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Our main purpose is to describe the category of isotropic cellular spectra over flexible fields. Guided by Gheorghe, Wang and Xu (*Acta Math.* 226 (2021) 319–407), we show that it is equivalent, as a stable ∞ -category equipped with a t -structure, to the derived category of left comodules over the dual of the classical topological Steenrod algebra. In order to obtain this result, the category of isotropic cellular modules over the motivic Brown–Peterson spectrum is also studied, and isotropic Adams and Adams–Novikov spectral sequences are developed. As a consequence, we also compute hom sets in the category of isotropic Tate motives between motives of isotropic cellular spectra.

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A list of symbols can be found on page 2046.

1 Introduction

Isotropic categories are local versions of motivic categories, obtained by, roughly speaking, killing all anisotropic varieties. Although they often have a handier structure than their global versions, they exhibit some key characteristics of both motivic and classical topological phenomena. In [21], Vishik introduced the isotropic triangulated category of motives and computed the isotropic motivic cohomology of the point, which is strongly related to the Milnor subalgebra. By following this lead, we studied in [19] the isotropic stable motivic homotopy category. In particular, we identified the isotropic motivic homotopy groups of the sphere spectrum with the cohomology of the topological Steenrod algebra, ie the E_2 -page of the classical Adams spectral sequence. These results are quite surprising since they show that topological objects naturally arise from isotropic environments, which could lead to a fruitful exchange between topology and isotropic motivic theory.

Motivic categories, constructed by Morel and Voevodsky (see [16; 23]) in order to study algebraic varieties by topological means, are extremely rich categories. Even

over an algebraically closed field they are more complex than the respective topological counterparts. For example, while every object in the classical stable homotopy category is cellular (built up by attaching spheres), not every motivic spectrum is cellular, since many algebrogeometric phenomena come into the picture. In spite of this, it is still interesting to understand the structure of the category of cellular objects in motivic stable homotopy theory. This project was initiated by Dugger and Isaksen in [3] and much attention has been dedicated to it since then. Our work, in particular, is concerned with understanding the structure of the subcategories of cellular objects in isotropic categories, which we believe could shed light on the deep interconnection with topology.

We have already highlighted that motivic categories are particularly challenging to study. For example, one of the difficulties that one does not encounter in classical topology is the presence of an object τ that appears in various incarnations throughout motivic homotopy theory, sometimes as an element of the motivic cohomology of the ground field and sometimes as a map in the 2-complete motivic stable homotopy groups of spheres. Hence, the principal task is to first find some substitutes for the original motivic categories and tools which could help in the process of analyzing them. In the case of algebraically closed fields, for example, topological realization is a very helpful tool since it allows us to study the initial motivic category by looking at its deformation $\tau = 1$, which happens to be just the classical stable homotopy theory; see Dugger and Isaksen [4]. However, in this process part of the information is lost, so one can try to recover it by studying other deformations, for example $\tau = 0$. This was done by Isaksen in [9], Gheorghe in [5] and Gheorghe, Wang and Xu in [6]. More precisely, in [9] the stable motivic homotopy groups of $C\tau$, the cofiber of τ , are identified with the E_2 -page of the classical Adams–Novikov spectral sequence, while in [5] the motivic spectrum $C\tau$ is provided with an E_∞ -ring structure inducing an isomorphism of rings with higher products between $\pi_{**}(C\tau)$ and the classical Adams–Novikov E_2 -page. A parallel result for isotropic categories was obtained in [19], where the isotropic sphere spectrum \mathfrak{X} was equipped with an E_∞ -ring structure inducing an isomorphism of rings with higher products between $\pi_{**}(\mathfrak{X})$ and the classical Adams E_2 -page. Moreover, in [6] the category of $C\tau$ -cellular spectra is described, and is proved to be equivalent as a stable ∞ -category equipped with a t -structure (see Lurie [13]) to the derived category of left $\mathrm{BP}_*\mathrm{BP}$ -comodules concentrated in even degrees, where BP is the Brown–Peterson spectrum and $\mathrm{BP}_*\mathrm{BP}$ its BP -homology.

We intend to follow a similar path for isotropic categories. Recall that a field k is called flexible if it is a purely transcendental extension of countable infinite degree over

some other field. In our situation it is really essential to work over flexible fields since, as highlighted in [21], these are the ground fields over which the isotropic categories behave particularly well. For example, over algebraically closed fields, due to the lack of anisotropic varieties, the isotropic category would be just the same as the original motivic category, so in this case the isotropic localization produces nothing new. We are encouraged by the evident parallel between the computations of $\pi_{**}(C\tau)$ over complex numbers (see [5; 9]) on the one hand, and of $\pi_{**}(\mathfrak{X})$ over flexible fields (see [19]) on the other. More precisely, we have been guided by the idea that studying the isotropic stable motivic homotopy category over a flexible field is similar in some sense to studying the stable ∞ -category of $C\tau$ -cellular spectra in the motivic stable homotopy category over complex numbers. Indeed, they obviously share some common features which is highlighted by our main theorem:

Theorem 1.1 *Let k be a flexible field of characteristic different from 2. Then there exists a t -exact equivalence of stable ∞ -categories*

$$\mathcal{D}^b(A_*\text{-Comod}_*) \xrightarrow{\cong} \mathfrak{X}\text{-Mod}_{\text{cell}, \mathbb{H}\mathbb{Z}/2}^b,$$

where A_* is the classical dual Steenrod algebra and $\mathfrak{X}\text{-Mod}_{\text{cell}, \mathbb{H}\mathbb{Z}/2}^b$ is the stable ∞ -category of $\mathbb{H}\mathbb{Z}/2$ -complete \mathfrak{X} -cellular modules having MBP-homology nontrivial in only finitely many Chow–Novikov degrees (the superscript “b” stands for “bounded”; see Definition 8.4).

As a consequence, we obtain that the category of isotropic cellular spectra is completely algebraic, which makes it easier to study. Moreover, it is deeply related to classical topology, as foreseeable from results in [19; 21].

In order to achieve our main results, we need several tools. In particular, it is necessary to develop and study isotropic versions of both the Adams spectral sequence and the Adams–Novikov spectral sequence. This requires a focus on the motivic Brown–Peterson spectrum MBP (see Vezzosi [20]) from an isotropic point of view. In particular, we note that the isotropic Brown–Peterson spectrum is an E_∞ -ring spectrum, in contrast to the topological picture where BP has been shown not to admit an E_∞ -ring structure by Lawson in [11]. Then we use techniques developed by Gheorghe, Wang and Xu in [6], based on Lurie’s results (see [13]), to first describe in algebraic terms the category of isotropic MBP-cellular modules, and then the category of all isotropic cellular spectra. Finally, we are also able to provide some results about the cellular subcategory of the isotropic triangulated category of motives, ie the category of isotropic Tate motives.

Outline We now briefly present the contents of each section. In [Section 2](#), we provide our main notation. Then we move on to [Section 3](#) by recalling isotropic categories and their main properties, mostly referring to results in [\[19; 21\]](#). Since we are mainly interested in cellular objects, we recall in [Section 4](#) definitions and some of the main results from [\[3\]](#), which are useful in the rest of the paper. [Section 5](#) is devoted to a deep analysis of the isotropic motivic Adams spectral sequence, which was already initiated in [\[19\]](#). These results are used in [Section 6](#) to study the motivic Brown–Peterson spectrum from an isotropic perspective. In particular, we compute its isotropic stable homotopy groups. Sections [7](#) and [8](#) are modeled on Sections 3, 4 and 5 of [\[6\]](#). More precisely, in [Section 7](#) we endow the isotropic motivic Brown–Peterson spectrum with an E_∞ –ring structure, and then identify, as a triangulated category, the category of isotropic MBP–cellular spectra with the category of bigraded \mathbb{F}_2 –vector spaces. In [Section 8](#), after developing an isotropic Adams–Novikov spectral sequence, we describe the category of isotropic cellular spectra in algebraic terms as the derived category of comodules over the dual of the Steenrod algebra equipped with a t –structure. Finally, in [Section 9](#), we provide an algebraic description of the hom sets in the category of isotropic motives between motives of isotropic cellular spectra, which is a step forward in understanding the category of isotropic Tate motives.

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2 Notation

We denote hom sets in $\mathcal{SH}(k)$ by $[\cdot, \cdot]$ and the suspension $S^{p,q} \wedge X$ of a motivic spectrum X by $\Sigma^{p,q} X$. Moreover, if E is a motivic E_∞ –ring spectrum, the stable ∞ –category of E –modules (see [\[13\]](#)) is denoted by $E\text{-Mod}$, its smash product by $\cdot \wedge_E \cdot$ and hom sets in its homotopy category by $[\cdot, \cdot]_E$.

If R is an algebra and C a coalgebra, then we denote by $R\text{-Mod}$ and $C\text{-Comod}$ the categories of left R –modules and left C –comodules, respectively. Hom sets in these categories are both denoted by $\text{Hom}_R(\cdot, \cdot)$ and $\text{Hom}_C(\cdot, \cdot)$, and it will be clear from context if they are meant to be hom of modules or comodules. For a bigraded

object M_{**} (resp. M^{**}) we denote by $\Sigma^{p,q} M_{**}$ (resp. $\Sigma^{p,q} M^{**}$) its suspension, the bigraded object defined by $\Sigma^{p,q} M_{a,b} = M_{a-p,b-q}$ (resp. $\Sigma^{p,q} M^{a,b} = M^{a+p,b+q}$). The convention for bigraded homomorphisms between bigraded objects is

$$\text{Hom}^{p,q}(M_{**}, N_{**}) = \text{Hom}^{0,0}(\Sigma^{p,q} M_{**}, N_{**})$$

and

$$\text{Hom}^{p,q}(M^{**}, N^{**}) = \text{Hom}^{0,0}(\Sigma^{p,q} M^{**}, N^{**}),$$

where $\text{Hom}^{0,0}(\cdot, \cdot)$ denotes the bidegree-preserving homomorphisms. Moreover, the bounded derived categories of $R\text{-Mod}$ and $C\text{-Comod}$ are denoted by $\mathcal{D}^b(R\text{-Mod})$ and $\mathcal{D}^b(C\text{-Comod})$, respectively.

3 Isotropic motivic categories

In this section we want to introduce the main categories we consider, namely isotropic motivic categories. These categories are built from the respective motivic ones by killing all anisotropic varieties. We refer to [19, Section 2; 21, Section 2] for more details on the construction and properties of isotropic categories.

Let us recall first the definition of flexible field from [21]:

Definition 3.1 A field k is called flexible if it is a purely transcendental extension of countable infinite degree: $k = k_0(t_1, t_2, \dots)$ for some other field k_0 .

Henceforth we assume k is a flexible base field of characteristic different from 2. We proceed by recalling the definition of a fundamental object in $\mathcal{SH}(k)$ for the construction of the isotropic stable motivic homotopy category $\mathcal{SH}(k/k)$.

Definition 3.2 Denote by Q the disjoint union of all connected anisotropic (mod 2) varieties over k , ie varieties which do not have closed points of odd degree, and by $\check{C}(Q)$ its Čech simplicial scheme $\check{C}(Q)_n = Q^{n+1}$ with face and degeneracy maps given by partial projections and partial diagonals, respectively. We define the isotropic sphere spectrum \mathfrak{X} as $\text{Cone}(\Sigma_+^\infty \check{C}(Q) \rightarrow S)$ in $\mathcal{SH}(k)$.

We recall from [19, Section 2] that \mathfrak{X} is an idempotent monoid, that is, there is an equivalence $\mathfrak{X} \wedge \mathfrak{X} \cong \mathfrak{X}$ induced by the map $S \rightarrow \mathfrak{X}$, and so it is an E_∞ -ring spectrum; see [19, Proposition 6.1].

Definition 3.3 The full triangulated subcategory $\mathfrak{X} \wedge \mathcal{SH}(k)$ of $\mathcal{SH}(k)$ will be called the isotropic stable motivic homotopy category and denoted by $\mathcal{SH}(k/k)$.

This triangulated category has very nice properties. In particular it is both localizing and colocalizing; see [19, Section 2]. The very same construction was done first for $\mathcal{DM}(k)$ by Vishik in [21] by tensoring the triangulated category of motives with the idempotent $M(\mathfrak{X})$, where $M : \mathcal{SH}(k) \rightarrow \mathcal{DM}(k)$ is the motivic functor.

Definition 3.4 The full triangulated subcategory $M(\mathfrak{X}) \otimes \mathcal{DM}(k)$ of $\mathcal{DM}(k)$ will be called the isotropic category of motives and denoted by $\mathcal{DM}(k/k)$.

The following result tells us that the isotropic stable motivic homotopy category is nothing but the stable ∞ -category of \mathfrak{X} -modules:

Proposition 3.5 *There is an equivalence between the isotropic stable motivic homotopy category $\mathcal{SH}(k/k)$ and the stable ∞ -category $\mathfrak{X}\text{-Mod}$ of modules over the motivic E_∞ -ring spectrum \mathfrak{X} .*

Proof This follows immediately from [13, Proposition 4.8.2.10]. □

Remark 3.6 Since by construction \mathfrak{X} kills all anisotropic varieties, it kills in particular nontrivial quadratic extensions. Consider an element x in k such that neither x nor $-x$ is a square. Then $\mathfrak{X} \wedge \Sigma_+^\infty \text{Spec}(k(\sqrt{x}))$ and $\mathfrak{X} \wedge \Sigma_+^\infty \text{Spec}(k(\sqrt{-x}))$ are both zero. This implies that the Euler characteristics of $\text{Spec}(k(\sqrt{x}))$ and $\text{Spec}(k(\sqrt{-x}))$, which are equal to $\langle 2 \rangle(1 + \langle x \rangle)$ and $\langle 2 \rangle(1 + \langle -x \rangle)$, respectively, in $\pi_{0,0}(\mathbb{S}) \cong \text{GW}(k)$ (see [12, Corollary 11.2; 15, Theorem 6.2.2]), vanish in $\pi_{0,0}(\mathfrak{X})$. It follows that $1 + \langle x \rangle$ and $1 + \langle -x \rangle$ vanish in $\pi_{0,0}(\mathfrak{X})$ and so does their sum

$$2 + \langle x \rangle + \langle -x \rangle = 2 + \langle 1 \rangle + \langle -1 \rangle = 3 + \langle -1 \rangle.$$

Hence, $-3 = \langle -1 \rangle$, and so $9 = 1$ (so $8 = 0$) in $\pi_{0,0}(\mathfrak{X})$. From all this one deduces that \mathfrak{X} is 2-power torsion.¹

We are now ready to define isotropic motivic homotopy groups and isotropic motivic homology and cohomology.

Definition 3.7 Let X be a motivic spectrum in $\mathcal{SH}(k)$. Then the isotropic stable motivic homotopy groups of X are defined by

$$\pi_{**}^{\text{iso}}(X) = [\mathbb{S}^{**}, \mathfrak{X} \wedge X] = \pi_{**}(\mathfrak{X} \wedge X).$$

¹I am grateful to Tom Bachmann for this argument.

