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**A smooth compactification of the space
of genus two curves in projective space:
via logarithmic geometry and Gorenstein curves**

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We construct a modular desingularisation of $\overline{\mathcal{M}}_{2,n}(\mathbb{P}^r, d)^{\text{main}}$. The geometry of Gorenstein singularities of genus two leads us to consider maps from prestable admissible covers; with this enhanced logarithmic structure, it is possible to desingularise the main component by means of a logarithmic modification. Both isolated and nonreduced singularities appear naturally. Our construction gives rise to a notion of reduced Gromov–Witten invariants in genus two.

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1 Introduction

Modern enumerative geometry is based on a series of compactifications of the moduli space of smooth embedded curves of genus g and curve class $\beta \in H_2^+(X, \mathbb{Z})$ in a smooth projective variety X , and on their virtual intersection theory. The boundary of M Kontsevich’s space of stable maps represents maps from nodal curves, including multiple covers and contracted components. The genus zero theory of projective space provides a smooth compactification with normal crossing boundary. In higher genus, instead, contracted subcurves and finite covers may give rise to boundary components of excess dimension. Even though this moduli space satisfies R Vakil’s Murphy’s law,

a desingularisation of the main component certainly exists after H Hironaka’s work on resolution of singularities. The main application of the methods developed in this paper is an *explicit modular desingularisation* of the main component of the moduli space of stable maps to projective space in genus two and degree $d \geq 3$.

Theorem *There exists a logarithmically smooth and proper DM stack $\mathcal{VZ}_{2,n}(\mathbb{P}^r, d)$ over \mathbb{C} , with locally free logarithmic structure (and therefore smooth) and a birational forgetful morphism to $\overline{\mathcal{M}}_{2,n}(\mathbb{P}^r, d)^{\text{main}}$, parametrising the following data:*

- A pointed admissible hyperelliptic cover

$$\psi : (C, D_R, x_1, \dots, x_n) \rightarrow (T, D_B, \psi(x_1), \dots, \psi(x_n)),$$

where C is a prestable curve of arithmetic genus two, D_R and D_B are length six (ramification and branch) divisors, and $(T, D_B, \psi(x))$ is a stable rational tree.

- A map $f : C \rightarrow \mathbb{P}^r$.
- A bubbling destabilisation $C \leftarrow \tilde{C}$ and a contraction $\tilde{C} \rightarrow \bar{C}$ to a curve with Gorenstein singularities such that f factors through $\tilde{f} : \bar{C} \rightarrow \mathbb{P}^r$, and the latter is not special on any subcurve of \bar{C} .

More generally, for any smooth projective variety X , we construct a proper moduli space $\mathcal{VZ}_{2,n}(X, \beta)$ admitting a perfect obstruction theory and defining *reduced Gromov–Witten invariants*. When X is a projective complete intersection, they satisfy the *quantum Lefschetz hyperplane principle*. In general, we expect them to have a simpler enumerative content compared to standard Gromov–Witten invariants.

1.1 Main and boundary components

The moduli space of stable maps to X (see Kontsevich [47]) represents $f : C \rightarrow X$, where C is a *prestabile* (nodal and reduced) curve such that every rational component of C contracted by f contains at least three *special points* (markings and nodes).

When $X = \mathbb{P}^r$, f is equivalently determined by the data of a line bundle L on C and $r + 1$ sections of L that do not vanish simultaneously on C . Forgetting the sections, we obtain a morphism to the universal Picard stack, parametrising pairs (C, L) of a prestable curve and a line bundle on it,

$$\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \rightarrow \mathfrak{Pic}_{\mathcal{C}_{g,n}/\mathfrak{M}_{g,n}}.$$

Obstructions lie in $H^1(C, L)$; see Wang [78]. This implies that $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ is smooth. On the other hand, for $g \geq 1$, boundary components may arise where L restricts to

a special line bundle on a subcurve of C . When $d > 2g - 2$, we can identify a *main component*: the closure of the locus of maps from a smooth source curve.

In genus one, the generic point of a boundary component has a contracted subcurve of genus one, together with $k \leq r$ tails of positive degree; see Vakil [74].

In genus two, two types of boundary phenomena occur:

- a subcurve of positive genus is contracted, or
- the restriction of f to the minimal subcurve of genus two (*core*) is the hyperelliptic cover of a line in \mathbb{P}^r .

As an example of the second phenomenon, the main component of $\overline{\mathcal{M}}_2(\mathbb{P}^3, 5)$ has dimension 20; but the locus of maps from a reducible curve $C = Z \cup_q R$, with Z of genus two covering a line two-to-one and $R \simeq \mathbb{P}^1$ parametrisng a twisted cubic, has dimension 21.

We stress that $\overline{\mathcal{M}}_{2,n}(\mathbb{P}^r, d)^{\text{main}}$ has no natural modular interpretation, since it is obtained by taking the closure of the nice locus. Moreover, its singularities along the boundary can be nasty; see Vakil [75]. In [12], we draw the consequences of the present construction, analysing the locus of smoothable maps in $\overline{\mathcal{M}}_2(\mathbb{P}^2, 4)$, a moduli space with more than twenty irreducible components.

1.2 Good factorisation through a Gorenstein curve

Our approach to the desingularisation of the main component takes off from a simple observation: a line bundle of degree at least $2g - 1$ on a *minimal Gorenstein* curve of genus g has vanishing h^1 ; we have already used this implicitly for nodal curves in the discussion of the irreducible components of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$. The Gorenstein assumption makes this fact a straightforward consequence of Serre duality. Minimality is a weaker notion than irreducibility, but it is needed to ensure that the line bundle has sufficiently positive degree on every subcurve.

This observation raises a natural question: is it possible to replace every $f: C \rightarrow \mathbb{P}^r$ with a “more positive”/“less obstructed” $\tilde{f}: \overline{C} \rightarrow \mathbb{P}^r$ by “contracting” the higher-genus subcurves on which $f^* \mathcal{O}_{\mathbb{P}^r}(1)$ is special? The answer is “no”, since every such map is *smoothable*, ie it can be deformed into a map from a smooth curve.¹

¹The answer could also be “yes”: it has been shown by M Viscardi [77] in genus one that maps from Gorenstein curves satisfying certain stability conditions give rise to irreducible compactifications of $\mathcal{M}_{1,n}(\mathbb{P}^r, d)$; yet, their deformation theory is hard to grasp, because such is the deformation theory of the singularities that are involved [68].

This, on the other hand, gives us a strategy to study the main component:

- (1) Construct a universal contraction $\mathcal{C} \rightarrow \bar{\mathcal{C}}$ to a Gorenstein curve, by collapsing the subcurves of C on which f has low degree. This Gorenstein curve will not be readily available on $\bar{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$, because the contraction map $\mathcal{C} \rightarrow \bar{\mathcal{C}}$ has moduli (called *crimping spaces* by van der Wyck [80] and *moduli of attaching data* by DI Smyth [68]); we first need to introduce a modification of $\bar{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ along the boundary.
- (2) Take only those maps that admit a factorisation through $\bar{\mathcal{C}}$: as we have discussed, these are all smoothable, so the moduli space is at the very least irreducible. It provides a birational model of $\bar{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^{\text{main}}$, with the advantage of admitting a modular interpretation.

Moreover, this space is unobstructed over a base that can be assumed to be smooth in the low-genus examples that we have at our disposal: those of D Ranganathan, K Santos-Parker and J Wise [63], where the base is a logarithmic modification of the moduli space of prestable curves; and those of this paper, where the base is a logarithmic modification of the moduli space of hyperelliptic admissible covers. Once such a moduli space is constructed, the proof of smoothness for target \mathbb{P}^r is entirely conceptual; furthermore, the same methods can be employed to approach the study of different targets (eg products of projective spaces, toric varieties, flag varieties) and stability conditions (eg quasimaps).

While point (2) was essentially established for us by Ranganathan, Santos-Parker and Wise, point (1) is at all open for $g \geq 2$. Here we make a hopefully meaningful step in this direction.

1.3 Logarithmic geometry and singularities

The moduli space of prestable curves has a natural logarithmic structure induced by its boundary divisor, thus keeping track of the nodes and their smoothing parameters; see F Kato [43]. This induces a logarithmic structure on the moduli space of maps as well. It is a natural question whether the desingularisation of the main component can be achieved by means of a logarithmic modification; it is indeed the case in genus one [63], but not in our construction. We think that the following are truly high-genus phenomena.

1.3.1 Augmenting the logarithmic structure: admissible covers Our first most relevant finding is that, instead, in genus two it is necessary to enrich the logarithmic

structure of the base by passing to a moduli space of admissible covers; see J Harris and D Mumford [37].

Every smooth curve of genus two is hyperelliptic, ie the canonical class induces a two-to-one cover of a line branched along six points. The hyperelliptic cover is essentially unique, but, when the curve becomes nodal and in the presence of markings, the uniqueness is lost. It can be restored by making it part of the moduli functor. The resulting space of admissible covers is as nice as the moduli space of curves, but it has the advantage of encoding the Brill–Noether theory of the curve in the logarithmic structure; see S Mochizuki [55]. The necessity of such enrichment can be understood already by looking at the isolated Gorenstein singularities of genus two (see Battistella [11]): there are two families of these, basically corresponding to the choice of either a Weierstrass point or a conjugate pair in the semistable model. In order to tell these two cases apart logarithmically, in the construction of $\mathcal{C} \rightarrow \bar{\mathcal{C}}$, we start by considering stable maps from the source of an admissible cover.

1.3.2 Realisable tropical canonical divisors This points to our second finding. The line bundle giving the contraction $\mathcal{C} \rightarrow \mathcal{C}'$ (the first step towards $\bar{\mathcal{C}}$) is a vertical twist of the relative dualising line bundle. The twist is indeed logarithmic, and the piecewise-linear function on $\text{trop}(\mathcal{C})$ determining it is nothing but a *tropical canonical divisor* satisfying certain requirements of compatibility with the admissible cover, and in particular realisable; see Möller, Ulirsch and Werner [57]. Pulling back from the target of the admissible cover, which is a rational curve, allows us to simplify several computations. The subdivision of the tropical moduli space according to the domain of linearity of such function should therefore be thought of as parametrising all possible Gorenstein contractions of \mathcal{C} compatible with the degree of f .

1.3.3 Nonisolated singularities Finally, we underline that nonreduced curves appear as fibres of $\bar{\mathcal{C}}$, which requires a careful study of what we call *tailed ribbons*, and further distinguishes our work from its genus one ancestor. This should not come as a surprise: first, nonreduced curves can still be Gorenstein if the nilpotent structure is supported along one-dimensional components, rather than isolated points; second, *ribbons* were introduced in the 1990s as limits of smooth canonical curves when the latter tend to the hyperelliptic locus in moduli; see Fong [31]. They appear naturally in our work at the intersection of the main component with the hyperelliptic ones. It is possibly less expected that they show up as well when the core is contracted, as a way of interpolating between isolated singularities whose special branches differ. The construction of $\bar{\mathcal{C}}$ is

concluded by gluing in \mathcal{C}' the hyperelliptic cover of a genus two subcurve supported along a boundary divisor.

For future investigation, we remark that we perform both the contraction $\mathcal{C} \rightarrow \mathcal{C}'$ and the pushout $\mathcal{C}' \rightarrow \bar{\mathcal{C}}$ in sufficiently general families. We wonder whether a pointwise construction might be possible, realising both steps as special instances of a more general pushout of logarithmic subcurves, in the spirit of S Bozlee [18].

1.4 Relation to other work and further directions

1.4.1 Local equations and resolution by blowing up It is always possible to find a local embedding of $\bar{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ in a smooth ambient space by looking at

$$\text{Tot}(\pi_* L)^{\oplus r+1} \subseteq \text{Tot}(\pi_* L \otimes \mathcal{O}_C(n))^{\oplus r+1}$$

over the Picard stack, where $\mathcal{O}_C(1)$ is a relative polarisation for the universal curve $\pi : \mathcal{C} \rightarrow \mathfrak{Pic}_{g,n}$ [25, Section 3.2]. If the polarisation is chosen carefully, the embedding is given by $\approx g$ local equations, repeated $r + 1$ times. An approach to the desingularisation of the main component is by blowing up according to these local equations; see Y Hu, J Li and J Niu [38; 39; 40]. Possibly the hardest task is to provide a compatible global description of the blow-up centres and their sequence. This method is close in spirit to the original construction of Vakil and A Zinger [84; 76] (in particular, it involves an iterated blow-up procedure and a good deal of bookkeeping), and it has the advantage of simultaneously “desingularising” the sheaves $\pi_* \mathcal{L}^{\otimes k}$, $k \geq 1$, on the main component, making them into vector bundles. This in turn makes the theory of projective complete intersections accessible via torus localisation; see Zinger [83] and Popa [62].

Hu, Li and Niu [40] carry out this strategy in genus two. For the reader’s benefit, we provide a brief conversion chart: in round 3, phase 3 of their routine, they blow up loci in the Picard stack, where the line bundle is required to restrict to the canonical bundle on the core; this part is subsumed in our work by starting from a birational model of the moduli space of weighted curves, namely the space of admissible covers. In round 1 they blow up loci in $\mathfrak{M}_2^{\text{wt}}$ that correspond explicitly to the boundary phenomena that we have summarised in Section 1.1. In rounds 2 and 3 they further blow up loci contained in the exceptional divisor of the previous rounds; these loci often depend on the attaching points of some rational tails being Weierstrass, or conjugate. In the language we have adopted, all of the blow-up centres are encoded in *the domain of linearity of a certain piecewise-linear function* — a tropical canonical divisor — while the Brill–Noether theory of the curve is built in the moduli space at the level of admissible covers

already. Our construction is thus more intrinsic. We remark, for example, that the genus one phenomena (corresponding to round 1, phases 2–4 in [40]) are addressed uniformly within this language, with the tropical canonical divisor resembling the function “distance from the core” that is at the base of [63]. The desingularisation of Hu, Li and Niu seems more efficient than ours, in the (vague) sense that the information encoded by the admissible cover *far from the special branch* is useless for the rest of our construction. Their procedure appears to select a “special branch” to start with in round 1, and then continue blowing up accordingly only “around that branch”; the price to be paid seems to be a certain noncanonicity of the blow-up procedure, with subdivisions appearing that are not quite natural from our perspective (when the core is reducible; see [Example 3.39](#)).

In [41], Hu and Niu reconstruct the blow-up by gluing projective bundle strata indexed by *treelike structures* and *level trees*; these data are reminiscent of piecewise-linear functions on tropical curves, though missing both the slope and the metric data. These authors have already pointed out the similarities between their indexing set and the combinatorial data appearing in the moduli space of multiscale differentials due to Bainbridge, Chen, Gendron, Grushevsky and Möller [9]. This relation certainly deserves further attention; we think that canonical divisors on tropical curves could provide the right language to talk about it, and the geometry of Gorenstein curves could be the informing principle of further investigations. We expect a crucial ingredient for an all-genus desingularisation will be a good compactification of canonical curves.

1.4.2 Computations in Gromov–Witten theory Naive computations of what we may now expect to coincide with our reduced invariants have made seldom appearances in the literature, for example, with Zinger’s enumeration of genus two curves with a fixed complex structure in \mathbb{P}^2 and \mathbb{P}^3 [81], and the computation of characteristic numbers of plane curves due to T Graber, J Kock and R Pandharipande [33]. To make the relation with the latter work precise, we should first extend our methods to the analysis of relative and logarithmic stable maps (compare with Battistella, Nabijou and Ranganathan [13] and Ranganathan, Santos-Parker and Wise [64] in genus one).

$\mathcal{VZ}_{2,n}(X, \beta)$ is only the main component of a moduli space of aligned admissible maps $\mathcal{A}_{2,n}(X, \beta)$, which dominates $\overline{\mathcal{M}}_{2,n}(X, \beta)$ and is virtually birational to it. The virtual class of $\mathcal{A}_{2,n}(X, \beta)$ is expected to split as the sum of its main and boundary components; the contribution of the latter should be expressible in terms of genus zero and reduced genus one invariants via virtual pushforward; see Manolache [52].

This would deliver an extension of Li and Zinger’s formula to genus two (see Li and Zinger [51], Zinger [82] and Coates and Manolache [26]). Together with a localisation computation of reduced invariants, as mentioned in the previous section, this would provide an alternative proof of the genus two mirror theorem for the quintic threefold; see Guo, Janda and Ruan [35] and Chang, Guo and Li [24]. Along the same lines, it would ease the computation of genus two Gromov–Witten invariants of Fano and Calabi–Yau complete intersections in projective space.

1.4.3 Logarithmic maps to toric varieties and tropical realisability In [64], Ranganathan, Santos-Parker and Wise apply their techniques to desingularising the space of genus one logarithmic maps to a toric variety with respect to its boundary. As an application, they are able to solve the *realisability problem* for tropical maps of genus one (see Speyer [69]): which of them arise as the tropicalisation of a map from a smooth genus one curve to an algebraic torus? With Ranganathan, we are working towards a similar result in genus two — where, as far as we know, there is at the moment no reasonable guess as to what the full answer should be, although a clear understanding of the moduli space of tropical curves has been obtained by Cueto and Markwig [27].

1.4.4 Birational geometry of moduli spaces of curves In [11], the first author produces a sequence of alternative compactifications of $\mathcal{M}_{2,n}$ based on replacing genus one and two subcurves with few special points by isolated Gorenstein singularities. Although we do not discuss it here, the techniques we develop also provide a resolution of the rational maps among these spaces. Moreover, the universal Gorenstein curve constructed here unveils the possibility of defining new birational models of $\overline{\mathcal{M}}_{2,n}$ by including nonreduced curves as well — contrary to $\overline{\mathcal{M}}_{2,n}(m)$, these models could respect the \mathfrak{S}_n -symmetry in the markings. It would be interesting to compare them with the work of Johnson and Polishchuk [42], and to establish their position in the Hassett–Keel program of Alper, Fedorchuk, Smyth and van der Wyck [68; 7]. Recently, Bozlee, B Kuo and A Neff [19] have classified all the compactifications of $\mathcal{M}_{1,n}$ in the stack of Gorenstein curves with distinct markings — it turns out that there are many more than envisioned by Smyth, although the numerosity arises more from combinatorial than geometric complications. It would be interesting if all compactifications of $\mathcal{M}_{2,n}$ could be classified by a mixture of our techniques.

Plan of the paper

In Section 2, we establish some language and background material concerning *logarithmic curves*, their tropicalisation and the use of piecewise-linear functions; *admissible*

covers and their logarithmic structure; and *Gorenstein curves*, with both isolated singularities and nonreduced structures, including a number of useful properties and formulas, the classification of isolated singularities of genus one (due to Smyth) and two (due to the first author), and a novel study of the nonisolated singularities of genus two with nonnegative canonical class.

In [Section 3](#), we introduce the key player: a subdivision of the tropical moduli space of weighted admissible covers based on aligning (ordering) the vertices of the tropical curve with respect to a piecewise-linear function constructed from tropical canonical divisors. This subdivision induces a logarithmically étale model of the moduli space of weighted admissible covers. It is on this model that we are able to construct a universal family of Gorenstein curves. This process consists of two steps: first, a birational contraction $\mathcal{C} \rightarrow \mathcal{C}'$; then, a pushout/normalisation $\mathcal{C}' \rightarrow \bar{\mathcal{C}}$.

In [Section 4](#), we apply these methods to desingularise $\overline{\mathcal{M}}_{2,n}(\mathbb{P}^r, d)^{\text{main}}$. To be more precise, we repeat the contraction/pushout process twice: first, to ensure that any curve of positive genus has weight at least one; then, to exclude the possibility that the map is hyperelliptic on a genus two subcurve. The intermediate space, although not being smooth, is already interesting in that its invariants satisfy quantum Lefschetz; see [Lee \[50\]](#).

Notation and conventions

We work throughout over \mathbb{C} .

By *trait* we mean the spectrum \mathcal{Y} of a discrete valuation ring. It only has two points; the closed one is often denoted by s (or 0), and the generic one by η .

Curves will always be projective and $S1$, ie without embedded points, but they may be nonreduced. Subcurves are not supposed to be irreducible, but they are usually connected. We call *core* the minimal subcurve of genus two. We may refer to subcurves $C' \subseteq C$ that intersect the rest of C and the markings in only one (resp. two) at worst nodal point(s) — or markings — as *tails* (resp. *bridges*).

The *dual graph* Γ of a nodal curve C has a vertex for every irreducible component, an edge for every node, and a leg for every marking (labelled or not); it is endowed with a genus function $g: V(\Gamma) \rightarrow \mathbb{Z}$, and the genus of the graph is given by

$$g(\Gamma) = h^1(\Gamma) + \sum_{v \in V(\Gamma)} g(v).$$

The graph Γ may be further weighted with a function $w: V(\Gamma) \rightarrow \mathbb{Z}$, which should be thought of as recording the degree of a line bundle on C .

A graph with no loops is called a *tree*; its valence one vertices are called *leaves* (sometimes one of them plays a different role and it is therefore named the *root*). A tree with only two leaves is a *chain*.

A *tropical curve* is a graph metrised in a monoid \overline{M} (most basically, $\mathbb{R}_{\geq 0}$), ie a graph Γ as above together with a length function $\ell: E(\Gamma) \rightarrow \overline{M}$ on the set of edges (legs are considered to be infinite, instead).

A *logarithmic space* is denoted by $X = (\underline{X}, M_X)$ with logarithmic structure $\alpha: M_X \rightarrow (\mathcal{O}_X, \cdot)$ (also simply indicated by exponential/logarithmic notation). M_X is a sheaf in the étale topology of \underline{X} . We assume that logarithmic structures are *FS*, ie fine (they admit an atlas of finitely generated and integral charts) and saturated. Let \overline{M}_X denote the characteristic sheaf $M_X/\alpha^{-1}(\mathcal{O}_X^*)$. This is a constructible sheaf of abelian groups.

A logarithmic space X gives rise to a cone stack $\text{trop}(X)$ via *tropicalisation*. In the case of logarithmically smooth curves, the tropicalisation is a tropical curve in the above sense. We abide to the rule that the tropicalisation of C should be denoted by the corresponding piecewise-linear character \square .

We have quite a few families of curves; usually we adopt the following notation:

- $\pi: C \rightarrow S$ will denote a prestable curve, \tilde{C} (with $\tilde{\pi}$) a partial destabilisation of C , C' (with π') a (not necessarily Gorenstein) contraction of \tilde{C} , and \overline{C} (with $\overline{\pi}$) a (not necessarily reduced) Gorenstein curve dominated by C' .
- $f: C \rightarrow X$ will denote a (stable) map to (a smooth and projective) X .
- $\psi: C \rightarrow T$ will denote an admissible cover from a genus two to a rational curve, $\psi': C' \rightarrow T'$ a double cover (where the curves are not necessarily prestable).

