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Derived equivalences of hyperkähler varieties

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We show that the Looijenga–Lunts–Verbitsky Lie algebra acting on the cohomology of a hyperkähler variety is a derived invariant, and obtain from this a number of consequences for the action on cohomology of derived equivalences between hyperkähler varieties.

This includes a proof that derived equivalent hyperkähler varieties have isomorphic $\mathbb{Q}$–Hodge structures, the construction of a rational "Mukai lattice" functorial for derived equivalences, and the computation (up to index 2) of the image of the group of auto-equivalences on the cohomology of certain Hilbert squares of K3 surfaces.

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1 Introduction

1.1 Background

We briefly recall the background to our results. We refer to Huybrechts [24] for more details. For a smooth projective complex variety X, we denote by $\mathcal{D}X$ the bounded derived category of coherent sheaves on X. By a theorem of Orlov [37] any (exact, $\mathbb{C}$–linear) equivalence $\Phi\colon \mathcal{D}X_1 \xrightarrow{\sim} \mathcal{D}X_2$ comes from a Fourier–Mukai kernel $\mathcal{P} \in \mathcal{D}(X_1 \times X_2)$, and convolution with the Mukai vector $v(\mathcal{P}) \in \mathrm{H}(X_1 \times X_2, \mathbb{Q})$ defines an isomorphism

$$\Phi^{\mathrm{H}}\colon \mathrm{H}(X_1, \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}(X_2, \mathbb{Q})$$

between the total cohomology of X_1 and X_2. This isomorphism is not graded, and respects the Hodge structures only up to Tate twists. Nonetheless, Orlov has conjectured [38] that if X_1 and X_2 are derived equivalent, then for every i there exist (noncanonical) isomorphisms $\mathrm{H}^i(X_1, \mathbb{Q}) \cong \mathrm{H}^i(X_2, \mathbb{Q})$ of $\mathbb{Q}$–Hodge structures.

For every X we have a representation

$$\rho_X\colon \mathrm{Aut}(\mathcal{D}X) \to \mathrm{GL}(\mathrm{H}(X, \mathbb{Q})), \quad \Phi \mapsto \Phi^{\mathrm{H}}.$$

Its image is known for varieties with ample or antiample canonical class (in which case $\mathrm{Aut}(\mathcal{D}X)$ is small and well understood; see Bondal and Orlov [9]), for abelian varieties — see Golyshev, Lunts and Orlov [18] — and for K3 surfaces. To place our results in context, we recall the description of the image for K3 surfaces.

Let X be a K3 surface. Consider the *Mukai lattice*

$$\widetilde{\mathrm{H}}(X,\mathbb{Z}) := \mathrm{H}^0(X,\mathbb{Z}) \oplus \mathrm{H}^2(X,\mathbb{Z}(1)) \oplus \mathrm{H}^4(X,\mathbb{Z}(2)).$$

This is a Hodge structure of weight 0, and it comes equipped with a perfect bilinear form b of signature $(4,20)$. For convenience, we denote by α and β the natural generators of $\mathrm{H}^0(X,\mathbb{Z})$ and $\mathrm{H}^4(X,\mathbb{Z}(2))$ respectively, so that $\widetilde{\mathrm{H}}(X,\mathbb{Z}) = \mathbb{Z}\alpha \oplus \mathrm{H}^2(X,\mathbb{Z}(1)) \oplus \mathbb{Z}\beta$. The pairing b is the orthogonal sum of the intersection pairing on $\mathrm{H}^2(X,\mathbb{Z}(1))$ and the pairing on $\mathbb{Z}\alpha \oplus \mathbb{Z}\beta$ given by $b(\alpha,\alpha) = b(\beta,\beta) = 0$ and $b(\alpha,\beta) = -1$.

It was observed by Mukai [35] that if $\Phi\colon \mathcal{D}X_1 \xrightarrow{\sim} \mathcal{D}X_2$ is a derived equivalence between K3 surfaces, then Φ^{H} restricts to an isomorphism $\Phi^{\widetilde{\mathrm{H}}}\colon \widetilde{\mathrm{H}}(X_1,\mathbb{Z}) \to \widetilde{\mathrm{H}}(X_2,\mathbb{Z})$ respecting the pairing and Hodge structures. Denote by $\mathrm{Aut}(\widetilde{\mathrm{H}}(X,\mathbb{Z}))$ the group of isometries of $\widetilde{\mathrm{H}}(X,\mathbb{Z})$ respecting the Hodge structure, and by $\mathrm{Aut}^+(\widetilde{\mathrm{H}}(X,\mathbb{Z}))$ the subgroup (of index 2) consisting of those isometries that respect the orientation on a four-dimensional positive definite subspace of $\widetilde{\mathrm{H}}(X,\mathbb{R})$.

Theorem 1.1 [22; 26; 35; 36; 39] *Let X be a* K3 *surface. Then the image of ρ_X is* $\mathrm{Aut}^+(\widetilde{\mathrm{H}}(X,\mathbb{Z}))$. □

In this paper, we prove Orlov's conjecture on $\mathbb{Q}$–Hodge structures for hyperkähler varieties, construct a rational version of the Mukai lattice for hyperkähler varieties, and compute (up to index 2) the image of ρ_X for certain Hilbert squares of K3 surfaces. The main tool in these results is the Looijenga–Lunts–Verbitsky Lie algebra.

1.2 The LLV Lie algebra and derived equivalences

Let X be a smooth projective complex variety. By the hard Lefschetz theorem, every ample class $\lambda \in \mathrm{NS}(X)$ determines a Lie algebra $\mathfrak{g}_\lambda \subset \mathrm{End}(\mathrm{H}(X,\mathbb{Q}))$ isomorphic to $\mathfrak{sl}_2$. More generally, this holds for every cohomology class $\lambda \in \mathrm{H}^2(X,\mathbb{Q})$ (algebraic or not) satisfying the conclusion of the hard Lefschetz theorem. Looijenga and Lunts [33] and Verbitsky [46] have studied the Lie algebra $\mathfrak{g}(X) \subset \mathrm{End}(\mathrm{H}(X,\mathbb{Q}))$ generated by the collection of the Lie algebras $\mathfrak{g}_\lambda$. We will refer to this as the LLV Lie algebra. See Section 2.1 for more details.

We say that X is *holomorphic symplectic* if it admits a nowhere degenerate holomorphic symplectic form $\sigma \in \mathrm{H}^0(X, \Omega_X^2)$.

Theorem A (Section 2.4) *Let X_1 and X_2 be holomorphic symplectic varieties. Then for every equivalence $\Phi\colon \mathcal{D}X_1 \xrightarrow{\sim} \mathcal{D}X_2$ there exists a canonical isomorphism of rational Lie algebras*

$$\Phi^{\mathfrak{g}}\colon \mathfrak{g}(X_1) \xrightarrow{\sim} \mathfrak{g}(X_2)$$

with the property that the map $\Phi^{\mathrm{H}}\colon \mathrm{H}(X_1, \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}(X_2, \mathbb{Q})$ is equivariant with respect to $\Phi^{\mathfrak{g}}$.

Note that $\mathfrak{g}(X)$ is defined in terms of the grading and the cup product on $\mathrm{H}(X, \mathbb{Q})$, neither of which are preserved under derived equivalences.

To prove Theorem A we introduce a complex Lie algebra $\mathfrak{g}'(X)$ whose definition is similar to the rational Lie algebra $\mathfrak{g}(X)$, but where the action of $\mathrm{H}^2(X, \mathbb{Q})$ on $\mathrm{H}(X, \mathbb{Q})$ is replaced with a natural action of the Hochschild cohomology group $\mathrm{HH}^2(X)$ on Hochschild homology $\mathrm{HH}_\bullet(X)$. Since Hochschild cohomology and its action on Hochschild homology is known to be invariant under derived equivalences, it follows that $\mathfrak{g}'(X)$ is a derived invariant. We show that if X is holomorphic symplectic, then the isomorphism $\mathrm{HH}_\bullet(X) \to \mathrm{H}(X, \mathbb{C})$ (coming from the Hochschild–Kostant–Rosenberg isomorphism) maps $\mathfrak{g}'(X)$ to $\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$. This is closely related to Verbitsky's "mirror symmetry" for hyperkähler varieties [46; 47]. From this we deduce that the rational Lie algebra $\mathfrak{g}(X)$ is a derived invariant.

1.3 A rational Mukai lattice for hyperkähler varieties

A *hyperkähler* (or irreducible holomorphic symplectic) variety is a simply connected smooth projective variety X for which $\mathrm{H}^0(X, \Omega_X^2)$ is spanned by a nowhere degenerate form.

