Prime-localized Weinstein subdomains

OLEG LAZAREV
ZACHARY SYLVAN

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For any high-dimensional Weinstein domain and finite collection of primes, we construct a Weinstein subdomain whose wrapped Fukaya category is a localization of the original wrapped Fukaya category away from the given primes. When the original domain is a cotangent bundle, these subdomains form a decreasing lattice whose order cannot be reversed.

Furthermore, we classify the possible wrapped Fukaya categories of Weinstein subdomains of a cotangent bundle of a simply connected, spin manifold, showing that they all coincide with one of these prime localizations. In the process, we describe which twisted complexes in the wrapped Fukaya category of a cotangent bundle of a sphere are isomorphic to genuine Lagrangians.

1 Introduction

1.1 Main results

One of the main problems in symplectic topology is to understand the set of Lagrangians L in a symplectic manifold X. For example, Arnold's nearby Lagrangian conjecture states that any closed exact Lagrangian L in $T^*M^n_{\mathrm{std}}$ is Hamiltonian isotopic to the zero section $M \subset T^*M_{\mathrm{std}}$; by work of Abouzaid [3], Fukaya, Seidel and Smith [16] and Kragh [25] on this conjecture, all such Lagrangians are homotopy equivalent to M^n. Each closed exact Lagrangian $L \subset X$ gives a Liouville subdomain T^*L of X, and the *skeleton* of T^*L, the stable set of its Liouville vector field, is precisely L. More generally, any *Weinstein* domain V deformation retracts to a possibly singular Lagrangian skeleton. Therefore a Weinstein *subdomain* $V \subset X$ can be considered a singular Lagrangian in X. In this paper, we consider the problem of constructing and classifying Weinstein subdomains of a fixed Weinstein domain, as well as the

wrapped Fukaya categories $\mathcal{W}(V; R)$ of such subdomains (here, R is a commutative coefficient ring). We will only consider Weinstein subdomains $V \subset X$ with the stronger property that $X \setminus V$ is also a Weinstein cobordism, ie V is the sublevel set of an ambient Weinstein Morse function on X; see Cieliebak and Eliashberg [10] for background on the geometry of Weinstein domains.

There is a (cohomologically) fully faithful embedding of $\mathcal{W}(X; R)$ into $\mathrm{Tw}\, \mathcal{W}(X; R)$, the category of *twisted complexes* on $\mathcal{W}(X; R)$. Since $\mathrm{Tw}\, \mathcal{W}(X; R)$ is a formal algebraic enlargement of a geometric category, this functor is usually not a quasi-equivalence. To understand which A_∞–categories *actually arise* from Weinstein subdomains, it turns out we will have to understand which twisted complexes come from actual geometric Lagrangians. In other words, we will largely be concerned with understanding the image of this embedding. We give examples when this functor is a quasi-equivalence (Proposition 2.2) and describe its image when $X = T^* S^n_{\mathrm{std}}$ (Example 1.9); see Section 1.2. This type of question about the geometricity of twisted complexes has previously been studied by Auroux and Smith [8] and Haiden, Katzarkov and Kontsevich [22].

Given a small A_∞–category $\mathcal{C}$ over $\mathbb{Z}$ and set of objects $\mathcal{A}$ of $\mathcal{C}$, one can form the quotient A_∞–category $\mathcal{C}/\mathcal{A}$, which comes with a localization functor $\mathcal{C} \to \mathcal{C}/\mathcal{A}$; see Lyubashenko and Manzyuk [30] and Lyubashenko and Ovsienko [31]. In particular, given a collection of prime numbers $P \subset \mathbb{Z}$, one can form

$$(1\text{-}1) \qquad \mathcal{C}\left[\frac{1}{P}\right] := \mathcal{C}/\{\mathrm{cone}(p \cdot \mathrm{Id}_L) \mid p \in P, L \in \mathcal{C}\},$$

the localization of $\mathcal{C}$ away from the primes P. Quotienting by $\mathrm{cone}(p \cdot \mathrm{Id}_L)$ kills the object $\mathrm{cone}(p \cdot \mathrm{Id}_L)$, which has the effect of making the morphism $p \cdot \mathrm{Id}_L$ a quasi-isomorphism, ie inverting p. Hence if $\hom^*_{\mathcal{C}}(L, K)$ is a cochain complex of free Abelian groups, then $\hom^*_{\mathcal{C}[1/P]}(L, K)$ is quasi-isomorphic to $\hom^*_{\mathcal{C}}(L, K) \otimes_{\mathbb{Z}} \mathbb{Z}[1/P]$, which explains our notation $\mathcal{C}[1/P]$. We will also allow P to be empty or contain 0, in which case $\mathcal{C}[1/P]$ is the original category $\mathcal{C}$ or the trivial category, respectively.

Our first result is that any high-dimensional Weinstein domain has Weinstein subdomains whose Fukaya categories are localizations away from any finite collection of primes P. Furthermore, these subdomains are *almost* symplectomorphic, ie their symplectic forms are homotopic through nondegenerate 2–forms, and hence indistinguishable from the point of view of classical smooth topology. We note that by Gromov's h-principle [21] for open symplectic manifolds, any two almost symplectomorphic

Weinstein domains are actually homotopic through symplectic structures (but may not be symplectomorphic).

Theorem 1.1 *For any Weinstein domain X^{2n} with $n \geq 5$ and finite collection of prime numbers P, which is possibly empty or contains 0, there is a Weinstein subdomain $X_P \subset X$ such that* $\operatorname{Tw} \mathcal{W}(X_P; \mathbb{Z}) \cong \operatorname{Tw} \mathcal{W}(X; \mathbb{Z})[1/P]$ *and the Viterbo transfer functor*

$$\mathcal{V} \colon \operatorname{Tw} \mathcal{W}(X; \mathbb{Z}) \to \operatorname{Tw} \mathcal{W}(X_P; \mathbb{Z})$$

is localization away from P. In particular, $\operatorname{Tw} \mathcal{W}(X_P; \mathbb{F}_p) = 0$ *if $p \in P$ or $0 \in P$, and* $\operatorname{Tw} \mathcal{W}(X_P; \mathbb{F}_p) \cong \operatorname{Tw} \mathcal{W}(X; \mathbb{F}_p)$ *otherwise. Furthermore, we can arrange that:*

(1) *The Weinstein cobordism $X \backslash X_P$ is smoothly trivial, and hence X_P is almost symplectomorphic to X.*

(2) *If $Q \subset P$ or $0 \in P$, we can exhibit a Weinstein embedding $\varphi_{P,Q} \colon X_P \hookrightarrow X_Q$ with the property that if $R \subset Q \subset P$, then $\varphi_{P,Q} \circ \varphi_{Q,R}$ is Weinstein homotopic to $\varphi_{Q,R}$.*

(3) *If P is empty, then X_P is X. If $0 \in P$, then X_P is Weinstein homotopic to the flexibilization X_{flex} of X defined in [10].*

Remark 1.2 For us, the objects of $\mathcal{W}(X; R)$ are graded exact spin Lagrangian submanifolds (branes) in X that are closed or have conical Legendrian boundary in a collar of ∂X. We will usually not specify what type of grading data our Lagrangian should have, except when X is a cotangent bundle and we will use the canonical $\mathbb{Z}$–grading. The result does not hold without using twisted complexes since, for example, the Viterbo transfer functor is only defined on the Fukaya category of twisted complexes; for example, see Proposition 1.29 of Ganatra, Pardon and Shende [19].

More precisely, there is a Weinstein homotopy of the Weinstein structure on X to a different structure X' such that X_P is a sublevel set of the Weinstein Morse function on X'. That is, X_P is itself a Weinstein domain *and* $X' \backslash X_P$ is a Weinstein cobordism. We also note that Theorem 1.1 holds for any grading of X (and the induced grading on its subdomains).

Our construction is related to a result of Abouzaid and Seidel [5], who also showed that any Weinstein domain X^{2n} with $n \geq 6$ can be modified to a produce a new Weinstein domain X'_P, almost symplectomorphic to X, with the property that $SH^*(X_P; \mathbb{F}_q) \cong SH^*(X; \mathbb{F}_q)$ if $q \notin P$ and $SH(X_P; \mathbb{F}_q) = 0$ otherwise. Theorem 1.1 proves this property

on the level of Fukaya categories, which implies the result on the level of symplectic cohomology due to the isomorphism between Hochschild homology and symplectic cohomology due to Ganatra [17]; for example, by [17],

$$SH(X_P; \mathbb{F}_q) \cong HH(\text{Tw } \mathcal{W}(X_P) \otimes \mathbb{F}_q; \mathbb{F}_q),$$

and if $q \in P$, then

$$\text{Tw } \mathcal{W}(X_P) \otimes \mathbb{F}_q \cong \text{Tw } \mathcal{W}(X) \otimes \mathbb{Z}\left[\frac{1}{P}\right] \otimes \mathbb{F}_q = 0,$$

so that $HH(\text{Tw } \mathcal{W}(X_P) \otimes \mathbb{F}_q; \mathbb{F}_q) \cong 0$. The other main difference between our domain X_P and the domain X'_P produced by Abouzaid and Seidel in [5] is that X_P is manifestly a subdomain of X while X'_P is an abstract Weinstein domain. The construction of Abouzaid and Seidel involves modifying a Lefschetz fibration for X by enlarging the fiber and adding new vanishing cycles, and there is no obvious map between X and X'_P. Our construction involves removing a certain regular Lagrangian disk (which also appears in Abouzaid and Seidel's work) so that X_P is automatically a subdomain of X; constructing these regular disks requires $n \geq 5$, hence the restriction on n in Theorem 1.1. Both our construction and that of Abouzaid and Seidel require many choices, but we conjecture that one can make these choices so that the resulting Weinstein domains X_P and X'_P agree.

Remark 1.3 An analog of Theorem 1.1 is true for Weinstein domains with Weinstein stops. For example, in Theorem 2.3 we prove that there is a Legendrian sphere $\Lambda_P \subset \partial B^{2n}_{\text{std}}$ such that

$$\text{Tw } \mathcal{W}(B^{2n}_{\text{std}}, \Lambda_P) \cong \text{Tw } \mathcal{W}(B^{2n}_{\text{std}}, \Lambda_\varnothing)\left[\frac{1}{P}\right] \cong \text{Tw } \mathbb{Z}\left[\frac{1}{P}\right],$$

where $\Lambda_\varnothing$ is the Legendrian unknot, and there is a smoothly trivial Lagrangian cobordism $L \subset \partial B^{2n}_{\text{std}} \times [0, 1]$ whose positive and negative ends $\partial_\pm L$ coincide with $\Lambda_\varnothing$ and Λ_P, respectively. $(B^{2n}_{\text{std}}, \Lambda_\varnothing)$ is the standard Weinstein handle of index n; we call $(B^{2n}_{\text{std}}, \Lambda_P)$ a Weinstein P–handle of index n. The construction of the Weinstein subdomain X_P in Theorem 1.1 can be viewed as replacing all standard Weinstein handles of index n with Weinstein P–handles. This is similar to the classical rationalization of a CW complex, in which all standard cells are replaced with "rational" cells.

Next we consider Weinstein subdomains of the cotangent bundle T^*M_{std} of a smooth manifold M. Using Theorem 1.1 and the additional fact that $\text{Tw } \mathcal{W}(T^*M_{\text{std}}; \mathbb{F}_p)$ is nontrivial for any p, we show that T^*M_{std} has many infinitely different Weinstein subdomains.

Corollary 1.4 *If $n \geq 5$, then for any finite collection P of primes numbers, possibly empty or containing zero, there is a Weinstein subdomain $T^*M_P^n \subset T^*M_{\mathrm{std}}^n$ almost symplectomorphic to T^*M_{std} such that $\mathrm{Tw}\,\mathcal{W}(T^*M_P; \mathbb{Z}) \cong \mathrm{Tw}\,\mathcal{W}(T^*M; \mathbb{Z})[1/P]$. Furthermore, we can arrange for T^*M_P to be a Weinstein subdomain of T^*M_Q if and only if $Q \subset P$ or $0 \in P$, ie the product of primes in P divides the product of those in Q.*

The claim here is stronger than in Theorem 1.1: here T^*M_P is a Weinstein subdomain of T^*M_Q if and *only if* $Q \subset P$, or $0 \in P$ (in fact, "Weinstein" subdomain can be replaced with "Liouville" subdomain). The proof of Corollary 1.4 carries over to any Weinstein domain X for which $\mathcal{W}(X; \mathbb{F}_p)$ is nontrivial for all p, eg if X has a closed exact Lagrangian. Furthermore, by the "only if" part of the claim, our subdomains form a decreasing lattice whose order cannot be reversed. For example, there is an infinite decreasing sequence

$$T^*M_{\mathrm{std}} \supsetneq T^*M_2 \supsetneq T^*M_{2,3} \supsetneq T^*M_{2,3,5} \supsetneq \cdots \supsetneq T^*M_{P_k} \supsetneq \cdots \supsetneq T^*M_0 = T^*M_{\mathrm{flex}},$$

where P_k is the set of the first k primes; the other subdomains T^*M_P where $P \neq P_k$, eg $T^*M_{7,13}$, contain $T^*M_{P_k}$ for sufficiently large k. In particular, T^*M_{std} has many *singular* Lagrangians given by the skeleta of T^*M_P. These skeleta are not Hamiltonian isotopic since otherwise we could find a Liouville embedding of T^*M_Q into T^*M_P for $P \supset Q$. We contrast this with the nearby Lagrangian conjecture, which claims that all closed exact *smooth* Lagrangians of T^*M_{std} are Hamiltonian isotopic. Finally, we note that T^*M_P has no closed exact smooth Lagrangians if P is nonempty, since its Fukaya category over $\mathbb{F}_p$ vanishes.

Our second main result about subdomains of T^*M_{std} is a converse to Corollary 1.4: the Fukaya category of *any* Weinstein subdomain of T^*M_{std} is a localization of $\mathrm{Tw}\,\mathcal{W}(T^*M_{\mathrm{std}}; \mathbb{Z})$ away from some finite collection of primes. Here we use the $\mathbb{Z}$–grading on T^*M_{std} and its subdomains induced by the Lagrangian fibration by cotangent fibers.

Theorem 1.5 *If M^n is a closed, simply connected, spin manifold and $i : X \hookrightarrow T^*M_{\mathrm{std}}$ is a Weinstein subdomain, then $\mathrm{Tw}\,\mathcal{W}(X; \mathbb{Z}) \cong \mathrm{Tw}\,\mathcal{W}(T^*M_{\mathrm{std}}; \mathbb{Z})[1/P]$ for some finite collection of primes P, which is possibly empty or contains 0, and is unique (unless P contains 0). Under this equivalence, the Viterbo transfer functor $\mathrm{Tw}\,\mathcal{W}(T^*M; \mathbb{Z}) \to \mathrm{Tw}\,\mathcal{W}(X; \mathbb{Z})$ is localization away from P. Furthermore, either the restriction map $i^* : H^n(T^*M^n; \mathbb{Z}) \to H^n(X; \mathbb{Z})$ is an isomorphism, or $\mathcal{W}(X; \mathbb{Z}) \cong 0$ (or both).*

For $n \geq 5$, Theorem 1.5 combined with Corollary 1.4 completely classify which categories appear as Fukaya categories (with integer coefficients) of Weinstein subdomains of cotangent bundles of closed, simply connected, spin manifolds. For $n \leq 4$, the question remains open whether the categories $\mathrm{Tw}\,\mathcal{W}(T^*M_{\mathrm{std}}^n; \mathbb{Z})[1/P]$ actually appear as Fukaya categories of subdomains. Indeed, in the $n = 1$ case, the only subdomains of $T^*S_{\mathrm{std}}^1$ are $T^*S_{\mathrm{std}}^1$ or B_{std}^2, which algebraically correspond to the cases $P = \varnothing$ and $P = 0$. We note that the condition on the map i^* shows that any Weinstein ball $\Sigma \subset T^*M_{\mathrm{std}}$ has trivial $\mathcal{W}(\Sigma)$. There are no restrictions on i^* in degrees *less than* n, as in the case of $T^*M_{\mathrm{std}} \cup H^{n-1} \subset T^*M_{\mathrm{std}}$. Finally, we note that the "both" case does occur in the case of $T^*M_{\mathrm{flex}}^n \subset T^*M_{\mathrm{std}}^n$.