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2 Preliminaries and background material

2.1 Logarithmically smooth curves and their tropicalisations

Let (S, M_S) be a logarithmic scheme. A family of *logarithmically smooth curves* over S is a proper and logarithmically smooth morphism $\pi : (C, M_C) \rightarrow (S, M_S)$ with connected one-dimensional (geometric) fibres such that π is integral and saturated. These hypotheses guarantee flatness and that the fibres are reduced. Logarithmically smooth curves naturally provide a compactification of the moduli space of smooth curves.

2.1.1 Characterisation of logarithmically smooth curves Kato proved in [43] that logarithmically smooth curves have at worst nodal singularities; moreover, he provided the following local description. For every geometric point $p \in C$, there exists an étale local neighbourhood of p in C with a strict étale morphism to:

- **Smooth point** \mathbb{A}_S^1 with the log structure pulled back from the base.
- **Marking** \mathbb{A}_S^1 with the log structure generated by the zero section and $\pi^* M_S$.
- **Node** $\mathcal{O}_S[x, y]/(xy = t)$ for some $t \in \mathcal{O}_S$, with semistable log structure induced by the multiplication map $\mathbb{A}_S^2 \rightarrow \mathbb{A}_S^1$ and $t : S \rightarrow \mathbb{A}^1$.

In the last case, the class of $\log(t)$ in \overline{M}_S is called a *smoothing parameter* for the node.

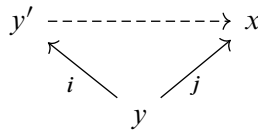
At times, we may have to consider more general logarithmic orbicurves [60].

2.1.2 Minimal logarithmic structures For every prestable curve $\underline{\pi} : \underline{C} \rightarrow \underline{S}$, there is a *minimal* logarithmic structure $M_{\underline{S}}^{\text{can}}$ on \underline{S} together with a logarithmically smooth enhancement $\pi : (C, M_C^{\text{can}}) \rightarrow (S, M_S^{\text{can}})$ such that any other logarithmically smooth

enhancement of $\underline{\pi}$ is pulled back from the minimal one. If $S = \{s\}$ is a geometric point, the characteristic sheaf of the minimal structure is freely generated by the smoothing parameters of the nodes of C_s :

$$\overline{M}_S^{\text{can}} = \mathbb{N}^{\#E(\Gamma(C_s))}.$$

More generally, the concept of minimality — introduced by WD Gillam [32] — serves the purpose of describing which stacks \mathcal{X} over (LogSch) are induced by ordinary stacks \mathcal{X} over (Sch) endowed with a logarithmic structure $\alpha: M_{\mathcal{X}} \rightarrow (\mathcal{O}_{\mathcal{X}}, \cdot)$. Gillam defines a *minimal object* x of \mathcal{X} to be one such that, for every solid diagram



with i and j over the identity of underlying schemes, there exists a unique dashed arrow in \mathcal{X} making the diagram commutative. He then shows that \mathcal{X} is induced by an $(\mathcal{X}, M_{\mathcal{X}})$ if and only if

- (1) for every object z of \mathcal{X} , there exist a minimal x , and $z \rightarrow x$ covering the identity of underlying schemes;
- (2) for every $w \rightarrow x$ with x minimal, the corresponding morphism of logarithmic schemes is strict if and only if w is minimal as well.

From this discussion, it follows that the stack of logarithmically smooth curves over (LogSch) is induced by a logarithmic stack over (Sch). It can be identified as the Artin stack of prestable (marked) curves endowed with the divisorial logarithmic structure corresponding to its normal crossings boundary.

2.1.3 Tropicalisation The combinatorial structure of an FS logarithmic space X is encoded by its *Artin fan* \mathcal{A}_X [2; 5]. An Artin fan is a logarithmic stack that looks like the quotient of a toric variety by its dense torus $[V/T]$ locally in the strict étale topology. The morphism $\underline{X} \rightarrow \text{Log}$ classifying the logarithmic structure on X [59] factors through $X \rightarrow \mathcal{A}_X$. The 2–category of Artin fans is equivalent to the 2–category of *cone stacks*, ie stacks over the category of rational polyhedral cone complexes [21, Theorem 6.11]. The latter should be thought of as a generalisation of the category of rational polyhedral cone complexes, where cones are allowed to be glued to themselves via automorphisms. In this way, we can associate to the logarithmic stack X a cone stack $\text{trop}(X)$ known as its *tropicalisation*, which is nothing but another incarnation

of the Artin fan. When X is a scheme with Zariski logarithmic structure (without monodromy), it is the generalised (standard) cone complex

$$\text{trop}(X) = \left(\coprod_{x \in X} \text{Hom}(\overline{M}_{X,x}, \mathbb{R}_{\geq 0}) \right) / \sim,$$

where x runs through the schematic points of X and gluing takes place along the face inclusions induced by the specialisation morphisms [34, Appendix B]. The language developed in [21] allows us to talk about the *tropicalisation map* within the category of logarithmic stacks. See also [1; 72; 73].

For a logarithmically smooth curve (C, M_C) over a geometric point (S, M_S) , the tropicalisation \square consists of its dual graph $\Gamma(C)$ metrised in \overline{M}_S ; the length function $\ell: E(\Gamma(C)) \rightarrow \overline{M}_S$ associates to each edge the smoothing parameter of the corresponding node. More precisely, vertices correspond to irreducible components weighted by their geometric genus, and there are (labelled) infinite legs corresponding to the markings. The tropicalisation of C is thus a family of classical (ie metrised in \mathbb{R}) tropical curves over the cone $\text{Hom}(\overline{M}_S, \mathbb{R}_{\geq 0})$. The construction can be generalised to more general base logarithmic schemes [21, Section 7]. It induces a (strict, smooth, surjective) morphism

$$\mathfrak{M}_{g,n}^{\text{log}} \rightarrow \widetilde{\mathfrak{M}}_{g,n}^{\text{trop}},$$

where the latter is the lift of the stack of tropical curves to the category of logarithmic schemes through the tropicalisation map.

2.1.4 Piecewise-linear functions and line bundles The characteristic monoid at a node $q \in C$ is the amalgamated sum

$$\overline{M}_{C,q} = \overline{M}_{S,\pi(q)} \oplus_{\mathbb{N}} \mathbb{N}^{\oplus 2},$$

where the map $\mathbb{N} \rightarrow \overline{M}_{S,\pi(q)}$ is the smoothing parameter δ_q and $\mathbb{N} \rightarrow \mathbb{N}^{\oplus 2}$ is the diagonal. It has been noticed in [34] that

$$\overline{M}_{C,q}^{\text{gp}} \simeq \{(\lambda_1, \lambda_2) \in \overline{M}_{S,\pi(q)}^{\text{gp},\oplus 2} \mid \lambda_2 - \lambda_1 \in \mathbb{Z}\delta_q\}.$$

If \square is a tropical curve metrised in \overline{M} , a *piecewise-linear (PL) function* on \square with values in \overline{M}^{gp} is a function λ from the vertex set of \square to \overline{M}^{gp} such that, for any two adjacent vertices v_1 and v_2 connected by an edge e_q , we can write

$$\lambda(v_2) - \lambda(v_1) = s(\lambda, e_q)\delta_q$$

for some $s(\lambda, e_q) \in \mathbb{Z}$ (called the *slope* of λ along e_q , outgoing from v_1), where δ_q is the smoothing parameter of the node q in \overline{M} .

Over a geometric point (or, more generally, if \overline{M}_S and \square are constant over S — in particular, if no nodes are smoothed out) the previous observation, together with a similar analysis at the markings, shows (see [21, Remark 7.3]) that

$$H^0(C, \overline{M}_C^{\text{gp}}) = \{\text{PL functions on } \square \text{ with values in } \overline{M}_S^{\text{gp}}\}.$$

Similarly, every section of $\pi_* \overline{M}_C / \overline{M}_S$ is described by the collection of slopes of a rational function on \square .

The exact sequence

$$0 \rightarrow \mathcal{O}_C^\times \rightarrow M_C^{\text{gp}} \rightarrow \overline{M}_C^{\text{gp}} \rightarrow 0$$

shows that to every section $\lambda \in \Gamma(C, \overline{M}_C^{\text{gp}})$ there is an associated \mathcal{O}_C^\times -torsor (or, equivalently, a line bundle) of lifts of λ to M_C^{gp} . Under the assumption that \overline{M}_S and \square are constant over S , every vertex v of \square determines an irreducible component of C , and the restriction of $\mathcal{O}_C(-\lambda)$ to C_v is given explicitly by [63, Proposition 2.4.1]

$$\mathcal{O}_{C_v}(\lambda) \simeq \mathcal{O}_{C_v} \left(\sum s(\lambda, e_q) q \right) \otimes \pi^* \mathcal{O}_S(\lambda(v)),$$

where $s(\lambda, e_q)$ denotes the outgoing slope of λ along the edge corresponding to q (either a marking or a node of C).

If we started from $\lambda \in \Gamma(C, \overline{M}_C)$, the associated line bundle $\mathcal{O}_C(-\lambda)$ would come with a cosection $\mathcal{O}_C(-\lambda) \rightarrow \mathcal{O}_C$ (induced by the logarithmic structure). Such a cosection is not always injective, but when it is it defines an effective Cartier divisor on C ; when it is not, it will vanish along components of C . Nonetheless, the association of this *generalised effective Cartier divisor* to λ behaves well under pullbacks, and in fact a useful point of view on logarithmic structures is to consider them as a functorial system of generalised effective Cartier divisors indexed by \overline{M}_C (see [17] for the details). See also [18, page 9] for a description in local coordinates.

Finally, we briefly recall the theory of divisors on tropical curves [36]. For this, let \square be a tropical curve over $\mathbb{R}_{\geq 0}$, thought of as its metric realisation. A *divisor* on \square is a finite \mathbb{Z} -linear combination of points on \square ; it is said to be *effective* if all the coefficients are nonnegative. The *tropical canonical divisor* is

$$K_\square = \sum_{v \in \square} (2g(v) - 2 + \text{val}(v))v.$$

It is effective unless the curve has rational tails. To a piecewise-linear function λ on (a subdivision of) \square , we can associate a divisor supported on its nonlinearity locus,

$$\text{div}(\lambda) = \sum_{v \in \square} \left(\sum_{e \in \text{Star}(v)} s(\lambda, e) \right) v.$$

Notice that the dependence on λ is only up to a global translation by \mathbb{R} . Divisors of the form $\text{div}(\lambda)$ are called *principal*. Two divisors are *linearly equivalent* if their difference is principal. We can define the *divisor class group* of \square by taking equivalence classes of divisors on \square modulo linear equivalence. The *linear system* of a divisor is given by

$$|D| = \{D' \sim_{\text{lin}} D \mid D' \geq 0\}.$$

We will loosely refer to a member of $|K_{\square}|$ as a tropical canonical divisor.

Notice that a tropical linear system is *tropically convex*, ie it is closed under taking max and real translations (see [36, Lemma 4]).

Everything here can be restated for tropical curves over a more general base. Subdivisions, though, may only make sense after enlarging the base monoid, often corresponding to a subdivision of its dual cone.

If $\psi : \square \rightarrow T$ is a harmonic morphism of tropical curves [8], both divisors and piecewise-linear functions can be pulled back from T to \square — in both cases, a coefficient accounting for the expansion factor of ψ along edges must be included.

2.1.5 Alignments and blow-ups The monoid \overline{M} induces a partial order on \overline{M}^{gp} ,

$$m_1 \leq m_2 \iff m_2 - m_1 \in \overline{M}.$$

Given a logarithmic scheme (S, M_S) with a logarithmic ideal $K = (m_1, \dots, m_h)$, the logarithmic subfunctor of S defined by requiring that there always be a minimum among the m_i for $i = 1, \dots, h$ — ie that the ideal be locally principal — is represented by the blow-up of S in the ideal $\alpha(K)$ [61, Section 3.4]. Tropically, it corresponds to a subdivision of the cone $\text{Hom}(\overline{M}_S, \mathbb{R}_{\geq 0})$, and vice versa.

This simple observation has had many fruitful applications in moduli theory, starting from [53].

2.2 Admissible covers and their logarithmic structure

Admissible covers have been introduced by Harris and Mumford [37] as a compactification of Hurwitz spaces. A fully fledged moduli theory for them has been developed only later by Mochizuki with the introduction of logarithmic techniques [55], and by D Abramovich, A Corti and A Vistoli with the introduction of twisted curves [3]. They have also been generalised by B Kim [44, Sections 5.2 and 7.2] (he calls them log-stable μ -ramified maps). We will only be concerned with the case of double covers of \mathbb{P}^1 , ie hyperelliptic curves.

In order to motivate the construction, we observe that every smooth curve of genus two is hyperelliptic (under the canonical map) with six Weierstrass points (ie ramification points for the hyperelliptic cover; this follows from the Riemann–Hurwitz formula). When the curve is allowed to become nodal, there still exists a degree two map to the projective line, but it is no longer finite. In this case, it is appropriate to define Weierstrass points to be limit of Weierstrass points in a smoothing family; the drawback is that whether a point on a (contracted) rational subcurve is Weierstrass or not may depend on the choice of the smoothing family. To resolve this ambiguity within the definition of the moduli problem, a solution is to expand the target, so as to consider the space of *finite* morphisms from nodal curves of genus two to rational *trees*. Here is a formal definition:

Definition 2.1 A family of *admissible hyperelliptic covers* over S is a finite morphism $\psi : (C, D_R) \rightarrow (T, D_B)$ over S such that

- (1) (C, D_R) and (T, D_B) are prestable curves with (unlabelled) smooth disjoint multisections D_R and D_B of length $2g + 2$, C has arithmetic genus g , and (T, D_B) is a stable rational tree;
- (2) ψ is a double cover on an open $U \subseteq T$ dense over S ;
- (3) ψ is étale on $C^{\text{sm}} \setminus D_R$, it maps D_R to D_B with simple ramification, and it maps nodes of C to nodes of T so that, in local² coordinates,

$$\psi^\# : \mathcal{O}_S[u, v]/(uv - s) \rightarrow \mathcal{O}_S[x, y]/(xy - t)$$

maps $u \mapsto x^i, v \mapsto y^i$ and $s \mapsto t^i$ for $i = 1$ or 2 .

Mochizuki shows that condition (3) can be replaced by requiring that ψ lift to a *logarithmically étale* morphism of logarithmic schemes $(C, M_C) \rightarrow (T, M_T)$ over (S, M_S) , so that the image of a smoothing parameter of T is either a smoothing parameter of C or its double. Moreover there is a minimal logarithmic structure over S that makes this possible: if p is a node of C that ψ maps to the node q of T ,

$$M_S^{\psi\text{-can}} = M_S^{C\text{-can}} \oplus_{\mathcal{O}_S^*} M_S^{T\text{-can}} / \sim, \quad (0, \delta_q) \sim (i\delta_p, 0),$$

where i is the local multiplicity defined at the end of (3). In the case of double covers, this simply means that the minimal logarithmic structure for the admissible cover is the same as that of the source curve, except that the smoothing parameters of two nodes

²We refer the reader to [55], and in particular Remark 2 of Section 3.9, for a detailed discussion.

have been identified if ψ matches them both with the same node of the target. We thus obtain a logarithmic Deligne–Mumford stack.

Theorem 2.2 [55, Section 3.22] *The moduli stack of admissible hyperelliptic covers $\mathcal{A}_{g,0,2}$ is a logarithmically smooth with locally free logarithmic structure (and therefore smooth), proper Deligne–Mumford stack, with a logarithmically étale morphism to $\overline{\mathcal{M}}_{0,2g+2}/\mathfrak{S}_{2g+2}$.*

To an admissible hyperelliptic cover we can associate a harmonic morphism of degree two to a metric tree $\psi: \square \rightarrow \mathbb{T}$. The tropical geometry of hyperelliptic and admissible covers has been analysed, for instance, in [23; 22].

2.3 Gorenstein curves

2.3.1 Tools and formulas Curves shall always be assumed Cohen–Macaulay, ie $S1$; they might still be nonreduced along some subcurve, but they have no embedded points.

Definition 2.3 A curve X is Gorenstein if its dualising sheaf ω_X is a line bundle.

A fundamental role in the study of singularities is played by the conductor ideal.

Definition 2.4 Let $\nu: \tilde{X} \rightarrow X$ be a finite and birational morphism. The *conductor ideal* of ν is

$$\mathfrak{c} = \text{Ann}_{\mathcal{O}_X}(\nu_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X) = \mathcal{H}om_X(\nu_*\mathcal{O}_{\tilde{X}}, \mathcal{O}_X).$$

The conductor is the largest \mathcal{O}_X -ideal sheaf that is also a $\nu_*\mathcal{O}_{\tilde{X}}$ -ideal sheaf. When X is a reduced curve with finite normalisation — which is always the case over a field — blowing up the conductor ideal recovers the normalisation [79]. The conductor ideal admits a further characterisation in terms of duality theory as

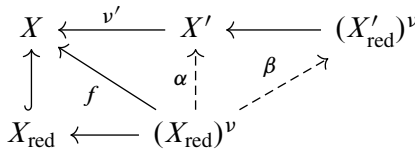
$$\mathfrak{c} = \nu^!\mathcal{O}_X = \omega_\nu.$$

Remark 2.5 Let us restrict to schemes of finite type over a field. For such an X , the normalisation $X^\nu \rightarrow X$ has two universal properties (see eg [16, Tag 035Q]):

- (1) It is final in the category of dominant morphisms $Y \rightarrow X$ with Y normal.
- (2) For every finite and birational morphism $Y \rightarrow X$, the normalisation of X factors uniquely through a morphism $X^\nu \rightarrow Y$.

Now suppose that X is a Cohen–Macaulay curve. The smooth (possibly disconnected) curve $(X_{\text{red}})^\nu$ is what is normally referred to as the normalisation of X . Notice that

$(X_{\text{red}})^\nu$ can be obtained as a (sequence of) blow-up(s) along a subscheme (points) δ of X_{red} . Set $X' = \text{Bl}_\delta X$ and consider the diagram



The arrow α exists by the universal property of blowing up (because $f^{-1}\mathcal{I}_{\delta/X}$ is principal in $\mathcal{O}_{(X_{\text{red}})^\nu}$). Since $(X_{\text{red}})^\nu$ is reduced and normal, α factors through $(X'_{\text{red}})^\nu$; hence, the arrow β exists. Since it is a finite and birational morphism of normal curves, β is an isomorphism. Abusing notation, we are going to call X' the *normalisation* of X . Note that the underlying reduced curve of every irreducible component is regular. Moreover, the sheaf $v'_*\mathcal{O}_{X'}/\mathcal{O}_X$ is supported along the singular locus of X_{red} . Suppose that X' contains a ribbon (see Definition 2.17) or, more generally, a *multiple curve* in the sense of [29]; the reader should be aware that, if we only ask of a morphism $g: Y \rightarrow X$ that the sheaf $g_*\mathcal{O}_Y/\mathcal{O}_X$ be supported at points, we can construct infinitely many more Y by blowing up X' successively at a number of smooth points of X_{red} . Indeed, for ribbons, these are essentially all the morphisms that restrict to the identity on the underlying \mathbb{P}^1 yet alter the square-zero ideal that defines the double structure (see [14, Theorem 1.10]).

It is known that, for a nodal curve X , the conductor ideal relates the dualising line bundle of X to that of its normalisation X^ν ; more generally [20, Proposition 1.2]:

Proposition 2.6 (Noether’s formula) *Let $v: \tilde{X} \rightarrow X$ be a finite birational morphism of Gorenstein curves. Then, viewing \mathfrak{c} as an ideal sheaf on \tilde{X} ,*

$$\omega_{\tilde{X}} = v^*\omega_X \otimes \mathfrak{c}.$$

Corollary 2.7 *Let X be a Gorenstein curve with Gorenstein normalisation $v: \tilde{X} \rightarrow X$. The conductor is a principal ideal on \tilde{X} .*

Definition 2.8 For a finite birational morphism $v: \tilde{X} \rightarrow X$ of curves, the coherent \mathcal{O}_X -module $v_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X$ has finite support; its length is called the δ -invariant of v . When v is the normalisation, we simply call it the δ -invariant of X . When X is reduced, it is the sum of the local contributions of all the isolated singularities of X .

We review a result of J-P Serre [66, Section 4, Proposition 7] for possibly nonreduced curves.

Lemma 2.9 *If X is a Gorenstein curve and \tilde{X} its normalisation, then*

$$(1) \quad \text{length}(\mathcal{O}_{\tilde{X}}/\mathfrak{c}) = 2\delta.$$

Proof Since the normalisation is finite and birational, we may work locally around the support of $\nu_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X$, which we may therefore assume is a single point. Let $X = \text{Spec}(R)$ and $\tilde{X} = \text{Spec}(\tilde{R})$. Using $\mathfrak{c} \subseteq R \subseteq \tilde{R}$, we see that (1) is equivalent to

$$\dim_{\mathbb{C}}(R/\mathfrak{c}) = \delta.$$

Let ω be a local generator of Ω_R ; given $f \in R$, Noether’s formula implies that $f\omega \in \Omega_{\tilde{R}}$ if and only if $f \in \mathfrak{c}$. We therefore get

$$R/\mathfrak{c} \hookrightarrow \Omega_R/\Omega_{\tilde{R}}.$$

Since Ω_R is free of rank one and ω a generator, the above map is surjective as well. Finally, applying $\text{Hom}_R(-, \Omega_R)$ to the normalisation exact sequence, we obtain $\Omega_R/\Omega_{\tilde{R}} = \text{Ext}_R^1(\tilde{R}/R, \Omega_R)$, and therefore $\dim_{\mathbb{C}} \Omega_R/\Omega_{\tilde{R}} = \delta$. □

Remark 2.10 The converse is true when X is reduced, but may fail otherwise, an example of which is provided by the subalgebra of $\mathbb{C}[[s, \epsilon]] \times \mathbb{C}[[t]]$ generated by $\langle s, \epsilon, t \rangle$, ie the transverse union of a ribbon with a line, which, though satisfying (1), is not Gorenstein as a consequence of the following lemma. What seems to be lacking in the nonreduced case is a good theory of dualising sheaves as rational differential forms on the normalisation satisfying some residue condition.

Definition 2.11 A curve is *decomposable* if it is obtained by gluing two curves along a (reduced closed) point.

Remark 2.12 This is equivalent to [70, Definition 2.1]. Indeed, the fibre product over \mathbb{C} of $\mathbb{C}[x_1, \dots, x_m]/I_1$ and $\mathbb{C}[y_1, \dots, y_n]/I_2$ is isomorphic to

$$(2) \quad \underbrace{\mathbb{C}[x_1, \dots, x_m, y_1, \dots, y_n]}_S / \underbrace{(I_1(\mathbf{x}), y_1, \dots, y_n)}_{J_1} \cap \underbrace{(x_1, \dots, x_m, I_2(\mathbf{y}))}_{J_2}.$$

On the other hand, with notation as in (2), there is a short exact sequence

$$0 \rightarrow S/J_1 \cap J_2 \rightarrow S/J_1 \times S/J_2 \rightarrow S/J_1 + J_2 \simeq \mathbb{C} \rightarrow 0,$$

which is exact in the middle because every element there can be represented as a pair of polynomials in \mathbf{x} only (resp. in \mathbf{y} only), which is easy to lift.