Recall that motivic cohomology with $\mathbb{Z}/2$ -coefficients is represented by the motivic Eilenberg–Mac Lane spectrum $\mathbb{H}\mathbb{Z}/2$. Then we define isotropic motivic cohomology as the cohomology theory represented by the motivic E_∞ -ring spectrum $\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2$.

Definition 3.8 For any X in $\mathcal{SH}(k)$, we define the isotropic motivic cohomology of X as

$$H_{\text{iso}}^{**}(X) = [X, \Sigma^{**}(\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2)]$$

and the isotropic motivic homology of X as

$$H_{**}^{\text{iso}}(X) = [S^{**}, \mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2 \wedge X] = H_{**}(\mathfrak{X} \wedge X).$$

The isotropic motivic cohomology of the point was computed by Vishik:

Theorem 3.9 [21, Theorem 3.7] *Let k be a flexible field. Then for any $i \geq 0$ there exists a unique cohomology class r_i of bidegree $(-2^i + 1)[-2^{i+1} + 1]$ such that*

$$H^{**}(k/k) \cong \Lambda_{\mathbb{F}_2}(r_i)_{i \geq 0}$$

and $Q_j r_i = \delta_{ij}$, where the Q_j are the Milnor operations.

At this point, we want to introduce the isotropic motivic Steenrod algebra $\mathcal{A}^{**}(k/k)$ and its dual $\mathcal{A}_{**}(k/k)$. They are defined as the isotropic motivic cohomology and homology, respectively, of the motivic Eilenberg–Mac Lane spectrum.

Definition 3.10 The isotropic motivic Steenrod algebra is defined by

$$\mathcal{A}^{**}(k/k) = H_{\text{iso}}^{**}(\mathbb{H}\mathbb{Z}/2) = [\mathbb{H}\mathbb{Z}/2, \Sigma^{**}(\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2)] \cong [\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2, \Sigma^{**}(\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2)]$$

and its dual by

$$\mathcal{A}_{**}(k/k) = H_{**}^{\text{iso}}(\mathbb{H}\mathbb{Z}/2) = [S^{**}, \mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2 \wedge \mathbb{H}\mathbb{Z}/2].$$

The structure of $\mathcal{A}^{**}(k/k)$ was studied in [19, Section 3]. We summarize the main results:

Proposition 3.11 [19, Propositions 3.5, 3.6 and 3.7] *Let k be a flexible field. Then there exists an isomorphism of $H^{**}(k/k) - \mathcal{M}^{**}$ -bimodules*

$$\mathcal{A}^{**}(k/k) \cong H^{**}(k/k) \otimes_{\mathbb{F}_2} \mathcal{G}^{**} \otimes_{\mathbb{F}_2} \mathcal{M}^{**},$$

where \mathcal{M}^{**} is the Milnor subalgebra $\Lambda_{\mathbb{F}_2}(Q_i)_{i \geq 0}$ and \mathcal{G}^{**} is the bigraded topological Steenrod algebra, ie $\mathcal{G}^{2n,n} = \mathcal{A}^n$.

By projecting the motivic Cartan formulas (see [24, Propositions 9.7 and 13.4]) to the isotropic category, one gets a coproduct on $\mathcal{A}^{**}(k/k)$ given by

$$\Delta(\mathrm{Sq}^{2n}) = \sum_{i+j=n} \mathrm{Sq}^{2i} \otimes \mathrm{Sq}^{2j}, \quad \Delta(Q_i) = Q_i \otimes 1 + 1 \otimes Q_i.$$

This coproduct structures $\mathcal{A}^{**}(k/k)$ as a coalgebra whose dual is described as an $H_{**}(k/k)$ -algebra by

$$\mathcal{A}_{**}(k/k) \cong \frac{H_{**}(k/k)[\tau_i, \xi_j]_{i \geq 0, j \geq 1}}{(\tau_i^2)},$$

where τ_i is the dual of the Milnor operation Q_i and ξ_j is the dual of the motivic cohomology operation $\mathrm{Sq}^{2^j} \cdots \mathrm{Sq}^2$. The coproduct in $\mathcal{A}_{**}(k/k)$ is given by (see [24, Lemma 12.11])

$$\psi(\xi_k) = \sum_{i=0}^k \xi_{k-i}^{2^i} \otimes \xi_i, \quad \psi(\tau_k) = \sum_{i=0}^k \xi_{k-i}^{2^i} \otimes \tau_i + \tau_k \otimes 1.$$

Remark 3.12 By Proposition 3.11, the projection from $\mathcal{A}^{**}(k/k)$ to its quotient by the left ideal generated by Milnor operations provides a homomorphism

$$\mathcal{A}^{**}(k/k) \rightarrow H^{**}(k/k) \otimes_{\mathbb{F}_2} \mathcal{G}^{**}.$$

This map induces a left $\mathcal{A}^{**}(k/k)$ -action on $H^{**}(k/k) \otimes_{\mathbb{F}_2} \mathcal{G}^{**}$ and, dually, a left $\mathcal{A}_{**}(k/k)$ -coaction on $H_{**}(k/k) \otimes_{\mathbb{F}_2} \mathcal{G}_{**}$, where \mathcal{G}_{**} is the subalgebra $\mathbb{F}_2[\xi_1, \xi_2, \dots]$.

4 Cellular motivic spectra

We are mostly interested in cellular objects of isotropic motivic categories. We recall from [3, Remark 7.4] that the category of cellular motivic spectra, which we denote by $\mathcal{SH}(k)_{\mathrm{cell}}$, is the localizing subcategory of $\mathcal{SH}(k)$ generated by the spheres $\Sigma^{p,q}\mathbb{S}$. Similarly, the category of Tate motives, which we denote by $\mathcal{DM}(k)_{\mathrm{Tate}}$, is the localizing subcategory of $\mathcal{DM}(k)$ generated by the Tate motives $T(q)[p]$. If E is a motivic E_∞ -ring spectrum, then we denote by $E\text{-Mod}_{\mathrm{cell}}$ the stable ∞ -category of E -cellular modules, meaning the localizing subcategory of $E\text{-Mod}$ generated by $\Sigma^{p,q}E$.

Definition 4.1 The category of \mathfrak{X} -cellular modules will be called the category of isotropic cellular motivic spectra, and is denoted by $\mathcal{SH}(k/k)_{\mathrm{cell}}$. In the same way, the full localizing subcategory of $\mathcal{DM}(k/k)$ generated by the objects $M(\mathfrak{X})(q)[p]$ will be called the category of isotropic Tate motives, and is denoted by $\mathcal{DM}(k/k)_{\mathrm{Tate}}$.

A fundamental property of the category of cellular objects is that isomorphisms can be detected by motivic homotopy groups:

Proposition 4.2 [3, Corollary 7.2 and Section 7.9] *Let E be a motivic E_∞ -ring spectrum and $X \rightarrow Y$ be a map of E -cellular motivic spectra that induces isomorphisms on $\pi_{p,q}$ for all p and q in \mathbb{Z} . Then the map is a weak equivalence.*

Another essential advantage of dealing with cellular objects is that they allow the construction of very useful convergent spectral sequences.

Proposition 4.3 [3, Propositions 7.7 and 7.10] *Let E be a motivic E_∞ -ring spectrum and N a left E -module. If M is a right E -cellular spectrum then there is a strongly convergent spectral sequence*

$$E_{s,t,u}^2 \cong \text{Tor}_{s,t,u}^{\pi_{**}(E)}(\pi_{**}(M), \pi_{**}(N)) \Rightarrow \pi_{s+t,u}(M \wedge_E N).$$

If M is a left E -cellular motivic spectrum then there is a conditionally convergent spectral sequence

$$E_2^{s,t,u} \cong \text{Ext}_{\pi_{**}(E)}^{s,t,u}(\pi_{**}(M), \pi_{**}(N)) \Rightarrow [\Sigma^{t-s,u} M, N]_E.$$

5 The isotropic motivic Adams spectral sequence

In this section we recall the construction of the isotropic motivic Adams spectral sequence; see [19, Section 4]. Moreover, we study the circumstances under which the E_2 -page is expressible in terms of Ext-groups over the isotropic motivic Steenrod algebra.

Definition 5.1 Let Y be an isotropic motivic spectrum (an object in $\mathfrak{X}\text{-Mod}$). Then the standard isotropic motivic Adams resolution of Y consists of the Postnikov system

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \overline{(\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2)}^{\wedge s} \wedge Y & \longrightarrow & \cdots & \longrightarrow & \overline{\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2} \wedge Y \longrightarrow Y \\ & \swarrow [1] & \downarrow & & \swarrow [1] & \downarrow & \swarrow [1] & \downarrow \\ & \mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2 \wedge \overline{(\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2)}^{\wedge s} \wedge Y & & & \mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2 \wedge \overline{\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2} \wedge Y & & \mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2 \wedge Y \end{array}$$

where $\overline{\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2}$ is defined by the exact triangle in $\mathcal{SH}(k)$

$$\overline{\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2} \rightarrow \mathbb{S} \rightarrow \mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2 \rightarrow \Sigma^{1,0} \overline{\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2}.$$

By applying motivic homotopy groups functors π_{**} to the previous Postnikov system we get an unrolled exact couple, which induces in turn a spectral sequence with E_1 -page described by

$$E_1^{s,t,u} \cong \pi_{t-s,u}(\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2 \wedge \overline{(\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2)}^{\wedge s} \wedge Y)$$

and first differential

$$d_1^{s,t,u} : \pi_{t-s,u}(\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2 \wedge \overline{(\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2)}^{\wedge s} \wedge Y) \rightarrow \pi_{t-s-1,u}(\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2 \wedge \overline{(\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2)}^{\wedge s+1} \wedge Y).$$

In general, differentials on the E_r -page have tridegrees given by

$$d_r^{s,t,u} : E_r^{s,t,u} \rightarrow E_r^{s+r,t+r-1,u}.$$

We call this spectral sequence the isotropic motivic Adams spectral sequence.

The isotropic Adams spectral sequence converges to the homotopy groups of a motivic spectrum closely related to Y , namely its $\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2$ -nilpotent completion, which we denote by $Y_{\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2}^\wedge$. Before proceeding, let us recall from [2, Section 5] how to construct the E -nilpotent completion of a spectrum Y for a homotopy ring spectrum E .

Definition 5.2 Let E be a homotopy ring spectrum and Y a motivic spectrum in $\mathcal{SH}(k)$. First, define \bar{E} by the distinguished triangle in $\mathcal{SH}(k)$

$$\bar{E} \rightarrow S \rightarrow E \rightarrow \Sigma^{1,0}\bar{E}.$$

Then define \bar{E}_n as $\text{Cone}(\bar{E}^{\wedge n+1} \rightarrow S)$ in $\mathcal{SH}(k)$. This way one gets an inverse system

$$\dots \rightarrow \bar{E}_n \wedge Y \rightarrow \dots \rightarrow \bar{E}_1 \wedge Y \rightarrow \bar{E}_0 \wedge Y,$$

and the E -nilpotent completion of Y is the motivic spectrum $Y_E^\wedge = \text{holim}(\bar{E}_n \wedge Y)$.

Note that, by [19, Proposition 2.3], if Y is an isotropic motivic spectrum so is Y_E^\wedge .

Proposition 5.3 Let Y be an isotropic motivic spectrum. If $\varprojlim_r^1 E_r^{s,t,u} = 0$ for any s, t and u , then the isotropic motivic Adams spectral sequence for Y is strongly convergent to the stable motivic homotopy groups of the $\mathbb{H}\mathbb{Z}/2$ -nilpotent completion of Y .

Proof By [2, Proposition 6.3; 4, Remark 6.11], under the vanishing hypothesis on $\varprojlim_r^1 E_r^{s,t,u}$, the isotropic motivic Adams spectral sequence strongly converges to $\pi_{**}(Y_{\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2}^\wedge)$. It only remains to notice that, since Y is an \mathfrak{X} -module, its $\mathbb{H}\mathbb{Z}/2$ -nilpotent and $\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2$ -nilpotent completions coincide. In fact, after smashing the

morphism of distinguished triangles

$$\begin{array}{ccccccc}
 \overline{\mathbb{H}\mathbb{Z}/2} & \longrightarrow & \mathbb{S} & \longrightarrow & \mathbb{H}\mathbb{Z}/2 & \longrightarrow & \Sigma^{1,0}\overline{\mathbb{H}\mathbb{Z}/2} \\
 \downarrow & & \parallel & & \downarrow & & \downarrow \\
 \overline{\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2} & \longrightarrow & \mathbb{S} & \longrightarrow & \mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2 & \longrightarrow & \Sigma^{1,0}\overline{\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2}
 \end{array}$$

with \mathfrak{X} , one gets

$$\begin{array}{ccccccc}
 \mathfrak{X} \wedge \overline{\mathbb{H}\mathbb{Z}/2} & \longrightarrow & \mathfrak{X} & \longrightarrow & \mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2 & \longrightarrow & \Sigma^{1,0}\mathfrak{X} \wedge \overline{\mathbb{H}\mathbb{Z}/2} \\
 \downarrow & & \parallel & & \downarrow \cong & & \downarrow \\
 \mathfrak{X} \wedge \overline{\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2} & \longrightarrow & \mathfrak{X} & \longrightarrow & \mathfrak{X} \wedge \overline{\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2} & \longrightarrow & \Sigma^{1,0}\mathfrak{X} \wedge \overline{\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2}
 \end{array}$$

since \mathfrak{X} is an idempotent in $\mathcal{SH}(k)$. It follows that $\mathfrak{X} \wedge \overline{\mathbb{H}\mathbb{Z}/2} \cong \mathfrak{X} \wedge \overline{\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2}$, and so $\mathfrak{X} \wedge \overline{\mathbb{H}\mathbb{Z}/2}_n \cong \mathfrak{X} \wedge (\overline{\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2})_n$ for any n . Therefore, since $Y \cong \mathfrak{X} \wedge Y$,

$$\begin{aligned}
 Y_{\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2}^\wedge &= \text{holim}((\overline{\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2})_n \wedge Y) \cong \text{holim}(\mathfrak{X} \wedge (\overline{\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2})_n \wedge Y) \\
 &\cong \text{holim}(\mathfrak{X} \wedge \overline{\mathbb{H}\mathbb{Z}/2}_n \wedge Y) \cong \text{holim}(\overline{\mathbb{H}\mathbb{Z}/2}_n \wedge Y) = Y_{\mathbb{H}\mathbb{Z}/2}^\wedge. \quad \square
 \end{aligned}$$

Remark 5.4 By [14, Section 5.2 and Theorem 1.0.3], the $\mathbb{H}\mathbb{Z}/2$ -completion of a connective motivic spectrum coincides with its $(2, \eta)$ -completion. Since all isotropic motivic spectra are 2-power torsion (see Remark 3.6) and so 2-complete, the previous result establishes the convergence of the isotropic Adams spectral sequence for a connective isotropic spectrum to the motivic stable homotopy groups of its η -completion.