Let X be a hyperkähler variety. Consider the $\mathbb{Q}$–vector space

$$\widetilde{\mathrm{H}}(X, \mathbb{Q}) := \mathbb{Q}\alpha \oplus \mathrm{H}^2(X, \mathbb{Q}) \oplus \mathbb{Q}\beta$$

equipped with the bilinear form b which is the orthogonal sum of the Beauville–Bogomolov form on $\mathrm{H}^2(X, \mathbb{Q})$ and a hyperbolic plane $\mathbb{Q}\alpha \oplus \mathbb{Q}\beta$ with α and β isotropic and $b(\alpha, \beta) = -1$. By analogy with the case of a K3 surface, we will call $\widetilde{\mathrm{H}}(X, \mathbb{Q})$ the (rational) *Mukai lattice* of X. Looijenga and Lunts [33] and Verbitsky [46] have shown that the Lie algebra $\mathfrak{g}(X)$ can be canonically identified with $\mathfrak{so}(\widetilde{\mathrm{H}}(X, \mathbb{Q}))$;

see Section 3.1 for a precise statement. Moreover, Verbitsky [46] has shown that the subalgebra $\mathrm{SH}(X,\mathbb{Q})$ of $\mathrm{H}(X,\mathbb{Q})$ generated by $\mathrm{H}^2(X,\mathbb{Q})$ forms an irreducible sub-$\mathfrak{g}(X)$–module. Using this, we show that Theorem A implies:

Theorem B (Section 4.2) *Let X_1 and X_2 be hyperkähler varieties and*

$$\Phi\colon \mathcal{D}X_1 \xrightarrow{\sim} \mathcal{D}X_2$$

an equivalence. Then the induced isomorphism Φ^{H} restricts to an isomorphism $\Phi^{\mathrm{SH}}\colon \mathrm{SH}(X_1,\mathbb{Q}) \xrightarrow{\sim} \mathrm{SH}(X_2,\mathbb{Q})$.

Taking $X_1 = X_2 = X$ in Theorem B we obtain a homomorphism

$$\rho_X^{\mathrm{SH}}\colon \mathrm{Aut}(\mathcal{D}X) \to \mathrm{GL}(\mathrm{SH}(X,\mathbb{Q})).$$

The complex structure on a hyperkähler variety X induces a Hodge structure of weight 0 on $\widetilde{\mathrm{H}}(X,\mathbb{Q})$ given by

$$\widetilde{\mathrm{H}}(X,\mathbb{Q}) = \mathbb{Q}\alpha \oplus \mathrm{H}^2(X,\mathbb{Q}(1)) \oplus \mathbb{Q}\beta.$$

Denote by $\mathrm{Aut}\,\widetilde{\mathrm{H}}(X,\mathbb{Q})$ the group of Hodge isometries of $\widetilde{\mathrm{H}}(X,\mathbb{Q})$.

Theorem C (Section 4.2) *Let X be a hyperkähler variety of dimension $2d$ and second Betti number b_2. Assume that b_2 is odd or d is odd. Then ρ_X^{SH} factors over a map $\rho_X^{\widetilde{\mathrm{H}}}\colon \mathrm{Aut}(\mathcal{D}(X)) \to \mathrm{Aut}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$.*

See Sections 3.2 and 4.2 for an explicit description of the implicit map

$$\mathrm{Aut}(\widetilde{\mathrm{H}}(X,\mathbb{Q})) \to \mathrm{GL}(\mathrm{SH}(X,\mathbb{Q})).$$

Note that all known hyperkähler varieties satisfy the parity conditions in the theorem: there are two infinite series of deformation classes with odd b_2 (generalized Kummers and Hilbert schemes of points), and three exceptional deformation classes with odd d (K3, OG6, OG10).

1.4 Hodge structures of derived equivalent hyperkähler varieties

Another application of Theorem A is the following:

Theorem D (Section 5) *Let X_1 and X_2 be derived equivalent hyperkähler varieties. Then for every i the $\mathbb{Q}$–Hodge structures $\mathrm{H}^i(X_1,\mathbb{Q})$ and $\mathrm{H}^i(X_2,\mathbb{Q})$ are isomorphic.*

This confirms Orlov's conjecture for hyperkähler varieties. The proof is inspired by Soldatenkov [43].

1.5 Auto-equivalences of the Hilbert square of a K3 surface

In the second half of the paper we consider the problem of determining the image of ρ_X for certain hyperkähler varieties. An important difference with the first half of the paper is that *integral* structures (lattices, arithmetic subgroups, ...) will play an important role here.

As a first approximation to determining the image of ρ_X, we consider a variation of this problem which is deformation invariant. Let X be a smooth projective complex variety. If X' and X'' are smooth deformations of X (parametrized by paths in the base), and if $\Phi\colon \mathcal{D}X' \xrightarrow{\sim} \mathcal{D}X''$ is an equivalence, then we obtain an isomorphism as the composition

$$\mathrm{H}(X,\mathbb{Q}) \to \mathrm{H}(X',\mathbb{Q}) \xrightarrow{\Phi^{\mathrm{H}}} \mathrm{H}(X'',\mathbb{Q}) \to \mathrm{H}(X,\mathbb{Q}).$$

We define the *derived monodromy group* of X to be the subgroup $\mathrm{DMon}(X)$ of $\mathrm{GL}(\mathrm{H}(X,\mathbb{Q}))$ generated by all these isomorphisms. This group contains both the usual monodromy group of X and the image of $\rho_X\colon \mathrm{Aut}(\mathcal{D}X) \to \mathrm{GL}(\mathrm{H}(X,\mathbb{Q}))$.

If S is a K3 surface, then the result of Huybrechts, Macrì and Stellari [26] implies $\mathrm{DMon}(S) = \mathrm{O}^+(\widetilde{\mathrm{H}}(S,\mathbb{Z}))$, and that the image of ρ_S consists of those elements of $\mathrm{DMon}(S)$ that respect the Hodge structure on $\widetilde{\mathrm{H}}(S,\mathbb{Z})$. Similarly, for an abelian variety A, the results of [18] imply $\mathrm{DMon}(A) = \mathrm{Spin}(\mathrm{H}^1(A,\mathbb{Z}) \oplus \mathrm{H}^1(A^\vee,\mathbb{Z}))$, and that the image of ρ_A consists of those elements of $\mathrm{DMon}(A)$ that respect the Hodge structure on $\mathrm{H}^1(A,\mathbb{Z}) \oplus \mathrm{H}^1(A^\vee,\mathbb{Z})$.

Now let X be a hyperkähler variety of type $\mathrm{K3}^{[2]}$. We have $\mathrm{H}(X,\mathbb{Q}) = \mathrm{SH}(X,\mathbb{Q})$ and hence by Theorem C the action of $\mathrm{Aut}(\mathcal{D}X)$ on $\mathrm{H}(X,\mathbb{Q})$ factors over a subgroup $\mathrm{O}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$ of $\mathrm{GL}(\mathrm{H}(X,\mathbb{Q}))$.

For an integral lattice $\Lambda \subset \widetilde{\mathrm{H}}(X,\mathbb{Q})$ we denote by $\mathrm{O}^+(\Lambda) \subset \mathrm{O}(\Lambda)$ the subgroup consisting of those Hodge isometries that respect the orientation of a positive 4–plane in $\Lambda_{\mathbb{R}}$.

Theorem E (Section 9.4) *Let X be a hyperkähler variety deformation equivalent to the Hilbert square of a* K3 *surface. There is an integral lattice $\Lambda \subset \widetilde{\mathrm{H}}(X,\mathbb{Q})$ such that*

$$\mathrm{O}^+(\Lambda) \subset \mathrm{DMon}(X) \subset \mathrm{O}(\Lambda)$$

inside $\mathrm{O}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$.

See Section 9.4 for a precise description of Λ. As an abstract lattice, Λ is isomorphic to $\mathrm{H}^2(X,\mathbb{Z}) \oplus U$, but its image in $\widetilde{\mathrm{H}}(X,\mathbb{Q})$ is *not* $\mathbb{Z}\alpha \oplus \mathrm{H}^2(X,\mathbb{Z}) \oplus \mathbb{Z}\beta$.

Crucial in the proof of Theorem E is the *derived McKay correspondence* due to Bridgeland, King and Reid [11] and Haiman [21]. It provides an ample supply of elements of $\mathrm{DMon}(X)$: every deformation of X to the Hilbert square $S^{[2]}$ of a K3 surface S induces an inclusion $\mathrm{DMon}(S) \to \mathrm{DMon}(X)$. As part of the proof, we explicitly compute this inclusion.

We denote by $\mathrm{Aut}(\Lambda)$ the group of isometries of $\Lambda \subset \widetilde{\mathrm{H}}(X,\mathbb{Q})$ that respect the Hodge structure on $\widetilde{\mathrm{H}}(X,\mathbb{Q})$. It follows from Theorem E that $\mathrm{im}(\rho_X)$ is contained in $\mathrm{Aut}(\Lambda)$ for every X which is deformation equivalent to the Hilbert square of a K3 surface. For some X we can show that the upper bound in the above corollary is close to being sharp. Denote by $\mathrm{Aut}^+(\Lambda) \subset \mathrm{Aut}(\Lambda)$ the subgroup consisting of those Hodge isometries that respect the orientation of a positive 4–plane in $\Lambda_{\mathbb{R}}$.