We review [6, Proposition 2.1] for not necessarily reduced curves.

Lemma 2.13 *A decomposable curve may be Gorenstein only if it is a node.*

Proof Let us assume that X is the decomposable union of X_1 and X_2 .

Notice first that, if X is Gorenstein, then so are X_1 and X_2 . Indeed, say we can find $(a_1, \dots, a_n) \subseteq \mathfrak{m}_X$ a regular sequence such that $\mathcal{O}_{X^0} = \mathcal{O}_X / (a_1, \dots, a_n)$ is Gorenstein of dimension zero, ie $\text{Hom}_{X^0}(\mathbb{C}, \mathcal{O}_{X^0}) \cong \mathbb{C}$ (and all higher Ext groups vanish, which is automatic by [54, Theorem 18.1]). Since $\mathfrak{m}_X = \mathfrak{m}_{X_1} \oplus \mathfrak{m}_{X_2}$, this defines a regular sequence in the latter as well, and $\mathcal{O}_{X_1} \oplus \mathcal{O}_{X_2} / (a_1, \dots, a_n) \cong \mathcal{O}_{X_1^0} \oplus \mathcal{O}_{X_2^0}$ is an extension of k with \mathcal{O}_{X^0} . From this we see that $\dim_{\mathbb{C}} \text{Hom}_{X_i^0}(\mathbb{C}, \mathcal{O}_{X_i^0}) = 1$ for $i = 1, 2$, which is to say that X_1 and X_2 are Gorenstein as well.

Now, X and $X_1 \sqcup X_2$ have the same normalisation \tilde{X} , and

$$(3) \quad \delta_X = \delta_{X_1} + \delta_{X_2} + 1.$$

On the other hand, $\mathcal{O}_{\tilde{X}} / \mathcal{O}_X \twoheadrightarrow \mathcal{O}_{\tilde{X}} / \mathcal{O}_{X_1 \sqcup X_2}$ implies that $\mathfrak{c}_X \subseteq \mathfrak{c}_{X_1 \sqcup X_2}$ (as ideals of $\mathcal{O}_{\tilde{X}}$). We claim that, if X is Gorenstein, the reduced curves underlying X_1 and X_2 are both smooth:

- If they are both singular, then $\mathfrak{c}_{X_1 \sqcup X_2} \subseteq \mathfrak{m}_{X_1} \oplus \mathfrak{m}_{X_2} = \mathfrak{m}_X$, so $\mathfrak{c}_X = \mathfrak{c}_{X_1 \sqcup X_2}$. By Lemma 2.9, we obtain $2\delta_X = \dim_{\mathbb{C}}(\mathcal{O}_{\tilde{X}} / \mathfrak{c}_X) = \dim_{\mathbb{C}}(\mathcal{O}_{\tilde{X}} / \mathfrak{c}_{X_1 \sqcup X_2}) = 2\delta_{X_1} + 2\delta_{X_2}$, which contradicts (3).
- If X_1 is singular and X_2 smooth, (3) reduces to $\delta_X = \delta_{X_1} + 1$. On the other hand, $\mathfrak{c}_{X_2} = \mathcal{O}_{X_2}$, and the contradiction comes from $2\delta_X = \dim_{\mathbb{C}}(\mathcal{O}_{\tilde{X}} / \mathfrak{c}_X) = \dim_{\mathbb{C}}(\mathcal{O}_{\tilde{X}} / \mathfrak{c}_{X_1}) = 2\delta_{X_1}$.

Finally, since both underlying curves are smooth, we see from the definition of decomposability that

$$\hat{\mathcal{O}}_X = \mathbb{C} \llbracket x, \epsilon_i, y, \eta_j \rrbracket / (xy, x\eta_j, \epsilon_i y, \epsilon_i \eta_j, \epsilon_i^{m_i}, \eta_j^{n_j})_{i=1, \dots, h, j=1, \dots, k}.$$

If X_1 and X_2 are reduced, we recover the node. In all other cases, one can verify that $\dim_{\mathbb{C}} \text{Hom}(\mathbb{C}, \mathcal{O}_X) \geq 2$ (it is generated by $xy, x\epsilon_1^{m_1-1}, \dots$). □

2.3.2 Isolated singularities Let (X, x) be (the germ of) a reduced curve with a unique singular point x , with normalisation $\nu: \tilde{X} \rightarrow X$. The following is a measure of how much of the arithmetic genus of a projective curve is hiding in its singularities:

Definition 2.14 [67] If X has m branches at x , the *genus* of (X, x) is

$$g = \delta - m + 1.$$

The classification of isolated Gorenstein singularities of genus one has been carried out by Smyth [67, Proposition A.3].

Proposition 2.15 An (X, x) of genus one with m branches is locally isomorphic to:

- $m = 1$ The cusp $V(y^2 - x^3) \subseteq \mathbb{A}_{x,y}^2$.
- $m = 2$ The tacnode $V(y^2 - yx^2) \subseteq \mathbb{A}_{x,y}^2$.
- $m \geq 3$ The union of m general lines through the origin of \mathbb{A}^{m-1} .

All of these singularities are smoothable. Choosing a one-parameter smoothing and passing to a regular semistable model, the *semistable tail*, ie the subcurve contracted to the singularity, admits a simple description: if we mark the semistable tail by the intersection with the rest of the central fibre, then it is a *balanced* nodal curve of genus one. This means that it consists of a genus one core — which can be either smooth, or a circle of rational curves — together with some rational trees supporting the markings, and the distance between a marking and the core — ie the length of the corresponding rational chain — is independent of the chosen marking [67, Proposition 2.12].

The classification of isolated Gorenstein singularities of genus two has been carried out by the first author [11, Section 2].

Proposition 2.16 The unique unibranch Gorenstein singularity of genus two is the **ramphoidal cusp** or A_4 -singularity $V(y^2 - x^5) \subseteq \mathbb{A}_{x,y}^2$. For every $m \geq 2$, there are exactly two isomorphism classes of germs of isolated Gorenstein singularities of genus two; see Table 1.

m	type I	type II
	$x_1 = t_1 \oplus 0 \oplus \dots \oplus t_m^3$ $x_2 = 0 \oplus t_2 \oplus \dots \oplus t_m^3$ \vdots $x_{m-1} = 0 \oplus \dots \oplus t_{m-1} \oplus t_m^3$ $x_m = 0 \oplus \dots \oplus 0 \oplus t_m^2$	$x_1 = t_1 \oplus 0 \oplus \dots \oplus t_m$ $x_2 = 0 \oplus t_2 \oplus \dots \oplus t_m^2$ \vdots $x_{m-1} = 0 \oplus \dots \oplus t_{m-1} \oplus t_m^2$ ($y = 0 \oplus t_2^3$ if $m = 2$)
1	$x^5 - y^2$ (A_4)	
2	$x_2(x_2^3 - x_1^2)$ (D_5)	$y(y - x_1^3)$ (A_5)
3	$\langle x_3(x_1 - x_2), x_3^3 - x_1x_2 \rangle$	$x_1x_2(x_2 - x_1^2)$ (D_6)
≥ 4	$\langle x_i(x_j - x_k), x_m(x_i - x_j), x_m^3 - x_1x_2 \rangle$ ($i \neq j \neq k \in \{1, \dots, m-1\}$)	$\langle x_3(x_1^2 - x_2), x_i(x_j - x_k) \rangle$ ($1 \leq i < j < k \leq m-1$ or $1 < j < k < i \leq m-1$)

Table 1: The isolated Gorenstein singularities of genus two. The first row gives the parametrisation; the second gives the equation(s) for different values of m .

It is not hard to see that, for $m \geq 3$, every type I singularity is the union of a cusp with $m - 1$ lines, and every type II singularity is the union of a tacnode with $m - 2$ lines; in each case, we refer to the components of the genus one subcurve as the *special branches*.

The description of semistable tails is a bit more cumbersome, but enlightening: assuming that the genus two core is smooth, the special branches are closer to the core and attached to a Weierstrass point (resp. two conjugate points) in type I (resp. II); all other branches are equidistant and further away from the core. In fact, the ratio between the length of the special rational chain and the others is fixed to $\frac{1}{3}$ (resp. $\frac{1}{2}$) in type I (resp. II). For a more detailed statement when the core is not smooth, see [11, Propositions 4.3–4.6].

2.3.3 Nonreduced structures Multiple curves were investigated in the 1990s in connection to Green’s conjecture [14]. Ribbons, in particular, were understood to arise as limits (in the Hilbert scheme of \mathbb{P}^{g-1}) of canonical curves, as the curve becomes hyperelliptic [31].

Definition 2.17 A *ribbon* is a double structure on \mathbb{P}^1 , ie a nonreduced curve R with $R_{\text{red}} = \mathbb{P}^1$ defined by a square-zero ideal $\mathcal{I}_{R_{\text{red}}/R}$ that is a line bundle on R_{red} .

Example 2.18 There is only one ribbon of genus two, R_2 , up to isomorphism: it is the first infinitesimal neighbourhood of the zero section in $\text{Tot}_{\mathbb{P}^1}(\mathcal{O}(3))$. The short exact sequence

$$0 \rightarrow \mathcal{S}_{\mathbb{P}^1/R_2} \simeq \mathcal{O}_{\mathbb{P}^1}(-3) \rightarrow \mathcal{O}_{R_2} \rightarrow \mathcal{O}_{\mathbb{P}^1} \rightarrow 0$$

is split by restricting the projection $\text{Tot}_{\mathbb{P}^1}(\mathcal{O}(3)) \rightarrow \mathbb{P}^1$ to R_2 . R_2 is therefore called a *split ribbon*. In fact, all ribbons of genus at most two are split.

Automorphisms and moduli of multiple curves have been studied in [29]. We are going to encounter nonreduced structures along singular curves as well.

Example 2.19 The simplest example of a Gorenstein nonreduced structure with singular underlying curve is given by the union of a ribbon and a line *along a double point*. Local equations are $\mathbb{C}[[x, y]]/(x^2y)$. It is easy to see that, for such a curve to have genus two, the ribbon needs to have ideal $\mathcal{S}_{\mathbb{P}^1/R} \simeq \mathcal{O}_{\mathbb{P}^1}(-2)$. Such a curve can be realised in the linear system $|2D_+ + F|$ on the Hirzebruch surface \mathbb{F}_2 , where D_+ denotes the class of the positive section and F the class of a fibre.

Example 2.20 More generally, the union of a ribbon, R , with a rational k -fold point along a double point (representing a generic tangent vector to the k -fold point) is Gorenstein. Local equations are given by

$$\mathbb{C}\langle x_1, \dots, x_k, y \rangle / (x_i x_j, (x_i - x_j)y)_{1 \leq i < j \leq k}.$$

To see that this is Gorenstein, it is enough to find a regular element ξ such that the quotient is Gorenstein of dimension zero. Let $\xi = \sum_{i=1}^k x_i - y$. The quotient A is a graded finite-dimensional algebra, with $A_0 = \mathbb{C}$, $A_1 = \mathbb{C}\langle x_1, \dots, x_k \rangle$ and $A_2 = \mathbb{C}\langle x_1^2 = \dots = x_k^2 \rangle$, having one-dimensional socle A_2 .

Now we can obtain a Gorenstein projective curve of genus two, C , by gluing a ribbon together with some k_i -fold points ($i = 1, \dots, r$) at distinct (closed) points of the ribbon, by iterating the local construction above. From the short exact sequence

$$0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_R \oplus \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}^{\oplus k_i} \rightarrow \bigoplus_{i=1}^r (\mathbb{C}^{k_i-1} \oplus \mathbb{C}[\epsilon]) \rightarrow 0,$$

we can compute the Euler characteristic of R , whose structure sheaf thus satisfies

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^1}(r-3) \rightarrow \mathcal{O}_R \rightarrow \mathcal{O}_{\mathbb{P}^1} \rightarrow 0,$$

depending only on the total number of “noded” points r and not on the number of branches of each k_i -fold point.

Definition 2.21 We call C as in [Example 2.20](#) a (k_1, \dots, k_r) -tailed ribbon of genus two.

Remark 2.22 We can employ Noether’s formula ([Proposition 2.6](#)) and adjunction (on a surface containing the ribbon, ie $\text{Tot}(\mathcal{O}_{\mathbb{P}^1}(3-r))$) to compute the restriction of the dualising sheaf to every component of a (k_1, \dots, k_r) -tailed ribbon C . First, the “normalisation” is given by

$$\begin{aligned} \mathbb{C}\langle x_1, \dots, x_k, y \rangle / (x_i x_j, (x_i - x_j)y)_{1 \leq i < j \leq k} &\rightarrow \mathbb{C}\langle s \rangle[\epsilon] / (\epsilon^2) \times \mathbb{C}\langle t_1 \rangle \times \dots \times \mathbb{C}\langle t_k \rangle \\ x_i &\mapsto (\epsilon, 0, \dots, t_i, \dots, 0), \quad i = 1, \dots, k, \\ y &\mapsto (s, 0, \dots, 0). \end{aligned}$$

From this we compute the conductor $\mathfrak{c} = \langle x_1^2, \dots, x_k^2, y \rangle$. The restriction to a tail is

$$\omega_C|_T \simeq \omega_T \otimes \mathcal{O}(2q) \simeq \mathcal{O}_T.$$

The restriction to the \mathbb{P}^1 underlying the ribbon is

$$\omega_C|_{R_{\text{red}}} \simeq \omega_{R_{\text{red}}} \otimes \mathcal{I}_{R_{\text{red}}/R}^\vee \otimes \mathfrak{c}_{R_{\text{red}}}^\vee \simeq \mathcal{O}_{R_{\text{red}}}(-2 + (3-r) + r) \simeq \mathcal{O}_{R_{\text{red}}}(1).$$

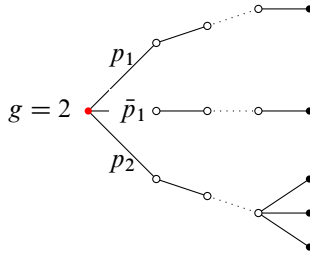


Figure 1: Dual graph of the central fibre of a regular semistable model for a $(2, 3)$ -tailed ribbon with p_1 and \bar{p}_1 conjugate.

For a nodal curve C with core Z , we say that a component D of C cleaves to $q \in Z$ if D is joined to Z at q through a chain of rational curves.

Lemma 2.23 *Regular semistable models of a (k_1, \dots, k_r) -tailed ribbon can be classified. If the core is a smooth curve of genus two, Z , the sets of tails $\{T_1^i, \dots, T_{k_i}^i\}_{i=1, \dots, r}$ cleave to the same (or to conjugate) point(s) of Z , and are all equidistant from the core (independently of i and j in T_j^i). The configuration of attaching points on Z is such that the hyperelliptic cover maps it to the corresponding configuration of noded points on R_{red} , up to reparametrisation.*

Proof This is a word-by-word repetition of the argument of [11, Proposition 4.3]. \square

See Figure 1 for an example of a semistable tail of a tailed ribbon.

Remark 2.24 The description of the semistable models of a (k_1, \dots, k_r) -tailed ribbon (Lemma 2.23) in case the former has reducible core is more cumbersome. Indeed, the length of a tree depends on which component of the core it is attached to. See Proposition 2.26 and Figure 2 for a precise description in terms of piecewise-linear functions on the tropicalisation. A ribbon arises when the special component (which we denote pictorially by a red dot) belongs to the core. The piecewise-linear function then has constant slope 1 on any tree emanating from the core, and in this sense it is reminiscent of the genus one situation.

Example 2.25 A semistable model of the 1-tailed ribbon C (Example 2.19) can be computed by taking the pencil

$$u_+^2 f_1 - tu_-^2 p_5(f_1, f_2) = 0$$

on $\mathbb{F}_2 \times \mathbb{A}_t^1$, with p_5 a generic homogeneous polynomial of degree 5 in two variables. The resulting smoothing family is singular at six points along the central fibre, including

the node of C . After blowing them up and normalising, we obtain a genus two curve covering the ribbon two-to-one; the tail of C is attached to a Weierstrass point. This appears to be an accident due to the restriction that the smoothing comes from a pencil on \mathbb{F}_2 .

In order to describe semistable tails in complete generality, there is no better way than expressing the problem in terms of existence of a certain piecewise-linear function on the dual graph. We refer the reader to Sections 2.1.4 and 2.2 for background and notation.

Proposition 2.26 *Let C be a Gorenstein curve of arithmetic genus two. Let $\mathcal{C} \rightarrow Y$ be a one-parameter smoothing of C . Let $\phi: \mathcal{C}^{\text{ss}} \rightarrow \mathcal{C}$ be a semistable model with exceptional locus Z . Then, up to destabilising \mathcal{C}^{ss} , there exists a hyperelliptic admissible cover $\psi: \mathcal{C}^{\text{ss}} \rightarrow \mathcal{T}$ and a tropical canonical divisor $\lambda: \square \rightarrow \mathbb{R}$ pulled back along $\text{trop}(\psi)$ such that Z is cut out by $\mathcal{O}_{\mathcal{C}^{\text{ss}}}(-\lambda)$, and we have $\omega_{\mathcal{C}^{\text{ss}}} = \phi^* \omega_{\mathcal{C}}(\lambda)$.*

Proof The proof in the isolated case has appeared in [11, Section 4.4]. The argument is insensitive to rational tails away from the core, so we may as well assume that C is minimal (see Definition 2.29 below). In this case, the canonical linear series gives a two-to-one map $\bar{\psi}: \mathcal{C} \rightarrow \mathbb{P}_{\mathcal{T}}(\pi_* \omega_{\mathcal{C}}) =: \mathbb{P}$. The general fibre is the hyperelliptic cover of a smooth curve of genus two. Applying semistable reduction to the target and branch divisor, we may lift $\bar{\psi}$ to a map of nodal curves, thanks to the properness of the moduli space of admissible covers

$$\begin{array}{ccc} \mathcal{C}^{\text{ss}} & \xrightarrow{\psi} & \mathcal{T} \\ \downarrow \phi & & \downarrow \phi_{\mathcal{T}} \\ \mathcal{C} & \xrightarrow{\bar{\psi}} & \mathbb{P} \end{array}$$

The line bundles $\phi_{\mathcal{T}}^* \mathcal{O}_{\mathbb{P}}(1)$ and $\omega_{\mathcal{T}}(\frac{1}{2} D_B)$ have the same total degree on \mathcal{T} , so, since the latter is a tree, their difference is associated to a piecewise-linear function $\lambda_{\mathcal{T}}$ on \mathcal{T} . Pulling back via \mathcal{C} , on the other hand, we find

$$\phi^* \omega_{\mathcal{C}} = \omega_{\mathcal{C}^{\text{ss}}}(-\text{trop}(\psi)^* \lambda_{\mathcal{T}}). \quad \square$$

For the reader’s benefit, Figure 2 provides a pictorial description of such piecewise-linear functions in case the core is a genus two configuration of rational curves, including the slopes along the nodes (or rational chains). This is where the admissible functions of Section 3.2 originate from. The two columns correspond to the two maximally

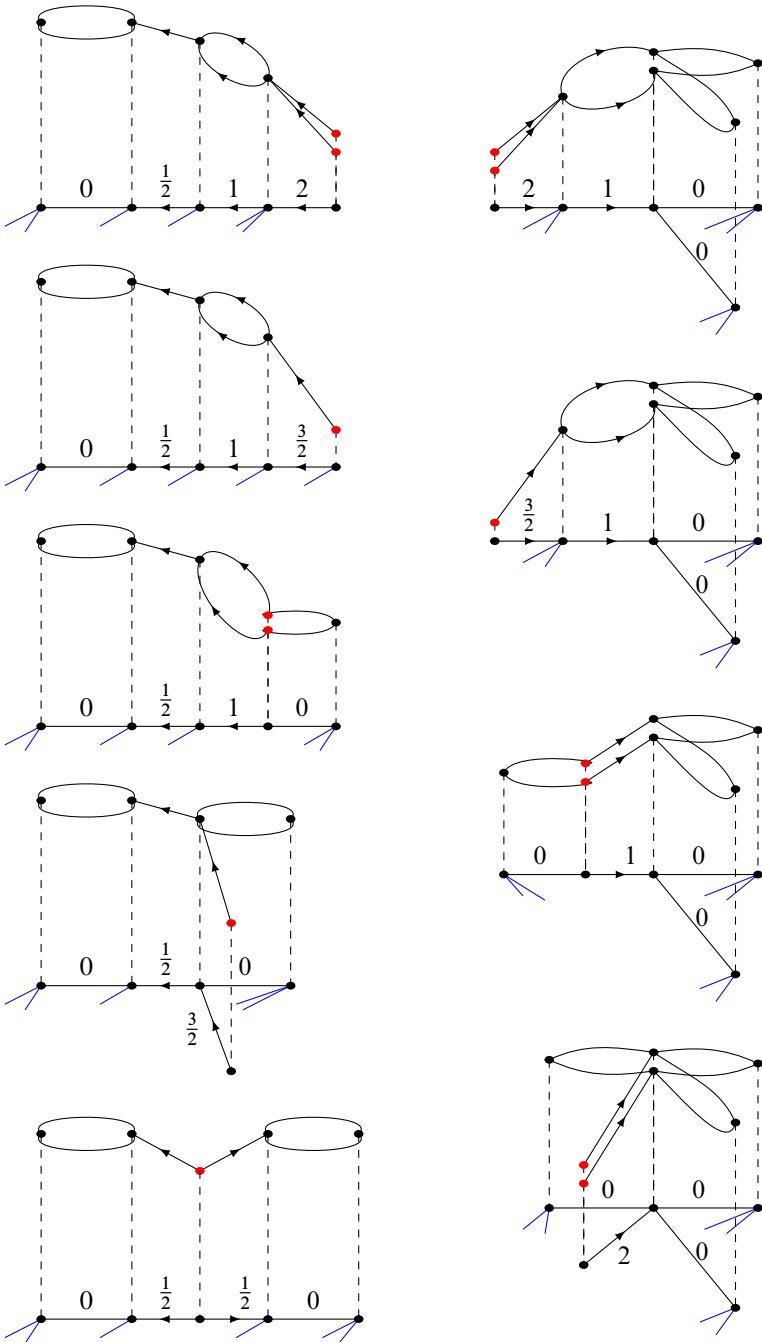


Figure 2: Admissible functions on maximally degenerated tropical hyper-elliptic curves: the dumbbell (left) and the theta graph (right).

degenerate (stable) tropical curves of genus two. Here, each red vertex corresponds to a component mapping to a special component of the singularity/ribbon. Blue legs represent branch points of the admissible cover. Tails are not drawn, but can be understood by a tropical modification: the corresponding vertices would lie on trees of slope 1 towards the core, all at the same height, and lower than each red vertex. See also the conventions set out in [Section 3.2](#).

Remark 2.27 The nonreduced structures of genus two are numerous; indeed, as opposed to the case of isolated singularities, where the δ -invariant is always related to the number of branches via the genus, in the nonreduced case a very large δ -invariant can always be compensated by lowering the genus of the ribbon (which can be negative).

