n \rightarrow \infty} a_n = a$ and $H_\Omega(a_n, a'_n) \leq \delta_3$, Proposition 3.13 implies that $x \in F_\Omega(a)$. Similar reasoning shows that $y \in F_\Omega(b)$.

Since $[x, y] \subset \partial_1 C$, Observation 3.12(4) implies that $[a, b] \subset \partial_1 C$. This is a contradiction since $c \in (a, b) \cap C \neq \emptyset$. □

Corollary 13.12 *If $S \in \mathcal{S}$, \mathcal{H} is a set of S -supporting hyperplanes, $R > 0$, $x, y \in C$ and $H_\Omega(L_{S, \mathcal{H}}(x), L_{S, \mathcal{H}}(y)) \geq R + 2\delta_4$, then*

- (1) *there exists $[x_0, y_0] \subset [x, y]$ such that $[x_0, y_0] \subset \mathcal{N}_\Omega(S; \delta_4)$,*
- (2) *$[L_{S, \mathcal{H}}(x), L_{S, \mathcal{H}}(y)] \subset \mathcal{N}_\Omega([x, y]; \delta_4)$, and*
- (3) *$\text{diam}_\Omega(\mathcal{N}_\Omega(S; \delta_4) \cap [x, y]) \geq R$.*

Proof Since $H_\Omega(L_{S, \mathcal{H}}(x), L_{S, \mathcal{H}}(y)) > \delta_4$, Proposition 13.11 implies that there exist $x_0, y_0 \in [x, y]$ such that

$$H_\Omega(L_{S, \mathcal{H}}(x), x_0) \leq \delta_4 \quad \text{and} \quad H_\Omega(L_{S, \mathcal{H}}(y), y_0) \leq \delta_4.$$

By Proposition 3.14,

$$H_\Omega^{\text{Haus}}([x_0, y_0], [L_{S, \mathcal{H}}(x), L_{S, \mathcal{H}}(y)]) \leq \delta_4$$

and, by convexity, $[L_{S, \mathcal{H}}(x), L_{S, \mathcal{H}}(y)] \subset S$. This proves parts (1) and (2). To prove part (3), observe that

$$H_\Omega(x_0, y_0) \geq H_\Omega(L_{S, \mathcal{H}}(x), L_{S, \mathcal{H}}(y)) - H_\Omega(L_{S, \mathcal{H}}(x), x_0) - H_\Omega(L_{S, \mathcal{H}}(y), y_0) \geq R.$$

Then $\text{diam}_\Omega(\mathcal{N}_\Omega(S; \delta_4) \cap [x, y]) \geq H_\Omega(x_0, y_0) \geq R$. □

Using the properties of linear projections established so far, we prove that Π_S is an almost-projection system.

Lemma 13.13 *If $S \in \mathcal{S}$, \mathcal{H} a set of S -supporting hyperplanes, $x \in \mathcal{C}$ and $z \in S$, then*

$$H_\Omega(x, z) \geq H_\Omega(x, L_{S, \mathcal{H}}(x)) + H_\Omega(L_{S, \mathcal{H}}(x), z) - 2\delta_3.$$

Proof By Proposition 13.10, there exists $q \in [x, z]$ such that $H_\Omega(L_{S, \mathcal{H}}(x), q) \leq \delta_3$. Then,

$$H_\Omega(x, z) = H_\Omega(x, q) + H_\Omega(q, z) \geq H_\Omega(x, L_{S, \mathcal{H}}(x)) + H_\Omega(L_{S, \mathcal{H}}(x), z) - 2\delta_3. \quad \square$$

Lemma 13.14 *There exists $\delta_5 \geq 0$ such that, if $S \neq S' \in \mathcal{S}$ and \mathcal{H} is a set of S -supporting hyperplanes, then*

$$\text{diam}_\Omega(L_{S, \mathcal{H}}(S')) \leq \delta_5.$$

Proof Since \mathcal{S} is strongly isolated, for every $r > 0$ there exists $D(r) > 0$ such that

$$(15) \quad \text{diam}_\Omega(\mathcal{N}_\Omega(S_1; r) \cap \mathcal{N}_\Omega(S_2, r)) \leq D(r)$$

for all $S_1, S_2 \in \mathcal{S}$ distinct.

Let $\delta_5 := D(\delta_4) + 2\delta_4 + 1$. Fix $x, y \in S'$ and suppose for a contradiction that $H_\Omega(L_{S, \mathcal{H}}(x), L_{S, \mathcal{H}}(y)) > \delta_5$. Then, by Corollary 13.12,

$$\text{diam}_\Omega(\mathcal{N}_\Omega(S; \delta_4) \cap S') \geq \text{diam}_\Omega(\mathcal{N}_\Omega(S; \delta_4) \cap [x, y]) \geq D(\delta_4) + 1,$$

which contradicts (15). □

Lemma 13.15 *If $x \in \mathcal{C}$, $S \in \mathcal{S}$, \mathcal{H} is a set of S -supporting hyperplanes and $R := H_\Omega(x, S)$, then*

$$\text{diam}_\Omega(L_{S, \mathcal{H}}(B_\Omega(x; R) \cap \mathcal{C})) \leq 8(\delta_4 + \delta_1).$$

Proof Fix $y \in B_\Omega(x; R) \cap \mathcal{C}$. We claim that

$$H_\Omega(L_{S, \mathcal{H}}(x), L_{S, \mathcal{H}}(y)) \leq 4(\delta_4 + \delta_1).$$

It is enough to consider the case when $H_\Omega(L_{S, \mathcal{H}}(x), L_{S, \mathcal{H}}(y)) \geq \delta_4$. Then, by Proposition 13.11, there exists $x' \in [x, y]$ such that $H_\Omega(L_{S, \mathcal{H}}(x), x') \leq \delta_4$. By Proposition 13.7,

$$H_\Omega(x, y) \leq R = H_\Omega(x, \pi_S(x)) \leq H_\Omega(x, L_{S, \mathcal{H}}(x)) + \delta_1.$$

Then

$$\begin{aligned} H_\Omega(x', y) &= H_\Omega(x, y) - H_\Omega(x, x') \leq H_\Omega(x, L_{S, \mathcal{H}}(x)) - H_\Omega(x, x') + \delta_1 \\ &\leq H_\Omega(L_{S, \mathcal{H}}(x), x') + \delta_1 \\ &\leq \delta_4 + \delta_1. \end{aligned}$$

Thus,

$$H_\Omega(L_{S, \mathcal{H}}(x), y) \leq H_\Omega(L_{S, \mathcal{H}}(x), x') + H_\Omega(x', y) \leq 2\delta_4 + \delta_1.$$

Since $L_{S, \mathcal{H}}(x) \in S$, using [Proposition 13.7](#) again,

$$H_\Omega(y, L_{S, \mathcal{H}}(y)) \leq H_\Omega(y, \pi_S(y)) + \delta_1 \leq H_\Omega(y, L_{S, \mathcal{H}}(x)) + \delta_1 \leq 2(\delta_4 + \delta_1).$$

Finally,

$$\begin{aligned} H_\Omega(L_{S, \mathcal{H}}(x), L_{S, \mathcal{H}}(y)) &\leq H_\Omega(L_{S, \mathcal{H}}(x), x') + H_\Omega(x', y) + H_\Omega(y, L_{S, \mathcal{H}}(y)) \\ &\leq 4(\delta_4 + \delta_1). \end{aligned} \quad \square$$

Proof of Theorem 13.3 This follows from [Lemmas 13.13, 13.14 and 13.15](#). □

13.2 \mathcal{S} is asymptotically transverse-free relative to \mathcal{G}

Theorem 13.16 *The family \mathcal{S} is asymptotically transverse-free relative to the geodesic path system \mathcal{G} .*

Proof Let δ_4 be the constant in [Proposition 13.11](#). We will show that there exists $\lambda > 0$ such that, for each $\Delta \geq 1$ and $\kappa \geq 2\delta_4$, if $\mathcal{T} \subset \mathcal{C}$ is a geodesic triangle whose sides are in \mathcal{G} and which is \mathcal{S} -almost-transverse with constants κ and Δ , then \mathcal{T} is $(\lambda\Delta)$ -thin.