We emphasize that Theorem 1.5 classifies *Weinstein* subdomains of $X \subset T^*M_{\mathrm{std}}$; namely, X is itself a Weinstein domain *and* $T^*M_{\mathrm{std}} \backslash X$ is a Weinstein cobordism (after Weinstein homotopy of T^*M_{std}). We do not know if our result holds for more general Liouville subdomains $X \subset T^*M_{\mathrm{std}}$, for which either X is not a Weinstein domain or $T^*M_{\mathrm{std}} \backslash X$ is not a Weinstein cobordism. However, in the only known examples of subdomains $X \subset T^*M_{\mathrm{std}}$ for which $T^*M_{\mathrm{std}} \backslash X$ is not a Weinstein cobordism, X is a flexible domain (see Eliashberg and Murphy [15]) and hence has trivial Fukaya category. Furthermore, our classification is quite special to cotangent bundles: for a general Weinstein domain X, there are subdomains X_0 for which $\mathrm{Tw}\,\mathcal{W}(X_0)$ is different from $\mathrm{Tw}\,\mathcal{W}(X)[1/P]$ for any collection of primes P. For example, the boundary connected sum $T^*M_{\mathrm{std}} \natural T^*N_{\mathrm{std}}$ of two cotangent bundles T^*M and T^*N has a natural collection of subdomains indexed by *pairs* of collections of primes P, Q, namely $T^*M_P \natural T^*N_Q$.

1.2 Outline of proofs

We now outline the proofs of our two main results: Theorems 1.1 and 1.5. We focus primarily on the latter result, whose proof involves describing which twisted complexes in $\mathrm{Tw}\,\mathcal{W}(T^*M_{\mathrm{std}})$ are quasi-isomorphic to actual Lagrangians, ie the image of the functor $\mathcal{W}(T^*M_{\mathrm{std}}) \hookrightarrow \mathrm{Tw}\,\mathcal{W}(T^*M_{\mathrm{std}})$.

To see the connection, consider a Weinstein subdomain $X_0^{2n} \subset X^{2n}$. The Weinstein cobordism $X \setminus X_0$ has index n Lagrangian cocore disks $D_1, \ldots, D_k$, which are objects of $\mathcal{W}(X)$. Ganatra, Pardon and Shende [19, Proposition 8.15] proved that

$$\mathrm{Tw}\,\mathcal{W}(X_0) \cong \mathrm{Tw}\,\mathcal{W}(X)/(D_1, \ldots, D_k),$$

and the localization functor

$$\mathrm{Tw}\,\mathcal{W}(X) \to \mathrm{Tw}\,\mathcal{W}(X_0)$$

has a geometric interpretation and is called the *Viterbo transfer functor*. See Sylvan [38] for results when X, X_0 are both Weinstein but $X \setminus X_0$ is not necessarily a Weinstein cobordism. So to describe $\operatorname{Tw} \mathcal{W}(X_0)$, it suffices to describe the quasi-isomorphism classes of the Lagrangian disks $D_1, \ldots, D_k$ in $\operatorname{Tw} \mathcal{W}(X)$. To prove Theorem 1.1, we construct a disjoint collection of disks $D_1, \ldots, D_k \subset X^{2n}$ with $n \geq 5$ so that $\operatorname{Tw} \mathcal{W}(X; \mathbb{Z})/(D_1, \ldots, D_k) \cong \operatorname{Tw} \mathcal{W}(X; \mathbb{Z})[1/P]$. By removing the Weinstein handles associated to these disks, we get the subdomain X_P with the desired property $\operatorname{Tw} \mathcal{W}(X_P; \mathbb{Z}) \cong \operatorname{Tw} \mathcal{W}(X; \mathbb{Z})/(D_1, \ldots, D_k) \cong \operatorname{Tw} \mathcal{W}(X; \mathbb{Z})[1/P]$.

Remark 1.6 In fact, the localization $\mathcal{C}/\mathcal{A}$ by some objects $\mathcal{A} \subset \mathcal{C}$ depends only on the *split-closure* of $\mathcal{A}$ in $\mathcal{C}$, see Corollary 3.14 of Ganatra, Pardon and Shende [20], which is the kernel of the localization $\mathcal{C} \to \mathcal{C}/\mathcal{A}$. A subcategory $\mathcal{C}' \subset \mathcal{C}$ is split-closed if for any two objects A, B of $\mathcal{C}$ for which $A \oplus B$ is an object of $\mathcal{C}'$, it holds that A and B are also objects of $\mathcal{C}'$. More generally, there is a correspondence between localizing functors $\mathcal{C} \to \mathcal{D}$ and split-closed subcategories of $\mathcal{C}$.

Any Weinstein domain $X \subset T^* M_{\mathrm{std}}$ has $\operatorname{Tw} \mathcal{W}(X) \cong \operatorname{Tw} \mathcal{W}(T^* M_{\mathrm{std}})/(D_1, \ldots, D_k)$ for some collection of Lagrangian disks $D_1, \ldots, D_k$ in $T^* M_{\mathrm{std}}$, so to prove Theorem 1.5 we need to classify the objects of $\operatorname{Tw} \mathcal{W}(T^* M_{\mathrm{std}})$ that are quasi-isomorphic to embedded Lagrangian disks. By work of Abouzaid [1], any object of $\operatorname{Tw} \mathcal{W}(T^* M_{\mathrm{std}})$ is quasi-isomorphic to a twisted complex of the cotangent fibers $T_q^* M$; after taking boundary connected sums of these cotangent fibers along isotropic arcs, we can replace this twisted complex with a single embedded Lagrangian disk *equipped with a bounding cochain*. However for Theorem 1.5, we need to consider Lagrangian disks without bounding cochains and as we will see in Theorem 1.7 below, not every twisted complex in $\operatorname{Tw} \mathcal{W}(T^* M_{\mathrm{std}})$ is quasi-isomorphic to such a disk.

In the following key result, we characterize those twisted complexes in $\operatorname{Tw} \mathcal{W}(T^* M_{\mathrm{std}})$ that are quasi-isomorphic to Lagrangian disks. To make this precise, we fix some notation. Let A be an object of some pretriangulated A_∞–category $\mathcal{C}$ over $\mathbb{Z}$. A homotopy unit $e \in \operatorname{end}_\mathcal{C}(A)$ of A gives an A_∞ homomorphism $\mathbb{Z} \to \operatorname{end}_\mathcal{C}(A)$, which induces a functor $\operatorname{Tw} \mathbb{Z} \to \operatorname{Tw} \operatorname{end}_\mathcal{C}(A)$. Applying this to $\mathcal{C} = \operatorname{Tw} \mathcal{W}(T^* M_{\mathrm{std}}; \mathbb{Z})$ and $A = T_q^* M$, we get the composition of functors

$$(1\text{-}2) \qquad \otimes T_q^* M : \operatorname{Tw} \mathbb{Z} \to \operatorname{Tw} \operatorname{end}(T_q^* M) \xrightarrow{\sim} \operatorname{Tw} \mathcal{W}(T^* M_{\mathrm{std}}; \mathbb{Z}).$$

Note here that $\operatorname{Tw} \mathbb{Z}$ is the category of finite cochain complexes, ie those $\mathbb{Z}$–cochain complexes whose underlying graded Abelian group is free and finitely generated. The

functor $\otimes T_q^* M$ sends such a twisted complex on $\mathbb{Z}$ to the corresponding twisted complex on $T_q^* M$. In particular, the differential consists entirely of morphisms that are all integer multiples of the unit. By Abouzaid's theorems [4; 1], the second functor is actually a quasi-equivalence, meaning that every object of $\text{Tw}\, \mathcal{W}(T^* M_{\text{std}}; \mathbb{Z})$ is a twisted complex of $T_q^* M$ with differential given by *arbitrary* elements of $\text{end}(T_q^* M)$. As we will see, the composite functor $\otimes T_q^* M$ is not essentially surjective, but for nice M every Lagrangian disk is contained in its essential image. More generally, we have the following result.

Theorem 1.7 *For M^n a closed, simply connected, spin manifold, let $i : L^n \hookrightarrow T^* M_{\text{std}}^n$ be an exact Lagrangian brane. If $i : L^n \hookrightarrow T^* M_{\text{std}}^n$ is null-homotopic as a continuous map, then L is in the image of $\otimes T_q^* M$. More precisely, L is quasi-isomorphic to*

$$CW^*(M, L; \mathbb{Z}) \otimes T_q^* M$$

in $\text{Tw}\, \mathcal{W}(T^* M_{\text{std}}; \mathbb{Z})$, where the cochain complex $CW^*(M, L; \mathbb{Z})$ is considered an object of $\text{Tw}\, \mathbb{Z}$.

Combining this result with the construction of the Lagrangian disks in the Theorem 1.1, we have the following description of the image of $\otimes T_q^* M$.

Corollary 1.8 *If M^n is a closed, simply connected, spin manifold and $L \subset T^* M_{\text{std}}$ is a Lagrangian disk, then L is in the essential image of $\otimes T_q^* M$. If $n \geq 5$, then every object of $\text{Tw}\, \mathcal{W}(T^* M_{\text{std}})$ in the image of $\otimes T_q^* M$ is quasi-isomorphic to a Lagrangian disk.*

Theorem 1.7 translates the purely topological condition that the Lagrangian is null-homotopic into the Floer-theoretic condition on its quasi-isomorphism class in the Fukaya category. The proof of Theorem 1.7 actually shows that this topological condition can be weakened to the algebraic condition that the restriction homomorphism $i^* : C^*(T^* M; \mathbb{Z}) \to C^*(L; \mathbb{Z})$ on singular cochain algebras is homotopic as an A_∞ homomorphism to a map that factors through $\mathbb{Z}$. In Proposition 3.3, we prove a generalization of Theorem 1.7 for arbitrary Lagrangians $i : L \hookrightarrow T^* M_{\text{std}}$ that are not necessarily null-homotopic: we prove that the $CW^*(M, M)$–module $CW^*(M, L)$ is in the image of the composition

$$\text{Mod}_{C^*(L)} \xrightarrow{i^\vee} \text{Mod}_{C^*(T^* M)} \cong \text{Mod}_{CW^*(M, M)},$$

where $i^\vee$ is the pullback functor on modules induced by the restriction homomorphism $i^* : C^*(T^* M) \to C^*(L)$.

The proof of Corollary 1.8 uses the Koszul duality between the wrapped Floer cochains of a cotangent fiber and those of the zero section of a cotangent bundle, the fact that the zero section M is homotopy equivalent to the ambient manifold T^*M_{std}, and a certain commutativity property of the closed–open map that holds for arbitrary Liouville domains; see Proposition 3.3 and Remark 3.5. Consequently, Corollary 1.8 is quite special to cotangent bundles and analogous results do not hold for general Weinstein domains. Even if X^{2n} has a single index n handle with cocore D^n, then it is not true that any Lagrangian disk $L \subset X$ is isomorphic to $C^* \otimes D$ for some cochain complex C^* over $\mathbb{Z}$ (but since D is a generator of $\text{Tw}\,\mathcal{W}(X)$, L is isomorphic to a twisted complex of D whose differential has arbitrary morphisms). For example, this is the case if X^{2n} is one of the exotic cotangent bundles constructed in [28] that have many closed regular Lagrangians with different topology.

In the following example, we illustrate the above results when $M = S^n$. We describe the image of the functor $\mathcal{W}(T^*S^n_{\text{std}}) \hookrightarrow \text{Tw}\,\mathcal{W}(T^*S^n_{\text{std}})$ and give examples of Lagrangians that are not in image of the functor $\otimes T^*_q S^n : \text{Tw}\,\mathbb{Z} \to \text{Tw}\,\mathcal{W}(T^*S^n_{\text{std}})$.

Example 1.9   We first Floer-theoretically classify all exact Lagrangian branes in $T^*S^n_{\text{std}}$. If $L \subset T^*S^n$ is closed, then it is quasi-isomorphic to the zero section $S^n \subset T^*S^n$ by [16]; if $L^n \subset T^*S^n$ has nonempty boundary, any embedding $i : L^n \hookrightarrow T^*S^n$ is automatically null-homotopic and so in the image of $\otimes T^*_q M : \text{Tw}\,\mathbb{Z} \to \text{Tw}\,\mathcal{W}(T^*S^n_{\text{std}}; \mathbb{Z})$; this implies that L is quasi-isomorphic to a disk if $n \geq 5$. However, there are many exact Lagrangians $L \subset T^*S^n_{\text{std}}$ that are not *homotopy equivalent* to a disk or n–sphere: any smooth n–manifold L with nonempty boundary and trivial complexified tangent bundle has an exact Lagrangian embedding into T^*S^n for $n \geq 3$; see [14; 27]. Using the above classification, one can check that for any Lagrangian $L \subset T^*S^n_{\text{std}}$ with nonempty boundary, the wrapped Floer cohomology $HW^*(L, L)$ is either trivial or infinite-dimensional (over some field $\mathbb{F}_p$). This implies the following new case of the Arnold chord conjecture: any Legendrian $\Lambda \subset ST^*S^n_{\text{std}}$ that bounds an exact Lagrangian brane (so graded, spin) in $T^*S^n_{\text{std}}$ has at least one Reeb chord for any contact for any contact form; see [24; 35; 32] for existing results.

Although all Lagrangians with nonempty boundary are in the image of the functor $\otimes T^*_q S^n$, we now show that the zero section $S^n \subset T^*S^n$ is not; this is compatible with the fact that $i : S^n \hookrightarrow T^*S^n$ is not null-homotopic. Indeed, any Lagrangian L that is in the image of

$$\otimes T^*_q M : \text{Tw}\,\mathbb{Z} \to \text{Tw}\,\mathcal{W}(T^*S^n_{\text{std}})$$

represents something in the image of the pullback functor $i^\vee \colon \mathrm{Mod}_{\mathbb{Z}} \to \mathrm{Mod}_{C^*(S)}$ and so has the property that the product

$$CW^*(S^n, L) \otimes CW^n(S^n, S^n) \to CW^{*+n}(S^n, L)$$

must vanish on cohomology (since it factors through the restriction map $CW^*(S^n, S^n) \cong C^*(S^n) \to \mathbb{Z}$, which vanishes in degree n).

Since this product does not vanish for $L = S^n$, this Lagrangian is not in the image of $\otimes T_q^* S^n$. However, since $T_q^* S^n$ generates $\mathrm{Tw}\, \mathcal{W}(T^* S^n_{\mathrm{std}})$, the zero section S^n is still some twisted complex of $T_q^* S^n$. It turns out that S^n is quasi-isomorphic to $T_q^* S^n[n] \xrightarrow{\gamma} T_q^* S^n$, where γ is the generator of

$$CW^1(T_q^* S^n[n], T_q^* S^n) = CW^{1-n}(T_q^* S^n, T_q^* S^n) \cong C_{n-1}(\Omega S^n) \cong \mathbb{Z}.$$

Note that γ is not a multiple of the unit.

In all, we have shown that if $n \geq 5$, the image of the full and faithful embedding $\mathcal{W}(T^* S^n_{\mathrm{std}}) \hookrightarrow \mathrm{Tw}\, \mathcal{W}(T^* S^n_{\mathrm{std}}) \cong \mathrm{Tw}\, \{T_q^* S^n\}$ is quasi-isomorphic to the subcategory

$$\{C^* \otimes T_q^* S^n \mid C^* \text{ is a cochain complex over } \mathbb{Z}\} \cup \{T_q^* S^n[n] \xrightarrow{\gamma} T_q^* S^n\}.$$

For more general manifolds M, $\mathcal{W}(T^* M)$ has other objects besides the zero section and Lagrangian disks, eg the surgery of the zero section and a cotangent fiber.

Finally, we use Corollary 1.8 to prove Theorem 1.5 classifying the wrapped Fukaya categories of subdomains of $T^* M_{\mathrm{std}}$.