Yet, if we require $\omega_C \geq 0$, we find a characterisation of tailed ribbons.

Proposition 2.28 *Let C be a Gorenstein curve of genus two, consisting of a ribbon glued in some way to a number of (possibly singular) rational curves. Assume that the dualising sheaf is nonnegative, and positive on the ribbon. Then C is a (k_1, \dots, k_r) -tailed ribbon.*

Proof We may assume that the normalisation sequence is

$$0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_R \oplus \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}^{\oplus k_i} \rightarrow \bigoplus_{i=1}^r \mathcal{F}_i \rightarrow 0,$$

where the \mathcal{F}_i are sheaves supported at the gluing points; call δ_i their lengths. We claim that $\delta_i \geq k_i$; indeed, this is the minimum number of conditions that we need to impose to ensure that the values of the functions on different branches agree at the preimage of the singular point. But in fact we may even assume $\delta_i \geq k_i + 1$, by the Gorenstein assumption and [Lemma 2.13](#).

Now it is easy to compute that, for the genus of C to be two, we need

$$\mathcal{S}_{R_{\text{red}}/R} \simeq \mathcal{O}_{\mathbb{P}^1} \left(\sum_{i=1}^r (\delta_i - k_i) - 3 \right).$$

So, by applying Noether’s formula and adjunction as in [Remark 2.22](#), we find

$$\omega_C|_{R_{\text{red}}} \simeq \omega_{\mathbb{P}^1} \otimes \mathcal{S}_{R_{\text{red}}/R}^{\vee} \otimes (\mathfrak{c}|_{R_{\text{red}}})^{\vee} \simeq \mathcal{O}_{\mathbb{P}^1} \left(1 + \sum_{i=1}^r (k_i - \deg(\mathfrak{c}_i|_{R_{\text{red}}}) - \delta_i) \right).$$

We study the local contributions $k_i - \deg(\mathfrak{c}_i |_{R_{\text{red}}}) - \delta_i$ around each singular point. We are then looking for Gorenstein subalgebras A of $\tilde{A} = \mathbb{C}[[s]][[\epsilon]](\epsilon^2) \times \mathbb{C}[[t_1]] \times \dots \times \mathbb{C}[[t_k]]$. By [Corollary 2.7](#), we know that \mathfrak{c} is principal as an ideal of \tilde{A} ; up to rescaling the generators, we may assume that \mathfrak{c} is generated by the element

$$(s^c + \epsilon s^d, t_1^{c_1}, \dots, t_k^{c_k}).$$

Indeed, under our assumptions, the degree of ω_C has to be one on R_{red} and zero on every tail; thus, it is easy to see that $c_1 = \dots = c_k = 2$. Now, if $d \geq c$, then $s^c \in \mathfrak{c}$; if $d < c$, then $s^{2c-d} \in \mathfrak{c}$. In the second case,

$$\tilde{A}/\mathfrak{c} = \langle 1, s, \epsilon, \dots, \epsilon s^d (= -s^c), s^{d+1}, \dots, s^{c-1}, \epsilon s^{c-1} (= s^{2c-d-1}); 1, t_1; \dots; 1, t_k \rangle.$$

In any case we find $2\delta = \dim_{\mathbb{C}} \tilde{A}/\mathfrak{c} = 2c + 2k$ (by [Lemma 2.9](#)), and $\delta_i = k_i - \deg(\mathfrak{c}_i |_{R_{\text{red}}})$ for all i .

By assumption (that the reduced subcurve underlying R is smooth), we must have a generator y of A whose linear part contains $s + \epsilon p(s)$. By [Lemma 2.13](#) and $t_i^2 \in \mathfrak{c}$, there must be generators with nontrivial linear part in t_i . Up to taking polynomial combinations of these generators, we may assume they take the form

$$y = s + \epsilon p(s), \quad x_i = \epsilon q_i(s) \oplus t_i, \quad i = 1, \dots, k,$$

where p and q_i are monomial. Since $1 + \epsilon \tilde{p}(s)$ is invertible in \tilde{A} , we may assume that $y = s$. Furthermore, by looking at the conductor ideal, we see that for x_i we may assume $q_i(s) = q_i s^{c-1}$, with $q_i \in \mathbb{C}^\times$. Up to blowing \tilde{A} down (see [Remark 2.5](#)), we may reduce to the case $c = 1$. Thus, we find the singularities of [Example 2.20](#), with local equations

$$\mathbb{C}[[x_1, \dots, x_k, y]] / (x_i x_j, x_i y - x_j y)_{1 \leq i < j \leq k}$$

and ideal sheaf $\mathcal{S}_{R_{\text{red}}/R} = \mathcal{O}_{R_{\text{red}}}(r - 3)$. □

2.4 h^1 -vanishing

First we describe all the possible cores. Recall the following:

Definition 2.29 A curve is *minimal* if there is no proper subcurve of the same arithmetic genus.

Remark 2.30 Up to replacing nodes with rational bridges, minimal curves of genus one are either irreducible or elliptic m -fold points with rational branches ($m \geq 2$).

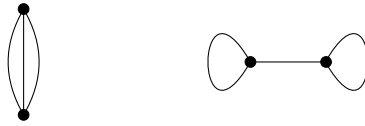


Figure 3: Minimal theta and dumbbell configurations (dual graphs).

Proposition 2.31 *Up to replacing nodes with rational bridges, a **minimal** Gorenstein curve of genus two with isolated singularities is one of the following:*

- (1) Irreducible.
- (2) The nodal union of two minimal curves of genus one (including a dumbbell configuration of \mathbb{P}^1 s; see Figure 3).
- (3) A theta configuration of \mathbb{P}^1 s (see Figure 3).
- (4) A genus one singularity with a minimal genus one branch.
- (5) A type I singularity with rational branches.
- (6) A type II singularity with rational branches.

Lemma 2.32 *With numbering as in the previous proposition, the dualising line bundle of a minimal Gorenstein curve of genus two has*

- (4) degree 2 on the genus one branch, or on the common branch of the two elliptic m -fold points;
- (5) degree 2 on the special branch;
- (6) multidegree $(1, 1)$ on the special branches.

Proof This follows from Noether’s formula (Proposition 2.6) and an explicit calculation. □

Now we describe sufficient conditions for the vanishing of higher cohomology of a line bundle on a minimal curve of genus two. This will allow us to prove the unobstructedness of the λ -aligned admissible maps satisfying the factorisation property. The proof of the following lemmas boils down to a simple albeit tedious application of the normalisation exact sequence.

Lemma 2.33 *Let C be a minimal Gorenstein curve of genus two with isolated singularities. A line bundle L on C having nonnegative multidegree, positive degree on every subcurve of genus one, degree at least two and $L \neq \omega_C$ has vanishing h^1 .*

Lemma 2.34 *Let C be a (k_1, \dots, k_r) -tailed ribbon and L a nonnegative line bundle on it. Then $h^1(C, L) = 0$ if*

- L has positive degree on at least two k -fold points; or
- L restricts to $\mathcal{O}_{\mathbb{P}^1}(1)$ on R_{red} (where R denotes the component of multiplicity two), and it has positive degree on at least one k -fold point.

3 Gorenstein curves over aligned admissible covers

Definition 3.1 A weighted admissible cover consists of a hyperelliptic admissible cover

$$\psi : (C, D_R) \rightarrow (T, D_B)$$

with prestable target, together with a weight function $w : V(\square) \rightarrow \mathbb{N}$, such that, with the induced weight function $w_T : V(T) \rightarrow \mathbb{N}$ given by

$$w_T(v) = \sum_{v' \in \psi^{-1}(v)} w(v'),$$

the rational tree T is *weighted-stable* (every weight zero component has at least three nodes or branch points). We denote by $\mathcal{A}_2^{\text{wt}}$ the smooth Artin stack of weighted admissible covers where C has arithmetic genus two.

3.1 The intuition and strategy

Our goal is to produce a morphism to a Gorenstein curve $\mathcal{C} \rightarrow \bar{\mathcal{C}}$ such that a line bundle on $\bar{\mathcal{C}}$ of degree as prescribed by the weight function w would have vanishing higher cohomology. This would ensure the unobstructedness of the space of maps to projective space.

Classically, we would look for a vertical divisor \mathcal{Z} , supported on the exceptional locus of $\mathcal{C} \rightarrow \bar{\mathcal{C}}$, such that the contraction is associated to a line bundle of the form $\omega_{\mathcal{C}}(\mathcal{Z})$ — this guarantees that $\bar{\mathcal{C}}$ is Gorenstein. Tropically, \mathcal{Z} is replaced by a PL function λ on \square . The relevant information encoded by λ is a collection of slopes along the edges of \square . These are the objects of the stack **Div**, introduced in [53, Section 4.2].

Finding λ such that $\omega_{\mathcal{C}}(\lambda)$ is trivial on the exceptional locus of $\mathcal{C} \rightarrow \bar{\mathcal{C}}$ reduces to a simple calculation of degrees — as opposed to a more complicated equality in the

Picard group of a genus two curve — because we impose that λ is pulled back from the target of the admissible cover.

When the weight of the core is zero, basically any of the positive-weight tails can be elected as the special branch of the Gorenstein singularity \overline{C} , be it isolated or a ribbon. The choice of λ can be thought of as the choice of a degree one divisor (ie a point) on the target of the admissible cover. The support of this divisor corresponds to the special component of the singularity.

Given a standard tropical weighted admissible cover over $\mathbb{R}_{\geq 0}$, the function λ is uniquely determined. It is a member of the tropical canonical linear series on the maximal subcurve $\Delta \subseteq \square$ (making it into a *level graph* in the sense of [9]) such that

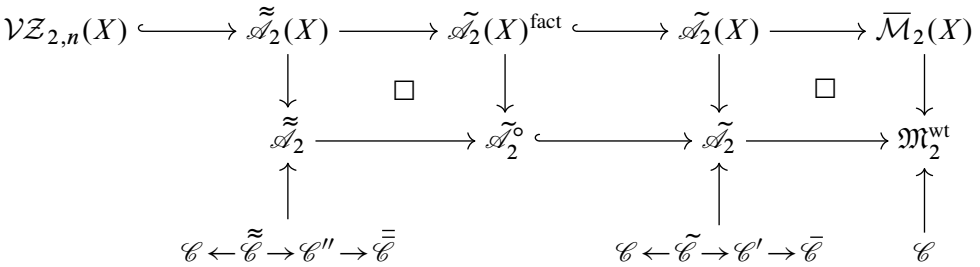
- the interior Δ° contains every subcurve of arithmetic genus g and $w \leq 2g - 2$;
- every connected component Ω of Δ° has weight at most $2p_a(\Omega) - 2$, and $\overline{\Omega}$ has weight at least $2p_a(\Omega) - 1$.

For a more general family of tropical curves, this determines a (polyhedral and simplicial) subdivision of the base cone. In order to prove this, we actually define λ by interpolating among finitely many piecewise-linear functions, which we call *admissible*. Our definition of admissible functions has been inspired by that of Bozlee's *mesa curves* [18, Section 3]. It could be said that it constitutes a Copernican revolution with respect to [63].

When the special component has weight 2, if this represents the degree of a map, the only possibility is that the map restrict on the core to the hyperelliptic cover of a line. In this case, factorisation through the Gorenstein singularity is not enough to ensure smoothability. If we again replace the core by a ribbon, imposing a second factorisation requires the map to satisfy some ramification condition on the nearby branches, which turns out to be sufficient for the obstructions to vanish.

In order to avoid complications (and nasty singularities) arising from nonfactoring situations — ie when the core has weight 2, but it cannot possibly be the weight of the dualising bundle of a Gorenstein singularity — we proceed in two steps: First, in Section 3.2, we make sure that every subcurve of positive genus has positive weight. Then, in Section 3.4, we discard the locus where the weight of the core is 1 or $1 + 1$. After doing this, we can safely replace any weight two core with a ribbon. We sum up

the construction of this and the next sections in the diagram of moduli spaces



3.2 Admissible functions and aligned admissible covers

Let $\psi: (C, D_R) \rightarrow (T, D_B)$ be a weighted admissible cover over S , endowed with the minimal logarithmic structure. Let T denote the tropicalisation of the target. T is therefore a weighted tree, metrised in the monoid \overline{M}_S . The metric structure can be disregarded for the minute.

It follows from the Riemann–Hurwitz formula that the $\frac{1}{2}\mathbb{Z}$ -divisor, supported on the vertices of T ,

$$(4) \quad D'(v) = \text{val}(v) - 2 + \frac{1}{2} \deg(D_B|_v),$$

where $\text{val}(v)$ denotes the valence (number of adjacent edges) of the vertex v and D_B is the branch divisor of the cover, pulls back to the canonical divisor of \square (see Section 2.1.4). The degree of D' on T is 1.

Let D be another $\frac{1}{2}\mathbb{Z}$ -divisor of degree 1 on T . Since T is a tree, its Jacobian is trivial, so any two divisors of the same degree are linearly equivalent (this carries over to $\frac{1}{2}\mathbb{Z}$ -divisors). Therefore, there exists a unique collection $\bar{\lambda}_T$ of half-integral slopes on the edges of T such that

$$D = D' + \text{div}(\bar{\lambda}_T).$$

If D has integral coefficients, we observe that half-integral slopes may only appear along edges over which $\text{trop}(\psi)$ has expansion factor two, so it is possible to think of $\bar{\lambda}_T$ as a piecewise-linear function on T with values in $\overline{M}_S^{\text{gp}}$, up to a global translation of $\overline{M}_S^{\text{gp}}$. In fact, it may be more accurate to think of $\bar{\lambda}_T$ as a PL function on the tropicalisation of the orbicurve $[C/\mathfrak{S}_2]$.

If D is effective, the pullback $\bar{\lambda}$ of $\bar{\lambda}_T$ along $\text{trop}(\psi)$, or, rather, the divisor $K_{\square} + \text{div}(\bar{\lambda})$, is an element of the tropical canonical linear system.

Definition 3.2 Let S be a geometric point. An *admissible function* $\bar{\lambda}$ on \square is a piecewise-linear function with integral slopes (with values in $\overline{M}_S^{\text{gp}}$, but defined only up to a global translation by $\overline{M}_S^{\text{gp}}$) such that

- $\bar{\lambda}$ is the pullback along $\text{trop}(\psi)$ of a (possibly half-integral) $\bar{\lambda}_T$ on T ;
- $K_{\square} + \text{div}(\bar{\lambda}) \geq 0$ defines an element of the tropical canonical system, which is the pullback of an effective, degree 1 divisor D supported on *exactly one* vertex of T .

We are going to call *special* the vertex of T supporting D , or its preimage(s) in \square . With an eye to the future, it will correspond to the special branch(es) of the Gorenstein curve of genus two.

Remark 3.3 There are only *finitely many* admissible functions compatible with a given ψ . Indeed, they are in bijection with the vertices of T .

When the core is maximally degenerate, ie a configuration of rational curves, all the possible admissible functions are depicted in [Figure 2](#). The red vertices represent the ones supporting $K_{\square} + \text{div}(\bar{\lambda})$. Missing from the picture is what happens outside the core, but this is easily explained: there may be any number of rational trees, on which $\bar{\lambda}$ has constant slope 1 towards the core.

For the sake of concreteness, we now look at some examples, adopting the:

Convention 3.4 With the application to stable maps in mind, we think of the source of an admissible cover as the destabilisation of a weighted-stable curve. In the following, we represent a component of positive weight by a black circle, and a component of weight zero by a white one, unless it is unstable (either a rational tail introduced as the conjugate of a rational tail of positive weight, or a rational bridge introduced by slicing \square), in which case it is represented by a cross and the corresponding edge is dotted. A red vertex represents the component supporting the divisor D , or its preimages in \square . The blue legs (*B-legs*) represent the branching divisor D_B of ψ (see [Definition 2.1](#)): the number of *B-legs* attached to a vertex v of T is the degree of $D_B \cap T_v$, where T_v is the irreducible component of T corresponding to v .

Example 3.5 Assume the core is irreducible of weight zero, and there are two tails of (large) positive weight: one of them (R) is attached to a Weierstrass point of the

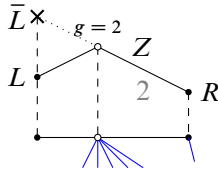


Figure 4: A weighted admissible cover with weight zero core.

core; the other one (L) is attached to a general point, so that a weight zero tail (\bar{L}) must be sprouted from the conjugate point in order for the admissible cover to exist (Figure 4). The tropical cover has expansion factor 2 (in grey) along the rightmost edge, corresponding to the ramification order of the algebrogeometric map at each branch of the node.

There are three admissible functions $\bar{\lambda}_i$ compatible with the given weighted admissible cover; see Figure 5.

Note that the pullback of D always has degree 2—although the tropical cover is injective on the rightmost edge, the expansion factor 2 provides the correct multiplicity. It may be instructive to compute the multidegree of various tropical divisors on \square —we represent them as vectors in \mathbb{Z}^4 by ordering the components from left to right, (\bar{L}, L, Z, R) :

	$\text{div}(\bar{\lambda}_i)$	$\text{trop}(\psi)^* D = K_{\square} + \text{div}(\bar{\lambda}_i)$
(i)	$(1, 1, -3, 1)$	$(0, 0, 2, 0)$
(ii)	$(1, 1, -5, 3)$	$(0, 0, 0, 2)$
(iii)	$(2, 2, -3, 1)$	$(1, 1, 0, 0)$

The canonical K_{\square} gives $(-1, -1, 5, -1)$.

Definition 3.6 A *prealigned* admissible cover over a logarithmic scheme (S, M_S) is one for which the values

$$(5) \quad \{\bar{\lambda}(v) - \bar{\lambda}(v') \mid v, v' \in V(\square)\} \subseteq (\bar{M}_S, \geq)$$

are comparable for every admissible function $\bar{\lambda}$ compatible with ψ (Definition 3.2).

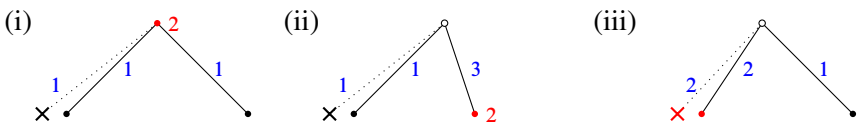


Figure 5: Admissible functions compatible with Figure 4.

Prealigned admissible covers form a subfunctor of weighted admissible covers over (LogSch). The next lemma follows from the definitions and Section 2.1.2.

Lemma 3.7 *The minimal logarithmic structure of a prealigned admissible cover is obtained, starting from the minimal logarithmic structure of the admissible cover, by adding in the elements of $\overline{M}_S^{\text{gp}}$ indicated in (5) and sharpifying.*

Remark 3.8 Aligning determines a subdivision Σ of the tropical moduli space $\sigma = \text{Hom}(\overline{M}_S, \mathbb{R}_{\geq 0})$ or, equivalently, a logarithmic modification of $\mathcal{A}_2^{\text{wt}}$, which will be denoted by $\mathcal{A}_2^{\text{pre}}$. See Section 2.1.5.

Let ψ be a prealigned admissible cover, and $\{\bar{\lambda}_1, \dots, \bar{\lambda}_s\}$ the set of admissible functions compatible with ψ . We are going to choose genuine PL functions λ_i on \square with values in $\overline{M}_S^{\text{gp}}$ lifting the $\bar{\lambda}_i$. A lift determines and is determined by the set of vertices mapping to 0 in $\overline{M}_S^{\text{gp}}$; let us denote by $\square_{\geq 0}$ (resp. $\square_{>0}$) the set of vertices with values in $\overline{M}_S \subseteq \overline{M}_S^{\text{gp}}$ (resp. $\overline{M}_S \setminus \overline{M}_S^*$; note that we may assume $\overline{M}_S^* = \{0\}$). In the end, we are going to define the object of interest as an interpolation/truncation of these lifts.

Definition 3.9 Let S be a geometric point. Define a lift $\lambda_i \in \Gamma(S, \pi_* \overline{M}_C^{\text{gp}})$ of $\bar{\lambda}_i$ by requiring that $w(\square_{>0}) \leq 0$ and $w(\square_{\geq 0}) \geq 1$.

Definition 3.10 Define a PL function on a subdivision $\tilde{\square}$ of \square by

$$\lambda = \max\{0, \lambda_1, \dots, \lambda_s\}.$$

We denote by Δ° the support of λ (appeared as $\square_{>0}$ before), and by Δ the minimal subcurve of arithmetic genus two in $\tilde{\square}$ containing the closure of Δ° .

Remark 3.11 Parallel to the above definition, we denote by Δ_T° the support of λ_T , and by Δ_T the image of Δ under $\text{trop}(\psi)$.

Since tropical linear systems are tropically convex (see Section 2.1.4), it is still true that λ is the pullback of a PL function λ_T (with slopes in $\frac{1}{2}\mathbb{Z}$) on \mathbb{T} , such that $D' + \text{div}(\lambda_T)$ is an effective divisor of degree 1 on Δ_T (notice that here, in the definition of D' , the valence of a vertex is the one in Δ_T and not the one in \mathbb{T}). This is clear away from 0, where λ coincides with one of the admissible functions above; at 0, on the other hand, all the slopes of λ_T are nonnegative, so the only doubt is for a vertex of valence 1 supporting exactly one B -leg; but such a vertex corresponds to a rational tail in Δ ,

so it can be included in Δ_T only if the slope of λ_T along the unique adjacent edge is strictly positive.

The divisor $D = D' + \text{div}(\lambda_T)$ need not be supported on a vertex of T , and could be the sum of two half-points in the locus over which $\text{trop}(\psi)$ has expansion factor 2.

Since by Riemann–Hurwitz we have $\text{trop}(\psi)^*(D') = K_\Delta$,

$$(6) \quad \text{trop}(\psi)^*(D) = K_\Delta + \text{div}(\lambda).$$

Lemma 3.12 *On every cone of $\text{trop}(\mathcal{A}_2^{\text{pre}})$, λ is well defined as a PL function on a subdivision of \square .*

Proof For λ to be well defined, we need to argue that, for every vertex v of \square , the values $\{\lambda_i(v) \mid i = 1, \dots, s\}$ are comparable, and there is a unique way of subdividing \square in order to make λ piecewise linear. Notice that, a priori, we only know that the values of a single λ_i at the vertices of \square are ordered.

It is convenient to work on T . The leaves of T have positive weight due to stability, so λ_T necessarily takes the value 0 on them.

In order to prove that the maximum is well defined at every vertex, we can proceed inductively from the leaves and run through all of T . Assume that $\lambda_T(v_1)$ has been determined. We need to establish the behaviour of λ_T on the edge e between v_1 and v_2 , and its value on v_2 . Upon relabelling, we may suppose that λ_1 is the function with maximal slope along e among the ones with $\lambda_i(v_1) = \lambda_T(v_1)$; so λ_T coincides with λ_1 in a neighbourhood of v_1 . If λ_1 is also the function of maximal slope along e among all the admissible functions, then $\lambda_T(v_2) = \lambda_1(v_2)$, and we are done.