Suppose such a $\lambda > 0$ does not exist. Then, for every $n \geq 1$, there exist $\kappa_n \geq 2\delta_4$, $\Delta_n \geq 1$ and a \mathcal{S} -almost-transverse triangle $\mathcal{T}_n \subset \mathcal{C}$ with constants κ_n and Δ_n such that \mathcal{T}_n is not $(n\Delta_n)$ -thin. Let a_n, b_n and c_n be the vertices of \mathcal{T}_n , labeled in a such a way that there exists $u_n \in [a_n, b_n] \subset \mathcal{T}_n$ with

$$(16) \quad H_\Omega(u_n, [a_n, c_n] \cup [c_n, b_n]) > n\Delta_n \geq n.$$

Note that [Observation 4.14](#) implies that the geodesic triangles \mathcal{T}_n are also \mathcal{S} -almost-transverse with constants $2\delta_4$ and Δ_n since $\kappa_n \geq 2\delta_4$.

Since Λ acts cocompactly on \mathcal{C} , by translating by elements of Λ and passing to a subsequence, we can assume that $u := \lim_{n \rightarrow \infty} u_n$ exists and $u \in \mathcal{C}$. By passing to

a further subsequence, we can assume that $a := \lim_{n \rightarrow \infty} a_n$, $b := \lim_{n \rightarrow \infty} b_n$ and $c := \lim_{n \rightarrow \infty} c_n$ exist in \bar{C} . By (16),

$$[a, c] \cup [c, b] \subset \partial_i \mathcal{C},$$

whereas, by construction, $u \in (a, b) \subset \mathcal{C}$. Thus, the points a, b, c form a half triangle. Then, by Theorem 11.2, there exists $S \in \mathcal{S}$ such that $a, b, c \in F_\Omega(\partial S)$.

Fix a set of S -supporting hyperplanes \mathcal{H} . Let $a'_n := L_{S, \mathcal{H}}(a_n)$, $b'_n := L_{S, \mathcal{H}}(b_n)$ and $c'_n := L_{S, \mathcal{H}}(c_n)$. Up to passing to a subsequence, we can assume that the limits $a' := \lim_{n \rightarrow \infty} a'_n$, $b' := \lim_{n \rightarrow \infty} b'_n$ and $c' := \lim_{n \rightarrow \infty} c'_n$ exist. By Lemma 13.4 and Proposition 6.6,

$$a' = \lim_{n \rightarrow \infty} L_{S, \mathcal{H}}(a_n) = L_{S, \mathcal{H}}(a) \in F_\Omega(a).$$

Similarly,

$$b' = L_{S, \mathcal{H}}(b) \in F_\Omega(b) \quad \text{and} \quad c' = L_{S, \mathcal{H}}(c) \in F_\Omega(c).$$

Using Observation 3.12(4), $(a', b') \subset \Omega$ and $[a', c'] \cup [b', c'] \subset \partial_i \mathcal{C}$. Then Observation 3.12(4) implies that the faces $F_\Omega(a')$, $F_\Omega(b')$, and $F_\Omega(c')$, are pairwise disjoint. Then, by Proposition 3.13,

$$\lim_{n \rightarrow \infty} H_\Omega(a'_n, b'_n) = \infty.$$

Thus, for n large enough, Corollary 13.12(2)–(3) implies

$$(17) \quad [a'_n, b'_n] \subset \mathcal{N}_\Omega([a_n, b_n]; \delta_4)$$

and

$$(18) \quad \text{diam}_\Omega(\mathcal{N}_\Omega(S; \delta_4) \cap [a_n, b_n]) \geq H_\Omega(a'_n, b'_n) - 2\delta_4.$$

Since \mathcal{T}_n is S -almost-transverse with constants $2\delta_4$ and Δ_n , by (18),

$$(19) \quad H_\Omega(a'_n, b'_n) \leq \Delta_n + 2\delta_4.$$

Similarly, for n large enough,

$$(20) \quad [b'_n, c'_n] \subset \mathcal{N}_\Omega([b_n, c_n]; \delta_4) \quad \text{and} \quad H_\Omega(b'_n, c'_n) \leq \Delta_n + 2\delta_4,$$

$$(21) \quad [c'_n, a'_n] \subset \mathcal{N}_\Omega([c_n, a_n]; \delta_4) \quad \text{and} \quad H_\Omega(c'_n, a'_n) \leq \Delta_n + 2\delta_4.$$

Let m_n^{ab} , m_n^{bc} and m_n^{ca} be the Hilbert distance midpoints of $[a'_n, b'_n]$, $[b'_n, c'_n]$ and $[c'_n, a'_n]$, respectively. By (17), (20) and (21), there exist w_n^{ab} , w_n^{bc} and w_n^{ca} in $[a_n, b_n]$, $[b_n, c_n]$ and $[c_n, a_n]$, respectively, such that

$$H_\Omega(w_n^{ab}, m_n^{ab}) \leq \delta_4, \quad H_\Omega(w_n^{bc}, m_n^{bc}) \leq \delta_4 \quad \text{and} \quad H_\Omega(w_n^{ca}, m_n^{ca}) \leq \delta_4.$$

Then

$$\begin{aligned}
 H_{\Omega}(w_n^{ab}, w_n^{bc}) &\leq H_{\Omega}(w_n^{ab}, m_n^{ab}) + H_{\Omega}(m_n^{ab}, m_n^{bc}) + H_{\Omega}(m_n^{bc}, w_n^{bc}) \\
 &\leq \delta_4 + H_{\Omega}(m_n^{ab}, b'_n) + H_{\Omega}(b'_n, m_n^{bc}) + \delta_4 \\
 &= 2\delta_4 + \frac{1}{2}(H_{\Omega}(a'_n, b'_n) + H_{\Omega}(b'_n, c'_n)) \\
 &\leq 4\delta_4 + \Delta_n
 \end{aligned}
 \tag{by (19)–(20)}.$$

Similarly,

$$(22) \quad H_{\Omega}(w_n^{bc}, w_n^{ca}) \leq \Delta_n + 4\delta_4 \quad \text{and} \quad H_{\Omega}(w_n^{ca}, w_n^{ab}) \leq \Delta_n + 4\delta_4.$$

Then, for n large enough, the triangles \mathcal{T}_n are $(\Delta_n + 4\delta_4)$ -thin, since

$$\begin{aligned}
 H_{\Omega}^{\text{Haus}}([a_n, w_n^{ab}], [a_n, w_n^{ca}]) &\leq \Delta_n + 4\delta_4, \\
 H_{\Omega}^{\text{Haus}}([b_n, w_n^{bc}], [b_n, w_n^{ab}]) &\leq \Delta_n + 4\delta_4, \\
 H_{\Omega}^{\text{Haus}}([c_n, w_n^{ca}], [c_n, w_n^{bc}]) &\leq \Delta_n + 4\delta_4.
 \end{aligned}$$

Since $\Delta_n \geq 1$, we have $\Delta_n + 4\delta_4 \leq (1 + 4\delta_4)\Delta_n$. Thus, for n large enough, \mathcal{T}_n is $(\lambda\Delta_n)$ -thin for $\lambda := 1 + 4\delta_4$, which contradicts the assumption that \mathcal{T}_n is not $(n\Delta_n)$ -thin. □

14 Proof of Theorem 1.13

For the rest of the section suppose that $(\Omega, \mathcal{C}, \Lambda)$ is a naive convex cocompact triple.