Proof of Theorem 1.5 Let $X^{2n} \subset T^* M_{\mathrm{std}}$ be a Weinstein subdomain and $C^{2n} := T^* M_{\mathrm{std}} \backslash X^{2n}$ the complementary Weinstein cobordism. Then we have

$$C = C_{\mathrm{sub}} \cup H_1^n \cup \cdots \cup H_k^n,$$

where all handles of C_{sub} are subcritical, ie have index less than n. The Viterbo restriction induces an equivalence $\mathrm{Tw}\, \mathcal{W}(X \cup C_{\mathrm{sub}}; \mathbb{Z}) \cong \mathrm{Tw}\, \mathcal{W}(X; \mathbb{Z})$ on the subcritical cobordism; see Corollary 1.21 of [19]. Also by Proposition 8.15 of [19],

$$\mathrm{Tw}\, \mathcal{W}(X \cup C_{\mathrm{sub}}; \mathbb{Z}) \cong \mathrm{Tw}\, \mathcal{W}(T^* M_{\mathrm{std}} \backslash (D_1 \sqcup \cdots \sqcup D_k); \mathbb{Z})$$

$$\cong \mathrm{Tw}\, \mathcal{W}(T^* M_{\mathrm{std}}; \mathbb{Z})/(D_1, \ldots, D_k),$$

where $D_1, \ldots, D_k \subset T^* M_{\mathrm{std}}$ are the Lagrangian cocores of $H_1^n, \ldots, H_k^n$; recall that this quotient category depends just on the subcategory split-generated by these disks by Remark 1.6. Now by Corollary 1.8, $D_i \cong CW^*(M, D_i) \otimes T_q^* M$ in $\mathrm{Tw}\, \mathcal{W}(T^* M; \mathbb{Z})$,

where $CW^*(M, D_i)$ is considered as an object of Tw $\mathbb{Z}$, or equivalently a cochain complex over $\mathbb{Z}$. Any cochain complex of free Abelian groups splits as a direct sum of twisted complexes of the form $\mathbb{Z}[1] \xrightarrow{m} \mathbb{Z}$ for some integer m and free groups $\mathbb{Z}$ (and their shifts); see Exercise 43 in Section 2.2 of [23]. If $CW^*(M, D_i)$ has a $\mathbb{Z}$–summand, then D_i split-generates and hence Tw $\mathcal{W}(T^*M_{\mathrm{std}}; \mathbb{Z})/(D_1, \ldots, D_k)$ is trivial. Otherwise, let $p_1, \ldots, p_j$ be the collection of primes dividing m in the summand $\mathbb{Z}[1] \xrightarrow{m} \mathbb{Z}$. Then the split-closure of $\mathbb{Z}[1] \xrightarrow{m} \mathbb{Z}$ coincides with that of the objects $\mathbb{Z}[1] \xrightarrow{p_1} \mathbb{Z}, \ldots, \mathbb{Z}[1] \xrightarrow{p_j} \mathbb{Z}$. So if P denotes the set of primes obtained this way over all $D_1, \ldots, D_k$, the split-closure of $(D_1, \ldots, D_k)$ coincides with that of $T_q^*M[1] \xrightarrow{p} T_q^*M \cong \mathrm{cone}(p \cdot \mathrm{Id}_{T_q^*M})$, where $p \in P$. Since T_q^*M generates Tw $\mathcal{W}(T^*M_{\mathrm{std}}; \mathbb{Z})$, the subcategory split-generated by $(D_1, \ldots, D_k)$ coincides with that split-generated by

$$\{\mathrm{cone}(p \cdot \mathrm{Id}_L) \mid p \in P, L \in \mathrm{Tw}\,\mathcal{W}(T^*M_{\mathrm{std}}; \mathbb{Z})\},$$

and so Tw $\mathcal{W}(X; \mathbb{Z}) \cong$ Tw $\mathcal{W}(T^*M_{\mathrm{std}}; \mathbb{Z})[1/P]$ as desired. Also, P is unique since $\mathcal{W}(X; \mathbb{F}_q)$ vanishes if $q \in P$, and Tw $\mathcal{W}(X; \mathbb{F}_q) \cong$ Tw $\mathcal{W}(T^*M_{\mathrm{std}}; \mathbb{F}_q)$ is nontrivial if $q \notin P$ and $0 \notin P$.

Finally, if $i^* \colon H^n(T^*M; \mathbb{Z}) \to H^n(X; \mathbb{Z})$ is not an isomorphism, then for some D_i, $[D_i] \in H^n(T^*M; \mathbb{Z}) \cong \mathbb{Z}$ is nonzero, so the algebraic intersection number $M \cdot D_i \in \mathbb{Z}$ is nonzero. Since this intersection number is the Euler characteristic $\chi(CW^*(M, D_i))$ of the Floer cochains $CW^*(M, D_i)$, the direct sum decomposition of $CW^*(M, D_i)$ discussed above must contain a free group $\mathbb{Z}$, implying that Tw $\mathcal{W}(X; \mathbb{Z})$ is trivial. $\square$

Remark 1.10 Abouzaid observed that Corollary 1.8, and hence Theorem 1.5, extends to the case where M has finite fundamental group and spin universal cover. Indeed, in that case any Lagrangian disk $L \subset T^*M$ lifts to a disk $\widetilde{L} \subset T^*\widetilde{M}$. Applying Corollary 1.8 to $\widetilde{L}$, we obtain an isomorphism

$$\widetilde{L} \cong K^* \otimes T_q^*\widetilde{M}$$

for some complex $K^* \in$ Tw $\mathbb{Z}$. Presenting the upstairs category $\mathcal{W}(T^*\widetilde{M})$ using pulled-back Floer data, we can push this isomorphism back down to $\mathcal{W}(T^*M)$ to conclude that

$$L \cong K^* \otimes T_q^*M.$$

The authors expect the same to hold if $\pi_1(M)$ is infinite, but that requires extending Theorem 1.7 to the noncompact case.

As we have seen, any Weinstein subdomain $X \subset T^*M_{\mathrm{std}}$ induces a localization (Viterbo) functor $\mathrm{Tw}\,\mathcal{W}(T^*M_{\mathrm{std}}; \mathbb{Z}) \to \mathrm{Tw}\,\mathcal{W}(X; \mathbb{Z})$, and hence, by Remark 1.6, is associated to a split-closed subcategory of $\mathrm{Tw}\,\mathcal{W}(T^*M_{\mathrm{std}}; \mathbb{Z})$. So Theorem 1.5 can be viewed as a classification of the split-closed subcategories of $\mathrm{Tw}\,\mathcal{W}(T^*M_{\mathrm{std}}; \mathbb{Z})$ coming from this geometric setting. The fact that these correspond to subsets of prime integers stems from the corresponding fact for $\mathrm{Tw}\,\mathbb{Z}$ (and the crucial Corollary 1.8). More generally, Hopkins and Neeman [34] proved that split-closed subcategories of $D^b\mathrm{Mod}_R$ correspond to certain subsets of $\mathrm{Spec}(R)$; in the global setting, Thomason [39] proved that split-closed subcategories of $D^b\mathrm{Coh}(X)$ that are closed under the tensor product correspond to certain closed subsets of X. Although the wrapped Fukaya category does not generally have a monoidal structure, we pose the open problem of classifying Fukaya categories of Weinstein subdomains of arbitrary Weinstein domains as a way of extending these results to the symplectic setting.

2   Proof of results

2.1   Constructing Lagrangian disks

Our construction of Weinstein subdomains of a Weinstein domain X^{2n} depends on the existence of certain Lagrangian disks near the index n cocores of X^{2n}. First we note that a neighborhood of an index n cocore C is the (unit disk) cotangent bundle T^*D^n. More precisely, we view T^*D^n as a compact sector in the sense of [20, Example 2.3], with sectorial boundary $T^*\partial D^n \times [0, 1]$, or as the stopped domain $(B^{2n}, \partial D^n)$ in the sense of [37], where B^{2n} is the standard Weinstein ball and $\partial D^n \subset \partial B^{2n}$ is the Legendrian unknot that is the boundary of Lagrangian unknot $D^n \subset B^{2n}$. The Liouville completion of the sector T^*D^n is precisely the (completion of the) stopped domain $(B^{2n}, \partial D^n)$, see [20, Section 2.8], and so we will treat these two interchangeably. A parametrized neighborhood of the cocore gives a proper inclusion of (completions of) sectors $\varphi \colon T^*D^n \to X$ mapping the cotangent fiber $T_0^*D^n$ to the cocore C. Note that the zero section $D^n \subset T^*D^n$ maps to a subset of the stable *core* of the critical point associated to C, not to the cocore C.

Next, we consider Lagrangians L in the sector T^*D^n that are properly embedded, or in the language of stops, disjoint from the stop $\partial D^n \subset \partial B^{2n}$. Then the proper inclusion φ maps such Lagrangians to Lagrangians in X with Legendrian boundary in the contact boundary ∂X of X. So to construct Lagrangians in X, it suffices to construct Lagrangians in T^*D^n. More generally, Section 3 of [20] associates to a sector the *partially* wrapped category, whose objects are properly embedded Lagrangians in the sector with Legendrian boundary (or disjoint from the stop), and morphism spaces are partially wrapped Floer cochains, which on the cohomology level is a direct limit of unwrapped Floer cohomology of a cofinal sequence of Lagrangians positively wrapped via Hamiltonians supported away from the stop. Section 3 of [20] also proves that proper inclusions of sectors induce a functor on partially wrapped categories.

In our setting, we consider the partially wrapped category $\mathcal{W}(T^*D^n, \partial D^n; \mathbb{Z})$ of T^*D^n and also its category of twisted complexes $\mathrm{Tw}\,\mathcal{W}(T^*D^n, \partial D^n; \mathbb{Z})$; we add the term ∂D^n to the notion to emphasize the existence of the stop/sectorial boundary. We use the canonical $\mathbb{Z}$–grading of T^*D^n via the Lagrangian fibration by cotangent fibers. By [19, Theorem 1.10] and [9], the category $\mathrm{Tw}\,\mathcal{W}(T^*D^n, \partial D^n)$ is generated by the cotangent fiber $T_0^*D^n \subset T^*D^n$ at the origin $0 \in D^n$. Let $D_-^n \subset T^*D^n$ be a negative perturbation of the zero section D^n, ie the result of applying the negative wrapping $-\sum q_i \partial_{p_i}$ to D^n so that ∂D_-^n is disjoint from the stop ∂D^n. Note that D_-^n is Lagrangian isotopic to $T_0^*D^n$ in the complement of ∂D^n by further applying the negative wrapping $-\sum q_i \partial_{p_i}$; therefore D_-^n and $T_0^*D^n$ are quasi-isomorphic in $\mathrm{Tw}\,\mathcal{W}(T^*D^n, \partial D^n)$. Furthermore, since D_-^n admits a positive wrapping into the stop ∂D^n, it is already cofinal with respect to the wrapping in T^*D^n and so $CW^*(D_-^n, D_-^n) = CF^*(D_-^n, D_-^n)$, the *unwrapped* Floer cochain complex; see Section 6.7 and Proposition 6.7 of [18], where D_-^n is called a *forward stopped* Lagrangian. Since D_- and its small positive pushoff intersect transversely in a single point (at the origin), the Floer cochain complex $CF(D_-^n, D_-^n)$ is just $\mathbb{Z}$. The same argument shows that if L is a Lagrangian in T^*D^n, then $CW^*(D_-^n, L) = CF^*(D_-^n, L)$; see Proposition 6.17 of [18]. Combining the generation by $T_0^*D^n$, the quasi-isomorphism between $T_0^*D^n$ and D_-^n, and the computation of $CW^*(D_-^n, D_-^n) = \mathbb{Z}$, there is a cohomologically full and faithful A_∞–functor

$$CF^*(D_-^n, \_): \mathrm{Tw}\,\mathcal{W}(T^*D^n, \partial D^n) \to \mathrm{Mod}_{\mathbb{Z}}.$$

Here $\mathrm{Mod}_{\mathbb{Z}}$ denotes the dg-category of right $\mathbb{Z}$–modules. This functor has image $\mathrm{Tw}\,\mathbb{Z}$, the category of cochain complexes whose underlying graded Abelian group is free and finitely generated, since $CF^*(D_-^n, L)$ is always of this form.

Next we review certain regular Lagrangian disks in T^*D^n introduced by Abouzaid and Seidel in [5, Section 3b] and study their isomorphism class in Tw $\mathcal{W}(T^*D^n, \partial D^n)$. Let $U \subset S^{n-1}$ be a compact codimension-zero submanifold with smooth boundary. Let $g \colon S^{n-1} \to \mathbb{R}$ be a C^1–small function so that g is strictly negative in the interior of U, zero on ∂U, strictly positive on $S^{n-1} \backslash U$, and has zero as a regular value. Next, we consider S^{n-1} as the radius $\frac{1}{2}$ sphere $S^{n-1}_{1/2}$ in the unit disk D^n and extend g to a smooth Morse function $f \colon D^n \to \mathbb{R}$ so that f is C^0–small in the $\frac{1}{2}$–radius disk and satisfies $f(tq) = |t|^2 g(q)$ for $q \in S^n_{1/2}$ and $t \geq 1$. Let $\Gamma(df)$ be the graph of df in T^*D^n and let $D_U = \Gamma(df) \cap T^*D^n$. Since f is homogeneous for $|q| \geq \frac{1}{2}$ and 0 is a regular value of g, D_U has Legendrian boundary (with respect to the standard radial Liouville vector field on B^{2n}) which is disjoint from ∂D^n. Furthermore, there is a Lagrangian isotopy $\Gamma(d(sf))$ from D_U to the zero section $D \subset T^*D^n$ (which intersects the stop ∂D precisely when $s = 0$). After fixing a grading on D, the isotopy $\Gamma(d(sf))$ induces a preferred grading on D_U. In particular, D_U with this $\mathbb{Z}$–grading is an object of Tw $\mathcal{W}(T^*D^n, \partial D^n)$.

We now compute the isomorphism class of D_U in Tw $\mathcal{W}(T^*D^n, \partial D^n)$, following [5]. Namely, as noted in Lemma 3.3 of [5], we can scale f so that the intersection points of D^n_U and D^n_- have small action, and then Floer trajectories are in bijection with trajectories of gradient flow of the function f; so $CF(D_-, D_U)$ is quasi-isomorphic to Morse cochains of $f \colon D^n \to \mathbb{R}$. Since D^n is contractible, this is quasi-isomorphic to $\widetilde{C}^{*-1}(U)$, reduced Morse cochains on U. Hence, under the equivalence $CF^*(D^n_-, \_)$ between Tw $\mathcal{W}(T^*D^n, \partial D; \mathbb{Z})$ and Tw $\mathbb{Z}$, the image of the disk D_U in $\mathrm{Mod}_\mathbb{Z}$ is quasi-isomorphic to $\widetilde{C}^{*-1}(U)$. Note that since $CF^*(T_0^* D^n, T_0^* D^n) \cong \mathbb{Z}$, the image of the twisted complex $\widetilde{C}^{*-1}(U) \otimes D^n_-$ under the functor $CF^*(D^n_-, \_)$ is also $\widetilde{C}^{*-1}(U)$. Since the $CF^*(D^n_-, \_)$ functor is cohomologically full and faithful, the disk D_U is quasi-isomorphic to the twisted complex $\widetilde{C}^{*-1}(U) \otimes D_- \cong \widetilde{C}^{*-1}(U) \otimes T_0^* D^n$ in Tw $\mathcal{W}(T^*D^n, \partial D; \mathbb{Z})$.

Remark 2.1 Our definition of the disk D_U agrees with that in Abouzaid and Seidel [5]. However, they make an inconsequential misidentification of the Floer complex with the Morse complex to obtain $\widetilde{C}^{*-1}(U)$ as $CF^*(D_U, D)$ instead of $CF^*(D, D_U)$.

Using the disks D_U, we now show that for sufficiently large n any Lagrangian in T^*D^n (or twisted complex of Lagrangians) is quasi-isomorphic to a Lagrangian disk. Note that this is stronger than the statement that any Lagrangian is a twisted complex of disks, which follows from the fact that $T_0^* D^n$ generates Tw $\mathcal{W}(T^*D^n, \partial D^n; \mathbb{Z})$.