If not, there must be a function λ_2 with slope along e greater than that of λ_1 but $\lambda_2(v_1) \leq \lambda_1(v_1)$. First assume that $\bar{\lambda}_1$ and $\bar{\lambda}_2$ differ only along e , ie $D(\lambda_1, v_2) = 1$ and $D(\lambda_2, v_1) = 1$. Then there is a unique way to interpolate between λ_1 and λ_2 along e , namely by solving the equations

$$\begin{cases} \ell(e) = \ell(e') + \ell(e''), \\ \lambda_2(v_2) - \lambda_1(v_1) = s(\lambda_1, e)\ell(e') + s(\lambda_2, e)\ell(e''). \end{cases}$$

This uniquely determines the subdivision of e on which λ_T is piecewise linear, and the values of λ_T on both v_2 and the newly introduced vertex $v_{1,2}$.

More generally, λ_1 and λ_2 will differ along the chain R connecting the support of D_1 to that of D_2 , and containing the edge e . Then $\lambda_2 - \lambda_1 = D_2 - D_1$ is a rational function

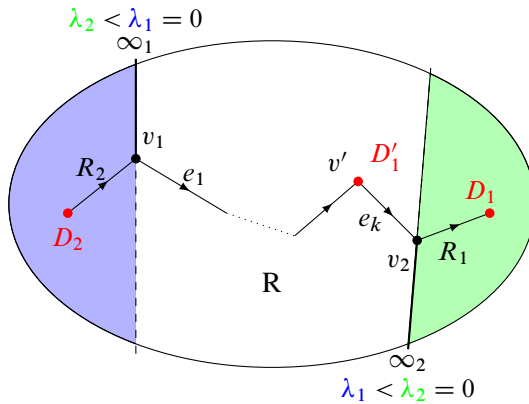


Figure 6: Comparison of λ_1 and λ_2 .

with constant slope 1 along R . In order to define λ_T , we need to interpolate between λ_1 and λ_2 along R ; see Figure 6.

We will argue that this is possible because we can always inductively reduce to the above case, for which we have an explicit formula for the interpolation.

We may assume that R_2 has zero length, because changing λ_2 on R_2 increases it without affecting $\lambda_2 \leq \lambda_1$ (in the blue-shaded region), and similarly for R_1 .

Now let $\bar{\lambda}'_1$ be the function that differs from $\bar{\lambda}_1$ only on the last edge e_k of R . If $\lambda'_1(\infty_1) = 0$, then $\lambda'_1 \geq \lambda_1$, so that we can instead compare λ'_1 and λ_2 , which is possible by the inductive assumption. If instead $\lambda'_1(\infty_1) < 0$, then $\lambda'_1(\infty'_1) = 0$ for some ∞'_1 in the green-shaded region (since $\lambda'_1 < \lambda_1$ on the complement). In fact, it is easy to see that $\lambda'_1(\infty_2) = 0$, so $\lambda_2(v') = \lambda'_1(v') < \lambda_1(v')$, and we conclude that λ_T interpolates between λ_1 and λ_2 (equivalently λ'_1) only along e_k — which we have already solved. \square

Remark 3.13 Over a geometric point, there is no obstruction to defining the lifts λ_i of $\bar{\lambda}_i$. On the other hand, they are stable under generisation (edge contraction), because the weight of $\square_{>0}$ stays the same, and the weight of $\square_{\geq 0}$ can only go up. So both the λ_i and λ define sections of $\pi_* \bar{M}_C$ over any base S .

Definition 3.14 Let Σ' be the subdivision of $\sigma = \text{trop}(\mathcal{A}_2^{\text{wt}})$ determined by applying universal semistable reduction [56, Theorem 2.4.2] to the morphism $\tilde{\square} \rightarrow \sigma$.

Let $a: \tilde{\mathcal{A}}_2 \rightarrow \mathcal{A}_2^{\text{wt}}$ be the corresponding logarithmically étale model.

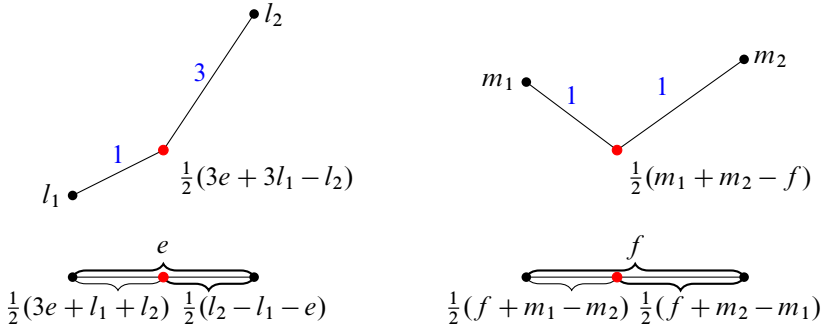


Figure 7: The two situations in which half-integral lengths occur.

It follows from the construction and from [56; 4] that there is a logarithmically smooth curve $\tilde{\mathcal{C}} \rightarrow \tilde{\mathcal{A}}_2$, which is a partial destabilisation of $a^*\mathcal{C}$, such that $\tilde{\mathcal{C}}$ is its tropicalisation, $\lambda \in \Gamma(\tilde{\mathcal{C}}, \overline{M}_{\tilde{\mathcal{C}}})$ and all of its values at vertices of $\tilde{\mathcal{C}}$ are comparable.

Remark 3.15 Σ' is neither finer nor coarser than the subdivision Σ previously defined in Definition 3.6. It is not finer, because λ is insensitive to the comparison of values of λ_i below 0. It is not coarser, because cones of $\tilde{\mathcal{C}}$ do not map surjectively to cones of Σ ; see Example 3.39.

Notice also that a is not simply a logarithmic blow-up in general. Indeed, due to the existence of some simplicial but not smooth cones in the subdivision, in order for the lattice map to be surjective it may be necessary to perform a Kummer extension of the base logarithmic structure [17, Section 4]:

$$M_{\mathcal{A}} \subseteq M_{\tilde{\mathcal{A}}} \subseteq \frac{1}{2} M_{\mathcal{A}}^{\text{gp}},$$

as prescribed by the subdivision $\tilde{\mathcal{C}}$ and the slopes of λ ; see Figure 7. This operation should be thought of as a generalised root stack construction, or generalising the presentation of a simplicial affine toric variety as the coarse moduli of $[\mathbb{A}^n/G]$, where G is the finite abelian group encoding the difference between the two lattices at stake. This is different from the genus one case (it has to do with the appearance of slopes other than 1 in λ), but it is nothing new with respect to [56].

Theorem 3.16 *The moduli space $\tilde{\mathcal{A}}_2$ of aligned admissible covers is a logarithmically smooth stack with locally free logarithmic structure (and therefore smooth) admitting a log étale morphism to $\mathcal{A}_2^{\text{wt}}$.*

Proof The construction of the minimal logarithmic structure can be traced back to Definition 3.14. The fact that it is locally free can be justified as follows: λ provides us

with a PL map from $\tilde{\square}$ to a polyhedral subdivision of $\mathbb{R}_{\geq 0}$. The rank of the logarithmic structure is the number of finite edges in the polyhedral subdivision of $\mathbb{R}_{\geq 0}$. For every level, there is at most one edge of $\tilde{\square}$ mapping to it with expansion factor strictly greater than 1; the smoothing parameter corresponding to the said edge of $\tilde{\square}$ can be taken as a generator of the minimal logarithmic structure.

Logarithmic smoothness follows from that of $\mathcal{A}_2^{\text{wt}}$, since $a: \tilde{\mathcal{A}}_2 \rightarrow \mathcal{A}_2$ is logarithmically étale.

Finally, a scheme logarithmically smooth over the trivial log point is smooth if and only if the stalks of its characteristic sheaf are locally free [58, Lemma 5.2]. \square

3.3 Combinatorial properties of λ

In the following, we work over the standard log point, so \square is a graph metrised in \mathbb{R} . To an open subcurve $\square_* \subseteq \square$ there corresponds a proper subcurve $C_* \subseteq C$; for a closed tropical subcurve, we take its algebrogeometric counterpart to be the closure of the preimage under tropicalisation. By abuse of notation, we will refer to \square_* or to C_* interchangeably.

Lemma 3.17 *Every component Ω of Δ° has positive genus.*

Proof Restricting (6) to Ω , we find

$$0 \leq \deg(\text{trop}(\psi)^*(D)|_\Omega) = 2p_a(\Omega) - 2 + \sum_{i=1}^h (1 + s(\lambda, e_i)),$$

where e_1, \dots, e_h are the edges outgoing from Ω . Since by assumption $s(\lambda, e_i) \leq -1$ for every i , we see that $p_a(\Omega)$ cannot be 0. \square

Lemma 3.18 *If $D = \frac{1}{2}(D_1 + D_2)$, then $\lambda(D_1) = \lambda(D_2) = 0$.*

Proof Since by definition every λ_i has D supported on a single vertex, this situation may only occur after interpolating with the zero function.

We first explain how to reduce to the path between D_1 and D_2 . Consider an edge e of Δ_T outside this path, and denote by Δ_- and Δ_+ the connected components of $\Delta_T \setminus \{e\}$. Assume that D is entirely supported on Δ_+ . Restricting (6) to Δ_- , we find

$$0 = \text{val}(\Delta_-) - 2 + \frac{1}{2} \#D_B|_{\Delta_-} + \text{div}(\lambda_T|_{\Delta_-}).$$

We notice that $\text{val}(\Delta_-) = 1$ and that $\text{div}(\lambda_T|_{\Delta_-})$ is simply the slope s of λ_T along e , oriented away from Δ_- . Then $s = 1 - \frac{1}{2} \#D_B|_{\Delta_-}$. Examining the restriction of (6) to Δ_+ , we notice that it is unaltered by contracting e and all the edges in Δ_- .

Assume by contradiction that $\lambda_T(D_1) < \lambda_T(D_2)$. By the previous reduction, we may assume that Δ_T consists of the path between D_1 and D_2 only. By assumption, there must be an edge e on this path along which λ_T has nonzero slope. Contract all the rest. Once again by (6) it is easy to see that the slope can be $\frac{1}{2}$ (resp. 1) from D_1 to D_2 if and only if the distribution of B -legs is $(2, 4)$ (resp. $(1, 5)$). Now, in order to see the half-points, the slope should be $\frac{1}{2}$; on the other hand, it has to be the slope of one of the λ_i , and we notice that they have integral slope along e if the distribution is $(2, 4)$. This is a contradiction, so we conclude that $\lambda_T(D_1) = \lambda_T(D_2)$. The value has to be 0 because half-points can only appear after interpolating with the zero function. \square

Corollary 3.19 *In the situation of the previous lemma, let e be any edge of Δ_T between D_1 and D_2 . Then each of the connected components of $\text{trop}(\psi)^{-1}(\Delta_T \setminus e^\circ)$ has genus one.*

Proof Once we know that λ_T has slope 0 between D_1 and D_2 , this follows from a straightforward application of (6). \square

We can now make the following:

Definition 3.20 Let ρ_1 denote the value of λ on the special vertex/vertices.

Lemma 3.21 *The subcurve $\square_{\geq \rho_1}$ is connected.*

Proof Suppose that there are two connected components Δ_1 and Δ_2 . Since \square is path-connected, we can find an oriented path P from Δ_1 to Δ_2 . The slope of λ must be negative at the beginning and positive at the end of P , so $\text{div}(\lambda)|_P \geq 2$. Restricting (6) to P , we find

$$0 = \text{trop}(\psi)^*(D)|_P = 2p_a(P) - 2 + (\text{div}(\lambda)|_P + 2),$$

or $p_a(P) \leq -1$, which is a contradiction. \square

Lemma 3.22 *The subcurve $\square_{\geq \rho_1}$ contains the core.*

Proof Equivalently, $p_a(\square_{\geq \rho_1}) = 2$. This follows from inspection of Figure 2, but we give a more formal argument; the following has been suggested by the referee. If $\rho_1 = 0$, there is nothing to prove, because $\square_{\geq 0} = \square$, so let us assume that $\rho_1 > 0$. Let $\square_{< \rho_1, \bullet}$

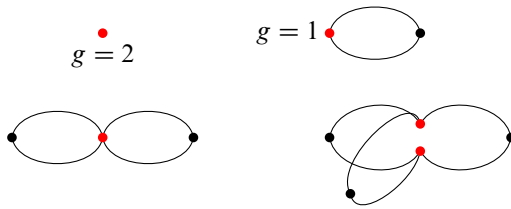


Figure 8: The top level of Δ over $\mathcal{D}_1 \setminus \mathcal{D}_2$.

be one of the h connected components of $\square \setminus \square_{\geq \rho_1}$, let e_1, \dots, e_k be the edges between $\square_{< \rho_1, \bullet}$ and $\square_{\geq \rho_1}$, with positive slope s_1, \dots, s_k . Restricting (6) to $\square_{< \rho_1, \bullet}$, we find

$$0 = 2p_a(\square_{< \rho_1, \bullet}) - 2 + k + \sum_{i=1}^k s_i,$$

which holds if and only if $p_a(\square_{< \rho_1, \bullet}) = 0, k = 1$ and $s_1 = 1$.

Restricting (6) to $\square_{\geq \rho_1}$ we find

$$2 = 2p_a(\square_{\geq \rho_1}) - 2 + h - \sum_{j=i}^h 1,$$

which gives the desired result. □

Corollary 3.23 *When λ is constant on $\square_{\geq \rho_1}$, every positive genus subcurve of $\square_{\geq \rho_1}$ supports D . The possible shapes of $\square_{\geq \rho_1}$ are depicted in Figure 8.*

Proof Let F be any positive genus subcurve on which $\lambda|_F \equiv \rho_1$. Restricting (6) to F , we find

$$\deg(\text{trop}(\psi)^*(D)|_F) = 2p_a(F) - 2 + \sum_{i=1}^k (1 + s(\lambda, e_i)),$$

where e_1, \dots, e_k are the edges from $\square \setminus F$ to F . It follows from the assumption and the proof above that the slope of λ along any e_i can only be 0 or -1 . The only possibility for $\deg(\text{trop}(\psi)^*(D)|_F)$ to be zero remains that of a genus one curve with all outgoing slopes -1 . But then λ would not be constant on $\square_{\geq \rho_1}$. □

By construction, there is always at least one vertex of positive weight on the boundary of Δ_T° . We make some stronger statements that we will need below.

Lemma 3.24 *Suppose that Δ° is connected of genus two. If there is only one vertex of positive weight on the boundary of Δ_T° , then it is the special one.*

Proof Let us call v_1 the vertex in the statement. Observe that, by [Lemma 3.18](#) and [Corollary 3.19](#), there has to be a single point of T supporting D . We argue by contradiction, showing that if v_1 did not support D we could find a different (interpolation of) admissible function(s) greater than λ ; but the latter is defined as the maximum. Extend λ with slope 1 outside Δ . Let v' denote the first vertex of T (not supporting D) encountered on the path from D to v_1 , and let ℓ_1 denote the distance from D to v' . Consider the function μ_1 that is 0 from v_1 to v' , has slope 1 from v' to D , and has value ℓ_1 on D and further away from v_1 ; extend it to T by making it locally constant outside the path from v_1 to D . The function $\lambda + \mu_1 > \lambda$ should still be cut at $\lambda + \mu_1(v_1) = 0$, unless one (call it v_2) of the vertices of positive weight beyond D has distance $\ell_2 < \ell_1$ from $\partial\Delta_T$. In this case, let ℓ_2 denote the minimal such length. Then consider the function μ_2 that has value ℓ_2 on D and beyond, and decreases with slope 1 towards v_1 , until it reaches 0 (which happens before v' since $\ell_2 < \ell_1$). We observe that $\lambda + \mu_2 > \lambda$ should still be cut at $\lambda + \mu_2(v_1) = \lambda + \mu_2(v_2) = 0$, and it has two vertices of positive weight on $\partial\Delta_T$ (on two different sides of D). This is a contradiction unless $\ell_1 = 0$. □

Example 3.25 [Figure 9](#) illustrates the proof of [Lemma 3.24](#) in one example. In these pictures, the circle delimits the locus where λ is positive. Outside the circle, λ can be extended with slope 1. Conjugate branches are not drawn; we leave it to the reader to complete the picture as necessary.

In the left-most picture — which is the one we start our argument by contradiction with — the values taken by λ are

$$\lambda(v_1) = 0, \quad \lambda(\text{core}) = \ell, \quad \lambda(v') = \ell - 2\ell' = m + 2\ell_1, \quad \lambda(w) = m, \quad \lambda(v_2) = -\ell_2,$$

where we have denoted by w the vertex supporting D (it might be arising from the subdivision of an edge) and by m the distance from w to the circle.

Now we start “moving” D towards v_1 , as shown in the second picture from the left; if ℓ_2 is long enough, D may reach v' before v_2 reaches $\partial\Delta_T$. The value of λ stays the same on the core, v_1 , v' and v_3 , but it increases otherwise:

$$\lambda(w) = m + \ell_1, \quad \lambda(v_2) = \ell_1 - \ell_2.$$

We can keep pushing D towards v_1 until either D reaches v_1 , or v_2 reaches $\partial\Delta_T$, in which case we are no longer in the hypotheses of the lemma. These are the two possibilities illustrated in the right-most pictures of [Figure 9](#).

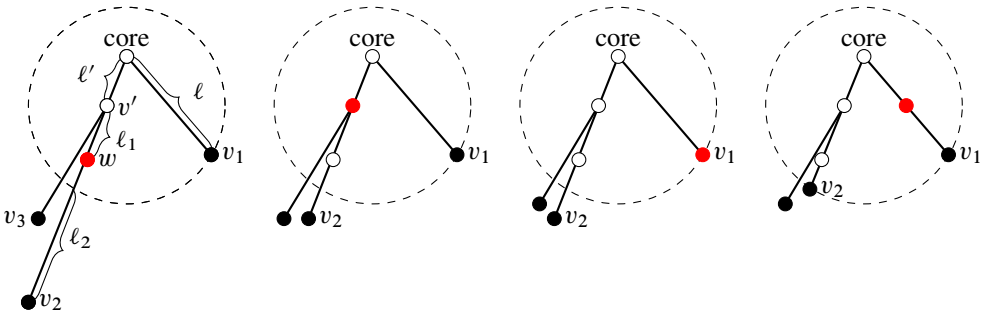


Figure 9: A graphical proof of Lemma 3.24.

The following statements are byproducts of the proof given above:

Corollary 3.26 *Suppose that Δ° is connected of genus two. If D is supported on the boundary of Δ_T° , the corresponding vertex has positive weight.*

Corollary 3.27 *Suppose that Δ° is connected of genus two. If D is supported in Δ_T° , then there is at least one vertex of positive weight on the boundary of Δ_T° on either side of D .*

Finally, we deal with a genus one situation.

Lemma 3.28 *Let Δ_1° be a connected component of Δ° of genus one. Then there is at least one vertex of positive weight on its boundary.*

Proof Let E_1 denote the genus one core of Δ_1° . There is an admissible function $\bar{\lambda}_i$ (at least one) behaving like the distance from E_1 : it is the one with D supported on (any vertex of) the other genus one subcurve E_2 of \square . Then the vertex of positive weight closest to E_1 can be at most as far as E_2 , because otherwise Δ° would be connected of genus two. Now λ looks like λ_i in a neighbourhood of E_1 . □

For the sake of concreteness, we notice that, in the situation of the above lemma, either E_2 is on the boundary of Δ_1° , supports D and has positive weight (essentially by Corollary 3.26), or the path from E_1 to E_2 crosses the boundary of Δ_1° at a rational vertex supporting $\frac{1}{2}D$, and proceeds with slope 0 in a neighbourhood.

3.4 The secondary alignment

Lemma 3.29 *The locus V where $w(\Delta)$ is 1, or it is 2 but supported on two non-conjugate vertices of \square , is closed in \mathcal{A}_2^\sim .*

Proof The statement is of a topological nature. The loci where the combinatorial type of \square is constant form a constructible stratification of $\tilde{\mathcal{A}}_2$. By Noetherianity, it is enough to check that the described locus is closed under specialisation for any trait, ie if we have an edge contraction $\square_s \rightarrow \square_\eta$ such that the latter is in V , then the former must be as well. This is easily checked. \square

Definition 3.30 Let $\tilde{\mathcal{A}}_2^\circ$ be the open substack of $\tilde{\mathcal{A}}_2$ defined as the complement of the locus V described in the previous lemma.

Definition 3.31 Let Σ'' be the subdivision of $\text{trop}(\tilde{\mathcal{A}}_2^\circ)$ defined by

- formally contracting Δ to a point z ;
- aligning the rest of the curve with respect to the distance from z .

Let $(\tilde{\mathcal{A}}_2^\circ)^{\text{pre}}$ denote the corresponding logarithmic blow-up.

On $\tilde{\mathcal{A}}_2^\circ$ the weight of Δ is at least 2. If it is 2, it is supported on a single vertex v_1 of Δ_T . In this case, by Lemma 3.24, λ has D supported on v_1 . By relabelling, assume that $\bar{\lambda}_1$ was the admissible function (Definition 3.2) determined by having v_1 as the special vertex. By Lemma 3.22, the slope of $\bar{\lambda}_1$ is always 1 below the level of v_1 ; hence, the alignment of the previous definition makes sense. We extend λ to reach the next vertex of positive weight.

Definition 3.32 Let S be a geometric point of $(\tilde{\mathcal{A}}_2^\circ)^{\text{pre}}$, and suppose that $w(\Delta) = 2$. Let $\lambda'_1 \in \Gamma(S, \pi_* \bar{M}_C)$ be the lift of $\bar{\lambda}_1$ determined by $w(\square_{>0}) \leq 2$ and $w(\square_{\geq 0}) \geq 3$. Let

$$\tilde{\lambda} = \max\{0, \lambda'_1\}.$$

Lemma 3.33 If a specialisation $\eta \rightsquigarrow s$ in $(\tilde{\mathcal{A}}_2^\circ)^{\text{pre}}$ induces an edge contraction $\square_s \rightarrow \square_\eta$ such that $w(\square_{s, \geq 0}) = 2$ but $w(\square_{\eta, \geq 0}) \geq 3$, then $\tilde{\lambda}$ generises to λ .

Proof By Lemma 3.24, on the cone of $\text{trop}(\tilde{\mathcal{A}}_2^\circ)$ corresponding to the point s , the function λ was a lift of $\bar{\lambda}_1$, with a cutoff at the first vertex v_1 of positive weight. Extending it to the second vertex v_2 of positive weight makes sense thanks to Definition 3.31. When an edge contraction makes $w(\Delta)$ grow, it means in particular that v_2 and v_1 become at the same height, so $\tilde{\lambda}$ coincides with the original λ . \square

This shows that $\tilde{\lambda}$ extends λ as a well-defined (continuous) PL function λ on a subdivision $\tilde{\tilde{\mathcal{C}}}$ of $\tilde{\mathcal{C}}$ for any S -point of $(\mathcal{A}_2^\circ)^{\text{pre}}$, which we think of as an extension of λ on certain cones of $\text{trop}((\mathcal{A}_2^\circ)^{\text{pre}})$.