Theorem F (Section 10.2) *Let S be a complex* K3 *surface and $X = S^{[2]}$. Assume that* $\mathrm{NS}(X)$ *contains a hyperbolic plane. Then* $\mathrm{Aut}^+(\Lambda) \subset \mathrm{im}(\rho_X) \subset \mathrm{Aut}(\Lambda)$.

Remark 1.2 To determine $\mathrm{im}\,\rho_X$ up to index 2 for a general hyperkähler of type $\mathrm{K3}^{[2]}$ new constructions of derived equivalences will be needed.

Remark 1.3 Theorems E and F leave an ambiguity of index 2, related to orientations on a maximal positive subspace of $\widetilde{\mathrm{H}}(X,\mathbb{R})$. In the case of K3 surfaces, it was conjectured by Szendrői [44] that derived equivalences must respect such orientation, and this was proven by Huybrechts, Macrì, and Stellari [26]. Their method is based on deformation to generic (formal or analytic) K3 surfaces of Picard rank 0, and on a complete understanding of the space of stability conditions on those [25]. It is far from clear if such a strategy can be used to remove the index 2 ambiguity for hyperkähler varieties of type $\mathrm{K3}^{[2]}$.

Remark 1.4 That a lattice of signature $(4, b_2-2)$ should play a role in describing the image of ρ_X for hyperkähler varieties X was expected from the physics literature — see Dijkgraaf [16] — but it is not clear where the lattice should come from, nor what its precise description should be for general hyperkähler varieties. In the above results, the lattice Λ arises in a rather implicit way, and one may hope for a more concrete interpretation of its elements.

Remark 1.5 It is tempting to try to conjecture a description of the group $\mathrm{Aut}(\mathcal{D}X)$ in terms of an action on a space of stability conditions on X, generalizing Bridgeland's work on K3 surfaces [10]. However, there is a representation-theoretic obstruction against doing this naively. The central charge of a hypothetical stability condition on X

takes values in $\mathrm{H}(X,\mathbb{C})$, yet Theorems E and F suggest the central charge should take values in $\widetilde{\mathrm{H}}(X,\mathbb{C})$. If X is of type $\mathrm{K3}^{[2]}$, then $\mathrm{H}(X,\mathbb{C})$ and $\widetilde{\mathrm{H}}(X,\mathbb{C})$ are nonisomorphic irreducible $\mathrm{DMon}(X)$–modules, so this would require a modification of the notion of stability condition.

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2 The LLV Lie algebra of a smooth projective variety

In this section we recall the construction of Looijenga and Lunts [33] and Verbitsky [46] of a Lie algebra acting naturally on the cohomology of algebraic varieties. For holomorphic symplectic varieties we show that this Lie algebra is a derived invariant.

2.1 The LLV Lie algebra

Let F be a field of characteristic zero and M be a $\mathbb{Z}$–graded F–vector space of finite F–dimension. Denote by h the endomorphism of M that is multiplication by n on M_n.

Let e be an endomorphism of M of degree 2. We say that *e has the hard Lefschetz property* if for every $n \geq 0$ the map $e^n\colon M_{-n} \to M_n$ is an isomorphism. This is equivalent to the existence of an $f \in \mathrm{End}(M)$ such that the relations

$$[h,e]=2e, \quad [h,f]=-2f, \quad [e,f]=h \tag{1}$$

hold in $\mathrm{End}(M)$. Thus, (e,h,f) forms an $\mathfrak{sl}_2$–triple and defines a Lie homomorphism $\mathfrak{sl}_2 \to \mathrm{End}(M)$.

Proposition 2.1 *Assume that e has the hard Lefschetz property. Then the element f satisfying* (1) *is unique, and if e and h lie in a semisimple sub-Lie algebra $\mathfrak{g} \subset \mathrm{End}(M)$, then so does f.*

Proof The action of $\mathrm{ad}\, e$ on $\mathrm{End}(M)$ has the hard Lefschetz property for the grading defined by $\mathrm{ad}\, h$. In particular,

$$(\mathrm{ad}\, e)^2\colon \mathrm{End}(M)_{-2} \xrightarrow{\sim} \mathrm{End}(M)_2$$

is an isomorphism. It sends f to $-2e$, so f is indeed uniquely determined.

If e and h lie in $\mathfrak{g}$, then $\mathfrak{g} \subset \mathrm{End}(M)$ is graded and the above map restricts to an injective map

$$(\mathrm{ad}\, e)^2 \colon \mathfrak{g}_{-2} \hookrightarrow \mathfrak{g}_2.$$

Since h is diagonalizable, it is contained in a Cartan subalgebra of $\mathfrak{g}$. The symmetry of the resulting root system implies that $\dim \mathfrak{g}_{-n} = \dim \mathfrak{g}_n$ for all n. In particular, the map $(\mathrm{ad}\, e)^2$ defines an isomorphism between $\mathfrak{g}_{-2}$ and $\mathfrak{g}_2$; thus f lies in $\mathfrak{g}$. □

Let $\mathfrak{a}$ be an abelian Lie algebra and $e \colon \mathfrak{a} \to \mathfrak{gl}(M)$, defined by $a \mapsto e_a$, a Lie homomorphism. We say that *e has the hard Lefschetz property* if $e(\mathfrak{a}) \subset \mathfrak{gl}(M)_2$ and if there exists some $a \in \mathfrak{a}$ such that e_a has the hard Lefschetz property. Note that this is a Zariski open condition on $a \in \mathfrak{a}$.

If $e \colon \mathfrak{a} \to \mathfrak{gl}(M)$ has the hard Lefschetz property, then we denote by $\mathfrak{g}(\mathfrak{a}, M)$ the Lie algebra generated by the $\mathfrak{sl}_2$–triples (e_a, h, f_a) for $a \in \mathfrak{a}$ such that e_a has the hard Lefschetz property. We say that $(\mathfrak{a}, M)$ is a *Lefschetz module* if $\mathfrak{g}(\mathfrak{a}, M)$ is semisimple.

Now let X be a smooth projective complex variety of dimension d. Denote by $M := \mathrm{H}(X, \mathbb{Q})[d]$ the shifted total cohomology of X (with middle cohomology in degree 0). For a class $\lambda \in \mathrm{H}^2(X, \mathbb{Q})$, consider the endomorphism $e_\lambda \in \mathrm{End}(M)$ given by cup product with λ. If λ is ample, then e_λ has the hard Lefschetz property, so the map $e \colon \mathrm{H}^2(X, \mathbb{Q}) \to \mathfrak{gl}(M)$ has the hard Lefschetz property. We denote the corresponding Lie algebra by $\mathfrak{g}(X) := \mathfrak{g}(\mathrm{H}^2(X, \mathbb{Q}), M)$.

Proposition 2.2 [33, 1.6, 1.9] *$(\mathrm{H}^2(X, \mathbb{Q}), M)$ is a Lefschetz module.* □

In other words, $\mathfrak{g}(X)$ is a semisimple Lie algebra over $\mathbb{Q}$.

2.2 Hochschild homology and cohomology

Let X be a smooth projective variety of dimension d with canonical bundle $\omega_X := \Omega_X^d$. Its *Hochschild cohomology* is defined as

$$\mathrm{HH}^n(X) := \mathrm{Ext}^n_{X \times X}(\Delta_* \mathcal{O}_X, \Delta_* \mathcal{O}_X)$$

and its *Hochschild homology* is defined as

$$\mathrm{HH}_n(X) := \mathrm{Ext}^{d-n}_{X \times X}(\Delta_* \mathcal{O}_X, \Delta_* \omega_X).$$

Composition of extensions defines maps

$$\mathrm{HH}^n \otimes \mathrm{HH}^m \to \mathrm{HH}^{n+m}, \quad \mathrm{HH}^n \otimes \mathrm{HH}_m \to \mathrm{HH}_{m-n},$$

making $\mathrm{HH}_\bullet(X)$ into a graded module over the graded ring $\mathrm{HH}^\bullet(X)$.

The Hochschild–Kostant–Rosenberg isomorphism (twisted by the square root of the Todd class as in [30; 15]) defines isomorphisms

$$I^n : \mathrm{HH}^n(X) \xrightarrow{\sim} \bigoplus_{i+j=n} \mathrm{H}^i\big(X, \textstyle\bigwedge^j T_X\big), \quad I_n : \mathrm{HH}_n(X) \xrightarrow{\sim} \bigoplus_{j-i=n} \mathrm{H}^i(X, \Omega_X^j).$$

Under these isomorphisms, multiplication in $\mathrm{HH}^\bullet(X)$ corresponds to the operation induced by the product in $\bigwedge^\bullet T_X$, and the action of $\mathrm{HH}^\bullet(X)$ on $\mathrm{HH}_\bullet(X)$ corresponds to the action induced by the contraction action of $\bigwedge^\bullet T_X$ on $\Omega_X^\bullet$; see [12; 13].

