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We show that properly and cocompactly cubulated relatively hyperbolic groups are virtually special, provided the peripheral subgroups are virtually special in a way that is compatible with the cubulation. This extends Agol’s result for cubulated hyperbolic groups, and applies to a wide range of peripheral subgroups. In particular, we deduce virtual specialness for properly and cocompactly cubulated groups that are hyperbolic relative to virtually abelian groups. As another consequence, by using a theorem of Martin and Steenbock we obtain virtual specialness for groups obtained as a quotient of a free product of finitely many virtually compact special groups by a finite set of relators satisfying the classical $C'(\frac{1}{6})$ –small cancellation condition.

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1 Introduction

CAT(0) cube complexes were introduced by Gromov in his seminal paper [17] as examples of singular metric spaces with nonpositive curvature. This notion has played a prominent role in the last decades, and has shown to have important connections with other aspects of topology and group theory. In particular, the class of *special cube complexes* introduced by Haglund and Wise [22; 23] was key to Agol’s proof of the Virtually Haken Conjecture, a result which turned out to be a consequence of the following property of cubulated hyperbolic groups [1, Theorem 1.1]:

Theorem 1.1 (Agol) *Let G be a hyperbolic group acting properly and cocompactly on a CAT(0) cube complex X . Then G has a finite index subgroup G' acting freely on X such that X/G' is special.*

For a good exposition about this result see Agol [2] and Shepherd [38]. Virtually special groups have finite-index subgroups embedding nicely into right-angled Artin groups, and so they inherit some of their properties, mainly in terms of subgroup separability.

In particular, hyperbolic groups satisfying Theorem 1.1 are residually finite, large, linear over \mathbb{Z} , and their quasiconvex subgroups are separable (in fact they are virtual retracts); see Haglund and Wise [22].

The assumption of hyperbolicity in Theorem 1.1 is in some sense necessary, since there are examples of infinite simple groups acting properly and cocompactly on products of trees; see Burger and Mozes [8]. The goal of this manuscript is to extend Agol's result to relatively hyperbolic groups, but for that we need to restrict our class of peripheral subgroups, since any countable group is hyperbolic relative to the whole group. The following is the main result of the paper.¹

Theorem 1.2 (Main Theorem) *Let G be a group acting properly and cocompactly on the CAT(0) cube complex X , and suppose G is hyperbolic relative to compatible virtually special subgroups. Then there exists a finite-index subgroup $G' < G$ acting freely on X such that X/G' is a special cube complex.*

The formal definition of *compatible virtually special peripheral subgroups* is given in Section 2.4, and it essentially means that if $P < G$ is a peripheral subgroup, then there exists *some* P -invariant convex subcomplex $Z \subset X$, and there is a finite-index subgroup $P' < P$ such that Z/P' is a compact and special cube complex. This compatibility is flexible enough so that Theorem 1.2 applies when G is cubulated and hyperbolic relative to virtually abelian subgroups.

Corollary 1.3 *If G acts properly and cocompactly on a CAT(0) cube complex and is hyperbolic relative to virtually abelian subgroups, then G is virtually compact special.*

The preceding corollary recovers some remarkable results, such as virtual specialness of fundamental groups of noncompact finite-volume hyperbolic 3-manifolds and of limit groups due to Wise [40]; see also Cooper and Futer [9] and Groves and Manning [20]. Another consequence of our main result depends on the combination theorem for cubulations in small cancellation theory due to Martin and Steenbock [29, Theorem 1.1].

Theorem 1.4 (Martin–Steenbock) *Let F be the free product of finitely many groups G_1, \dots, G_r , and assume each G_i acts properly and cocompactly on the CAT(0) cube complex X_i . If G is a quotient of F by a finite set of relators that satisfies the classical $C'(\frac{1}{6})$ -small cancellation condition over F , then G acts properly and cocompactly on a CAT(0) cube complex X .*

¹After this paper was written, Groves and Manning [21, Theorem A] gave a proof of Theorem 1.2 via different methods.

Moreover, this complex is constructed in such a way that for each i , there is a G_i -equivariant combinatorial isometric embedding $\dot{X}_i \hookrightarrow X$, where \dot{X}_i is the cubical barycentric subdivision of X_i .

The “moreover” part of the previous theorem is implicit in the construction of X ; see [29, Remark 3.43] and compare with Jankiewicz and Wise [27, Corollary 4.5]. The group G is hyperbolic relative to the free factors G_1, \dots, G_r (see Example (II) of Osin [33, page 2]), and since a compact nonpositively curved complex is virtually special if and only if its cubical barycentric division is virtually special (see Corollary A.3), the existence of equivariant isometric embeddings $\dot{X}_i \hookrightarrow X$ implies that the cubulation (G, X) satisfies the compatibility condition provided each of the cubulations (G_i, X_i) is virtually special. Hence Theorem 1.2 implies the following combination result:

Proposition 1.5 *Let F be the free product of finitely many virtually compact special groups. If G is a quotient of F by a finite set of relators satisfying the classical $C'(\frac{1}{6})$ -small cancellation condition over F , then G is also virtually compact special.*

By the work of Schreve [37, Theorem 1.2], virtually compact special groups satisfy the strong Atiyah conjecture. Martin and Steenbock used this result, together with Theorems 1.1 and 1.4 to deduce the strong Atiyah conjecture for quotients of free products of cubulated hyperbolic groups satisfying the $C'(\frac{1}{6})$ -small cancellation condition. If we use Proposition 1.5 instead, we can remove the hyperbolicity assumption on the free factors:

Corollary 1.6 *Let G be as in the assumptions of Proposition 1.5. Then G satisfies the strong Atiyah conjecture.*

A cubulated malnormal hierarchy A key tool in Agol’s proof of Theorem 1.1 is Wise’s quasiconvex hierarchy theorem (see Theorem 2.22), which says that a hyperbolic group is virtually compact special if and only if it can be obtained from finite groups after a finite sequence of (virtual) amalgamations or HNN extensions over quasiconvex subgroups. Our proof of Theorem 1.2 follows that approach, and for that we require a notion of hierarchy that is compatible with relatively hyperbolic groups.

Definition 1.7 Let \mathcal{CMVH} denote the smallest class of cubulated and relatively hyperbolic groups (G, X) (here G acts properly and cocompactly on the cubulation X) relative to compatible virtually special subgroups, which is closed under the following operations:

- (i) $(\{1\}, X) \in \mathcal{CMVH}$ for any finite CAT(0) cube complex X .

- (ii) If G splits as a finite graph of groups (Γ, \mathcal{G}) such that
- each edge/vertex group is convex in (G, X) ,
 - if v is a vertex of Γ then the collection $\mathcal{A}_v := \{G_e : e \text{ an edge attached to } v\}$ is *relatively malnormal* in G_v , and
 - if G_v is a vertex group, then it has a convex core $X_v \subset X$ with $(G_v, X_v) \in \mathcal{CMVH}$,

then $(G, X) \in \mathcal{CMVH}$.

- (iii) If $H < G$ with $|G : H| < \infty$ and $(H, X) \in \mathcal{CMVH}$, then $(G, X) \in \mathcal{CMVH}$.

The notation \mathcal{CMVH} is meant to be an abbreviation of “cubulated (relatively) malnormal virtual hierarchy”, and the main concepts involved in this definition are explained in Section 2. In Section 4 we give a relative version of Wise’s quasiconvex hierarchy theorem and prove:

Theorem 1.8 *If $(G, X) \in \mathcal{CMVH}$, then there is a finite-index subgroup $G' < G$ acting freely on X such that X/G' is special.*

Note that the compatibility between the peripheral structure on G and the splitting (Γ, \mathcal{G}) in condition (ii) of the definition above is only reflected in the relative quasiconvexity of edge/vertex subgroups (this is implied by convexity, see Theorem 2.4) and the relative malnormality of edge groups in the vertex groups. This seems to be a weak assumption when we compare, for instance, with the quasiconvex, malnormal and \mathcal{P} -fully elliptic hierarchy that requires fully relative quasiconvexity of edge/vertex groups (see eg [12, Section 3]), and is extremely important in our argument, particularly in Proposition 4.4.

The second main ingredient in Agol’s proof of Theorem 1.1 is the “coloring” trick to produce a hierarchy for a cubulated hyperbolic group. By adapting this technique to a cubulated relatively hyperbolic group, we prove the theorem below, which together with Theorem 1.8 and Corollary A.3 implies Theorem 1.2.

Theorem 1.9 *If (G, X) is cubulated and hyperbolic relative to compatible virtually special subgroups, then $(G, \dot{X}) \in \mathcal{CMVH}$, where \dot{X} is the cubical barycentric subdivision of X .*

While the theorem above is enough to deduce our main result, virtual compact specialness for relatively hyperbolic groups implies the existence of strong virtual hierarchies, as was proven by Einstein [12, Theorem 1]. In consequence, Theorem 1.2 implies:

Corollary 1.10 *Let (G, \mathcal{P}) be a cubulated relatively hyperbolic group with compatible virtually special subgroups. Then there exists a finite-index subgroup $G_0 < G$ with induced relatively hyperbolic structure (G_0, \mathcal{P}_0) so that G_0 has a quasiconvex, malnormal and fully \mathcal{P}_0 -elliptic hierarchy terminating in groups isomorphic to elements of \mathcal{P}_0 .*

Organization of the paper In Section 2 we present the required background, recalling the main properties about relatively hyperbolic groups, cubulations, and special cube complexes. There we also define compatibility of virtually special peripheral subgroups, deduce Corollary 1.3 and Proposition 1.5, and state Proposition 2.24, which shows that the compatibility of the peripherals is independent on the convex core.

In Section 3 we introduce Dehn filling, the main tool in the proof of Theorem 1.8, as well as extend some properties of H -wide fillings in Theorems 3.9 and 3.21. We also introduce Einstein's relative malnormal special quotient theorem, a fundamental result that, by means of Wise's quasiconvex hierarchy theorem, will allow us to obtain hyperbolic virtually special fillings for relatively hyperbolic groups in \mathcal{CMVH} . All these results will be used in Section 4 to prove Theorem 1.8.

The rest of the paper consists in proving Theorem 1.9, by adapting Agol's proof that cubulated hyperbolic groups are in \mathcal{QVH} . In Section 5 we construct a quotient cube complex \mathcal{X} for X with finite embedded walls, which will be used to model the desired hierarchy. We color the walls of \mathcal{X} in Section 6, and in Section 7 we use this coloring to start the construction of the *cubical polyhedra*, an inductively defined collection of (disconnected) cube complexes with *boundary walls* that codify the description of (G, X) as a cubulated group in \mathcal{CMVH} . We study these cubical polyhedra in more detail in Section 8, and conclude the proof of Theorem 1.9 in Section 9, where we show how to perform the inductive construction.

An appendix is included at the end of the article, in which we prove Theorem A.1, which is used in the proof of Theorem 1.8 and says that double cosets of convex subgroups of virtually compact special groups are separable. This is done by studying the functoriality of the canonical completion, and implies Proposition 2.24.

2 Preliminaries

We start by introducing the main concepts and notation about relative hyperbolicity, $\text{CAT}(0)$ cube complexes and special cube complexes, which will be used in the paper.

We expect the reader to be familiar with hyperbolic groups [17] and CAT(0) metric spaces [7].

2.1 Relatively hyperbolic groups

There are several equivalent definitions for relatively hyperbolic groups [24], some of them valid for arbitrary countable groups. For convenience we will restrict to the finitely generated case and introduce relative hyperbolicity in terms of *cusped spaces*, which will be useful in Section 3. See [18] or [1, Appendix A] for more details about cusped spaces associated to relatively hyperbolic groups.

Let G be a group generated by a finite symmetric set S , and let $\mathcal{P} = \{P_1, \dots, P_n\}$ be a finite collection of subgroups of G such that $S \cap P_i$ generates P_i for every i .

For each $P \in \mathcal{P}$, let $S_0 = S \cap P \setminus \{1\}$, and for $n > 0$ let

$$S_n := S_{n-1} \cup \{s_1 s_2 \neq 1 : s_1, s_2 \in S_{n-1}\}.$$

Given a left coset gP with $g \in G$, define the 1-complex $\mathcal{H}(gP)$ as the vertex set $\mathcal{H}(gP)^{(0)} = gP \times \mathbb{Z}_{\geq 0}$ and edges given by:

- (i) **Vertical** If $(v, n) \in \mathcal{H}(gP)^{(0)}$, then an edge joins (v, n) and $(v, n + 1)$.
- (ii) **Horizontal** If $(v, n) \in \mathcal{H}(gP)^{(0)}$ and $s \in S_n$, there is an edge from (v, n) to (vs, n) ; so that if for instance s has order 2, then there are two different edges between (v, n) and (vs, n) .

Note that there is a natural way to glue $\mathcal{H}(gP)$ and the Cayley graph $\Gamma(G, S)$ along $gP = gP \times \{0\}$.

Definition 2.1 The *cusped space* $X(G, \mathcal{P}, S)$ is obtained from $\Gamma(G, S)$ by gluing all the complexes $\mathcal{H}(gP)$ for $g \in G$ and $P \in \mathcal{P}$ in the previously mentioned way. The group G is then *hyperbolic relative to \mathcal{P}* if the cusped space $X(G, \mathcal{P}, S)$ is Gromov hyperbolic [1, Remark A.13]. In that case we say that \mathcal{P} is a *peripheral structure* on G .

Remark 2.2 With the definition of cusped space given above, the natural isometric action of G on $X(G, \mathcal{P}, S)$ is *free*, and the distance function on vertices coincides with the one constructed in [18]. Therefore, the coarse geometry is unchanged, and the classical results about cusped spaces also hold for this slightly different construction; see [1, Remark A.13].

If (G, \mathcal{P}) is relatively hyperbolic, then a *parabolic subgroup* will be any subgroup of G that can be conjugated into a member of \mathcal{P} , and a maximal parabolic subgroup will be called *peripheral subgroup*. An element of G is *loxodromic* if it has infinite order and the group it generates is not parabolic.

A *horoball* of $X(G, \mathcal{P}, S)$ is a full subgraph on the vertices $gP \times \mathbb{Z}_{\geq R}$ for some $R \geq 0$ and some left coset gP with $P \in \mathcal{P}$. Note that peripheral subgroups correspond to stabilizers in G of horoballs.

An infinite subgroup of G is nonparabolic if and only if it contains a loxodromic element. Namely, if $H < G$ is infinite and with no loxodromics, then any H -orbit of a point in $X(G, \mathcal{P}, S)$ is unbounded and H must fix a point in the Gromov boundary $\partial X(G, \mathcal{P}, S)$; see eg [10, Theorem 6.2.3]. The action of G on $\partial X(G, \mathcal{P}, S)$ is geometrically finite [24, Section 3], so this fixed point corresponds to a horoball by [39, Theorem 3A(a)].

2.2 Relative quasiconvexity

Let $H < G$ be a finitely generated subgroup of a relatively hyperbolic group (G, \mathcal{P}) with cusped space $X = X(G, \mathcal{P}, S)$, and suppose there exists a finite set \mathcal{D} of representatives for H -conjugacy classes of the infinite groups of the form $H \cap P^g$ with $g \in G$ and $P \in \mathcal{P}$. Given $D \in \mathcal{D}$, there is a unique $P_D \in \mathcal{P}$ and some $c_D \in G$ of shortest word-length such that $D = H \cap P_D^{c_D}$. Also, assume that each group in \mathcal{D} is finitely generated, and let X_H be a combinatorial cusped space for the pair (H, \mathcal{D}) with respect to some compatible finite generating set S' of H , in the sense that S' is symmetric and $S' \cap D$ generates D for each $D \in \mathcal{D}$. We extend the inclusion $\iota: H \hookrightarrow G$ to an H -equivariant Lipschitz map $\check{\iota}: X_H^{(0)} \rightarrow X$ as follows: a vertex in a horoball of X_H is a tuple (sD, h, n) with $s \in H$, $D \in \mathcal{D}$, $h \in sD$ and $n \in \mathbb{Z}_{\geq 0}$, so define its image by $\check{\iota}(sD, h, n) := (sc_D P_D, hc_D, n)$.

Definition 2.3 The pair (H, \mathcal{D}) is *relatively quasiconvex* in (G, \mathcal{P}) if the image $\check{\iota}(X_H^{(0)}) \subset X$ is λ -quasiconvex for some λ , which will be called a *quasiconvexity constant* for (H, \mathcal{D}) in (G, \mathcal{P}) . Sometimes we will omit the peripheral structures and simply say that H is relatively quasiconvex in G .

As noted in [20, Definition 2.9], this definition is equivalent to other notions of relative quasiconvexity existing in the literature, at least in the finitely generated case [24]. In particular, if $H < G$ is relatively quasiconvex, then the collection \mathcal{D} defined above makes (H, \mathcal{D}) into a relatively hyperbolic group [24, Theorem 9.1], and we call \mathcal{D} an *induced relatively hyperbolic structure* (or *peripheral structure*) on H .

The characterization of relative quasiconvexity given above will be helpful in Section 3, while for the rest of the paper we will use the criterion stated below. A subgroup $H < G$ of a finitely generated group G is *undistorted* if it is finitely generated and the inclusion map $H \hookrightarrow G$ is a quasi-isometric embedding with respect to some (hence any) choice of finite sets of generators for H and G . Hruska proved the following criterion for relative quasiconvexity valid for finitely generated relatively hyperbolic groups [24, Theorem 1.5].

Theorem 2.4 (Hruska) *If $H < G$ is an undistorted subgroup, then H is relatively quasiconvex in (G, \mathcal{P}) with respect to any possible peripheral structure \mathcal{P} on G .*

An important class of relatively quasiconvex subgroups is given by those which are full, and in a sense this is the correct generalization to quasiconvex subgroups of hyperbolic groups.

Definition 2.5 A relatively quasiconvex subgroup $H < G$ of a relatively hyperbolic group (G, \mathcal{P}) is *fully relatively quasiconvex* if for any peripheral subgroup P of G , the group $H \cap P$ is either finite or finite-index in P .

2.3 Cube complexes and cubulated groups

A *cube complex* is a metric polyhedral complex in which all polyhedra are unit-length Euclidean cubes. Such a complex is *nonpositively curved* (NPC) if its universal cover is a CAT(0) metric space when endowed with the induced length distance. There is a combinatorial description of this property, due to Gromov [7, Theorem II.5.2].

Proposition 2.6 *A cube complex X is NPC if and only if the link of each vertex is a flag complex.*

Recall that the *link* of a vertex v is the complex $\text{Lk}_X(v)$ with vertices the edges of X incident to v , and in which n such edges are the 1-skeleton of an $(n-1)$ -face in $\text{Lk}_X(v)$ if and only if they are incident to a common n -cell at v . Also, a *flag simplicial complex* is a complex determined by its 1-skeleton: for every complete subgraph of the 1-skeleton there is a simplex with 1-skeleton equal to that subgraph. For more references about the geometry of NPC cube complexes, see [7; 35]. An n -cube $C = [0, 1]^n \subset \mathbb{R}^n$ has n -midcubes obtained by setting one coordinate to $\frac{1}{2}$. Since the face of a midcube of C is the midcube of a face of C , the set of midcubes of a cube complex X has a cube complex structure, called the *wall complex*. A *wall* of X is a connected component of the wall complex. When X is a CAT(0) cube complex any

wall W is 2-sided and embeds as a convex subspace, so we will not make distinction between a wall and its embedded image. Also, the complement of a wall in X has exactly two connected components, whose closures are called *half-spaces* of X . Since walls are convex subcomplexes of the cubical barycentric subdivision of X , they are CAT(0) cube complexes as well. For an edge e in the cube complex X , let $W(e)$ denote the unique wall it intersects, and say that e is *dual* to $W(e)$.

We are interested in group actions on CAT(0) cube complexes.

Definition 2.7 Suppose G is a group acting properly and cocompactly by cubical isometries on the CAT(0) cube complex X with CAT(0) distance d . In that case we say that (G, X) is a *cubulated group*, and that X is a *cubulation* of G .

Remark 2.8 Although in this paper we will restrict to proper and cocompact actions, when G is relatively hyperbolic we can consider cubulations that are not necessarily cocompact, but cosparsely [26; 36]. Also, there has been some progress in understanding nonproper actions of hyperbolic groups [19]. Recently, Einstein and Groves introduced relatively geometric cubulations of relatively hyperbolic groups [13].

If (G, X) is a cubulated group, then X is finite-dimensional, locally finite, and G is finitely generated and quasi-isometric to X [38, Proposition 4.2]. Cocompactness and properness also imply the following lemma (cf [1, page 1052]):

Lemma 2.9 *If (G, X) is a cubulated group, then:*

- (i) *There are only finitely many conjugacy classes of torsion elements in G .*
- (ii) *If $Y \subset X$ is a nonempty subset and $H < G$ preserves Y and acts cocompactly on it, then for any $R > 0$ the set of double cosets*

$$A_{Y,H,R} := \{HgH : g \in G, d(Y, gY) \leq R\}$$

is finite.

Proof Part (i) holds for arbitrary proper and cocompact actions by isometries on CAT(0) spaces [7, Corollary II.2.8(2)].

For part (ii), let $g \in G$ be such that $d(gY, Y) \leq R$, and let γ be a geodesic of length $\leq R + 1$ joining Y and gY . Consider $D \subset Y$ a compact subset with $H \cdot D = Y$ and $w_1, w_2 \in H$ such that γ intersects $w_1 D$ in Y and $gw_2 D$ in gY . Then

$$d(w_1^{-1}gw_2D, D) \leq R + 1,$$

and so by local finiteness and properness of the action, there is a finite set $F \subset G$ such that $w_1^{-1} g w_2 \in F$. Therefore $g \in w_1 F w_2^{-1} \subset HFH$. \square

Let (G, X) be a cubulated group, and for W a wall of X , let $G_W < G$ denote its setwise stabilizer in G . The group G_W acts properly and cocompactly on W [38, Remark 2.3], and since W is convex in X , (G_W, W) is a cubulated group and G_W is undistorted in G .

Definition 2.10 If (G, X) is a cubulated group, we say that a subgroup $H < G$ is *convex* in (G, X) if it acts cocompactly on a convex subcomplex $Z \subset X$. Such a subcomplex Z will be called a *convex core* for H .

Definition 2.11 If Y is a NPC cube complex and $S \subset Y$ is any subset, then the *cubical neighborhood* of S is the subcomplex $\mathcal{N}(S) \subset Y$ consisting of the cubes of Y intersecting S .

If (G, X) is a cubulated group and W is a wall of X , then $\mathcal{N}(W)$ is a convex subcomplex of X [22, Lemma 13.4], and hence it is a convex core for G_W . If in addition G is relatively hyperbolic, and since G quasi-isometric to X , by Theorem 2.4 every convex subgroup of G is relatively quasiconvex. As a partial converse, Sageev and Wise proved the following [36, Theorem 1.1].

Theorem 2.12 (Sageev–Wise) *If (G, X) is a cubulated group with G relatively hyperbolic, and $H < G$ is fully relatively quasiconvex, then for any compact subset $B \subset X$ there exists a convex core $Z \subset X$ for H that contains B . In particular, any peripheral subgroup of G is convex.*

Remark 2.13 When (G, X) is cubulated with G relatively hyperbolic, and $P < G$ is a peripheral subgroup with convex core $Z \subset X$, then for any wall $W \subset X$ with $W \cap Z \neq \emptyset$ we have $P_{W \cap Z} = P \cap G_W$. The inclusion $P \cap G_W \subset P_{W \cap Z}$ is evident, and if e is an edge of Z dual to W , then e is also dual to the wall $W \cap Z$ of Z . Therefore, for any $g \in P$ with $g(W \cap Z) = W \cap Z$, the edge ge is dual to $W \cap Z \subset W$, and hence $gW = gW(e) = W(ge) = W$, implying $P_{W \cap Z} = P \cap G_W$.

From the previous remark we see that the intersection of a peripheral subgroup with a wall stabilizer is a convex subgroup. In general, we have the following lemma.

Lemma 2.14 *If (G, X) is a cubulated group and $H_1, H_2 < G$ are convex subgroups with convex cores $Y_1, Y_2 \subset X$ respectively and such that $Y_1 \cap Y_2 \neq \emptyset$, then $H_1 \cap H_2$ is a convex subgroup with convex core $Y_1 \cap Y_2$.*

Proof Let x_0 be a vertex in the convex subcomplex $Y_1 \cap Y_2$, and let $R \geq 0$ be such that if $D \subset X^{(0)}$ is the combinatorial R -ball around x_0 , then $Y_1 \subset H_1 \cdot D$ and $Y_2 \subset H_2 \cdot D$. Since $X^{(0)}$ is a proper metric space with the combinatorial distance and G acts properly on $X^{(0)}$, it can be proven that for any $K \geq 0$ there exists some $L = L(K)$ such that

$$N_K(H_1 \cdot x_0) \cap N_K(H_2 \cdot x_0) \subset N_L((H_1 \cap H_2) \cdot x_0),$$

with $N_r(S)$ denoting the combinatorial r -neighborhood of $S \subset X^{(0)}$; compare with [30, Lemma 4.2]. In particular, we have

$$\begin{aligned} Y_1 \cap Y_2 \subset H_1 \cdot D \cap H_2 \cdot D &\subset N_R(H_1 \cdot x_0) \cap N_R(H_2 \cdot x_0) \\ &\subset N_{L(R)}((H_1 \cap H_2) \cdot x_0), \end{aligned}$$

implying that $H_1 \cap H_2$ acts cocompactly on $Y_1 \cap Y_2$ by the local finiteness of X . \square

The lemma above requires two convex cores to intersect, which can be always assumed after enlarging the convex cores.