(1) implies (2) and (3) This is [Theorem 1.18](#).

(3) implies (2) Suppose that Λ is a relatively hyperbolic group with respect to a collection of subgroups $\{H_1, \dots, H_k\}$ each of which is virtually abelian of rank at least two. For each $1 \leq j \leq k$, let $A_j \leq H_j$ be a finite-index abelian subgroup with rank at least two. Then, by definition, Λ is a relatively hyperbolic group with respect to $\{A_1, \dots, A_k\}$.

Fix a word metric d_{Λ} on Λ . Then, for $U \subset \Lambda$ and $r > 0$, define

$$\mathcal{N}_{\Lambda}(U; r) := \{g \in \Lambda : d_{\Lambda}(g, U) < r\}$$

and

$$\text{diam}_{\Lambda}(U) = \sup\{d_{\Lambda}(g_1, g_2) : g_1, g_2 \in U\}.$$

Next, for each $1 \leq j \leq k$, let \hat{A}_j be a maximal abelian subgroup of Λ that contains A_j . By [Theorem 1.23](#), there exists a properly embedded simplex $S_j \subset \mathcal{C}$ such that $\hat{A}_j \leq$

$\text{Stab}_\Lambda(S_j)$, \widehat{A}_j acts cocompactly on S_j , and \widehat{A}_j has a finite-index subgroup isomorphic to $\mathbb{Z}^{\dim(S_j)}$. Since A_j (and hence \widehat{A}_j) has rank at least two, this implies that $\dim S_j \geq 2$.

We claim that $A_j \leq \widehat{A}_j$ has finite index and hence A_j also acts cocompactly on S_j . By **Observation 5.3**, the metric space (S_j, H_Ω) is quasi-isometric to $\mathbb{R}^{\dim S_j}$. So, by the fundamental lemma of geometric group theory [8, Chapter I, Proposition 8.19], $(\widehat{A}_j, d_\Lambda)$ is also quasi-isometric to $\mathbb{R}^{\dim S_j}$. Since $\dim S_j \geq 2$, **Theorem 4.9** implies that there exist $r_1 > 0$, $g_j \in \Lambda$, and $1 \leq i_j \leq k$ such that

$$\widehat{A}_j \subset \mathcal{N}_\Lambda(g_j A_{i_j}; r_1).$$

Then

$$\text{diam}_\Lambda(\mathcal{N}_\Lambda(g_j A_{i_j}; r_1) \cap \mathcal{N}_\Lambda(A_j; r_1)) \geq \text{diam}_\Lambda(A_j) = \infty.$$

So **Theorem 4.6** implies that $g_j A_{i_j} = A_j$. Then

$$\widehat{A}_j \subset \mathcal{N}_\Lambda(A_j; r_1)$$

and hence $A_j \leq \widehat{A}_j$ has finite index.

Fix some $x_0 \in \mathcal{C}$ and consider the orbit map

$$F : (\Lambda, d_\Lambda) \rightarrow (\mathcal{C}, H_\Omega), \quad F(g) = gx_0.$$

By the fundamental lemma of geometric group theory [8, Proposition 8.19], this is a quasi-isometry. Let $G : \mathcal{C} \rightarrow \Lambda$ be a quasi-inverse. Then, using the fact that A_j acts cocompactly on S_j , there exists $r_2 > 0$ such that

$$F(gA_j) \subset \mathcal{N}_\Omega(gS_j; r_2) \quad \text{and} \quad G(gS_j) \subset \mathcal{N}_\Lambda(gA_j; r_2)$$

for all $g \in \Lambda$ and $1 \leq j \leq k$. Then, by definition and **Theorem 4.8**, (\mathcal{C}, H_Ω) is relatively hyperbolic with respect to the family of properly embedded simplices of dimension at least two

$$\mathcal{S} := \{gS_j : g \in \Lambda, 1 \leq j \leq k\}.$$

(2) implies (1) Suppose that (\mathcal{C}, H_Ω) is a relatively hyperbolic space with respect to a family \mathcal{S}_0 of properly embedded simplices in \mathcal{C} of dimension at least two. It is fairly easy to show that \mathcal{S}_0 is isolated and coarsely complete, but we will have to modify \mathcal{S}_0 to construct a Λ -invariant family.

By **Theorem 4.6**, for any $r > 0$, there exists $Q_1(r) > 0$ such that

$$\text{diam}_\Omega(\mathcal{N}_\Omega(S_1; r) \cap \mathcal{N}_\Omega(S_2; r)) \leq Q_1(r)$$

when $S_1, S_2 \in \mathcal{S}_0$ are distinct. Further, by [Observation 5.3](#) and [Theorem 4.9](#), there exists $Q_2 > 0$ such that, if $S \subset \mathcal{C}$ is a properly embedded simplex of dimension at least two, then there exists $S' \in \mathcal{S}_0$ such that

$$(23) \quad S \subset \mathcal{N}_\Omega(S'; Q_2).$$

Lemma 14.1 *If $S \in \mathcal{S}_0$ and $v \in \partial S$ is a vertex, then*

$$H_{F_\Omega(v)}^{\text{Haus}}(\{v\}, \bar{\mathcal{C}} \cap F_\Omega(v)) \leq Q_2.$$

Proof Suppose v, v_2, \dots, v_p are the vertices of S . If $w \in \bar{\mathcal{C}} \cap F_\Omega(v)$, then, by [Lemma 3.18](#),

$$\tilde{S} := \Omega \cap \mathbb{P}(\text{Span}\{w, v_2, \dots, v_p\})$$

is a properly embedded simplex in \mathcal{C} with

$$H_\Omega^{\text{Haus}}(S, \tilde{S}) \leq H_{F_\Omega(v)}(v, w).$$

Then there exists $S' \in \mathcal{S}_0$ such that

$$\tilde{S} \subset \mathcal{N}_\Omega(S'; Q_2).$$

Then, when $r > Q_2 + H_{F_\Omega(v)}(v, w)$,

$$\text{diam}_\Omega(\mathcal{N}_\Omega(S; r) \cap \mathcal{N}_\Omega(S'; r)) \geq \text{diam}_\Omega(\tilde{S}) = \infty.$$

So $S = S'$ and

$$\tilde{S} \subset \mathcal{N}_\Omega(S; Q_2).$$

Then [Proposition 3.13](#) implies that there exists $v' \in \bar{S} \cap F_\Omega(v)$ with

$$H_{F_\Omega(v)}(v', w) \leq Q_2.$$

But, by [Observation 5.4](#),

$$\{v\} = F_S(v) = \bar{S} \cap F_\Omega(v)$$

and so $v = v'$. So

$$H_{F_\Omega(v)}(v, w) \leq Q_2.$$

Finally, since $w \in \bar{\mathcal{C}} \cap F_\Omega(v)$ was arbitrary, this proves the lemma. □

Now we repeat part of the argument from [Section 10](#). In particular, for each simplex $S \in \mathcal{S}_0$ we construct a new simplex $\Phi(S)$ as follows. Let v_1, \dots, v_p be the vertices of S . Then, by the previous lemma,

$$H_{F_\Omega(v_j)}^{\text{Haus}}(\bar{C} \cap F_\Omega(v_j), \{v_j\}) \leq Q_2.$$