Definition 3.34 Let $\tilde{\mathcal{A}}_2$ denote the logarithmic blow-up of \mathcal{A}_2° induced by applying universal semistable reduction to $\tilde{\tilde{\mathcal{C}}} \rightarrow \text{trop}((\mathcal{A}_2^\circ)^{\text{pre}})$.

Remark 3.35 The spaces $(\mathcal{A}_2^\circ)^{\text{pre}}$ and $\tilde{\mathcal{A}}_2$ are analogous to, respectively, the space of radially aligned and the space of centrally aligned logarithmic curves in [63].

Similarly to the previous section, we have the following result:

Theorem 3.36 *The moduli space $\tilde{\mathcal{A}}_2$ is a logarithmically smooth stack with locally free logarithmic structure (and therefore smooth). There is a logarithmically smooth curve $\tilde{\tilde{\mathcal{C}}} \rightarrow \tilde{\mathcal{A}}_2$, which is a partial destabilisation of $\tilde{a}^*\mathcal{C}$, such that $\tilde{\tilde{\mathcal{C}}}$ is its tropicalisation, $\tilde{\lambda} \in \Gamma(\tilde{\tilde{\mathcal{C}}}, \overline{M}_{\tilde{\tilde{\mathcal{C}}}})$ and all of its values at vertices of $\tilde{\tilde{\mathcal{C}}}$ are comparable.*

3.5 Examples of subdivisions

In this section we collect a few examples to show what the subdivisions Σ and Σ' of a cone $\sigma \in \text{trop}(\mathcal{A}_2^{\text{wt}})$, the combinatorics of λ on the various cones, and the associated singularities in $\tilde{\mathcal{C}}$ look like. The construction of $\tilde{\mathcal{C}}$ will be carried out in the next section.

Example 3.37 See Figure 10. The stabilisation of C has smooth core of weight zero, and two rational tails of high weight, one of which is attached to a Weierstrass point (compare with Example 3.5). In this case, the subdivisions Σ and Σ' coincide. Notice that there is a simplicial nonsmooth cone σ_3 ; correspondingly, half edge-lengths occur in a Kummer extension of $M_{\tilde{\mathcal{A}}}$.

Example 3.38 See Figure 11. The stabilisation of C has a smooth core of weight zero, and three rational tails attached to general points. Aligning with respect to all the admissible functions produces the nonsimplicial subdivision (a); the subdivisions (b) and (c) are the coarsening in case of high-weight, respectively weight two, tails. We also represent λ_T and the associated singularity on some cones of the subdivision; the other ones can be derived by symmetry. In case (c) the singularity over σ_2 is replaced by a (sprouted) ribbon.

Example 3.39 See Figure 12. The core of C consists of two elliptic curves of weight zero, meeting in a node; each elliptic curve is attached to a high-degree rational tail.

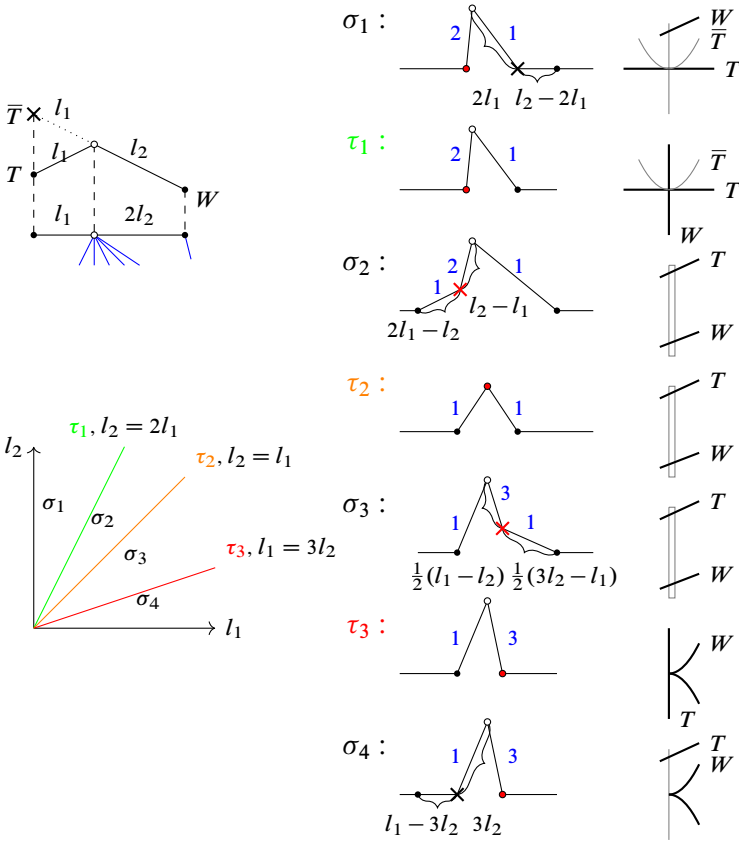


Figure 10: Tropical admissible cover (top left); the subdivision of $\text{trop}(\mathcal{A}_2^{\text{wt}})$ (bottom left); λ_T and the Gorenstein singularities (right).

Notice that the central cone σ_2 is not smooth; again, we need a Kummer extension (half-lengths). We also remark that the hyperplane $l_1 + l_2 = m$ does not come from the alignment, but from the procedure of Definition 3.14.

3.6 The primary construction

We construct a universal morphism $\tilde{\mathcal{C}} \rightarrow \bar{\mathcal{C}}$ over $\tilde{\mathcal{A}}_2$, where $\bar{\mathcal{C}}$ is a family of Gorenstein singularities (both isolated and ribbons) with core of positive weight. We do so in two steps: first, a contraction informed by λ , producing a possibly non-Gorenstein curve. The image in $\tilde{\mathcal{A}}_2$ of the non-Gorenstein locus is contained in a divisor, which we name \mathcal{D}_1 below. We complete the construction of $\bar{\mathcal{C}}$ by gluing in a portion of ψ over this locus, thus producing a nonreduced structure along the fibres.

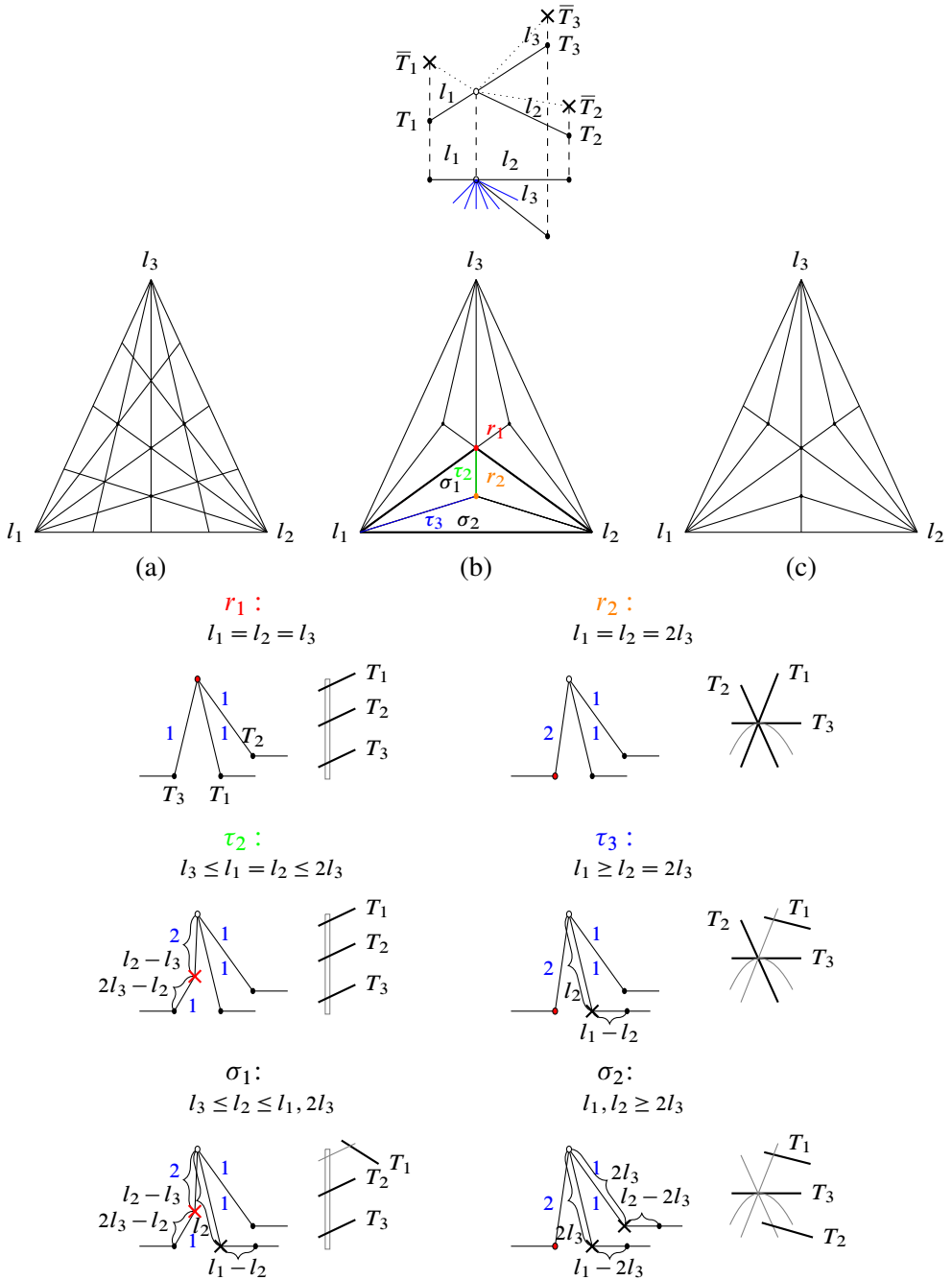


Figure 11: (a) Subdivision Σ (nonsimplicial). (b) Subdivision Σ' for high-degree tails. (c) Subdivision Σ' for degree two tails.

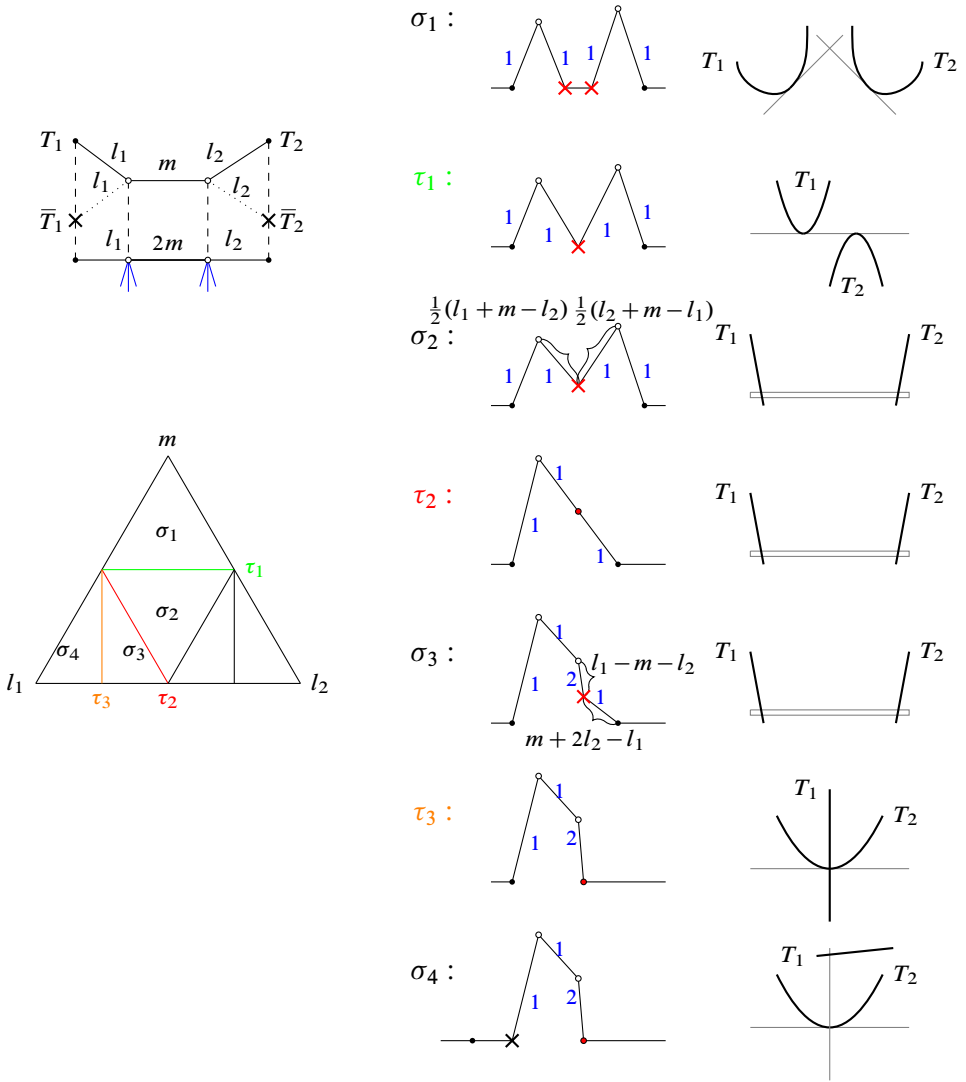


Figure 12: Tropical admissible cover (top left); the subdivision of $\text{trop}(\mathcal{A}_2^{\text{wt}})$ (bottom left); λ_T and the Gorenstein singularities (right).

Definition 3.40 Let ρ_{\max} denote the maximum value attained by λ on \square , and let $\mathcal{D} \subseteq \tilde{\mathcal{A}}_2$ be the Cartier divisor determined by the ideal sheaf $\mathcal{O}_{\tilde{\mathcal{A}}_2}(-\rho_{\max}) \hookrightarrow \mathcal{O}_{\tilde{\mathcal{A}}_2}$.

Definition 3.41 Recall from Definition 3.20 that ρ_1 denotes the value attained by λ at the vertex(ices) supporting $K_{\Delta} + \text{div}(\lambda)$. Let $\mathcal{D}_1 \subseteq \tilde{\mathcal{A}}_2$ be the Cartier divisor cut by the ideal sheaf $\mathcal{O}_{\tilde{\mathcal{A}}_2}(-\rho_1) \hookrightarrow \mathcal{O}_{\tilde{\mathcal{A}}_2}$.

Note that $\rho_1 \leq \rho_{\max}$ implies $\mathcal{D}_1 \subseteq \mathcal{D}$. By Lemma 3.22, the subcurve $\square_{\geq \rho_1}$ always contains the core, so, in particular, $p_a(\Delta^\circ) = 2$ over \mathcal{D}_1 . Ribbons will appear in $\bar{\mathcal{C}}$ precisely over \mathcal{D}_1 .

Definition 3.42 Let \mathcal{D}_2 be cut by $\rho_{\max} - \rho_1$, so that we have an exact sequence

$$0 \rightarrow \mathcal{O}_{\mathcal{D}_2}(-\rho_1) \rightarrow \mathcal{O}_{\mathcal{D}} \rightarrow \mathcal{O}_{\mathcal{D}_1} \rightarrow 0.$$

The Gorenstein curve $\bar{\mathcal{C}}$ will have only isolated singularities over \mathcal{D}_2 . They may have genus one or two according to $p_a(\Delta^\circ)$.

Definition 3.43 Let \mathcal{Z} be the Cartier divisor on $\bar{\mathcal{C}}$, supported over \mathcal{D} , which is determined by the inclusion $\mathcal{O}_{\bar{\mathcal{C}}}(-\lambda) \hookrightarrow \mathcal{O}_{\bar{\mathcal{C}}}$.

Remark 3.44 Locally, we construct a line bundle on $\bar{\mathcal{C}}$ that is trivial on \mathcal{Z} except when it contains the special component, and relatively ample elsewhere, as follows. We pick smooth disjoint sections p_1, \dots, p_{2d} of $\bar{\mathcal{C}}$ according to the weight function, namely so that $\#\{i \mid p_i \in \tilde{C}_v\} = 2w(v)$. These sections exist only locally, but we will show that our construction does not depend on this choice; therefore, it glues on the whole of $\tilde{\mathcal{S}}_2$.

Notation Let $\sigma: \bar{\mathcal{C}} \rightarrow \bar{\mathcal{C}}$ be the hyperelliptic involution. Let \mathbf{p}_i denote the multisection $p_i + \sigma(p_i)$, and $\mathbf{p} = \sum \mathbf{p}_i$. We denote by \mathcal{L} the line bundle $\omega_{\bar{\mathcal{C}}|\tilde{\mathcal{S}}_2}(\mathbf{p})(\lambda)$ on $\bar{\mathcal{C}}$.

Theorem 3.45 The line bundle \mathcal{L} is π -semiample. In the diagram

$$\begin{array}{ccc} \bar{\mathcal{C}} & \xrightarrow{\phi} & \mathcal{C}' := \underline{\text{Proj}}_{\tilde{\mathcal{S}}_2}(\pi_* \bigoplus_{k \geq 0} \mathcal{L}^{\otimes k}) \\ & \searrow \pi & \swarrow \pi' \\ & \tilde{\mathcal{S}}_2 & \end{array}$$

the morphism π' is a flat family of reduced, projective, Cohen–Macaulay curves of arithmetic genus two, with Gorenstein fibres outside \mathcal{D}_1 . Moreover, we can perform a parallel contraction $\mathcal{T} \rightarrow \mathcal{T}'$ so that $\psi': \mathcal{C}' \rightarrow \mathcal{T}'$ remains finite. Neither \mathcal{C}' nor \mathcal{T}' depend on the choice of sections respecting the weight function, as per Remark 3.44.

Remark 3.46 \mathcal{L} is symmetric under the hyperelliptic involution, ie it is pulled back from the target of the admissible cover. Any weight zero branch appearing as the conjugate of a rational tail (or bridge) of positive weight is therefore not contracted under ϕ . This ensures that \mathcal{C}' remains “hyperelliptic”, ie it admits a double cover to a rational curve, possibly with ordinary m -tuple points. As far as the factorisation of

the stable map is concerned, all those unstable components of weight zero (different from the special branches) do not interfere. We could have just as well carried out the construction without symmetrising the line bundle.

Theorem 3.47 *Let \mathcal{Z}' denote the image of \mathcal{Z} under ϕ . Over \mathcal{D}_1 , it is a flat family of Gorenstein curves of arithmetic genus two, and the image of \mathcal{Z}' under ψ' is a rational curve $\mathcal{F}'_{\mathcal{Z}}$. Let $\bar{\mathcal{C}}$ be obtained as the pushout of*

$$\begin{array}{ccc} \mathcal{Z}'_{\mathcal{D}_1} & \hookrightarrow & \mathcal{C}' \\ \downarrow \psi' & & \\ \mathcal{F}'_{\mathcal{Z}} & & \end{array}$$

Then $\bar{\mathcal{C}} \rightarrow \tilde{\mathcal{A}}_2$ is a flat family of projective, Gorenstein curves of arithmetic genus two. Moreover, the weight of any subcurve of positive genus is at least one.

The proof of the two theorems above occupies the next two sections.

3.7 First step: the contraction

Lemma 3.48 *For $k \geq 2$, $R^1\pi_*\mathcal{L}^{\otimes k}$ is supported along \mathcal{D}_2 . Moreover, it admits a two-term resolution that remains such after pullback to a sufficiently generic base; in particular, if $f : S \rightarrow \tilde{\mathcal{A}}_2$ is a morphism such that $\mathcal{O}_S(-\rho_{\max}) \rightarrow \mathcal{O}_S$ remains injective, then*

$$f^*\pi_*(\mathcal{L}^{\otimes k}) \rightarrow (\pi_S)_*(\mathcal{L}_S^{\otimes k})$$

is an isomorphism.

Proof There are three things to show:

Support Since $\mathcal{L}^{\otimes k}$ is flat on the base, by cohomology and base change it is enough to show the vanishing of $H^1(\tilde{\mathcal{C}}_s, \mathcal{L}^{\otimes k})$ for $s \in \tilde{\mathcal{A}}_2 \setminus \mathcal{D}_2$.

By weighted stability, \mathcal{L} is relatively ample outside of \mathcal{Z} , and in particular over $\tilde{\mathcal{A}}_2 \setminus \mathcal{D}$.

For $s \in \mathcal{D} \setminus \mathcal{D}_2$ we have $0 \neq \rho_1 = \rho_{\max}$. We note that $\mathcal{L} \geq 0$ and, by Corollary 3.23, it has degree two on every positive genus subcurve of $\tilde{\mathcal{C}}_s$. Then it is clear for degree reasons that $h^1(\mathcal{L}^{\otimes k}) = 0$ for $k \geq 2$.

Resolution Note first that the rank of $R^1\pi_*\mathcal{L}^{\otimes k}$ is not constant along \mathcal{D}_2 .

Indeed, \mathcal{D}_2 has two types of irreducible components:

- $\mathcal{D}_{2,1}$, where generically Δ° has genus one, and $R^1\pi_*\mathcal{L}^{\otimes k}$ has rank one;
- $\mathcal{D}_{2,2}$, where Δ° has genus two, and $R^1\pi_*\mathcal{L}^{\otimes k}$ has rank two.

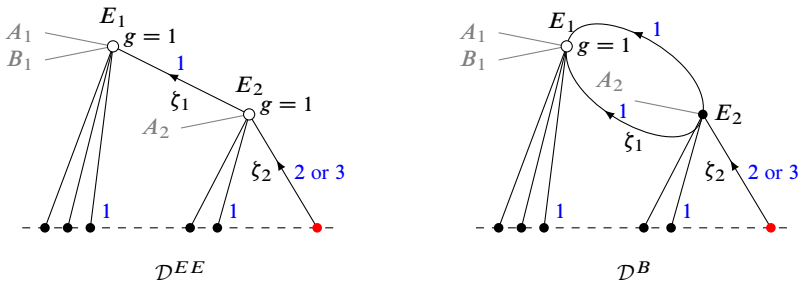


Figure 13: The generic points of $\mathcal{D}_{2,1} \cap \mathcal{D}_{2,2}$.

Away from their intersection, it is easy to find the desired *local* resolution

$$0 \rightarrow \mathcal{O}_U^{\oplus 2} \xrightarrow{\begin{pmatrix} e^{\rho_{\max}} & 0 \\ 0 & e^{\rho_{\max}} \end{pmatrix}} \mathcal{O}_U^{\oplus 2} \rightarrow \mathcal{O}_{\mathcal{D}_{2,2} \cap U}^{\oplus 2} \rightarrow 0$$

on $U \subseteq \tilde{\mathcal{A}}_2$ a neighbourhood of a generic point of $\mathcal{D}_{2,2}$ (assuming $\rho_1 = 0$), where $e^{\rho_{\max}}$ denotes a local equation for \mathcal{D} . Similarly,

$$0 \rightarrow \mathcal{O}_U \xrightarrow{e^{\rho_{\max}}} \mathcal{O}_U \rightarrow \mathcal{O}_{\mathcal{D}_{2,1} \cap U} \rightarrow 0$$

around a generic point in $\mathcal{D}_{2,1}$.