Together with the degeneration of the Hodge–de Rham spectral sequence, the isomorphism $I_\bullet$ defines an isomorphism

$$\mathrm{HH}_\bullet(X) \xrightarrow{\sim} \mathrm{H}(X, \mathbb{C}).$$

This map does not respect the grading; rather it maps HH_i to the i^{th} column of the Hodge diamond (normalized so that the 0^{th} column is the central column $\bigoplus_p \mathrm{H}^{p,p}$). Combining with the action of $\mathrm{HH}^\bullet$ on $\mathrm{HH}_\bullet$, we obtain an action of the ring $\mathrm{HH}^\bullet(X)$ on $\mathrm{H}(X, \mathbb{C})$.

Theorem 2.3 *Let $\Phi : \mathcal{D}X_1 \xrightarrow{\sim} \mathcal{D}X_2$ be a derived equivalence between smooth projective complex varieties. Then we have natural graded isomorphisms*

$$\Phi^{\mathrm{HH}^\bullet} : \mathrm{HH}^\bullet(X_1) \xrightarrow{\sim} \mathrm{HH}^\bullet(X_2), \quad \Phi^{\mathrm{HH}_\bullet} : \mathrm{HH}_\bullet(X_1) \xrightarrow{\sim} \mathrm{HH}_\bullet(X_2),$$

compatible with the ring structure on $\mathrm{HH}^\bullet$ and the module structure on $\mathrm{HH}_\bullet$, and such that the square

$$\begin{array}{ccc} \mathrm{HH}_\bullet(X_1) & \xrightarrow{I} & \mathrm{H}(X_1, \mathbb{C}) \\ \downarrow{\scriptstyle \Phi^{\mathrm{HH}_\bullet}} & & \downarrow{\scriptstyle \Phi^{\mathrm{H}}} \\ \mathrm{HH}_\bullet(X_2) & \xrightarrow{I} & \mathrm{H}(X_2, \mathbb{C}) \end{array}$$

commutes.

Proof See [13; 34]. □

2.3 The Hochschild Lie algebra of a holomorphic symplectic variety

Now assume that X is holomorphic symplectic of dimension $2d$. That is, we assume that there exists a symplectic form $\sigma \in \mathrm{H}^0(X, \Omega_X^2)$. Note that this implies that a Zariski-dense collection of $\sigma \in \mathrm{H}^0(X, \Omega_X^2)$ will be nowhere degenerate.

Through the isomorphism $I : \mathrm{HH}_\bullet(X) \to \mathrm{H}(X, \mathbb{C})$, the vector space $\mathrm{H}(X, \mathbb{C})$ becomes a module under the ring $\mathrm{HH}^\bullet(X)$.

Lemma 2.4 $\mathrm{HH}^\bullet(X) \cong \mathrm{H}^\bullet(X,\mathbb{C})$ *as graded rings, and* $\mathrm{H}(X,\mathbb{C})$ *is free of rank one as an* $\mathrm{HH}^\bullet(X)$*–module.*

Proof A symplectic form σ defines an isomorphism $\Omega^1_X \xrightarrow{\sim} T_X$, and hence an isomorphism of algebras $\bigwedge^\bullet \Omega^1_X \xrightarrow{\sim} \bigwedge^\bullet T_X$. Combining this with the Hochschild–Kostant–Rosenberg isomorphism I and the degeneration of the Hodge–de Rham spectral sequence, we obtain a chain of isomorphisms of graded rings

$$\mathrm{HH}^\bullet(X) \xrightarrow{\sim} \mathrm{H}^\bullet\big(X, \textstyle\bigwedge^\bullet T_X\big) \xrightarrow{\sim} \mathrm{H}^\bullet(X,\Omega^\bullet_X) \xrightarrow{\sim} \mathrm{H}^\bullet(X,\mathbb{C}).$$

This proves the first assertion. For the second it suffices to observe that the module $\mathrm{HH}_\bullet(X,\mathbb{C})$ is generated by $\sigma^d \in \mathrm{HH}_{2d}(X) = \mathrm{H}^0(X,\Omega^{2d}_X)$. □

Consider the endomorphisms $h_p, h_q \in \mathrm{End}(\mathrm{H}(X,\mathbb{C}))$ given by

$$h_p = p-d, \quad h_q = q-d \quad \text{on } \mathrm{H}^{p,q}.$$

These define the Hodge bigrading on $\mathrm{H}(X,\mathbb{C})$, normalized to be symmetric along the central part $\mathrm{H}^{d,d}$. Note that $h = h_p + h_q$. The action of $\mathrm{HH}^n(X)$ on $\mathrm{H}(X,\mathbb{C})$ has degree n for the grading defined by $h' = h_q - h_p$.

Lemma 2.4 and hard Lefschetz imply:

Corollary 2.5 *For a Zariski-dense collection of* $\mu \in \mathrm{HH}^2(X)$*, the action by* μ*,*

$$e'_\mu : \mathrm{H}(X,\mathbb{C}) \to \mathrm{H}(X,\mathbb{C}),$$

has the hard Lefschetz property with respect to the grading defined by h'*.* □

In particular, for every such μ we have a complex subalgebra $\mathfrak{g}_\mu \subset \mathrm{End}(\mathrm{H}(X,\mathbb{C}))$ isomorphic to $\mathfrak{sl}_2$, and the collection of such algebras generates a Lie algebra which we denote by $\mathfrak{g}'(X) \subset \mathrm{End}(\mathrm{H}(X,\mathbb{C}))$. From Lemma 2.4 we also obtain:

Corollary 2.6 *The complex Lie algebras* $\mathfrak{g}'(X)$ *and* $\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$ *are isomorphic.* □

In the next section, we will show something stronger: that $\mathfrak{g}'(X)$ and $\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$ coincide as sub-Lie algebras of $\mathrm{End}(\mathrm{H}(X,\mathbb{C}))$. Theorem A then follows by combining this with the following proposition:

Proposition 2.7 *Assume that* X_1 *and* X_2 *are holomorphic symplectic varieties. Then for every equivalence* $\Phi\colon \mathcal{D}X_1 \xrightarrow{\sim} \mathcal{D}X_2$ *there exists a canonical isomorphism of complex Lie algebras*

$$\Phi^{\mathfrak{g}'} : \mathfrak{g}'(X_1) \xrightarrow{\sim} \mathfrak{g}'(X_2).$$

It has the property that the map $\Phi^{\mathrm{H}}\colon \mathrm{H}(X_1,\mathbb{C}) \xrightarrow{\sim} \mathrm{H}(X_2,\mathbb{C})$ *is equivariant with respect to* $\Phi^{\mathfrak{g}'}$.

Proof This follows immediately from Theorem 2.3. □

2.4 Comparison of the two Lie algebras and proof of Theorem A

The remainder of this section is devoted to the proof of the following:

Proposition 2.8 *If* X *is holomorphic symplectic, then* $\mathfrak{g}(X)\otimes_{\mathbb{Q}}\mathbb{C} = \mathfrak{g}'(X)$ *as sub-Lie algebras of* $\mathrm{End}(\mathrm{H}(X,\mathbb{C}))$.

Let X be holomorphic symplectic. If $\mathcal{F}$ is a coherent $\mathcal{O}_X$–module then we will simply write $\mathrm{H}^i(\mathcal{F})$ for $\mathrm{H}^i(X,\mathcal{F})$. We have decompositions

$$\mathrm{H}^2(X,\mathbb{C}) = \mathrm{H}^2(\mathcal{O}_X)\oplus \mathrm{H}^1(\Omega^1_X)\oplus \mathrm{H}^0(\Omega^2_X)$$

and

$$\mathrm{HH}^2(X) = \mathrm{H}^2(\mathcal{O}_X)\oplus \mathrm{H}^1(T_X)\oplus \mathrm{H}^0\big(\textstyle\bigwedge^2 T_X\big).$$

We will use the same symbol λ to denote an element $\lambda\in \mathrm{H}^2(X,\mathbb{C})$ and the endomorphism of $\mathrm{End}(\mathrm{H}(X,\mathbb{C}))$ given by cup product with λ. Note that $\lambda\in\mathfrak{g}(X)\otimes_{\mathbb{Q}}\mathbb{C}$ by construction. Similarly, we will use the same symbol for $\mu\in\mathrm{HH}^2(X)$ and the resulting $\mu\in\mathrm{End}(\mathrm{H}(X,\mathbb{C}))$, given by contraction with μ. We have $\mu\in\mathfrak{g}'(X)$.