Definition 5.5 A spectral sequence $\{E_r^{s,t,u}\}$ is called Mittag-Leffler if for each s, t and u there exists r_0 such that $E_r^{s,t,u} \cong E_\infty^{s,t,u}$ whenever $r > r_0$.

Note that every Mittag-Leffler spectral sequence satisfies the condition $\varprojlim_r^1 E_r^{s,t,u} = 0$ for any s, t and u ; see [2, after Proposition 6.3]. We will see that in many important cases the isotropic Adams spectral sequence is Mittag-Leffler, which guarantees strong convergence.

Now, we would like to understand what conditions we need to impose on Y in order to be able to express the E_2 -page of the isotropic Adams spectral sequence in terms of Ext-groups over the isotropic motivic Steenrod algebra. In order to do so, we need the following lemmas.

Lemma 5.6 Let k be a flexible field and Y an object in $\mathfrak{X}\text{-Mod}$. Then there exists an isomorphism of left $H_{**}(k/k)$ -modules

$$H_{**}^{\text{iso}}(\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2 \wedge Y) \cong \mathcal{A}_{**}(k/k) \otimes_{H_{**}(k/k)} H_{**}^{\text{iso}}(Y).$$

Proof Since by [8, Theorem 5.10] $\mathbb{H}\mathbb{Z}/2 \wedge \mathbb{H}\mathbb{Z}/2$ is a split $\mathbb{H}\mathbb{Z}/2$ -module, ie it is equivalent to a wedge sum of the form $\bigvee_{\alpha \in A} \Sigma^{p_\alpha, q_\alpha} \mathbb{H}\mathbb{Z}/2$,

$$\begin{aligned} \mathcal{A}_{**}(k/k) &\cong \pi_{**}(\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2 \wedge \mathbb{H}\mathbb{Z}/2) \cong \pi_{**}\left(\bigvee_{\alpha \in A} \Sigma^{p_\alpha, q_\alpha} (\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2)\right) \\ &\cong \bigoplus_{\alpha \in A} \Sigma^{p_\alpha, q_\alpha} \pi_{**}(\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2) \cong \bigoplus_{\alpha \in A} \Sigma^{p_\alpha, q_\alpha} H_{**}(k/k). \end{aligned}$$

Now, let Y be any object in $\mathfrak{X}\text{-Mod}$. Then

$$\begin{aligned} H_{**}^{\text{iso}}(\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2 \wedge Y) &\cong \pi_{**}(\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2 \wedge \mathbb{H}\mathbb{Z}/2 \wedge Y) \\ &\cong \pi_{**}\left(\bigvee_{\alpha \in A} \Sigma^{p_\alpha, q_\alpha} (\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2 \wedge Y)\right) \\ &\cong \bigoplus_{\alpha \in A} \Sigma^{p_\alpha, q_\alpha} \pi_{**}(\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2 \wedge Y) \\ &\cong \bigoplus_{\alpha \in A} \Sigma^{p_\alpha, q_\alpha} H_{**}^{\text{iso}}(Y) \cong \mathcal{A}_{**}(k/k) \otimes_{H_{**}(k/k)} H_{**}^{\text{iso}}(Y). \quad \square \end{aligned}$$

Remark 5.7 By the previous lemma, the map $Y \rightarrow \mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2 \wedge Y$ induces in isotropic motivic homology a coaction $H_{**}^{\text{iso}}(Y) \rightarrow \mathcal{A}_{**}(k/k) \otimes_{H_{**}(k/k)} H_{**}^{\text{iso}}(Y)$, which structures $H_{**}^{\text{iso}}(Y)$ as a left $\mathcal{A}_{**}(k/k)$ -comodule.

Next we show that, if the homology of an isotropic cellular spectrum Y is free over $H_{**}(k/k)$, then the motivic spectrum $\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2 \wedge Y$ is a split $\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2$ -module.

Lemma 5.8 *Let k be a flexible field and Y an object in $\mathfrak{X}\text{-Mod}_{\text{cell}}$ such that $H_{**}^{\text{iso}}(Y)$ is a free left $H_{**}(k/k)$ -module generated by a set of elements $\{x_\alpha\}_{\alpha \in A}$, where x_α has bidegree $(q_\alpha)[p_\alpha]$. Then there exists an isomorphism of spectra*

$$\bigvee_{\alpha \in A} \Sigma^{p_\alpha, q_\alpha} (\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2) \xrightarrow{\cong} \mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2 \wedge Y.$$

Proof Since $H_{**}^{\text{iso}}(Y) \cong \pi_{**}(\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2 \wedge Y)$, we can represent each generator x_α as a map $\Sigma^{p_\alpha, q_\alpha} \mathbb{S} \rightarrow \mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2 \wedge Y$, where $(q_\alpha)[p_\alpha]$ is the bidegree of x_α . For all $\alpha \in A$, this map corresponds bijectively to a map $\Sigma^{p_\alpha, q_\alpha} (\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2) \rightarrow \mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2 \wedge Y$ of $\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2$ -cellular modules. Hence, we get a map

$$\bigvee_{\alpha \in A} \Sigma^{p_\alpha, q_\alpha} (\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2) \rightarrow \mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2 \wedge Y$$

of $\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2$ -cellular modules. In order to check that it is an isomorphism, by Proposition 4.2 it is enough to look at the induced morphisms on homotopy groups.

Indeed, we have, on the one hand,

$$\pi_{**} \left(\bigvee_{\alpha \in A} \Sigma^{p_\alpha, q_\alpha} (\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2) \right) \cong \bigoplus_{\alpha \in A} \Sigma^{p_\alpha, q_\alpha} \pi_{**} (\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2) \cong \bigoplus_{\alpha \in A} \Sigma^{p_\alpha, q_\alpha} H_{**}(k/k)$$

and, on the other,

$$\pi_{**} (\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2 \wedge Y) \cong \bigoplus_{\alpha \in A} H_{**}(k/k) \cdot x_\alpha,$$

by hypothesis. By construction, the map we are considering induces in homotopy groups the homomorphism of $H_{**}(k/k)$ -modules

$$\pi_{**} \left(\bigvee_{\alpha \in A} \Sigma^{p_\alpha, q_\alpha} (\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2) \right) \rightarrow \pi_{**} (\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2 \wedge Y)$$

which sends $1 \in \Sigma^{p_\alpha, q_\alpha} H_{**}(k/k)$ to x_α for any $\alpha \in A$, so it is an isomorphism. \square

The next lemma provides us with a condition under which the isotropic cohomology of a spectrum is dual to its isotropic homology.

Lemma 5.9 *Let k be a flexible field and Y an object in $\mathfrak{X}\text{-Mod}$ such that there is an isomorphism $\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2 \wedge Y \cong \bigvee_{\alpha \in A} \Sigma^{p_\alpha, q_\alpha} (\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2)$ for some set A . Then for any bidegree $(q)[p]$ there is an isomorphism*

$$H_{\text{iso}}^{p,q}(Y) \cong \text{Hom}_{H_{**}(k/k)}^{-p,-q}(H_{**}^{\text{iso}}(Y), H_{**}(k/k)).$$

Proof Since $\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2 \wedge Y \cong \bigvee_{\alpha \in A} \Sigma^{p_\alpha, q_\alpha} (\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2)$ by hypothesis,

$$\begin{aligned} H_{**}^{\text{iso}}(Y) &= [S^{**}, \mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2 \wedge Y] \cong \left[S^{**}, \bigvee_{\alpha \in A} \Sigma^{p_\alpha, q_\alpha} (\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2) \right] \\ &\cong \bigoplus_{\alpha \in A} \Sigma^{p_\alpha, q_\alpha} H_{**}(k/k), \end{aligned}$$

from which it follows that

$$\text{Hom}_{H_{**}(k/k)}^{-p,-q}(H_{**}^{\text{iso}}(Y), H_{**}(k/k)) \cong \prod_{\alpha \in A} H_{p_\alpha - p, q_\alpha - q}(k/k).$$

On the other hand, we have the chain of isomorphisms

$$\begin{aligned} H_{\text{iso}}^{p,q}(Y) &= [Y, \Sigma^{p,q} (\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2)] \cong [\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2 \wedge Y, \Sigma^{p,q} (\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2)]_{\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2} \\ &\cong \left[\bigvee_{\alpha \in A} \Sigma^{p_\alpha, q_\alpha} (\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2), \Sigma^{p,q} (\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2) \right]_{\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2} \\ &\cong \left[\bigvee_{\alpha \in A} S^{p_\alpha, q_\alpha}, \Sigma^{p,q} (\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2) \right] \cong \prod_{\alpha \in A} H_{p_\alpha - p, q_\alpha - q}(k/k). \quad \square \end{aligned}$$

We now define a certain concept of finiteness which suits the isotropic environment:

Definition 5.10 A set of bidegrees $\{(q_\alpha)[p_\alpha]\}_{\alpha \in A}$ is isotropically finite type if, for any bidegree $(q)[p]$, there are only finitely many $\alpha \in A$ such that $p - p_\alpha \geq 2(q - q_\alpha) \geq 0$. Moreover, we say that a set of bigraded elements $\{x_\alpha\}_{\alpha \in A}$ is isotropically finite type if the corresponding set of bidegrees is so.

Lemma 5.11 Let k be a flexible field and $\{(q_\alpha)[p_\alpha]\}_{\alpha \in A}$ an isotropically finite type set of bidegrees. Then for any bidegree $(q)[p]$, the obvious map

$$\pi_{p,q} \left(\mathfrak{X} \wedge \bigvee_{\alpha \in A} \Sigma^{p_\alpha, q_\alpha} \mathbb{H}\mathbb{Z}/2 \right) \rightarrow \text{Hom}_{\mathcal{A}^{**}(k/k)}^{p,q} \left(H_{\text{iso}}^{**} \left(\bigvee_{\alpha \in A} \Sigma^{p_\alpha, q_\alpha} \mathbb{H}\mathbb{Z}/2 \right), H^{**}(k/k) \right)$$

is an isomorphism.

Proof First note that, for any bidegree $(q)[p]$, one has the commutative diagram

$$\begin{array}{ccc} \pi_{p,q} \left(\mathfrak{X} \wedge \bigvee_{\alpha \in A} \Sigma^{p_\alpha, q_\alpha} \mathbb{H}\mathbb{Z}/2 \right) & & \\ \downarrow & \searrow & \\ \text{Hom}_{\mathbb{F}_2}^{p,q} \left(\bigoplus_{\alpha \in A} \Sigma^{-p_\alpha, -q_\alpha} \mathbb{F}_2, H^{**}(k/k) \right) & & \text{Hom}_{\mathcal{A}^{**}(k/k)}^{p,q} \left(H_{\text{iso}}^{**} \left(\bigvee_{\alpha \in A} \Sigma^{p_\alpha, q_\alpha} \mathbb{H}\mathbb{Z}/2 \right), H^{**}(k/k) \right) \\ & \searrow & \downarrow \\ & & \text{Hom}_{\mathcal{A}^{**}(k/k)}^{p,q} \left(\bigoplus_{\alpha \in A} \Sigma^{-p_\alpha, -q_\alpha} \mathcal{A}^{**}(k/k), H^{**}(k/k) \right) \end{array}$$

The left vertical arrow is the isomorphism described by the chain of equivalences

$$\begin{aligned} \pi_{p,q} \left(\mathfrak{X} \wedge \bigvee_{\alpha \in A} \Sigma^{p_\alpha, q_\alpha} \mathbb{H}\mathbb{Z}/2 \right) &\cong \bigoplus_{\alpha \in A} \pi_{p,q} \left(\mathfrak{X} \wedge \Sigma^{p_\alpha, q_\alpha} \mathbb{H}\mathbb{Z}/2 \right) \\ &\cong \bigoplus_{\alpha \in A} H^{p_\alpha - p, q_\alpha - q}(k/k) \cong \prod_{\alpha \in A} H^{p_\alpha - p, q_\alpha - q}(k/k) \\ &\cong \prod_{\alpha \in A} \text{Hom}_{\mathbb{F}_2}^{p,q} \left(\Sigma^{-p_\alpha, -q_\alpha} \mathbb{F}_2, H^{**}(k/k) \right) \\ &\cong \text{Hom}_{\mathbb{F}_2}^{p,q} \left(\bigoplus_{\alpha \in A} \Sigma^{-p_\alpha, -q_\alpha} \mathbb{F}_2, H^{**}(k/k) \right), \end{aligned}$$

where the identification

$$\bigoplus_{\alpha \in A} H^{p_\alpha - p, q_\alpha - q}(k/k) \cong \prod_{\alpha \in A} H^{p_\alpha - p, q_\alpha - q}(k/k)$$

is due to the fact that the set $\{(q_\alpha)[p_\alpha]\}_{\alpha \in A}$ is isotropically finite type, so for any bidegree $(q)[p]$ the group $H^{p_\alpha-p, q_\alpha-q}(k/k)$ is nonzero only for a finite number of $\alpha \in A$ by [Theorem 3.9](#). The bottom horizontal map is obviously an isomorphism since $\mathcal{A}^{**}(k/k)$ is an \mathbb{F}_2 -vector space. The right vertical map is an isomorphism since

$$\begin{aligned} \text{Hom}_{\mathcal{A}^{**}(k/k)}^{p,q} \left(H_{\text{iso}}^{**} \left(\bigvee_{\alpha} \Sigma^{p_\alpha, q_\alpha} \mathbb{H}\mathbb{Z}/2 \right), H^{**}(k/k) \right) \\ \cong \text{Hom}_{\mathcal{A}^{**}(k/k)}^{p,q} \left(\prod_{\alpha} H_{\text{iso}}^{**}(\Sigma^{p_\alpha, q_\alpha} \mathbb{H}\mathbb{Z}/2), H^{**}(k/k) \right) \\ = \text{Hom}_{\mathcal{A}^{**}(k/k)}^{p,q} \left(\prod_{\alpha} \Sigma^{-p_\alpha, -q_\alpha} \mathcal{A}^{**}(k/k), H^{**}(k/k) \right) \\ \cong \text{Hom}_{\mathcal{A}^{**}(k/k)}^{p,q} \left(\bigoplus_{\alpha} \Sigma^{-p_\alpha, -q_\alpha} \mathcal{A}^{**}(k/k), H^{**}(k/k) \right), \end{aligned}$$

where the last isomorphism comes from the fact that the set of bidegrees $\{(q_\alpha)[p_\alpha]\}_{\alpha \in A}$ is isotropically finite type, so for any bidegree $(q)[p]$ the group

$$\text{Hom}_{\mathcal{A}^{**}(k/k)}^{p,q}(\Sigma^{-p_\alpha, -q_\alpha} \mathcal{A}^{**}(k/k), H^{**}(k/k)) \cong H^{p_\alpha-p, q_\alpha-q}(k/k)$$

is nontrivial only for finitely many $\alpha \in A$ by [Theorem 3.9](#). □

At this point, we are ready to present the structure of the E_2 -page of the isotropic Adams spectral sequence, which behaves as in the classical case.