Proposition 2.2 *If $n \geq 5$, every object of* $\mathrm{Tw}\,\mathcal{W}(T^*D^n, \partial D^n; \mathbb{Z})$ *is quasi-isomorphic to an exact Lagrangian disk. In particular,*

$$\mathcal{W}(T^*D^n, \partial D^n; \mathbb{Z}) \to \mathrm{Tw}\,\mathcal{W}(T^*D^n, \partial D^n; \mathbb{Z})$$

is a quasi-equivalence.

Proof An arbitrary object of $\mathrm{Tw}\,\mathcal{W}(T^*D^n, \partial D^n; \mathbb{Z})$ can be identified with some finite-dimensional cochain complex of free Abelian groups via the quasi-equivalence $CW^*(T_0^*D^n, \_)$. Every such cochain complex (C^*, ∂) splits as a direct sum of twisted complexes of the form $\mathbb{Z}[d+1] \xrightarrow{m} \mathbb{Z}[d]$ for some integer m, or complexes $\mathbb{Z}[d]$ with no differential. To see this, we use the fact that the short exact sequence

$$0 \to \ker\,\partial_n \to C_n \to \mathrm{im}\partial_n \to 0$$

splits since $\mathrm{im}\partial_n$ is free; see Exercise 43 in [23, Section 2.2].

Next, we recall that given two exact Lagrangians $L, K \subset X$ and a framed isotropic arc between their Legendrian boundaries $\partial L, \partial K \subset \partial X$, one can form a new exact Lagrangian $L \natural K \subset X$, the isotropic boundary connected sum of L and K. If L and K are $\mathbb{Z}$–graded Lagrangians, then there is a choice of framing for the isotropic arc (the space of such choices up to homotopy is a $\mathbb{Z}$–torsor) so that $L \natural K$ also has a $\mathbb{Z}$–grading that restricts to the $\mathbb{Z}$–grading of L and K and hence $L \natural K$ is quasi-isomorphic to $L \sqcup K \cong L \oplus K$ in $\mathrm{Tw}\,\mathcal{W}(X)$. This follows from Proposition 1.29 of [19] since for an isotropic connected sum the primitive of the Liouville form is zero on both components $\partial L \sqcup \partial K$; note that this is false for the more general connected sums along short Reeb chords considered in Proposition 1.30 of [19]. Whenever we discuss the isotropic connected sum of two Lagrangians, we mean the sum using any isotropic arc with this framing. The actual geometric disk will depend on the homotopy class of the arc, but since we are only concerned with the resulting object of $\mathcal{W}(X)$ we will ignore the distinction.

Returning to $X = T^*D^n$, we can assume that any two Lagrangians $L, K \subset T^*D^n$ are disjoint since we can view T^*D^n as the result of gluing two copies of T^*D^n together, and place L in one copy and K in the other copy. So in light of the above discussion and the splitting from the previous paragraph, it suffices to prove that the twisted complexes $\mathbb{Z}[d+1] \xrightarrow{m} \mathbb{Z}[d]$ and $\mathbb{Z}[d]$ are quasi-isomorphic to embedded Lagrangian disks. The latter complex is quasi-isomorphic to $T_0^*D^n$ with the appropriate grading, so it suffices to prove that $\mathbb{Z}[d+1] \xrightarrow{m} \mathbb{Z}[d]$ is quasi-isomorphic to a disk.

As noted in Abouzaid and Seidel [6], for $n \geq 5$ and any $m \geq 0$ there is a codimension-0 Moore space $U_m \subset S^{n-1}$ with $\widetilde{C}^*(U_m) \cong \mathbb{Z}[-1] \xrightarrow{m} \mathbb{Z}[-2]$. For example, consider the CW complex V obtained by attaching D^2 to S^1 along a degree m map $S^1 \to S^1$; then V embeds into S^{n-1} for $n \geq 6$ by the Whitney trick, and for $n = 5$ by the explicit map $D^2 \to \mathbb{C}^2$ given by $z \to ((1 - |z|^2)z, z^m)$. Let U_m be a neighborhood of V in S^{n-1}. Then D_{U_m} is quasi-isomorphic to $\mathbb{Z}[-2] \xrightarrow{m} \mathbb{Z}[-3]$. Finally, we shift the grading on D_{U_m} by $d + 3$ and the resulting disk $D_{U_m}[d+3]$ is quasi-isomorphic to $\mathbb{Z}[d+1] \xrightarrow{m} \mathbb{Z}[d]$, as desired. $\qquad\square$

We observe that not every object of $\operatorname{Tw} \mathcal{W}(T^*D^n, \partial D^n; \mathbb{Z})$ is quasi-isomorphic to a disk D_U. This is because $CW^*(D_-, D_U)$ is a cochain complex that is supported between degrees 0 and $n-1$ (since $U \subset S^{n-1}$) or a shift thereof (if we shift the grading on D_U^n), while a general cochain complex can have arbitrarily wide support. However, Proposition 2.2 shows that every object of $\operatorname{Tw} \mathcal{W}(T^*D^n, \partial D^n; \mathbb{Z})$ is quasi-isomorphic to the boundary connected sum of possibly several different D_U, with possibly different gradings.

2.2 Constructing subdomains

Now we use the Lagrangian disks from the previous section to construct Weinstein subdomains of a Weinstein domain X and prove Theorem 1.1. As stated in Remark 1.3, the construction of subdomains also holds when the ambient Weinstein domains has stops. The most important case for us is when $X = (T^*D^n, \partial D^n)$, the stopped domain considered in the previous section. As we will see, Theorem 1.1 for arbitrary Weinstein domains follows from this case.

In the following, we say a stopped Weinstein domain (X_0, Λ_0) is a Weinstein subdomain of (X, Λ) if $X = X_0 \cup C$ for some Weinstein cobordism C which is trivial along $\Lambda_0 = \Lambda$. In particular, there is a smoothly trivial regular Lagrangian cobordism between Λ_0 and Λ_1 in $X \setminus X_0$ which allows us to identify the linking disk of Λ_0 in X_0 with the linking disk of Λ in X. We say that this cobordism is flexible if the attaching spheres of the index n handles are loose in the complement of Λ_0. We also say that two Weinstein subdomains $X_0, X_1 \subset X$ are Weinstein homotopic if the following holds: there is a homotopy of Weinstein Morse functions f_t with $0 \leq t \leq 1$, possibly with birth–death singularities at isolated moments, on X that have c as a regular level set for all t and such that X_0 and X_1 are the c–sublevel sets of f_0 and f_1, respectively.

Theorem 2.3 *Let $n \geq 5$. For any finite collection of prime numbers P, which is possibly empty or contains 0, there is a Legendrian sphere $\Lambda_P \subset \partial B^{2n}_{\mathrm{std}}$ formally isotopic to the standard unknot $\Lambda_\varnothing$ so that $(B^{2n}_{\mathrm{std}}, \Lambda_P)$ embeds as a Weinstein subdomain of $(B^{2n}_{\mathrm{std}}, \Lambda_\varnothing)$, which has associated sector $(T^*D^n, \partial D^n)$, with the following properties:*

(1) *The Viterbo restriction functor*

$$\mathrm{Tw}\,\mathcal{W}(B^{2n}_{\mathrm{std}}, \Lambda_\varnothing; \mathbb{Z}) \to \mathrm{Tw}\,\mathcal{W}(B^{2n}_{\mathrm{std}}, \Lambda_P; \mathbb{Z})$$

induces an equivalence

$$\mathrm{Tw}\,\mathcal{W}(B^{2n}_{\mathrm{std}}, \Lambda_P; \mathbb{Z}) \cong \mathrm{Tw}\,\mathcal{W}(B^{2n}_{\mathrm{std}}, \Lambda_\varnothing; \mathbb{Z})\left[\frac{1}{P}\right] \cong \mathrm{Tw}\,\mathbb{Z}\left[\frac{1}{P}\right].$$

(2) *$(B^{2n}_{\mathrm{std}}, \Lambda_P)$ embeds as a Weinstein subdomain of $(B^{2n}_{\mathrm{std}}, \Lambda_Q)$ if and only if $Q \subset P$ or $0 \in P$. In such cases, we can construct such an embedding with the property that the Weinstein cobordism between unstopped domains is trivial, ie $\partial B^{2n} \times [0, 1]$, and if $R \subset Q \subset P$, the composition*

$$(B^{2n}_{\mathrm{std}}, \Lambda_P) \subset (B^{2n}_{\mathrm{std}}, \Lambda_Q) \subset (B^{2n}_{\mathrm{std}}, \Lambda_R)$$

is Weinstein homotopic to $(B^{2n}_{\mathrm{std}}, \Lambda_P) \subset (B^{2n}_{\mathrm{std}}, \Lambda_R)$ obtained by viewing $P \subset R$.

(3) *There is a smoothly trivial regular Lagrangian cobordism $L \subset \partial B^{2n}_{\mathrm{std}} \times [0, 1]$ with $\partial_- L = \Lambda_P$ and $\partial_+ L = \Lambda_Q$ if and only if $Q \subset P$ or $0 \in P$. Furthermore, for two disjoint subsets of primes P_1 and P_2, the Legendrian sphere $\Lambda_{P_1 \sqcup P_2}$ is the isotropic connected sum $\Lambda_{P_1} \natural \Lambda_{P_2}$ of Λ_{P_1} and Λ_{P_2} embedded in disjoint Darboux balls in $\partial B^{2n}_{\mathrm{std}}$.*

(4) *If $0 \in P$, then $\Lambda_P \subset \partial B^{2n}_{\mathrm{std}}$ is loose.*

In particular, we have a sequence of Legendrians

$$\Lambda_{\mathrm{unknot}} = \Lambda_\varnothing, \Lambda_2, \Lambda_{2,3}, \Lambda_{2,3,5}, \Lambda_{2,3,5,7}, \dots, \Lambda_0 = \Lambda_{\mathrm{loose}}$$

in $\partial B^{2n}_{\mathrm{std}}$, and Lagrangian cobordisms in $\partial B^{2n}_{\mathrm{std}} \times [0, 1]$ connecting consecutive Legendrians interpolating between $\Lambda_{\mathrm{unknot}}$ and Λ_{loose}, analogous to the sequence of subdomains in Theorem 1.1. We note that such Legendrians do not exist for $n = 2$ as proven in [11]: if L^2 is a *decomposable* Lagrangian cobordism (a condition similar to regularity) with negative end Λ and positive end $\Lambda_\varnothing$, then either $\Lambda = \Lambda_\varnothing$ or Λ is stabilized in the sense of [33], so $\mathrm{Tw}\,\mathcal{W}(B^4_{\mathrm{std}}, \Lambda; \mathbb{Z}) \cong \mathrm{Tw}\,\mathbb{Z}$ or $\mathrm{Tw}\,\mathcal{W}(B^4_{\mathrm{std}}, \Lambda; \mathbb{Z}) \cong 0$ are the only possibilities.

Remark 2.4 The construction of the isotropic connected sum $\Lambda_1 \natural \Lambda_2$ of two Legendrians Λ_1 and Λ_2 in the statement of Theorem 2.3 above is similar to the boundary connected sum of two Lagrangians discussed in the proof of Proposition 2.2 (the former happens on the boundary of the latter) and also depends on a framed isotropic arc between Λ_1 and Λ_2. However if Λ_2 is contained in a Darboux chart of (Y, ξ) disjoint from Λ_1, then the isotropic connected sum $\Lambda_1 \natural \Lambda_2$ is actually independent of the isotropic arc and its framing: this is because we can isotope Λ_2 to a small neighborhood of Λ_1 via the original isotropic arc and then isotope it back to its original position using the new isotropic arc.

More precisely, we can identify the Darboux chart containing Λ_2 with the cotangent bundle of the frame-thickening of the isotropic arc and use this to produce a family of Darboux charts. If the two arcs have the same framing at the endpoints, the resulting family of Darboux charts is a loop, which means that Λ_2 returns to itself.

Proof of Theorem 2.3 We prove this theorem in several stages: first we construct Λ_p when p is a single prime and prove that it has the claimed geometric properties, then we construct Λ_P for a general set of primes P, and finally we prove our claims about the Fukaya category of $(B_{\mathrm{std}}^{2n}, \Lambda_P)$.

2.2.1 Λ_p for a single prime p

We first consider the case when the collection of primes P consists of a single prime p. As discussed in the previous section, let $U_p \subset S^{n-1}$ be a fixed p–Moore space. Then the Lagrangian disk $D_p := D_{U_p} \subset (T^* D^n, \partial D^n)$ is isomorphic to $T_0^* D^n[1] \xrightarrow{p} T_0^* D^n$ in $\mathrm{Tw}\,\mathcal{W}(T^* D^n, \partial D^n; \mathbb{Z})$. Also, if $p = 0$, we set $U = S^{n-1}$ (as a full subset of S^{n-1}) and form $D_0 := D_{S^{n-1}}^n$, which is Lagrangian isotopic in $(T^* D^n, \partial D^n)$ to the cotangent fiber $T_0^* D^n$. If p is the empty set, we set U to be a ball $B^{n-1} \subset S^{n-1}$ and form $D_\varnothing := D_{B^{n-1}}^n$, which is a small Lagrangian disk that is disjoint from the zero section $D^n \subset T^* D^n$; note that any two such small Lagrangian disks are isotopic in $(T^* D^n, \partial D^n)$. In particular, $D_\varnothing$ is the zero object in $\mathrm{Tw}\,\mathcal{W}(T^* D^n, \partial D^n; \mathbb{Z})$. To construct $(B_{\mathrm{std}}^{2n}, \Lambda_P)$, we will *carve out* these Lagrangian disks, as we now explain.

In general, given a Liouville domain X^{2n} and an exact Lagrangian disk $D^n \subset X^{2n}$ with Legendrian boundary, there is a Liouville subdomain $X_0 \subset X$ (which we say is obtained by *carving out* D^n from X) and a Legendrian sphere $\Lambda \subset \partial X_0$ so that $X = X_0 \cup H_\Lambda^n$ and the cocore of H_Λ^n is D^n; see [14] for details. If X is a Weinstein domain and $D^n \subset X$ is a regular Lagrangian, then $X_0 \subset X$ is a Weinstein subdomain.

The disks $D_p \subset T^*D^n$ we consider are indeed regular; in fact $D_p = \Gamma(df)$ is isotopic through Lagrangians with Legendrian boundary $(D_p)_s = \Gamma(sdf) \cap T^*D^n$ to the zero section $D^n \subset T^*D^n$. Therefore, $T^*D^n \backslash D_p$ is homotopic to the Weinstein domain $T^*D^n \backslash D^n$, which is actually the subcritical domain $T^*(S^{n-1} \times D^1) = B_{\text{std}}^{2n} \cup H^{n-1}$. Since D_p is disjoint from ∂D^n, we can consider $(T^*D^n, \partial D^n) \backslash D_p$ as $T^*(S^{n-1} \times D^1)$ with some stop, namely the image of ∂D^n.

Since the subdomain $T^*D^n \backslash D_p$ is obtained by carving out D_p, there is a Legendrian $\Lambda \subset \partial(T^*D^n \backslash D_p)$ disjoint from ∂D^n so that $(T^*D^n, \partial D^n) \backslash D_p \cup H_\Lambda^n = (T^*D^n, \partial D^n)$ and the cocore of H_Λ^n is D_p. Because $n \geq 5$, there is a unique loose Legendrian $\Lambda_{\text{loose}} \subset \partial(T^*D^n \backslash D_p)$ that is formally isotopic to Λ and is loose in the complement of ∂D^n; see [33]. Next we form the stopped domain $(T^*D^n, \partial D^n) \backslash D_p \cup H_{\text{flex}}^n$ by attaching the handle H_{flex}^n along Λ_{loose}. We note that the ambient Weinstein domain $T^*D^n \backslash D_p \cup H_{\text{flex}}^n$ is flexible since $T^*D^n \backslash D_p$ is subcritical and H_{flex}^n is attached along a loose Legendrian. Furthermore, it is formally symplectomorphic to the standard Weinstein ball since Λ_{loose} is formally isotopic to Λ (and attaching a handle to Λ reproduces B_{std}^{2n}). Therefore by the h-principle for flexible Weinstein domains [10, Theorem 14.3], $T^*D^n \backslash D_p \cup H_{\text{flex}}^n$ is Weinstein homotopic to B_{std}^{2n}. Under this identification with B_{std}^{2n}, the stop $\partial D^n \subset T^*D^n \backslash D_p \cup H_{\text{flex}}^n$ becomes some Legendrian in $\partial B_{\text{std}}^{2n}$, which we call Λ_p. That is, we set

$$(B_{\text{std}}^{2n}, \Lambda_p) := (T^*D^n, \partial D^n) \backslash D_p \cup H_{\text{flex}}^n.$$

We will show that $(B_{\text{std}}^{2n}, \Lambda_p)$ satisfies the claimed properties.