For a symplectic form $\sigma\in\mathrm{H}^0(\Omega^2_X)$, we denote by $\check{\sigma}\in\mathrm{H}^0\big(\bigwedge^2 T_X\big)$ the image of the form $\sigma\in\mathrm{H}^0(\Omega^2_X)$ under the isomorphism $\Omega^2_X\to\bigwedge^2 T_X$ defined by σ. In suitable local coordinates, we have

$$\sigma = \mathrm{d}u_1\wedge\mathrm{d}v_1+\cdots+\mathrm{d}u_d\wedge\mathrm{d}v_d$$

and

$$\check{\sigma} = \frac{\partial}{\partial u_1}\wedge\frac{\partial}{\partial v_1}+\cdots+\frac{\partial}{\partial u_d}\wedge\frac{\partial}{\partial v_d}.$$

Lemma 2.9 *If* σ *is a nowhere degenerate symplectic form then* $(\sigma,h_p,\check{\sigma})$ *is an* $\mathfrak{sl}_2$–*triple in* $\mathrm{End}(\mathrm{H}(X,\mathbb{C}))$.

Proof Clearly σ has degree 2 and $\check{\sigma}$ has degree -2 for the grading given by h_p, so $[h_p,\sigma]=2\sigma$ and $[h_p,\check{\sigma}]=-2\check{\sigma}$.

We need to show that $[\sigma,\check{\sigma}]=h_p$. This follows immediately from a local computation: in the above local coordinates, one verifies that on the standard basis of Ω^p the commutator $[\sigma,\check{\sigma}]$ acts as $p-d$. □

Note that the existence of one nowhere degenerate σ implies that a Zariski-dense collection of $\sigma \in \mathrm{H}^0(\Omega_X^2)$ is nowhere degenerate.

Lemma 2.10 *For a Zariski-dense collection $\alpha \in \mathrm{H}^2(X, \mathcal{O}_X)$, there is $\check{\alpha} \in \mathrm{End}(\mathrm{H}(X, \mathbb{C}))$ such that $(\alpha, h_q, \check{\alpha})$ is an $\mathfrak{sl}_2$–triple.*

Proof This follows from Lemma 2.9 and Hodge symmetry. □

Lemma 2.11 *For all $\tau \in \mathrm{H}^0\big(X, \bigwedge^2 T_X\big)$ the endomorphism τ lies in $\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$.*

Proof It suffices to show that this holds for a Zariski-dense collection of τ; hence we may assume without loss of generality that $\tau = \check{\sigma}$ with σ and $\check{\sigma}$ as in Lemma 2.9. Let α and $\check{\alpha}$ be as in Lemma 2.10. Because σ and h_p commute with both α and h_q, we have that every element of the $\mathfrak{sl}_2$–triple $(\sigma, h_p, \check{\sigma})$ commutes with every element of the $\mathfrak{sl}_2$–triple $(\alpha, h_q, \check{\alpha})$. From this, it follows that

$$(\alpha + \sigma, h, \check{\alpha} + \check{\sigma}) \quad \text{and} \quad (\alpha - \sigma, h, \check{\alpha} - \check{\sigma})$$

are $\mathfrak{sl}_2$–triples. Since the elements $\alpha \pm \sigma$ lie in $\mathrm{H}^2(X, \mathbb{C})$, and apparently have the hard Lefschetz property, we conclude that the endomorphisms $\check{\alpha} \pm \check{\sigma}$ lie in $\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$; hence also $\tau = \check{\sigma}$ lies in $\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$. □

Corollary 2.12 *h_p and h_q lie in $\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$.*

Proof By Lemma 2.9 we have $h_p = [\sigma, \check{\sigma}]$, which by Lemma 2.11 lies in $\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$. Since $h_q = h - h_p$ we also have that h_q lies in $\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$. □

Fix a $\tau \in \mathrm{H}^0\big(X, \bigwedge^2 T_X\big)$ that is nowhere degenerate as an alternating form on Ω_X^1. This defines isomorphisms $c_\tau \colon \Omega_X^1 \to T_X$ and $c_\tau \colon \mathrm{H}^1(\Omega_X^1) \to \mathrm{H}^1(T_X)$ given by contracting sections of Ω_X^1 with τ.

Lemma 2.13 *For all $\eta \in \mathrm{H}^1(\Omega_X^1)$, we have $[\tau, \eta] = c_\tau(\eta)$ in $\mathrm{End}(\mathrm{H}(X, \mathbb{C}))$.*

Proof This is again a local computation. If η is a local section of Ω_X^1, then a computation on a local basis shows $[\tau, \eta] = c_\tau(\eta)$ as maps $\Omega_X^p \to \Omega_X^{p-1}$. □

Corollary 2.14 *Every element η' of $\mathrm{H}^1(X, T_X)$ lies in $\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$.*

Proof (See also [19, 4.5] for the case of a hyperkähler variety.) Every such η' is of the form $c_\tau(\eta)$ for a unique $\eta \in \mathrm{H}^1(\Omega_X^1)$, and hence the corollary follows from Lemmas 2.13 and 2.11 and the fact that η lies in $\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$. □

We can now finish the comparison of the two Lie algebras.

Proof of Proposition 2.8 By Corollary 2.6 it suffices to show that $\mathfrak{g}'(X)$ is contained in $\mathfrak{g}(X)\otimes_{\mathbb{Q}}\mathbb{C}$. By Proposition 2.1 it suffices to show that h' is contained in $\mathfrak{g}(X)\otimes_{\mathbb{Q}}\mathbb{C}$, and that for almost every $a\in \mathrm{HH}^2(X)$ we have that the action of a on $\mathrm{H}(X,\mathbb{C})$ is contained in $\mathfrak{g}(X)\otimes_{\mathbb{Q}}\mathbb{C}$. This follows from Lemma 2.11, Corollaries 2.12 and 2.14, and the fact that the action of any $\alpha\in \mathrm{H}^2(\mathcal{O}_X)$ lies in $\mathfrak{g}(X)\otimes_{\mathbb{Q}}\mathbb{C}$. □

Together with Proposition 2.7, this proves Theorem A.

3 Rational cohomology of hyperkähler varieties

3.1 The BBF form and the LLV Lie algebra

Let X be a complex hyperkähler variety of dimension $2d$. We denote by

$$b=b_X\colon \mathrm{H}^2(X,\mathbb{Q})\times \mathrm{H}^2(X,\mathbb{Q})\to\mathbb{Q}$$

its Beauville–Bogomolov–Fujiki, and by c_X its Fujiki constant. These are related by

$$\int_X \lambda^{2d}=\frac{(2d)!}{2^d d!}c_X b(\lambda,\lambda)^d \tag{2}$$

for $\lambda\in \mathrm{H}^2(X,\mathbb{Q})$; see eg [41].

We extend b to a bilinear form on

$$\widetilde{\mathrm{H}}(X,\mathbb{Q}):=\mathbb{Q}\alpha\oplus \mathrm{H}^2(X,\mathbb{Q})\oplus\mathbb{Q}\beta,$$

by declaring α and β to be orthogonal to $\mathrm{H}^2(X,\mathbb{Q})$, and setting $b(\alpha,\beta)=-1$, $b(\alpha,\alpha)=0$ and $b(\beta,\beta)=0$. We equip $\widetilde{\mathrm{H}}(X,\mathbb{Q})$ with a grading satisfying $\deg\alpha=-2$ and $\deg\beta=2$, and for which $\mathrm{H}^2(X,\mathbb{Q})$ sits in degree 0. This induces a grading on the Lie algebra $\mathfrak{so}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$.

For $\lambda\in \mathrm{H}^2(X,\mathbb{Q})$ we consider the endomorphism $e_\lambda\in\mathfrak{so}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$ given by $e_\lambda(\alpha)=\lambda$, $e_\lambda(\mu)=b(\lambda,\mu)\beta$ for all $\mu\in \mathrm{H}^2(X,\mathbb{Q})$, and $e_\lambda(\beta)=0$.

Theorem 3.1 (Looijenga–Lunts, Verbitsky) *There is a unique isomorphism of graded Lie algebras*

$$\mathfrak{so}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))\xrightarrow{\sim}\mathfrak{g}(X)$$

that maps e_λ to e_λ for every $\lambda\in \mathrm{H}^2(X,\mathbb{Q})$.