Theorem 5.12 *Let k be a flexible field and Y an object in $\mathfrak{X}\text{-Mod}_{\text{cell}}$ such that $H_{**}^{\text{iso}}(Y)$ is a free left $H_{**}(k/k)$ -module generated by an isotropically finite type set of elements $\{x_\alpha\}_{\alpha \in A}$. Then the E_2 -page of the isotropic motivic Adams spectral sequence is described by*

$$E_2^{s,t,u} \cong \text{Ext}_{\mathcal{A}^{**}(k/k)}^{s,t,u}(H_{\text{iso}}^{**}(Y), H^{**}(k/k)).$$

Proof First, we want to prove by induction that $H_{**}^{\text{iso}}(\overline{(\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2)^{\wedge s}} \wedge Y)$ is a free left $H_{**}(k/k)$ -module generated by an isotropically finite type set of elements $\{x_\alpha\}_{\alpha \in A_s}$ for any $s \geq 0$. The induction basis is guaranteed by hypothesis after setting $A_0 = A$. Suppose the statement is true at the $s - 1$ stage, ie

$$H_{**}^{\text{iso}}(\overline{(\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2)^{\wedge s-1}} \wedge Y) \cong \bigoplus_{\alpha \in A_{s-1}} \Sigma^{1-s,0} H_{**}(k/k) \cdot x_\alpha.$$

Then by Lemma 5.6, the map $(\overline{\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2})^{\wedge s-1} \wedge Y \rightarrow \mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2 \wedge (\overline{\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2})^{\wedge s-1} \wedge Y$ induces in isotropic motivic homology the monomorphism

$$\bigoplus_{\alpha \in A_{s-1}} \Sigma^{1-s,0} H_{**}(k/k) \cdot x_\alpha \rightarrow \bigoplus_{\alpha \in A_{s-1}} \Sigma^{1-s,0} A_{**}(k/k) \cdot x_\alpha.$$

Hence, the standard Adams resolution induces, for any p and q , a short exact sequence

$$0 \rightarrow H_{p,q}^{\text{iso}}((\overline{\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2})^{\wedge s-1} \wedge Y) \rightarrow H_{p,q}^{\text{iso}}(\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2 \wedge (\overline{\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2})^{\wedge s-1} \wedge Y) \rightarrow H_{p-1,q}^{\text{iso}}((\overline{\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2})^{\wedge s} \wedge Y) \rightarrow 0.$$

Now note that, by the very structure of the dual of the isotropic motivic Steenrod algebra, $A_{**}(k/k)$ is freely generated over $H_{**}(k/k)$ by a set of generators $\{1, y_\beta\}_{\beta \in B}$ which is finite in each bidegree and such that $p_\beta \geq 2q_\beta \geq 0$ for any $\beta \in B$, where $(q_\beta)[p_\beta]$ is the bidegree of y_β . Hence, the set $\{y_\beta x_\alpha\}_{\beta \in B, \alpha \in A_{s-1}}$ is isotropically finite type and freely generates $H_{**}^{\text{iso}}((\overline{\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2})^{\wedge s} \wedge Y)$ over $H_{**}(k/k)$:

$$H_{**}^{\text{iso}}((\overline{\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2})^{\wedge s} \wedge Y) \cong \bigoplus_{\beta \in B, \alpha \in A_{s-1}} \Sigma^{-s,0} H_{**}(k/k) \cdot y_\beta x_\alpha.$$

Therefore, Lemma 5.8 implies that all $\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2 \wedge (\overline{\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2})^{\wedge s} \wedge Y$ are wedges of appropriately shifted $\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2$. More precisely, for any $s \geq 0$, there exists an isomorphism

$$\mathfrak{X} \wedge \bigvee_{\alpha \in A_s} \Sigma^{p_\alpha - s, q_\alpha} \mathbb{H}\mathbb{Z}/2 \cong \mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2 \wedge (\overline{\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2})^{\wedge s} \wedge Y,$$

where $A_s = B \times A_{s-1}$, from which we deduce, using Lemma 5.11, that the E_1 -page of the isotropic Adams spectral sequence can be described by

$$\begin{aligned} E_1^{s,t,u} &\cong \pi_{t-s,u}(\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2 \wedge (\overline{\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2})^{\wedge s} \wedge Y) \\ &\cong \text{Hom}_{A_{**}(k/k)}^{t,u} \left(\bigoplus_{\alpha \in A_s} \Sigma^{-p_\alpha, -q_\alpha} A_{**}(k/k), H_{**}(k/k) \right). \end{aligned}$$

Moreover, note that

$$0 \leftarrow H_{\text{iso}}^{**}(Y) \leftarrow \bigoplus_{\alpha \in A_0} \Sigma^{-p_\alpha, -q_\alpha} A_{**}(k/k) \leftarrow \bigoplus_{\alpha \in A_1} \Sigma^{-p_\alpha, -q_\alpha} A_{**}(k/k) \leftarrow \dots$$

is a free $A_{**}(k/k)$ -resolution of $H_{\text{iso}}^{**}(Y)$. Thus, for any s, t and u , we have an isomorphism

$$E_2^{s,t,u} \cong \text{Ext}_{A_{**}(k/k)}^{s,t,u}(H_{\text{iso}}^{**}(Y), H_{**}(k/k)). \quad \square$$

By using the isotropic motivic Adams spectral sequence, in [19] we computed the isotropic motivic homotopy groups of the sphere spectrum, which can be identified with the E_2 -page of the classical Adams spectral sequence.

Theorem 5.13 [19, Theorem 5.7] *Let k be a flexible field. Then the stable motivic homotopy groups of the $H\mathbb{Z}/2$ -completed isotropic sphere spectrum are completely described by*

$$\pi_{*,*'}(\mathfrak{X}_{H\mathbb{Z}/2}^\wedge) \cong \text{Ext}_{\mathcal{G}^{**}}^{2*'-*, 2*', *'}(\mathbb{F}_2, \mathbb{F}_2) \cong \text{Ext}_{\mathcal{A}^*}^{2*'-*, *'}(\mathbb{F}_2, \mathbb{F}_2).$$

6 The motivic Brown–Peterson spectrum

In this section, we recall from [20] the construction of the motivic Brown–Peterson spectrum. Moreover, we compute its isotropic homology and homotopy, which will be useful later on for the construction of the isotropic motivic Adams–Novikov spectral sequence, and so for the proofs of our main results.

Definition 6.1 Suppose $\text{MGL}_{(2)}$ is the motivic algebraic cobordism spectrum (see [22, Section 6.3]) localized at 2. Then following [20, Section 5] one defines the motivic Brown–Peterson spectrum at the prime 2 as the colimit of the diagram in $\mathcal{SH}(k)$

$$\cdots \rightarrow \text{MGL}_{(2)} \xrightarrow{e_{(2)}} \text{MGL}_{(2)} \xrightarrow{e_{(2)}} \text{MGL}_{(2)} \rightarrow \cdots,$$

where $e_{(2)}$ is the motivic Quillen idempotent.

Note, in particular, that MBP is a homotopy commutative ring spectrum and a direct summand of $\text{MGL}_{(2)}$.

Proposition 6.2 *Let k be a flexible field. Then there is an isomorphism of $H^{**}(k/k)$ -modules*

$$H_{\text{iso}}^{**}(\text{MGL}) \cong H_{\text{iso}}^{**}(\text{BGL}) \cong H^{**}(k/k)[c_1, c_2, \dots]$$

and an isomorphism of $H_{**}(k/k)$ -algebras

$$H_{**}^{\text{iso}}(\text{MGL}) \cong H_{**}^{\text{iso}}(\text{BGL}) \cong H_{**}(k/k)[b_1, b_2, \dots],$$

where c_i is the i^{th} Chern class in $H_{\text{iso}}^{2i,i}(\text{BGL})$ and $b_i \in H_{2i,i}^{\text{iso}}(\text{BGL})$ is the dual of c_1^i with respect to the monomial basis for any i .

Proof First, note that the maps $P^1 \rightarrow P^\infty$ and $\mathrm{HZ}/2 \rightarrow \mathfrak{X} \wedge \mathrm{HZ}/2$ induce a commutative square

$$\begin{CD} H^{**}(P^\infty) @>>> H_{\mathrm{iso}}^{**}(P^\infty) \\ @VVV @VVV \\ H^{**}(P^1) @>>> H_{\mathrm{iso}}^{**}(P^1) \end{CD}$$

where the left vertical morphism is the projection $H^{**}(k)[c] \rightarrow H^{**}(k)[c]/(c^2)$ and c is the only nonzero class in $H^{2,1}(P^\infty) \cong H^{2,1}(P^1) \cong \mathbb{Z}/2$. If we also denote by c the images of c under the horizontal maps in isotropic motivic cohomology, then the right vertical homomorphism is given by the projection

$$H^{**}(k/k)[c] \rightarrow H^{**}(k/k)[c]/(c^2).$$

Hence, $\mathfrak{X} \wedge \mathrm{HZ}/2$ is an oriented motivic spectrum (see [20, Definition 3.1]) and the statement follows immediately from [17, Proposition 6.2]. □

Following [8, Section 6], let $h: L \rightarrow \mathbb{F}_2[b_1, b_2, \dots]$ be the homomorphism from the Lazard ring L classifying the formal group law on $\mathbb{F}_2[b_1, b_2, \dots]$ which is isomorphic to the additive one via the exponential $\sum_{n \geq 0} b_n x^{n+1}$. Lazard’s theorem implies that $h(L)$ is a polynomial subring $\mathbb{F}_2[b'_n \mid n \neq 2^r - 1]$, where $b'_n \equiv b_n$ modulo decomposables. Denote by $\pi: \mathbb{F}_2[b_1, b_2, \dots] \rightarrow h(L)$ a retraction of the inclusion.

In the next proposition, we give a description of isotropic homology and cohomology of the algebraic cobordism spectrum MGL .

Proposition 6.3 *Let k be a flexible field. Then the coaction*

$$\Delta: H_{**}^{\mathrm{iso}}(\mathrm{MGL}) \rightarrow \mathcal{A}_{**}(k/k) \otimes_{H_{**}(k/k)} H_{**}^{\mathrm{iso}}(\mathrm{MGL})$$

factors through $H_{**}(k/k) \otimes_{\mathbb{F}_2} \mathcal{G}_{**} \otimes_{\mathbb{F}_2} \mathbb{F}_2[b_1, b_2, \dots]$ and the composition

$$\begin{aligned} H_{**}^{\mathrm{iso}}(\mathrm{MGL}) &\xrightarrow{\Delta} H_{**}(k/k) \otimes_{\mathbb{F}_2} \mathcal{G}_{**} \otimes_{\mathbb{F}_2} \mathbb{F}_2[b_1, b_2, \dots] \\ &\xrightarrow{\mathrm{id} \otimes \pi} H_{**}(k/k) \otimes_{\mathbb{F}_2} \mathcal{G}_{**} \otimes_{\mathbb{F}_2} h(L) \end{aligned}$$

is an isomorphism of left $\mathcal{A}_{**}(k/k)$ -comodule algebras. Dually, the map

$$H^{**}(k/k) \otimes_{\mathbb{F}_2} \mathcal{G}^{**} \otimes_{\mathbb{F}_2} h(L)^\vee \rightarrow H_{\mathrm{iso}}^{**}(\mathrm{MGL})$$

is an isomorphism of left $\mathcal{A}^{**}(k/k)$ -module coalgebras.

Proof From [8, Lemma 5.2], since $\mathbb{H}\mathbb{Z}/2 \wedge \text{MGL}$ is a split $\mathbb{H}\mathbb{Z}/2$ -module (see the remark after [8, Definition 5.4]), we deduce that

$$\begin{aligned} H_{**}^{\text{iso}}(\text{MGL}) &\cong \pi_{**}(\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2) \otimes_{\pi_{**}(\mathbb{H}\mathbb{Z}/2)} \pi_{**}(\mathbb{H}\mathbb{Z}/2 \wedge \text{MGL}) \\ &\cong H_{**}(k/k) \otimes_{H_{**}(k)} H_{**}(\text{MGL}) \end{aligned}$$

as an $H_{**}(k/k)$ -algebra. From [8, Theorem 6.5] we know that the coaction

$$\Delta: H_{**}(\text{MGL}) \rightarrow \mathcal{A}_{**}(k) \otimes_{H_{**}(k)} H_{**}(\text{MGL})$$

factors through $\mathcal{P}_{**} \otimes_{\mathbb{F}_2} \mathbb{F}_2[b_1, b_2, \dots]$ and the composition

$$H_{**}(\text{MGL}) \xrightarrow{\Delta} \mathcal{P}_{**} \otimes_{\mathbb{F}_2} \mathbb{F}_2[b_1, b_2, \dots] \xrightarrow{\text{id} \otimes \pi} \mathcal{P}_{**} \otimes_{\mathbb{F}_2} h(L)$$

is an isomorphism of left $\mathcal{A}_{**}(k)$ -comodule algebras, where \mathcal{P}_{**} is the subalgebra of $\mathcal{A}_{**}(k)$ defined by $H_{**}(k)[\xi_1, \xi_2, \dots]$. By tensoring the previous composition with $H_{**}(k/k)$ over $H_{**}(k)$ we get the desired isomorphism, which completes the first part. The second part follows easily, since $\mathcal{G}_{**} \otimes_{\mathbb{F}_2} h(L)$ is isotropically finite type, from Lemmas 5.8 and 5.9 by dualizing the homology isomorphism. \square

The next result provides us with the structure of isotropic homology and cohomology of the motivic Brown–Peterson spectrum MBP.