Lemma 2.15 *If (G, X) is a cubulated group and $H < (G, X)$ is a convex subgroup, then for any compact set $B \subset X$ there is a convex core $Y \subset X$ for H such that $B \subset Y$. In addition, for any two convex cores $Y_1, Y_2 \subset X$ for the subgroup H , there exists a third convex core $Y_3 \subset X$ containing both Y_1 and Y_2 .*

Proof Let Y' be any convex core for H . Since X is finite-dimensional, it follows from [22, Lemma 13.15] that the iterated cubical neighborhoods $\mathcal{N}^k(Y') = \mathcal{N}(\mathcal{N}^{k-1}(Y'))$ are convex cores for H for all $k \geq 1$, and we can choose k such that $Y := \mathcal{N}^k(Y')$ contains B . The second statement follows by considering a compact set $B_i \subset Y_i$ such that $H \cdot B_i \supset Y_i$ for each $i = 1, 2$, and then finding a convex core Y_3 containing $B_1 \cup B_2$. \square

We need one more result, which is key in the proof of Proposition 8.12. We use the notation $[p, q]$ for the geodesic segment joining the points p and q .

Proposition 2.16 *If (G, \mathcal{P}) is relatively hyperbolic and cubulated by X , then there exists some $\delta \geq 0$ such that if $h \in G$ is loxodromic and preserves two axes $\gamma_1, \gamma_2 \subset X$, then $d(\gamma_1, \gamma_2) \leq \delta$.*

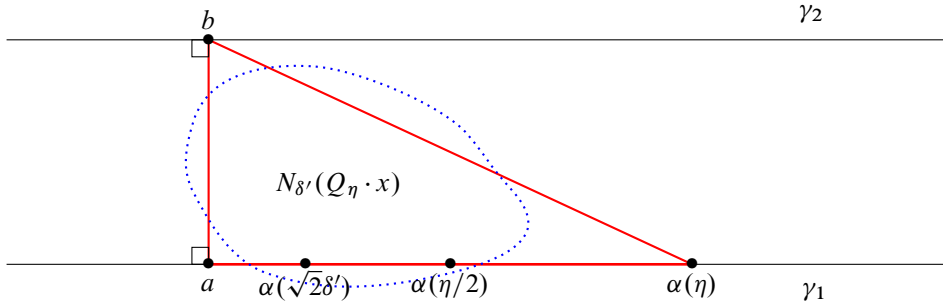


Figure 1: Proof of Proposition 2.16.

Proof Let $x \in X$ be a basepoint. The map $G \xrightarrow{\mu} X, g \mapsto gx$ is a quasi-isometry for X considered with the CAT(0) metric, so there exists some $\delta' \geq 0$ such that for any geodesic triangle $\Delta \subset X$ with vertices a, b and c there is some peripheral left coset $Q_\Delta = g_\Delta P_\Delta$ with $g_\Delta \in G$ and $P_\Delta \in P$, such that for any point $p \in \Delta$, either:

- (i) p lies in the δ' -neighborhood of the union of the sides of Δ not containing p , or
- (ii) $p \in N_{\delta'}(Q_\Delta \cdot x)$.

See eg [36, Theorem 4.1 and Proposition 4.2] or [11, Section 8.1.3]. In addition, for such a δ' there exists some $\lambda \geq 0$ such that if gP and $g'P'$ are distinct peripheral left cosets, then

$$(1) \quad \text{diam}(N_{\delta'}(gP \cdot x) \cap N_{\delta'}(g'P' \cdot x)) < \lambda,$$

where $N_R(M)$ denotes the R -neighborhood of M ; see [12, 10, Corollary 2.3].

Let $\delta := 4\delta'$, and suppose for the sake of contradiction that $h \in G$ is loxodromic and preserves two axes $\gamma_1, \gamma_2 \subset X$ with $r = d(\gamma_1, \gamma_2) > \delta$. We claim that there is some peripheral left coset gP such that $\gamma_1 \subset N_{\delta'}(gP \cdot x)$.

By [7, Theorem II.2.13], γ_1 and γ_2 are asymptotic and bound a flat strip isometric to $\mathbb{R} \times [0, r]$. Let $a \in \gamma_1$, and let b be its closest-point projection into γ_2 . Choose an isometry $\alpha: \mathbb{R} \rightarrow \gamma_1$ sending 0 to a , and for $\eta > r$ consider the geodesic triangle Δ_η with vertices a, b , and $\alpha(\eta)$. After using some Euclidean trigonometry we can prove that the segment $[\alpha(\sqrt{2}\delta'), \alpha(\eta/2)]$ lies outside the δ' -neighborhood of $[a, b] \cup [\alpha(\eta), b]$, so by condition (ii) above there exists some peripheral left coset $Q_\eta = g_\eta P_\eta$ with $[\alpha(\sqrt{2}\delta'), \alpha(\eta/2)] \subset N_{\delta'}(Q_\eta \cdot x)$. Also, by (1) we have $Q_\eta = Q_{\eta'}$ for any $\eta \geq \eta' := \max(2\lambda + 2\sqrt{2}\delta', r)$, implying $\alpha([\sqrt{2}\delta', +\infty)) \subset N_{\delta'}(Q_{\eta'} \cdot x)$. In fact, by a completely analogous argument we can prove $\gamma_1 \subset N_{\delta'}(Q_{\eta'} \cdot x)$, and so the claim follows with $gP := Q_{\eta'}$.

Now, let $L = d(hx, \gamma_1)$, which equals $d(h^n x, \gamma_1)$ for any $n \in \mathbb{Z}$. By our previous claim we have $d(h^n x, gP \cdot x) \leq L + \delta'$ for all n , so by means of the quasi-isometry μ we can find a constant $C \geq 0$ such that

$$d_S(h^n, gP) \leq C$$

for any $n \in \mathbb{Z}$, for d_S the word-metric with respect to some finite generating set S of G . This is our desired contradiction, since in that case the infinite cyclic group generated by h would be bounded in G for the word-metric $d_{(S \cup \mathcal{P})}$, contradicting that h is loxodromic [25, Lemma 8.3(1)]. \square

2.4 Virtually special groups

We will not work directly with the definition of special cube complex [22]. Instead, we present some of their main properties and some criteria for virtual specialness in the case of cubulated hyperbolic groups. We will extend some of these results in Section 4 when we prove Theorem 1.8.

Definition 2.17 We say that a cubulated group (G, X) is *special* if G acts freely on X and the (compact) quotient X/G is a special cube complex, and that (G, X) is *virtually special* if there is a finite-index subgroup $G' < G$ such that (G', X) is special.

By abuse of notation, sometimes we will simply say that G is (virtually) special without mentioning the cubulation X , and it will be implicit that the quotient X/G is compact.

Definition 2.18 A subset $S \subset G$ of a group G is *separable* in G (or simply *separable* if the ambient group G is understood) if for any $a \in G \setminus S$ there exists a finite quotient of G in which the image of a does not lie in the image of S . In particular, a subgroup $H < G$ is separable if it is the intersection of finite-index subgroups of G . The group G is *residually finite* if $\{1\}$ is separable.

One of the main properties of virtually special groups is the following result due to Haglund and Wise [22, Theorem 1.3, Corollary 7.9 and Proposition 13.7].

Theorem 2.19 (Haglund–Wise) *If G is hyperbolic and virtually special, then every quasiconvex subgroup of G is separable.*

In general, if (G, X) is a virtually special group, then every convex subgroup of (G, X) is separable.

As a partial converse, they also proved the following characterization of virtual specialness [22, Theorem 9.19].

Theorem 2.20 (Haglund–Wise) *The cubulated group (G, X) is virtually special if and only if*

- (i) G_W is separable for every wall $W \subset X$, and
- (ii) the double coset $G_{W_1}G_{W_2}$ is separable for any pair W_1, W_2 of intersecting walls of X .

The second characterization of virtual specialness is due to Wise [40, Theorem 13.3], and is in terms of the *quasiconvex virtual hierarchy* $Q\mathcal{VH}$ defined below; see also [3, Theorem 10.2].

Definition 2.21 Let $Q\mathcal{VH}$ be the smallest class of hyperbolic groups closed under the following operations:

- (i) $\{1\} \in Q\mathcal{VH}$.
- (ii) If $G = A *_B C$ with $A, C \in Q\mathcal{VH}$ and with B quasiconvex in G , then $G \in Q\mathcal{VH}$.
- (iii) If $G = A *_B$ with $A \in Q\mathcal{VH}$ and with B quasiconvex in G , then $G \in Q\mathcal{VH}$.
- (iv) If $H < G$ with $|G : H| < \infty$ and $H \in Q\mathcal{VH}$, then $G \in Q\mathcal{VH}$.

Theorem 2.22 (Wise) *A hyperbolic group is virtually special if and only if it is in $Q\mathcal{VH}$.*

As mentioned in the introduction, we will require the following compatibility condition for cubulated relatively hyperbolic groups.

Definition 2.23 A cubulated and relatively hyperbolic group (G, X) with peripheral structure \mathcal{P} is said to be *hyperbolic relative to compatible virtually special subgroups* if for any peripheral subgroup $P < G$, there exists some convex core $Z \subset X$ for P such that the cubulated group (P, Z) is virtually special.

From Theorems 2.12 and 2.19, it follows that if (G, X) is relatively hyperbolic and virtually special, then (G, X) is hyperbolic relative to compatible virtually special subgroups. By Theorem 1.4, this compatibility also holds under the assumptions in Proposition 1.5. The same applies to Corollary 1.3 since any subgroup or double coset of a finitely generated virtually abelian group is separable, so Theorem 2.20 gives us that any cubulated group that is hyperbolic relative to virtually abelian subgroups satisfies Definition 2.23.

In general, the existence of a virtually special convex core cubulation for a convex subgroup implies that any other convex core gives a virtually special cubulation. This follows from the next proposition, which will be proven in the appendix as a consequence of Theorem A.1.

Proposition 2.24 *Let (G, X) be a cubulated group and let $H < G$ be a convex subgroup with convex core $Y \subset X$. If (H, Y) is virtually special, then (H, Y') is virtually special for any other convex core $Y' \subset X$ for H .*

We end this subsection by proving that all the relevant subgroups mentioned in points (i) and (iii) of Definition 1.7 are hyperbolic relative to compatible virtually special subgroups.

Lemma 2.25 *If (G, X) is a cubulated group such that (G, \mathcal{P}) is hyperbolic relative to compatible virtually special subgroups, and $H < G$ is a convex subgroup with convex core $Y \subset X$, then (H, Y) is also hyperbolic relative to compatible virtually special subgroups, when endowed with its induced peripheral structure.*

Proof By Theorem 2.4, H is relatively quasiconvex in G . Let $P < G$ be a peripheral subgroup such that $H \cap P$ is infinite, and let $U \subset Y$ be a convex core for $H \cap P$. We claim that $(H \cap P, U)$ is virtually special. If $Z \subset X$ is any convex core for $P < G$, by Proposition 2.24 the cubulation (P, Z) is virtually special, and so by the characterization of special cube complexes given in [23, Definition 2.4] it is enough to show that there is a convex core Z with $U \subset Z$. But $U/(H \cap P)$ is compact, so there is a compact subset $B \subset Z$ such that $(H \cap P) \cdot B = U$. Therefore by Theorem 2.12 we can choose Z containing B , and hence containing U . \square

3 Dehn filling, relative height, and a weak separation theorem

In this section we introduce Dehn filling, the malnormal special quotient theorem, and prove some results which will be used in the remaining sections. The main results of this section are Theorems 3.9 and 3.21.

3.1 Wide Dehn fillings

Dehn filling was introduced independently by Groves and Manning [18] and Osin [34] as a group-theoretical generalization of the corresponding concept to cusped hyperbolic 3-manifolds.

Definition 3.1 Let $(G, \mathcal{P} = \{P_1, \dots, P_n\})$ be a relatively hyperbolic group, and consider subgroups $N_i \leq P_i$. The *group-theoretic Dehn filling* of G (or simply the *filling*) is the quotient map

$$\phi: G \rightarrow G(N_1, \dots, N_n) = G(\mathcal{N}) := G / \left\langle\left\langle \bigcup N_i \right\rangle\right\rangle_G,$$

where $\mathcal{N} = \{N_1, \dots, N_n\}$ is the collection of *filling kernels* of ϕ . Let $\bar{\mathcal{P}}$ denote the set of images of the subgroups P_i under ϕ .

If $H < (G, \mathcal{P})$ is a relatively quasiconvex subgroup, we need further conditions on a filling to guarantee good properties for the image of H . In [20], Groves and Manning introduced *H-wide fillings*, as a generalization of *H-fillings*, but with enough flexibility to behave nicely even when H is not necessarily full. Let \mathcal{D} be an induced peripheral structure on H such that every $D \in \mathcal{D}$ is of the form $D = H \cap P_{i_D}^{c_D}$ for some $P_{i_D} \in \mathcal{P}$ and some $c_D \in G$ of shortest length (with respect to a fixed compatible generating set of G).

Definition 3.2 If $S \subset \bigcup \mathcal{P} \setminus \{1\}$ is a finite set, then a filling $G \rightarrow G(N_1, \dots, N_n)$ is *(H, S)-wide* if for any $D \in \mathcal{D}$, and for any $d \in D$ and $w \in S \cap P_{i_D}$, we have $c_D w c_D^{-1} \in D$ whenever $d c_D w c_D^{-1} \in N_{i_D}^{c_D}$.

More generally, if $\mathcal{H} = \{(H_1, \mathcal{D}_1), \dots, (H_k, \mathcal{D}_k)\}$ is a collection of relatively quasiconvex subgroups of G , then a filling is *(H, S)-wide* if it is *(H_j, S)-wide* for each $1 \leq j \leq k$.

Given a collection $\{N_i \leq P_i\}_i$ of filling kernels of (G, \mathcal{P}) , the *induced filling kernels* of (H, \mathcal{D}) are the groups of the collection $\mathcal{N}_H = \{K_D := D \cap N_{i_D}^{c_D}\}_{D \in \mathcal{D}}$. These groups define the *induced filling*

$$\phi_H: (H, \mathcal{D}) \rightarrow (H(\mathcal{N}_H), \bar{\mathcal{D}}),$$

with $\bar{\mathcal{D}}$ being the set of images of the elements of \mathcal{D} in $H(\mathcal{N}_H) := H / \left\langle\left\langle \bigcup_D K_D \right\rangle\right\rangle_H$. Note that there is a natural map from $H(\mathcal{N}_H)$ into $\bar{G} = G(N_1, \dots, N_n)$.

If $G \rightarrow \bar{G}$ is a Dehn filling of (G, \mathcal{P}) with kernel K , and X is a cusped space for (G, \mathcal{P}) defined in Section 2.1, then a combinatorial cusped space for $(\bar{G}, \bar{\mathcal{P}})$ is obtained from $\bar{X} = X/K$ by removing self-loops. This removing process does not affect the metric on the 0-skeleton, so we will ignore this ambiguity and simply set $\bar{X} = X/K$ [20, page 4].

Definition 3.3 A property P holds for all sufficiently long and H -wide fillings if there is a finite set $S \subset \bigcup \mathcal{P} \setminus \{1\}$ such that P holds for any (H, S) -wide filling $G \rightarrow G(N_1, \dots, N_n)$ with $S \cap (\bigcup_i N_i) = \emptyset$. In general, if \mathcal{H} is a collection of relatively quasiconvex subgroups of G , then P holds for all sufficiently long and \mathcal{H} -wide fillings if there is a finite set S such that P holds for any (\mathcal{H}, S) -wide filling with $S \cap (\bigcup_i N_i) = \emptyset$.

The next result summarizes the main required properties about wide Dehn filling of relatively hyperbolic groups; see [3, Section 7] and [20, Sections 3 and 4].

Theorem 3.4 Let (G, \mathcal{P}) be a relatively hyperbolic group and let X be a cusped space for (G, \mathcal{P}) . Let $H < G$ be a relatively quasiconvex subgroup with quasiconvexity constant λ with respect to X , and let $A \subset G$ be a finite set. Then there exist positive numbers $\delta' = \delta'(\delta)$ and $\lambda' = \lambda'(\lambda, \delta)$ such that for all sufficiently long and H -wide fillings $\phi: G \rightarrow \bar{G} := G(N_1, \dots, N_n)$, we have:

- (i) $(\bar{G}, \bar{\mathcal{P}})$ is relatively hyperbolic and \bar{X} is δ' -hyperbolic.
- (ii) $\phi(A) \cap \phi(H) = \phi(A \cap H)$.
- (iii) $\bar{H} := \phi(H)$ is relatively quasiconvex in $(\bar{G}, \bar{\mathcal{P}})$, and λ' is a quasiconvexity constant for \bar{H} with respect to \bar{X} . In addition, if H is fully relatively quasiconvex, then \bar{H} is fully relatively quasiconvex.
- (iv) \bar{H} is canonically isomorphic to the induced filling $H(\mathcal{N}_H)$.

Definition 3.5 Let (H, \mathcal{D}) be a relatively quasiconvex subgroup of the relatively hyperbolic group (G, \mathcal{P}) . We say that H is *peripherally separable* in (G, \mathcal{P}) if D is separable in $P_{i_D}^{cD}$ for every $D \in \mathcal{D}$. We say that H is *strongly peripherally separable* if D' is separable in $P_{i_D}^{cD}$ for any finite-index subgroup $D' < D$ and for any $D \in \mathcal{D}$.

The existence of wide fillings is guaranteed by the following lemma [20, Lemma 5.2].

Lemma 3.6 Let (G, \mathcal{P}) be relatively hyperbolic and let $\mathcal{H} = \{(H_1, \mathcal{D}_1), \dots, (H_k, \mathcal{D}_k)\}$ be a finite collection of relatively quasiconvex and peripherally separable subgroups of G . Then for any finite set $S \subset \bigcup \mathcal{P} \setminus \{1\}$ there exist finite-index subgroups $\dot{N}_i \trianglelefteq P_i$ such that any filling $G \rightarrow G(N_1, \dots, N_n)$ with $N_i < \dot{N}_i$ is (\mathcal{H}, S) -wide.

In our situation, strong peripheral separability follows from convexity, as is shown in the following lemma.

Lemma 3.7 *Let (G, X) be a cubulated group that is hyperbolic relative to compatible virtually special subgroups. Then any convex subgroup of (G, X) is strongly peripherally separable.*

Proof Let $H < G$ be a convex subgroup with convex core $Y \subset X$, and let $P < G$ be a peripheral subgroup with $H \cap P$ infinite. We claim that if $D' < H \cap P$ is a finite-index subgroup, then it is separable in P . To prove the claim, let $Z \subset X$ be a convex core for P , which we can assume to have nontrivial intersection with Y by Theorem 2.12. In this case, Lemma 2.14 implies that D' is a convex subgroup of (P, Z) . But the cubulated group (P, Z) is virtually special by Proposition 2.24, and so the claim follows from Theorem 2.19. □

3.2 Relative height and R -hulls

The relative height of a relatively quasiconvex subgroup of a relatively hyperbolic group was introduced by Hruska and Wise in [25, Definition 8.1] as the appropriate analog of the height for quasiconvex subgroups of hyperbolic groups [16]. In particular, they proved that the relative height is always finite. In this subsection we introduce the corresponding version for a finite collection of relatively quasiconvex subgroups, we show that it is finite, and prove that it is nonincreasing under Dehn filling, extending the results in [20, Section 7]; cf [1, Remark A.3].

Definition 3.8 Let (G, \mathcal{P}) be a relatively hyperbolic group and consider a set $\mathcal{H} = \{H_1, \dots, H_k\}$ of *distinct* relatively quasiconvex subgroups of G . The *relative height* of \mathcal{H} in G is the maximum number $n \geq 0$ so that there are distinct left cosets $\{g_1 H_{\sigma(1)}, \dots, g_n H_{\sigma(n)}\}$ with $\bigcap_{i=1}^n H_{\sigma(i)}^{g_i}$ containing a loxodromic element of G .

The following is the main result of the subsection.

Theorem 3.9 *For all sufficiently long and \mathcal{H} -wide fillings, the groups in $\bar{\mathcal{H}} = \{\bar{H}_1, \dots, \bar{H}_k\}$ are all distinct and the relative height of $\bar{\mathcal{H}}$ in $(\bar{G}, \bar{\mathcal{P}})$ is at most the relative height of \mathcal{H} in (G, \mathcal{P}) .*

When \mathcal{H} consists of a single subgroup H , Theorem 3.9 reduces to [3, Theorem 7.12] for H -fillings, and to [20, Theorem 7.15] for general H -wide fillings.

Definition 3.10 A collection $\{H_1, \dots, H_k\}$ of distinct subgroups of (G, \mathcal{P}) is *relatively malnormal* if, whenever $H_i \cap H_j^g$ contains a loxodromic element, then $i = j$ and $g \in H_i$. The collection is *almost malnormal* if $H_i \cap H_j^g$ is finite unless $i = j$ and $g \in H_i$.

Note that relative malnormality is equivalent to relative height at most 1, and that relative malnormality coincides with almost malnormality when \mathcal{P} is empty or consists of finite groups (in which case G is hyperbolic). These observations together with Theorem 3.9 imply the following corollary. Recall that a filling $G \rightarrow G(N_1, \dots, N_n)$ is *peripherally finite* if $N_i \trianglelefteq P_i$ is finite-index for each $1 \leq i \leq n$.

Corollary 3.11 *Suppose that $\mathcal{H} = \{H_1, \dots, H_k\}$ is a relatively malnormal collection of relatively quasiconvex subgroups of (G, \mathcal{P}) . Then for all sufficiently long and \mathcal{H} -wide peripherally finite fillings, the collection $\{\bar{H}_1, \dots, \bar{H}_k\}$ is almost malnormal in \bar{G} .*

The proof of Theorem 3.9 will take us the rest of the subsection, in which we basically refine the techniques from [3] and [20] about R -hulls over cusped spaces and relative multiplicity.

Let X be a cusped space for (G, \mathcal{P}) , and let $\delta \geq 0$ be given by Theorem 3.4(i), so that X and \bar{X} are δ -hyperbolic for all sufficiently long fillings. The *depth* of a vertex of X is its distance to the Cayley graph $\Gamma(G, S)$. Note that the action of G on X is depth-preserving.

Definition 3.12 (cf [20, Definition 7.3]) Let $\tilde{*} = 1 \in G \subset X$, and consider $(H, \mathcal{D}) < (G, \mathcal{P})$ a relatively quasiconvex subgroup and $R \geq 0$. An R -hull for H acting on X is a connected H -invariant full subgraph $\tilde{Z}_H \subset X$ satisfying:

- (i) $\tilde{*} \in \tilde{Z}_H$.
- (ii) If $\gamma \subset X$ is a geodesic with endpoints in the limit set $\Lambda(H)$, it holds that $N_R(\gamma) \cap N_R(G) \subset \tilde{Z}_H$.
- (iii) If A is a horoball in X containing a vertex a at depth greater than 0 in the image $\check{i}(X_H)$, then $\tilde{Z}_H \cap A^{(0)}$ contains every vertex of the maximal vertical ray in A containing a .
- (iv) The action of (H, \mathcal{D}) on \tilde{Z}_H is *weakly geometrically finite* in the sense of [1, Definition A.27].

By [20, Lemma 7.6], R -hulls for the action of H on X exist for any $R \geq 0$, and moreover we can construct them in such a way that they have only finitely many H -orbits of vertices at depth 0. Therefore, anytime we consider an R -hull, implicitly we will assume it satisfies this property. The relevance of R -hulls is that they allow us to extract some algebraic information about their corresponding relatively quasiconvex subgroups, as we will see in Proposition 3.16 below.

Let $Y = X/G$, and consider an R -hull \tilde{Z}_H for the action of H on X with quotient $Z_H := \tilde{Z}_H/H$. The natural map $\mathcal{J}_H : Z_H \rightarrow Y$ induces the inclusion $H \rightarrow G$ in the following way. If $*_H \in Z_H$ and $* \in Y$ are the respective projections of $\tilde{*}$, we obtain canonical surjections $s : \pi_1(Z_H, *_H) \rightarrow H$ and $s : \pi_1(Y, *) \rightarrow G$ such that the diagram

$$(2) \quad \begin{array}{ccc} \pi_1(Z_H, *_H) & \xrightarrow{(\mathcal{J}_H)_*} & \pi_1(Y, *) \\ \downarrow s & & \downarrow s \\ H & \longrightarrow & G \end{array}$$

commutes, with the bottom map being the inclusion.

Let $\mathcal{H} = \{H_1, \dots, H_k\}$ be a collection of relatively quasiconvex subgroups, and assume that $H_i \neq H_j$ whenever $i \neq j$. By Theorem 3.4(iii), let λ be a quasiconvexity constant for each H_j and \bar{H}_j for all sufficiently long and \mathcal{H} -wide fillings. Consider R -hulls $\tilde{Z}_j = \tilde{Z}_{H_j}$ for each $1 \leq j \leq k$, with quotients $Z_j = \tilde{Z}_j/H_j$ and maps $\mathcal{J}_j = \mathcal{J}_{H_j} : Z_j \rightarrow Y$ as above.