Proof See [33, Proposition 4.5] or [46, Theorem 1.4] for the theorem over the real numbers. This readily descends to $\mathbb{Q}$; see [43, Proposition 2.9] for more details. □

The representation of $\mathfrak{so}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$ on $\mathrm{H}(X,\mathbb{Q})$ integrates to a representation of the group $\mathrm{Spin}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$ on $\mathrm{H}(X,\mathbb{Q})$. Let $\lambda\in\mathrm{H}^2(X,\mathbb{Q})$. Then e_λ is nilpotent, and hence $B_\lambda := \exp e_\lambda$ is an element of $\mathrm{Spin}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$. It acts on $\widetilde{\mathrm{H}}(X,\mathbb{Q})$ by

$$B_\lambda(r\alpha+\mu+s\beta) = r\alpha + (\mu + r\lambda) + \big(s + b(\mu,\lambda) + r\tfrac{1}{2}b(\lambda,\lambda)\big)\beta \tag{3}$$

for all $r,s\in\mathbb{Q}$ and $\mu\in\mathrm{H}^2(X,\mathbb{Q})$. The action on the total cohomology of X is given by:

Proposition 3.2 *B_λ acts as multiplication by $\mathrm{ch}(\lambda)$ on $\mathrm{H}(X,\mathbb{Q})$.* □

In particular, if $\mathcal{L}$ is a line bundle on X and $\Phi\colon \mathcal{D}X\to\mathcal{D}X$ is the equivalence that maps $\mathcal{F}$ to $\mathcal{F}\otimes\mathcal{L}$, then $\Phi^{\mathrm{H}} = B_{c_1(\mathcal{L})}$.

3.2 The Verbitsky component of cohomology

Let X be a complex hyperkähler variety of dimension $2d$. We define the *even cohomology* of X as the graded $\mathbb{Q}$–algebra

$$\mathrm{H}^{\mathrm{ev}}(X,\mathbb{Q}) := \bigoplus_n \mathrm{H}^{2n}(X,\mathbb{Q}),$$

and the *Verbitsky component* of the cohomology of X as the sub-$\mathbb{Q}$–algebra $\mathrm{SH}(X,\mathbb{Q})$ of $\mathrm{H}^{\mathrm{ev}}(X,\mathbb{Q})$ generated by $\mathrm{H}^2(X,\mathbb{Q})$. Clearly, $\mathrm{SH}(X,\mathbb{Q})[2d]$ is a sub-Lefschetz module of $\mathrm{H}^{\mathrm{ev}}(X,\mathbb{Q})[2d]$ for $\mathrm{H}^2(X,\mathbb{Q})$.

Lemma 3.3 (Verbitsky [8; 45]) *The kernel of the $\mathbb{Q}$–algebra homomorphism*

$$\mathrm{Sym}^\bullet \mathrm{H}^2(X,\mathbb{Q}) \twoheadrightarrow \mathrm{SH}(X,\mathbb{Q})$$

is generated by the elements λ^{d+1} with $\lambda\in\mathrm{H}^2(X,\mathbb{Q})$ satisfying $b(\lambda,\lambda)=0$. □

Lemma 3.4 (Verbitsky) *$\mathrm{SH}(X,\mathbb{Q})[2d]$ is an irreducible Lefschetz module.*

Proof It is the smallest sub-Lefschetz module of $\mathrm{H}^{\mathrm{ev}}(X,\mathbb{Q})[2d]$ having a nontrivial component of degree $-2d$. □

Verbitsky also describes the space $\mathrm{SH}(X,\mathbb{Q})$ explicitly. Below we normalize this description, and use it to compute the Mukai pairing on $\mathrm{SH}(X,\mathbb{Q})$.

Proposition 3.5 *There is a unique map*

$$\Psi\colon \mathrm{SH}(X,\mathbb{Q})[2d] \to \mathrm{Sym}^d \widetilde{\mathrm{H}}(X,\mathbb{Q})$$

satisfying

(i) Ψ *is morphism of Lefschetz modules*,

(ii) $\Psi(1) = \alpha^d/d!$.

Note that the Lefschetz module structure on $\mathrm{Sym}^d \widetilde{\mathrm{H}}(X,\mathbb{Q})$ is given by the Leibniz rule

$$e_\lambda(x_1\cdots x_d) := \sum_i x_1\cdots e_\lambda(x_i)\cdots x_d.$$

Proof Uniqueness is clear. For existence, consider the map

$$\widetilde{\Psi}\colon \mathrm{Sym}^\bullet \mathrm{H}^2(X,\mathbb{Q}) \to \mathrm{Sym}^d \widetilde{\mathrm{H}}(X,\mathbb{Q}),$$

given by

$$\lambda_1\cdots\lambda_n \mapsto e_{\lambda_1}\cdots e_{\lambda_n}(\alpha^d/d!).$$

This map is well defined since the e_{λ_i} commute. Moreover, the map is graded and satisfies $\widetilde{\Psi}(\lambda x) = e_\lambda \widetilde{\Psi}(x)$ for all $\lambda \in \mathrm{H}^2(X,\mathbb{Q})$ and $x \in \mathrm{Sym}^\bullet \mathrm{H}^2(X,\mathbb{Q})$. To show that $\widetilde{\Psi}$ induces a morphism of Lefschetz modules with the desired properties it now suffices to verify that it vanishes on the ideal generated by the λ^{d+1} for $\lambda \in \mathrm{H}^2(X,\mathbb{Q})$ satisfying $b(\lambda,\lambda) = 0$. Equivalently, it suffices to show that for every $x \in \mathrm{Sym}^d \widetilde{\mathrm{H}}(X,\mathbb{Q})$ and for every $\lambda \in \mathrm{H}^2(X,\mathbb{Q})$ with $b(\lambda,\lambda) = 0$ we have $e_\lambda^{d+1}(x) = 0$.

Without loss of generality, we may assume that x is a monomial of the form

$$x = \alpha^i\beta^j\lambda_1\cdots\lambda_m, \quad i+j+m = d, \quad \lambda_i \in \mathrm{H}^2(X,\mathbb{Q}).$$

For degree reasons, we have $e_\lambda^k(\beta^j\lambda_1\cdots\lambda_m) = 0$ for $k > m$. Moreover, it follows from $b(\lambda,\lambda) = 0$ that $e_\lambda^k(\alpha^i) = 0$ for $k > i$. Combining these, one concludes that $e_\lambda^{d+1}(x) = 0$, which is what we had to prove. □

Lemma 3.6 $\Psi(\mathrm{pt}_X) = \beta^d/c_X.$

Proof Choose $\lambda \in \mathrm{H}^2(X,\mathbb{Q})$ with $b(\lambda,\lambda) \neq 0$. Then we have

$$\Psi(\lambda^{2d}) = e_\lambda^{2d}\left(\frac{\alpha^d}{d!}\right) = \frac{(2d)!}{2^d d!} b(\lambda,\lambda)^d \beta^d. \tag{4}$$

Dividing by (2) gives the claimed identity. □

Consider the contraction (or Laplacian) operator

$$\Delta\colon \mathrm{Sym}^d \widetilde{\mathrm{H}}(X,\mathbb{Q}) \to \mathrm{Sym}^{d-2} \widetilde{\mathrm{H}}(X,\mathbb{Q}),$$

given by

$$x_1 \dots x_d \mapsto \sum_{i<j} b(x_i, x_j) x_1 \cdots \hat{x}_i \cdots \hat{x}_j \cdots x_d.$$

This is a morphism of Lefschetz modules, or equivalently of $\mathfrak{so}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$–modules.

Lemma 3.7 *The sequence of Lefschetz modules*

$$0 \to \mathrm{SH}(X,\mathbb{Q})[2d] \xrightarrow{\Psi} \mathrm{Sym}^d \widetilde{\mathrm{H}}(X,\mathbb{Q}) \xrightarrow{\Delta} \mathrm{Sym}^{d-2} \widetilde{\mathrm{H}}(X,\mathbb{Q}) \to 0$$

is exact.

Proof Since $\Delta\Psi(1) = 0$, we have $\Delta \circ \Psi = 0$. The map Δ is well known to be a surjective map of $\mathfrak{so}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$–modules with irreducible kernel. Since Ψ is nonzero and $\mathrm{SH}(X,\mathbb{Q})$ is irreducible, it follows that the sequence is exact. □

The *Mukai pairing* [14] on $\mathrm{H}^{\mathrm{ev}}(X,\mathbb{Q})$ restricts to a pairing b_{SH} on $\mathrm{SH}(X,\mathbb{Q})$. It pairs elements of degree m with elements of degree $2d-m$, according to the formula

$$b_{\mathrm{SH}}(\lambda_1 \cdots \lambda_m,\, \mu_1 \cdots \mu_{2d-m}) = (-1)^m \int_X \lambda_1 \cdots \lambda_m \mu_1 \cdots \mu_{2d-m}.$$

Note that $b_{\mathrm{SH}}(e_\lambda x, y) + b_{\mathrm{SH}}(x, e_\lambda y) = 0$ for all $x, y \in \mathrm{SH}(X,\mathbb{Q})$ and $\lambda \in \mathrm{H}^2(X,\mathbb{Q})$, so b_{SH} is $\mathfrak{so}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$–invariant.