Proposition 6.4 *Let k be a flexible field. Then the isotropic motivic homology of MBP is described as a left $\mathcal{A}_{**}(k/k)$ -comodule by*

$$H_{**}^{\text{iso}}(\text{MBP}) \cong H_{**}(k/k) \otimes_{\mathbb{F}_2} \mathcal{G}_{**}.$$

*Dually, the isotropic motivic cohomology of MBP is described as a left $\mathcal{A}^{**}(k/k)$ -module by*

$$H_{\text{iso}}^{**}(\text{MBP}) \cong H^{**}(k/k) \otimes_{\mathbb{F}_2} \mathcal{G}^{**}.$$

Proof From [8, Remark 6.20], one knows that MBP is equivalent to $\text{MGL}_{(2)}/x$, where x is any maximal h -regular sequence (a sequence of homogeneous elements in L such that $h(x)$ is a regular sequence in $h(L)$ which generates the maximal ideal). Therefore, Theorem 6.11 of [8] implies that there exists an isomorphism of $\mathcal{A}_{**}(k)$ -comodules

$$H_{**}(\text{MBP}) \cong \mathcal{P}_{**}.$$

Since $\mathbb{H}\mathbb{Z}/2 \wedge \text{MBP}$ is a split $\mathbb{H}\mathbb{Z}/2$ -module, we deduce from [8, Lemma 5.2] that

$$\begin{aligned} H_{**}^{\text{iso}}(\text{MBP}) &\cong H_{**}(k/k) \otimes_{H_{**}(k)} H_{**}(\text{MBP}) \cong H_{**}(k/k) \otimes_{H_{**}(k)} \mathcal{P}_{**} \\ &\cong H_{**}(k/k) \otimes_{\mathbb{F}_2} \mathcal{G}_{**}, \end{aligned}$$

which proves the first part. The second part follows again from dualization, since \mathcal{G}_{**} is isotropically finite type, by Lemmas 5.8 and 5.9. \square

Later on, we will also need the isotropic homology and cohomology of $\text{MBP} \wedge \text{MBP}$:

Proposition 6.5 *Let k be a flexible field. Then the isotropic motivic homology of $\text{MBP} \wedge \text{MBP}$ is described as a left $\mathcal{A}_{**}(k/k)$ -comodule by*

$$H_{**}^{\text{iso}}(\text{MBP} \wedge \text{MBP}) \cong H_{**}(k/k) \otimes_{\mathbb{F}_2} \mathcal{G}_{**} \otimes_{\mathbb{F}_2} \mathcal{G}_{**}.$$

*Dually, the isotropic motivic cohomology of $\text{MBP} \wedge \text{MBP}$ is described as a left $\mathcal{A}^{**}(k/k)$ -module by*

$$H_{\text{iso}}^{**}(\text{MBP} \wedge \text{MBP}) \cong H^{**}(k/k) \otimes_{\mathbb{F}_2} \mathcal{G}^{**} \otimes_{\mathbb{F}_2} \mathcal{G}^{**}.$$

Proof Since $\text{HZ}/2 \wedge \text{MBP}$ is a split $\text{HZ}/2$ -module,

$$\begin{aligned} H_{**}^{\text{iso}}(\text{MBP} \wedge \text{MBP}) &\cong (H_{**}(k/k) \otimes_{\mathbb{F}_2} \mathcal{G}_{**}) \otimes_{H_{**}(k/k)} (H_{**}(k/k) \otimes_{\mathbb{F}_2} \mathcal{G}_{**}) \\ &\cong H_{**}(k/k) \otimes_{\mathbb{F}_2} \mathcal{G}_{**} \otimes_{\mathbb{F}_2} \mathcal{G}_{**} \end{aligned}$$

by [8, Lemma 5.2] and Proposition 6.4. The description of the isotropic cohomology follows again by dualizing the homology isomorphism. \square

Now, we compute the isotropic stable homotopy groups of MBP by using the isotropic Adams spectral sequence developed in the previous section.

Theorem 6.6 *Let k be a flexible field. Then the isotropic motivic homotopy groups of MBP are described by*

$$\pi_{**}^{\text{iso}}(\text{MBP}) \cong \mathbb{F}_2.$$

Proof Note that, by Proposition 6.4, $H_{**}^{\text{iso}}(\text{MBP})$ is freely generated over $H_{**}(k/k)$ by \mathcal{G}_{**} , which is isotropically finite type. Hence, Theorem 5.12 implies that the E_2 -page of the isotropic motivic Adams spectral sequence for $\mathfrak{X} \wedge \text{MBP}$ is given by

$$E_2^{s,t,u} \cong \text{Ext}_{\mathcal{A}^{**}(k/k)}^{s,t,u}(H_{\text{iso}}^{**}(\text{MBP}), H^{**}(k/k)).$$

Now, we deduce from Proposition 6.4 and [19, Theorem 5.4] that

$$\begin{aligned} \text{Ext}_{\mathcal{A}^{**}(k/k)}^{s,t,u}(H_{\text{iso}}^{**}(\text{MBP}), H^{**}(k/k)) &\cong \text{Ext}_{\mathcal{A}^{**}(k/k)}^{s,t,u}(H^{**}(k/k) \otimes_{\mathbb{F}_2} \mathcal{G}^{**}, H^{**}(k/k)) \\ &\cong \text{Ext}_{\mathcal{G}^{**}}^{s,t,u}(\mathcal{G}^{**}, \mathbb{F}_2) \cong \text{Ext}_{\mathbb{F}_2}^{s,t,u}(\mathbb{F}_2, \mathbb{F}_2) \\ &\cong \begin{cases} \mathbb{F}_2 & \text{if } s = t = u = 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore, the E_2 -page of the isotropic Adams spectral sequence for $\mathfrak{X} \wedge \text{MBP}$ is concentrated just in the tridegree $(0, 0, 0)$, from which it follows that all differentials

from the second on are trivial. Thus, the Mittag-Leffler condition is clearly satisfied, and so strong convergence holds by Proposition 5.3. Then it immediately follows from Remark 5.4 and the fact that MBP is η -complete that

$$\pi_{**}^{\text{iso}}(\text{MBP}) \cong \pi_{**}(\mathfrak{X} \wedge \text{MBP}) \cong \mathbb{F}_2. \quad \square$$

In the following sections it will be also useful to know the isotropic homotopy groups of $\text{MBP} \wedge \text{MBP}$, which we compute in the next result.

Theorem 6.7 *Let k be a flexible field. Then the isotropic motivic homotopy groups of $\text{MBP} \wedge \text{MBP}$ are described by*

$$\pi_{**}^{\text{iso}}(\text{MBP} \wedge \text{MBP}) \cong \mathcal{G}_{**}.$$

Proof The proof of this theorem goes along the lines of the previous one. Since $H_{**}^{\text{iso}}(\text{MBP} \wedge \text{MBP}) \cong H_{**}(k/k) \otimes_{\mathbb{F}_2} \mathcal{G}_{**} \otimes_{\mathbb{F}_2} \mathcal{G}_{**}$ by Proposition 6.5 and $\mathcal{G}_{**} \otimes_{\mathbb{F}_2} \mathcal{G}_{**}$ is isotropically finite type, by Theorem 5.12 the E_2 -page of the isotropic Adams spectral sequence for $\mathfrak{X} \wedge \text{MBP} \wedge \text{MBP}$ is provided by

$$E_2^{s,t,u} \cong \text{Ext}_{\mathcal{A}^{**}(k/k)}^{s,t,u}(H_{\text{iso}}^{**}(\text{MBP} \wedge \text{MBP}), H^{**}(k/k)).$$

Again, we note that by [19, Theorem 5.4],

$$\begin{aligned} \text{Ext}_{\mathcal{A}^{**}(k/k)}^{s,t,u}(H_{\text{iso}}^{**}(\text{MBP} \wedge \text{MBP}), H^{**}(k/k)) &\cong \text{Ext}_{\mathcal{A}^{**}(k/k)}^{s,t,u}(H^{**}(k/k) \otimes_{\mathbb{F}_2} \mathcal{G}^{**} \otimes_{\mathbb{F}_2} \mathcal{G}^{**}, H^{**}(k/k)) \\ &\cong \text{Ext}_{\mathcal{G}^{**}}^{s,t,u}(\mathcal{G}^{**} \otimes_{\mathbb{F}_2} \mathcal{G}^{**}, \mathbb{F}_2) \cong \text{Ext}_{\mathbb{F}_2}^{s,t,u}(\mathcal{G}^{**}, \mathbb{F}_2) \\ &\cong \begin{cases} \mathcal{G}_{t,u} & \text{if } s = 0, \\ 0 & \text{if } s \neq 0. \end{cases} \end{aligned}$$

In particular, since \mathcal{G}_{**} is concentrated on the slope 2 line, all differentials from the second on are trivial by degree reasons. Hence, the Mittag-Leffler condition is met, which implies that the spectral sequence is strongly convergent. From all this, it follows as above that

$$\pi_{**}^{\text{iso}}(\text{MBP} \wedge \text{MBP}) \cong \pi_{**}(\mathfrak{X} \wedge \text{MBP} \wedge \text{MBP}) \cong \mathcal{G}_{**}. \quad \square$$

7 The category of isotropic cellular MBP-modules

In this section we start by providing $\mathfrak{X} \wedge \text{MBP}$ with an E_∞ -ring structure. This allows us to talk about the stable ∞ -category of $\mathfrak{X} \wedge \text{MBP}$ -modules $\mathfrak{X} \wedge \text{MBP}\text{-Mod}$ and its

cellular part $\mathfrak{X} \wedge \text{MBP}\text{-Mod}_{\text{cell}}$. Our aim is to focus on the category of isotropic cellular MBP-modules, which is the same as that of cellular $\mathfrak{X} \wedge \text{MBP}$ -modules. In particular, we completely describe the category $\mathfrak{X} \wedge \text{MBP}\text{-Mod}_{\text{cell}}$ in algebraic terms. This section is structured along the lines of [6, Section 3]. Therefore, before each result we indicate the one from [6] it corresponds to. We hope this will clearly shed light on the deep parallelism between [6] and this work.

Proposition 7.1 *The homotopy commutative ring structure on $\mathfrak{X} \wedge \text{MBP}$ extends to an E_∞ -ring structure.*

Proof It follows from [13, Proposition 1.4.4.11] that there exists a t -structure on $\mathfrak{X}\text{-Mod}$ with nonnegative part generated by $\mathfrak{X}^{2n,n}$ for any $n \in \mathbb{Z}$. By [1, Theorem A.1], $\mathfrak{X} \wedge \text{MGL}$ belongs to the nonnegative part of this t -structure, and so $\mathfrak{X} \wedge \text{MBP}$ does also. On the other hand, one deduces from Theorem 6.6 and [1, Lemma 2.4] that $\mathfrak{X} \wedge \text{MBP}$ belongs to the nonpositive part too. Hence, $\mathfrak{X} \wedge \text{MBP}$ is a homotopy commutative ring spectrum in the heart of the abovementioned t -structure, which means that it is an E_∞ -ring spectrum.² □

Once we know that $\mathfrak{X} \wedge \text{MBP}$ is a motivic E_∞ -ring spectrum, we can consider the stable ∞ -category of $\mathfrak{X} \wedge \text{MBP}$ -modules and its homotopy category which is tensor triangulated. In particular, we focus on its cellular part.

Proposition 7.2 *Let k be a flexible field and Y an object in $\mathfrak{X} \wedge \text{MBP}\text{-Mod}_{\text{cell}}$ such that $\pi_{**}(Y)$ is isomorphic to the \mathbb{F}_2 -vector space $\bigoplus_{\alpha \in A} \Sigma^{p_\alpha, q_\alpha} \mathbb{F}_2$. Then there exists an isomorphism of spectra*

$$\bigvee_{\alpha \in A} \Sigma^{p_\alpha, q_\alpha} (\mathfrak{X} \wedge \text{MBP}) \xrightarrow{\cong} Y.$$

Proof We follow the lines of the proof of Lemma 5.8. Each generator of $\pi_{**}(Y)$ represents a map $\Sigma^{p_\alpha, q_\alpha} S \rightarrow Y$. For all $\alpha \in A$, this map corresponds bijectively to a map $\Sigma^{p_\alpha, q_\alpha} (\mathfrak{X} \wedge \text{MBP}) \rightarrow Y$ of $\mathfrak{X} \wedge \text{MBP}$ -cellular modules. Hence, we get a map

$$\bigvee_{\alpha \in A} \Sigma^{p_\alpha, q_\alpha} (\mathfrak{X} \wedge \text{MBP}) \rightarrow Y$$

of $\mathfrak{X} \wedge \text{MBP}$ -cellular modules that induces an isomorphism on homotopy groups since $\pi_{**}(\mathfrak{X} \wedge \text{MBP}) \cong \mathbb{F}_2$ by Theorem 6.6. Therefore, it follows from Proposition 4.2 that the above map is an isomorphism of spectra. □

²I am grateful to Tom Bachmann for this argument.

This result implies the following corollary, which corresponds to [6, Corollary 3.3]:

Corollary 7.3 *Let k be a flexible field and X and Y be objects in $\mathfrak{X} \wedge \text{MBP-Mod}_{\text{cell}}$. Then*

$$[X, Y]_{\mathfrak{X} \wedge \text{MBP}} \cong \text{Hom}_{\mathbb{F}_2}^{0,0}(\pi_{**}(X), \pi_{**}(Y)).$$

Proof It follows from Proposition 7.2 that

$$X \cong \bigvee_{\alpha \in A} \Sigma^{p_\alpha, q_\alpha} (\mathfrak{X} \wedge \text{MBP}) \quad \text{and} \quad Y \cong \bigvee_{\beta \in B} \Sigma^{p_\beta, q_\beta} (\mathfrak{X} \wedge \text{MBP})$$

for some sets A and B . Then

$$\begin{aligned} [X, Y]_{\mathfrak{X} \wedge \text{MBP}} &\cong \left[\bigvee_{\alpha \in A} \Sigma^{p_\alpha, q_\alpha} S, \bigvee_{\beta \in B} \Sigma^{p_\beta, q_\beta} (\mathfrak{X} \wedge \text{MBP}) \right] \\ &\cong \prod_{\alpha \in A} \bigoplus_{\beta \in B} \pi_{p_\alpha - p_\beta, q_\alpha - q_\beta} (\mathfrak{X} \wedge \text{MBP}) \cong \prod_{\alpha \in A} \bigoplus_{\beta \in B} \Sigma^{p_\alpha - p_\beta, q_\alpha - q_\beta} \mathbb{F}_2 \\ &\cong \text{Hom}_{\mathbb{F}_2}^{0,0} \left(\bigoplus_{\alpha \in A} \Sigma^{p_\alpha, q_\alpha} \mathbb{F}_2, \bigoplus_{\beta \in B} \Sigma^{p_\beta, q_\beta} \mathbb{F}_2 \right) \\ &\cong \text{Hom}_{\mathbb{F}_2}^{0,0} (\pi_{**}(X), \pi_{**}(Y)). \quad \square \end{aligned}$$

The next theorem, which corresponds to [6, Theorem 3.8], identifies $\mathfrak{X} \wedge \text{MBP-Mod}_{\text{cell}}$ with the category of bigraded \mathbb{F}_2 -vector spaces, which we denote by $\mathbb{F}_2\text{-Mod}_{**}$.

Theorem 7.4 *Let k be a flexible field. Then the functor*

$$\pi_{**} : \mathfrak{X} \wedge \text{MBP-Mod}_{\text{cell}} \xrightarrow{\cong} \mathbb{F}_2\text{-Mod}_{**}$$

is an equivalence of categories.