Definition 3.13 (cf [1, Definition A.15]) Given $m > 0$ and $\sigma = (\sigma(1), \dots, \sigma(m)) \in \{1, \dots, k\}^m$, we write $|\sigma| = m$ and define

$$S_\sigma := \{(z_1, \dots, z_m) \in Z_{\sigma(1)} \times \dots \times Z_{\sigma(m)} : \mathcal{J}_{\sigma(1)}(z_1) = \dots = \mathcal{J}_{\sigma(m)}(z_m)\} \setminus \Delta_\sigma,$$

where

$$\Delta_\sigma = \{(z_1, \dots, z_m) : \exists i \neq j \text{ such that } \sigma(i) = \sigma(j) \text{ and } z_i = z_j\}$$

should be understood as the fat diagonal.

Points in S_σ have a well-defined depth, which is the depth of the image in Y . Let C be a component of S_σ containing a basepoint $p = (p_1, \dots, p_m)$ at depth 0, and fix maximal trees T_j in Z_j inducing preferred paths β_v between $*_j = *_{H_j}$ and the vertex $v \in Z_j$. The paths $\beta_i = \beta_{p_i}$ give isomorphisms $\pi_1(Z_{\sigma(i)}, p_i) \rightarrow \pi_1(Z_{\sigma(i)}, *_\sigma(i))$, and composing them with the pushforwards of the projections $\omega_i : (C, p) \rightarrow (Z_{\sigma(i)}, p_i)$, we obtain homomorphisms $(\omega_i)_* : \pi_1(C, p) \rightarrow \pi_1(Z_{\sigma(i)}, *_\sigma(i))$. Define then

$$\tau_{C,i} := s \circ (\omega_i)_* : \pi_1(C, p) \rightarrow H_{\sigma(i)},$$

where $s : \pi_1(Z_{\sigma(i)}, *_\sigma(i)) \rightarrow H_{\sigma(i)}$ is as in (2).

Definition 3.14 (cf [20, Definition 7.7]) The *relative multiplicity* of the set of maps $\{\mathcal{J}_j : Z_j \rightarrow Y\}_{1 \leq j \leq k}$ is the largest m for which there is some σ with $|\sigma| = m$ and S_σ containing a component C such that the group $\tau_{C,i}(\pi_1(C, p))$ is infinite nonparabolic for all $1 \leq i \leq m$.

Remark 3.15 As we mentioned after Definition 3.12, we are considering R -hulls with only finitely many orbits of vertices at depth 0. This implies that for any j , the number r_j of vertices in Z_j at depth 0 is finite. In particular, if $m > r_1 + \dots + r_k$ and $|\sigma| = m$, then any tuple in $Z_{\sigma(1)} \times \dots \times Z_{\sigma(m)}$ at depth 0 lies in Δ_σ , and consequently the relative multiplicity of $\{\mathcal{J}_j : Z_j \rightarrow Y\}_{1 \leq j \leq k}$ is bounded by $r_1 + \dots + r_k$.

The main property of relative multiplicity is given by the following proposition.

Proposition 3.16 (cf [20, Theorem 7.8]) *For sufficiently large R , depending only on δ and λ , if each \tilde{Z}_j is an R -hull for the action of H_j on X , then the relative height of $\{H_1, \dots, H_k\}$ in G is equal to the relative multiplicity of $\{\mathcal{J}_j : Z_j \rightarrow Y\}_{1 \leq j \leq k}$.*

We need a preliminary lemma; cf [1, Lemma A.37].

Lemma 3.17 *Let C be a fixed component of S_σ based at the point $p = (p_1, \dots, p_m)$ at depth 0. Then for any $1 \leq i, j \leq m$, there is some $g_{i,j} \in G$ such that*

$$g_{i,j} \tau_{C,j}([\alpha]) g_{i,j}^{-1} = \tau_{C,i}([\alpha])$$

for any homotopy class $[\alpha] \in \pi_1(C, p)$.

Proof Recall the paths β_i, β_j in the construction of the maps $\tau_{C,i}, \tau_{C,j}$. In virtue of the commutative diagram (2), the element $\tau_{C,i}([\alpha]) \in G$ coincides with

$$s([\mathcal{J}_i \circ \beta_i] \cdot [\mathcal{J}_i \circ \omega_i \circ \alpha] \cdot [\mathcal{J}_i \circ \bar{\beta}_i]).$$

But α is a loop in C , so $(\mathcal{J}_i \circ \omega_i \circ \alpha) = (\mathcal{J}_j \circ \omega_j \circ \alpha)$, and in particular $(\mathcal{J}_i \circ \beta_i) \cdot (\mathcal{J}_j \circ \bar{\beta}_j)$ is a loop in Y based at $*$. This loop defines the element $g_{i,j} \in G$, for which it is easy to check it satisfies the requirement. □

Proof of Proposition 3.16 We proceed as in the proof of [20, Theorem 7.8]. First we prove that the relative height is at most the relative multiplicity. So suppose that the relative height is at least m , and let $\sigma = (\sigma(1), \dots, \sigma(m))$ and $g_1 H_{\sigma(1)}, \dots, g_m H_{\sigma(m)}$ be distinct left cosets such that $\bigcap_{i=1}^m H_{\sigma(i)}^{g_i}$ contains a loxodromic element, say h . In [20] it was proven that by choosing an appropriate R , and by replacing h by a power, we may suppose that h has a quasigeodesic axis $\tilde{\gamma}$ contained in $\bigcap_{i=1}^m g_i \tilde{Z}_{\sigma(i)}$. In this case the path $\tilde{\gamma}$ induces a loop $\gamma : \mathbb{R}/\mathbb{Z} \rightarrow Z_{\sigma(1)} \times \dots \times Z_{\sigma(m)}$ defined by

$$\gamma(t) = (\pi_{\sigma(1)}(g_1^{-1} \tilde{\gamma}(t)), \dots, \pi_{\sigma(m)}(g_m^{-1} \tilde{\gamma}(t))).$$

The cosets $g_i H_{\sigma(i)}$ are distinct and G acts freely on X , so γ misses the fat diagonal Δ_σ , and hence defines a loop in a connected component C of S_σ containing a vertex p at depth 0. It is not hard to see that $\tau_{C,i}(\pi_1(C, p))$ contains a conjugate of h for each i , so the relative multiplicity is at least m .

For the converse, suppose that the relative multiplicity of $\{\mathcal{J}_j: Z_j \rightarrow Y\}_j$ is m , and let $\sigma = (\sigma(1), \dots, \sigma(m))$ and C be a component of S_σ containing a point p at depth 0, such that for each $1 \leq i \leq m$ the group $A_i = \tau_{C,i}(\pi_1(C, p)) < H_{\sigma(i)} < G$ contains a loxodromic element. Consider the elements $g_i = g_{1,i} \in G$ given by Lemma 3.17 so that $A_i^{g_i} = A_1$ for each i , which implies $A_1 \subset \bigcup_{i=1}^m H_{\sigma(i)}^{g_i}$.

It is only left to show that the cosets $g_i H_{\sigma(i)}$ are all different. Indeed, let $p = (p_1, \dots, p_m)$ and suppose there exist $i \neq j$ with $g_i H_{\sigma(i)} = g_j H_{\sigma(j)}$, implying that $\sigma(i) = \sigma(j)$ and $g_i = g_j h$ for some $h \in H_{\sigma(i)}$. Let $\tilde{\beta}_i$ and $\tilde{\beta}_j$ be the unique liftings of β_i and β_j to $\tilde{Z}_{\sigma(i)} \subset X$ starting at $\tilde{*}$, respectively, and note that $g_j^{-1} g_i$ is represented by the loop $(\mathcal{J}_{\sigma(j)} \circ \beta_j) \cdot (\mathcal{J}_{\sigma(i)} \circ \tilde{\beta}_i)$ in $\pi_1(Y, *)$. This implies $\tilde{p}_j = h \tilde{p}_i$, where \tilde{p}_i and \tilde{p}_j are the corresponding endpoints of $\tilde{\beta}_i$ and $\tilde{\beta}_j$, and by projecting back into $Z_{\sigma(i)}$ we get $p_i = p_j$, contradicting $p \notin \Delta_\sigma$. □

The previous proposition together with Remark 3.15 implies the following corollary, generalizing [25, Theorem 1.4].

Corollary 3.18 *If G is a relatively hyperbolic group and \mathcal{H} is a finite collection of distinct relatively quasiconvex subgroups of G , then the relative height of \mathcal{H} in G is finite.*

Before starting the proof of Theorem 3.9 we state the following lemma, proven in [20, Lemmas 7.12 and 7.14].

Lemma 3.19 *Suppose (G, \mathcal{P}) is relatively hyperbolic and $\mathcal{H} = \{H_1, \dots, H_k\}$ is a collection of relatively quasiconvex subgroups of G . Then for all R there is some R' satisfying the following: for all sufficiently long and \mathcal{H} -wide fillings $\phi: G \rightarrow G/K$, if \tilde{Z}_j is an R' -hull for the action of H_j on X for each $1 \leq j \leq k$, then its image $\tilde{\tilde{Z}}_j \subset \bar{X}$ under ϕ is an R -hull for the action of \bar{H}_j on \bar{X} and is the embedded image of $\tilde{Z}_j / (H_j \cap K)$.*

Proof of Theorem 3.9 Let δ be a constant such that the cusped spaces X and \bar{X} are both δ -hyperbolic for all sufficiently long fillings, and let λ be a common quasiconvexity

constant for the groups H_j , which we may also assume is a quasicconvexity constant for each \bar{H}_j in any sufficiently long and \mathcal{H} -wide filling $\phi: G \rightarrow G/K$. For these fillings, by Theorem 3.4(ii)–(iv) we may assume $\bar{H}_i \neq \bar{H}_j$ for all $i \neq j$ (just take a set A containing elements in $H_i \setminus H_j$ for any pair $i \neq j$), and also that each \bar{H}_j is naturally isomorphic to the image of the induced filling of H_j . Let R be sufficiently large that Proposition 3.16 applies for δ and λ , and let $R' = R'(R)$ be given by Lemma 3.19.

For each $1 \leq j \leq k$ consider the commutative diagram

$$\begin{array}{ccc} H_j \hookrightarrow \tilde{Z}_j & \longrightarrow & X \curvearrowright G \\ \downarrow & & \downarrow \\ H_j/K_j \hookrightarrow \tilde{\tilde{Z}}_j & \longrightarrow & \bar{X} \curvearrowright G/K \end{array}$$

where \tilde{Z}_j is an R' -hull for the action of H_j on X , and

$$\bar{X} = X/K, \quad K_j = K \cap H_j \quad \text{and} \quad \tilde{\tilde{Z}}_j = \tilde{Z}_j/K_j.$$

From Lemma 3.19, $\tilde{\tilde{Z}}_j$ embeds in \bar{X} and is an R -hull for the action of H_j/K_j on \bar{X} .

Taking quotients under the respective groups we obtain the diagram

$$(3) \quad \begin{array}{ccc} Z_j & \xrightarrow{\mathcal{J}_j} & Y \\ \downarrow & & \downarrow \\ \bar{Z}_j & \xrightarrow{\bar{\mathcal{J}}_j} & \bar{Y} \end{array}$$

the vertical maps being homeomorphisms. By our choice of R , Proposition 3.16 implies that the relative heights of \mathcal{H} in G and of $\bar{\mathcal{H}}$ in \bar{G} coincide with the relative multiplicities of $\{\mathcal{J}_j: Z_j \rightarrow Y\}_j$ and $\{\bar{\mathcal{J}}_j: \bar{Z}_j \rightarrow \bar{Y}\}_j$, respectively.

The vertical maps of the diagram (3) induce depth-preserving homeomorphisms between

$$S_\sigma = \{(z_1, \dots, z_m) \in Z_{\sigma(1)} \times \dots \times Z_{\sigma(m)} : \mathcal{J}_{\sigma(1)}(z_1) = \dots = \mathcal{J}_{\sigma(m)}(z_m)\} \setminus \Delta_\sigma$$

and

$$\bar{S}_\sigma = \{(z_1, \dots, z_m) \in \bar{Z}_{\sigma(1)} \times \dots \times \bar{Z}_{\sigma(m)} : \bar{\mathcal{J}}_{\sigma(1)}(z_1) = \dots = \bar{\mathcal{J}}_{\sigma(m)}(z_m)\} \setminus \bar{\Delta}_\sigma$$

for any $\sigma = (\sigma(1), \dots, \sigma(m))$ with $|\sigma| = m > 0$, where Δ_σ and $\bar{\Delta}_\sigma$ are the corresponding fat diagonals.

If $\pi_\sigma : S_\sigma \rightarrow \bar{S}_\sigma$ is this homeomorphism, since $\bar{H}_j \cong H_j / K_j$ for all j , then for any component C of S_σ with basepoint p at depth 0 and for any $1 \leq i \leq m$, the diagram

$$\begin{array}{ccc} \pi_1(C, p) & \xrightarrow{\tau_{C,i}} & H_{\sigma(i)} \\ \downarrow (\pi_\sigma)_* & & \downarrow \phi_{H_{\sigma(i)}} \\ \pi_1(\bar{C}, \bar{p}) & \xrightarrow{\tau_{\bar{C},i}} & \bar{H}_{\sigma(i)} \end{array}$$

commutes, where $\phi : G \rightarrow \bar{G}$ is the filling map. In particular, if $\tau_{\bar{C},i}(\pi_1(\bar{C}, \bar{p})) < \bar{G}$ contains a loxodromic element of $(\bar{G}, \bar{\mathcal{P}})$, then $\tau_{C,i}(\pi_1(C, p))$ contains a loxodromic element of (G, \mathcal{P}) as well, and therefore the relative height of \mathcal{H} in G is at least the relative height of $\bar{\mathcal{H}}$ in \bar{G} . □

3.3 Weak separability of double cosets

In this subsection we study the behavior of double cosets of relatively quasiconvex subgroups under Dehn filling. Similarly to the peripheral separability we require for a single relatively quasiconvex subgroup, we need some assumptions on the double cosets.

Definition 3.20 Let H and L be relatively quasiconvex subgroups of the relatively hyperbolic group (G, \mathcal{P}) . The pair H, L is said to be *doubly peripherally separable* if the double coset $(H^{g_1} \cap P)(L^{g_2} \cap P)$ is separable in P for any $g_1, g_2 \in G$ and any peripheral subgroup $P < G$ such that $H^{g_1} \cap P$ and $L^{g_2} \cap P$ are infinite.

Our next theorem establishes weak separability for double cosets of doubly peripherally separable pairs of relatively quasiconvex subgroups under Dehn fillings. This extends [20, Proposition 6.2], where the peripheral subgroups are assumed to be abelian, and hence peripheral separability and double peripheral separability trivially hold. It also generalizes [19, Theorem 6.4], where it is assumed that the relatively quasiconvex subgroups are full (though they proved a result for several subgroups, and also allowing some finite subsets removed).

Theorem 3.21 Let $(G, \mathcal{P} = \{P_1, \dots, P_n\})$ be relatively hyperbolic and let $H, L < G$ be relatively quasiconvex and peripherally separable subgroups of (G, \mathcal{P}) such that the pair H, L is doubly peripherally separable. Also, let $S \subset \bigcup \mathcal{P} \setminus \{1\}$ be a finite set and consider $a \in G \setminus H L$.

Then there exist finite-index subgroups $K_i \trianglelefteq P_i$ such that for any subgroups $N_i < K_i$, the filling $G \rightarrow G(N_1, \dots, N_n) = G/K$ is $(\{H, L\}, S)$ -wide, and moreover $a \notin K H L$.

The lemmas below are [28, Proposition A.6] and [20, Lemmas 4.1 and 4.4], respectively, where for [20, Lemma 4.1] we cite the statement for the case $L_1 = 10\delta$.

Proposition 3.22 *Let H be a relatively quasiconvex subgroup of a relatively hyperbolic group (G, \mathcal{P}) . For a cusped space X for (G, \mathcal{P}) , there exists a positive constant R such that for any horoball A in X , if there is a geodesic in X with endpoints in H and intersecting A at a point at depth R , then $H \cap \text{Stab}(A)$ is infinite.*

Lemma 3.23 *Let (G, \mathcal{P}) be with cusped space X , and let δ be a hyperbolicity constant for X and for the induced cusped space of any sufficiently long filling. Then for any $L_2 \geq 10\delta$, and for all sufficiently long fillings $\phi: G \rightarrow G/K$, the following holds: if γ is a geodesic in X such that $\phi(\gamma)$ is a loop in \bar{X} , then*

- *there is a horoball A in X such that γ intersects A in a segment $[x, y]$ with $x, y \in G$ and containing a point at depth L_2 , and*
- *there is some $k \in K \cap \text{Stab}(A)$ such that $d_X(x, ky) \leq 20\delta + 3$.*

Proof of Theorem 3.21 Almost all the work has been done in [20, Proposition 6.2], so we will only check the details where the doubly peripheral separability assumption is required.

By Lemma 3.6 there exist finite-index subgroups $\dot{N}_i \trianglelefteq P_i$ such that any filling $G \rightarrow G(N_1, \dots, N_n)$ with $N_i < \dot{N}_i$ is $(\{H, L\}, S)$ -wide. The rest of the proof is devoted to finding finite-index subgroups $\dot{K}_i \trianglelefteq P_i$ such that a is separated from HL in any filling $G \rightarrow G(N_1, \dots, N_n)$ with $N_i < \dot{K}_i$. In that case, the theorem follows by defining $K_i := \dot{N}_i \cap \dot{K}_i$.

Consider an induced peripheral structure \mathcal{D} on H , with each $D \in \mathcal{D}$ of the form $D = H \cap P_{j_D}^{c_D}$ for some $1 \leq j_D \leq n$ and some shortest $c_D \in G$. Similarly, let \mathcal{E} be an induced peripheral structure on L with each $E \in \mathcal{E}$ of the form $E = L \cap P_{k_E}^{d_E}$ for some $1 \leq k_E \leq n$ and some shortest $d_E \in G$. Also, let X, X_H and X_L be cusped spaces for $(G, \mathcal{P}), (H, \mathcal{D})$ and (L, \mathcal{E}) , respectively, with induced maps $\check{\iota}_H: X_H^{(0)} \rightarrow X$ and $\check{\iota}_L: X_L^{(0)} \rightarrow X$. Let $\delta \geq 1$ be a hyperbolicity constant for X and for the induced cusped space of any sufficiently long filling of G , and let $\lambda \geq 2\delta + 1$ be such that the images $\check{\iota}_H(X_H^{(0)})$ and $\check{\iota}_L(X_L^{(0)})$ are λ -quasiconvex in X . We assume that δ and λ are integer numbers.

For H and L , consider the constants R_H and R_L given by Proposition 3.22, and define

$$M = d_X(1, a) \quad \text{and} \quad L_2 = \max\{20\delta + M + \lambda + 4, R_H, R_L\}.$$

Given $1 \leq i \leq n$, let $S_i \subset P_i$ be a finite set containing

$$\{p \in P_i : d_X(1, p) \leq 48\delta + 8M + 16\lambda + 5\}.$$

Also, consider the collection \mathcal{C}_i of pairs of subgroups of P_i of the form $(B_1, B_2) = (D^{\alpha c_D^{-1}}, E^{\beta d_E^{-1}})$ with $D \in \mathcal{D}$ and $E \in \mathcal{E}$ such that $j_D = k_E = i$, and $\alpha, \beta \in S_i$. Note that there are finitely many such pairs, and that by assumption, for each of them the double coset $B_1 B_2$ is separable in P_i . This implies the existence of finite-index subgroups $\dot{K}_{B_1, B_2} \trianglelefteq P_i$ such that

$$(4) \quad S_i \cap \dot{K}_{B_1, B_2} B_1 B_2 \subset B_1 B_2.$$

Set $\dot{K}_i := \bigcap_{(B_1, B_2) \in \mathcal{C}_i} \dot{K}_{B_1, B_2}$, for which we expect the separability conditions to hold.

Let $N_i < \dot{K}_i$ be filling kernels inducing the filling $\phi: G \rightarrow G(N_1, \dots, N_n) = G/K$, for which we claim that $a \notin KHL$. Suppose for the sake of contradiction that there is some $g \in K$ of the form $g = hla^{-1}$ for some $h \in H$ and $l \in L$, and assume $d_X(1, g)$ is minimal among the elements of $K \cap HLa^{-1}$.

Consider a geodesic quadrilateral in X with vertices $1, h, hl, g$, and let ξ_1, ξ_2, η and ρ be the geodesics from 1 to h , from h to hl , from hl to g and from 1 to g , respectively. By assumption $g \neq 1$.

The map $X \rightarrow \bar{X}$ induced by ϕ sends ρ to a loop, so by Lemma 3.23 there is a horoball A in X intersecting ρ in a geodesic segment $[x, y]$ (with x between 1 and y), such that $[x, y]$ contains points at depth L_2 , and there is some $k \in K \cap \text{Stab}(A)$ with $d_X(x, ky) \leq 20\delta + 3$. This implies $d_X(1, kg) < d_X(1, g)$, since $d_X(x, ky) \leq 20\delta + 3$ and $d_X(x, y) \geq 2L_2 > 20\delta + 4$. Our contradiction will be obtained by showing that $kg \in HLa^{-1}$, contrary to our minimality assumption on g .

By [18, Lemma 3.10] we can assume that the segment $[x, y]$ is a geodesic through A consisting of a vertical segment down from x , a horizontal segment of length at most 3 , and then a vertical segment with endpoint y . We will make a similar assumption in the case that A intersects ξ_1 or ξ_2 . Let x' and y' be the points on $[x, y]$ directly below x and y , respectively, at depth $3\delta + M + \lambda$. The δ -hyperbolicity condition applied to the quadrilateral $\xi_1 \cup \xi_2 \cup \eta \cup \rho$ implies that x' and y' lie in the 2δ -neighborhood of $\xi_1 \cup \xi_2 \cup \eta$, but since η is of length M and has extreme points at depth 0 , in fact x' and y' are within 2δ of $\xi_1 \cup \xi_2$.

The geodesics ξ_1 and ξ_2 join points in H and hL , respectively, and so λ -quasiconvexity gives us points $u_0, v_0 \in \check{y}_H(X_H^{(0)}) \cup h\check{y}_L(X_L^{(0)})$ with $d_X(u_0, x'), d_X(v_0, y') \leq 2\delta + \lambda$.

The points u_0 and v_0 lie in A since x' and y' have depth greater than $2\delta + \lambda$. Let $u, v \in A$ be the points at depth 0 directly above u_0 and v_0 , respectively. We have $d_X(u, x) \leq d_X(u, u_0) + d_X(u_0, x') + d_X(x', x) \leq 2(d_X(x, x') + d_X(u_0, x')) \leq 10\delta + 2M + 4\lambda$, and similarly $d_X(v, y) \leq 10\delta + 2M + 4\lambda$.

This gives us four cases, depending on whether each of u_0 and v_0 is contained in $\check{y}_H(X_H^{(0)})$ or $h\check{y}_L(X_L^{(0)})$. The cases where u_0 and v_0 are both contained in $\check{y}_H(X_H^{(0)})$ or $h\check{y}_L(X_L^{(0)})$ are completely analogous, and the proof given in [20, Proposition 6.2] still works here since it only requires peripheral separability. As noted at the end of that proof, the case where u_0 is contained in $h\check{y}_L(X_L^{(0)})$ and v_0 is contained in $\check{y}_H(X_H^{(0)})$ essentially becomes the case where u_0 and v_0 are contained in $\check{y}_H(X_H^{(0)})$. Therefore, we are only left to check the case where u_0 is contained in $\check{y}_H(X_H^{(0)})$ and v_0 is contained in $h\check{y}_L(X_L^{(0)})$.

We can write $u_0 = (sc_D P_{j_D}, h_u c_D, n)$ for some $s \in H, h_u \in sD$ and $n \in \mathbb{N}$, where $D \in \mathcal{D}$, and similarly $v_0 = (ht d_E P_{k_E}, hl_v d_E, m)$ for $t \in L, l_v \in tE, m \in \mathbb{N}$ and $E \in \mathcal{E}$. This implies $u = h_u c_D$ and $v = hl_v d_E$. We write $c = c_D, d = d_E$ and $j_D = k_E = i$, where A is the horoball based on the coset $scP_i = ht dP_i$.

The geodesic ξ_1 intersects A in a segment with extreme point h_1 closer to h . Since we may assume that x' lies within 2δ of ξ_1 , there exists a point h'_1 in $\xi_1 \cap A$ directly below h_1 and at depth $\delta + M + \lambda$. By λ -quasiconvexity of $\check{y}_H(X_H)$ we can find some $w' \in \check{y}_H(X_H)$ with $d_X(w', h'_1) \leq \lambda$, which indeed lies in A , so that group element $w \in A$ directly above w' is of the form $w = h_w c \in sDc \subset scP_i$ with $h_w \in H$ and $d_X(w, h_1) \leq d_X(w, w') + d_X(w', h'_1) + d_X(h'_1, h_1) \leq 2\delta + 2M + 4\lambda$. In the same way, if g_2 is the extreme point of the segment $\xi_2 \cap A$ closer to h , then it is directly above the vertex $g'_2 \in A$ at depth $\delta + M + \lambda$ and there is some $z = hl_z d \in htEd \subset ht dP_i$ with $d_X(g_2, z) \leq 2\delta + 2M + 4\lambda$.