The pairing on $\widetilde{\mathrm{H}}(X,\mathbb{Q})$ induces a pairing on $\mathrm{Sym}^d \widetilde{\mathrm{H}}(X,\mathbb{Q})$ defined by

$$b_{[d]}(x_1 \cdots x_d, y_1 \cdots y_d) := (-1)^d \sum_{\sigma \in \mathfrak{S}_d} \prod_i b(x_i, y_{\sigma i}).$$

By construction, $b_{[d]}$ is $\mathfrak{so}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$–invariant. The map Ψ is almost an isometry, in the following sense:

Proposition 3.8 *For all* $x, y \in \mathrm{SH}(X,\mathbb{Q})$,

$$c_X b_{[d]}(\Psi x, \Psi y) = b_{\mathrm{SH}}(x, y).$$

Proof Both the Mukai form on $\mathrm{SH}(X,\mathbb{Q})[2d]$ and the pairing on $\mathrm{Sym}^d \widetilde{\mathrm{H}}(X,\mathbb{Q})$ are $\mathfrak{so}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$–invariant. Since $\mathrm{SH}(X,\mathbb{Q})$ is an irreducible $\mathfrak{so}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$–module, it suffices to verify the identity for some $x, y \in \mathrm{SH}(X,\mathbb{Q})$ with $b_{\mathrm{SH}}(x,y) \neq 0$.

Let $\lambda \in \mathrm{H}^2(X, \mathbb{Q})$ with $b(\lambda, \lambda) \neq 0$. We have

$$b_{\mathrm{SH}}(1, \lambda^{2d}) = \int_X \lambda^{2d} = \frac{(2d)!}{2^d d!} c_X b(\lambda, \lambda)^d \neq 0.$$

By (4),

$$\Psi(\lambda^{2d}) = \frac{(2d)!}{2^d d!} b(\lambda, \lambda)^d \beta^d,$$

and hence

$$c_X b_{[d]}(\Psi(1), \Psi(\lambda^{2d})) = \frac{c_X (2d)!}{2^d (d!)^2} b_{[d]}(\alpha^d, \beta^d) = \frac{c_X (2d)!}{2^d d!} b(\lambda, \lambda)^d,$$

which agrees with the above expression for $b_{\mathrm{SH}}(1, \lambda^{2d})$. □

Remark 3.9 If X is of type $\mathrm{K3}^{[d]}$ then $c_X = 1$ and Ψ is an isometry.

4 Action of derived equivalences on the Verbitsky component

In this section we prove Theorems B and C from the introduction.

4.1 A representation-theoretical construction

Let K be a field of characteristic different from 2, and let $V = (V, b)$ be a nondegenerate quadratic space over K. Let d be a positive integer and consider the space

$$S_{[d]} V := \ker(\mathrm{Sym}^d V \xrightarrow{\Delta} \mathrm{Sym}^{d-2} V).$$

The Lie algebra $\mathfrak{so}(V)$ acts faithfully on $S_{[d]} V$, inducing an inclusion

$$\mathfrak{so}(V) \subset \mathrm{End}(S_d V).$$

Consider the normalizer of $\mathfrak{so}(V)$ in $\mathrm{GL}(S_{[d]} V)$, that is, the group

$$N(V, d) := \{g \in \mathrm{GL}(S_{[d]} V) \mid g\, \mathfrak{so}(V) g^{-1} = \mathfrak{so}(V)\}.$$

Proposition 4.1 *Assume that K is separably closed. Then there is an exact sequence*

$$1 \to \{\pm 1\} \to \mathrm{O}(V) \times K^\times \to N(V, d) \to 1,$$

where the inclusion maps ϵ to (ϵ, ϵ^d) and the surjection maps (φ, λ) to $\lambda S_{[d]}(\varphi)$.

Proof The only nontrivial part is surjectivity of $\mathrm{O}(V) \times K^\times \to N(V, d)$. Denote by

$$\sigma\colon \mathrm{O}(V) \to N(V, d), \quad \varphi \mapsto S_{[d]}(\varphi),$$

the restriction of this map to the first component.

The representation $S_{[d]}V$ of $\mathfrak{so}(V)$ is irreducible, so by Schur's lemma the centralizer of $\mathfrak{so}(V)$ in $\mathrm{GL}(S_{[d]}V)$ is $K^\times$, and we have an exact sequence

$$1 \to K^\times \to N(V,d) \xrightarrow{\psi} \mathrm{Aut}(\mathfrak{so}(V)).$$

It therefore suffices to show that the image of ψ equals the image of $\psi \circ \sigma$.

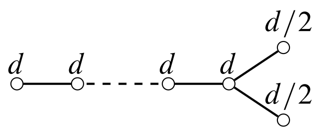
The adjoint group of $\mathfrak{so}(V)$ is $\mathrm{PSO}(V)$, and we have a short exact sequence

$$1 \to \mathrm{PSO}(V) \to \mathrm{Aut}(\mathfrak{so}(V)) \to \mathrm{Out}(\mathfrak{so}(V)) \to 1, \tag{5}$$

where $\mathrm{Out}(\mathfrak{so}(V))$ coincides with the group of symmetries of the Dynkin diagram.

If $\dim V = 2n+1$, then we have $\mathrm{PSO}(V) = \mathrm{SO}(V)$. The Dynkin diagram (of type B_n) has no nontrivial automorphisms, so $\mathrm{Aut}(\mathfrak{so}(V,b)) = \mathrm{SO}(V)$. The composition $\psi \circ \sigma$ maps $\mathrm{SO}(V)$ identically to $\mathrm{SO}(V)$, and we conclude that the image of ψ is the image of $\psi \circ \sigma$.

Now assume $\dim V = 2n$. Since K is algebraically closed, $\mathrm{PSO}(V) = \mathrm{SO}(V)/\{\pm 1\}$. The larger group $\mathrm{O}(V)/\{\pm 1\}$ embeds in $\mathrm{Aut}\,\mathfrak{so}(V)$, with elements of determinant -1 in $\mathrm{O}(V)$ inducing the reflection in the horizontal axis in the Dynkin diagram (of type D_n). For $n \neq 4$, this inclusion is an equality, while for $n = 4$ "triality" gives extra automorphisms. However, expressed on simple roots the highest weight of the representation $S_{[d]}V$ of $\mathfrak{so}(V)$ is



such that for $n = 4$ the extra automorphisms of $\mathfrak{so}(V)$ do not lift to automorphisms of $S_{[d]}V$. We conclude that the image of ψ is contained in $\mathrm{O}(V)/\{\pm 1\}$ and that the composition $\psi \circ \sigma$ is the natural map $\mathrm{O}(V) \to \mathrm{O}(V)/\{\pm 1\}$, so also in this case the image of ψ coincides with the image of $\psi \circ \sigma$. □

Remark 4.2 The condition that K is algebraically closed is needed in the case of even $\dim V$. If K is not algebraically closed, then one still has the exact sequence (5), but one should be careful to define $\mathrm{PSO}(V)$ as the group of K–points of the algebraic group $\mathbf{PSO}(V)$ over K. In general, this group is bigger than $\mathrm{SO}(V)/\{\pm 1\}$). In particular, not every element of $N(V,d)$ can be lifted to $\mathrm{O}(V) \times K^\times$.

Proposition 4.3 *Let V_1 and V_2 be nondegenerate quadratic spaces over K. Assume that there is a linear isomorphism $f : S_{[d]}V_1 \to S_{[d]}V_2$ such that $f\,\mathfrak{so}(V_1)\,f^{-1} = \mathfrak{so}(V_2)$*

as subspaces of $\mathrm{End}(V_2)$. *Then there exists a* $\mu \in K^\times$ *and a similitude* $\varphi\colon V_1 \to V_2$ *such that* $f = \mu S_{[d]}(\varphi)$.

Proof Let $\overline{K}$ be a separable closure of K. Consider the $\mathrm{Gal}(\overline{K}/K)$–sets

$$S := \{\varphi\colon V_{1,\overline{K}} \to V_{2,\overline{K}} \mid \varphi \text{ is a similitude}\}$$

and

$$N := \{g\colon S_{[d]}V_{1,\overline{K}} \to S_{[d]}V_{2,\overline{K}} \mid g\,\mathfrak{so}(V_{1,\overline{K}})g^{-1} = \mathfrak{so}(V_{2,\overline{K}})\}$$

and the Galois-equivariant map

$$\xi\colon \overline{K}^\times \times S \to N, \quad (\mu,\varphi) \mapsto \mu S_{[d]}(\varphi).$$

The map ξ is surjective. Indeed, since over a separably closed field the quadratic spaces are isometric, we may assume without loss of generality that $V_1 = V_2$. Then $N = N(V_{1,\overline{K}}, d)$ and the surjectivity follows from Proposition 4.1 (it suffices even to consider isometries instead of similitudes).