Proof This follows immediately from Proposition 7.2 and Corollary 7.3. □

Remark 7.5 The equivalence provided by Theorem 7.4 is actually an equivalence of triangulated categories, where $\mathbb{F}_2\text{-Mod}_{**}$ is structured as a triangulated category in the obvious way. More precisely, the translation functor is the suspension $\Sigma^{1,0}$ and distinguished triangles are of the form

$$V \xrightarrow{f} W \rightarrow \text{coker}(f) \oplus \Sigma^{1,0} \ker(f) \rightarrow \Sigma^{1,0} V,$$

where f is a morphism of bigraded \mathbb{F}_2 -vector spaces.

8 The category of isotropic cellular spectra

This section is devoted to the understanding of the structure of the category $\mathfrak{X}\text{-Mod}_{\text{cell}}$, that is, as we have already noticed, the category of cellular isotropic spectra $\mathcal{SH}(k/k)_{\text{cell}}$. We give a nice algebraic description of this category based on the dual of the topological Steenrod algebra. The results here are the isotropic versions of the ones in [6, Sections 4 and 5], therefore the proofs we provide are isotropic adaptations of the respective ones in [6].

In the next lemma, which corresponds to [6, Lemma 5.1], we compute the MBP-homology of isotropic MBP-cellular spectra.

Lemma 8.1 *Let k be a flexible field. Then for any $I \in \mathfrak{X} \wedge \text{MBP-Mod}_{\text{cell}}$ there is an isomorphism of left \mathcal{G}_{**} -comodules*

$$\text{MBP}_{**}(I) \cong \mathcal{G}_{**} \otimes_{\mathbb{F}_2} \pi_{**}(I).$$

Proof Since the motivic spectrum I is by hypothesis in $\mathfrak{X} \wedge \text{MBP-Mod}_{\text{cell}}$, we deduce from Theorem 7.4 that $I \cong \bigvee_{\alpha \in A} \Sigma^{p\alpha, q\alpha} (\mathfrak{X} \wedge \text{MBP})$ for some set A . Therefore, by Theorem 6.7,

$$\begin{aligned} \text{MBP}_{**}(I) &= \pi_{**}(\text{MBP} \wedge I) \cong \pi_{**} \left(\bigvee_{\alpha \in A} \Sigma^{p\alpha, q\alpha} (\mathfrak{X} \wedge \text{MBP} \wedge \text{MBP}) \right) \\ &\cong \bigoplus_{\alpha \in A} \Sigma^{p\alpha, q\alpha} \pi_{**}(\mathfrak{X} \wedge \text{MBP} \wedge \text{MBP}) \\ &\cong \bigoplus_{\alpha \in A} \Sigma^{p\alpha, q\alpha} \mathcal{G}_{**} \cong \mathcal{G}_{**} \otimes_{\mathbb{F}_2} V, \end{aligned}$$

where $V \cong \bigoplus_{\alpha \in A} \Sigma^{p\alpha, q\alpha} \mathbb{F}_2$. Now, note that by Theorem 6.6,

$$\pi_{**}(I) \cong \bigoplus_{\alpha \in A} \Sigma^{p\alpha, q\alpha} \pi_{**}(\mathfrak{X} \wedge \text{MBP}) \cong V.$$

It follows that

$$\text{MBP}_{**}(I) \cong \mathcal{G}_{**} \otimes_{\mathbb{F}_2} \pi_{**}(I). \quad \square$$

The following lemma, which corresponds to [6, Lemma 5.3], describes algebraically the hom sets from isotropic cellular spectra to isotropic MBP-cellular spectra.

Lemma 8.2 *Suppose that k is a flexible field. Then for any $X \in \mathfrak{X}\text{-Mod}_{\text{cell}}$ and $I \in \mathfrak{X} \wedge \text{MBP-Mod}_{\text{cell}}$ there is an isomorphism*

$$[X, I] \cong \text{Hom}_{\mathcal{G}_{**}}^{0,0}(\text{MBP}_{**}(X), \text{MBP}_{**}(I)).$$

Proof By [Theorem 7.4](#) and [Lemma 8.1](#), we have the sequence of isomorphisms

$$\begin{aligned} [X, I] &\cong [\mathfrak{X} \wedge \text{MBP} \wedge X, I]_{\mathfrak{X} \wedge \text{MBP}} \cong \text{Hom}_{\mathbb{F}_2}^{0,0}(\pi_{**}(\mathfrak{X} \wedge \text{MBP} \wedge X), \pi_{**}(I)) \\ &\cong \text{Hom}_{\mathcal{G}_{**}}^{0,0}(\pi_{**}(\mathfrak{X} \wedge \text{MBP} \wedge X), \mathcal{G}_{**} \otimes_{\mathbb{F}_2} \pi_{**}(I)) \\ &\cong \text{Hom}_{\mathcal{G}_{**}}^{0,0}(\text{MBP}_{**}(X), \text{MBP}_{**}(I)), \quad \square \end{aligned}$$

Before constructing the isotropic version of the Adams–Novikov spectral sequence we need:

Lemma 8.3 *Let k be a flexible field and Y an object in $\mathfrak{X}\text{-Mod}$. Then, for any $s \geq 0$, there exist isomorphisms*

$$\text{MBP}_{**}((\overline{\mathfrak{X} \wedge \text{MBP}})^{\wedge s} \wedge Y) \cong \Sigma^{-s,0} \overline{\mathcal{G}_{**}}^{\otimes s} \otimes_{\mathbb{F}_2} \text{MBP}_{**}(Y)$$

and

$$\text{MBP}_{**}(\mathfrak{X} \wedge \text{MBP} \wedge (\overline{\mathfrak{X} \wedge \text{MBP}})^{\wedge s} \wedge Y) \cong \Sigma^{-s,0} \mathcal{G}_{**} \otimes_{\mathbb{F}_2} \overline{\mathcal{G}_{**}}^{\otimes s} \otimes_{\mathbb{F}_2} \text{MBP}_{**}(Y).$$

Proof First note that, by arguments similar to the ones in [Lemma 5.6](#), we have an isomorphism

$$\text{MBP}_{**}(\mathfrak{X} \wedge \text{MBP} \wedge (\overline{\mathfrak{X} \wedge \text{MBP}})^{\wedge s} \wedge Y) \cong \mathcal{G}_{**} \otimes_{\mathbb{F}_2} \text{MBP}_{**}((\overline{\mathfrak{X} \wedge \text{MBP}})^{\wedge s} \wedge Y)$$

for any isotropic spectrum Y and any $s \geq 0$, so we only need to prove the first part of the statement. We achieve this by an induction argument, after noting that obviously the statement holds for $s = 0$.

Now, suppose the statement holds for $s - 1$, ie

$$\text{MBP}_{**}((\overline{\mathfrak{X} \wedge \text{MBP}})^{\wedge s-1} \wedge Y) \cong \Sigma^{1-s,0} \overline{\mathcal{G}_{**}}^{\otimes s-1} \otimes_{\mathbb{F}_2} \text{MBP}_{**}(Y)$$

and

$$\text{MBP}_{**}(\mathfrak{X} \wedge \text{MBP} \wedge (\overline{\mathfrak{X} \wedge \text{MBP}})^{\wedge s-1} \wedge Y) \cong \Sigma^{1-s,0} \mathcal{G}_{**} \otimes_{\mathbb{F}_2} \overline{\mathcal{G}_{**}}^{\otimes s-1} \otimes_{\mathbb{F}_2} \text{MBP}_{**}(Y).$$

Then the distinguished triangle in $\mathcal{SH}(k)$

$$\begin{aligned} (\overline{\mathfrak{X} \wedge \text{MBP}})^{\wedge s} \wedge Y &\rightarrow (\overline{\mathfrak{X} \wedge \text{MBP}})^{\wedge s-1} \wedge Y \rightarrow \mathfrak{X} \wedge \text{MBP} \wedge (\overline{\mathfrak{X} \wedge \text{MBP}})^{\wedge s-1} \wedge Y \\ &\rightarrow \Sigma^{1,0} (\overline{\mathfrak{X} \wedge \text{MBP}})^{\wedge s} \wedge Y \end{aligned}$$

induces in MBP–homology the short exact sequence

$$\begin{aligned} 0 \rightarrow \Sigma^{1-s,0} \overline{\mathcal{G}_{**}}^{\otimes s-1} \otimes_{\mathbb{F}_2} \text{MBP}_{**}(Y) &\rightarrow \Sigma^{1-s,0} \mathcal{G}_{**} \otimes_{\mathbb{F}_2} \overline{\mathcal{G}_{**}}^{\otimes s-1} \otimes_{\mathbb{F}_2} \text{MBP}_{**}(Y) \\ &\rightarrow \Sigma^{1,0} \text{MBP}_{**}((\overline{\mathfrak{X} \wedge \text{MBP}})^{\wedge s} \wedge Y) \rightarrow 0. \end{aligned}$$

It follows that

$$\text{MBP}_{**}((\overline{\mathfrak{X} \wedge \text{MBP}})^{\wedge^s} \wedge Y) \cong \Sigma^{-s,0} \overline{\mathcal{G}_{**}}^{\otimes s} \otimes_{\mathbb{F}_2} \text{MBP}_{**}(Y)$$

and

$$\text{MBP}_{**}(\mathfrak{X} \wedge \text{MBP} \wedge (\overline{\mathfrak{X} \wedge \text{MBP}})^{\wedge^s} \wedge Y) \cong \Sigma^{-s,0} \mathcal{G}_{**} \otimes_{\mathbb{F}_2} \overline{\mathcal{G}_{**}}^{\otimes s} \otimes_{\mathbb{F}_2} \text{MBP}_{**}(Y). \quad \square$$

We are now ready to construct the isotropic Adams–Novikov spectral sequence, which corresponds to [6, Theorem 5.6]. Before proceeding, we would like to fix some notation.

Definition 8.4 Let X be an isotropic spectrum. The Chow–Novikov degree of $\text{MBP}_{p,q}(X)$ is the integer $p - 2q$. We denote by $\mathfrak{X}\text{-Mod}_{\text{cell}}^b$ the category of bounded isotropic cellular spectra, that is, isotropic cellular spectra whose MBP–homology is nontrivial only for a finite number of Chow–Novikov degrees.

Theorem 8.5 Let k be a flexible field and X and Y objects in $\mathfrak{X}\text{-Mod}_{\text{cell}}^b$. Then there is a strongly convergent spectral sequence

$$E_2^{s,t,u} \cong \text{Ext}_{\mathcal{G}_{**}}^{s,t,u}(\text{MBP}_{**}(X), \text{MBP}_{**}(Y)) \Rightarrow [\Sigma^{t-s,u} X, Y \wedge_{\mathbb{H}\mathbb{Z}/2}].$$

Proof Consider the Postnikov system in $\mathfrak{X}\text{-Mod}_{\text{cell}}$

$$\begin{array}{ccccccc} \cdots & \longrightarrow & (\overline{\mathfrak{X} \wedge \text{MBP}})^{\wedge^s} \wedge Y & \longrightarrow & \cdots & \longrightarrow & \overline{\mathfrak{X} \wedge \text{MBP}} \wedge Y & \longrightarrow & Y \\ & \swarrow [1] & \downarrow & & \swarrow [1] & \downarrow & \swarrow [1] & \downarrow & \\ & & \mathfrak{X} \wedge \text{MBP} \wedge (\overline{\mathfrak{X} \wedge \text{MBP}})^{\wedge^s} \wedge Y & & & \mathfrak{X} \wedge \text{MBP} \wedge \overline{\mathfrak{X} \wedge \text{MBP}} \wedge Y & & \mathfrak{X} \wedge \text{MBP} \wedge Y & \end{array}$$

where $\overline{\mathfrak{X} \wedge \text{MBP}}$ is defined by the distinguished triangle in $\mathcal{SH}(k)$

$$\overline{\mathfrak{X} \wedge \text{MBP}} \rightarrow \mathbb{S} \rightarrow \mathfrak{X} \wedge \text{MBP} \rightarrow \Sigma^{1,0} \overline{\mathfrak{X} \wedge \text{MBP}}.$$

If we apply the functor $[\Sigma^{**} X, -]$ we get an unrolled exact couple

$$\begin{array}{ccccccc} \cdots & \longrightarrow & [\Sigma^{**} X, \overline{\mathfrak{X} \wedge \text{MBP}} \wedge Y] & \longrightarrow & \cdots & \longrightarrow & [\Sigma^{**} X, Y] \\ & \swarrow [1] & \downarrow & \swarrow [1] & \downarrow & \swarrow [1] & \downarrow \\ & & [\Sigma^{**} X, \mathfrak{X} \wedge \text{MBP} \wedge \overline{\mathfrak{X} \wedge \text{MBP}} \wedge Y] & & & [\Sigma^{**} X, \mathfrak{X} \wedge \text{MBP} \wedge Y] & \end{array}$$

that induces a spectral sequence with E_1 –page given by

$$E_1^{s,t,u} \cong [\Sigma^{t-s,u} X, \mathfrak{X} \wedge \text{MBP} \wedge (\overline{\mathfrak{X} \wedge \text{MBP}})^{\wedge^s} \wedge Y]$$

and first differential

$$d_1^{s,t,u} : E_1^{s,t,u} \rightarrow E_1^{s+1,t,u}.$$

This is what we call the isotropic Adams–Novikov spectral sequence. Note that by Lemmas 8.2 and 8.3 the E_1 -page has a nice description:

$$E_1^{s,t,u} \cong \text{Hom}_{\mathcal{G}_{**}}^{t,u}(\text{MBP}_{**}(X), \mathcal{G}_{**} \otimes_{\mathbb{F}_2} \overline{\mathcal{G}_{**}}^{\otimes s} \otimes_{\mathbb{F}_2} \text{MBP}_{**}(Y)).$$

Hence, the E_2 -page has the usual description given in terms of Ext-groups of left \mathcal{G}_{**} -comodules:

$$E_2^{s,t,u} \cong \text{Ext}_{\mathcal{G}_{**}}^{s,t,u}(\text{MBP}_{**}(X), \text{MBP}_{**}(Y)).$$

By standard formal reasons, this spectral sequence actually converges to the groups $[\Sigma^{t-s,u} X, Y_{\mathfrak{X} \wedge \text{MBP}}^\wedge]$. We only have to notice that

$$Y_{\mathfrak{X} \wedge \text{MBP}}^\wedge \cong Y_{\mathfrak{X} \wedge \text{HZ}/2}^\wedge \cong Y_{\text{HZ}/2}^\wedge.$$