Let $[h, h'_1] \cup [h, g'_2] \cup [h'_1, g'_2]$ be a geodesic triangle in X with $[h, h'_1] \subset \xi_1$ and $[h, g'_2] \subset \xi_2$. Since h'_1 and g'_2 are at depth $\delta + M + \lambda \geq 3\delta + 1$ (recall that $\lambda \geq 2\delta + 1$), it follows from [18, Lemma 3.26] that every point of $[h'_1, g'_2]$ is at depth at least $3\delta + 1$. By δ -hyperbolicity, this implies that the vertex in $[h, h'_1]$ directly below h_1 at depth $\delta + 1$ lies within δ of a point in $[h, g'_2]$, which must lie directly below g_2 and at depth at most $2\delta + 1$. In particular, $d_X(h_1, g_2) \leq 4\delta + 2$ and we have $d_X(w, z) \leq 8\delta + 4M + 8\lambda + 2$.

Let $p = (u^{-1}kv)(z^{-1}w) = (u^{-1}ku)(u^{-1}w)(z^{-1}v)w^{-1}z$. Since $k \in K \cap \text{Stab}(A) = K \cap P_i^{cD}$ and K is normal, we have $u^{-1}ku \in K \cap P_i = N_i \leq P_i$. Also, note that $u^{-1}w \in D^{c^{-1}}, z^{-1}v \in E^{d^{-1}}$ and $w^{-1}z \in P_i$, implying $p \in N_i D^{c^{-1}} E^{(w^{-1}z)d^{-1}}$.

In addition, $d_X(u, kv) \leq d_X(u, x) + d_X(x, ky) + d(y, v) \leq 40\delta + 4M + 8\lambda + 3$, so $d_X(1, p) \leq d_X(w, z) + 40\delta + 4M + 8\lambda + 3 \leq 48\delta + 8M + 16\lambda + 5$, and hence $p \in S_i$. The pair $(D^{c^{-1}}, E^{(w^{-1}z)^{d^{-1}}})$ is then in \mathcal{C}_i , so in virtue of (4) there exist $\hat{d} \in D^{c^{-1}}$ and $\hat{e} \in E^{d^{-1}}$ such that $p = \hat{d}(w^{-1}z)\hat{e}(z^{-1}w)$. In consequence $kh l$ can be written as

$$\begin{aligned} kh l &= u(u^{-1}k v z^{-1}w)(w^{-1}z)v^{-1}h l = u p (w^{-1}z)v^{-1}h l \\ &= u(\hat{d}(w^{-1}z)\hat{e}(z^{-1}w))(w^{-1}z)v^{-1}h l \\ &= (h_u c)\hat{d}(c^{-1}h_w^{-1}h l_z d)\hat{e}(d^{-1}l_v^{-1}h^{-1})h l \\ &= (h_u \hat{d}^c h_w^{-1} h)(l_z \hat{e}^d l_v^{-1} l), \end{aligned}$$

the last expression being in HL since $\hat{d}^c \in D$ and $\hat{e}^d \in E$. Therefore, $kg = kh l a^{-1} \in H L a^{-1}$, implying the desired contradiction and concluding the proof of this case and of the theorem. □

The next corollary roughly says that under some mild assumptions, double cosets of doubly peripherally separable pairs of relatively quasiconvex subgroups are “almost” the intersection of double cosets of fully relatively quasiconvex subgroups.

Corollary 3.24 *Let (G, \mathcal{P}) be relatively hyperbolic and let $H, L < G$ be relatively quasiconvex subgroups such that H, L is a doubly peripherally separable pair. Also, suppose that \dot{H}, \dot{L} and \dot{P} are separable in G for any finite-index subgroups $\dot{H} < H, \dot{L} < L$ or $\dot{P} < P$, with P being a peripheral subgroup of G .*

Then for any $a \in G \setminus HL$ there exist fully relatively quasiconvex subgroups \hat{H}, \hat{L} with $H \cap \hat{H} < H$ and $L \cap \hat{L} < L$ of finite-index, such that $a \notin H \hat{H} \hat{L} L$.

Proof For a, H and L as in the statement, let K_1, \dots, K_n be the filling kernels given by Theorem 3.21, and let $K = K(a, H, L) \trianglelefteq G$ be the kernel of the filling $G \rightarrow G/(K_1, \dots, K_n)$. Since by assumption each K_i is finite-index in P_i and separable in G , the subgroups $K_i \cap H$ are separable in H , and there exists a finite-index subgroup $\dot{H} \trianglelefteq H$ such that $\dot{H} \cap P_i < K_i$ for all i , implying $\dot{H} \cap P < K$ for any peripheral subgroup P . Similarly we can find $\dot{L} \trianglelefteq L$ of finite index so that $\dot{L} \cap P < K$ for any peripheral P .

We claim the existence of parabolic subgroups $Q_1, \dots, Q_s, R_1, \dots, R_t < G$, each of them a finite-index subgroup of the conjugate of some K_i , and such that

$$\hat{H} := \langle \dot{H}, Q_1, \dots, Q_s \rangle \quad \text{and} \quad \hat{L} := \langle \dot{L}, R_1, \dots, R_t \rangle$$

are fully relatively quasiconvex subgroups of G . By assuming this claim, the corollary follows since $a \notin KHL$ and

$$\begin{aligned} H\hat{H}\hat{L}L &= H\langle \dot{H}, Q_1, \dots, Q_s \rangle \langle \dot{L}, R_1, \dots, R_t \rangle L \\ &\subset H\langle \dot{H}, K \rangle \langle \dot{L}, K \rangle L = H\dot{H}K\dot{L}L = KHL. \end{aligned}$$

To prove the claim, we will only construct the groups Q_1, \dots, Q_s , since the groups R_1, \dots, R_t can be found in exactly the same way. Let us choose $\mathcal{D}_0 = \{D_1, \dots, D_s\}$ an induced peripheral structure on $\dot{H}_0 = \dot{H}$ with $D_i = \dot{H} \cap P_{j_i}^{c_i}$ for some $1 \leq j_i \leq n$ and $c_i \in G$, and inductively construct Q_1, \dots, Q_s with each $Q_i < K_{j_i}^{c_i}$ of finite index, so that $\mathcal{D}_i := \{Q_1, \dots, Q_i, D_{i+1}, \dots, D_s\}$ is an induced peripheral structure on $\dot{H}_i := \langle \dot{H}, Q_1, \dots, Q_i \rangle = \langle \dot{H}_{i-1}, Q_i \rangle$.

Assume we have found \dot{H}_{i-1} and Q_1, \dots, Q_{i-1} , and note that every parabolic subgroup of \dot{H}_{i-1} is a subgroup of K . If $D_i < K_{j_i}^{c_i}$ is finite index, define $Q_i = D_i$. Otherwise, by [30, Theorem 1.1] there exists a finite set $F \subset P_{j_i}^{c_i} \setminus (\dot{H}_{i-1} \cap P_{j_i}^{c_i})$ such that for any $P' < P_{j_i}^{c_i}$ containing $\dot{H}_{i-1} \cap P_{j_i}^{c_i}$ and disjoint from F , the group $\langle \dot{H}_{i-1}, P' \rangle$ is relatively quasiconvex and has $\{Q_1, \dots, Q_{i-1}, P'\}$ as induced peripheral structure. It is then enough to find a subgroup $P' < K_{j_i}^{c_i}$ of finite index, containing $\dot{H}_{i-1} \cap P_{j_i}^{c_i}$ and disjoint from F , so in that case we can define $Q_i = P'$.

To find such P' , note that each infinite parabolic subgroup of \dot{H}_{i-1} is \dot{H}_{i-1} -conjugate into some group in \mathcal{D}_{i-1} , implying $\dot{H}_{i-1} \cap P_{j_i}^{c_i} = D_i$. Also, since \dot{H} is separable in G , we have that D_i is separable in $P_{j_i}^{c_i}$. But by construction D_i is contained in K , so in fact D_i is separable in $K_{j_i}^{c_i}$, and there is a finite-index subgroup $P' < K_{j_i}^{c_i}$ which is disjoint from F and such that $P' \supset D_i$. This solves the claim, and since each subgroup in \mathcal{D}_s is finite index in some peripheral subgroup, the group $\hat{H} = \dot{H}_s$ is fully relatively quasiconvex. □

3.4 The malnormal special quotient theorem

We end this section by presenting a result that will be needed in the proofs of Theorem 5.1 and Proposition 8.14. It depends on the relative version of Wise’s malnormal special quotient theorem, recently obtained by Einstein [12, Theorem 2].

Theorem 3.25 (Einstein) *If $(G, \{P_1, \dots, P_n\})$ is a relatively hyperbolic group with G cubulated and virtually special, then there exist finite-index subgroups $\dot{P}_i \trianglelefteq P_i$ such that if $\bar{G} = G(N_1, \dots, N_n)$ is any filling with each $N_i < \dot{P}_i$ of finite index, then \bar{G} is hyperbolic and virtually special.*

By combining Theorem 3.4, Lemma 3.6, Theorem 3.9 and Einstein’s Theorem 3.25, we obtain:

Proposition 3.26 *Let (G, \mathcal{P}) be a relatively hyperbolic group with each $P \in \mathcal{P}$ residually finite, and let $\mathcal{H} = \{H_0, H_1, H_2, \dots, H_k\}$ be a collection of relatively quasiconvex subgroups of G . Assume that the groups H_1, \dots, H_k are all distinct and contained in H_0 , that H_l is peripherally separable for $0 \leq l \leq k$, and H_0 is strongly peripherally separable. Consider also a finite set $A \subset G$.*

Then there exist finite-index subgroups $\dot{P}_i \trianglelefteq P_i$ such that for any further finite-index subgroups $N_j < \dot{P}_j$, the filling $\phi: G \rightarrow \bar{G} = G(\mathcal{N}) = G(N_1, \dots, N_n)$ is such that:

- (i) \bar{G} is hyperbolic.
- (ii) $\bar{H}_l = \phi(H_l)$ is quasiconvex in \bar{G} and naturally isomorphic to the induced filling $H_l(\mathcal{N}_{H_l})$ for each $0 \leq l \leq k$.
- (iii) $\phi|_A: A \rightarrow \bar{G}$ is injective and $\phi(A \cap H_l) = \phi(A) \cap \bar{H}_l$ for all $0 \leq l \leq k$.
- (iv) For each l , if H_l is virtually special and strongly peripherally separable, then \bar{H}_l is also virtually special.
- (v) For $1 \leq l \leq k$, \bar{H}_l is isomorphic to the filling induced by $H_0 \rightarrow \bar{H}_0$.
- (vi) The relative height of $\{\bar{H}_1, \dots, \bar{H}_k\}$ in \bar{H}_0 is at most the relative height of $\{H_1, \dots, H_k\}$ in H_0 .

Proof First, consider a peripheral structure $\mathcal{D}_0 = \{D_{0,1}, \dots, D_{0,k_0}\}$ on H_0 induced by G so that $D_{0,i} = H_0 \cap P_{\alpha_{0,i}}^{c_{0,i}}$ for some $1 \leq \alpha_{0,i} \leq n$ and some shortest $c_{0,i} \in G$. Also, for any $1 \leq l \leq k$ consider a peripheral structure $\mathcal{D}_l = \{D_{l,1}, \dots, D_{l,k_l}\}$ on H_l induced by (H_0, \mathcal{D}_0) , so that $D_{l,i} = H_l \cap D_{0,\beta_{l,i}}^{d_{l,i}}$ for some $1 \leq \beta_{l,i} \leq k_0$ and some shortest $d_{l,i} \in H_0$. Since \mathcal{D}_l is also a peripheral structure induced by G , we have $D_{l,i} = H_l \cap P_{\alpha_{l,i}}^{c_{l,i}}$ for $1 \leq \alpha_{l,i} \leq n$ and $c_{l,i} \in G$. The equation

$$D_{l,i} = H_l \cap P_{\alpha_{l,i}}^{c_{l,i}} = H_l \cap D_{0,\beta_{l,i}}^{d_{l,i}} = H_l \cap P_{\alpha_{0,\beta_{l,i}}}^{d_{l,i}c_{0,\beta_{l,i}}}$$

then implies

$$(5) \quad \alpha_{l,i} = \alpha_{0,\beta_{l,i}} \quad \text{and} \quad (c_{l,i})^{-1}d_{l,i}c_{0,\beta_{l,i}} \in P_{\alpha_{l,i}}.$$

The relevance of this equation is that if $\phi: G \rightarrow G(N_1, \dots, N_n)$ induces the filling $\phi_0: H_0 \rightarrow H_0(K_{0,1}, \dots, K_{0,k_0})$, then for any $1 \leq l \leq k$ the filling $\phi_l: H_l \rightarrow H_l(K_{l,1}, \dots, K_{l,k_l})$ induced by ϕ is the same as the one induced by ϕ_0 . Indeed,

a filling kernel of ϕ_l induced by ϕ is of the form $K_{l,i} = D_{l,i} \cap N_{\alpha_{l,i}}^{c_{l,i}}$, while the one induced by ϕ_0 takes the form

$$K_{l,i}^{(0)} = D_{l,i} \cap K_{0,\beta_{l,i}}^{d_{l,i}} = D_{l,i} \cap N_{\alpha_{0,\beta_{l,i}}}^{d_{l,i}c_{0,\beta_{l,i}}}.$$

The equality $K_{l,i} = K_{l,i}^{(0)}$ then follows from $N_{\alpha_{l,i}}^{c_{l,i}} = N_{\alpha_{0,\beta_{l,i}}}^{d_{l,i}c_{0,\beta_{l,i}}}$, which is a consequence of (5) and the fact that $N_{\alpha_{l,i}}$ is normal in $P_{\alpha_{l,i}}$.

With that in mind, by Theorem 3.4 we can find a finite set $S \subset \bigcup \mathcal{P} \setminus \{1\}$ such that all (\mathcal{H}, S) -wide and peripherally finite fillings $\phi: G \rightarrow \bar{G} = G(N_1, \dots, N_n)$ with $S \cap (\bigcup N_i) = \emptyset$ satisfy the conditions (i)–(iii)—for the injectivity of $\phi|_A$, include the trivial group into \mathcal{H} and consider the finite set $\{ab^{-1} \neq 1 : a, b \in A\} \subset G$. For such S , Lemma 3.6 gives us finite-index subgroups $\dot{N}_j \trianglelefteq P_j$ such that any filling $G \rightarrow G(N_1, \dots, N_n)$ with $N_j < \dot{N}_j$ is (\mathcal{H}, S) -wide.

Now, let I be the set of $0 \leq l \leq k$ such that H_l is virtually special, and apply Theorem 3.25 to each pair (H_l, \mathcal{D}_l) with $l \in I$ to obtain finite-index subgroups $\dot{D}_{l,i} \trianglelefteq D_{l,i}$ such that for any further finite-index subgroups $\tilde{D}_{l,i} < \dot{D}_{l,i}$, the filling $H_l(\tilde{D}_{l,1}, \dots, \tilde{D}_{l,k_l})$ is virtually special.

Given $l \in I$ and $1 \leq i \leq k_l$, there is a chain of inclusions

$$\dot{D}_{l,i} \cap \dot{N}_{\alpha_{l,i}}^{c_{l,i}} \trianglelefteq D_{l,i} \cap \dot{N}_{\alpha_{l,i}}^{c_{l,i}} < \dot{N}_{\alpha_{l,i}}^{c_{l,i}},$$

the first one being of finite index. Since H_l is strongly peripherally separable, the group $\dot{D}_{l,i}$ is separable in $P_{\alpha_{l,i}}^{c_{l,i}}$, and hence the inclusion $\dot{D}_{l,i} \cap \dot{N}_{\alpha_{l,i}}^{c_{l,i}} < \dot{N}_{\alpha_{l,i}}^{c_{l,i}}$ is also separable. Therefore we can find a finite-index subgroup $\tilde{N}_{\gamma_{l,i}} \trianglelefteq P_{\alpha_{l,i}}$ contained in $\dot{N}_{\alpha_{l,i}}$ such that $\dot{D}_{l,i} \cap \tilde{N}_{\gamma_{l,i}}^{c_{l,i}} = D_{l,i} \cap \tilde{N}_{\gamma_{l,i}}^{c_{l,i}}$. We conclude that $K_{l,i} := D_{l,i} \cap \tilde{N}_{\gamma_{l,i}}^{c_{l,i}}$ is contained in $\dot{D}_{l,i}$ as a finite-index subgroup.

For $1 \leq j \leq n$, let I_j be the set of pairs (l, i) with $l \in I$ and $1 \leq i \leq k_l$, such that $j = \alpha_{l,i}$. Let $\tilde{P}_j := \bigcap_{(l,i) \in I_j} \tilde{N}_{\gamma_{l,i}}$ if $I_j \neq \emptyset$, and $\tilde{P}_j = \dot{N}_j$ otherwise. Since each \dot{P}_j is of finite index in P_j and by assumption the groups P_j are residually finite, we can find finite-index subgroups $\dot{P}_j^{(1)} \trianglelefteq P_j$ contained in \tilde{P}_j and disjoint from S . By construction any filling $\phi: G \rightarrow \bar{G} = G(N_1, \dots, N_n)$ with $N_j < \dot{P}_j^{(1)}$ of finite index satisfies the conclusions (i)–(iv).

To deal with (v)–(vi), note that the groups H_1, \dots, H_k are peripherally separable in (H_0, \mathcal{D}_0) , and that each group in \mathcal{D}_0 is residually finite. Therefore, by applying Theorems 3.4, 3.6 and 3.9, and in the same way we constructed the groups $\dot{P}_j^{(1)}$ above,

we obtain finite-index subgroups $\dot{K}_{0,i} \trianglelefteq D_{0,i}$ such that for any filling $\phi_0: H_0 \rightarrow H_0(K_{0,1}, \dots, K_{0,k_0})$ with $K_{0,i} < \dot{K}_{0,i}$, we have that

- (a) for any $1 \leq l \leq k$, the induced filling ϕ_l of H_l satisfies $\ker \phi_l = \ker \phi_0 \cap H_l$, and
- (b) the height of $\{\phi_0(H_1), \dots, \phi_0(H_k)\}$ in $\phi_0(H_0)$ is at most the relative height of $\{H_1, \dots, H_k\}$ in H_0 .

By our strong peripheral separability assumption, and by the same separability argument used to find the groups $\tilde{N}_{\gamma_l,i}$, there exist finite-index subgroups $\dot{P}_j \trianglelefteq P_j$ contained in $\dot{P}_j^{(1)}$ such that $D_{0,i} \cap \dot{P}_{\alpha_{0,i}}^{c_{0,i}} < \dot{K}_{0,i}$ for any $1 \leq i \leq k_0$.

By construction, if $\phi: G \rightarrow G(N_1, \dots, N_n)$ is a peripherally finite filling with $N_i < \dot{P}_j$, then it satisfies conclusions (i)–(iv). In addition, $\ker \phi \cap H_j = (\ker \phi_0 \cap H_0) \cap H_j = \ker \phi_j$ by (a), so it also satisfies (v). Finally, note that by the discussion after equation (5), the embedding $\phi_0(H_0) \hookrightarrow \phi(G)$ induces isomorphisms $\phi_0(H_j) \xrightarrow{\sim} \phi(H_j)$ for each j , and so the height of $\{\phi_0(H_1), \dots, \phi_0(H_k)\}$ in $\phi_0(H_0)$ coincides with the height of $\{\phi(H_1), \dots, \phi(H_k)\}$ in $\phi(H_0)$. This fact together with (b) proves (vi). □

4 Proof of Theorem 1.8

In this section we prove Theorem 1.8. We first recall the notion of graph of groups, referring to the work of Bass in [4].

Definition 4.1 A graph of groups is a pair (Γ, \mathcal{G}) consisting of

- (i) a connected, nonempty graph Γ with vertex set $V = V(\Gamma)$, and an oriented edge set $E = E(\Gamma)$ with an involution $e \mapsto \bar{e}$ that switches the orientation of each edge,
- (ii) an assignment $\mathcal{G}: V \sqcup E \rightarrow \mathbf{Grp}$ of a group $x \mapsto G_x = \mathcal{G}(x)$ for any vertex or edge x , such that $G_e = G_{\bar{e}}$ for any edge e , and
- (iii) a set of attachment monomorphisms $\psi_e: G_e \rightarrow G_{t(e)}$, where $t(e)$ is the terminal vertex of the edge e .

Given a graph of groups (Γ, \mathcal{G}) we consider the group

$$F(\Gamma, \mathcal{G}) = (*_{v \in V} G_v) * F(E) / N,$$

where $F(E)$ is the free group generated by the set E and N is the normal subgroup generated by the relations $e^{-1} = \bar{e}$ and $e\psi_e(g)e^{-1} = \psi_{\bar{e}}(g)$ for any edge e and any $g \in G_e$.

Definition 4.2 The *fundamental group* of (Γ, \mathcal{G}) based at $v_0 \in V$ is the subgroup

$$\pi_1(\Gamma, \mathcal{G}, v_0) < F(\Gamma, \mathcal{G})$$

consisting of the elements of the form

$$g = g_0 e_1 g_1 e_2 \cdots e_n g_n,$$

where e_1, e_2, \dots, e_n form a *circuit* in Γ based at v_0 (that is, $t(e_i) = t(\bar{e}_{i+1})$ for $1 \leq i \leq n-1$ and $t(e_n) = t(\bar{e}_1) = v_0$) and $g_i \in G_{t(\bar{e}_{i+1})}$ for any $0 \leq i \leq n$, with the convention $t(\bar{e}_{n+1}) = v_0$.

The isomorphism class of $\pi_1(\Gamma, \mathcal{G}, v_0)$ is independent of the basepoint, and the canonical maps from vertex groups into $F(\Gamma, \mathcal{G})$ are injective, so we can consider any vertex group as a subgroup of $\pi_1(\Gamma, \mathcal{G}, v_0)$ by means of choosing a maximal tree T of Γ containing v_0 .

Definition 4.3 We say that a group G *splits* as a graph of groups (Γ, \mathcal{G}) if G is isomorphic to $\pi_1(\Gamma, \mathcal{G}, v_0)$ for some (hence any) vertex v_0 of Γ .

The proposition below gives existence of hyperbolic and virtually special fillings for groups in \mathcal{CMVH} , and is the key ingredient in the proof of Theorem 1.8.

Proposition 4.4 Let $(G, \mathcal{P} = \{P_1, \dots, P_n\})$ be a relatively hyperbolic group with each P_i residually finite. Suppose G splits as a finite graph of groups (Γ, \mathcal{G}) so that

- each edge group is relatively quasiconvex in G and peripherally separable,
- if v is a vertex of Γ , then the collection $\mathcal{A}_v := \{G_e : e \text{ attached to } v\}$ is relatively malnormal in G_v , and
- each vertex group is virtually special and strongly peripherally separable.

Then:

- (i) Every relatively quasiconvex, peripherally separable subgroup of G is separable.
- (ii) Every double coset of a doubly peripherally separable pair of relatively quasiconvex and strongly peripherally separable subgroups of G is separable.

Proposition 4.5 Let (Γ, \mathcal{G}) be as in the statement of Proposition 4.4. Then there exist finite-index subgroups $\dot{P}_j \trianglelefteq P_j$ such that for any further finite-index subgroups $N_j < \dot{P}_j$, there are subgroups $M_j < N_j$ (normal in P_j) with $\bar{\mathbb{G}} = G(M_1, \dots, M_n)$ hyperbolic and virtually special.

Remark 4.6 Since we are restricting to finitely generated groups, relative quasiconvexity of edge groups implies relative quasiconvexity of vertex groups [6, Lemma 4.9].

Before proving Proposition 4.5, we need a lemma. Condition (ii) will be used in the proof of Proposition 8.14.

Lemma 4.7 *Let $\bar{\mathbb{G}}$ be a group splitting as a finite graph of groups $(\Gamma, \bar{\mathcal{G}})$ with hyperbolic vertex groups, and edge groups quasiconvex in their corresponding vertex groups. Suppose that either*

- (i) *for each vertex $v \in V(\Gamma)$, the collection $\bar{\mathcal{P}}_v := \{\bar{G}_e : e \text{ is an edge attached to } v\}$ is almost malnormal in \bar{G}_v , or*
- (ii) *Γ is bipartite, with $V(\Gamma) = V_1 \sqcup V_2$ and each edge joining vertices of V_1 and V_2 , such that for each $v \in V_1$ the collection $\bar{\mathcal{P}}_v := \{\bar{G}_e : e \text{ is an edge attached to } v\}$ is almost malnormal in \bar{G}_v .*

Then $\bar{\mathbb{G}}$ is hyperbolic and the edge groups are quasiconvex in $\bar{\mathbb{G}}$.