First we show that $(B_{\text{std}}^{2n}, \Lambda_p)$ is a Weinstein subdomain of $(T^*D^n, \partial D^n)$. Note that $(B_{\text{std}}^{2n}, \Lambda_\varnothing)$ is precisely $(T^*D^n, \partial D^n)$. This is because $(T^*D^n, \partial D^n) \backslash D_\varnothing = (T^*D^n, \partial D^n) \cup H^{n-1}$ and the Legendrian Λ from the previous paragraph intersects the belt sphere of H^{n-1} exactly once; so $H^{n-1} \cup H_{\text{flex}}^n$ are canceling handles and hence

$$(T^*D^n, \partial D^n) \backslash D_\varnothing \cup H_{\text{flex}}^n = (T^*D^n, \partial D^n) \cup H^{n-1} \cup H_{\text{flex}}^n = (T^*D^n, \partial D^n).$$

Now we consider the case when p is a (nonzero) prime. Clearly $(T^*D^n, \partial D^n) \backslash D_p$ is a subdomain of $(T^*D^n, \partial D^n)$ by construction; we claim that it is still a subdomain even after attaching the flexible handle H_{flex}^n to $(T^*D^n, \partial D^n) \backslash D_p$. To see this, let C be the Weinstein cobordism between $(T^*D^n, \partial D^n) \backslash D_p$ and $(T^*D^n, \partial D^n)$ given by the handle H_Λ^n (whose cocore is D_p). By [29], we can Weinstein homotope C, in the complement of ∂D^n, to a Weinstein cobordism $H_{\text{flex}}^n \cup H^{n-1} \cup H_{\Lambda'}^n$, where H_{flex}^n is attached along Λ_{loose} and $H^{n-1} \cup H_{\Lambda'}^n$ is a smoothly trivial Weinstein cobordism

whose attaching spheres are disjoint from ∂D^n. So, up to Weinstein homotopy, we have the equalities

$$(2\text{-}1) \qquad (B_{\text{std}}^{2n}, \Lambda_p) \cup H^{n-1} \cup H_{\Lambda'}^n = (T^* D^n, \partial D^n) \setminus D_p \cup H_{\text{flex}} \cup H^{n-1} \cup H_{\Lambda'}^n,$$

$$(2\text{-}2) \qquad\qquad\qquad\qquad\quad = (T^* D^n, \partial D^n) \setminus D_p \cup C = (T^* D^n, \partial D^n),$$

which show that $(B_{\text{std}}^{2n}, \Lambda_p)$ is a subdomain of $(B_{\text{std}}^{2n}, \Lambda_\varnothing) = (T^* D^n, \partial D^n)$. Furthermore, the construction in [29] shows that Λ' is loose (but not in the complement of Λ_{loose} or ∂D^n) since Λ is loose (but not in the complement of ∂D^n). So the Weinstein cobordism $H^{n-1} \cup H_{\Lambda'}^n$ is flexible (but not in the complement of the stop ∂D^n) and therefore is homotopic to $\partial B_{\text{std}}^{2n} \times [0, 1]$.

The attaching spheres of $H^{n-1}, H_{\Lambda'}^n$ are disjoint from ∂D^n, so we can view $\partial D^n \times [0, 1]$ as a trivial Lagrangian cobordism between ∂D^n in $T^* D^n \setminus D_p \cup H_{\text{flex}}^n$ and ∂D^n in $T^* D^n$. Under our identifications, this produces a smoothly trivial regular Lagrangian cobordism (regular in that the Liouville vector field can be made tangent to it) between Λ_p and $\Lambda_\varnothing$ in $\partial B_{\text{std}}^{2n} \times [0, 1]$, as desired. We also observe that Λ_p is formally Legendrian isotopic to $\Lambda_\varnothing$ in $\partial B_{\text{std}}^{2n}$ because the attaching spheres Λ and Λ_{loose} are formally Legendrian isotopic in the complement of ∂D^n. More precisely, note that $\partial D^n \sqcup \Lambda$ and $\partial D^n \sqcup \Lambda_{\text{loose}}$ are formally isotopic Legendrian links. Furthermore, there is a genuine Legendrian isotopy from Λ to Λ_{loose} (but not in the complement of ∂D^n) and so this extends to a Legendrian isotopy from $\partial D^n \sqcup \Lambda_{\text{loose}}$ to $\overline{\partial D^n} \sqcup \Lambda$, where $\overline{\partial D^n}$ is some other Legendrian that becomes Λ_p after handle attachment to Λ. Since a genuine Legendrian isotopy preserves formal Legendrian isotopies, $\partial D^n \sqcup \Lambda$ and $\overline{\partial D^n} \sqcup \Lambda$ are also formally Legendrian isotopic links. So when we attach a handle to Λ to get B_{std}^{2n}, ∂D^n and $\overline{\partial D^n}$ are still formally Legendrian isotopic in $\partial B_{\text{std}}^{2n}$, which is precisely the statement that $\Lambda_\varnothing$ and Λ_p are formally Legendrian isotopic.

Next we consider the case when $p = 0$. Recall that in this case, D_0 is the cotangent fiber $T_0^* D^n \subset T^* D^n$. Then $(T^* D^n, \partial D^n) \setminus D_p$ is $(T^* (S^{n-1} \times D^1), S^{n-1} \times \{0\})$. We note that $S^{n-1} \times \{0\} \subset \partial T^* (S^{n-1} \times D^1)$ is loose since this Legendrian crosses the belt sphere of the index $n-1$ handle (corresponding to the index $n-1$ Morse critical point of S^{n-1}) exactly once; see [10, Lemma 14.12, Type IIb] for this looseness criterion. To construct $(B_{\text{std}}^{2n}, \Lambda_0)$ from $(T^* D^n, \partial D^n) \setminus D_p = (T^* (S^{n-1} \times D^1), S^{n-1} \times \{0\})$, we attach an index n handle H_{flex}^n along the Legendrian Λ_{loose}, which is loose in the complement of $\partial D^n = S^{n-1} \times \{0\}$ (and is formally isotopic to the Legendrian Λ). Since $\partial D^n = S^{n-1} \times \{0\}$ is loose and Λ_{loose} is loose in the complement of ∂D^n, it follows that $\partial D^n = S^{n-1} \times \{0\}$ is in fact also loose in the complement of Λ_{loose}, ie $\partial D^n = S^{n-1} \times \{0\}$ and

Λ_{loose} form a loose link: to see this, we use the h-principle to Legendrian isotope Λ_{loose} to be disjoint from the loose chart of ∂D^n (which is a contractible disk), and by extending this Legendrian isotopy to an ambient contactomorphism, we see that ∂D^n must have a loose chart disjoint from Λ_{loose}. In particular, the loose chart of ∂D^n persists under attaching the handle H_{flex}^n along Λ_{loose} and so $\partial D^n \subset (T^*D^n, \partial D^n) \setminus D_p \cup H_{\text{flex}}^n$ is still loose. By definition, this means that the stop Λ_0 in $(B_{\text{std}}^{2n}, \Lambda_0)$ is loose, as desired. As in the previous paragraph, $(B_{\text{std}}^{2n}, \Lambda_0)$ is a Weinstein subdomain of $(B_{\text{std}}^{2n}, \Lambda_\varnothing)$. Hence there is a regular Lagrangian cobordism from a loose Legendrian to the Legendrian unknot, as originally proven in [15; 27].

This proves all of claims (2), (3) and (4) when P consists of a single element.

2.2.2 Λ_P for a collection of primes P Now we construct the Legendrian sphere Λ_P when $P = \{p_1, \ldots, p_k\}$ is a collection of primes with multiple elements. We consider disjoint Weinstein balls $B_{\text{std},i}^{2n}$ so that $\Lambda_{p_i} \subset \partial B_{\text{std},i}^{2n}$ and do a *simultaneous* boundary connected sum to the $B_{\text{std},i}^{2n}$ and Λ_i, as in the construction of regular Lagrangians [14]:

$$(B_{\text{std}}^{2n}, \Lambda_P) := (B_1^{2n}, \Lambda_{p_1}) \natural \cdots \natural (B_k^{2n}, \Lambda_{p_k}).$$

Namely, we attach index 1 Weinstein handles to the disjoint union of Weinstein balls

$$B_{\text{std},1}^{2n} \sqcup \cdots \sqcup B_{\text{std},k}^{2n}$$

so that the attaching spheres of these index 1 handles, ie two points, are on different Λ_i; we simultaneously do Legendrian surgery on the Λ_i via isotropic arcs in the 1–handles. The resulting Legendrian Λ_P is connected and in fact coincides with the usual isotropic connected sum of Legendrians $\Lambda_{p_1}, \ldots, \Lambda_{p_k}$ embedded in disjoint Darboux balls in a *single* $\partial B_{\text{std}}^{2n}$. This also shows that up to Legendrian isotopy, Λ_P does not depend on the order of the set P.

Next we show that $(B_{\text{std}}^{2n}, \Lambda_P)$ is a Weinstein subdomain of $(B_{\text{std}}^{2n}, \Lambda_Q)$ if $Q \subset P$. Via our previous identification, $(B_{\text{std}}^{2n}, \Lambda_P)$ is the same as

$$(2\text{-}3) \qquad ((T^*D^n, \partial D^n) \setminus D_{p_1} \cup H_{\text{flex}}^n) \natural \cdots \natural ((T^*D^n, \partial D^n) \setminus D_{p_k} \cup H_{\text{flex}}^n),$$

where we choose points on each ∂D^n to do the simultaneous boundary connected sum. So if $Q \subset P$, $(B_{\text{std}}^{2n}, \Lambda_P)$ differs from $(B_{\text{std}}^{2n}, \Lambda_Q)$ by a boundary connected sum with

$$(T^*D^n, \partial D^n) \setminus D_p \cup H_{\text{flex}}^n$$

for all $p \in P \setminus Q$. We saw previously that $(T^*D^n, \partial D^n) \setminus D_p \cup H_{\text{flex}}^n$ is a subdomain of $(T^*D^n, \partial D^n)$, and hence $(B_{\text{std}}^{2n}, \Lambda_P)$ is a subdomain of $(B_{\text{std}}^{2n}, \Lambda_Q)$ boundary connected sum with several copies of $(T^*D^n, \partial D^n)$, one for each $p \in P \setminus Q$. Since

doing boundary connected sum with $(T^*D^n, \partial D^n)$ does not change the Weinstein homotopy type, the latter domain is still $(B^{2n}_{\text{std}}, \Lambda_Q)$ and so $(B^{2n}_{\text{std}}, \Lambda_P)$ is a subdomain of $(B^{2n}_{\text{std}}, \Lambda_Q)$, as desired. This also shows that if $R \subset Q \subset P$, the Weinstein cobordism $(B^{2n}_{\text{std}}, \Lambda_R) \setminus (B^{2n}_{\text{std}}, \Lambda_P)$ is homotopic to the concatenation of Weinstein cobordisms $(B^{2n}_{\text{std}}, \Lambda_Q) \setminus (B^{2n}_{\text{std}}, \Lambda_P)$ and $(B^{2n}_{\text{std}}, \Lambda_R) \setminus (B^{2n}_{\text{std}}, \Lambda_Q)$. Since $(B^{2n}_{\text{std}}, \Lambda_P)$ is a subdomain of $(B^{2n}_{\text{std}}, \Lambda_Q)$, we have a Lagrangian cobordism in $\partial B^{2n}_{\text{std}} \times [0, 1]$ with negative boundary Λ_P and positive boundary Λ_Q by definition.

If $0 \in P$, then Λ_P is loose since it is the isotropic connected sum of $\Lambda_{P \setminus 0}$ and Λ_0, which we already saw to be loose. Let Q be another set of primes. Then Λ_P and $\Lambda_{P \cup Q}$ are both loose unknots (since $P \cup Q$ contains 0) and so Λ_P and $\Lambda_{P \cup Q}$ are Legendrian isotopic by the h-principle for loose Legendrians [33]. By the previous discussion, this implies that $(B^{2n}_{\text{std}}, \Lambda_P) = (B^{2n}_{\text{std}}, \Lambda_{P \cup Q})$ is a subdomain of $(B^{2n}_{\text{std}}, \Lambda_Q)$, since now $Q \subset P \cup Q$.

This proves all of claims (2), (3) and (4), except the "only if" part of claims (2) and (3).

2.2.3 Fukaya category of $(B^{2n}_{\text{std}}, \Lambda_P)$

Finally, we compute the partially wrapped Fukaya category of $(B^{2n}_{\text{std}}, \Lambda_P)$. By the description in equation (2-3), $(B^{2n}_{\text{std}}, \Lambda_P)$ is the result of carving out the disks $D_{p_1}, \ldots, D_{p_k}$ from $(B^{2n}_{\text{std}}, \Lambda_\varnothing) = (T^*D^n, \partial D^n)$ and then attaching some flexible handles; here the disks are embedded disjointly by viewing $(T^*D^n, \partial D^n)$ as the boundary connected sum of several disjoint copies of $(T^*D^n, \partial D^n)$. By [19, Proposition 8.15] and [36], there is a geometrically defined Viterbo transfer functor

$$\operatorname{Tw} \mathcal{W}(T^*D^n, \partial D) \to \operatorname{Tw} \mathcal{W}((T^*D^n, \partial D) \setminus D_p),$$

which is localization by D_p. That is,

$$\operatorname{Tw} \mathcal{W}((T^*D^n, \partial D) \setminus D_p) \cong \operatorname{Tw} \mathcal{W}(T^*D^n, \partial D)/D_p,$$

and the Viterbo functor is the algebraic localization by the object D_p. By construction, the Lagrangian D_p of $\operatorname{Tw} \mathcal{W}(T^*D^n, \partial D^n; \mathbb{Z})$ is isomorphic to the twisted complex

$$T_0^* D^n[1] \xrightarrow{p} T_0^* D^n = \operatorname{cone}(p \cdot \operatorname{Id}_{T_0^* D^n}).$$

So

$$\operatorname{Tw} \mathcal{W}((T^*D^n, \partial D) \setminus D_p) \cong \operatorname{Tw} \mathcal{W}(T^*D^n, \partial D)/\operatorname{cone}(p \cdot \operatorname{Id}_{T_0^* D^n}).$$

Furthermore, the localization by a collection of objects depends only on the split-closure of that collection of objects. Since T^*D_0 generates $\operatorname{Tw} \mathcal{W}(T^*D^n, \partial D^n)$, we have

the equivalence

$$\mathrm{Tw}\, \mathcal{W}(T^*D^n, \partial D)/\mathrm{cone}(p \cdot \mathrm{Id}_{T_0^*D^n})$$
$$\cong \mathrm{Tw}\, \mathcal{W}(T^*D^n, \partial D)/\{\mathrm{cone}(p \cdot \mathrm{Id}_L) \mid L \in \mathrm{Tw}\, \mathcal{W}(T^*D^n, \partial D^n)\}$$
$$=: \mathrm{Tw}\, \mathcal{W}(T^*D^n, \partial D; \mathbb{Z})\Big[\tfrac{1}{p}\Big].$$

Combining with the previous equivalence, we have

$$(2\text{-}4) \qquad \mathrm{Tw}\, \mathcal{W}((T^*D^n, \partial D)\backslash D_p; \mathbb{Z}) \cong \mathrm{Tw}\, \mathcal{W}(T^*D^n, \partial D; \mathbb{Z})\Big[\tfrac{1}{p}\Big].$$

Similarly, when we carve out multiple disks $D_{p_1}, \ldots, D_{p_k}$, we invert $p_1, \ldots, p_k$ in the Fukaya category. Attaching flexible handles does not affect the Fukaya category. To see this, suppose $X = X_0 \cup H^n_{\mathrm{flex}}$ and $C \subset X$ is the cocore of the flexible handle. Then the dga of wrapped Floer cochains of C is isomorphic to the Cieliebak–Eliashberg dga of the attaching sphere by [12], and this dga vanishes for loose Legendrians by Proposition 4.8 of [13]. So the cocore C is quasi-isomorphic to the trivial object in the Fukaya category of X; then by the localization result Proposition 8.15 of [19], we have $\mathrm{Tw}\, \mathcal{W}(X_0) \cong \mathrm{Tw}\, \mathcal{W}(X)/C$, and the latter is quasi-equivalent to $\mathrm{Tw}\, \mathcal{W}(X)$ since localizing by the trivial object does not affect the category. Therefore,

$$(2\text{-}5) \qquad \mathrm{Tw}\, \mathcal{W}(B^{2n}_{\mathrm{std}}, \Lambda_P; \mathbb{Z}) \cong \mathrm{Tw}\, \mathcal{W}(T^*D^n, \partial D^n; \mathbb{Z})\Big[\tfrac{1}{P}\Big],$$

as desired. If p is zero, then $D_p = T_0^*D^n$ and $\mathrm{Tw}\, \mathcal{W}(T^*D^n, \partial D)/T_0^*D^n \cong 0$, which is indeed the case for $(B^{2n}_{\mathrm{std}}, \Lambda_0)$ since Λ_0 is loose.