So $\bar{C} \cap F_\Omega(v_j)$ is a compact subset of $F_\Omega(v_j)$. Then let

$$w_j := \text{CoM}_{F_\Omega(v_j)}(\bar{C} \cap F_\Omega(v_j)).$$

Then [Lemma 3.18](#) implies that

$$\Phi(S) := \Omega \cap \mathbb{P}(\text{Span}\{w_1, \dots, w_p\})$$

is a properly embedded simplex with vertices w_1, \dots, w_p and

$$(24) \quad H_\Omega^{\text{Haus}}(S, \Phi(S)) \leq Q_2.$$

Then define

$$\mathcal{S} := \{\gamma\Phi(S) : \gamma \in \Lambda, S \in \mathcal{S}_0\}.$$

We will show that \mathcal{S} is isolated, coarsely complete and Λ -invariant, but first a preliminary lemma:

Lemma 14.2 *If $S \in \mathcal{S}$, $S' \in \mathcal{S}_0$ and*

$$S \subset \mathcal{N}_\Omega(S'; r)$$

for some $r > 0$, then $\Phi(S') = S$.

Proof Let v_1, \dots, v_p be the vertices of S . Then, by [Proposition 3.13](#), there exist $v'_1, \dots, v'_p \in \partial S'$ such that

$$v'_j \in F_\Omega(v_j)$$

for all $1 \leq j \leq p$. [Lemma 14.1](#) and the definition of Φ implies that $\bar{C} \cap F_\Omega(v_j)$ is a compact neighborhood of $\{v_j\}$ in $F_\Omega(v_j)$. Since

$$F_{S'}(v'_j) \subset \bar{C} \cap F_\Omega(v_j),$$

[Observation 5.4](#) implies that v'_j is a vertex of S' . Further, by [Lemma 3.18](#),

$$\text{ConvHull}_\Omega\{v'_1, \dots, v'_p\}$$

intersects Ω . Since S' is a properly embedded simplex, v'_1, \dots, v'_p must be all of the vertices of S' . Then, by definition, $\Phi(S') = S$. □

Lemma 14.3 \mathcal{S} is an isolated, coarsely complete and Λ -invariant family of maximal properly embedded simplices in \mathcal{C} of dimension at least two. Hence, $(\Omega, \mathcal{C}, \Lambda)$ has coarsely isolated simplices.

Proof By construction, \mathcal{S} is Λ -invariant.

We next argue that \mathcal{S} is isolated. Suppose $S_n \in \mathcal{S}$ converges to a closed set S in the local Hausdorff topology. Then S is a properly embedded simplex by [Observation 3.20](#). For each n , there exists $S'_n \in \mathcal{S}_0$ such that

$$S_n \subset \mathcal{N}_\Omega(S'_n; Q_2).$$

Since $\lim_{n \rightarrow \infty} S_n = S$, we have

$$\infty = \lim_{n \rightarrow \infty} \text{diam}_\Omega(\mathcal{N}_\Omega(S'_n; Q_2 + 1) \cap \mathcal{N}_\Omega(S'_{n+1}; Q_2 + 1)).$$

So there exists $N \geq 0$ such that $S'_n = S'_N$ for all $n \geq N$. Then, by [Lemma 14.2](#),

$$S_n = \Phi(S'_n) = \Phi(S'_N)$$

for $n \geq N$. So $S = S_N$. Since $S_n \in \mathcal{S}$ was an arbitrary convergent subsequence, the set S is closed and discrete in the local Hausdorff topology, and hence isolated.

Finally, we show that \mathcal{S} is coarsely complete. Since \mathcal{S}_0 is coarsely complete, if $S \subset \mathcal{C}$ is a properly embedded simplex of dimension at least two, then there exists $S' \in \mathcal{S}_0$ such that

$$S \subset \mathcal{N}_\Omega(S'; Q_2).$$

Then, by [\(24\)](#),

$$S \subset \mathcal{N}_\Omega(S''; 2Q_2),$$

where $S'' := \Phi(S') \in \mathcal{S}$. □

Part III The convex cocompact case

15 Lines and corners in the boundary

In this section we prove the following result, which we will use to verify properties (7) and (8) in [Theorem 1.8](#):

Proposition 15.1 Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $\Lambda \leq \text{Aut}(\Omega)$ is convex cocompact. Assume that the family \mathcal{S}_{\max} of all maximal properly embedded simplices in $\mathcal{C} := \mathcal{C}_\Omega(\Lambda)$ of dimension at least two satisfies the following:

- (1) \mathcal{S}_{\max} is strongly isolated.
- (2) If $S \in \mathcal{S}_{\max}$ and $x \in \partial S$, then $F_\Omega(x) = F_S(x)$.

Then:

- (a) If $\ell \subset \partial_i \mathcal{C}$ is a nontrivial line segment, then there exists $S \in \mathcal{S}_{\max}$ with $\ell \subset \partial S$.
- (b) If $y \in \partial_i \mathcal{C}$ is not a C^1 -smooth point of $\partial\Omega$, then there exists $S \in \mathcal{S}_{\max}$ with $y \in \partial S$.

Remark 15.2 In Section 16 we will show that if $\Lambda \leq \text{Aut}(\Omega)$ is convex cocompact and \mathcal{S}_{\max} is an isolated family of properly embedded simplices, then conditions (1) and (2) are automatically satisfied.

We will need the following observation about convex cocompact subgroups:

Proposition 15.3 [12, Lemma 4.1(1)] Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $\Lambda \leq \text{Aut}(\Omega)$ is convex cocompact. If $x \in \partial_i \mathcal{C}_\Omega(\Lambda)$, then $F_\Omega(x) \subset \partial_i \mathcal{C}_\Omega(\Lambda)$.

We start the proof of Proposition 15.1 with some general lemmas.

Lemma 15.4 Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $\Lambda \leq \text{Aut}(\Omega)$ is convex cocompact and $\mathcal{C} := \mathcal{C}_\Omega(\Lambda)$. Assume $\ell \subset \partial_i \mathcal{C}$ is a nontrivial open line segment, $m \in \ell$, $q \in \mathcal{C}$ and $V = \text{Span}\{\ell, q\}$. For any $r > 0$ and $\epsilon > 0$, there exists a neighborhood U of m in $\mathbb{P}(V)$ such that, if $p \in U \cap \mathcal{C}$, then there exists a properly embedded simplex $S = S(p) \subset \mathcal{C}$ of dimension at least two such that

$$(25) \quad B_\Omega(p; r) \cap \mathbb{P}(V) \subset \mathcal{N}_\Omega(S; \epsilon).$$

Proof The argument is very similar to the proof of Lemma 11.4.

Fix $r > 0$ and $\epsilon > 0$. Suppose for a contradiction that such a neighborhood U does not exist. Then we can find $p_n \in \mathcal{C} \cap \mathbb{P}(V)$ such that $\lim_{n \rightarrow \infty} p_n = m$ and p_n does not satisfy (25) for any properly embedded simplex in \mathcal{C} of dimension at least two.

By replacing ℓ with the maximal open line segment containing, it we can assume that $\ell = (a, b)$, where $a, b \in \partial F_\Omega(m)$.