Proof Since edge groups are quasiconvex in the vertex groups, by almost malnormality the pair $(\bar{G}_v, \bar{\mathcal{P}}_v)$ is relatively hyperbolic for each $v \in V$ (resp. $v \in V_1$). Also, for $v \in V_2$ consider the trivial peripheral structure $(\bar{G}_v, \{\bar{G}_v\})$. With these conventions, each edge group is maximal parabolic in vertex groups of V (resp. V_1), and no two of them are conjugate into a common vertex group \bar{G}_v with $v \in V$ (resp. $v \in V_1$), unless they are finite. Therefore we are in the assumptions of [6, Corollary 4.6] (resp. [6, Corollary 1.5]), and hence $\bar{\mathbb{G}}$ is hyperbolic relative to $\bigcup_{v \in V} \bar{\mathcal{P}}_v - \{\text{repeats}\}$ (resp. $\{\bar{G}_v : v \in V_2\}$), and the edge groups are quasiconvex in some peripheral subgroup of $\bar{\mathbb{G}}$. In both cases $\bar{\mathbb{G}}$ will be hyperbolic relative to hyperbolic groups, hence hyperbolic as well (see for instance [14, Theorem 3.8]), and edge groups will be quasiconvex since the peripheral subgroups are quasiconvex in $\bar{\mathbb{G}}$. □

Proof of Proposition 4.5 Let us fix a vertex $v_0 \in V = V(\Gamma)$, an isomorphism $G \cong \pi_1(\Gamma, \mathcal{G}, v_0)$, and a maximal tree T of Γ containing v_0 that induces embeddings of the vertex/edge groups of (Γ, \mathcal{G}) into G . For each vertex, apply Proposition 3.26 to the collection \mathcal{A}_v and the group G_v , to find finite-index subgroups $\dot{P}_j(v) \trianglelefteq P_j$ such that for any choice of peripherally finite filling kernels $\mathcal{N} = \{N_1, \dots, N_n\}$ with $N_i < \dot{P}_i(v)$, the filling $\phi: G \rightarrow \bar{G} = G(N_1, \dots, N_n)$ satisfies:

- \bar{G} is hyperbolic.
- $\bar{G}_v := \phi(G_v)$ is virtually special, quasiconvex (hence hyperbolic) in \bar{G} , and isomorphic to the image of the induced filling $\phi_v: G_v \rightarrow G_v(\mathcal{N}_v)$.

- The collection $\bar{\mathcal{A}}_v$ of images under ϕ of groups in \mathcal{A}_v is almost malnormal in \bar{G}_v .
- Each $\bar{G}_e := \phi(G_e)$ in $\bar{\mathcal{A}}_v$ is naturally isomorphic to the image of the filling $\phi_e: G_e \rightarrow G_e(\mathcal{N}_e)$ induced by both ϕ and ϕ_v ; that is, $\ker \phi_e = \ker \phi_v \cap G_e = \ker \phi \cap G_e$.

For $1 \leq j \leq n$ define $\dot{P}_j := \bigcap_{v \in V} \dot{P}_j(v)$, which is a finite-index normal subgroup of P_j , and consider finite-index subgroups $N_j < \dot{P}_j$ inducing the filling $\phi: G \rightarrow \bar{G} = G(N_1, \dots, N_n)$.

To construct $\bar{\mathbb{G}}$, consider a new graph of groups $(\Gamma, \bar{\mathcal{G}})$ with the same underlying graph Γ , with $\bar{\mathcal{G}}$ assigning the group \bar{G}_x to each $x \in V \sqcup E$, and with attaching maps being the inclusions $\bar{\psi}_e: \bar{G}_e \hookrightarrow \bar{G}_{t(e)}$ induced by ϕ . Define $\bar{\mathbb{G}}$ as $\pi_1(\Gamma, \bar{\mathcal{G}}, v_0)$, and choose embeddings of the vertex groups according to the same maximal tree T .

This choice of embeddings induces commuting diagrams

$$\begin{array}{ccc} G_e & \xrightarrow{\phi_e} & \bar{G}_e \\ \downarrow \phi_e & & \downarrow \bar{\phi}_e \\ G_{t(e)} & \xrightarrow{\phi_{t(e)}} & \bar{G}_{t(e)} \end{array}$$

all of them together inducing a homomorphism of graphs of groups (see eg Bass [4])

$$\Phi: G \rightarrow \bar{\mathbb{G}}$$

such that $\Phi(x) = \phi_v(x)$ for any vertex v and for any $x \in G_v$.

By our choice of ϕ , the splitting $(\Gamma, \bar{\mathcal{G}})$ of $\bar{\mathbb{G}}$ satisfies the assumptions of Lemma 4.7(i), and so $\bar{\mathbb{G}}$ is hyperbolic and the edge groups $\Phi(G_e) = \bar{G}_e$ are quasiconvex in $\bar{\mathbb{G}}$. In addition, by assumption the vertex groups of $(\Gamma, \bar{\mathcal{G}})$ are hyperbolic and virtually special, and so Theorem 2.22 implies that $\bar{\mathbb{G}}$ is virtually special.

Finally, we need to show that $\ker \Phi = \langle\langle \bigcup_j M_j \rangle\rangle_G$ for some $M_j \trianglelefteq P_j$ contained in N_j . To do this, first note that $\ker \Phi = \langle\langle \bigcup_{v \in V} \ker \phi_v \rangle\rangle_G$, and that for each v we have the equality $\ker \phi_v = \langle\langle \bigcup_{D \in \mathcal{D}_j} K_D \rangle\rangle_{G_v}$, with \mathcal{D}_v being a peripheral structure on G_v induced by (G, \mathcal{P}) , and $\{K_D\}_{D \in \mathcal{D}_v}$ being the filling kernels induced by ϕ .

Suppose each $D \in \mathcal{D}_v$ is of the form $D = G_v \cap P_{i_D}^{c_D}$ with $1 \leq i_D \leq n$ and $c_D \in G$, and hence $K_D = H \cap D \cap N_{i_D}^{c_D}$. For $1 \leq j \leq n$, let \mathcal{D}_j denote the set of all $D \in \bigcup_v \mathcal{D}_v$ such that $i_D = j$, and define

$$M_j := \begin{cases} \langle\langle \bigcup_{D \in \mathcal{D}_j} K_D^{c_D^{-1}} \rangle\rangle_{P_j} & \text{if } \mathcal{D}_j \text{ is nonempty,} \\ \{1\} & \text{otherwise.} \end{cases}$$

Note that $M_j < N_j$ for each j . We claim that $\langle\langle \bigcup_j M_j \rangle\rangle_G = \langle\langle \bigcup_{v \in V} \ker \phi_v \rangle\rangle_G$ for these choices of M_j .

For the inclusion \subset , it is enough to show $M_j \subset \langle\langle \bigcup_{v \in V} \ker \phi_v \rangle\rangle_G$ for any j , which holds because when \mathcal{D}_j is nonempty, we have

$$\begin{aligned} M_j &= \left\langle\left\langle \bigcup_{D \in \mathcal{D}_j} K_D^{c_D^{-1}} \right\rangle\right\rangle_{P_j} \subset \left\langle\left\langle \bigcup_{D \in \mathcal{D}_j} K_D^{c_D^{-1}} \right\rangle\right\rangle_G = \left\langle\left\langle \bigcup_{D \in \mathcal{D}_j} K_D \right\rangle\right\rangle_G \\ &\subset \left\langle\left\langle \bigcup_{v \in V} \bigcup_{D \in \mathcal{D}_v} K_D \right\rangle\right\rangle_G \subset \left\langle\left\langle \bigcup_{v \in V} \ker \phi_v \right\rangle\right\rangle_G. \end{aligned}$$

On the other hand, for any $v \in V$ we obtain

$$\begin{aligned} \ker \phi_v &= \left\langle\left\langle \bigcup_{D \in \mathcal{D}_v} K_D \right\rangle\right\rangle_{G_v} \subset \left\langle\left\langle \bigcup_{D \in \mathcal{D}_v} K_D \right\rangle\right\rangle_G \subset \left\langle\left\langle \bigcup_{D \in \mathcal{D}_v} K_D^{c_D^{-1}} \right\rangle\right\rangle_G \\ &\subset \left\langle\left\langle \bigcup_j \bigcup_{D \in \mathcal{D}_j} K_D^{c_D^{-1}} \right\rangle\right\rangle_G = \left\langle\left\langle \bigcup_j M_j \right\rangle\right\rangle_G, \end{aligned}$$

which proves \supset . □

Proof of Proposition 4.4 To prove conclusion (i), let H be a relatively quasiconvex and peripherally separable subgroup of G and consider $a \in G \setminus H$, which we want to separate from H in a finite quotient of G . Instead of using Dehn filling directly on H , we will use [36, Corollary 6.3] and the peripheral separability of H to find a fully relatively quasiconvex subgroup $\hat{H} < G$ containing H and such that $a \notin \hat{H}$, so now it is enough to separate a from \hat{H} . By Theorem 3.4(i)–(iii) there exists a finite set $S \subset \bigcup \mathcal{P} \setminus \{1\}$ such that for any (\hat{H}, S) –wide filling $\phi: G \rightarrow \bar{G} = G(N_1, \dots, N_n)$ with $S \cap \bigcup_j N_j = \emptyset$, the pair $(\bar{G}, \bar{\mathcal{P}})$ is relatively hyperbolic with $\bar{H} = \phi(\hat{H})$ fully relatively quasiconvex in $(\bar{G}, \bar{\mathcal{P}})$ and $\phi(a) \notin \bar{H}$. Since the groups P_j are residually finite, there exist finite-index subgroups $Q_j \trianglelefteq P_j$ with each Q_j being disjoint from S . Also, since \hat{H} is fully relatively quasiconvex, it is peripherally separable and by Lemma 3.6 there exist finite-index subgroups $\dot{N}_j \trianglelefteq P_j$ such that any filling $G \rightarrow G(N_1, \dots, N_n)$ with $N_j < \dot{N}_j$ for each j is (\hat{H}, S) –wide.

Let $\dot{P}_j \trianglelefteq P_j$ be given by Proposition 4.5, and set $N_j := \dot{N}_j \cap \dot{P}_j \cap Q_j \trianglelefteq P_j$. By construction, $\phi: G \rightarrow \bar{G} = G(N_1, \dots, N_n)$ is an (\hat{H}, S) –wide filling factoring through the (\hat{H}, S) –wide filling $\Phi: (G, \mathcal{P}) \rightarrow (\bar{G}, \bar{\mathcal{P}})$, with $(\bar{G}, \bar{\mathcal{P}})$ relatively hyperbolic, $\Phi(\hat{H})$ fully relatively quasiconvex in $(\bar{G}, \bar{\mathcal{P}})$ and $\Phi(a) \notin \Phi(\hat{H})$ (here $\bar{\mathcal{P}}$ denotes the set of images of groups in \mathcal{P} under Φ). Also, \bar{G} is hyperbolic and virtually special, and therefore, to solve the problem it is enough to show that $\Phi(\hat{H})$ is separable in \bar{G} .

By Theorem 2.19 it suffices to show that $\Phi(\hat{H})$ is quasiconvex in \bar{G} , ie that $\Phi(\hat{H})$ is relatively quasiconvex in (\bar{G}, \emptyset) . But (\bar{G}, \bar{P}) is an extended peripheral structure for (\bar{G}, \emptyset) , and so by [41, Theorem 1.3] it is enough to show that $\Phi(\hat{H}) \cap \Phi(P)$ is quasiconvex in \bar{G} for any peripheral subgroup $P \subset G$. This last statement follows from the quasiconvexity of $\Phi(P)$ in \bar{G} [11, Lemma 4.15], and the full relative quasiconvexity of $\Phi(\hat{H})$ in \bar{G} , since quasiconvexity is stable under finite-index inclusions.

Now, let $H, L < G$ be relatively quasiconvex and strongly peripherally separable subgroups such that H, L is doubly peripherally separable, and consider $a \in G \setminus HL$. Conclusion (i) implies that the peripheral subgroups of G are separable, as well as any of their finite-index subgroups. In addition, by our strong peripheral separability assumption, any finite-index subgroup of H or L will be peripherally separable, hence also separable. Therefore, H and L satisfy the assumptions of Corollary 3.24, and so there are fully relatively quasiconvex subgroups \hat{H} and \hat{L} with $H \cap \hat{H} < H$ and $L \cap \hat{L} < L$ of finite index, and such that $a \notin H\hat{H}\hat{L}L$.

Thus, to prove conclusion (ii) we only need to verify that $H\hat{H}\hat{L}L$ is separable, and since this set is a finite union of translates of $\hat{H}\hat{L}$, it is enough to show that double cosets of fully relatively quasiconvex subgroups are separable. This can be done in the same way as in the proof of conclusion (i), by using Proposition 4.5 together with Theorem 3.21 and the separability of double cosets of quasiconvex subgroups of hyperbolic virtually special groups [32, Theorem 1.1]. Details are left to the reader. \square

Remark 4.8 The last part in the proof of conclusion (ii) can also be deduced directly from [31, Corollary 4.4].

Proof of Theorem 1.8 Let (G, X) be a cubulated group in \mathcal{CMVH} . We will prove that (G, X) is virtually special by induction on the minimal number of operations (i)–(iii) used in a description of (G, X) (see Definition 1.7), where clearly $(\{1\}, X)$ is virtually special if X is a finite CAT(0) cube complex.

First, suppose that $H < G$ is of finite index with $(H, X) \in \mathcal{CMVH}$. Our inductive assumption implies that (H, X) is virtually special, so clearly (G, X) is also virtually special.

Now, let G be splitting as a finite graph of groups (Γ, \mathcal{G}) such that

- if $u \in V(\Gamma) \sqcup E(\Gamma)$, then G_u has convex core $X_u \subset X$,

- if $v \in V(\Gamma)$, then $(G_v, X_v) \in \mathcal{CMVH}$ (and hence (G_v, X_v) is virtually special by our inductive assumption and Lemma 2.25), and
- \mathcal{A}_v is relatively malnormal in G_v for any $v \in V(\Gamma)$.

To show that (G, X) is virtually special we will use Theorem 2.20, but first we need to show that G satisfies the hypotheses of Proposition 4.4. This follows because by our compatibility assumption the peripheral subgroups are residually finite, and because by Lemma 3.7, all convex subgroups of (G, X) are strongly peripherally separable.

Therefore, (G, X) satisfies Proposition 4.4, so any relatively quasiconvex and peripherally separable subgroup of G is separable, as well as any double coset of a doubly peripherally separable pair of relatively quasiconvex and strongly peripherally separable subgroups. In particular, the wall stabilizers of (G, X) are separable, so by Theorem 2.20 it is enough to show that the pair G_{W_1}, G_{W_2} is doubly peripherally separable for any pair of walls if $W_1, W_2 \subset X$. To prove this, let $g_1, g_2 \in G$ and let $P < G$ be a peripheral subgroup such that $G_{g_1^{-1}W_1} \cap P = (G_{W_1})^{g_1} \cap P$ and $G_{g_2^{-1}W_2} \cap P = (G_{W_2})^{g_2} \cap P$ are both infinite. By Theorem 2.12 choose a convex core $Z \subset X$ for P such that $g_1^{-1}W_1 \cap Z$ and $g_2^{-1}W_2 \cap Z$ are both nonempty, so that (P, Z) is virtually special by Proposition 2.24. The subgroups $G_{g_1^{-1}W_1} \cap P$ and $G_{g_2^{-1}W_2} \cap P$ are then convex in (P, Z) by Remark 2.13, and the conclusion follows from Theorem A.1. □

5 Construction of the complex with finite walls

Now we start the proof of Theorem 1.9, by induction on the dimension of X ; the 0-dimensional case is evident. Henceforth, let (G, X) be a cubulated relatively hyperbolic group with compatible virtually special peripherals, and assume that

$(H, \dot{Y}) \in \mathcal{CMVH}$ for every cubulated and relatively hyperbolic group (H, Y) with compatible virtually special peripheral subgroups and such that $\dim Y < \dim \dot{X} = \dim X$.

From Corollary A.3, (G, \dot{X}) is also hyperbolic relative to compatible virtually special subgroups.

Fix $R \geq \delta + 2\sqrt{\dim \dot{X}}$, and let W_1, \dots, W_m be a complete set of representatives of G -orbits of walls in \dot{X} . The goal of this section is to prove the following.

Theorem 5.1 *There exists a torsion-free normal subgroup $K \trianglelefteq G$ such that the quotient cube complex $\mathcal{X} := \dot{X}/K$ satisfies:*

- (i) G acts cocompactly on \mathcal{X} .
- (ii) All walls of \mathcal{X} are finite.
- (iii) If W is a wall of \dot{X} then the R -neighborhood $N_R(W)$ quotiented by $K \cap G_W$ embeds in \mathcal{X} . In particular, all walls of \mathcal{X} are embedded, and distinct walls in \dot{X} which are less than R apart map to distinct walls in \mathcal{X} .

We will need the following weak separability result for virtually special quasiconvex subgroups of hyperbolic groups, which is the main result in the appendix of [1].

Theorem 5.2 (Agol–Groves–Manning) *Let G be a hyperbolic group and $H < G$ be a quasiconvex virtually special subgroup. Then for any $g \in G \setminus H$ there is a hyperbolic group \mathcal{G} and a surjective homomorphism $\psi : G \rightarrow \mathcal{G}$ such that $\psi(g) \notin \psi(H)$ and $\psi(H)$ is finite.*

Proof of Theorem 5.1 For every $1 \leq i \leq m$, consider a set \mathcal{A}_i of representatives of double cosets $G_{W_i}gG_{W_i}$ with $g \notin G_{W_i}$ and $d(W_i, gW_i) \leq 2R$. Also, choose a set $\mathcal{T} \subset G$ of representatives of conjugacy classes of nontrivial torsion elements of G . These sets are finite by Lemma 2.9. Set $\mathcal{H} = \{G_{W_1}, \dots, G_{W_m}\}$.

As a first step, since each subgroup G_{W_i} is convex in (G, \dot{X}) , Lemma 3.7 and Proposition 3.26 imply that we can find a peripherally finite Dehn filling $\phi : G \rightarrow \bar{G} = G(\dot{P}_1, \dots, \dot{P}_n)$ so that \bar{G} is hyperbolic, each $\bar{G}_i := \phi(G_{W_i})$ is quasiconvex, virtually special and disjoint from $\phi(\mathcal{A}_i)$, and $1 \notin \phi(\mathcal{T})$.

Our second step is to find a quasiconvex and virtually special subgroup $\bar{H} < \bar{G}$ such that $\bar{H} \cap \bar{G}_i < \bar{G}_i$ is finite-index for $1 \leq i \leq m$. We will do this by inducting on k , and finding for each $1 \leq j \leq m$ a quasiconvex and virtually special subgroup $\bar{H}_j < \bar{G}$ such that $\bar{H}_j \cap \bar{G}_i < \bar{G}_i$ is finite-index for $1 \leq i \leq j$. For the base case we choose $\bar{H}_1 = \bar{G}_1$.

Suppose we have found \bar{H}_j , and consider the intersection $L = \bar{H}_j \cap \bar{G}_{j+1}$. By Gitik’s ping-pong theorem [15, Theorem 1], there exists a finite set $F \subset (\bar{H}_j \cup \bar{G}_{j+1}) \setminus L$ such that if $\tilde{H}_j < \bar{H}_j$ and $\tilde{G}_{j+1} < G_{j+1}$ are finite-index subgroups with $\tilde{H}_j \cap \tilde{G}_{j+1} = L$, and $\tilde{H}_j, \tilde{G}_{j+1}$ disjoint from F , then $\langle \tilde{H}_j \cup \tilde{G}_{j+1} \rangle$ is quasiconvex in \bar{G} and isomorphic to $\tilde{H}_j *_L \tilde{G}_{j+1}$. The existence of finite-index subgroups \tilde{H}_j and \tilde{G}_{j+1} disjoint from F

is guaranteed by subgroup separability, since by assumption \bar{H}_j and \bar{G}_{j+1} are virtually special, and L is quasiconvex in both groups, so we can apply Theorem 2.19.

Define $\bar{H}_{j+1} := \langle \tilde{H}_j \cup \tilde{G}_{j+1} \rangle$. By construction $\bar{H}_{j+1} \cap \bar{G}_{W_i}$ is finite-index in \bar{G}_{W_i} for $1 \leq i \leq j + 1$. The virtual specialness of \bar{H}_{j+1} follows from Theorem 2.22 and the isomorphism $\bar{H}_{j+1} \cong \tilde{H}_j *_L \tilde{G}_{j+1}$, where \tilde{H}_j and \tilde{G}_{j+1} are virtually special (finite-index subgroups of virtually special groups), and L is quasiconvex in \bar{H}_{j+1} since both \tilde{H}_j and \tilde{G}_{j+1} are quasiconvex subgroups of \bar{G} . The induction is then complete, and set $\bar{H} = \bar{H}_m$.

Before the end of the proof, and after possibly replacing \bar{H} by a finite-index subgroup, we can assume that \bar{H} does not intersect $\phi(\mathcal{T})$. This is because \bar{H} is virtually special, hence residually finite, and $\phi(\mathcal{T})$ is finite. This modification does not affect the expected properties for \bar{H} .

For $1 \leq i \leq m$, let $\bar{N}_i = \bar{H} \cap \bar{G}_i$, and define $N_i := \phi^{-1}(\bar{N}_i) \cap G_{W_i} < G_{W_i}$. Note that N_i is finite index in G_{W_i} for each i , so N_i acts cocompactly on W_i , and by Lemma 2.9(ii) there is a finite set $\tilde{\mathcal{A}}_i$ of representatives of double cosets $N_i g N_i$ with $g \notin N_i$ and $d(gW_i, W_i) \leq 2R$. We claim that $\phi(\tilde{\mathcal{A}}_i) \cap \bar{N}_i = \emptyset$ for each i . Indeed, let $a \in \tilde{\mathcal{A}}_i$, and suppose first that $a \notin G_{W_i}$. In that case, $a = g_1 b g_2$ for some $b \in \mathcal{A}_i$ and $g_1, g_2 \in G_{W_i}$, implying $\phi(a) = \phi(g_1)\phi(b)\phi(g_2) \notin \bar{G}_i \supset \bar{N}_i$ since by construction $\phi(b) \notin \bar{G}_i$. In the case $a \in G_{W_i}$, the conclusion follows from the definition of N_i . Our claim proven, the groups \bar{N}_i are quasiconvex in \bar{H} , hence separable by Theorem 2.19, and we can find finite-index subgroups $\hat{H}_i < \bar{H}$ such that $\hat{H}_i \cap \bar{G}_i = \bar{N}_i$ and $\hat{H}_i \cap \phi(\tilde{\mathcal{A}}_i) = \emptyset$ for all i . Note that the groups \hat{H}_i are quasiconvex in \bar{G} and virtually special.

Now we construct K . For each $1 \leq i \leq m$ and each $g \in \phi(\tilde{\mathcal{A}}_i)$ we use Theorem 5.2 to find a quotient homomorphism φ_g of \bar{G} with $\varphi_g(\hat{H}_i)$ finite and $\varphi_g(g) \notin \varphi_g(\hat{H}_i)$. Similarly, for each $g \in \phi(\mathcal{T})$ we construct a quotient homomorphism τ_g of \bar{G} with $\tau_g(g) \notin \tau_g(\bar{H})$ and $\tau_g(\bar{H})$ finite.

Define $K := \phi^{-1}((\bigcap_{g \in \cup_i \tilde{\mathcal{A}}_i} \ker \varphi_g) \cap (\bigcap_{g \in \mathcal{T}} \ker \tau_g)) \trianglelefteq G$ and $\mathcal{X} := \dot{X}/K$. Note that K is disjoint from \mathcal{T} , hence torsion-free, so \mathcal{X} is a cube complex. Clearly G acts cocompactly on \mathcal{X} , so (i) holds, and by construction $K \cap G_W$ is finite-index in G_W for each wall W , implying (ii). Finally, if $x, y \in N_R(W_i)$ and $k \in K$ are such that $x = ky$, then $d(kW_i, W_i) \leq 2R$. So if $k \notin K_i$, there is some $g \in \tilde{\mathcal{A}}_i$ with $k = h_1 g h_2$ for some $h_1, h_2 \in N_i$. But this is impossible since it would imply

$$\begin{aligned} \varphi_{\phi(g)}(\phi(g)) &= (\varphi_{\phi(g)} \circ \phi)(h_1^{-1} k h_2^{-1}) \\ &= (\varphi_{\phi(g)} \circ \phi)(h_1^{-1} h_2^{-1}) \in \varphi_{\phi(g)}(\bar{N}_i) \subset \varphi_{\phi(g)}(\hat{H}_i). \end{aligned}$$

Therefore $k \in N_i \subset G_{W_i}$, and the map $N_R(W_i)/(K \cap G_{W_i}) \rightarrow \mathcal{X}$ is an embedding for all $1 \leq i \leq m$. Property (iii) then follows from the normality of K and by considering translates of the W_i . \square

The point of working with \dot{X} instead of X , is that for every wall W of \dot{X} , the subgroup G_W does not exchange the sides of W . This allows us to find a G -equivariant *coorientation* on the walls of \dot{X} , that is, a labeling W^+ and W^- for the half-spaces of each wall W of \dot{X} , such that $(gW)^\pm = g(W^\pm)$ for any $g \in G$ and for any wall $W \subset \dot{X}$.