The second isomorphism comes from the same argument as the proof of Proposition 5.3. Regarding the first isomorphism, we may consider, following [4, Section 7.3], the bicompletion $Y_{\{\mathfrak{X} \wedge \text{MBP}, \mathfrak{X} \wedge \text{HZ}/2\}}^\wedge$. This spectrum may be obtained by computing the homotopy limit of the cosimplicial spectrum

$$\begin{aligned} (\mathfrak{X} \wedge \text{HZ}/2 \wedge Y)_{\mathfrak{X} \wedge \text{MBP}}^\wedge &\rightrightarrows ((\mathfrak{X} \wedge \text{HZ}/2)^{\wedge 2} \wedge Y)_{\mathfrak{X} \wedge \text{MBP}}^\wedge \\ &\rightrightarrows ((\mathfrak{X} \wedge \text{HZ}/2)^{\wedge 3} \wedge Y)_{\mathfrak{X} \wedge \text{MBP}}^\wedge \rightrightarrows \cdots \end{aligned}$$

or, equivalently, by computing the homotopy limit of the cosimplicial spectrum

$$\begin{aligned} (\mathfrak{X} \wedge \text{MBP} \wedge Y)_{\mathfrak{X} \wedge \text{HZ}/2}^\wedge &\rightrightarrows ((\mathfrak{X} \wedge \text{MBP})^{\wedge 2} \wedge Y)_{\mathfrak{X} \wedge \text{HZ}/2}^\wedge \\ &\rightrightarrows ((\mathfrak{X} \wedge \text{MBP})^{\wedge 3} \wedge Y)_{\mathfrak{X} \wedge \text{HZ}/2}^\wedge \rightrightarrows \cdots \end{aligned}$$

Since $\text{HZ}/2$ is a motivic MBP -module, for any n ,

$$((\mathfrak{X} \wedge \text{HZ}/2)^{\wedge n} \wedge Y)_{\mathfrak{X} \wedge \text{MBP}}^\wedge \cong (\mathfrak{X} \wedge \text{HZ}/2)^{\wedge n} \wedge Y,$$

from which it follows that the first homotopy limit is just $Y_{\mathfrak{X} \wedge \text{HZ}/2}^\wedge$. On the other hand, we know that $\mathfrak{X} \wedge \text{MBP}$ is $\text{HZ}/2$ -complete; thus, for any n ,

$$((\mathfrak{X} \wedge \text{MBP})^{\wedge n} \wedge Y)_{\mathfrak{X} \wedge \text{HZ}/2}^\wedge \cong (\mathfrak{X} \wedge \text{MBP})^{\wedge n} \wedge Y,$$

and the second homotopy limit gives back $Y_{\mathfrak{X} \wedge \text{MBP}}^\wedge$. This implies $Y_{\mathfrak{X} \wedge \text{MBP}}^\wedge \cong Y_{\mathfrak{X} \wedge \text{HZ}/2}^\wedge$.

It only remains to prove the strong convergence. The arguments are the same as in [6, Theorem 3.2] and we report them here only for completeness. First, suppose that $\text{MBP}_{**}(X)$ is concentrated in Chow–Novikov degrees $[a, b]$ and $\text{MBP}_{**}(Y)$ is concentrated in Chow–Novikov degrees $[c, d]$. Then the E_1 -page, and so all the following pages, are trivial outside the range $c - b + 2u \leq t \leq d - a + 2u$. Now, note that the differential on the E_r -page has, as usual, the tridegree $(r, r - 1, 0)$, which

means in particular that it is trivial when $r - 1 > d - a - c + b$. This amounts to saying that the spectral sequence collapses at the $E_{d-a-c+b+2}$ -page, and so it is strongly convergent. □

Definition 8.6 Let $\mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}$ be the full triangulated subcategory of $\mathfrak{X}\text{-Mod}_{\text{cell}}$ consisting of $\mathbb{H}\mathbb{Z}/2$ -complete cellular isotropic spectra. Denote by $\mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^{b,\geq 0}$ the full subcategory of $\mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^b$ whose objects have MBP-homology concentrated in nonnegative Chow–Novikov degrees, and by $\mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^{b,\leq 0}$ the full subcategory of $\mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^b$ whose objects have MBP-homology concentrated in nonpositive Chow–Novikov degrees. Finally, let $\mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^\heartsuit$ be the full subcategory whose objects are in $\mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^{b,\geq 0}$ and $\mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^{b,\leq 0}$, i.e. the objects have MBP-homology concentrated in Chow–Novikov degree 0.

We want to point out that, since $\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2$ is a $\mathfrak{X} \wedge \text{MBP}$ -module and $\mathfrak{X} \wedge \text{MBP}$ is $\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2$ -complete, the subcategories of $\mathbb{H}\mathbb{Z}/2$ -complete and MBP-complete isotropic spectra coincide.

The next corollary, which corresponds to [6, Corollary 4.7], computes hom sets from $\mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^{b,\geq 0}$ to $\mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^{b,\leq 0}$ in algebraic terms.

Corollary 8.7 Let k be a flexible field, X an object in $\mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^{b,\geq 0}$ and Y in $\mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^{b,\leq 0}$. Then the functor MBP_{**} provides an isomorphism

$$[X, Y] \cong \text{Hom}_{\mathcal{G}_{**}}^{0,0}(\text{MBP}_{**}(X), \text{MBP}_{**}(Y)).$$

Proof As we have already pointed out, the E_1 -page of the isotropic Adams–Novikov spectral sequence is given by

$$E_1^{s,t,u} \cong \text{Hom}_{\mathcal{G}_{**}}^{t,u}(\text{MBP}_{**}(X), \mathcal{G}_{**} \otimes_{\mathbb{F}_2} \overline{\mathcal{G}_{**}}^{\otimes s} \otimes_{\mathbb{F}_2} \text{MBP}_{**}(Y)).$$

Since we are interested in the group $[X, Y]$, the part of the E_1 -page that is involved consists of the groups in tridegrees $(t, t, 0)$. By hypothesis, X is in $\mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^{b,\geq 0}$ while Y is in $\mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^{b,\leq 0}$, so, among these groups, only $E_1^{0,0,0}$ is nontrivial. Since in this tridegree all differentials from the second on are trivial by degree reasons,

$$\begin{aligned} [X, Y] &\cong E_2^{0,0,0} \cong \text{Ext}_{\mathcal{G}_{**}}^{0,0,0}(\text{MBP}_{**}(X), \text{MBP}_{**}(Y)) \\ &\cong \text{Hom}_{\mathcal{G}_{**}}^{0,0}(\text{MBP}_{**}(X), \text{MBP}_{**}(Y)), \end{aligned} \quad \square$$

By using the isotropic Adams–Novikov spectral sequence we also get a corollary, which corresponds to [6, Corollary 4.8] and is a generalization of [19, Theorem 5.7]:

Corollary 8.8 *Let k be a flexible field and X and Y objects in $\mathfrak{X}\text{-Mod}_{\text{cell}, \mathbb{H}\mathbb{Z}/2}^\heartsuit$. Then there is an isomorphism*

$$[\Sigma^{t,u} X, Y] \cong \text{Ext}_{\mathcal{G}_{**}}^{2u-t, 2u,u}(\text{MBP}_{**}(X), \text{MBP}_{**}(Y)).$$

Proof This follows because the differentials $d_r^{s,t,u} : E_r^{s,t,u} \rightarrow E_r^{s+r,t+r-1,u}$ of the isotropic Adams–Novikov spectral sequence are trivial for $r \geq 2$ since $E_2^{s,t,u}$ is trivial for $t \neq 2u$. Hence, the spectral sequence is strongly convergent and collapses at the second page, from which we get that

$$[\Sigma^{t,u} X, Y] \cong E_2^{2u-t, 2u,u} \cong \text{Ext}_{\mathcal{G}_{**}}^{2u-t, 2u,u}(\text{MBP}_{**}(X), \text{MBP}_{**}(Y)). \quad \square$$

Before proceeding, we also need the following lemma which essentially corresponds to [6, Lemma 4.10].

Lemma 8.9 *Let k be a flexible field and M a \mathcal{G}_{**} -comodule concentrated in Chow–Novikov degree 0 which is finitely generated as an \mathbb{F}_2 -vector space. Then there exists an object X in $\mathfrak{X}\text{-Mod}_{\text{cell}, \mathbb{H}\mathbb{Z}/2}^\heartsuit$ such that $M \cong \text{MBP}_{**}(X)$.*

Proof Since by hypothesis M is a finite-dimensional \mathbb{F}_2 -vector space, according to [10, Theorem 3.3] one has a finite filtration of subcomodules

$$0 \cong M_0 \subset M_1 \subset \dots \subset M_n \cong M$$

such that, for any i , M_i/M_{i-1} is stably isomorphic to \mathbb{F}_2 , ie $M_i/M_{i-1} \cong \Sigma^{2q_i, q_i} \mathbb{F}_2$ for some integer q_i . We want to prove the statement by induction on i . First, note that by Theorem 6.6 the comodule $\Sigma^{2q_i, q_i} \mathbb{F}_2$ is the MBP-homology of the isotropic spectrum $\Sigma^{2q_i, q_i} \mathfrak{X}_{\mathbb{H}\mathbb{Z}/2}^\wedge$ for any i . Now, suppose that there exists an object X_{i-1} in $\mathfrak{X}\text{-Mod}_{\text{cell}, \mathbb{H}\mathbb{Z}/2}^\heartsuit$ such that $M_{i-1} \cong \text{MBP}_{**}(X_{i-1})$. Then the short exact sequence

$$0 \rightarrow M_{i-1} \rightarrow M_i \rightarrow \Sigma^{2q_i, q_i} \mathbb{F}_2 \rightarrow 0$$

represents an element of $\text{Ext}_{\mathcal{G}_{**}}^{1,0,0}(\Sigma^{2q_i, q_i} \mathbb{F}_2, M_{i-1})$, namely, by Corollary 8.8, a morphism f_i in $[\Sigma^{2q_i-1, q_i} \mathfrak{X}_{\mathbb{H}\mathbb{Z}/2}^\wedge, X_{i-1}]$. Let us define X_i as $\text{Cone}(f_i)$. Then we have a long exact sequence in MBP-homology

$$\dots \rightarrow \Sigma^{2q_i-1, q_i} \mathbb{F}_2 \xrightarrow{0} M_{i-1} \rightarrow \text{MBP}_{**}(X_i) \rightarrow \Sigma^{2q_i, q_i} \mathbb{F}_2 \xrightarrow{0} \Sigma^{1,0} M_{i-1} \rightarrow \dots$$

Note that the connecting homomorphism

$$g_{i*} : \text{Ext}^{0,0,0}(\Sigma^{2q_i, q_i} \mathbb{F}_2, \Sigma^{2q_i, q_i} \mathbb{F}_2) \rightarrow \text{Ext}^{1,0,0}(\Sigma^{2q_i, q_i} \mathbb{F}_2, M_{i-1}),$$

described as the Yoneda product with the element g_i of $\text{Ext}_{\mathcal{G}_{**}}^{1,0,0}(\Sigma^{2q_i,q_i}\mathbb{F}_2, M_{i-1})$ corresponding to the short exact sequence

$$0 \rightarrow M_{i-1} \rightarrow \text{MBP}_{**}(X_i) \rightarrow \Sigma^{2q_i,q_i}\mathbb{F}_2 \rightarrow 0,$$

converges to the map

$$f_{i*} : [\Sigma^{2q_i-1,q_i} \mathfrak{X}_{\mathbb{H}\mathbb{Z}/2}^\wedge, \Sigma^{2q_i-1,q_i} \mathfrak{X}_{\mathbb{H}\mathbb{Z}/2}^\wedge] \rightarrow [\Sigma^{2q_i-1,q_i} \mathfrak{X}_{\mathbb{H}\mathbb{Z}/2}^\wedge, X_{i-1}]$$

induced by f_i in isotropic homotopy groups; see [18, Theorem 2.3.4]. By Corollary 8.8 the isotropic Adams–Novikov spectral sequence collapses at the second page, so $g_{i*} = f_{i*}$. It follows that the extensions g_i and f_i coincide, which implies that $\text{MBP}_{**}(X_i) \cong M_i$. □

The next result is the isotropic equivalent of [6, Lemma 4.2].

Lemma 8.10 *Let k be a flexible field and X_α be a filtered system in $\mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^\heartsuit}$. Then the colimit $\text{colim } X_\alpha$ in $\mathfrak{X}\text{-Mod}_{\text{cell}}$ also belongs to $\mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^\heartsuit}$.*

Proof First note that, since $\text{MBP}_{**}(\text{colim } X_\alpha) \cong \text{colim } \text{MBP}_{**}(X_\alpha)$, $\text{colim } X_\alpha$ has MBP–homology concentrated in Chow–Novikov degree 0. Moreover, recall from [18, Corollary A1.2.12] that $\text{Ext}_{\mathcal{G}_{**}}(\mathbb{F}_2, -)$ may be computed as the homology of the cobar complex for the second variable. Since the cobar complex preserves filtered colimits, so does $\text{Ext}_{\mathcal{G}_{**}}(\mathbb{F}_2, -)$. Then Corollary 8.8 implies that

$$\begin{aligned} \pi_{t,u}(\text{colim } X_\alpha) &\cong \text{colim } \pi_{t,u}(X_\alpha) \cong \text{colim } \text{Ext}_{\mathcal{G}_{**}}^{2u-t,2u,u}(\mathbb{F}_2, \text{MBP}_{**}(X_\alpha)) \\ &\cong \text{Ext}_{\mathcal{G}_{**}}^{2u-t,2u,u}(\mathbb{F}_2, \text{colim } \text{MBP}_{**}(X_\alpha)) \\ &\cong \text{Ext}_{\mathcal{G}_{**}}^{2u-t,2u,u}(\mathbb{F}_2, \text{MBP}_{**}(\text{colim } X_\alpha)) \cong \pi_{t,u}((\text{colim } X_\alpha)_{\mathbb{H}\mathbb{Z}/2}^\wedge) \end{aligned}$$

from which it follows that $\text{colim } X_\alpha$ is $\mathbb{H}\mathbb{Z}/2$ –complete. □

We are now ready to identify $\mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^\heartsuit}$ with the abelian category of left \mathcal{G}_{**} –comodules concentrated in Chow–Novikov degree 0 that we denote by $\mathcal{G}_{**}\text{-Comod}_{**}^0$. The following proposition is an isotropic version of [6, Proposition 4.11]:

Proposition 8.11 *Let k be a flexible field. Then the functor*

$$\text{MBP}_{**} : \mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^\heartsuit} \xrightarrow{\cong} \mathcal{G}_{**}\text{-Comod}_{**}^0$$

is an equivalence of categories.

Proof First, note that [Corollary 8.7](#) guarantees that the functor MBP_{**} is fully faithful. We just need to show that it is essentially surjective. Recall from [[7](#), Propositions 1.4.10, 1.4.4 and 1.4.1] that any left \mathcal{G}_{**} -comodule M is a filtered colimit of comodules M_α which are finitely generated as \mathbb{F}_2 -vector spaces. By [Lemma 8.9](#) all M_α are expressible as $\text{MBP}_{**}(X_\alpha)$ for some X_α in $\mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^\heartsuit$. Therefore, $M \cong \text{MBP}_{**}(X)$, where $X = \text{colim } X_\alpha$. \square

Remark 8.12 $\mathcal{G}_{**}\text{-Comod}_{**}^0$ is equivalent to the category of left \mathcal{A}_* -comodules, where \mathcal{A}_* is the dual of the topological Steenrod algebra. Hence, the previous result can be rephrased by saying that $\mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^\heartsuit$ is equivalent to the abelian category of left \mathcal{A}_* -comodules.