Remark 2.5 The above discussion does not automatically show that the equivalence in equation (2-5) is given by the Viterbo functor induced by the Weinstein embedding of $(B^{2n}_{\mathrm{std}}, \Lambda_P)$ into $(B^{2n}_{\mathrm{std}}, \Lambda_\varnothing) = (T^*D^n, \partial D^n)$, due to the presence of the extra flexible handles. However, this is indeed the case. Recall that the Weinstein cobordism between these two domains is $H^{n-1} \cup H^n_{\Lambda'}$, which comes from a construction in [29; 26]. The proof there shows that the cocore of $H^n_{\Lambda'}$ is $D_p \natural \overline{D}_p \subset (T^*D^n, \partial D^n)$ and so the Viterbo functor between these two domains is localization by $D_p \natural \overline{D}_p$. Now $D_p \natural \overline{D}_p \cong D_p \oplus D_p[1]$ and D_p have the same split-closure, so localization by $D_p \natural \overline{D}_p$ is the same as localization by D_p, as in equation (2-5).

Finally, we prove the "only if" part of claims (2) and (3). Suppose that $(B^{2n}_{\mathrm{std}}, \Lambda_P)$ is a Weinstein subdomain of $(B^{2n}_{\mathrm{std}}, \Lambda_Q)$ but $Q \not\subset P$ and $0 \notin P$. There would be a localization functor from the Fukaya category of $(B^{2n}_{\mathrm{std}}, \Lambda_Q)$ to that of $(B^{2n}_{\mathrm{std}}, \Lambda_P)$

over any coefficient ring R. However, if we take $R = \mathbb{F}_q$ for any $q \in Q \setminus P$, we have $D_q \cong \mathrm{cone}(0_{T_0^* D^n}) \cong T_0^* D^n[1] \oplus T_0^* D^n$ in $\mathrm{Tw}\,\mathcal{W}(B_{\mathrm{std}}^{2n}, \Lambda_\varnothing; \mathbb{F}_q)$ since $q \equiv 0$ in $\mathbb{F}_q$. This object split-generates $\mathrm{Tw}\,\mathcal{W}(B_{\mathrm{std}}^{2n}, \Lambda_\varnothing; \mathbb{F}_q)$ and so

$$\mathrm{Tw}\,\mathcal{W}(B_{\mathrm{std}}^{2n}, \Lambda_Q; \mathbb{F}_q) \cong \mathrm{Tw}\,\mathcal{W}(B_{\mathrm{std}}^{2n}, \Lambda_\varnothing; \mathbb{F}_q)/D_q \cong 0.$$

On the other hand, all $p \in P$ are invertible in $\mathbb{F}_q$ because $q \in Q \setminus P$ by assumption and $p \neq 0$. Therefore $D_p \cong \mathrm{cone}(p \cdot \mathrm{Id}_{T_0^* D^n}) \cong 0$ in $\mathrm{Tw}\,\mathcal{W}(B_{\mathrm{std}}^{2n}, \Lambda_\varnothing; \mathbb{F}_p)$ for all $p \in P$, and so

$$\mathrm{Tw}\,\mathcal{W}(B_{\mathrm{std}}^{2n}, \Lambda_P; \mathbb{F}_q) \cong \mathrm{Tw}\,\mathcal{W}(B_{\mathrm{std}}^{2n}, \Lambda_\varnothing; \mathbb{F}_q)/0 \cong \mathrm{Tw}\,\mathcal{W}(B_{\mathrm{std}}^{2n}, \Lambda_\varnothing; \mathbb{F}_q) \cong \mathrm{Tw}\,\mathbb{F}_q,$$

which is nontrivial. Since there cannot be a localization functor from the trivial category to $\mathrm{Tw}\,\mathbb{F}_q$, $(B_{\mathrm{std}}^{2n}, \Lambda_P)$ cannot be a Weinstein subdomain of $(B_{\mathrm{std}}^{2n}, \Lambda_Q)$. This proves the "only if" part of claim (2). If there is a smoothly trivial regular Lagrangian cobordism from Λ_P to Λ_Q in $\partial B_{\mathrm{std}}^{2n} \times [0,1]$, then $(B_{\mathrm{std}}^{2n}, \Lambda_P)$ is a Weinstein subdomain of $(B_{\mathrm{std}}^{2n}, \Lambda_Q)$ and so the "only if" part of claim (3) follows from that for claim (2). $\square$

Now we show that Theorem 2.3 implies Theorem 1.1 concerning Weinstein subdomains of an *arbitrary* Weinstein domain. Recall that an index n Weinstein handle can be viewed as the stopped domain $(T^* D^n, \partial D^n) = (B_{\mathrm{std}}^{2n}, \Lambda_\varnothing)$. We will consider the stopped domains $(B_{\mathrm{std}}^{2n}, \Lambda_P)$ in Theorem 2.3 as generalized Weinstein handles.

Definition 2.6 A P–Weinstein handle of index n is the stopped domain $(B_{\mathrm{std}}^{2n}, \Lambda_P)$.

Here our model for the P–Weinstein handle uses explicit embeddings of Moore spaces into S^{n-1} and hence is well-defined. When attaching Weinstein handles, one implicitly uses the canonical parametrization of $\partial D^n \subset T^* D^n$. Via the construction in the proof of Theorem 2.3, this parametrization gives the Legendrians $\Lambda_P \subset \partial B^{2n}$ a parametrization as well. Therefore, given a parametrized Legendrian sphere Λ in a contact manifold (Y, ξ), we can attach a P–Weinstein handle $(B_{\mathrm{std}}^{2n}, \Lambda_P)$ to it and produce a Weinstein cobordism, just like we do for usual Weinstein handles. To prove Theorem 1.1, we replace all standard Weinstein n–handles $(B_{\mathrm{std}}^{2n}, \Lambda_\varnothing)$ with Weinstein P–handles $(B_{\mathrm{std}}^{2n}, \Lambda_P)$.

Proof of Theorem 1.1 Let X^{2n} be a Weinstein domain with $n \geq 5$ and $C_1^n, \ldots, C_k^n \subset X^{2n}$ the Lagrangian cocore disks of its index n handles $H_1^n, \ldots, H_k^n$. Hence there is a subcritical Weinstein domain $X_0 \subset X$ and Legendrian spheres $\Lambda_1, \ldots, \Lambda_k \subset \partial X_0$

such that $X = X_0 \cup H^n_{\Lambda_1} \cup \cdots \cup H^n_{\Lambda_1}$ and the cocore of $H^n_{\Lambda_i}$ is $C_i \subset X$. That is, X_0 is obtained from X by carving out the Lagrangian disks $C_1, \ldots, C_k$. This gives the decomposition

$$(2\text{-}6) \qquad X = (X_0, \Lambda_1, \ldots, \Lambda_k) \cup_{\Lambda_1 = \Lambda_\varnothing} (B^{2n}_{\mathrm{std}}, \Lambda_\varnothing) \cup \cdots \cup_{\Lambda_k = \Lambda_\varnothing} (B^{2n}_{\mathrm{std}}, \Lambda_\varnothing)$$

of X, where the ith copy of $(B^{2n}_{\mathrm{std}}, \Lambda_\varnothing)$ is glued to X_0 by identifying $\Lambda_\varnothing$ with Λ_i. Now we define X_P to be the Weinstein domain

$$(2\text{-}7) \qquad X_P := (X_0, \Lambda_1, \ldots, \Lambda_k) \cup_{\Lambda_1 = \Lambda_P} (B^{2n}_{\mathrm{std}}, \Lambda_P) \cup \cdots \cup_{\Lambda_k = \Lambda_P} (B^{2n}_{\mathrm{std}}, \Lambda_P).$$

Namely, we replace each standard Weinstein n–handle $(B^{2n}_{\mathrm{std}}, \Lambda_\varnothing)$ by a P–Weinstein handle $(B^{2n}_{\mathrm{std}}, \Lambda_P)$.

Remark 2.7 Attaching P–Weinstein handles $(B^{2n}_{\mathrm{std}}, \Lambda_P)$ to $(X_0, \Lambda_1, \ldots, \Lambda_k)$ is the same as attaching standard Weinstein handles $(B^{2n}_{\mathrm{std}}, \Lambda_\varnothing)$ to X_0 with some modified attaching Legendrian $\Lambda^P_i \subset \partial X_0$. In fact, Λ^P_i is the isotropic connected sum $\Lambda_i \natural \Lambda_P$ of $\Lambda_i \subset \partial X_0$ and $\Lambda_P \subset \partial B^{2n}_{\mathrm{std}}$, which we place into a Darboux chart in ∂X_0 disjoint from Λ_i. To see this, note that gluing $(B^{2n}_{\mathrm{std}}, \Lambda_P)$ to (X_0, Λ_i) by identifying Λ_P with $\Lambda_i \subset \partial X_0$ is the same as gluing a cylinder $T^*(S^{n-1} \times D^1)$ to $(X_0, \Lambda_i) \sqcup (B^{2n}_{\mathrm{std}}, \Lambda_P)$ by identifying $S^{n-1} \times 0$ with Λ_i and $S^{n-1} \times 1$ with Λ_P. The cylinder can be decomposed into a standard Weinstein index 1 handle and a standard Weinstein index n handle. So we first do simultaneous index 1 handle attachment to (X_0, Λ_i) and $(B^{2n}_{\mathrm{std}}, \Lambda_P)$, with attaching sphere a point in Λ_i and a point in Λ_P, to produce $(X_0 \natural B^{2n}_{\mathrm{std}}, \Lambda_i \natural \Lambda_P)$. If we identify $X_0 \natural B^{2n}$ with X_0, then Λ_P becomes a Legendrian in ∂X_0 (in a Darboux chart disjoint from Λ_i) and $\Lambda_i \natural \Lambda_P$ is precisely the isotropic connected sum of Λ_i and Λ_P in ∂X_0. Then we attach the (standard) index n Weinstein handle of the cylinder $T^*(S^{n-1} \times D^1)$ along $\Lambda_i \natural \Lambda_P$. Thus, the decomposition of X_P in equation (2-7) can alternatively be described as

$$(2\text{-}8) \quad (X_0, \Lambda_1 \natural \Lambda_P, \ldots, \Lambda_k \natural \Lambda_P) \cup_{\Lambda_1 \natural \Lambda_P = \Lambda_\varnothing} (B^{2n}_{\mathrm{std}}, \Lambda_\varnothing) \cup \cdots$$
$$\cup_{\Lambda_k \natural \Lambda_P = \Lambda_\varnothing} (B^{2n}_{\mathrm{std}}, \Lambda_\varnothing).$$

In particular, the attaching spheres for the (standard) index n handles for X and X_P differ by a purely local modification, namely an isotropic connected sum with Λ_P.

Now claims (1)–(3) in Theorem 1.1 follow from the analogous claims in Theorem 2.3. For example, $X_\varnothing = X$ since $(B^{2n}_{\mathrm{std}}, \Lambda_\varnothing)$ is the standard Weinstein handle $(T^* D^n, \partial D^n)$. Also, $(B^{2n}_{\mathrm{std}}, \Lambda_P)$ is a Weinstein subdomain of $(B^{2n}_{\mathrm{std}}, \Lambda_Q)$ for $Q \subset P$, so X_P is a

Weinstein subdomain of X_Q and this Weinstein embedding is also functorial with respect to inclusions of various subsets of primes. If $0 \in P$, then X_P is flexible. To see this, recall that $\Lambda_P \subset \partial B_{\mathrm{std}}^{2n}$ is loose by Theorem 2.3; this implies that the attaching spheres $\Lambda_i^P \subset \partial X_0$ for X_P are also loose since by Remark 2.7, Λ_i^P is the isotropic connected sum of Λ_i with Λ_P, which is a loose Legendrian loosely embedded in a Darboux chart disjoint from Λ_i. If $0 \in Q \subset P$, then the cobordism between X_P and X_Q is flexible since the cobordism between $(B_{\mathrm{std}}^{2n}, \Lambda_P)$, and $(B_{\mathrm{std}}^{2n}, \Lambda_Q)$ is also flexible (in the complement of Λ_P).

Finally, we compute $\operatorname{Tw} \mathcal{W}(X_P; \mathbb{Z})$. Since X_P is a Weinstein subdomain of X, there is a Viterbo transfer functor

$$\operatorname{Tw} \mathcal{W}(X; \mathbb{Z}) \to \operatorname{Tw} \mathcal{W}(X_P; \mathbb{Z}).$$

As in the proof of Theorem 2.3, this functor is localization by $D_p \subset (T^*D^n, \partial D^n)$ — or equivalently, by $D_p \natural \overline{D}_p$ — and $D_p \cong \operatorname{cone}(p \cdot \operatorname{Id}_{T_0^* D^n})$. On the other hand, $T_0^* D^n \subset (T^*D^n, \partial D^n) = (B_{\mathrm{std}}^{2n}, \Lambda_\varnothing)$ is precisely the cocore C_i^n of $H_{\Lambda_i}^n$ under the decomposition of X in equation (2-6) and so D_p is isomorphic to $\operatorname{cone}(p \cdot \operatorname{Id}_{C_i^n})$. By [19, Theorem 1.10] and [9], the cocores C_i^n of all the $H_{\Lambda_i}^n$ generate $\operatorname{Tw} \mathcal{W}(X)$. So localizing by $\operatorname{cone}(p \cdot \operatorname{Id}_{C_i^n})$ for all i is the same as localizing by $\operatorname{cone}(p \cdot \operatorname{Id}_L)$ for all $L \in \operatorname{Tw} \mathcal{W}(X; \mathbb{Z})$. That is, $\operatorname{Tw} \mathcal{W}(X_P; \mathbb{Z}) \cong \operatorname{Tw} \mathcal{W}(X; \mathbb{Z})[1/P]$, as desired. $\qquad\square$

We observe that our construction of X_P depends on many choices. For example, it depends on the choice of initial Weinstein presentation for X. There are Weinstein homotopic presentations for X with different numbers of index n handles; hence in this case, our construction would involve carving out different numbers of Lagrangian disks (and then attaching the appropriate flexible cobordism). There are also choices to be made in constructing the P–handles $(B_{\mathrm{std}}^{2n}, \Lambda_P)$. We fixed a p–Moore space $U \subset S^{n-1}$ so that $\widetilde{C}^*(U) = \mathbb{Z}[-2] \xrightarrow{p} \mathbb{Z}[-3]$, and used this to construct $D_p := D_U$ and then form $(B_{\mathrm{std}}^{2n}, \Lambda_P)$. In fact, we could have taken any $U \subset S^{n-1}$ so that $\widetilde{C}^*(U)$ is quasi-isomorphic to $\bigoplus_i (\mathbb{Z}[k_i + 1] \xrightarrow{p} \mathbb{Z}[k_i])$ for any k_i. Repeating the construction for such U, we would have $\operatorname{Tw} \mathcal{W}(B_{\mathrm{std}}^{2n}, \Lambda_P; \mathbb{Z}) \cong \operatorname{Tw} \mathcal{W}(B_{\mathrm{std}}^{2n}, \Lambda_\varnothing; \mathbb{Z})[1/P]$ as well.