Let $q: \dot{X} \rightarrow \mathcal{X}$ denote the quotient map from Theorem 5.1. This map will send a vertex x (resp. an edge e and wall W) of \dot{X} to a vertex \bar{x} (resp. an edge \bar{e} and wall \bar{W}).

6 Coloring walls in \mathcal{X}

Now we proceed to color the walls of \mathcal{X} , in the same way as in [1, Section 5].

Definition 6.1 Let $\Gamma(\mathcal{X})$ be the simplicial graph with vertices the walls of \mathcal{X} and with an edge joining the walls \bar{W}_1 and \bar{W}_2 if and only $d(\bar{W}_1, \bar{W}_2) \leq R$, with d being the induced distance on \mathcal{X} and R as in Theorem 5.1.

There is a natural action of G on $\Gamma(\mathcal{X})$, and since \mathcal{X} is locally finite and with finite walls, there are only finitely many G -orbits of vertices in $\Gamma(\mathcal{X})$. This implies that there exists some k such that the degree of any vertex of $\Gamma(\mathcal{X})$ is bounded above by k , and also that G acts cocompactly on $\Gamma(\mathcal{X})$.

Definition 6.2 A *coloring* of $\Gamma(\mathcal{X})$ is a map $c: V(\Gamma(\mathcal{X})) \rightarrow \{1, \dots, k + 1\}$ such that if $\bar{W}_1, \bar{W}_2 \in V(\Gamma(\mathcal{X}))$ are adjacent walls, then $c(\bar{W}_1) \neq c(\bar{W}_2)$. Let $C_{k+1}(\Gamma(\mathcal{X}))$ denote the set of colorings, which is nonempty since vertices of $\Gamma(\mathcal{X})$ have degree $\leq k$.

The action of G on $\Gamma(\mathcal{X})$ induces an action on $C_{k+1}(\Gamma(\mathcal{X}))$ via pullback, given by $g: c \mapsto gc := c \circ g^{-1}$ for each $g \in G$.

We define several equivalence classes related to $C_{k+1}(\Gamma(\mathcal{X}))$, following the notation of [38, Definition 6.2].

- (i) If W is a wall of \dot{X} , define the equivalence class $[c]_W$ of $c \in C_{k+1}(\Gamma(\mathcal{X}))$ as

$$[c]_W := \{c' \in C_{k+1}(\Gamma(\mathcal{X})) : c = c' \text{ on the ball of radius } c(\bar{W}) \text{ in } \Gamma(\mathcal{X}) \text{ centered at } \bar{W}\}.$$

- (ii) If e is an edge of \dot{X} which is dual to the wall W , then $[c]_e := [c]_W$ for any $c \in C_{k+1}(\Gamma(\mathcal{X}))$.
- (iii) If x is a vertex of \dot{X} we define $[c]_x := \bigcap \{[c]_e : e \text{ incident to } x\}$.
- (iv) We define equivalence classes on $V(\dot{X}) \times C_{k+1}(\Gamma(\mathcal{X}))$ and $E(\dot{X}) \times C_{k+1}(\Gamma(\mathcal{X}))$ according to $[e, c] := \{e\} \times [c]_e$ and $[x, c] := \{x\} \times [c]_x$.

There are natural actions of G on these sets of equivalence classes, given by $g[e, c] := [ge, gc]$ and $g[x, c] := [gx, gc]$. For each edge e , the class $[-]_e$ depends only on the colors of vertices in some $(k + 1)$ -ball of $\Gamma(\mathcal{X})$, so there are only finitely many equivalence classes $[-]_e$, and similarly for $[-]_x$. Since there are finitely many G -orbits of edges and vertices in \dot{X} , there are only finitely many G -orbits of equivalence classes on $E(\dot{X}) \times C_{k+1}(\Gamma(\mathcal{X}))$ and $V(\dot{X}) \times C_{k+1}(\Gamma(\mathcal{X}))$.

7 Cubical polyhedra and the gluing construction

We introduce the main construction used to prove Theorem 1.9. As in [38, Sections 7 and 8], we will mainly work with the universal covers instead of the ‘‘cubical polyhedra’’ used in [1], so our notation will be similar to that used in [38].

Inductively, we will construct nonempty sets $\mathcal{V}_{k+1}, \dots, \mathcal{V}_0$, where each \mathcal{V}_j is a finite collection of triplets $(Z, H, (c_x))$, where

- $Z \subset \dot{X}$ is a nonempty intersection of half-spaces (thus closed and convex),
- for each vertex $x \in Z$ we have a coloring $c_x \in C_{k+1}(\Gamma(\mathcal{X}))$,
- $H < G$ acts freely and cocompactly on Z , and $c_{hx} = hc_x$ for each $h \in H$ and vertex $x \in Z$.

The subset $Z \subset \dot{X}$ is not a subcomplex of \dot{X} , but H also acts cocompactly on its cubical neighborhood $\mathcal{N}(Z)$. Since Z is intersection of half-spaces, the vertices inside Z span a convex subcomplex. The cubical neighborhood of this subcomplex is therefore convex, and also equals $\mathcal{N}(Z)$, so we have proven:

Lemma 7.1 $\mathcal{N}(Z)$ is a convex subcomplex of (G, \dot{X}) , hence a convex core for H .

We permit \mathcal{V}_j to contain duplicates of some triplets, and sometimes we will write $Z \in \mathcal{V}_j$ instead of $(Z, H, (c_x)) \in \mathcal{V}_j$, and also $(Z, H, (c_x); \alpha) \in \mathcal{V}_j$ to specify that there are exactly $\alpha \in \mathbb{N}$ duplicates of $(Z, H, (c_x))$ in \mathcal{V}_j .

Definition 7.2 We require each triplet $(Z, H, (c_x)) \in \mathcal{V}_j$ to satisfy four conditions:

- (i) If $e \in E(\dot{X})$ joins vertices $x, y \in Z \in \mathcal{V}_j$, then $[e, c_x] = [e, c_y]$.
- (ii) If $e \in E(\dot{X})$ joins the vertices $x \in Z \in \mathcal{V}_j$ and $y \in \dot{X}$, then $y \in Z$ if and only if $c_x(\overline{W}(e)) > j$.
- (iii) If $(Z, H, (c_x)) \in \mathcal{V}_j$, then $(H, \mathcal{N}(Z)) \in \mathcal{CMVH}$, when H is endowed with the induced peripheral structure; see Lemma 2.25.
- (iv) For $e \in E(\dot{X})$ with endpoints $x_+ \in W(e)^+$ and $x_- \in W(e)^-$, and $c \in C_{k+1}(\Gamma(\mathcal{X}))$, define

$$\mathcal{V}_j^\pm(e, c) := \{(H \cdot x, Z) : x \in Z \in \mathcal{V}_j \text{ and } \exists g \in G \text{ such that } gx = x_\pm, [e, gc_x] = [e, c]\},$$

where duplicates of $Z \in \mathcal{V}_j$ are counted separately. The collection \mathcal{V}_j must satisfy the *gluing equations*

$$|\mathcal{V}_j^+(e, c)| = |\mathcal{V}_j^-(e, c)|$$

for any $e \in E(\dot{X})$ and $c \in C_{k+1}(\Gamma(\mathcal{X}))$.

Remark 7.3 By Property (i), for an edge e intersecting Z we can consider a coloring $c_e \in C_{k+1}(\Gamma(\mathcal{X}))$ (in fact an equivalence class) such that if e is incident to $x \in Z$ then $[e, c_e] = [e, c_x]$.

Let us see how the existence of \mathcal{V}_0 implies Theorem 1.9. Consider an arbitrary triplet $(Z, H, (c_x))$ in \mathcal{V}_0 . By conditions (i)–(ii) of \mathcal{V}_0 , any vertex of \dot{X} is contained in Z , and since Z is intersection of half-spaces, we must have $\mathcal{N}(Z) = Z = \dot{X}$. But then $H < G$ acts cocompactly on \dot{X} , implying that H is of finite index in G . Condition (iii) implies that $(H, \dot{X}) \in \mathcal{CMVH}$, so (G, \dot{X}) is also in \mathcal{CMVH} .

The rest of the paper concerns the construction of the sequence $\mathcal{V}_{k+1}, \mathcal{V}_k, \dots, \mathcal{V}_0$ by induction. In the hyperbolic case, the existence of \mathcal{V}_{k+1} was given by Agol by means of an ingenious argument regarding invariant measures on $\Gamma(\mathcal{X})$; see [1, Section 7] and also [38, Lemma 7.1]. Definition 7.2(iii) differs from that in [38] since it uses \mathcal{CMVH} rather than \mathcal{QVH} , but the proof of [38, Lemma 7.1] still applies to \mathcal{CMVH} since the groups H are finite. Therefore we obtain:

Proposition 7.4 *There exists a \mathcal{V}_{k+1} satisfying all conditions (i)–(iv) of Definition 7.2.*

In the next sections we will need to modify our collections \mathcal{V}_j , and for that we will use the following lemma:

Lemma 7.5 *Let \mathcal{V}_j consist of the weighted triplets $(Z, H, (c_x)); \alpha_Z$ and for each Z consider a finite-index normal subgroup $H_0 \trianglelefteq H$ of index i_Z . Then after replacing each $(Z, H, (c_x); \alpha_Z)$ by $(Z, H_0, (c_x); (\prod_{Z' \neq Z} i_{Z'}) \alpha_Z)$, the collection \mathcal{V}_j still satisfies properties (i)–(iv).*

Proof Properties (i) and (ii) are immediate. To verify (iv), let e be an edge of \dot{X} , $c \in C_{k+1}(\Gamma(\mathcal{X}))$ be a coloring, and let $\tilde{\mathcal{V}}_j^\pm(e, c)$ be the set of pairs $(H_0 \cdot x, Z)$ such that $(H \cdot x, Z)$ is in $\mathcal{V}_j^\pm(e, c)$. The contribution of a triplet $(Z, H_0, (c_x))$ to $\tilde{\mathcal{V}}_j^\pm(e, c)$ is i_Z times the contribution of a triplet $(Z, H, (c_x))$ to $\mathcal{V}_j^\pm(e, c)$. Let \tilde{C}_Z^\pm and C_Z^\pm denote these contributions, respectively. Then if we choose

$$\tilde{\alpha}_Z := \prod_{Z' \neq Z} i_{Z'} \alpha_Z,$$

we have

$$\begin{aligned} |\tilde{\mathcal{V}}_j^+(e, c)| &= \sum_Z \tilde{\alpha}_Z \tilde{C}_Z^+ = \left(\prod_Z i_Z \right) \sum_Z \alpha_Z C_Z^+ \\ &= \left(\prod_Z i_Z \right) \sum_Z \alpha_Z C_Z^- = \sum_Z \tilde{\alpha}_Z \tilde{C}_Z^- \\ &= |\tilde{\mathcal{V}}_j^-(e, c)|, \end{aligned}$$

so the gluing equations are also satisfied by the modified \mathcal{V}_j . Finally, property (iii) follows from the lemma below, which will also be used in Section 9. \square

Lemma 7.6 *If $(H, Y) \in \mathcal{CMVH}$ and $H_0 \trianglelefteq H$ is a finite-index normal subgroup, then $(H_0, Y) \in \mathcal{CMVH}$.*

Proof The lemma follows by induction on the minimal number of operations (i)–(iii) (Definition 1.7) used in a description of (H, Y) as a group in \mathcal{CMVH} , after noting that finite-index subgroups of convex subgroups are convex, and that if (H, Y) splits as a graph of groups satisfying the properties of condition (iii) in Definition 1.7, then the induced splitting of (H_0, Y) also satisfies (iii) when H_0 is considered with the induced peripheral structure; see eg [3, Proposition 3.18]. \square

Definition 7.7 Any change of \mathcal{V}_j by first considering finite-index subgroups $H_0 < H$ for each $(Z, H, (c_x))$ and then duplicating the triplets $(Z, H_0, (c_x))$ as in the previous lemma will be called a *virtual modification* of \mathcal{V}_j .

8 Boundary walls and a graph of groups

For this section fix $(Z, H, (c_x)) \in \mathcal{V}_j$. We will introduce the main definitions that will be used in Section 9 to define \mathcal{V}_{j-1} from \mathcal{V}_j .

Definition 8.1 A *boundary wall* of Z is a wall W dual to an edge e crossing out of Z . By property (ii) and Remark 7.3, W is a boundary wall if and only if $W = \overline{W(e)}$ for an edge e intersecting Z and $c_e(\overline{W(e)}) \leq j$.

The next lemma is implicit in [1, page 1062], and is stated as the *zipping lemma* in [38, Lemma 8.4].

Lemma 8.2 If $W = W(e_1) = W(e_2)$ is a boundary wall of Z with e_1, e_2 edges crossing out of Z , then $[c_{e_1}]_W = [c_{e_2}]_W$.

Remark 8.3 By the lemma above, the color $c_e(\overline{W})$ is independent of the choice of e , and should be thought of as the *color* of the boundary wall W .

Definition 8.4 Let W be a boundary wall of Z with color j in the sense of the previous remark. We say that W is a j -*boundary wall* of Z , and that $P(W) := W \cap Z$ is the *portal of W leading to Z* . If an edge e dual to W crosses out of Z , then we say that e is *dual* to $P(W)$. Let $\partial_j Z$ be the union of all portals leading to Z .

The next lemma is [38, Lemma 8.6].

Lemma 8.5 A vertex in $Z \in \mathcal{V}_j$ cannot be incident to distinct edges dual to j -boundary walls.

Definition 8.6 For a wall \overline{W} in \mathcal{X} and coloring $c \in C_{k+1}(\Gamma(\mathcal{X}))$, let $B(\overline{W}, c) := \overline{W} \cap c^{-1}([1, j])$ be the intersection of \overline{W} with other walls in \mathcal{X} colored $\leq j$ by c (in this section, j is fixed). Define \overline{W} *split along c* by $\overline{W} - c := \overline{W} - B(\overline{W}, c)$, and for a vertex \bar{x} in \overline{W} , let $(\overline{W} - c)(\bar{x})$ denote the component of $\overline{W} - c$ containing \bar{x} .

Lemma 8.7 (cf [38, Lemma 8.9]) Let W be a j -boundary wall with portal $P = Z \cap W$ and let e be an edge dual to P with midpoint x_0 . Let $\overset{\circ}{P}$ denote the interior of P as a subspace of W . Then the following statements hold:

- (i) The quotient map $q: \dot{X} \rightarrow \mathcal{X}$ restricts to a universal covering map

$$q|_{\overset{\circ}{P}}: \overset{\circ}{P} \rightarrow (\overline{W} - c_e)(\bar{x}_0).$$

- (ii) $(\overline{W} - c_e)(\overline{x}_0) = (\overline{W} - c)(\overline{x})$ for any other vertex $x \in P$ of W and any $c \in [c_e]_W$.
- (iii) The group of deck transformations of $q|_P$ is $K_P := \{g \in K : gx_0 \in P\} = \text{Stab}_K(P)$ (where K is from Theorem 5.1), hence K_P acts cocompactly on P .

If P is a portal leading to $(Z, H, (c_x))$, let H_P denote its setwise stabilizer in H . Suppose P lies in the wall W and let $x \in P$ and $h \in H$ be such that $hx \in P$. If x' is the vertex closest to x in P , then x' is the midpoint of the edge e dual to P with he also dual to $P \subset W$, implying $hx' \in P$, $hW = W$ and $hP = h(Z \cap W) = Z \cap W = P$. We conclude that $h \in H_P$, so the map $P/H_P \rightarrow Z/H$ is an embedding, and also that H_P acts properly and cocompactly on P because Z/H is compact.

This last observation and properness of the action of G on \dot{X} , together with the previous two lemmas, imply that $H_P \cap K$ is finite-index in H_P for any portal P leading to $(Z, H, (c_x))$. Thus we can use Lemma 7.5 to modify our set \mathcal{V}_j .

Corollary 8.8 We can virtually modify \mathcal{V}_j (in the sense of Lemma 7.5) so that $H_P < K$ for any portal P leading to Z .

Proof Let P_1, \dots, P_k be a set of representatives of H -orbits for portals of Z , and note that by Lemma 7.5 it is enough to replace H by a finite-index normal subgroup $H_0 \trianglelefteq H$ such that $H_0 \cap H_{P_i} \cap K = H_0 \cap H_{P_i}$ for each i . But each $H_{P_i} \cap K$ is finite-index in H_{P_i} , so we just need each subgroup $H_{P_i} \cap K$ to be separable in H , which is true by Theorems 2.19 and 1.8 since $(H, \mathcal{N}(Z)) \in \mathcal{CMVH}$ and all the subgroups $H_{P_i} \cap K$ are convex in $(H, \mathcal{N}(Z))$ (they preserve the respective convex subcomplexes $\mathcal{N}(P_i) \cap \mathcal{N}(Z)$). □

Definition 8.9 Two portals P and P' leading to $(Z, H, (c_x)), (Z', H', (c'_x)) \in \mathcal{V}_j$ respectively are said to be *compatible* if there are edges e and e' , dual to P and P' respectively, such that $[e, c_e] \in G \cdot [e', c'_{e'}]$.

Let P and P' be compatible portals as above, lying in walls W and W' , say. Take $g \in G$ and edges e and e' dual to P and P' such that $[e, c_e] = g[e', c'_{e'}]$, and let x_0 and x'_0 be the midpoints of e and e' . So $e = ge'$, $W = gW'$ and $x_0 = gx'_0$. At the level of \mathcal{X} the action of g translates to

$$g(\overline{W}' - c'_{e'})(\overline{x}'_0) = (\overline{W} - gc'_{e'})(\overline{x}_0) = (\overline{W} - c_e)(\overline{x}_0),$$

where we used Lemma 8.7(ii) and the fact that $[c_e]_W = [gc'_{e'}]_W$.

Since q restricts to coverings for \mathring{P} and \mathring{P}' , we deduce that g restricts to a cube isomorphism $P' \rightarrow P$ which is equivariant with respect to the group isomorphism $K_{P'} \rightarrow K_P$ given by $k \mapsto gkg^{-1}$. This induces an isomorphism $P'/K_{P'} \xrightarrow{\sim} P/K_P$.

Lemma 8.10 (cf [38, Lemma 9.2]) *Two portals P and P' leading to $(Z, H, (c_x))$ and $(Z', H', (c'_x))$ in \mathcal{V}_j are compatible if and only if there exists $g \in G$ such that*

$$\{[e, c_e] : e \text{ is dual to } P\} = g\{[e', c'_{e'}] : e' \text{ is dual to } P'\}.$$

In particular, compatibility of portals is an equivalence relation.

Following terminology of [38], in the case of P, P' and g as above we will say that P is a g -teleport of P' .

8.1 The graph of groups $(\mathbb{A}, \mathcal{A})$

As we saw previously, compatible portals P and P' leading to $(Z, H, (c_x))$ and $(Z', H', (c'_x))$ in \mathcal{V}_j are isomorphic and induce an isomorphism $P'/K_{P'} \rightarrow P/K_P$. However, we would like to glue Z'/H' and Z/H along $P'/H'_{P'}$ and P/H_P , which are only isomorphic up to a finite-sheeted cover by Corollary 8.8. If we want isomorphisms $g: P'/H'_{P'} \rightarrow P/H_P$, we need to virtually modify \mathcal{V}_j again, and for that we will construct a graph of groups.

Definition 8.11 Let $(\mathbb{A}, \mathcal{A})$ be the finite bipartite (and possibly disconnected) graph of groups defined as follows.

- *Type I* vertices of \mathbb{A} are triplets $(Z, H, (c_x)) \in \mathcal{V}_j$ with corresponding vertex group H . Here repeated triplets are counted separately.
- *Type II* vertices of \mathbb{A} are portals $\{P_i\}$ forming a complete set of representatives for the compatibility classes of portals, with corresponding vertex groups K_{P_i} .
- Edges attached to the type I vertex $(Z, H, (c_x))$ will be portals P leading to $(Z, H, (c_x)) \in \mathcal{V}_j$, such that we choose just one P from each H -orbit of portals (repeated triplets will have the same conjugacy representatives). The edge P will be attached to the type II vertex in its compatibility class of portals, and its edge group will be H_P .

For a portal P leading to $(Z, H, (c_x))$, the injection of the edge H_P into its type I vertex group is just the inclusion $H_P \hookrightarrow H$, while the map into a type II vertex group is

the composition $H_P \hookrightarrow K_P \xrightarrow{g(-)g^{-1}} K_{P_i}$, where $g \in G$ is such that P_i is a g -teleport of P (for the case $g: P_i \rightarrow P_i$ we take $g = 1$, and the same portals corresponding to repeated triplets will have the same g).

The next proposition may be thought of as a relative version of the acylindricity of the graph of groups \mathbb{A} proven in the absolute case; cf [1, page 1063] and [38, Lemma 8.7]. The proof is practically the same, the only difference is that we require Proposition 2.16.

Proposition 8.12 *If $(Z, H, (c_x))$ is a type I vertex group of $(\mathbb{A}, \mathcal{A})$, then the collection*

$$\{H_P : P \text{ is an edge attached to } Z\}$$

is relatively malnormal in H . That is, if P_1 and P_2 are edges attached to Z and $h \in H$ is such that $H_{P_1} \cap H_{P_2}^h$ contains a loxodromic, then $P_1 = P_2$ and $h \in H_{P_1}$.

Proof Assume $\lambda \in H_{P_1} \cap H_{P_2}$ is a loxodromic element, in which case we claim that $P_1 = P_2$. If W_1 and W_2 are the walls containing P_1 and P_2 respectively, then it is enough to prove that $W_1 = W_2$, since that implies $P_1 = Z \cap W_1 = Z \cap W_2 = P_2$.

The element λ acts freely on P_1 and P_2 , so it acts hyperbolically on them, and there exist axes $\gamma_i \subset P_i \subset W_i$ on which λ acts by nontrivial translation. These axes are asymptotic in Z , thus γ_1 and γ_2 bound a flat strip in Z of width $r \geq 0$. Since λ is loxodromic, by Proposition 2.16 we have $r \leq \delta$. Let us assume $W_1 \neq W_2$, so that $\overline{W}_1 \neq \overline{W}_2$ and $d(\overline{W}_1, \overline{W}_2) \leq \delta \leq R$, and get a contradiction by showing that \overline{W}_1 and \overline{W}_2 are colored equal by some coloring.

If p is any point in P_1 , then it is contained in a cube C of X and we can find a vertex $x \in Z$ incident to an edge dual to P_1 , with $d(p, x) \leq \frac{1}{2}\sqrt{\dim C} \leq \frac{1}{2}\sqrt{\dim X}$. The same is true for P_2 , so there are vertices $x_1, x_2 \in Z$ with each x_i being incident to an edge dual to P_i so that the geodesic segment α joining x_1 and x_2 has length at most $\delta + \sqrt{\dim X}$. By considering the sequence of cubes that α travels through, we can find an edge path β in Z from x_1 to x_2 with $\beta \subset N_{\sqrt{\dim X}}(\alpha)$. Let e_1, \dots, e_s be the edges of β , and $x_1 = y_1, y_2, \dots, y_{s+1} = x_2$ be their vertices, with e_i joining y_i and y_{i+1} . Since $R \geq \delta + 2\sqrt{\dim X}$ (see the beginning of Section 5) we have $d(\overline{W}(e_i), \overline{W}_1) \leq d(W(e_i), W_1) \leq R$ for all $1 \leq i \leq s$, and so $\overline{W}(e_i)$ and \overline{W}_1 are adjacent vertices in $\Gamma(\mathcal{X})$. For each $1 \leq i \leq s$, from property (i) of \mathcal{V}_j we deduce $[c_{y_i}]_{e_i} = [c_{y_{i+1}}]_{e_i}$, so $c_{y_i}(\overline{W}_1) = c_{y_{i+1}}(\overline{W}_1)$ and hence $c_{x_1}(\overline{W}_1) = c_{x_2}(\overline{W}_1) = j$, because W_1 and W_2 are j -boundary walls. Thus c_{x_2} is a coloring with $c_{x_2}(\overline{W}_1) = c_{x_2}(\overline{W}_2)$, contradicting $W_1 \neq W_2$.

To finish the proposition, let P_1 and P_2 be edges attached to $(Z, H, (c_x))$ and $h \in H$ be such that $H_{P_2} \cap H_{P_2}^h = H_{P_1} \cap H_{P_2}^h$ contains a loxodromic. By our previous claim we obtain $P_1 = hP_2$, and since different edges correspond to distinct representatives of H -orbits of portals, we must have $P_1 = P_2$, implying $h \in H_{P_1}$. \square

Definition 8.13 Let $\mathbb{G} = \mathbb{G}_c$ be the fundamental group of a connected component $(\mathbb{A}_c, \mathcal{A}_c)$ of $(\mathbb{A}, \mathcal{A})$ with respect to some vertex w_0 of \mathbb{A}_c , and fix a maximal subtree T of \mathbb{A}_c containing w_0 that fixes inclusions of the edge/vertex groups of $(\mathbb{A}_c, \mathcal{A}_c)$ into \mathbb{G} .

Proposition 8.14 Let G_{e_0} be an edge group of $(\mathbb{A}_c, \mathcal{A}_c)$ attached to the type II vertex group G_{v_0} , and let $a \in G_{v_0} \setminus G_{e_0} \subset \mathbb{G}$. Then there exists a finite-index subgroup $\dot{\mathbb{G}}_{e_0,a} < \mathbb{G}$ containing G_{e_0} with $a \notin \dot{\mathbb{G}}_{e_0,a}$.