The next proposition, corresponding to [[6](#), Proposition 4.12], provides $\mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^b$ with a t -structure.

Proposition 8.13 *Let k be a flexible field. Then $(\mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^{b,\geq 0}, \mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^{b,\leq 0})$ defines a bounded t -structure on $\mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^b$.*

Proof Just by the definition of $\mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^{b,\geq 0}$ and $\mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^{b,\leq 0}$ the first is closed under suspensions, the second under desuspensions and both under extensions. Clearly

$$\mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^b = \bigcup_{n \in \mathbb{Z}} \mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^{b,\geq n},$$

where $\mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^{b,\geq n}$ is the n^{th} suspension of $\mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^{b,\geq 0}$. Next, we consider objects X and Y in $\mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^{b,\geq 0}$ and $\mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^{b,\leq -1}$ (the first desuspension of $\mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^{b,\leq 0}$), respectively. Then by [Corollary 8.7](#)

$$[X, Y] \cong \text{Hom}_{\mathcal{G}_{**}}^{0,0}(\text{MBP}_{**}(X), \text{MBP}_{**}(Y)) \cong 0,$$

since $\text{MBP}_{**}(Y)$ is concentrated in negative Chow–Novikov degrees while $\text{MBP}_{**}(X)$ is concentrated in nonnegative Chow–Novikov degrees. Finally, let X be an object in $\mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^{b,\geq 0}$, then $\text{MBP}(X)$ is concentrated in nonnegative Chow–Novikov degrees. Consider the projection $\text{MBP}(X) \rightarrow \text{MBP}(X)_0$ that kills all the elements in positive Chow–Novikov degrees, and note that there exists an object X_0 in $\mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^\heartsuit$ such that $\text{MBP}(X_0) \cong \text{MBP}(X)_0$. Now, by [Corollary 8.7](#), this morphism comes from a map $f: X \rightarrow X_0$ such that $\Sigma^{-1,0} \text{Cone}(f)$ belongs to $\mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^{b,\geq 1}$. Therefore, by [[6](#), Proposition 3.6], the pair $(\mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^{b,\geq 0}, \mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^{b,\leq 0})$ defines a bounded t -structure on $\mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^b$. \square

We are now ready to prove the main result of this section, which corresponds to [6, Theorem 4.13]. In this theorem we identify $\mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^b$ with the derived category of left \mathcal{G}_{**} -comodules concentrated in Chow–Novikov degree 0.

Theorem 8.14 *Let k be a flexible field. Then there exists a t -exact equivalence of stable ∞ -categories*

$$\mathcal{D}^b(\mathcal{G}_{**}\text{-Comod}_{**}^0) \cong \mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^b.$$

Proof First, by Propositions 8.11 and 8.13, $(\mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^{b,\geq 0}, \mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^{b,\leq 0})$ defines a bounded t -structure on $\mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^b$ whose heart is equivalent to the category of left \mathcal{G}_{**} -comodules concentrated in Chow–Novikov degree 0, so has enough injectives. Now, let X and Y be objects in $\mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^\heartsuit$ such that $\text{MBP}_{**}(Y)$ is an injective \mathcal{G}_{**} -comodule. In this case the isotropic Adams–Novikov spectral sequence

$$E_2^{s,t,u} \cong \text{Ext}_{\mathcal{G}_{**}}^{s,t,u}(\text{MBP}_{**}(X), \text{MBP}_{**}(Y)) \Rightarrow [\Sigma^{t-s,u} X, Y]$$

collapses at the second page since the E_2 -page is trivial for $s \neq 0$. Hence,

$$\begin{aligned} [\Sigma^{-i} X, Y] &\cong \text{Ext}_{\mathcal{G}_{**}}^{0,-i,0}(\text{MBP}_{**}(X), \text{MBP}_{**}(Y)) \\ &\cong \text{Hom}_{\mathcal{G}_{**}}^{-i,0}(\text{MBP}_{**}(X), \text{MBP}_{**}(Y)) \cong 0 \end{aligned}$$

for any $i > 0$ since both $\text{MBP}_{**}(X)$ and $\text{MBP}_{**}(Y)$ are concentrated in Chow–Novikov degree 0. It follows by [6, Proposition 2.12], which is based on Lurie’s recognition criterion [13, Proposition 1.3.3.7], that there exists a t -exact equivalence of stable ∞ -categories

$$\mathcal{D}^b(\mathcal{G}_{**}\text{-Comod}_{**}^0) \cong \mathfrak{X}\text{-Mod}_{\text{cell},\mathbb{H}\mathbb{Z}/2}^b$$

extending the equivalence on the hearts. □

Remark 8.15 Given the identification $\mathcal{G}_{**} \cong \mathcal{A}_*$, Theorem 8.14 identifies as triangulated categories the category of bounded isotropic $\mathbb{H}\mathbb{Z}/2$ -complete cellular spectra with the derived category of left \mathcal{A}_* -comodules, namely $\mathcal{D}^b(\mathcal{A}_*\text{-Comod}_*)$.

By using the same argument as in [6, Corollary 1.2] one is able to obtain an unbounded version of the previous theorem, identifying the whole $\mathfrak{X}_{\mathbb{H}\mathbb{Z}/2}^\wedge\text{-Mod}_{\text{cell}}$ with Hovey’s unbounded derived category $\text{Stable}(\mathcal{G}_{**}\text{-Comod}_{**}^0)$, which is the same as $\text{Stable}(\mathcal{A}_*\text{-Comod}_*)$; see [7, Section 6].

Corollary 8.16 *Let k be a flexible field. Then there exists an equivalence of stable ∞ -categories*

$$\mathfrak{X}_{\mathbb{H}\mathbb{Z}/2}^\wedge\text{-Mod}_{\text{cell}} \cong \text{Stable}(\mathcal{G}_{**}\text{-Comod}_{**}^0).$$

9 The category of isotropic Tate motives

We finish in this section by applying previous results in order to obtain information on the category of isotropic Tate motives $\mathcal{DM}(k/k)_{\text{Tate}}$. In particular, we get an easy algebraic description for the hom sets in $\mathcal{DM}(k/k)_{\text{Tate}}$ between motives of isotropic cellular spectra.

First, we prove the following lemma, which tells us that the isotropic motivic homology of an isotropic spectrum is always a free $H_{**}(k/k)$ -module.

Lemma 9.1 *Let k be a flexible field and X an object in $\mathfrak{X}\text{-Mod}$. Then there exists an isomorphism of left $H_{**}(k/k)$ -modules*

$$H_{**}^{\text{iso}}(X) \cong H_{**}(k/k) \otimes_{\mathbb{F}_2} \text{MBP}_{**}(X).$$

Proof The Hopkins–Morel equivalence (see [8, Theorem 7.12]) implies in particular that $\text{HZ}/2$ is a quotient spectrum of MBP. It follows that $\text{HZ}/2$ can be obtained from MBP by applying cones and homotopy colimits, and so it is an MBP-cellular module, from which we get by Theorem 7.4 that

$$\mathfrak{X} \wedge \text{HZ}/2 \cong \bigvee_{\alpha \in A} \Sigma^{p_\alpha, q_\alpha} (\mathfrak{X} \wedge \text{MBP})$$

for some set A . Now note that, by Theorem 6.6,

$$\begin{aligned} H_{**}(k/k) &\cong \pi_{**}(\mathfrak{X} \wedge \text{HZ}/2) \cong \pi_{**}\left(\bigvee_{\alpha \in A} \Sigma^{p_\alpha, q_\alpha} (\mathfrak{X} \wedge \text{MBP})\right) \\ &\cong \bigoplus_{\alpha \in A} \Sigma^{p_\alpha, q_\alpha} \pi_{**}(\mathfrak{X} \wedge \text{MBP}) \cong \bigoplus_{\alpha \in A} \Sigma^{p_\alpha, q_\alpha} \mathbb{F}_2. \end{aligned}$$

At this point, let X be an object in $\mathfrak{X}\text{-Mod}$. Then

$$\begin{aligned} H_{**}^{\text{iso}}(X) &\cong \pi_{**}(\mathfrak{X} \wedge \text{HZ}/2 \wedge X) \cong \pi_{**}\left(\bigvee_{\alpha \in A} \Sigma^{p_\alpha, q_\alpha} (\mathfrak{X} \wedge \text{MBP} \wedge X)\right) \\ &\cong \bigoplus_{\alpha \in A} \Sigma^{p_\alpha, q_\alpha} \pi_{**}(\mathfrak{X} \wedge \text{MBP} \wedge X) \cong \bigoplus_{\alpha \in A} \Sigma^{p_\alpha, q_\alpha} \text{MBP}_{**}(X) \\ &\cong H_{**}(k/k) \otimes_{\mathbb{F}_2} \text{MBP}_{**}(X). \end{aligned} \quad \square$$

In the next proposition we compute hom sets in the isotropic triangulated category of motives between motives of isotropic cellular spectra. They happen to be isomorphic to hom sets of left $H_{**}(k/k)$ -modules between the respective isotropic homology.

Proposition 9.2 *Let k be a flexible field and X and Y objects in $\mathfrak{X}\text{-Mod}_{\text{cell}}$. Then there exists an isomorphism*

$$\text{Hom}_{\mathcal{DM}(k/k)_{\text{Tate}}}(M(X), M(Y)) \cong \text{Hom}_{H_{**}(k/k)}(H_{**}^{\text{iso}}(X), H_{**}^{\text{iso}}(Y)).$$

Proof Consider the functor

$$H_{**}^{\text{iso}} : \mathcal{DM}(k/k)_{\text{Tate}} \rightarrow H_{**}(k/k)\text{-Mod}_{**}$$

which sends each isotropic Tate motive to the respective isotropic motivic homology, and let X and Y be motivic spectra in $\mathfrak{X}\text{-Mod}_{\text{cell}}$. Then, by [Theorem 7.4](#), [Lemma 9.1](#) and [[19](#), Proposition 2.4],

$$\begin{aligned} \text{Hom}_{\mathcal{DM}(k/k)_{\text{Tate}}}(M(X), M(Y)) &\cong [X, \mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2 \wedge Y] \cong [\mathfrak{X} \wedge \text{MBP} \wedge X, \mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2 \wedge Y]_{\mathfrak{X} \wedge \text{MBP}} \\ &\cong \text{Hom}_{\mathbb{F}_2}(\pi_{**}(\mathfrak{X} \wedge \text{MBP} \wedge X), \pi_{**}(\mathfrak{X} \wedge \mathbb{H}\mathbb{Z}/2 \wedge Y)) \\ &\cong \text{Hom}_{\mathbb{F}_2}(\text{MBP}_{**}(X), H_{**}^{\text{iso}}(Y)) \\ &\cong \text{Hom}_{H_{**}(k/k)}(H_{**}(k/k) \otimes_{\mathbb{F}_2} \text{MBP}_{**}(X), H_{**}^{\text{iso}}(Y)) \\ &\cong \text{Hom}_{H_{**}(k/k)}(H_{**}^{\text{iso}}(X), H_{**}^{\text{iso}}(Y)). \quad \square \end{aligned}$$

Remark 9.3 The last result suggests that isotropic Tate motives that come from $\mathcal{SH}(k/k)_{\text{cell}}$ are very special in the sense that hom sets in $\mathcal{DM}(k/k)_{\text{Tate}}$ between them are described simply in terms of hom sets of free $H_{**}(k/k)$ -modules. This property does not hold in general, so the next task should be to understand hom sets in $\mathcal{DM}(k/k)_{\text{Tate}}$ between general isotropic Tate motives and try to describe them in algebraic terms. Unfortunately, since $H_{**}(k/k)$ is not concentrated in Chow–Novikov degree 0, the strategy used in [[6](#)] and adapted in Sections [7](#) and [8](#) does not immediately apply. Hence, some new ideas are needed and the hope is to develop them in future work.

List of symbols

k	flexible field with $\text{char}(k) \neq 2$
$\mathcal{SH}(k)$	stable motivic homotopy category over k
$\mathcal{SH}(k/k)$	isotropic stable motivic homotopy category over k
$\mathcal{DM}(k)$	triangulated category of motives with $\mathbb{Z}/2$ -coefficients over k
$\mathcal{DM}(k/k)$	isotropic triangulated category of motives with $\mathbb{Z}/2$ -coefficients over k
$\pi_{**}(\cdot)$	stable motivic homotopy groups

$\pi_{**}^{\text{iso}}(\cdot)$	isotropic stable motivic homotopy groups
$H_{**}(\cdot), H^{**}(\cdot)$	motivic homology and cohomology with $\mathbb{Z}/2$ -coefficients
$H_{**}^{\text{iso}}(\cdot), H_{\text{iso}}^{**}(\cdot)$	isotropic motivic homology and cohomology with $\mathbb{Z}/2$ -coefficients
$H_{**}(k), H^{**}(k)$	motivic homology and cohomology with $\mathbb{Z}/2$ -coefficients of $\text{Spec}(k)$
$H_{**}(k/k), H^{**}(k/k)$	isotropic motivic homology and cohomology with $\mathbb{Z}/2$ -coefficients of $\text{Spec}(k)$
$\mathcal{A}^{**}(k), \mathcal{A}_{**}(k)$	mod 2 motivic Steenrod algebra and its dual
$\mathcal{A}^{**}(k/k), \mathcal{A}_{**}(k/k)$	mod 2 isotropic motivic Steenrod algebra and its dual
$\mathcal{A}^*, \mathcal{A}_*$	mod 2 topological Steenrod algebra and its dual
$\mathcal{G}^{**}, \mathcal{G}_{**}$	bigraded mod 2 topological Steenrod algebra and its dual, ie $\mathcal{G}^{2q,q} = \mathcal{A}^q$ and $\mathcal{G}^{p,q} = 0$ for $p \neq 2q$, similar for the dual
\mathcal{M}^{**}	Milnor subalgebra $\Lambda_{\mathbb{F}_2}(Q_i)_{i \geq 0}$ of $\mathcal{A}^{**}(k/k)$ where the Q_i are the Milnor operations in bidegrees $(2^i - 1)[2^{i+1} - 1]$
S	motivic sphere spectrum
$H\mathbb{Z}/2$	motivic Eilenberg–Mac Lane spectrum with $\mathbb{Z}/2$ -coefficients
MGL	motivic algebraic cobordism spectrum
MBP	motivic Brown–Peterson spectrum at the prime 2
\mathfrak{X}	isotropic sphere spectrum

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