After passing to a subsequence, we can find $\gamma_n \in \Lambda$ such that $\gamma_n p_n \rightarrow p_\infty \in \mathcal{C}$. Passing to a further subsequence, we can suppose that $\gamma_n a \rightarrow a_\infty$, $\gamma_n b \rightarrow b_\infty$ and $\gamma_n q \rightarrow q_\infty$. Then $[a_\infty, b_\infty] \subset \partial_i \mathcal{C}$. Since $a, b \in \partial F_\Omega(m)$, we have

$$\infty = \lim_{n \rightarrow \infty} H_\Omega(p_n, (a, q) \cup (q, b)) = \lim_{n \rightarrow \infty} H_\Omega(\gamma_n p_n, (\gamma_n a, \gamma_n q) \cup (\gamma_n q, \gamma_n b)).$$

So $[a_\infty, q_\infty] \cup [q_\infty, b_\infty] \subset \partial_i \mathcal{C}$. Thus, $a_\infty, b_\infty, q_\infty$ are the vertices of a properly embedded simplex $S \subset \mathcal{C}$. However, for n sufficiently large, we have

$$B_\Omega(\gamma_n p_n; r) \cap \gamma_n \mathbb{P}(V) \subset \mathcal{N}_\Omega(S; \epsilon)$$

and so

$$B_\Omega(p_n; r) \cap \mathbb{P}(V) \subset \mathcal{N}_\Omega(\gamma_n^{-1} S; \epsilon).$$

Hence, we have a contradiction. □

Lemma 15.5 *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $\Lambda \leq \text{Aut}(\Omega)$ is convex cocompact and $\mathcal{C} := \mathcal{C}_\Omega(\Lambda)$. Assume $z \in \partial_i \mathcal{C}$ is not a C^1 -smooth point of $\partial\Omega$ and $q \in \mathcal{C}$. For any $r > 0$ and $\epsilon > 0$, there exists $q_{r,\epsilon} \in (z, q]$ such that, if $p \in (z, q_{r,\epsilon})$, then there exists a properly embedded simplex $S = S(p) \subset \mathcal{C}$ of dimension at least two such that*

$$(26) \quad B_\Omega(p; r) \cap (z, q] \subset \mathcal{N}_\Omega(S; \epsilon).$$

Proof Once again, the argument is very similar to the proof of Lemma 11.4.

Fix $r > 0$ and $\epsilon > 0$. Suppose for a contradiction that such a $q_{r,\epsilon} \in (z, q]$ does not exist. Then we can find $p_n \in (z, q]$ such that $\lim_{n \rightarrow \infty} p_n = z$ and p_n does not satisfy (26) for any properly embedded simplex in \mathcal{C} of dimension at least two.

We can find a 3-dimensional linear subspace V such that $(z, q] \subset \mathbb{P}(V)$ and $z \in \partial_i \mathcal{C}$ is not a C^1 -smooth boundary point of $\mathbb{P}(V) \cap \Omega$. By changing coordinates, we can suppose that

$$\begin{aligned} \mathbb{P}(V) &= \{[w : x : y : 0 : \cdots : 0] : w, x, y \in \mathbb{R}\}, \\ \mathbb{P}(V) \cap \Omega &\subset \{[1 : x : y : 0 : \cdots : 0] : x \in \mathbb{R}, y > |x|\}, \\ z &= [1 : 0 : 0 : \cdots : 0], \\ q &= [1 : 0 : 1 : 0 : \cdots : 0]. \end{aligned}$$

We may also assume that $\mathbb{P}(V) \cap \Omega$ is bounded in the affine chart

$$\{[1 : x : y : 0 : \cdots : 0] : x, y \in \mathbb{R}\}$$

of $\mathbb{P}(V)$.

Then

$$p_n = [1 : 0 : y_n : 0 : \cdots : 0],$$

where $0 < y_n < 1$ and y_n converges to 0. Let

$$L_n := \{[1 : x : y_n : 0 : \cdots : 0] : x \in \mathbb{R}\} \cap \Omega.$$

By passing to a subsequence, we can suppose that $(y_n)_{n \geq 1}$ is a decreasing sequence and

$$(27) \quad \lim_{n \rightarrow \infty} H_\Omega(p_n, L_{n-1}) = \infty.$$

Then

$$\lim_{n \rightarrow \infty} \frac{y_{n-1}}{y_n} = \infty.$$

Let $a_n, b_n \in \partial\Omega$ be the endpoints of $L_n = (a_n, b_n)$. We claim that

$$(28) \quad \lim_{n \rightarrow \infty} H_\Omega(p_n, (z, a_{n-1})) = \infty = \lim_{n \rightarrow \infty} H_\Omega(p_n, (z, b_{n-1})).$$

Consider $g_n \in \text{PGL}(V)$ defined by

$$g_n([w : x : y : 0 : \cdots : 0]) = \left[w : \frac{1}{y_n}x : \frac{1}{y_n}y : \cdots : 0 \right].$$

Since $(y_n)_{n \geq 1}$ is a decreasing sequence converging to zero, $D_n := g_n(\mathbb{P}(V) \cap \Omega)$ is an increasing sequence of properly convex domains in $\mathbb{P}(V)$ and

$$D := \bigcup_{n \geq 1} D_n \subset \{[1 : x : y : 0 : \cdots : 0] : x \in \mathbb{R}, y > |x|\}$$

is also a properly convex domain. Notice that H_{D_n} converges to H_D uniformly on compact subsets of D . Also, by construction, there exist $t \leq -1$ and $1 \leq s$ such that

$$D = \{[1 : x : y : 0 : \cdots : 0] : x \in \mathbb{R}, y > \max\{sx, tx\}\}.$$

Then $a_n = [1 : t_n^{-1}y_n : y_n : 0 : \cdots : 0]$, where $t_n \rightarrow t$.

Now pick $v_n \in (z, a_{n-1})$ such that

$$H_\Omega(p_n, (z, a_{n-1})) = H_\Omega(p_n, v_n).$$

Since

$$\lim_{n \rightarrow \infty} g_n a_{n-1} = \lim_{n \rightarrow \infty} \left[1 : t_{n-1}^{-1} \frac{y_{n-1}}{y_n} : \frac{y_{n-1}}{y_n} : 0 : \dots : 0 \right] = [0 : t^{-1} : 1 : 0 : \dots : 0],$$

any limit point of $g_n v_n$ is in

$$\{[0 : t^{-1} : 1 : 0 : \dots : 0]\} \cup \{[1 : r t^{-1} : r : 0 : \dots : 0] : r \geq 0\} \subset \partial D.$$

Then

$$\lim_{n \rightarrow \infty} H_\Omega(p_n, (z, a_{n-1})) = \lim_{n \rightarrow \infty} H_\Omega(p_n, v_n) = \lim_{n \rightarrow \infty} H_{D_n}(g_n p_n, g_n v_n) = \infty$$

since $g_n p_n \rightarrow [1 : 0 : 1 : 0 : \dots : 0] \in D$.

For the same reasons,

$$\lim_{n \rightarrow \infty} H_\Omega(p_n, (z, b_{n-1})) = \infty.$$

This establishes (28).

Next we can pass to a subsequence and find $\gamma_n \in \Lambda$ such that $\gamma_n p_n \rightarrow p_\infty \in \mathcal{C}$. Passing to a further subsequence, we can suppose that $\gamma_n a_{n-1} \rightarrow a_\infty$, $\gamma_n b_{n-1} \rightarrow b_\infty$, $\gamma_n z \rightarrow z_\infty$ and $\gamma_n q \rightarrow q_\infty$.