Now that we have described the subdomains X_P of X, we can explain the difference between our construction and that of Abouzaid and Seidel [6] more precisely. Abouzaid and Seidel [6] start with a Lefschetz fibration for X^{2n} whose fiber is a Weinstein domain F^{2n-2}. They then embed the Lagrangian disks D_p^{n-1} into F^{2n-2} so that they are in a neighborhood of the cocores C_i^{n-1} of the critical index $n-1$ handles H_i^{n-1}

of F^{2n-2}; using these disks, they build a larger fiber F' (which has F as a Weinstein subdomain) and add new vanishing cycles to create a new Lefschetz fibration, which is their space X'_P. On the other hand, the construction in Theorem 1.1 embeds the disks D_p^n into the *total space* X^{2n} so that they are in a neighborhood of the cocores C_i^n of the critical index n handles H_i^n of X^{2n}; we then carve out these disks. The construction of Abouzaid and Seidel holds only for $n \geq 6$. Because we work near the index n handles instead of the index $n-1$ handles, our construction improves this to hold for $n \geq 5$.

Next we complete the proof of Corollary 1.4 concerning subdomains of T^*M_{std}.

Proof of Corollary 1.4 The only extra feature of this result over Theorem 1.1 is the "only if" part of the statement: $T^*M_P \subset T^*M_Q$ if and only if $Q \subset P$ or $0 \in P$. To prove this, we repeat the proof in Theorem 2.3 that $(B_{\mathrm{std}}^{2n}, \Lambda_P)$ is a subdomain of $(B_{\mathrm{std}}^{2n}, \Lambda_Q)$ if and only if $Q \subset P$. Namely, suppose that $T^*M_P \subset T^*M_Q$ is a Weinstein subdomain but $Q \not\subset P$ and $0 \notin P$. Then there is a Viterbo localization functor on Fukaya categories over $\mathbb{F}_q$ for $q \in Q \setminus P$. However, $\mathrm{Tw}\,\mathcal{W}(T^*S_Q^n; \mathbb{F}_q) \cong 0$ but $\mathrm{Tw}\,\mathcal{W}(T^*S_P^n; \mathbb{F}_q) \cong \mathrm{Tw}\,\mathcal{W}(T^*S^n; \mathbb{F}_q) \cong \mathrm{Tw}\,C_*(\Omega S^n; \mathbb{F}_q)$ is nontrivial, so there cannot be such a localization functor. $\qquad\square$

Remark 2.8 A similar argument using the fact that the Viterbo map on symplectic cohomology is a unital ring map shows that $T^*S_P^n$ cannot be a *Liouville* subdomain of $T^*S_Q^n$ if $Q \not\subset P$ and $0 \notin P$.

3 Classifying Lagrangian disks

In this section we prove Theorem 1.7: if M is simply connected and spin, and the map $i: L \hookrightarrow T^*M$ is null-homotopic, then $L \cong CW^*(M, L) \otimes T_q^*M^n$ in $\mathrm{Tw}\,\mathcal{W}(T^*M; \mathbb{Z})$. To do this, we will apply Koszul duality to characterize objects of $\mathrm{Tw}\,\mathcal{W}(T^*M; \mathbb{Z})$ as modules over the A_∞ algebra $CW^*(M, M) \cong C^*(M)$. Here it is crucial that we work with the $\mathbb{Z}$–graded wrapped Fukaya category, where the $\mathbb{Z}$–grading comes from the Lagrangian fibration by cotangent fibers. Any Lagrangian disk, since it is contractible, can be $\mathbb{Z}$–graded; the zero section $M \subset T^*M_{\mathrm{std}}$ can also be $\mathbb{Z}$–graded for this grading. Hence these Lagrangians define objects of the $\mathbb{Z}$–graded Fukaya category.

3.1 $C_*(X)$–modules

We begin with a general discussion of how to view Floer complexes as modules over Morse cochain algebras. The outcome is Proposition 3.3, which says that the

module structures are unexpectedly topological. This is what will allow us to draw Floer-theoretic conclusions from the topological assumption of null-homotopy.

For now, we will work in a general Liouville domain X. Given two Lagrangian branes $K, L \subset X$, we can endow $CW^*(K, L)$ with the structure of a right $C^*(X)$–module in a number of ways. In each case, we model the A_∞ structure on our cochain algebras $C^*(X), C^*(K)$ and $C^*(L)$ with Morse complexes and perturbed gradient flow trees [2] associated to exhausting Morse functions f_X, f_K and f_L.

Let us fix some notation. The moduli space of domains controlling the A_∞ operations is the space

$$
\begin{array}{c}
\mathcal{T}^{d+1} \\
\downarrow \\
\mathcal{R}^{d+1}
\end{array}
$$

of metric ribbon trees with $d + 1$ infinite leaves and no finite leaves, labeled $x_0, \ldots, x_d$ in counterclockwise order. More explicitly, a point $p \in \mathcal{R}^{d+1}$ is an isomorphism class $[T_p]$, where T_p is a noncompact tree with

- $d + 1$ ends and no mono- or bivalent vertices,
- a ribbon structure, which for a tree is the same as a homotopy class of planar embeddings,
- an edge metric, meaning that we can measure the distance between any two points of T_p (not necessarily vertices), and
- a labeling of the ends by $x_0, \ldots, x_d$ in counterclockwise order with respect to the ribbon structure.

The fibration $\mathcal{T}^{d+1} \to \mathcal{R}^{d+1}$ is the tautological one, which over each p is a representative T_p. In what follows, we will imagine x_0 as the bottom of T_p and the other x_i as the top, which will allow us to use the prepositions "below" or "above" to mean "closer to x_0" or "closer to some other x_i", respectively.

The restriction homomorphisms $i_K^*: C^*(X) \to C^*(K)$ and $i_L^*: C^*(X) \to C^*(L)$ are controlled by the space

$$
\begin{array}{c}
\mathcal{G}^{d+1} \\
\downarrow \\
\mathcal{S}^{d+1}
\end{array}
$$

of *grafted trees*, which are metric ribbon trees T as above with the additional data of a (necessarily finite) subset $D \subset T$ which separates x_0 from the other leaves and whose

elements are equidistant from x_0. For $d \geq 2$, $\mathcal{S}^{d+1}$ has a natural $\mathbb{R}$–action which translates D, and the quotient is canonically identified with $\mathcal{R}^{d+1}$ (for $d = 1$, $\mathcal{S}^{1+1}$ is a single point). However, the natural compactification $\overline{\mathcal{R}}^{d+1}$ models the associahedron, while $\overline{\mathcal{S}}^{d+1}$ models the multiplihedron. The restriction homomorphism

$$\{F^d \mid d = 1, \dots, \infty\} \colon C^*(X) \to [C^*(K) \text{ or } C^*(L)]$$

is then given by counting isolated perturbed gradient flow trees of shape T_q for some $q \in \mathcal{S}^{d+1}$, where the portion of T_q above (resp. below) D maps into X (resp. K or L). Note that, because we work with a *perturbed* gradient flow, we do not need to require f_X to restrict to f_K or f_L. Of course, if we wanted to we could arrange that f_X restrict to one of these Morse functions, but generally it would impossible to achieve both. Fortunately, all the resulting homomorphisms are homotopic.

To make Floer complexes into $C^*(X)$–modules, we need chain-level PSS-type structures, which are built from *short trees* or *short grafted trees*. A short tree with d inputs is a rooted metric ribbon tree with d infinite leaves and no finite leaves (except possibly the root). The root is labeled y, while the leaves are labeled $x_1, \dots, x_d$ in counterclockwise order. A short grafted tree is a short tree equipped with the additional data of a dividing set D as above, either separating y from the x_i, or equal to $\{y\}$. We will denote the spaces of short trees and short grafted trees by $\mathcal{R}_s^{d+1}$ and $\mathcal{S}_s^{d+1}$, respectively. There are canonical piecewise smooth homeomorphisms

$$(3\text{-}1) \qquad\qquad \mathcal{R}_s^{d+1} \cong \mathcal{R}^{d+1} \times \mathbb{R}_{\geq 0} \quad \text{for } d \geq 2,$$

$$(3\text{-}2) \qquad\qquad \mathcal{S}_s^{d+1} \cong \mathcal{R}_s^{d+1} \times \mathbb{R}_{\geq 0} \quad \text{for } d \geq 1,$$

ie all d. In (3-1), the $\mathbb{R}_{\geq 0}$ factor measures the distance between the root y and the first vertex, while in (3-2), it measures the distance between y and the dividing set.

The PSS-type structures in question all come from moduli spaces of strips with some number of short Morse trees attached at marked points.

Definition 3.1 A *hedge* comprises

(1) a smooth function $f \colon \mathbb{R} \to [0, 1]$,

(2) a collection of k points $z_1, \dots, z_k$ on the graph $\Gamma(f) \subset \mathbb{R} \times [0, 1]$ with strictly increasing $\mathbb{R}$ components, and

(3) for each z_i, a short tree T_i.

Identifying z_i with the root y_i of the tree T_i induces a total lexicographic order of the leaves x_{ij} of the trees T_i, namely $x_{i,j} < x_{i'j'}$ if either $i < i'$ or both $i = i'$ and $j < j'$.
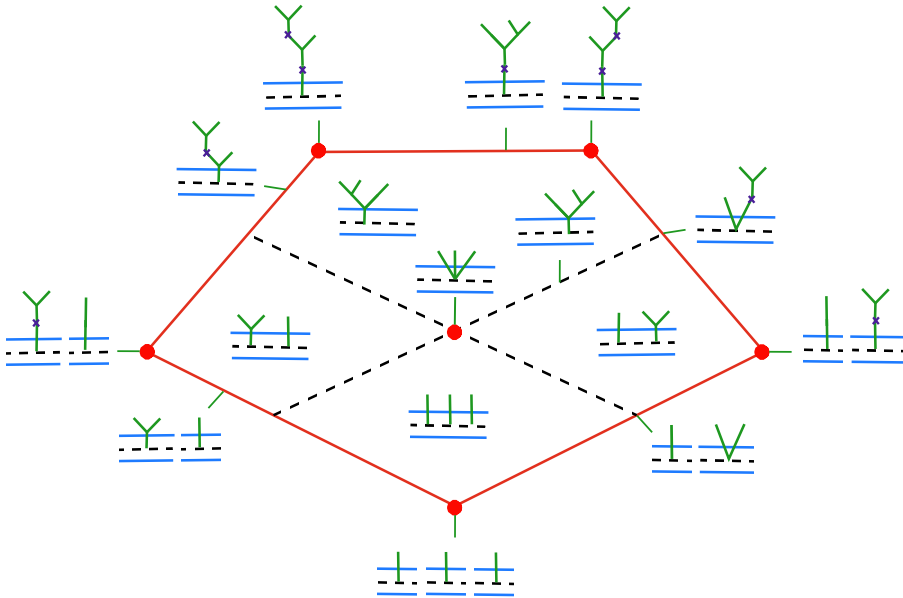
Figure 1: The space $\overline{\mathcal{H}}_c^3$. The boundary consists of Floer breaking (lower boundary in the diagram) and Morse breaking (upper boundary).

Fix a number $c \in [0, 1]$. The space $\mathcal{H}_c^d$ of hedges with d leaves x_{ij} and $f(s) = c$ comes a priori as a disjoint union of components indexed by partitions of the leaves into trees T_i. However, there is a natural way to glue the various components to build a connected moduli space. To see this, note that the boundary strata (before compactification) come from one or more roots y_i becoming multivalent, or in horticultural terms from some tree T_i becoming maximally short. Such configurations can also be achieved by having multiple smaller short trees attached to distinct marked points collide. The result is that we can make $\mathcal{H}_c^d$ into a *connected*, *smoothly stratified*, *topological manifold without boundary*; see Figure 1. This is good enough to construct operations in Floer theory.

$\mathcal{H}_c^d$ has a natural compactification $\overline{\mathcal{H}}^d$, where the codimension-1 boundary strata come in two types. The first is associated with Morse breaking, where a single short tree will break into a short tree and a (long) tree. The second is a type of Floer breaking associated with the marked points z_i moving apart, so that the limiting configuration is made up of two hedges.

An X–valued perturbation datum for a hedge $\boldsymbol{H}$ amounts to a perturbation datum for each short tree T_i, which is just an ε–parametrized family of vector fields on X for each edge ε of T_i which vanishes outside a compact subset of ε. Given a Morse–Smale

pair on X and a Floer datum for the pair (K, L), we can define a hedge map out of $\boldsymbol{H}$ to be a tuple $(u, \tau_1, \ldots, \tau_k)$, where

- u is a Floer trajectory with boundary on (K, L),
- τ_i is a perturbed gradient flow tree in X parametrized by T_i, and
- $\tau_i(y_i) = u(z_i)$.

If $\boldsymbol{H} \in \mathcal{H}_0^d$, we can analogously define a K–valued perturbation datum for $\boldsymbol{H}$ to be a family of vector fields on K, and a hedge map to involve gradient flow trees in K; if $\boldsymbol{H} \in \mathcal{H}_1^d$, we can do the same with L.

For generic Morse-Smale pairs, smooth, boundary-consistent, translation-invariant families of perturbation data on $\mathcal{H}_c^d$, and Floer data on X, the spaces of d–leaved hedge maps are smoothly stratified topological manifolds of the expected dimension. Counting such maps which are isolated up to translation makes $CW^*(K, L)$ into a right $C^*(X)$–module, which we'll denote by $CW^*(K, L)_{X,c}$. Similarly, when $c = 0$ or 1, we can make $CW^*(K, L)$ into a *right $C^*(K)$– or $C^*(L)$–module* $CW^*(K, L)_{K,0}$ or $CW^*(K, L)_{L,1}$, respectively.

Remark 3.2 Morse cochains form an E_∞ algebra, so we could use this same data to define left module structures, or even E_∞–module structures, but it will be convenient to use right modules so that we can use the standard Yoneda embedding instead of the co-Yoneda embedding. See Remark 3.5 for a discussion of the alternative.

The key holomorphic curve ingredient of our story is that these modules are all homotopy equivalent (and therefore quasi-isomorphic) when pulled back to $C^*(X)$:

Proposition 3.3 *For $c_1, c_2 \in [0, 1]$, there is a homotopy*

$$(3\text{-}3) \qquad CW^*(K, L)_{X,c_1} \simeq CW^*(K, L)_{X,c_2}$$

of right $C^*(X; \mathbb{Z})$–modules.

Similarly, for any c, there are homotopies

$$(3\text{-}4) \qquad CW^*(K, L)_{X,c} \simeq i_K^\vee CW^*(K, L)_{K,0},$$

$$(3\text{-}5) \qquad CW^*(K, L)_{X,c} \simeq i_L^\vee CW^*(K, L)_{L,1},$$

where

$$i_K^\vee \colon \mathrm{Mod}_{C^*(K;\mathbb{Z})} \to \mathrm{Mod}_{C^*(X;\mathbb{Z})}$$

is the pullback functor under the restriction homomorphism $i_K^* \colon C^*(X) \to C^*(K)$ of cochains, and similarly for $i_L^\vee$.

Remark 3.4 The key takeaway of Proposition 3.3 is not just that $CW^*(K, L)$ has a canonically defined $C^*(X)$–module structure, but that this module structure is *determined by* either the $C^*(K)$– or the $C^*(L)$–module structure.