Proof The proof is almost the same as the one given for Proposition 4.5, so we just give a sketch of it.

Recall that by Lemma 3.7, each vertex/edge group is convex and strongly peripherally separable in G , so by several applications of Proposition 3.26 we can find finite-index subgroups $\dot{P}_j < P_j$ such that the filling $\phi: G \rightarrow \bar{G} = G(\mathcal{N} = \{\dot{P}_1, \dots, \dot{P}_n\})$ satisfies:

- $\bar{G}_v := \phi(G_v)$ is hyperbolic and virtually special, and isomorphic to the image of the induced filling $\phi_v: G_v \rightarrow G_v(\mathcal{N}_v)$ for any vertex v of \mathbb{A}_c — images of type II vertex groups will be virtually special because every type II vertex is a finite-index extension of an edge group, which is virtually special by Theorem 1.8, Corollary A.3 and our inductive assumption; see the beginning of Section 5.
- The image $\bar{G}_e := \phi(G_e)$ of the edge group G_e of $(\mathbb{A}_c, \mathcal{A}_c)$ with terminal vertex of type I is naturally isomorphic to the image of the filling $\phi_e: G_e \rightarrow G_e(\mathcal{N}_e)$ induced by both ϕ and $\phi_{t(e)}$ — that is, $\ker \phi_e = \ker \phi_{t(e)} \cap G_e = \ker \phi \cap G_e$.
- The collection of images under ϕ of groups in $\{\bar{G}_e : e \text{ attached to } v\}$ is almost malnormal in \bar{G}_v for any type I vertex v of \mathbb{A}_c .
- $\phi(a) \notin \bar{G}_{e_0}$.

We then consider the graph of groups $(\mathbb{A}_c, \bar{\mathcal{A}}_c)$ with the same underlying graph \mathbb{A}_c , and $\bar{\mathcal{A}}_c$ assigning the group \bar{G}_x to each vertex/edge x of \mathbb{A}_c , and with attaching maps induced by ϕ and the attaching maps of $(\mathbb{A}_c, \mathcal{A}_c)$ — every attaching map of $(\mathbb{A}_c, \mathcal{A}_c)$ is composition of inclusions and conjugations in G . Let $\bar{\mathbb{G}} = \pi_1(\mathbb{A}_c, \bar{\mathcal{A}}_c, w_0)$ and choose embeddings of edge/vertex groups according to the same maximal subtree T of \mathbb{A}_c .

The homomorphism ϕ restricted to each edge/vertex group induces a homomorphism $\Phi: \mathbb{G} \rightarrow \overline{\mathbb{G}}$ such that $\Phi(x) = \phi_v(x)$ for any vertex v of \mathbb{A}_c and for any $x \in G_v$, and since the splitting $(\mathbb{A}_c, \overline{\mathcal{A}}_c)$ of $\overline{\mathbb{G}}$ satisfies the assumptions of Lemma 4.7 (2), $\overline{\mathbb{G}}$ is hyperbolic and \overline{G}_{e_0} is quasiconvex in $\overline{\mathbb{G}}$. Then Theorem 2.22 implies that $\overline{\mathbb{G}}$ is virtually special, and since $\Phi(a) = \phi(a) \notin \overline{G}_{e_0} = \Phi(G_{e_0})$, Theorem 2.19 gives us the separability of $\Phi(G_{e_0})$ in $\overline{\mathbb{G}}$, and hence the existence of $\dot{\mathbb{G}}_{a,e_0}$. \square

Corollary 8.15 *There is a finite-index subgroup $\mathbb{N}_c \trianglelefteq \mathbb{G}_c$ such that if $G_e < \mathbb{G}_c$ is an edge group attached to (hence contained in) the type II vertex group $G_v < \mathbb{G}_c$, then*

$$(6) \quad G_e \cap \mathbb{N}_c = G_v \cap \mathbb{N}_c.$$

Proof Recall that an edge group G_e is finite index in the type II vertex group G_v it is attached to. So, for each edge e let $S_e \subset G_v \setminus G_e$ be any finite set of representatives of nontrivial left cosets of G_e in G_v . By Proposition 8.14, for each $a \in S_e$ there is a finite-index subgroup $\dot{\mathbb{G}}_{a,e_0} < \mathbb{G}_c$ separating G_e from a , and so the intersection of the finitely many conjugates of $\dot{\mathbb{G}}_{a,e_0}$ in \mathbb{G}_c is a finite-index normal subgroup, which we denote by $\mathbb{N}_{e,a}$. The group $\mathbb{N} := \bigcap_{e \in E(\mathbb{A}_c)} \bigcap_{a \in S_v} \mathbb{N}_{e,a}$ satisfies the required equalities. \square

9 Constructing \mathcal{V}_{j-1} from \mathcal{V}_j

Finally we can start the construction of \mathcal{V}_{j-1} , which implies Theorem 1.9.

Let $(\mathbb{A}_c, \mathcal{A}_c)$ be a component of $(\mathbb{A}, \mathcal{A})$ with fundamental group \mathbb{G}_c , and let $\mathbb{N}_c \trianglelefteq \mathbb{G}_c$ be given by Corollary 8.15. For each type I vertex group H of $(\mathbb{A}_c, \mathcal{A}_c)$, define $\hat{H} := \mathbb{N}_c \cap H \trianglelefteq H$, and view these groups as subgroups of G . After doing this for each connected component $(\mathbb{A}_c, \mathcal{A}_c)$ of $(\mathbb{A}, \mathcal{A})$ we obtain a finite-index subgroup of H for each $(Z, H, (c_x)) \in \mathcal{V}_j$ and so Lemma 7.5 gives us a virtual modification $\hat{\mathcal{V}}_j$ of \mathcal{V}_j with triplets $(Z, \hat{H}, (c_x))$.

We will use $\hat{\mathcal{V}}_j$ to construct a (possibly disconnected) graph of spaces $(\mathbb{S}, \mathcal{S})$. As before, let $\{P_i\}$ be the set of type II vertices of \mathbb{A} , and for each $(Z, \hat{H}, (c_x)) \in \hat{\mathcal{V}}_j$ choose a set \mathbb{B}_Z of representatives of portals leading to Z , with exactly one portal P for each \hat{H} -orbit of translates of portals (with the same representatives for repeated triplets). Set $\mathbb{B} := \bigsqcup_{Z \in \hat{\mathcal{V}}_j} \mathbb{B}_Z$, and choose the representatives of portals in such a way that each edge of \mathbb{A} lies in \mathbb{B} .

As in [38, Definition 9.6], the *size* of an edge P of \mathbb{A} is defined by

$$sz(P) := |\{H_P \cdot e : e \text{ is an edge dual to } P\}|,$$

where P leads to $(Z, H, (c_x)) \in \mathcal{V}_j$. Similarly, for $P \in \mathbb{B}$ leading to $(Z, \hat{H}, (c_x)) \in \hat{\mathcal{V}}_j$, define the *size* of P as

$$\hat{sz}(P) := |\{\hat{H}_P \cdot e : e \text{ is an edge dual to } P\}|,$$

where $\hat{H}_P := \hat{H} \cap H_P$ is the stabilizer of P in \hat{H} . Note that $\hat{sz}(P) = |H_P : \hat{H}_P| sz(P)$ for any edge P of \mathbb{A} attached to $(Z, H, (c_x))$. Also, by equation (6), $\hat{sz}(P) = \hat{sz}(P')$ whenever $P, P' \in \mathbb{B}$ are compatible portals.

If P is a portal leading to Z and contained in the wall W , with Z in either \mathcal{V}_j or $\hat{\mathcal{V}}_j$, and P_i is a g -teleport of P contained in the wall W_i that is a type II vertex group of \mathbb{A} , we say that P is a P_i^+ -portal if $gZ \cap W_i^+ \neq \emptyset$, and a P_i^- -portal if $gZ \cap W_i^- \neq \emptyset$. For $Z \in \mathcal{V}_j$ define $\mathbb{A}_Z(P_i, \pm)$ as the set of P_i^\pm -portals P leading to Z which are an edge of A , and let $\mathbb{A}(P_i, \pm) := \bigsqcup_{Z \in \mathcal{V}_j} \mathbb{A}_Z(P_i, \pm)$. The sets $\mathbb{B}_Z(P_i, \pm)$ for $Z \in \hat{\mathcal{V}}_j$ and $\mathbb{B}(P_i, \pm)$ are defined similarly.

Before defining (\mathbb{S}, \mathbb{S}) we need some combinatorial results, the first one contained in [38, Lemma 9.7].

Lemma 9.1 *If P_i is a type II vertex of \mathbb{A} , then*

$$\sum_{P \in \mathbb{A}(P_i, +)} sz(P) = \sum_{P \in \mathbb{A}(P_i, -)} sz(P).$$

Corollary 9.2 *For each type II vertex P_i of \mathbb{A} , the number of P_i^+ -portals P in \mathbb{B} equals the number of P_i^- -portals P in \mathbb{B} .*

Proof Let $s := \hat{sz}(P)$ be the size of any portal P in \mathbb{B} compatible with P_i . From the proof of Lemma 7.5 it follows that there exists some positive integer d such that for each triplet $(Z, H, (c_x)) \in \mathcal{V}_j$ there are $d/|H : \hat{H}|$ triplets of $(Z, \hat{H}, (c_x)) \in \hat{\mathcal{V}}_j$. Also, if P is any portal leading to $(Z, H, (c_x))$, then the set of H -orbits of P is the disjoint union of $|H : \hat{H}|/|H_P : \hat{H}_P|$ \hat{H} -orbits of translates of portals. With this in mind we have

$$\begin{aligned} \sum_{P \in \mathbb{B}(P_i, +)} 1 &= \sum_{Z \in \hat{\mathcal{V}}_j} \sum_{P \in \mathbb{B}_Z(P_i, +)} 1 = \sum_{(Z, H, (c_x)) \in \mathcal{V}_j} \frac{d}{|H : \hat{H}|} \left(\sum_{P \in \mathbb{B}_Z(P_i, +)} 1 \right) \\ &= \sum_{(Z, H, (c_x)) \in \mathcal{V}_j} \frac{d}{|H : \hat{H}|} \left(\sum_{P \in \mathbb{A}_Z(P_i, +)} \frac{|H : \hat{H}|}{|H_P : \hat{H}_P|} \right) \end{aligned}$$

$$\begin{aligned}
 &= \frac{d}{s} \sum_{(Z, H, (c_x)) \in \mathcal{V}_j} \left(\sum_{P \in \mathbb{A}_Z(P_i, +)} \frac{\widehat{\text{sz}}(P)}{|H_P : \widehat{H}_P|} \right) \\
 &= \frac{d}{s} \sum_{(Z, H, (c_x)) \in \mathcal{V}_j} \left(\sum_{P \in \mathbb{A}_Z(P_i, +)} \text{sz}(P) \right) \\
 &= \frac{d}{s} \sum_{P \in \mathbb{A}(P_i, +)} \text{sz}(P).
 \end{aligned}$$

The same is true for the P_i^- -portals, and so the conclusion follows by Lemma 9.1. \square

Definition 9.3 Let $(\mathbb{S}, \mathcal{S})$ be the graph of spaces defined as follows.

- The vertices of \mathbb{S} will be the triplets $(Z, \widehat{H}, (c_x)) \in \widehat{\mathcal{V}}_j$, with corresponding vertex spaces Z/\widehat{H} .
- If P_i is a type II vertex of \mathbb{A} , by Corollary 9.2 there exists a perfect matching between $\mathbb{B}(P_i, +)$ and $\mathbb{B}(P_i, -)$. If $p := (P, P')$ is an oriented pair given by this matching with P and P' leading to $(Z, \widehat{H}, (c_x))$ and $(Z', \widehat{H}', (c'_x))$ respectively, then P is a g_p -teleport of P' for some fixed $g_p \in G$, and there are embeddings

$$(7) \quad P'/\widehat{H}'_P \hookrightarrow Z'/\widehat{H}' \quad \text{and} \quad P'/\widehat{H}'_{P'} \xrightarrow{g_p} P/\widehat{H}_P \hookrightarrow Z/\widehat{H},$$

where $P'/\widehat{H}'_{P'} \xrightarrow{g_p} P/\widehat{H}_P$ is an isomorphism for g_p chosen appropriately, due to Corollary 8.15.

- The edges of \mathbb{S} are oriented pairings $p := (\widehat{H}_P, \widehat{H}'_{P'})$ attached to Z and Z' as above, with edge spaces $P'/\widehat{H}'_{P'}$, and attaching maps given by (7). For the reverse pairing $\bar{p} = (P', P)$, the attaching maps are constructed in the same way with $g_{\bar{p}} = g_p^{-1}$.

Consider a component $(\mathbb{S}_c, \mathcal{S}_c)$ of $(\mathbb{S}, \mathcal{S})$, with underlying space \mathcal{T}_c obtained by gluing vertex spaces along images of attaching maps. We want to construct an embedding $\widetilde{\mathcal{T}}_c \hookrightarrow \dot{X}$ of the universal cover of \mathcal{T}_c into \dot{X} .

First of all, fix a basepoint $x \in Z$ for each $(Z, \widehat{H}, (c_x)) \in \widehat{\mathcal{V}}_j$. If Z/\widehat{H} and Z'/\widehat{H}' are vertex spaces joined by the oriented edge $p = (P, P')$ then, up to homotopy, there exists a unique path α_p in $Z \cup g_p Z'$ from x to $g_p x$ (this is because $Z \cup g_p Z'$ is simply connected). Let β_p be the projection of α_p into \mathcal{T}_c via $Z \rightarrow Z/\widehat{H}$ and $Z' \rightarrow Z'/\widehat{H}'$. We can choose our paths so that $\beta_{\bar{p}}$ is the reverse path of β_p . Also, for each $(Z, \widehat{H}, (c_x))$ and $h \in \widehat{H}$, let γ_h be a loop in Z/\widehat{H} lifting to a path from x to hx .

Fix a base vertex $(Z_0, \hat{H}_0, (c_x)_0)$ of \mathbb{S}_c . For p_1, \dots, p_n edges in \mathbb{S}_c forming a path through vertex spaces

$$Z_0/\hat{H}_0 \xrightarrow{p_1} Z_1/\hat{H}_1 \xrightarrow{p_2} \dots \xrightarrow{p_n} Z_n/\hat{H}_n,$$

and $h_i \in \hat{H}_i$ for $i = 0, 1, \dots, n$, consider the concatenation

$$(8) \quad \gamma = \gamma_{h_0} \cdot \beta_{p_1} \cdot \gamma_{h_{n-1}} \cdots \beta_{p_n} \cdot \gamma_{h_n},$$

which gives a path in \mathcal{T}_c . Note that every path in \mathcal{T} between (projections of) basepoints of vertex spaces and starting in Z_0/\hat{H}_0 is homotopic to a path in this form. For each such γ , we define

$$g(\gamma) = h_0 g_{p_1} h_1 \cdots g_{p_n} h_n,$$

and we take $\tilde{\mathcal{T}}_c \subset \dot{X}$ to be the union of all the possible G -translates $g(\gamma)Z_n$. The covering map $\mu: \tilde{\mathcal{T}}_c \rightarrow \mathcal{T}_c$ restricts to $g(\gamma)Z_n$ by

$$\mu: g(\gamma)Z_n \xrightarrow{g(\gamma)^{-1}} Z_n \rightarrow Z_n/\hat{H}_n \rightarrow \mathcal{T}_c.$$

The next lemma is [38, Lemmas 9.13 and 9.15].

Lemma 9.4 *The map $\mu: \tilde{\mathcal{T}}_c \rightarrow \mathcal{T}_c$ is a universal covering map and $\tilde{\mathcal{T}}_c \subset \dot{X}$ is a nonempty intersection of half-spaces.*

For a loop γ as in (8) and $g(\beta)Z$ in $\tilde{\mathcal{T}}_c$, we have that $g(\gamma)g(\beta) = g(\gamma \cdot \beta)$, hence $g(\gamma)g(\beta)Z = g(\gamma \cdot \beta)Z \subset \tilde{\mathcal{T}}_c$. This holds for all translates $g(\beta)Z$ in $\tilde{\mathcal{T}}_c$, thus $g(\gamma)\tilde{\mathcal{T}}_c = \tilde{\mathcal{T}}_c$ and $\mu \circ g(\gamma) = \mu$.

Define

$$H(\mathcal{T}_c) := \{g(\gamma) : \gamma \text{ is a loop of the form (8)}\} < G.$$

Since $g(\gamma)g(\beta) = g(\gamma \cdot \beta)$ for any loop γ and path β , $H(\mathcal{T}_c)$ is a subgroup of G preserving $\tilde{\mathcal{T}}_c$. Moreover, $\mu \circ g(\gamma) = \mu$ for every $g(\gamma) \in H(\mathcal{T}_c)$, so $H(\mathcal{T}_c)$ is a subgroup of the group of deck transformations of μ . In addition, by construction, the orbit of the basepoint x_0 of Z_0 is $H(\mathcal{T}_c)\mu^{-1}1(\mu(x_0))$, implying that $H(\mathcal{T}_c) \cong \pi_1(\mathcal{T}_c)$ is the full group of deck transformations of μ , and hence it acts freely and cocompactly on $\tilde{\mathcal{T}}_c$.

Finally, if $x \in Z$ is any vertex with $(Z, \hat{H}, (c_x)) \in \hat{\mathcal{V}}_j$ and γ is as in (8), then we endow $g(\gamma)x$ with the coloring $c_{g(\gamma)x}^{\mathcal{T}_c} := g(\gamma)c_x$. It is evident that $H(\mathcal{T}_c)$ preserves these colorings.

Definition 9.5 Let \mathcal{V}_{j-1} consist of the set of triplets $(\tilde{\mathcal{T}}_c, H(\mathcal{T}_c), (c_x^{\mathcal{T}_c}))$, with one triplet for each underlying space \mathcal{T}_c for a component $(\mathbb{S}_c, \mathcal{S}_c)$ of $(\mathbb{S}, \mathcal{S})$.

Proposition 9.6 \mathcal{V}_{j-1} satisfies all the desired properties (i)–(iv) of Definition 7.2.

Proof For properties (i) and (ii), the proof is the same as in [38, pages 30–31], so will be omitted.

To show property (iii), first note that by Lemma 2.25 each subgroup $H(\mathcal{T}_c)$ is convex in G with convex core $\mathcal{N}(\tilde{\mathcal{T}}_c)$ so is also hyperbolic relative to compatible virtually special subgroups by Lemma 7.1. In addition, since any edge space embedding into a vertex space of $(\mathbb{S}_c, \mathcal{S}_c)$ is π_1 –injective, by Van Kampen’s theorem there is an induced splitting $(\mathbb{S}_c, \mathcal{U}_c)$ of $H(\mathcal{T}_c) \cong \pi_1(\mathcal{T}_c)$ with vertex groups \hat{H} for $(Z, \hat{H}, (c_x)) \in \hat{\mathcal{V}}_j$, and with edge groups of the form $\hat{H}_P \cong \hat{H}'_{P'}$ for each edge $p = (P, P')$ with P and P' leading to $(Z, \hat{H}, (c_x))$ and $(Z', \hat{H}', (c'_x))$ respectively. By assumption and Lemma 7.6 each cubulated vertex group $(\hat{H}, \mathcal{N}(Z))$ of $(\mathbb{S}_c, \mathcal{U}_c)$ is in \mathcal{CMVH} , and by construction $\mathcal{N}(Z)$ is the G –translate of a convex subcomplex of $\mathcal{N}(\tilde{\mathcal{T}}_c)$. The same is true for the embedding of an edge group into $H(\mathcal{T}_c)$ since it acts cocompactly on a translate of the cubical neighborhood $\mathcal{N}(P)$ of a portal P , which is a convex subcomplex of $\mathcal{N}(\tilde{\mathcal{T}}_c)$. This implies that each vertex/edge group is convex in $(H(\mathcal{T}_c), \mathcal{N}(\tilde{\mathcal{T}}_c))$. Finally note that the conclusion of Proposition 8.12 holds also for vertex groups in $(\mathbb{S}_c, \mathcal{U}_c)$, and hence the collection of embeddings of edge groups into a vertex group of $(\mathbb{S}_c, \mathcal{U}_c)$ is relatively malnormal. Therefore $(H(\mathcal{T}_c), \mathcal{N}(\tilde{\mathcal{T}}_c)) \in \mathcal{CMVH}$.

For property (iv), note that since each \mathcal{T}_c is obtained by gluing quotients Z/\hat{H} for $(Z, \hat{H}, (c_x)) \in \hat{\mathcal{V}}_j$, with each triplet being used in exactly one component \mathcal{T}_c (repeated triplets are counted separately), there is a canonical bijection Λ from the set of vertices of $\bigsqcup_{(Z, \hat{H}, (c_x)) \in \hat{\mathcal{V}}_j} Z/\hat{H}$ onto the set of vertices of $\bigcup_c \mathcal{T}_c$, where c runs over the components of $(\mathbb{S}, \mathcal{S})$ —here by vertex we mean an image of a vertex of \dot{X} contained in some Z . Also, any vertex of some $\tilde{\mathcal{T}}_c$ takes the form $\tilde{x} = g(\gamma)x$ for some path γ as in (8) and some vertex $x \in (Z, \hat{H}, (c_x)) \in \hat{\mathcal{V}}_j$. Since by definition $c_{\tilde{x}}^{\mathcal{T}_c} = g(\gamma)c_x$, Λ restricts to a bijection from $\mathcal{V}_j^\pm(e, c)$ onto $\mathcal{V}_{j-1}^\pm(e, c)$ for any equivalence class $[e, c]$ with e an edge and c a coloring. As \mathcal{V}_j satisfies the gluing equations, so does \mathcal{V}_{j-1} . \square

Appendix Functoriality of the canonical completion

This appendix is devoted to proving the following theorem, which generalizes one of the implications of Theorem 2.20.

Theorem A.1 *Let (G, X) be a cubulated virtually special group. Then for any pair $H, K < G$ of convex subgroups, the double coset HK is separable in G .*

By using the fact that $gKg^{-1} < G$ is convex whenever $K < G$ is convex and $g \in G$, we deduce:

Corollary A.2 *Let (G, X) , H and K be as in the previous theorem. Then for any $g \in G$, the double coset HgK is separable in G .*

As before \dot{X} denotes the cubical barycentric subdivision of the CAT(0) cube complex X .

Corollary A.3 *Let (G, X) be a cubulated group. Then (G, X) is virtually special if and only if (G, \dot{X}) is virtually special.*

Proof If W' is a wall of \dot{X} , then $G_{W'}$ is a finite-index subgroup of G_W for some wall W of X , hence a convex subgroup of (G, X) . Therefore, by Theorems 2.19, A.1 and 2.20, if (G, X) is virtually special then (G, \dot{X}) is virtually special. The converse follows by a similar argument. □

To prove Theorem A.1, we will make use of the properties of the *canonical completion and retraction* introduced by Haglund and Wise [22, Section 6], so we first proceed to recall this construction.

Definition A.4 Let $f: A \rightarrow B$ be a local isometry of NPC special cube complexes, and assume that B is *fully clean* in the sense of [22, Definition 6.1], and has simplicial 1-skeleton. Then there exists a covering map $p: C(A, B) \rightarrow B$ (of finite degree if A is finite), an injection $j: A \rightarrow C(A, B)$ and a cellular map $r: C(A, B) \rightarrow A$ such that $rj = 1_A$ and $pj = f$. The covering is defined as follows:

The 0-skeleton of $C(A, B)$ is $A^{(0)} \times B^{(0)}$ with $j(a) = (a, f(a))$ for $a \in A^{(0)}$, and $r: A^{(0)} \times B^{(0)} \rightarrow A^{(0)}$ and $p: A^{(0)} \times B^{(0)} \rightarrow B^{(0)}$ are the projections to the first and second coordinate, respectively.

Since the 1-skeletons of A and B are simplicial, edges in A and B are determined by their endpoints. The edges of $C(A, B)$ are of two types:

- **Horizontal** These are pairs of the form $\{(a, b), (a, b')\}$ with $\{b, b'\}$ an edge of B and such that there is no edge e of A incident to a with $f(e)$ and $\{b, b'\}$ dual to the same wall.
- **Diagonal** These are pairs of the form $\{(a, b), (a', b')\}$ with $\{b, b'\}$ an edge of B and $e = \{a, a'\}$ an edge of A with $f(e)$ and $\{b, b'\}$ dual to the same wall — note that $\{(a', b), (a, b')\}$ is also a diagonal edge.

It follows from this definition that if $\{a, a'\}$ is an edge of A , then $j(\{a, a'\}) = \{(a, f(a)), (a', f(a'))\}$ is a diagonal edge of $C(A, B)$. Moreover, for an edge $e = \{(a, b), (a, b')\}$ as above, we define $p(e) = \{b, b'\}$ and $r(e) = a$ if $a = a'$ and e is horizontal, and $r(e) = \{a, a'\}$ if e is diagonal.