Equation (27) implies that $[a_\infty, b_\infty] \subset \partial\Omega$ and (28) implies that

$$[z_\infty, a_\infty] \cup [z_\infty, b_\infty] \subset \partial\Omega.$$

Thus, a_∞, b_∞ and z_∞ are the vertices of a properly embedded simplex $S \subset \Omega$ which contains p_∞ . Further, for n sufficiently large, we have

$$B_\Omega(\gamma_n p_n; r) \cap \gamma_n(z, q) \subset \mathcal{N}_\Omega(S; \epsilon)$$

and so

$$B_\Omega(p_n; r) \cap (z, q) \subset \mathcal{N}_\Omega(\gamma_n^{-1} S; \epsilon).$$

To obtain a contradiction, we have to show that $\gamma_n^{-1} S \subset \mathcal{C}$ for every n or, equivalently, that $S \subset \mathcal{C}$. By construction, $q_\infty \in \partial_i \mathcal{C} \cap (a_\infty, b_\infty)$. Then Proposition 15.3 implies that $[a_\infty, b_\infty] \subset \partial_i \mathcal{C}$. Since $z_\infty \in \partial_i \mathcal{C}$ and S has vertices a_∞, b_∞ and z_∞ , we then see that $S \subset \mathcal{C}$. □

The rest of the section is devoted to the proof of Proposition 15.1, so suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $\Lambda \leq \text{Aut}(\Omega)$ is convex cocompact and the family \mathcal{S}_{\max} of all maximal properly embedded simplices in $\mathcal{C} := \mathcal{C}_\Omega(\Lambda)$ of dimension at least two satisfies the hypotheses of the proposition.

Lemma 15.6 *If $\ell \subset \partial_1 \mathcal{C}$ is a nontrivial line segment, then there exists $S \in \mathcal{S}_{\max}$ with $\ell \subset \partial S$.*

Proof We can assume that ℓ is an open line segment. Then fix some $m \in \ell$ and $q \in \mathcal{C}$. Since \mathcal{S}_{\max} is strongly isolated, there exists some $D > 0$ such that, if $S_1, S_2 \in \mathcal{S}_{\max}$ are distinct, then

$$\text{diam}_\Omega(\mathcal{N}_\Omega(S_1; 1) \cap \mathcal{N}_\Omega(S_2; 1)) < D.$$

Let $V := \text{Span}\{\ell, q\}$. By Lemma 15.4, there exists a neighborhood U of m in $\mathbb{P}(V)$ such that, if $x \in U \cap \mathcal{C}$, then there exists a maximal properly embedded simplex $S_x \subset \mathcal{C}$ of dimension at least two such that

$$B_\Omega(x; D) \cap \mathbb{P}(V) \subset \mathcal{N}_\Omega(S_x; 1).$$

By shrinking U , we can assume that $U \cap \mathcal{C}$ is convex.

We claim that $S_x = S_y$ for every $x, y \in U \cap \mathcal{C}$. Since $U \cap \mathcal{C}$ is convex, it is enough to show this when $H_\Omega(x, y) \leq \frac{1}{2}D$. Then

$$B_\Omega(x; \frac{1}{2}D) \cap \mathbb{P}(V) \subset B_\Omega(y; D) \cap \mathbb{P}(V) \subset \mathcal{N}_\Omega(S_y; 1).$$

So

$$B_\Omega(x; \frac{1}{2}D) \cap \mathbb{P}(V) \subset \mathcal{N}_\Omega(S_x; 1) \cap \mathcal{N}_\Omega(S_y; 1)$$

and hence

$$\text{diam}_\Omega(\mathcal{N}_\Omega(S_x; 1) \cap \mathcal{N}_\Omega(S_y; 1)) \geq \text{diam}_\Omega(B_\Omega(x; \frac{1}{2}D) \cap \mathbb{P}(V)) = D.$$

So $S_x = S_y$.

Now let $S := S_x$ for some (and hence any) $x \in U \cap \mathcal{C}$. Then

$$U \cap \mathcal{C} \subset \mathcal{N}_\Omega(S; 1).$$

So, by Proposition 3.13, there exists $m' \in \partial S$ with $m \in F_\Omega(m')$. Then, since ℓ is an open line segment, $\ell \subset F_\Omega(m')$. Finally, by condition (2) of the hypotheses, $F_\Omega(m') = F_S(m') \subset \partial S$. Hence, $\ell \subset \partial S$. □

Lemma 15.7 *If $z \in \partial_1 \mathcal{C}$ is not a C^1 -smooth point of $\partial\Omega$, then there exists $S \in \mathcal{S}_{\max}$ with $z \in \partial S$.*

Proof Fix $q \in \mathcal{C}$. Arguing as in the proof of [Lemma 15.6](#) and using [Lemma 15.5](#) shows that there exist some $q_0 \in (z, q]$ and a maximal properly embedded simplex $S \subset \mathcal{C}$ of dimension at least two such that

$$(z, q_0] \subset \mathcal{N}_\Omega(S; 1).$$

Then, by [Proposition 3.13](#), there exists $z' \in \partial S$ with $z \in F_\Omega(z')$. Finally, by condition (2) of the hypotheses on \mathcal{S}_{\max} ,

$$z \in F_\Omega(z') = F_S(z') \subset \partial S. \quad \square$$

16 Proof of Theorems 1.7 and 1.8

For the rest of this section, suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $\Lambda \leq \text{Aut}(\Omega)$ is a convex cocompact subgroup (see [Definition 1.1](#)) and \mathcal{S}_{\max} is the family of all maximal properly embedded simplices in $\mathcal{C}_\Omega(\Lambda)$ of dimension at least two.

16.1 Proof of Theorem 1.7

We claim that the following are equivalent:

- (A) \mathcal{S}_{\max} is closed and discrete in the local Hausdorff topology induced by H_Ω .
- (B) $(\Omega, \mathcal{C}_\Omega(\Lambda), \Lambda)$ has coarsely isolated simplices.
- (C) $(\mathcal{C}_\Omega(\Lambda), H_\Omega)$ is a relatively hyperbolic space with respect to \mathcal{S}_{\max} .
- (D) $(\mathcal{C}_\Omega(\Lambda), H_\Omega)$ is a relatively hyperbolic space with respect to a family of properly embedded simplices in $\mathcal{C}_\Omega(\Lambda)$ of dimension at least two.
- (E) Λ is a relatively hyperbolic group with respect to a collection of virtually abelian subgroups of rank at least two.

By definition, (A) implies (B) and (C) implies (D). Further, [Theorem 1.13](#) implies that (B), (D) and (E) are all equivalent. So it is enough to assume (B) and show that (A) and (C) hold. We establish this using [Theorem 1.18](#) and the next lemma.

Lemma 16.1 *If $(\Omega, \mathcal{C}_\Omega(\Lambda), \Lambda)$ has coarsely isolated simplices, then \mathcal{S}_{\max} is strongly isolated, coarsely complete and Λ -invariant. Moreover, if $S \in \mathcal{S}_{\max}$ and $x \in \partial S$, then $F_\Omega(x) = F_S(x)$.*

Remark 16.2 A careful reading of the proof shows that \mathcal{S}_{\max} is actually the *unique* family of strongly isolated, coarsely complete and Λ -invariant maximal properly embedded simplices in $\mathcal{C}_\Omega(\Lambda)$ of dimension at least two.

Proof By [Theorem 1.17](#), there exists $\mathcal{S}_{\text{core}}$, a strongly isolated, coarsely complete and Λ -invariant family of maximal properly embedded simplices in $\mathcal{C}_\Omega(\Lambda)$ of dimension at least two.

We first claim that, if $S \in \mathcal{S}_{\text{core}}$ and $x \in \partial S$, then

$$(29) \quad F_\Omega(x) = F_S(x).$$

By definition, $F_S(x) \subset F_\Omega(x)$. To establish the other inclusion, it suffices to show that, if $e \in \partial F_\Omega(x)$ is an extreme point, then $e \in \partial F_S(x)$.