Proof Pick a smooth function $f : \mathbb{R} \to [0, 1]$ interpolating between $f(s) = c_1$ for s near $+\infty$ and $f(s) = c_2$ for s near $-\infty$. Write $\mathcal{H}_f^d$ for the space of hedges with marked points z_i on the graph of f. Counting isolated (no longer up to translation) hedge maps parametrized by $\mathcal{H}_f^d$ defines the homotopy (3-3).

For the second part, we prove (3-4), since the proof of (3-5) is identical. For this, we may apply the first part to assume $c = 0$, so we need only produce a homotopy between $CW^*(K, L)_{X,0}$ and $i_K^\vee CW^*(K, L)_{K,0}$. We do this by generalizing the notion of a hedge to that of a *grafted hedge*. This is the same as Definition 3.1, except $f \equiv 0$ and the short trees T_i are replaced by short grafted trees. When the inputs of the short tree are in X and the root is forced to be in K (by attaching it to the boundary of a holomorphic curve with boundary in K), these models result in the same hedge maps. A (ordinary) hedge can thus be viewed as a special case of a grafted hedge, where all the dividing points are at the root y_i. Using this identification, we can extend the definition of the spaces $\mathcal{H}_c^d$ to negative values of c. Concretely, we declare $\mathcal{H}_c^d$ to be the space of d–leaved grafted hedges, where each tree is attached to the strip at $t = 0$ and has dividing set at distance $|c|$ from the root. For negative c, $\mathcal{H}_c^d$ continues to have a natural compactification $\overline{\mathcal{H}}_c^d$, and there is a canonical diffeomorphism $\overline{\mathcal{H}}_c^d \cong \overline{\mathcal{H}}_{c'}^d$ for any $c, c' \in (-\infty, 1]$.

For $\boldsymbol{H}$ a grafted hedge, a hedge map out of $\boldsymbol{H}$ is a tuple $(u, \tau_1, \ldots, \tau_k)$, where

- u is a Floer trajectory with boundary on (K, L),
- τ_i is a perturbed grafted gradient flow tree with leaves in X and root in K parametrized by T_i, and
- $\tau_i(y_i) = u(z_i)$.

Now the diffeomorphism $\overline{\mathcal{H}}_c^d \cong \overline{\mathcal{H}}_{c'}^d$ is compatible with both the internal stratification and the boundary decompositions, so it follows that hedge maps parametrized by $\mathcal{H}_c^d$ continue to define $C^*(X)$–module structures $CW^*(K, L)_{X,c}$ for $c < 0$. Moreover, the same argument as for nonnegative c shows that these module structures are homotopic — just interpolate the dividing sets rather than the attaching points.

To conclude, observe that the pullback module $i_K^* CW^*(K, L)_{K,0}$ is what we get by sending the dividing set to infinity. While it is delicate to do that directly, it is enough

to move the dividing set close to infinity: below any given action bound, gluing theory establishes a bijection of spaces of hedge maps. This ensures first that the module structure maps stabilize to the pulled-back ones, and second that the homotopies eventually become trivial. □

Remark 3.5 A version of Proposition 3.3 remains true with $C^*(X)$ replaced by symplectic cochains $SC^*(X)$, $C^*(K \text{ or } L)$ replaced by $CW^*(K \text{ or } L)$, and the restriction maps replaced by closed–open maps. In that case, one is forced to use *left* $CW^*(L)$–modules. While we expect all of the resulting homotopies to be intertwined by the relevant A_∞ algebra homomorphisms, sticking to Morse cochains allows us to avoid a good deal of combinatorial messiness.

Recall that for a Weinstein domain X, $SC^*(X)$ is quasi-isomorphic to the Hochschild cochains $CC^*(\mathcal{W}(X))$ of $\mathcal{W}(X)$ [17]. Using this quasi-isomorphism, we note that Proposition 3.3 has a purely categorical analog. For any A_∞ category $\mathcal{A}$, there is an A_∞ homomorphism $CC^*(\mathcal{A}) \to \mathrm{hom}^*(X, X)$ and hence a pullback map on modules,

$$\pi_X : \mathrm{Mod}_{\mathrm{end}^*(X)} \to \mathrm{Mod}_{CC^*(\mathcal{A})}.$$

Since $CC^*(\mathcal{A})$ is an E_2 algebra, there is also an A_∞ homomorphism $CC^*(\mathcal{A}) \to \mathrm{hom}^*(X, X)^{\mathrm{op}}$ and hence a similar pullback functor

$$\bar{\pi}_X : \mathrm{Mod}_{\mathrm{end}^*(X)^{\mathrm{op}}} \to \mathrm{Mod}_{CC^*(\mathcal{A})}.$$

For any two objects $X, Y \in \mathcal{A}$, composition of morphisms in $\mathcal{A}$ makes $\mathrm{hom}(X, Y)$ an object of $\mathrm{Mod}_{\mathrm{end}(X)}$ and also of $\mathrm{Mod}_{\mathrm{end}(Y)^{\mathrm{op}}}$. the categorical analog of Proposition 3.3 is that the objects $\pi_X \mathrm{hom}(X, Y)$ and $\bar{\pi}_Y \mathrm{hom}(X, Y)$ are quasi-isomorphic in $\mathrm{Mod}_{CC^*(\mathcal{A})}$.

For the actual statement in Proposition 3.3, we work with $C^*(X)$, the low-action part of $CC^*(\mathcal{W}(X))$, and need to identify $CC^*(\mathcal{W}(X)) \to \mathrm{hom}^*(L, L)$ with the restriction map $C^*(X) \to C^*(L)$ on Morse cochains. Here it is essential that our Lagrangian L *is not* equipped with a bounding cochain, which destroys the action filtration on Floer cochains and hence our access to the low-energy, topological subcomplex.

While so far we have considered general A_∞ presentations of our Morse cochain complexes, the above constructions work just as well for their strict unitalizations $C_s^*(-)$. Indeed, suppose X is connected, and pick a positive exhausting Morse function f on X with a unique degree 0 critical point. Define

$$C_s^*(X) := CM^{\geq 1}(f) \oplus \mathbb{Z} \cdot \mathbf{1}$$

with the restricted A_∞ structure on $CM^{\geq 1}(f)$ — which is well-defined because μ^k increases reduced degree, which is nonnegative by assumption — and for which $\mathbf{1}$ is a strict unit. Any A_∞ homomorphism

$$C^*(X) \to \mathcal{A}$$

for $\mathcal{A}$ a strictly unital A_∞ algebra induces a strictly unital homomorphism

$$C_s^*(X) \to \mathcal{A}.$$

Because modules are just functors to the strictly unital dg-category **Ch**, we conclude:

Corollary 3.6 *If X, K and L are connected, then Proposition 3.3 continues to hold in the realm of strictly unital modules with $C^*(X)$ replaced with $C_s^*(X)$, and similarly with K and L.* $\square$

Corollary 3.7 *Let M be a closed connected manifold. If the restriction A_∞ homomorphism $i^*\colon C^*(T^*M;\mathbb{Z}) \to C^*(L)$ factors up to homotopy through the canonical augmentation,*

$$
\begin{array}{ccc}
C^*(T^*M;\mathbb{Z}) & \xrightarrow{\quad i^* \quad} & C^*(L;\mathbb{Z}) \\
\searrow{\scriptstyle \varepsilon_{\mathrm{can}}} & & \nearrow{\scriptstyle \eta} \\
& \mathbb{Z} &
\end{array}
$$

then $CW^*(M,L)_{M,0}$ is isomorphic to a module in the image of

$$\mathrm{Tw}\,\mathbb{Z} \subset \mathrm{Mod}_{\mathbb{Z}} \xrightarrow{\varepsilon_{\mathrm{can}}} \mathrm{Mod}_{C^*(M;\mathbb{Z})}.$$

Proof Replacing $C^*(-)$ by $C_s^*(-)$, we may assume all algebras and maps are strictly unital. In particular, the pullback functor

$$\eta^*\colon \mathrm{Mod}_{C^*(L;\mathbb{Z})} \to \mathrm{Mod}_{\mathbb{Z}}$$

preserves strict unitality of modules. Since a strictly unital $\mathbb{Z}$–module is just a chain complex, the $\mathbb{Z}$–module $\eta^*(CW(M,L)_{L,1})$ coincides with its underlying chain complex, which lies in $\mathrm{Tw}\,\mathbb{Z}$ because M is compact.

The result now follows from Corollary 3.6 (on each connected component of L), together with the observation that the restriction $C^*(T^*M) \to C^*(M)$ is an isomorphism. $\square$

3.2 Disks in cotangent bundles

In the previous section, we studied properties of Floer modules $CW^*(K,L)$ over various Morse cochain algebras. In this section, we restrict to the case of T^*M, where M is a simply connected, spin manifold. We use Koszul duality to show that the

module structure over $C^*(M)$ knows everything about the Fukaya category, and prove Theorem 1.7.

We first construct a presentation of the wrapped Fukaya category which is well-adapted to talking about modules over $C^*(M)$. First, write $\mathcal{C}$ for the semiorthogonally glued category

$$\langle M^{\mathrm{Morse}}, \mathcal{W}(T^*M) \rangle,$$

where $\mathrm{end}^*(M^{\mathrm{Morse}}) = C_s^*(M)$, and $\mathrm{hom}_{\mathcal{C}}^*(M^{\mathrm{Morse}}, L) = CW^*(M, L)$. The mixed A_∞ operations count generalized hedges, ie usual perturbed holomorphic disks whose first boundary lies geometrically on M, together with short perturbed gradient flow trees in M attached at boundary marked points. We will obtain our desired presentation by localizing $\mathcal{C}$:

Lemma 3.8 *Suppose that* $e \in \mathrm{hom}^0(M^{\mathrm{Morse}}, M)$ *is a cocycle representing the unit in* $CW^*(M, M)$. *Define*

(3-6) $$\mathcal{W}^{\mathrm{Morse}}(T^*M) := \mathcal{C}/\mathrm{cone}(e),$$

so that we have tautological functors

$$\mathcal{W}(T^*M) \xrightarrow{i_{\mathcal{W}}} \mathcal{W}^{\mathrm{Morse}}(T^*M) \xleftarrow{i_M} \mathrm{end}_{\mathcal{C}}^*(M^{\mathrm{Morse}}) = C_s^*(M).$$

Then $i_{\mathcal{W}}$ *is a quasi-equivalence and* i_M *is fully faithful.*

Proof For any object $X \in \mathcal{W}(T^*M)$, precomposition with the cocycle e induces a quasi-isomorphism

$$\mathrm{hom}_{\mathcal{C}}^*(M, X) \cong \mathrm{hom}_{\mathcal{C}}^*(M^{\mathrm{Morse}}, X).$$

This means that $\mathrm{cone}(e)$ is left-orthogonal to every $X \in \mathcal{W}(T^*M)$, which implies that $i_{\mathcal{W}}$ is fully faithful. Because $i_{\mathcal{W}}(M)$ is isomorphic to M^{Morse} in $\mathcal{W}^{\mathrm{Morse}}(T^*M)$, $i_{\mathcal{W}}$ is also essentially surjective, which means it's an equivalence.

The proof for i_M is identical, except $\mathrm{cone}(e)$ is right-orthogonal to M^{Morse} by the classical Lagrangian PSS isomorphism. $\square$

The benefit of $\mathcal{W}^{\mathrm{Morse}}(T^*M)$ is that it allows for direct Koszul duality between the Morse cochain algebra on the zero section and the wrapped Fukaya algebra of the fiber. In particular, we do not have to transfer Corollary 3.7 through Floer's isomorphism.

Proposition 3.9 *If M is a simply connected, spin manifold, then the restricted Yoneda functor*

$$\mathcal{Y} \colon \mathcal{W}^{\mathrm{Morse}}(T^*M) \xrightarrow{\text{Yoneda}} \mathrm{Mod}_{\mathcal{W}^{\mathrm{Morse}}(T^*M)} \xrightarrow{i_M^*} \mathrm{Mod}_{C^*(M)}$$

is fully faithful.

Remark 3.10 We have used the full-faithfulness of i_M from Lemma 3.8 to write $C^*(M)$ rather than $\mathrm{end}^*(M^{\mathrm{Morse}})$. At the level of objects, $\mathcal{Y}$ sends L to $CW^*(M, L)_{M,0}$.

Proof Lemma 3.8 and Abouzaid's theorems [4; 1] give us a chain of quasi-equivalences

$$\mathrm{Tw}\,\mathcal{W}^{\mathrm{Morse}}(T^*M) \xRightarrow{\cong} \mathrm{Tw}\,\mathcal{W}(T^*M) \xRightarrow{\cong} \mathrm{Tw}\,(\mathrm{end}^*(T_q^*M)) \xRightarrow{\cong} \mathrm{Tw}\,(C_{-*}(\Omega M)).$$

The resulting functor F sends the cotangent fiber T_q^*M to the rank 1 free module.

Let us study what happens to M^{Morse}. We know $CW^*(M^{\mathrm{Morse}}, T_q^*M) \cong \mathbb{Z}$, since the zero section and fiber have just one intersection point. This means that $F(M^{\mathrm{Morse}})$ is an augmentation, and in fact it is the canonical augmentation of $C_{-*}(\Omega M)$. Indeed, all $C_{-*}(\Omega M)$–modules whose cohomology is $\mathbb{Z}$ are quasi-isomorphic. To see this, use the homological perturbation lemma to replace $C_{-*}(\Omega M)$ with its cohomology $H_{-*}(\Omega M)$. This is supported in nonpositive degrees and, because M is simply connected, has $H_0(\Omega M) \cong \mathbb{Z}$. Since the A_∞–module operation

$$\mu^{k|1} \colon H_{-*}(\Omega M)^{\otimes k} \otimes \mathbb{Z} \to \mathbb{Z}$$

has degree $1-k$ and $H_{-*}(\Omega M)$ is supported in nonpositive degrees, the only nontrivial A_∞ operation is the product $\mu^{1|1} \colon H_0(\Omega M) \otimes \mathbb{Z} \to \mathbb{Z}$; this is the identity operation.

By [7], the standard augmentation $\mathbb{Z}$ and the rank 1 free module of $C_{-*}(\Omega M)$ are Koszul dual if M is simply connected, ie $C_{-*}(\Omega M)$ is quasi-isomorphic to hom^* of the augmentation $\mathrm{hom}^*_{\mathrm{Mod}_{C(\Omega M)}}(\mathbb{Z}, C_{-*}(\Omega M))$ of $C^*(M) \cong \mathrm{hom}^*_{\mathrm{Mod}_{C(\Omega M)}}(\mathbb{Z}, \mathbb{Z})$, and hence the restricted Yoneda functor $\mathrm{hom}^*_{\mathrm{Mod}_{C(\Omega M)}}(\mathbb{Z}, -)$ is a quasi-embedding. By our above identifications of M^{Morse} with the standard augmentation and T_q^*M with the rank 1 free module, the result follows. $\square$

Remark 3.11 Simple connectedness and $\mathbb{Z}$–grading are standard essential ingredients for Koszul duality. The spin condition also seems essential in our proof, but we do not have an example showing that Proposition 3.9 fails without it.

We now have the necessary ingredients to prove Theorem 1.7.

Proof of Theorem 1.7 We now turn to the Lagrangian $L \subset T^*M$. The hypothesis that L is null-homotopic in T^*M implies that the hypothesis of Corollary 3.7 is satisfied. This means that, up to isomorphism, $\mathcal{Y}(L)$ is the finite-dimensional cochain complex $CW^*(M, L)$, ie a complex of standard augmentations. On the other hand, the same reasoning (or a direct appeal to Corollary 3.6) shows that $\mathcal{Y}(T_q^*M)$ is itself a standard augmentation and hence $\mathcal{Y}(CW^*(M, L) \otimes T_q^*M)$ is also the finite-dimensional cochain complex $CW^*(M, L)$. Since $\mathcal{Y}$ is full and faithful by Proposition 3.9, L is quasi-isomorphic to $CW^*(M, L) \otimes T_q^*M$, as desired. $\qquad\square$

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Department of Mathematics, University of Massachusetts Boston
Boston, MA, United States

Columbia University
New York, NY, United States

olazarev1@gmail.com, zsylvan@gmail.com

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