The intersection of $\mathcal{D}_{2,1}$ with $\mathcal{D}_{2,2}$ has two types of irreducible components; see Figure 13. In order to obtain a local resolution, we adapt an argument of [40]. We sketch it here for the reader's benefit.

Locally on the base, there is a section p_{2d+1} such that $\omega_C(\lambda) = \mathcal{O}_C(p_{2d+1})$, where $p_{2d+1} = p_{2d+1} + \sigma(p_{2d+1})$, and therefore $\mathcal{L} = \mathcal{O}_C(\sum_{i=1}^{2d+1} p_i)$.

Again locally on the base, we may find disjoint generic sections A_1, A_2 and B such that A_1 and B pass through E_1 , and A_2 passes through E_2 . Then $\mathcal{L}(A_1 + A_2 - B)$ has vanishing h^1 on fibres, and therefore, by cohomology and base change, $\pi_* \mathcal{L}(A_1 + A_2 - B)$ is a vector bundle. Locally, we can write $\pi_* \mathcal{L} \simeq \mathcal{O}_U \oplus \pi_* \mathcal{L}(-B)$, and the second factor is the kernel of the evaluation map

$$\pi_* \mathcal{L}(A_1 + A_2 - B) \rightarrow \pi_*(\mathcal{O}_{A_1}(A_1) \oplus \mathcal{O}_{A_2}(A_2)).$$

The evaluation map can be studied fibrewise, since both sheaves in question have vanishing h^1 ; moreover, the source can be decomposed into

$$\pi_* \mathcal{L}(A_1 + A_2 - B) \cong \bigoplus_{i=1}^{2d+2} \pi_* \mathcal{O}(p_i + A_1 + A_2 - B),$$

and the evaluation map can be studied componentwise:

$$\text{ev}_{i,j} : \pi_* \mathcal{O}(p_i + A_1 + A_2 - B) \rightarrow \pi_*(\mathcal{O}_{A_j}(A_j)), \quad i = 1, \dots, 2d + 2, \quad j = 1, 2.$$

The cokernel of the latter is $H^1(C, \mathcal{O}_C(p_i + A_{2-j} - B))$. See [38, Section 4.2].

It follows from an argument similar to that of [38, Proposition 4.13] that the latter is nonzero precisely when p_i stays away from E_j ; therefore, in some local trivialisation of the line bundles involved,

$$\text{ev}_{i,j} = c_{i,j} \prod_{q \in [p_i, A_j]} \zeta_q,$$

where $c_{i,j} \in \mathcal{O}_U^*$, we denote by $[p_i, A_j]$ the set of nodes separating p_i from A_j , and $\zeta_q \in \mathcal{O}_U$ is the smoothing parameter of the node q . Thanks to the alignment,

- the smoothing parameter ζ_2 of the node separating E_2 from the component supporting D divides all the expressions of the form $\text{ev}_{i,2}$ for $i = 1, \dots, 2d + 2$;
- if ζ_1 denotes the smoothing parameter of the node separating E_1 from E_2 , the product $\zeta_1 \zeta_2$ divides all the expressions of the form $\text{ev}_{i,1}$ for $i = 1, \dots, 2d + 2$.

(Notice that ζ_1 and ζ_2 should be replaced by some products of smoothing parameters when the curve degenerates.) We can therefore use the column of the evaluation matrix associated to a marking p_i (up to relabelling, $i = 1$) on the component supporting D in order to put the matrix in triangular form. In order to diagonalise it, we need more refined information, which we borrow from [40, Sections 2.6–2.7] and we restate here in streamlined form:

Proposition (Hu, Li and Niu) *The determinant of the matrix*

$$\begin{pmatrix} c_{1,1} & c_{j,1} \\ c_{1,2} & c_{j,2} \end{pmatrix}$$

is invertible when the markings p_1 and p_j are not conjugate under ψ .

Since the component supporting D contains at least three markings (p_1, p_{2d+1} and p_{2d+2}), we can find two nonconjugate ones.

Summing up, the evaluation matrix can be put in the form

$$\begin{pmatrix} \zeta_1 & 0 \\ 0 & \zeta_1 \zeta_2 \end{pmatrix}$$

(the remaining columns are zero). Noticing that $\mathcal{D}_{2,i} = \{\zeta_i = 0\}$ for $i = 1, 2$, around the given point, we have thus found the desired local resolution of $R^1 \pi_* \mathcal{L}^k$. In particular, it follows that $\pi_* \mathcal{L}^{\otimes \geq 2}$ is a vector bundle on $\tilde{\mathcal{F}}_2$.

Base change See the proof of [63, Lemma 3.7.5.3]. Note that the evaluation matrix above is an explicit instance of the Grothendieck–Mumford complex for cohomology and base change, and $\zeta_1\zeta_2 = e^{\rho_{\max}}$ is a local equation for \mathcal{D} . \square

Proof of Theorem 3.45 This is analogous to [67, Lemma 2.13; 63, Proposition 3.7.6.1]. We recap for the reader’s convenience.

Flatness This is equivalent to $\pi_*\mathcal{L}^{\otimes k}$ being a vector bundle for $k \gg 0$ [16, Tag 0D4D].

Basepoint-freeness That is, existence of the morphism ϕ . This is clear outside \mathcal{D} , where $\mathcal{L}^{\otimes k}$ is π -ample. Even on \mathcal{D} , we have that $\mathcal{L}^{\otimes k}$ is π -ample outside of \mathcal{Z} . Moreover, from the short exact sequence

$$0 \rightarrow \mathcal{L}(-\lambda)^{\otimes k} \rightarrow \mathcal{L}^{\otimes k} \rightarrow \mathcal{L}|_{\mathcal{Z}}^{\otimes k} \rightarrow 0$$

and the vanishing of $R^1\pi_*\mathcal{L}(-\lambda)^{\otimes k}$ (stability), it is enough to show that, for any $x \in \mathcal{Z}$, there exists a section of $\pi_*\mathcal{L}|_{\mathcal{Z}}^{\otimes k}$ around $\pi(x)$ which does not vanish in x . By definition of λ , the line bundle $\mathcal{L}|_{\mathcal{Z}}$ is the pullback along ψ of a line bundle of degree one and nonnegative multidegree on $\mathcal{T}_{\mathcal{Z}}$; the latter has enough sections.

Properties of the fibres This can be studied after base change to a generic trait \mathcal{Y} . We may assume that the generic point corresponds to a smooth curve and the closed point maps to $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$. Thus,

$$\phi_{\mathcal{Y}} : C := \mathcal{C}_{\mathcal{Y}} \rightarrow \mathcal{C}'_{\mathcal{Y}} =: C'$$

is a birational contraction satisfying $\phi_*\mathcal{O}_C = \mathcal{O}_{C'}$. It follows that C' is a normal surface. In particular, the central fibre is $S1$; it is also generically reduced, being birational to that of C ; thus, it is reduced (and Cohen–Macaulay).

Finally, we want to argue that the fibres are Gorenstein outside \mathcal{D}_1 . We may assume that the special point of \mathcal{Y} maps to $\mathcal{D}_2 \setminus \mathcal{D}_1$. Then \mathcal{L} is trivial along Z , which is therefore contracted to a codimension two locus Z' of C' . Outside of Z , ϕ restricts to an isomorphism. The equality of line bundles

$$\mathcal{O}_{C'}(1)\left(-\sum p_i\right)\Big|_{C \setminus Z'} = \omega_{C/\mathcal{Y}}|_{C \setminus Z} = \omega_{C'/\mathcal{Y}}|_{C \setminus Z'},$$

together with the fact that $\omega_{C'/\mathcal{Y}}$ is an $S2$ sheaf, shows that the latter coincides with the line bundle $\mathcal{O}_{C'}(1)\left(-\sum p_i\right)$ on the whole of C' (Hartogs’ theorem). Thus, C' is Gorenstein over \mathcal{Y} .

Compatible contraction of \mathcal{T} This can be performed using the line bundle

$$\omega_{\mathcal{T}}(D_B)(\lambda_T)(\psi(\mathbf{p})),$$

whose pullback to $\tilde{\mathcal{C}}$ is \mathcal{L} .

Well-posedness The construction of \mathcal{C}' and \mathcal{T}' is independent of the choice of markings respecting the weight function by birational rigidity [28, Lemma 1.15]. \square

3.8 Second step: the pushout

Ribbons can be used to interpolate between isolated singularities over \mathcal{D}_1 — this is completely natural from the point of view of piecewise-linear functions on the tropical side, and serves as a correction of the failure of \mathcal{C}' at being Gorenstein.

Recall that \mathcal{Z}' was defined as the image of \mathcal{Z} under ϕ . First, we prove that the definition of \mathcal{Z}' commutes with base change to a generic trait; then, we will show that the pushout construction commutes with such a base change, and thus we may reduce to the case of surfaces in order to study the singularities of the fibres.

Proposition 3.49 *Let \mathcal{Z}' be the subscheme of \mathcal{C}' defined by the ideal sheaf $\phi_*\mathcal{O}_{\mathcal{C}}(-\lambda)$ and supported on \mathcal{D} . Then:*

- (1) $R^1\phi_*\mathcal{O}_{\mathcal{C}}(-\lambda) = 0$; in particular, $\phi_*\mathcal{O}_{\mathcal{C}}(-\lambda) = \text{Fitt}(\phi_*\mathcal{O}_{\mathcal{Z}})$.
- (2) For every generic trait $\Upsilon \xrightarrow{\sim} \tilde{\mathcal{A}}$ with generic point mapping to the smooth locus, the definition of \mathcal{Z}' commutes with base change, ie $\iota^*\phi_*\mathcal{O}_{\mathcal{C}}(-\lambda) = \phi_{\Upsilon*}\mathcal{O}_{\mathcal{C}_{\Upsilon}}(-\lambda)$.

The analogous statements about $\mathcal{T}'_{\mathcal{Z}}$ hold as well.

Proof The discussion has been somewhat inspired by [71, Section 1].

(1) Let C_s be a fibre on which ϕ_s is not an isomorphism; in particular, $\lambda_s \neq 0$. Working locally on the base, we can choose smooth and disjoint sections p_1, \dots, p_d of \mathcal{C} respecting the weight function. It is enough to prove that $R^1\phi_*\mathcal{O}_{\mathcal{C}}(-\lambda)(k\mathbf{p}) = 0$ for $k \gg 0$. Indeed, once we know the latter is vanishing, since ϕ is an isomorphism around \mathbf{p} , we have that $\mathcal{O}_{\mathcal{C}}(k\mathbf{p}) = \phi^*\mathcal{O}_{\mathcal{C}'}(k\mathbf{p}')$ and thus

$$\begin{aligned} 0 &= R^1\phi_*\mathcal{O}_{\mathcal{C}}(-\lambda)(k\mathbf{p}) = H^1(R\phi_*\mathcal{O}_{\mathcal{C}}(-\lambda)(k\mathbf{p})) \\ &= H^1(R\phi_*\mathcal{O}_{\mathcal{C}}(-\lambda) \otimes \phi^*\mathcal{O}_{\mathcal{C}'}(k\mathbf{p}')) = R^1\phi_*\mathcal{O}_{\mathcal{C}}(-\lambda) \otimes \mathcal{O}_{\mathcal{C}'}(k\mathbf{p}'), \end{aligned}$$

implying the desired vanishing.

From the spectral sequence computing $R\pi'_* \circ R\phi_*$, the five-term exact sequence

$$0 \rightarrow R^1\pi'_*\phi_*\mathcal{O}_{\mathcal{C}}(-\lambda)(k\mathbf{p}) \rightarrow R^1\pi_*\mathcal{O}_{\mathcal{C}}(-\lambda)(k\mathbf{p}) \rightarrow \pi'_*R^1\phi_*\mathcal{O}_{\mathcal{C}}(-\lambda)(k\mathbf{p})$$

ends there, because the next term would involve an $R^2\pi'_*(-)$, which vanishes as the fibre dimension is bounded by one.

Notice that if $R^1\pi_*\mathcal{O}_{\mathcal{C}}(-\lambda)(k\mathbf{p}) \otimes k(s) = 0$ for those s where ϕ_s is not an isomorphism, so is $\pi'_*R^1\phi_*\mathcal{O}_{\mathcal{C}}(-\lambda)(k\mathbf{p}) \otimes k(s)$.

On the other hand, $R^1\phi_*\mathcal{O}_{\mathcal{C}}(-\lambda)(k\mathbf{p})$ is a coherent sheaf supported on the closed locus V' in \mathcal{C}' over which the fibre of ϕ has positive dimension. The latter is finite over $\tilde{\mathcal{A}}$, and thus $\pi'_*R^1\phi_*\mathcal{O}_{\mathcal{C}}(-\lambda)(k\mathbf{p}) \otimes k(s) = 0$ implies $R^1\phi_*\mathcal{O}_{\mathcal{C}}(-\lambda)(k\mathbf{p}) \otimes k(x) = 0$ for each $x \in V'_s$.

Since $R^1\pi_*\mathcal{O}_{\mathcal{C}}(-\lambda)(k\mathbf{p})$ satisfies cohomology and base change, the vanishing can be checked after restricting to a fibre. Let Z_s denote the support of λ_s , C_i the trees rooted at the vertices of $\partial\Delta$, and e_i the first edges encountered in Δ° . Taking the normalisation of C_s at the nodes corresponding to the edges e_i , we get

$$0 \rightarrow \mathcal{O}_{C_s}(-\lambda)(k\mathbf{p}) \rightarrow \mathcal{O}_{Z_s}(-\lambda) \oplus \bigoplus_i \mathcal{O}_{C_i}(-\lambda)(k\mathbf{p}) \rightarrow \bigoplus_i \mathbb{C}_{e_i} \rightarrow 0.$$

The evaluation map on the nodes e_i is clearly surjective at the level of H^0 , as the line bundle restricted to C_i is very ample for k big enough, and the desired vanishing follows from that of $H^1(\mathcal{O}_{Z_s}(-\lambda_s))$, which can be argued by the definition of λ and Serre duality.

(2) We know by Lemma 3.48 that the construction of \mathcal{C}' commutes with the given base change, so the following diagram is Cartesian:

$$\begin{array}{ccc} \mathcal{C}_\gamma & \xrightarrow{\phi_\gamma} & \mathcal{C}'_\gamma \\ \downarrow \iota & \square & \downarrow \iota' \\ \mathcal{C} & \xrightarrow{\phi} & \mathcal{C}' \end{array}$$

Furthermore, since the source and target are smooth, $\gamma \hookrightarrow \tilde{\mathcal{A}}$ is an LCI morphism, and so are ι and ι' ; it thus follows from [49, Corollary 2.27] that

$$L\iota'^*R\phi_*\mathcal{O}_{\mathcal{C}}(-\lambda) = R\phi_\gamma*L\iota^*\mathcal{O}_{\mathcal{C}}(-\lambda).$$

On the other hand, the higher pushforward vanishes by the previous point, so

$$R\phi_*\mathcal{O}_{\mathcal{C}}(-\lambda) = \phi_*\mathcal{O}_{\mathcal{C}}(-\lambda),$$

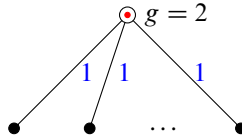


Figure 14: λ at the generic point of \mathcal{D}_1 .

and $Lt^* \mathcal{O}_{\mathcal{C}}(-\lambda) = \mathcal{O}_{\mathcal{C}\gamma}(-\lambda)$ because it is a line bundle; hence, the derived statement is equivalent to what we want. □

Proposition 3.50 *The restriction of \mathcal{Z}' to \mathcal{D}_1 is a flat family of Gorenstein curves of genus two. Similarly, $\mathcal{T}'_{\mathcal{Z}}$ is a flat family of Gorenstein curves of genus zero (ie at worst nodal).*

Notice that \mathcal{C}' is not always Gorenstein over \mathcal{D}_1 ; in particular, it can be the decomposable union of \mathcal{Z}' with some lines. For the proof we need the following:

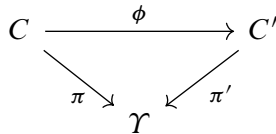
Lemma 3.51 *\mathcal{D}_1 is a reduced divisor.*

Proof \mathcal{D}_1 is a Cartier divisor in a smooth ambient space, so it is enough to check that it is generically reduced.

The generic point of \mathcal{D}_1 looks like in Figure 14.

Therefore, a generic trait with uniformiser t will intersect \mathcal{D}_1 in (t) . □

Proof of Proposition 3.50 We may change the base to a generic trait γ with closed point mapping to the given point of \mathcal{D}_1 , ie



so C is a smooth surface, ϕ is a birational contraction, which is an isomorphism outside the divisor Z defined by $\mathcal{I}_Z = \mathcal{O}_C(-\lambda)$. Notice that there is a subcurve of Z on which the line bundle defining the contraction is ample; therefore, $Z' \subseteq C'$ has pure codimension one. We want to show that $\omega_{Z'}$ is a line bundle and that $\chi(\omega_{Z'}) = 1$ (or equivalently $p_a(Z') = 2$).

Recall that, by Grothendieck duality,

$$\omega_{Z'} = \mathcal{E}xt_{C'}^1(\mathcal{O}_{Z'}, \omega_{C'}) = \mathcal{H}om_C(\mathcal{I}_{Z'}, \omega_{C'})|_{Z'}.$$

By adjunction for the Cartier divisor $Z \subseteq C$, there is a short exact sequence

$$(7) \quad 0 \rightarrow \omega_C \rightarrow \omega_C(\lambda) \rightarrow \omega_Z \rightarrow 0,$$

which stays exact after pushforward along ϕ by Grauert–Riemenschneider vanishing [46, Corollary 2.68]. By applying $\mathcal{H}om_{\mathcal{O}_{C'}}(-, \omega_{C'})$ to the exact sequence

$$0 \rightarrow \phi_* \mathcal{O}_C(-\lambda) \rightarrow \mathcal{O}_{C'} \rightarrow \mathcal{O}_{Z'} \rightarrow 0,$$

we obtain the bottom row of the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \phi_* \omega_C & \longrightarrow & \phi_* \omega_C(\lambda) & \longrightarrow & \phi_* \omega_Z \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \omega_{C'} & \longrightarrow & \mathcal{H}om_C(\phi_* \mathcal{O}_C(-\lambda), \omega_{C'}) & \longrightarrow & \omega_{Z'} \longrightarrow 0 \end{array}$$

By Grothendieck duality ($\phi_* \dashv \phi^!$ and $\phi^! \omega_{C'} = \omega_C$) and the snake lemma, we conclude that the vertical arrows are isomorphisms. This implies that $\omega_{Z'}$ is a line bundle if $\phi_* \omega_C(\lambda)$ is. Since the sections p are away from $\text{Exc}(\phi)$, by construction

$$\phi_* \omega_C(\lambda) = \mathcal{O}_{C'}(1)(-\phi(p))$$

is a line bundle. We have thus proved that Z' is Gorenstein.

Moreover, $\phi_* \omega_Z = \omega_{Z'}$. Now, to prove that $\chi(\omega_{Z'}) = 1$, it is enough to prove that $\chi(\omega_{Z'}) = \chi(\omega_Z)$. Indeed, over \mathcal{D}_1 , the definition of λ and adjunction (7) show that $\omega_C(\lambda)$ restricts to a line bundle of degree two on Z , which is therefore a curve of genus two. Since $\phi_* \mathcal{O}_Z = \mathcal{O}_{Z'}$, it is enough to prove the vanishing of $R^1 \phi_* \mathcal{O}_Z$. By Proposition 3.49, $R^1 \phi_* \mathcal{O}_Z \simeq R^1 \phi_* \mathcal{O}_C$; therefore, the desired statement is equivalent to the fact the C' has rational singularities. This follows from $\phi_* \omega_C = \omega_{C'}$; see for example [48]. We have thus proved that Z' has genus two.

In order to prove the flatness of $\mathcal{Z}'_{\mathcal{D}_1} \rightarrow \mathcal{D}_1$, it is sufficient to show that

$$\pi'_*(\mathcal{O}_{\mathcal{Z}'_{\mathcal{D}_1}} \otimes \mathcal{O}_{\mathcal{C}'}(n))$$

is a vector bundle on \mathcal{D}_1 for n large enough. Since \mathcal{D}_1 is a reduced divisor (Lemma 3.51), we only have to show that its rank is constant along \mathcal{D}_1 . It is easy to see that

$$\mathcal{O}_{\mathcal{C}'}(1)|_{\mathcal{Z}'_{\mathcal{D}_1}} \cong \omega_{\mathcal{Z}'_{\mathcal{D}_1}}.$$

Since $\mathcal{Z}'_{\mathcal{D}_1}$ is a curve of genus two, it follows from Riemann–Roch that

$$H^0(\mathcal{Z}'_s, \omega_{\mathcal{Z}'_s}^{\otimes n}) = 2n - 1,$$

so it is enough to show that $\bar{\mathcal{O}}_{\mathcal{C}'}(1)|_{\mathcal{Z}'_{\mathcal{D}_1}}$ satisfies cohomology and base change. For this, we observe the short exact sequence

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \pi'_*(\phi_*\mathcal{O}_{\mathcal{C}}(-\lambda) \otimes \mathcal{O}_{\mathcal{C}'}(n)) & \longrightarrow & \pi'_*\mathcal{O}_{\mathcal{C}'}(n) & \longrightarrow & \pi'_*\mathcal{O}_{\mathcal{C}'}(n)|_{\mathcal{Z}'} \longrightarrow 0 \\
 & & \downarrow \cong & & & & \\
 & & \pi_*(\omega_{\pi}^{\otimes n}(n\mathbf{p}) \otimes \mathcal{O}_{\mathcal{C}}((n-1)\lambda)) & & & &
 \end{array}$$

As \mathcal{C} and \mathcal{C}' are flat over the base, and both $\omega_{\pi}(\mathbf{p})$ and $\mathcal{O}_{\mathcal{C}'}(1)$ are relatively ample, the first two bundles satisfy cohomology and base change. It follows from a diagram chase that so does the third. We have thus proved that \mathcal{Z}' is flat on \mathcal{D}_1 .

The statement about $\mathcal{T}'_{\mathcal{Z}}$ can be proven in an analogous (but easier) fashion. □

Proof of Theorem 3.47 Noticing that $\mathcal{Z}' \rightarrow \mathcal{T}'_{\mathcal{Z}}$ is finite, the existence of the pushout as a scheme over $\tilde{\mathcal{A}}_2$ follows from results of D Ferrand [30]. We have already proved that the construction of \mathcal{C}' , \mathcal{Z}' and $\mathcal{T}'_{\mathcal{Z}}$ commutes with pullback to a generic trait. The pushout does as well in virtue of [16, Tag 0ECK]. So, in order to prove that the fibres of $\bar{\mathcal{C}}$ are Gorenstein, we may work with fibred surfaces, in which case we may apply some results of M Reid [65].