So let $e \in \partial F_\Omega(x)$ be an extreme point. [Theorem 1.19\(4\)](#) implies that there exists $D_1 > 0$ such that

$$H_{F_\Omega(x)}^{\text{Haus}}(\overline{\mathcal{C}_\Omega(\Lambda)} \cap F_\Omega(x), F_S(x)) \leq D_1.$$

By [Proposition 15.3](#), $F_\Omega(x) = \overline{\mathcal{C}_\Omega(\Lambda)} \cap F_\Omega(x)$. Thus,

$$(30) \quad H_{F_\Omega(x)}^{\text{Haus}}(F_\Omega(x), F_S(x)) \leq D_1.$$

Then, by [Proposition 3.13](#) and (30), there exists

$$e' \in \partial F_S(x) \cap F_{F_\Omega(x)}(e).$$

But, since e is an extreme point, $F_{F_\Omega(x)}(e) = F_\Omega(e) = \{e\}$. So $e = e' \in \partial F_S(x)$. This proves the claim.

Next we show that $\mathcal{S}_{\text{core}} = \mathcal{S}_{\max}$. By definition, $\mathcal{S}_{\text{core}} \subset \mathcal{S}_{\max}$, so it is enough to show that $\mathcal{S}_{\max} \subset \mathcal{S}_{\text{core}}$. Fix $S \in \mathcal{S}_{\max}$. Since $\mathcal{S}_{\text{core}}$ is coarsely complete, there exist $S' \in \mathcal{S}_{\text{core}}$ and $r > 0$ such that

$$S \subset \mathcal{N}_\Omega(S'; r).$$

Then, by [Proposition 3.13](#) and (29),

$$\partial S \subset \bigcup_{x \in \partial S'} F_\Omega(x) = \bigcup_{x \in \partial S'} F_{S'}(x) = \partial S'.$$

Hence, $S \subset S'$. Since S is a maximal properly embedded simplex, we then have $S = S' \in \mathcal{S}_{\text{core}}$.

Finally, the “moreover” part follows from (29) and the equality $\mathcal{S}_{\max} = \mathcal{S}_{\text{core}}$. □

16.2 Proof of Theorem 1.8

Now assume, in addition to the hypotheses at the beginning of Section 16, that \mathcal{S}_{\max} is closed and discrete in the local Hausdorff topology induced by H_Ω .

By Lemma 16.1, \mathcal{S}_{\max} is strongly isolated, coarsely complete and Λ -invariant. Then properties (1), (2), (3) and (5) follow immediately from Theorem 1.19. Property (6) holds since \mathcal{S}_{\max} is strongly isolated. Property (4) is the “moreover” part of Lemma 16.1. Finally, properties (7) and (8) follow from Proposition 15.1.

Appendix Remarks on Theorem 4.15

In this appendix we explain how to modify Sisto’s arguments in [31] to establish Theorem 4.15. In fact, we will explain why a more general result is true. Before stating the result, we introduce a generalization of the notion of *asymptotically transverse-free* obtained by replacing geodesics in Definition 4.13 with $(1, c)$ -quasigeodesics.

Definition A.1 Let (X, d) be a complete geodesic metric space, $\alpha \geq 1, \beta \geq 0$ and \mathcal{S} be a collection of subsets of X .

- (1) If $I \subset \mathbb{R}$ is an interval, then $\theta: I \rightarrow X$ is a (α, β) -*quasigeodesic* in (X, d) if

$$\frac{1}{\alpha}|t_1 - t_2| - \beta \leq d(\theta(t_1), \theta(t_2)) \leq \alpha|t_1 - t_2| + \beta$$

for all $t_1, t_2 \in I$.

- (2) An (α, β) -*quasigeodesic triangle* in (X, d) is a choice of three points in X and (α, β) -quasigeodesics connecting these points.
- (3) A quasigeodesic triangle \mathcal{T} in X is \mathcal{S} -*almost-transverse with constants κ and Δ* if

$$\text{diam}_X(\mathcal{N}_X(S; \kappa) \cap \gamma) \leq \Delta$$

for every $S \in \mathcal{S}$ and edge γ of \mathcal{T} .

- (4) The collection \mathcal{S} is *strongly asymptotically transverse-free* if there exist λ and σ such that, for each $c \geq 1, \Delta \geq 1, \kappa \geq \sigma$, if \mathcal{T} is a $(1, c)$ -quasigeodesic triangle in X which is \mathcal{S} -almost-transverse with constants κ and Δ , then \mathcal{T} is $(\lambda\Delta + \lambda c)$ -thin.

We will prove the following generalization of Theorem 4.15, which connects the three different notions of “asymptotically transverse-free”:

Proposition A.2 *Let (X, d) be a complete geodesic metric space and \mathcal{S} a collection of subsets of X . Then the following are equivalent:*

- (1) *\mathcal{S} is asymptotically transverse-free relative to a geodesic path system and there exists an almost-projection system for \mathcal{S} .*
- (2) *\mathcal{S} is asymptotically transverse-free and there exists an almost-projection system for \mathcal{S} .*
- (3) *\mathcal{S} is strongly asymptotically transverse-free and there exists an almost-projection system for \mathcal{S} .*

In [Proposition A.2](#), observe that (3) implies (2) and (2) implies (1) by definition. Thus, in order to prove [Proposition A.2](#), it suffices to prove (1) implies (3). Sisto [[31](#), Lemma 2.13] previously proved that (2) implies (3) and in the rest of this section we modify Sisto’s argument to show that (1) implies (3).

Fix (X, d) a complete geodesic metric space, \mathcal{G} a geodesic path system on X , \mathcal{S} a collection of subsets of X , and $\Pi_{\mathcal{S}} = \{\pi_S : X \rightarrow S : S \in \mathcal{S}\}$ an almost-projection system with constant C . Then fix a constant

$$\sigma_0 \geq \max\{10C, 1\}.$$

Finally, for any pair of distinct points $x, y \in X$, let $\gamma_{x,y}$ denote a path in \mathcal{G} connecting x and y .

The proof of [Proposition A.2](#) will require the following two lemmas. Informally, the first one says that, if θ is an “ \mathcal{S} -almost-transverse quasigeodesic”, then any geodesic joining points on θ is also “ \mathcal{S} -almost-transverse”.

Lemma A.3 [[31](#), page 176] *Suppose $c > 0$, $\Delta \geq 1$, $\kappa \geq c\sigma_0$, $\theta : [0, T] \rightarrow X$ is a $(1, c)$ -quasigeodesic and*

$$\text{diam}_X(\mathcal{N}_X(S; \kappa) \cap \theta) \leq \Delta$$

for every $S \in \mathcal{S}$. Then

$$\text{diam}_X(\mathcal{N}_X(S; c\sigma_0) \cap \gamma_{x,y}) \leq \Delta + 10\sigma_0 + 18c\sigma_0$$

for every $S \in \mathcal{S}$ and $x, y \in \theta$.

[Lemma A.3](#) follows from Sisto’s proof of Lemma 2.13 in [[31](#)]. For the reader’s convenience we will provide the argument at the end of this section.

We also need the following variant of the Morse lemma:

Lemma A.4 Suppose $c > 0$, $x, y \in X$ and $\theta: [0, T] \rightarrow X$ is a $(1, c)$ -quasigeodesic with $x = \theta(0)$ and $y = \theta(T)$. Moreover, suppose that there exists $\delta \geq 0$ such that any triangle with all its vertices on θ and all its edges in \mathcal{G} is δ -thin. Then

$$d^{\text{Haus}}(\theta, \gamma_{x,y}) \leq 4\delta + 10c.$$

Delaying the proof of [Lemma A.4](#), we prove [Proposition A.2](#).