Clearly we have $rj = 1_A$ and $pj = f$ at the level of 1–skeletons, and full cleanliness implies that p is a covering map from $C(A, B)^{(1)}$ into $B^{(1)}$. It can be proven that any lifting of the 1–skeleton of a square of B is a closed 4–circuit, so the 2–skeleton of $C(A, B)$ is constructed in such a way that the boundaries of squares coincide with liftings of boundaries of squares of B .

The maps p and j then naturally extend to the 2–skeleton, and since any wall of the 2–skeleton of $C(A, B)$ only consists of either horizontal or diagonal edges, we can extend r to this 2–skeleton by mapping a square $Q \subset C(A, B)$ either to a square (if the walls dual to Q are both diagonal), to an edge (if one wall dual to Q is diagonal and the other one is horizontal, then collapse the horizontal edge to a point), or to a point (if both walls dual to Q are horizontal).

There is a unique way to complete this 2–skeleton to produce a nonpositively curved cube complex $C(A, B)$ [22, Lemma 3.13], for which we can naturally extend j , r and p to maps satisfying the desired commuting properties; see [22, Corollary 6.7]. We call $C(A, B)$ the *canonical completion* of $f : A \rightarrow B$ with (*canonical*) *inclusion* j and (*canonical*) *retraction* r .

Remark A.5 In general, $C(A, B)$ may be a disconnected covering of B . Also, from the construction above, it follows that if e is an edge of A and e' is an edge of $C(A, B)$ dual to the wall $W(j(e))$, then e' is diagonal and $r(e')$ is dual to the wall $W(e) \subset A$. It is not hard to see that j maps distinct walls of A to distinct walls of $C(A, B)$.

To produce separable double cosets from the canonical completion, we will use the following separability criterion [22, Lemma 9.3]:

Proposition A.6 *Let G be a residually finite group, and let $\rho : G \rightarrow G$ be a retraction homomorphism with $H = \rho(G)$. Then $H < G$ is a separable subgroup, and if $K < G$ is separable with $\rho(K) \subset K$, then the double coset HK is separable in G .*

Our first lemma is just a generalization of [22, Proposition 9.7], following exactly the same proof.

Lemma A.7 *If (G, X) is a virtually special group, $H < (G, X)$ is a convex subgroup and $W \subset X$ is a wall, then $HG_W \subset G$ is separable.*

Proof By using Lemma 2.15 we can find a convex core $Y \subset X$ for H such that $Y \cap W$ is nonempty, and let a be an edge of Y dual to W and incident to the vertex $y \in Y$. Since G is residually finite, by Lemma 2.9 there is a finite-index subgroup $\widehat{G} < G$ acting freely on X such that $\overline{X} := X/\widehat{G}$ is fully clean and special with simplicial 1-skeleton, and such that the projection $X \rightarrow \overline{X}$ maps squares to squares [22, Remark 6.8]. In that case, if $\widehat{H} = H \cap \widehat{G}$, then $\overline{Y} = Y/\widehat{H}$ is compact and the composition $f: \overline{Y} \rightarrow X/\widehat{H} \rightarrow \overline{X}$ is a local isometry. We can also assume that $(W \cap Y)/\widehat{H}_W \rightarrow \overline{Y}$ and $W/\widehat{G}_W \rightarrow X/\widehat{G}$ embed as walls, which we respectively denote by $W_{\overline{a}}$ and $W_{f(\overline{a})}$, with \overline{a} being the image of a under the projection $Y \rightarrow \overline{Y}$. Let $p: C(\overline{Y}, \overline{X}) \rightarrow \overline{X}$ be the canonical completion induced by f , with retraction r and inclusion map j .

Let $\mathcal{N}_{f(\overline{a})} \subset \overline{X}$ denote the cubical neighborhood of $W_{f(\overline{a})}$, which is lifted by p to the cubical neighborhood $\mathcal{N}_{j(\overline{a})}$ of the wall $W_{j(\overline{a})} \subset C(\overline{Y}, \overline{X})$ dual to $j(\overline{a})$. By Remark A.5, r maps any edge of $C(\overline{Y}, \overline{X})$ dual to $W_{j(\overline{a})}$ to an edge dual to $W_{\overline{a}} \subset \overline{Y}$. In particular, since r maps cubes to cubes (possibly of lower dimension), we have $r(\mathcal{N}_{j(\overline{a})}) \subset \mathcal{N}_{\overline{a}}$, where $\mathcal{N}_{\overline{a}}$ is the cubical neighborhood of $W_{\overline{a}}$. On the other hand, since $X \rightarrow \overline{X}$ maps squares to squares, j and f also map squares to squares, and in fact we have $r(\mathcal{N}_{j(\overline{a})}) = \mathcal{N}_{\overline{a}}$. Therefore, if $\overline{y} \in \overline{Y}$ denotes the projection of y , there is a commutative diagram

$$\begin{array}{ccc}
 (\mathcal{N}_{j(\overline{a})}, j(\overline{y})) & \xrightarrow{\subset} & (C(\overline{Y}, \overline{X}), j(\overline{y})) \\
 \downarrow r & & \downarrow r \\
 (\mathcal{N}_{\overline{a}}, \overline{y}) & \xrightarrow{\subset} & (\overline{Y}, \overline{y})
 \end{array}$$

At the level of fundamental groups, and after considering the corresponding isomorphisms induced by $\widehat{G} \cong \pi_1(\overline{X}, f(\overline{y}))$, we obtain a retraction homomorphism $r_*: G' \rightarrow \widehat{H}$, where $G' < \widehat{G}$ is the subgroup corresponding to $\pi_1(C(\overline{Y}, \overline{X}), j(\overline{y}))$, such that $r_*(G'_W) \subset \widehat{H}_W$. It is not hard to see that the group $\widetilde{G}_W := G'_W \cap r_*^{-1}(G'_W) < G'$ satisfies $r_*(\widetilde{G}_W) \subset \widetilde{G}_W$, so by Proposition A.6 the double coset $\widehat{H}\widetilde{G}_W$ is separable in G' , and hence in G since $G' < G$ is of finite index. But $G'_W = G_W \cap G' < G_W$ and $\widehat{H} < H$ are of finite index, and hence $\widetilde{G}_W = G'_W \cap r_*^{-1}(G'_W) = G'_W \cap r_*^{-1}(G'_W \cap \widehat{H}) < G'_W \cap r_*^{-1}(G_W \cap \widehat{H}) = G'_W$ is also of finite-index. We conclude that HG_W is a finite union of translates of $\widehat{H}\widetilde{G}_W$, so it is also separable in G . \square

Corollary A.8 *Let (G, X) be a virtually special group, and let $H < G$ be a convex subgroup with convex core $Y \subset X$. Then there exists a finite-index subgroup $G' < G$ such that for any wall $W \subset X$ intersecting Y :*

- (i) *If $g \in G'$ satisfies $g\mathcal{N}(W) \cap Y \neq \emptyset$, then in fact $gW \cap Y \neq \emptyset$.*

- (ii) If W' is another wall of X intersecting Y and $g \in G'$ is such that $W' = gW$, then in fact $W' \cap Y = h'(W' \cap Y)$ for some $h' \in G' \cap H$.

Remark A.9 When we project to the corresponding quotients, conclusion (i) of the corollary above may be thought as a version of *no inter-osculation* for a wall and a locally convex subcomplex of a compact special cube complex.

Proof The proof closely follows the idea of [22, Lemma 9.14]. Let W_1, \dots, W_n be a complete set of representatives of H -orbits of walls intersecting Y , and for each i define the sets

$$I(G, Y, i) = \{g \in G : gW_i \cap Y \neq \emptyset\},$$

$$J(G, Y, i) = \{g \in G : g\mathcal{N}(W_i) \cap Y \neq \emptyset\}.$$

These sets are clearly (H, G_{W_i}) -invariant, and also $I(G, Y, i) \subset J(G, Y, i)$, so there are subsets $\mathcal{I}_i \subset \mathcal{J}_i \subset G$ such that $I(G, Y, i) = \bigsqcup_{g \in \mathcal{I}_i} HgG_{W_i}$ and $J(G, Y, i) = \bigsqcup_{g \in \mathcal{J}_i} HgG_{W_i}$.

Let us prove first that each of the sets \mathcal{J}_i is finite. Fix $1 \leq i \leq n$, and consider finite sets $D_i \subset \mathcal{N}(W_i)^{(0)}$ and $E \subset Y^{(0)}$ such that $\mathcal{N}(W_i)^{(0)} = G_{W_i} \cdot D$ and $Y^{(0)} = H \cdot E$. Given $g \in G$ such that $g\mathcal{N}(W_i) \cap Y$ is nonempty, there is a vertex v of $\mathcal{N}(W_i)$ with $gv \in Y$, and so there are group elements $w \in G_{W_i}$ and $h \in H$ satisfying $wv \in D_i$ and $hgv \in E$. In particular, since the action of G on X is proper, the composition hgw^{-1} lies in the finite set \mathcal{F}_i of group elements $g' \in G$ such that $g'D_i \cap E \neq \emptyset$, and hence we can choose $\mathcal{J}_i \subset \mathcal{F}_i$.

Next, note that since by assumption $W_i \cap Y \neq \emptyset$, we have $HG_{W_i} \subset I(G, Y, i)$, so we may assume $1 \in \mathcal{I}_i$. The finite set $\mathcal{J}_i \setminus 1$ is then disjoint from HG_{W_i} , which is separable in G by Lemma A.7. Therefore, there exists a finite-index normal subgroup $\widehat{G}_i \trianglelefteq G$ such that $(\bigcup_{g \in \mathcal{J}_i \setminus \{1\}} g\widehat{G}_i) \cap HG_{W_i} = \emptyset$. We claim that (any finite-index subgroup of) $\widehat{G} := \bigcap_i \widehat{G}_i$ satisfies conclusion (i).

Indeed, let $W \subset X$ be a wall intersecting Y , and let $1 \leq i \leq n$ and $h \in H$ be such that $W = hW_i$. Assume for the sake of contradiction that there is some $g \in \widehat{G}$ such that $g\mathcal{N}(W) \cap Y \neq \emptyset$ but $gW \cap Y = \emptyset$. Since \widehat{G} is normal, this implies $h^{-1}gh \in \widehat{G} \cap J(G, Y, i) \setminus I(G, Y, i) \subset \widehat{G}_i \cap J(G, Y, i) \setminus I(G, Y, i)$, and hence $h^{-1}gh = vg_iw$ for some $v \in H$, $g_i \in \mathcal{J}_i \setminus \mathcal{I}_i$ and $w \in G_{W_i}$. This is a contradiction, because otherwise we would have $g_i((wh^{-1})g^{-1}(wh^{-1})^{-1}) = v^{-1}w^{-1} \in g_i\widehat{G}_i \cap HG_{W_i}$, and so conclusion (i) follows.

For conclusion (ii), since (G, X) is virtually special we can assume that X/\widehat{G} is special, fully clean, and with simplicial 1–skeleton, so that the composition

$$f : Y/(H \cap \widehat{G}) \rightarrow X/(H \cap \widehat{G}) \rightarrow X/\widehat{G}$$

is a local isometry. But (H, Y) is also virtually special, so by using Lemma 2.9(ii) and the separability of wall stabilizers in H we can find a finite-index subgroup $H' < H$ such that for any further finite-index subgroup $H'' < H'$ and for any wall $W \subset X$ intersecting Y , the map $(W \cap Y)/(H'' \cap G_W) \rightarrow Y/H''$ is an embedding and the image is a wall of Y/H'' .

The group $H' < G$ is convex, hence separable by Theorem 2.19, so by a separability argument we may assume that $H' \cap \widehat{G} = H \cap \widehat{G}$. With this in mind, let C be the connected component of $C(Y/(H \cap \widehat{G}), X/\widehat{G})$ including $Y/(H \cap \widehat{G})$, and let $G' < \widehat{G}$ correspond to its fundamental group, which is finite-index in G since $Y/(H \cap \widehat{G})$ is compact. Also, we have $H \cap \widehat{G} < G'$, and so $H \cap G' = H \cap \widehat{G}$. Since the inclusion of $Y/(H \cap G')$ into C maps distinct walls to distinct walls (see Remark A.5), our assumptions about H' imply that the group G' satisfies conclusion (ii). \square

The key idea in the proof of Theorem A.1 is based on the following proposition, which says that under some mild assumptions, the canonical completion is functorial.

Proposition A.10 *Let $\overline{V}, \overline{X}, \overline{Y}, \overline{Z}$ be special cube complexes such that the following is a commutative diagram of local isometries:*

$$(9) \quad \begin{array}{ccc} \overline{V} & \xrightarrow{f} & \overline{Y} \\ \downarrow s & & \downarrow t \\ \overline{Z} & \xrightarrow{g} & \overline{X} \end{array}$$

In addition, assume that:

- (i) \overline{X} and \overline{Y} are fully clean and have simplicial 1–skeleton.
- (ii) The map t maps distinct walls to distinct walls.
- (iii) If e is an edge of \overline{X} incident to a vertex $t(y)$ of $t(\overline{Y})$ with e dual to a wall intersecting $t(\overline{Y})$, then $e = t(e')$ for some edge e' of \overline{Y} incident to y .
- (iv) For every vertex $v \in \overline{V}$, if there exist edges e of \overline{Y} and e' of \overline{Z} incident to $f(v)$ and $s(v)$, respectively, and such that $t(e) = g(e')$, then there is an edge e'' of \overline{V} incident to v and such that $e = f(e'')$ and $e' = s(e'')$.

Then there is a local isometry $\hat{t}: C(\bar{V}, \bar{Y}) \rightarrow C(\bar{Z}, \bar{X})$ of the canonical completions which commutes with the corresponding inclusions and projections, in the sense that the following diagrams commute:

$$(10) \quad \begin{array}{ccccc} \bar{V} & \xrightarrow{j} & C(\bar{V}, \bar{Y}) & \xrightarrow{r} & \bar{V} & & C(\bar{V}, \bar{Y}) & \xrightarrow{\hat{t}} & C(\bar{Z}, \bar{X}) \\ \downarrow s & & \downarrow \hat{t} & & \downarrow s & & \downarrow p & & \downarrow p' \\ \bar{Z} & \xrightarrow{j'} & C(\bar{Z}, \bar{X}) & \xrightarrow{r'} & \bar{Z} & & \bar{Y} & \xrightarrow{t} & \bar{X} \end{array}$$

Remark A.11 As we will see below, the conditions (i), (ii) and (iii) in the previous proposition can be obtained for a general commutative diagram of local isometries between compact special cube complexes after passing to finite coverings. Condition (iv) may be achieved if, for instance, the universal cover of \bar{V} coincides with the intersection of the universal covers of \bar{Y} and \bar{Z} , when we view these complexes as naturally embedded in the universal cover of \bar{X} .

Proof We first construct the map \hat{t} for lower-dimensional cubes of $C(\bar{V}, \bar{Y})$, starting with the 0–skeleton, where we define $\hat{t}: \bar{V}^{(0)} \times \bar{Y}^{(0)} \rightarrow \bar{Z}^{(0)} \times \bar{X}^{(0)}$ by $(v, y) \mapsto (s(v), t(y))$. In this way, \hat{t} clearly satisfies (10). For the 1–skeleton, we will check that the image under \hat{t} of a pair of vertices of $C(\bar{V}, \bar{Y})$ representing a horizontal (resp. diagonal) edge is a pair of vertices representing a horizontal (resp. diagonal) edge of $C(\bar{Z}, \bar{X})$. Let $e = \{(v, y), (v, y')\}$ be a horizontal edge of $C(\bar{V}, \bar{Y})$, for which we claim that the pair $\{(s(v), t(y)), (s(v), t(y'))\}$ represents a horizontal edge. Assume for the sake of contradiction that there exists an edge b of \bar{Z} incident to $s(v)$, with $g(b)$ dual to the wall $W(\{t(y), t(y')\}) \subset \bar{X}$ — note that $\{t(y), t(y')\}$ is an edge since \bar{X} has simplicial 1–skeleton and t is local isometry. This edge $g(b)$ is incident to $g(s(v)) = t(f(v))$, so by condition (iii), $g(b)$ equals $t(b')$ for an edge b' incident to $f(v)$. Condition (ii) then implies that b' is dual to the wall $W(\{y, y'\}) \subset \bar{Y}$, and condition (iv) gives us an edge b'' of \bar{V} incident to v with $f(b'') = b'$. But this would imply that e is not horizontal, and this contradiction proves the claim. The case of $e = \{(v, y), (v', y')\}$ diagonal is easier since t maps walls to walls, and hence $g(\{s(v), s(v')\})$ is dual to $W(\{t(y), t(y')\})$. Therefore, the image of a horizontal/diagonal edge of $C(\bar{V}, \bar{Y})$ is defined as the expected horizontal/diagonal edge of $C(\bar{Z}, \bar{X})$, and since the image of an edge under r or r' only depends on whether the edge is horizontal or vertical, \hat{t} also satisfies (10) at the level of the 1–skeleton.

Now, let Q be a square of $C(\bar{V}, \bar{Y})$, say with 1–skeleton determined by the vertices $\{(v_i, y_i)\}_{i=1}^4$. By definition, this means $p(Q)$ is also a square with 0–skeleton $\{y_i\}_{i=1}^4$,

and by condition (i) the vertices $\{t(y_i)\}_{i=1}^4$ are the 0-skeleton of the square $t(p(Q))$ of \bar{X} . As $C(\bar{Z}, \bar{X})$ is a covering, these vertices lift under p' to the set $\{(s(v_i), t(y_i))\}_{i=1}^4$ that is the 0-skeleton of a square Q' , which we define as the image of Q under \hat{t} . Again, since the image of a square under a retraction only depends on whether its 1-skeleton consists of horizontal/diagonal edges, the diagrams (10) still commute.

Finally, by [22, Lemma 2.5] there is a unique way to extend \hat{t} to a combinatorial map $C(\bar{V}, \bar{Y}) \rightarrow C(\bar{Z}, \bar{X})$, which is clearly a local isometry. Also, since the maps r, r', p and p' restricted to a higher-dimensional cube depend only on its 2-skeleton, by uniqueness of \hat{t} it must satisfy (10). \square

Remark A.12 In the proof of Theorem A.1 below, we will be interested in the commutative diagrams of fundamental groups induced by (10), so we will only require these diagrams to commute at the level of 2-skeletons.

Proof of Theorem A.1 Let us use Lemma 2.15 to find convex cores Y and Z , for H and K respectively, such that $V = Y \cap Z$ is nonempty. By Lemma 2.14 this will imply that V is a convex core for $H \cap K$. We will prove first that there exists a finite-index subgroup $\hat{G} < G$ such that if $\hat{H} = H \cap \hat{G}$ and $\hat{K} = K \cap \hat{G}$, then after defining $\bar{V} = V/(\hat{H} \cap \hat{K})$, $\bar{X} = X/\hat{G}$, $\bar{Y} = Y/\hat{H}$ and $\bar{Z} = Z/\hat{K}$, the induced diagram (9) is of local isometries and satisfies the conditions (i)–(iv) of Proposition A.10.

By [22, Corollary 8.9] we can find $\hat{G} < G$ of finite index such that condition (i) holds, and by possibly replacing \hat{G} by a further finite-index subgroup satisfying Corollary A.8, we can ensure that \hat{G} also satisfies (ii).

To prove condition (iii), let e be an edge of \bar{X} incident to $t(y)$ for a vertex $y \in \bar{Y}$, and let $\tilde{e} \subset X$ be a lifting of e incident to the lifting $\tilde{y} \in Y$ of y . If e is dual to the wall $W(t(b)) \subset \bar{X}$ for some edge $b \subset \bar{Y}$, then there exists a lifting $\tilde{b} \subset Y$ of b and some $g \in \hat{G}$ such that $W(\tilde{e}) = gW(\tilde{b})$. Since $\tilde{y} \in Y$, we have $W(\tilde{b}) \cap Y \neq \emptyset$ and $g\mathcal{N}(W(\tilde{b})) \cap Y \neq \emptyset$, so by conclusion (i) of Corollary A.8 we have $W(\tilde{e}) \cap Y \neq \emptyset$ and $\tilde{e} \subset Y$, implying condition (iii).

Finally, let $v \in \bar{V}$ be a vertex lifting to $\tilde{v} \in V$, and let \tilde{e} and \tilde{e}' be edges of Y and Z respectively, incident to \tilde{v} and such that there exists some $g \in \hat{G}$ with $g\tilde{e} = \tilde{e}'$. Since the action of \hat{G} is free, we have $g = 1$ and $\tilde{e} = \tilde{e}' \in V$. Projecting to the corresponding quotients we deduce (iv).

Therefore, we are in the assumptions of Proposition A.10, and there is a local isometry $\hat{t}: C(\bar{V}, \bar{Y}) \rightarrow C(\bar{Z}, \bar{X})$ such that the diagrams (10) commute. The proof now goes as

in Lemma A.7. If $H' < \widehat{H}$ and $G' < \widehat{G}$ are the finite-index subgroups representing the fundamental groups of the (appropriate connected components of the) canonical completions $C(\overline{V}, \overline{Y})$ and $C(\overline{Z}, \overline{X})$ respectively, then $H' < G'$ and there is a retraction homomorphism $r_*: G' \rightarrow G'$ with image \widehat{K} and such that $r_*(H') = \widehat{H} \cap \widehat{K}$. Again, we can check that $\widetilde{H} := H' \cap r_*^{-1}(H')$ satisfies $r_*(\widetilde{H}) \subset \widetilde{H}$, and so Proposition A.6 implies that $\widetilde{H}\widehat{K}$ is separable in G' . Since $G' < G$, $\widetilde{H} < H$ and $\widehat{K} < K$ are all of finite index, we conclude that HK is separable in G , completing the proof. \square

We now see how Theorem A.1 implies Proposition 2.24. In fact, by Lemma 2.15, Proposition 2.24 follows from the next proposition.

Proposition A.13 *Let (G, X) be a cubulated group, and let $Y \subset X$ be a G -invariant convex subcomplex. If the cubulated group (G, Y) is virtually special, then (G, X) is also virtually special.*

Before proving this result, we recall the definition of gate map projection [5, Section 2].

Definition A.14 Let X be a CAT(0) cube complex, and consider a convex subcomplex $Y \subset X$. The *gate map* is defined as the unique cubical map $g: X \rightarrow Y$ characterized by the following property: for any point $x \in X$, the wall $W \subset X$ separates x from $g(x)$ if and only if it separates x from Y .

The next lemma is well known by experts and is implicit, for instance, in [5], so we provide a proof in the absence of a precise reference.

Lemma A.15 *Let (G, X) be a cubulated group and let $Y \subset X$ be a G -invariant convex subcomplex. Then for any convex subcomplex $K \subset X$, its image $g(K) \subset Y$ is also a convex subcomplex. Moreover, if $H < G$ preserves K and acts cocompactly on it, then it also acts cocompactly on $g(K)$.*

Proof For the first assertion, it is enough to prove that if $x, y \in X$ are vertices and β is a combinatorial geodesic path joining $g(x)$ and $g(y)$, then $\beta = g(\alpha)$ for some geodesic α joining x and y . We will prove this by induction on the sum of combinatorial distances $d = d(x, g(x)) + d(y, g(y))$, where the case $d = 0$ holds since Y is convex. So, assume that the claim follows for $d \geq 0$, and let $x, y \in X$ be vertices with $d(x, g(x)) + d(y, g(y)) = d + 1$, for which we can assume $d(x, g(x)) > 0$. Thus, let γ be a combinatorial geodesic joining x and $g(x)$, and u be the vertex on this geodesic at distance 1 to x . Except for the wall dual to the edge e determined by x and u ,

any other wall dual to an edge of γ separates u and $g(x)$, so $g(u) = g(x)$, and by our inductive assumption there is a geodesic α' joining u and y , such that $g(\alpha') = \beta$.

If e separates x from y , the concatenation of e and α' defines a geodesic α projecting to β . Otherwise, there is an edge e' of α' dual to $W(e)$, say determined by the vertices p and q with p between u and q . In this case, the segment α'' of α' between u and p lies in one of the sides of $\mathcal{N}(W(e))$, which we know is a convex subcomplex, so every vertex of α'' lies in an edge dual to $W(e)$. If we follow the extreme points of these edges lying on the other side of $\mathcal{N}(W(e))$, we will obtain a geodesic path joining x and q . By concatenating this path with the segment of α between q and y , we will obtain a geodesic path α (there is no repetition in the walls dual to α), and it is easy to see that $g(\alpha) = \beta$.

The second assertion follows easily since Y is G -invariant, and since $g(K)/H$ is the image of the compact set K/H under the induced projection $g: X/G \rightarrow Y/H$. \square

Proof of Proposition A.13 Let $W_1, W_2 \subset X$ be walls with stabilizers G_1 and G_2 , respectively. By Theorem 2.20, to prove the proposition it is enough to show that G_1 and G_1G_2 are separable in G . Consider then the gate map $g: X \rightarrow Y$ and the projections $g(\mathcal{N}(W_1))$ and $g(\mathcal{N}(W_2))$, which by Lemma A.15 are convex subcomplexes of Y . This same lemma also implies that each subgroup G_i acts cocompactly on $g(\mathcal{N}(W_i))$, and so G_1 and G_2 are convex subgroups of (G, Y) , which by assumption is virtually special. The conclusion then follows by Theorems 2.19 and A.1. \square

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
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