Following [65, Section 2.1], $\bar{\phi}: C' \rightarrow \bar{C}$ is the normalisation, with conductor

$$\text{Ann}(\bar{\phi}_*\mathcal{O}_{C'}/\mathcal{O}_{\bar{C}}) = \text{Ann}(\psi_*\mathcal{O}_{\mathcal{Z}'}/\mathcal{O}_{T'_Z}).$$

Since ψ is a double cover, it is in particular flat, and $\psi_*\mathcal{O}_{\mathcal{Z}'}/\mathcal{O}_{T'_Z}$ is a line bundle; it follows from [65, Proposition 2.2] that \bar{C} is S_2 .

Moreover, we have seen in the proof of Proposition 3.50 that

$$\omega_{C'}(\mathcal{Z}') = \mathcal{H}om_{\mathcal{C}'}(\mathcal{I}_{\mathcal{Z}'}, \omega_{\mathcal{C}'})$$

is a line bundle on C' . Since $\psi: \mathcal{Z}' \rightarrow T'_Z$ is a double cover of a rational curve, it follows that the kernel of the canonical map $\psi_*\omega_{\mathcal{Z}'} \rightarrow \omega_{T'_Z}$ is a line bundle as well. The criterion of [65, Corollary 2.8(iv)] allows us to conclude that \bar{C} is Gorenstein.

Finally, the statement about weights is obvious from the construction of λ if its support is connected. If there are two components of Δ° , each of them necessarily of genus one by Lemma 3.17, the statement follows from Lemma 3.28. □

3.9 The secondary construction

We restrict now to $\tilde{\mathcal{A}}_2$, defined in Definition 3.34. Thanks to the alignment, we have:

Definition 3.52 Let $\mathcal{D}_1^2 \subseteq \tilde{\mathcal{A}}_2$ be the logarithmic divisor where the weight of the subcurve Δ determined by λ is two.

The curve $\bar{\mathcal{C}}|_{\tilde{\mathcal{A}}_2}$ will be manipulated further to insert ribbons over \mathcal{D}_1^2 . We could descend the line bundle $\mathcal{O}_{\mathcal{C}'}(1)$ to $\bar{\mathcal{C}}$ using the results of [30, Theorem 2.2] and proceed from there. For simplicity we chose instead to work on $\tilde{\mathcal{C}}$ from Theorem 3.36, so that the phrasing of the following two theorems is analogous to that of the main theorems in Section 3.6. The proof goes along the same lines too, so we leave it to the reader to figure out the details.

Let $\tilde{\mathcal{L}}$ denote the line bundle $\omega_{\tilde{\mathcal{C}}/\tilde{\mathcal{A}}_2}(\mathbf{p})(\tilde{\lambda})$ on $\tilde{\mathcal{C}}$, where \mathbf{p} is a local multisection compatible with the weight function; see the notation at the beginning of Section 3.6.

Theorem 3.53 The line bundle $\tilde{\mathcal{L}}$ is $\tilde{\pi}$ -semiample. In the diagram

$$\begin{array}{ccc}
 \tilde{\mathcal{C}} & \xrightarrow{\tilde{\phi}} & \mathcal{C}'' := \underline{\text{Proj}}_{\tilde{\mathcal{A}}_2}(\tilde{\pi}_* \bigoplus_{k \geq 0} \tilde{\mathcal{L}}^{\otimes k}) \\
 & \searrow \tilde{\pi} & \swarrow \pi'' \\
 & & \tilde{\mathcal{A}}_2
 \end{array}$$

the morphism π'' is a flat family of reduced, projective, Cohen–Macaulay curves of arithmetic genus two, with Gorenstein fibres outside $\mathcal{D}_1 \cup \mathcal{D}_1^2$. Moreover, we can perform a parallel contraction $\mathcal{T} \rightarrow \mathcal{T}''$ so that $\psi'' : \mathcal{C}'' \rightarrow \mathcal{T}''$ remains finite. Neither \mathcal{C}'' nor \mathcal{T}'' depend on the choice of local sections \mathbf{p} .

Let $\tilde{\mathcal{Z}}$ be the logarithmic divisor on $\tilde{\mathcal{C}}$ defined by $\mathcal{O}_{\tilde{\mathcal{C}}}(-\tilde{\lambda}) \hookrightarrow \mathcal{O}_{\tilde{\mathcal{C}}}$.

Theorem 3.54 Let \mathcal{Z}'' denote the image of $\tilde{\mathcal{Z}}$ under $\tilde{\phi}$. Over $\mathcal{D}_1 \cup \mathcal{D}_1^2$, it is a flat family of Gorenstein curves of arithmetic genus two, and the image of \mathcal{Z}'' under ψ'' is a rational curve $\mathcal{T}''_{\mathcal{Z}}$. Let $\bar{\mathcal{C}}$ be obtained as the pushout of

$$\begin{array}{ccc}
 \mathcal{Z}''_{\mathcal{D}_1 \cup \mathcal{D}_1^2} & \hookrightarrow & \mathcal{C}'' \\
 \downarrow \psi'' & & \\
 \mathcal{T}''_{\mathcal{Z}} & &
 \end{array}$$

Then $\bar{\mathcal{C}} \rightarrow \tilde{\mathcal{A}}_2$ is a flat family of projective, Gorenstein curves of arithmetic genus two. Moreover, the weight of the minimal subcurve of genus two is at least three.

Remark 3.55 \mathcal{D}_1 and \mathcal{D}_1^2 do not intersect; thus, the flatness of \mathcal{Z}'' can be checked independently on the two components. To this goal, we notice that $\mathcal{O}_{\varphi''}(1)$ restricts to $\omega_{\mathcal{Z}''_{\mathcal{D}_1}/\mathcal{D}_1}$ on $\mathcal{Z}''_{\mathcal{D}_1}$ (as seen in the previous section) and to $\omega_{\mathcal{Z}''_{\mathcal{D}_1^2}/\mathcal{D}_1^2}^{\otimes 3}$ on $\mathcal{Z}''_{\mathcal{D}_1^2}$. The pushout construction can be carried out independently on these two loci.

Remark 3.56 We could have extended λ to $\tilde{\lambda}$ over the whole $\tilde{\mathcal{S}}_2$ by choosing a different cutoff level whenever $w(\Delta) \leq 2$. In this case, though, we would have had to deal with the cases that the weight is 1, or $1 + 1$, or 2 but supported on a different vertex than v . In these cases, the construction above produces singularities worse than tailed ribbons. When there are two vertices in Δ° on which \mathcal{L} has positive degree, the singularity looks like a chain of two ribbons on an underlying node, with local equations $\mathbb{C}\llbracket x, y \rrbracket / (x^2 y^2)$. When there are three, if the contraction ϕ acts nontrivially, there may even be three ribbons with underlying curve an ordinary 3-fold point. We chose to keep the singularities under control by discarding the bad locus in Definition 3.30. We shall see below that this forces an intermediate step on us (which can actually have some interest of its own), but it does not affect the end result.

3.10 Markings

To avoid overloading the notation and the exposition, we have so far considered only weighted admissible covers without markings. However, with the application to stable maps in mind, markings are necessary to impose cohomological constraints using the evaluation maps. Our construction extends to the marked version essentially unchanged. We can consider:

Definition 3.57 A weighted admissible cover with markings consists of

$$(\psi : (C, D_R, \mathbf{x}) \rightarrow (T, D_B, \mathbf{y} = \psi(\mathbf{x})), w : V(\square) \rightarrow \mathbb{N})$$

such that D_R and \mathbf{x} are separately disjoint (multi)sections of C , and T is weighted-stable, ie every weight zero component has at least three special (marked, branch or nodal) points. We denote the moduli space of weighted admissible covers with n markings by $\mathcal{A}_{2,n}^{\text{wt}}$.

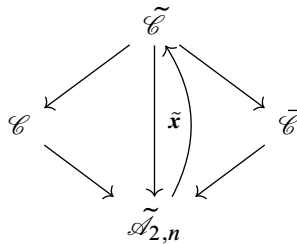
Markings are represented by infinitely long legs on \square . They play no role in the alignment: admissible functions will have slope 1 along them, and the infinite legs will be subdivided accordingly. In particular, if a marking is supported on a vertex of Δ° , then we may *sprout* (blow up) the marking as many times as is necessary for its strict

transform to be supported on a vertex of $\partial\Delta$. (In fact, only one blow-up is necessary if it is allowed to be weighted.)

Theorem 3.58 *There exists a logarithmically étale modification $\mathcal{A}_{2,n} \rightarrow \mathcal{A}_{2,n}^{\text{wt}}$ (resp. $\tilde{\mathcal{A}}_{2,n} \rightarrow \mathcal{A}_{2,n}^{\text{wt}}$) parametrising weighted admissible covers with markings and a primary (resp. secondary) alignment. The moduli space $\tilde{\mathcal{A}}_{2,n}$ (resp. $\tilde{\mathcal{A}}_{2,n}$) is a logarithmically smooth stack with locally free logarithmic structure (and therefore smooth).*

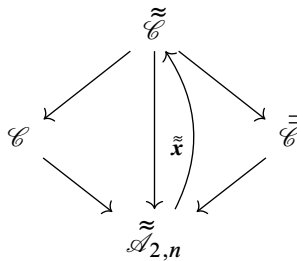
In particular, the strict transform of the markings never touches the singularity.

Theorem 3.59 *There is a diagram of flat families of projective, Gorenstein curves of arithmetic genus two, with n corresponding disjoint sections of the smooth locus:*



Moreover, the weight of every subcurve of positive genus in $\bar{\mathcal{C}}$ is at least one.

Theorem 3.60 *There is a diagram of flat families of projective, Gorenstein curves of arithmetic genus two, with n corresponding disjoint sections of the smooth locus:*



Moreover, the weight of the core of $\bar{\bar{\mathcal{C}}}$ is at least three.

4 A modular desingularisation of $\overline{\mathcal{M}}_{2,n}(\mathbb{P}^r, d)^{\text{main}}$

4.1 Irreducible components

We draw the weighted dual graph of the general member of all possible irreducible components of $\overline{\mathcal{M}}_2(\mathbb{P}^r, d)$. Our running convention is that a white vertex corresponds

to a contracted component, a grey one to a genus two subcurve covering a line two-to-one, and a black vertex to a nonspecial subcurve. Vertices are labelled with their genus and weight.

(1) *Main* is the closure of the locus of maps from a smooth curve of genus two.

$$(2) \mathcal{D}(d_1, \dots, d_k) = \left\{ \begin{array}{c} \bullet \text{ } g = 0, d_1 \\ \bullet \text{ } g = 0, d_2 \\ \dots \\ \bullet \text{ } g = 0, d_k \end{array} \right\}.$$

$$(3) \text{hyp}\mathcal{D}(d_1, \dots, d_k) = \left\{ \begin{array}{c} \bullet \text{ } g = 0, d_1 \\ \bullet \text{ } g = 0, d_2 \\ \dots \\ \bullet \text{ } g = 0, d_k \end{array} \right\}.$$

$$(4) d_0 \mathcal{E}(d_1, \dots, d_k) = \left\{ \begin{array}{c} g = 1, \\ d = 0 \\ \bullet \text{ } g = 0, d_1 \\ \bullet \text{ } g = 0, d_2 \\ \dots \\ \bullet \text{ } g = 0, d_k \end{array} \right\}.$$

$$(5) (d_{1,1}, \dots, d_{1,k_1}) \mathcal{E} d_0 \mathcal{E}(d_{2,1}, \dots, d_{2,k_2})$$

$$= \left\{ \begin{array}{c} \bullet \text{ } g = 0, d_{1,1} \\ \bullet \text{ } g = 0, d_{1,2} \\ \dots \\ \bullet \text{ } g = 0, d_{1,k_1} \end{array} \begin{array}{c} g = 1, \\ d = 0 \\ \bullet \text{ } g = 0, d_0 \end{array} \begin{array}{c} g = 1, \\ d = 0 \\ \bullet \text{ } g = 0, d_{2,1} \\ \bullet \text{ } g = 0, d_{2,2} \\ \dots \\ \bullet \text{ } g = 0, d_{2,k_2} \end{array} \right\}.$$

$$(6) \text{br} = d_0 \mathcal{E}(d_1, \dots, d_k) = \left\{ \begin{array}{c} g = 1, \\ d = 0 \\ \bullet \text{ } g = 0, d_1 \\ \bullet \text{ } g = 0, d_2 \\ \dots \\ \bullet \text{ } g = 0, d_k \end{array} \right\}.$$

This is taken from the first author’s PhD thesis [10], and is implicit in [40].

4.2 Factorisation through a Gorenstein curve

Definition 4.1 Let $(X, \mathcal{O}_X(1))$ be a polarised variety, and $\beta \in H_2^+(X, \mathbb{Z})$ an effective curve class. The moduli space $\mathcal{A}_{2,n}(X, \beta)$ of *admissible maps* to X is defined by the fibre diagram

$$\begin{array}{ccc} \mathcal{A}_{2,n}(X, \beta) & \longrightarrow & \overline{\mathcal{M}}_{2,n}(X, \beta) \\ \downarrow & \square & \downarrow \\ \mathcal{A}_{2,n}^{\text{wt}} & \longrightarrow & \mathfrak{M}_{2,n}^{\text{wt}} \end{array}$$

It can be described as a space of maps and admissible covers with the same source

$$\begin{array}{ccc} (C, D_R, \mathbf{x}) & \xrightarrow{f} & \mathbb{P}^r \\ \downarrow \psi & & \\ (T, D_B, \psi(\mathbf{x})) & & \end{array}$$

subject to the stability condition that, if $\sigma: C \rightarrow C$ is the hyperelliptic involution,

$$\omega_C(\mathbf{x}) \otimes f^* \mathcal{O}_X(2) \otimes \sigma^* f^* \mathcal{O}_X(2)$$

is relatively ample.

If we set $d = \mathcal{O}_X(1) \cdot \beta$, we can restrict to the component of the base where the weight is d .

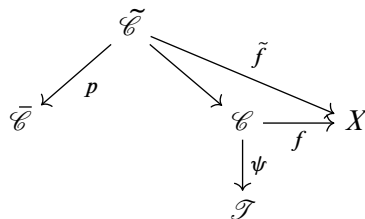
Remark 4.2 On the locus of maps from a smooth curve, $\mathcal{A}_{2,n}(X, \beta) \rightarrow \overline{\mathcal{M}}_{2,n}(X, \beta)$ is an isomorphism; therefore, for $X = \mathbb{P}^r$, the main components are birational.

Definition 4.3 Let $\tilde{\mathcal{A}}_{2,n}(X, \beta)$ denote the fibre product

$$\begin{array}{ccc} \tilde{\mathcal{A}}_{2,n}(X, \beta) & \longrightarrow & \mathcal{A}_{2,n}(X, \beta) \\ \downarrow & \square & \downarrow \\ \tilde{\mathcal{A}}_{2,n} & \longrightarrow & \mathcal{A}_{2,n}^{\text{wt}} \end{array}$$

We call it the moduli space of *aligned admissible maps*.

Remark 4.4 $\tilde{\mathcal{A}}_{2,n}(X, \beta)$ is logarithmically étale over $\mathcal{A}_{2,n}(X, \beta)$. It comes with universal structures



Definition 4.5 Let $\tilde{\mathcal{A}}_{2,n}(X, \beta)^{\text{fact}} \subseteq \tilde{\mathcal{A}}_{2,n}(X, \beta)$ be the locus of maps such that $\tilde{f}: \tilde{\mathcal{C}} \rightarrow X$ factors through a map $\bar{f}: \tilde{\mathcal{C}} \rightarrow X$. We call it the moduli space of aligned admissible maps satisfying the *first factorisation property*.

Remark 4.6 The map f need *not* factor through the admissible cover ψ .

Proposition 4.7 $\tilde{\mathcal{A}}_{2,n}(X, \beta)^{\text{fact}} \subseteq \tilde{\mathcal{A}}_{2,n}(X, \beta)$ is a closed substack. If X is smooth, there is a perfect obstruction theory

$$(R^* \pi_* \tilde{f}^* T_X)^\vee \rightarrow \mathbb{L}_{\tilde{\mathcal{A}}_{2,n}(X, \beta)^{\text{fact}} / \tilde{\mathcal{A}}_{2,n}}$$

endowing $\tilde{\mathcal{A}}_{2,n}(X, \beta)^{\text{fact}}$ with a virtual fundamental class in $A_{\text{vdim}}(\tilde{\mathcal{A}}_{2,n}(X, \beta)^{\text{fact}})$, where

$$\text{vdim}(\mathcal{VZ}_{2,n}(X, \beta)) = 3 - \dim(X) + n - K_X \cdot \beta.$$

Proof For the first claim, we refer the reader to [63, Theorem 4.3]. The second claim goes back to K Behrend and B Fantechi [15, Proposition 6.3]. □

Although $\tilde{\mathcal{A}}_{2,n}(\mathbb{P}^r, d)^{\text{fact}}$ is not necessarily smooth, as it may still have a hyperelliptic component, it can already be useful to the end of computing the invariants of a projective complete intersection.

Lemma 4.8 If $\iota: X \hookrightarrow \mathbb{P}^r$ is a complete intersection of degree (l_1, \dots, l_h) ($l_i \geq 2$ for all i),

$$\tilde{\mathcal{A}}_{2,n}(X, \beta)^{\text{fact}} \subseteq \tilde{\mathcal{A}}_{2,n}(\mathbb{P}^r, \iota_* \beta)^{\text{fact}}$$

is cut out by a section of the vector bundle

$$\mathcal{E}_{(l_1, \dots, l_k)} = \bar{\pi}_* \tilde{f}^* \left(\bigoplus_{i=1}^h \mathcal{O}_{\mathbb{P}^r}(l_i) \right).$$

In particular, the invariants satisfy the **quantum Lefschetz hyperplane principle**.

Proof Thanks to the last statement of Theorem 3.47, the pullback $f^* \mathcal{O}_{\mathbb{P}^r}(1)$ has degree at least one on any subcurve of positive genus. Then the degree has to be at least two on the minimal subcurve of genus two in order for a nonconstant map to exist. This implies that $\tilde{f}^* \mathcal{O}_{\mathbb{P}^r}(l_i)$ has vanishing h^1 along the fibres of $\tilde{\mathcal{C}}$ for $l_i \geq 2$. Therefore, $\mathcal{E}_{(l_1, \dots, l_k)}$ is a vector bundle by “cohomology and base change”. The virtual statement follows as in genus zero from [45]. □

Lemma 4.9 The projection $\tilde{\mathcal{A}}_{2,n}(\mathbb{P}^r, d)^{\text{fact}} \rightarrow \tilde{\mathcal{A}}_{2,n}$ factors through $\tilde{\mathcal{A}}_{2,n}^\circ$.

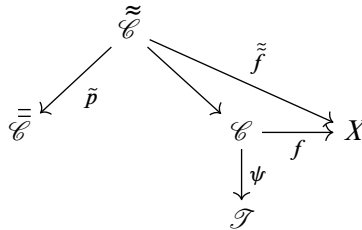
Proof See Definition 3.30 for the notation. The key observation is that the only (nonconstant) maps of degree two from a minimal Gorenstein curve of genus two are those which factor through the hyperelliptic cover of a line. Hence, we may discard the locus where the weight of Δ is two but (partly) supported away from D . □

Thanks to this lemma we can make the following:

Definition 4.10 Let $\tilde{\mathcal{A}}_{2,n}(X, \beta)$ be the moduli space of *admissible maps with a secondary alignment* defined by the fibre diagram

$$\begin{array}{ccc} \tilde{\mathcal{A}}_{2,n}(X, \beta) & \longrightarrow & \tilde{\mathcal{A}}_{2,n}(X, \beta)^{\text{fact}} \\ \downarrow & \square & \downarrow \\ \tilde{\mathcal{A}}_{2,n} & \longrightarrow & \tilde{\mathcal{A}}_{2,n}^{\circ} \end{array}$$

This space comes with universal structures



Let $\mathcal{VZ}_{2,n}(X, \beta) \subseteq \tilde{\mathcal{A}}_{2,n}(X, \beta)$ be the locus of maps such that $\tilde{f}: \tilde{\mathcal{C}} \rightarrow X$ factors through a map $\tilde{f}: \bar{\mathcal{C}} \rightarrow X$. We call it the moduli space of *aligned admissible maps* satisfying the *second factorisation property*.³

Definition 4.11 Let $\text{ev}_i: \mathcal{VZ}_{2,n}(X, \beta) \rightarrow X$ denote the evaluation map at the i^{th} marked point for $i = 1, \dots, n$. Let $\alpha_1, \dots, \alpha_n \in H^*(X)$ be cohomology classes on the target manifold. The *reduced genus two Gromov–Witten invariants* are defined as

$$\langle \alpha_1, \dots, \alpha_n \rangle_{2,\beta,n}^{X,\text{red}} = \int_{[\mathcal{VZ}_{2,n}(X,\beta)]^{\text{vir}}} \text{ev}_1^* \alpha_1 \cup \dots \cup \text{ev}_n^* \alpha_n.$$

Theorem 4.12 For $d \geq 3$, $\mathcal{VZ}_{2,n}(\mathbb{P}^r, d)$ is a desingularisation of $\overline{\mathcal{M}}_{2,n}(\mathbb{P}^r, d)^{\text{main}}$.

Proof Consider the factorisation

$$\mathcal{VZ}_{2,n}(\mathbb{P}^r, d) \rightarrow \mathfrak{Pic}_{\tilde{\mathcal{A}}_{2,n}} \rightarrow \tilde{\mathcal{A}}_{2,n}.$$

Obstructions to the first map can be found in $H^1(\bar{\mathcal{C}}, L)$, where $L = \tilde{f}^* \mathcal{O}_{\mathbb{P}^r}(1)$. If there are two disconnected subcurves of genus one, the degree must be positive on each of them by [Theorem 3.59](#). On the other hand, if the core is a minimal subcurve of genus two, it must have degree at least three by [Theorem 3.60](#), and either the special component has positive degree (at least two) by [Corollary 3.26](#), or the special component is a collapsed ribbon with at least two tails of positive degree by [Corollary 3.27](#). It follows from [Lemmas 2.33](#) and [2.34](#) that the obstructions vanish.

³The notation is reminiscent of the celebrated desingularisation of $\mathcal{M}_{1,n}(\mathbb{P}^r, d)^{\text{main}}$ due to Vakil and Zinger [\[76\]](#).

Obstructions to the second map lie in $H^2(\overline{C}, \mathcal{O})$, which vanishes for dimension reasons. The map is therefore unobstructed. The base is smooth by [Theorem 3.36](#). We conclude that $\mathcal{VZ}_{2,n}(\mathbb{P}^r, d)$ is smooth as well. Since it is proper and it contains the locus of maps from a smooth curve as an open dense, $\mathcal{VZ}_{2,n}(\mathbb{P}^r, d) \rightarrow \overline{\mathcal{M}}_{2,n}(\mathbb{P}^r, d)^{\text{main}}$ is birational (see [Remarks 4.2](#) and [4.4](#)). \square

Remark 4.13 A posteriori, we note that aligning and the factorisation property do not alter the main component of $\overline{\mathcal{M}}_2(\mathbb{P}^r, 2)$, which therefore is already smooth.

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