Proof of Proposition A.2 By the remarks above, it suffices to show that (1) implies (3). So suppose that \mathcal{S} is asymptotically transverse-free relative to the geodesic path system \mathcal{G} with constants $\lambda_{\mathcal{G}}$ and $\sigma_{\mathcal{G}}$.

By increasing σ_0 if necessary, we can assume that

$$\sigma_0 = \max\{10C, 1, \sigma_{\mathcal{G}}\}.$$

Then fix

$$\lambda_0 := \max\{9\lambda_{\mathcal{G}}(1 + 10\sigma_0), 20\sigma_0(1 + 9\lambda_{\mathcal{G}})\}.$$

Fix a $(1, c)$ -quasigeodesic triangle $\mathcal{T} := (\theta_1 \cup \theta_2 \cup \theta_3)$ that is \mathcal{S} -almost-transverse with constants κ and Δ where $c \geq 1$, $\kappa \geq c\sigma_0$, and $\Delta \geq 1$. We will show that \mathcal{T} is $(\lambda_0\Delta + \lambda_0c)$ -thin. Since \mathcal{T} is arbitrary, this will complete the proof that \mathcal{S} is strongly asymptotically transverse free and hence that (1) implies (3) in [Proposition A.2](#).

Let $\mathcal{T}_{\mathcal{G}}$ be a geodesic triangle with the same vertices as \mathcal{T} but edges in \mathcal{G} . Let γ_1, γ_2 and γ_3 be the edges of $\mathcal{T}_{\mathcal{G}}$ labeled so that the edge γ_i corresponds to the edge θ_i for all $1 \leq i \leq 3$. By [Lemma A.3](#), $\mathcal{T}_{\mathcal{G}}$ is \mathcal{S} -almost-transverse with constants $c\sigma_0$ and $\Delta + 10\sigma_0 + 18c\sigma_0$. Notice that $c\sigma_0 \geq \sigma_{\mathcal{G}}$ and $\Delta + 10\sigma_0 + 18c\sigma_0 \geq 1$. Since \mathcal{S} is asymptotically transverse-free relative to the geodesic path system \mathcal{G} , the triangle $\mathcal{T}_{\mathcal{G}}$ is δ -thin, where

$$(31) \quad \delta := \lambda_{\mathcal{G}}(\Delta + 10\sigma_0 + 18c\sigma_0).$$

[Lemma A.3](#) also show that, for each $1 \leq i \leq 3$, the $(1, c)$ -quasigeodesic θ_i and the geodesic $\gamma_i \in \mathcal{G}$ satisfy the hypothesis in [Lemma A.4](#) with δ as in (31). Thus,

$$(32) \quad \max_{1 \leq i \leq 3} d^{\text{Haus}}(\theta_i, \gamma_i) \leq 4\delta + 10c.$$

So \mathcal{T} is $(9\delta + 20c)$ -thin. Further,

$$\begin{aligned} 9\delta + 20c &= 9\lambda_{\mathcal{G}}(\Delta + 10\sigma_0 + 18c\sigma_0) + 20c \\ &< 9\lambda_{\mathcal{G}}(1 + 10\sigma_0)\Delta + 20\sigma_0(1 + 9\lambda_{\mathcal{G}})c \\ &\leq \lambda_0(\Delta + c). \end{aligned}$$

Thus, \mathcal{T} is $(\lambda_0\Delta + \lambda_0c)$ -thin. □

Proof of Lemma A.3 Before proving the lemma we need to recall two other estimates from Sisto’s paper.

Proposition A.5 (Sisto [31, Corollary 2.7]) *If $r \geq 2C$, $x_1, x_2 \in X$, $S \in \mathcal{S}$, $\pi_S \in \Pi_S$ and ξ is any geodesic in X connecting x_1 and x_2 , then*

$$\text{diam}_X(\xi \cap \mathcal{N}_X(S; r)) \leq d(\pi_S(x_1), \pi_S(x_2)) + 18r + 62C.$$

Proposition A.6 (Sisto [31, Lemma 2.10]) *If $x_1, x_2 \in X$, $S \in \mathcal{S}$, $\pi_S \in \Pi_S$, ξ is any geodesic in X connecting x_1 and x_2 , and $d(\pi_S(x_1), \pi_S(x_2)) \geq 8C + 1$, then ξ intersects $B_X(\pi_S(x_1); 10C)$, $B_X(\pi_S(x_2); 10C)$ and $\mathcal{N}_X(S; 2C)$.*

We now claim that

$$(33) \quad d(\pi_S(x), \pi_S(y)) \leq \Delta + 20C + 1.$$

If this is not true, then, by Proposition A.6, $\gamma_{x,y}$ intersects $B_\Omega(\pi_S(x); 10C)$ and $B_\Omega(\pi_S(y); 10C)$. Thus,

$$\text{diam}_X(\mathcal{N}_X(S; \kappa) \cap \gamma_{x,y}) \geq \text{diam}_X(\mathcal{N}_X(S; 10C) \cap \gamma_{x,y}) \geq \Delta + 1,$$

which is a contradiction. Hence, the estimate in (33) is true. Since $c\sigma_0 \geq 2C$, Proposition A.5 implies that

$$\text{diam}_X(\gamma_{x,y} \cap \mathcal{N}_X(S; c\sigma_0)) \leq \Delta + 18c\sigma_0 + 82C + 1 \leq \Delta + 10\sigma_0 + 18c\sigma_0. \quad \square$$

Proof of Lemma A.4 Let $M := 2\delta + 5c$. By a standard argument (see for instance [8, proof of Theorem 1.7, page 404]), it suffices to prove

$$\theta \subset \mathcal{N}_X(\gamma_{x,y}; M - c).$$

Fix $z \in \theta$ and consider the geodesic triangle $\gamma_{x,y} \cup \gamma_{x,z} \cup \gamma_{z,y}$. By hypothesis, this triangle is δ -thin. Next pick $a \in \gamma_{x,z}$ such that $d(z, a) = \delta + 4c$. If such a point does not exist, then $d(x, z) < \delta + 4c$, which implies that

$$z \in \mathcal{N}_X(\gamma_{x,y}; \delta + 4c) \subset \mathcal{N}_X(\gamma_{x,y}; M - c)$$

and we are done. Now, since $\gamma_{x,y} \cup \gamma_{x,z} \cup \gamma_{z,y}$ is δ -thin, there exists $b \in \gamma_{x,y} \cup \gamma_{z,y}$ such that $d(a, b) \leq \delta$. We will show that $b \in \gamma_{x,y}$. Since θ is a $(1, c)$ -quasigeodesic,

$$d(x, z) + d(z, y) \leq d(x, y) + 3c.$$

Then, for all $y' \in \gamma_{z,y}$, we have

$$\begin{aligned} d(a, y') &\geq d(x, y) - d(x, a) - d(y', y) \\ &\geq d(x, z) + d(z, y) - 3c - d(x, a) - d(y', y) \\ &= d(a, z) + d(y', z) - 3c \geq \delta + c. \end{aligned}$$

So we must have $b \in \gamma_{x,y}$. Then

$$d(z, \gamma_{x,y}) \leq d(z, a) + d(a, b) \leq 2\delta + 4c = M - c.$$

Since $z \in \theta$ was arbitrary,

$$\theta \subset \mathcal{N}_X(\gamma_{x,y}; M - c). \quad \square$$

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