The group $\overline{K}^\times$ acts on $\overline{K}^\times \times S$ by $\lambda(\mu,\varphi) := (\lambda^{-d}\mu, \lambda\varphi)$ and the fibers of ξ are principal homogenous spaces under this action.

The map f defines a Galois-invariant element $f \in N$, so its fiber $\xi^{-1}(f)$ carries a natural Galois action. By Hilbert 90, we have $\mathrm{H}^1(\mathrm{Gal}(\overline{K}/K, \overline{K}^\times) = \{1\}$, which implies that $\xi^{-1}(f)$ contains a Galois-invariant element (μ,φ). □

The bilinear form b on V induces a bilinear form $b_{[d]}$ on $S_{[d]}V$ defined as

$$b_{[d]}(x_1\cdots x_d, y_1\cdots y_d) := (-1)^d \sum_{\sigma\in S_n} \prod_i b(x_i, y_{\sigma i}),$$

Consider the group

$$G(V,d) := N(V,d) \cap \mathrm{O}(S_{[d]}, b_{[d]})$$

of isometries of $S_{[d]}V$ that preserve the subspace $\mathfrak{so}(V)$ of $\mathrm{End}\, S_{[d]}V$.

Proposition 4.4 *If d is odd, then the map*

$$\mathrm{O}(V) \to G(V,d), \quad \varphi \mapsto S_{[d]}(\varphi),$$

is an isomorphism. If d is even and $\dim V$ *is odd, then the map*

$$\mathrm{O}(V) \to G(V,d), \quad \varphi \mapsto \det(\varphi) S_{[d]}(\varphi),$$

is an isomorphism.

Proof Assume first that K is separably closed. The short exact sequence of Proposition 4.1 restricts to a short exact sequence

$$1 \to \{\pm 1\} \to \mathrm{O}(V) \times \{\pm 1\} \to G(V,d) \to 1,$$

from which one verifies directly that the given maps are isomorphisms. If K is not separably closed, then the result follows from taking Galois invariants. □

Remark 4.5 If both d and $\dim V$ are even, one obtains

$$G(V_{\overline{K}}, d) \cong \mathrm{O}(V_{\overline{K}})/\{\pm 1\} \times \{\pm 1\}.$$

Note, however, that in general there are more Galois-invariant elements than just those in $\mathrm{O}(V)/\{\pm 1\}$. See also Remark 4.2.

4.2 The Verbitsky component

Theorem 4.6 *Let X_1 and X_2 be hyperkähler varieties and $\Phi\colon \mathcal{D}X_1 \to \mathcal{D}X_2$ an equivalence. Then the induced isomorphism $\Phi^{\mathrm{H}}\colon \mathrm{H}(X_1,\mathbb{Q}) \to \mathrm{H}(X_2,\mathbb{Q})$ restricts to an isomorphism $\Phi^{\mathrm{SH}}\colon \mathrm{SH}(X_1,\mathbb{Q}) \to \mathrm{SH}(X_2,\mathbb{Q})$. Moreover:*

(i) Φ^{SH} *is an isometry with respect to the Mukai pairings.*

(ii) $\Phi^{\mathrm{SH}}\mathfrak{g}(X_1)(\Phi^{\mathrm{SH}})^{-1} = \mathfrak{g}(X_2)$ *in* $\mathrm{End}(\mathrm{SH}(X_2,\mathbb{Q}))$.

Proof Note that $\mathrm{SH}(X,\mathbb{Q})$ can be characterized as the minimal sub-$\mathfrak{g}(X)$–module of $\mathrm{H}(X,\mathbb{Q})$ whose Hodge structure attains the maximal possible level (width). It then follows from Theorem A and from Lemma 3.4 that Φ^{H} restricts to an isomorphism

$$\Phi^{\mathrm{SH}}\colon \mathrm{SH}(X_1,\mathbb{Q}) \xrightarrow{\sim} \mathrm{SH}(X_2,\mathbb{Q})$$

respecting the Lie algebras $\mathfrak{g}(X_1)$ and $\mathfrak{g}(X_2)$. By [14], the map Φ^{H} respects the Mukai pairings, and the theorem follows. □

Definition 4.7 For a complex hyperkähler variety we equip $\mathrm{SH}(X,\mathbb{Q})$ and $\widetilde{\mathrm{H}}(X,\mathbb{Q})$ with Hodge structures of weight 0, given by

$$\mathrm{SH}(X,\mathbb{Q}) \subset \mathrm{H}^{\mathrm{ev}}(X,\mathbb{Q}) = \bigoplus_n \mathrm{H}^{2n}(X,\mathbb{Q}(n))$$

and

$$\widetilde{\mathrm{H}}(X,\mathbb{Q}) = \mathbb{Q}\alpha \oplus \mathrm{H}^2(X,\mathbb{Q}(1)) \oplus \mathbb{Q}\beta.$$

Lemma 4.8 *Let X be a hyperkähler variety of dimension $2d$. Then the map*

$$\Psi\colon \mathrm{SH}(X,\mathbb{Q}) \to \mathrm{Sym}^d\, \widetilde{\mathrm{H}}(X,\mathbb{Q})$$

of Proposition 3.5 is a morphism of Hodge structures of weight 0.

Proof One verifies directly that the "action map"

$$\mathrm{H}^2(X,\mathbb{Q}(1))\otimes\widetilde{\mathrm{H}}(X,\mathbb{Q})\to\widetilde{\mathrm{H}}(X,\mathbb{Q}),$$

which maps (λ,x) to $e_\lambda(x)$ is a map of Hodge structures. From this it follows that the action map

$$\mathrm{H}^2(X,\mathbb{Q}(1))\otimes\mathrm{Sym}^d\,\widetilde{\mathrm{H}}(X,\mathbb{Q})\to\mathrm{Sym}^d\,\widetilde{\mathrm{H}}(X,\mathbb{Q})$$

is a map of Hodge structures, and that the map

$$\widetilde{\Psi}\colon \mathrm{Sym}^\bullet\,\mathrm{H}(X,\mathbb{Q}(1))\to\mathrm{Sym}^d\,\widetilde{\mathrm{H}}(X,\mathbb{Q})$$

from the proof of Proposition 3.5 is a morphism of Hodge structures.

Since multiplication in the cohomology of X preserves the Hodge structure, the quotient map $\mathrm{Sym}^\bullet\,\mathrm{H}(X,\mathbb{Q}(1))\to\mathrm{SH}(X,\mathbb{Q})$ is also a morphism of Hodge structures, and hence so is the map Ψ constructed in the proof of Proposition 3.5. □

Proposition 4.9 *Let X_1 and X_2 be derived equivalent hyperkähler varieties. Then there exists a Hodge similitude $\varphi\colon\widetilde{\mathrm{H}}(X_1,\mathbb{Q})\xrightarrow{\sim}\widetilde{\mathrm{H}}(X_2,\mathbb{Q})$ and a scalar $\lambda\in\mathbb{Q}^\times$ such that the square*

$$\begin{array}{ccc}\mathrm{SH}(X_1,\mathbb{Q}) & \xrightarrow{\Phi^{\mathrm{SH}}} & \mathrm{SH}(X_2,\mathbb{Q})\\ \big\downarrow{\scriptstyle\Psi} & & \big\downarrow{\scriptstyle\Psi}\\ \mathrm{Sym}^d\,\widetilde{\mathrm{H}}(X_1,\mathbb{Q}) & \xrightarrow{\lambda\,\mathrm{Sym}^d(\varphi)} & \mathrm{Sym}^d\,\widetilde{\mathrm{H}}(X_2,\mathbb{Q})\end{array}$$

commutes.

Proof Recall from Lemma 3.7 that the image of Ψ is precisely $S_{[d]}\widetilde{\mathrm{H}}\subset\mathrm{Sym}^d\,\widetilde{\mathrm{H}}$. It then follows from Theorem 4.6 and Proposition 4.3 that there exists a similitude φ and a scalar λ that make the square commute.

It remains to check that φ respects the Hodge structures. The Hodge structure on $\widetilde{\mathrm{H}}(X_i,\mathbb{Q})$ is given by a morphism $h_i:\mathbb{C}^\times\to\mathrm{O}(\widetilde{\mathrm{H}}(X_i,\mathbb{R}))$, and the preceding lemma implies that the Hodge structure on $\mathrm{SH}(X_i,\mathbb{Q})$ is given by composing h_i with the injective map $\mathrm{O}(\widetilde{\mathrm{H}}(X_i,\mathbb{R}))\to\mathrm{GL}(\mathrm{SH}(X_i,\mathbb{R}))$. Since φ maps the Hodge structure on $\mathrm{SH}(X_1,\mathbb{Q})$ to the Hodge structure on $\mathrm{SH}(X_2,\mathbb{Q})$, we conclude that φ maps h_1 to h_2. □

Theorem 4.10 (d odd) *Assume that d is odd, and that X_1 and X_2 are deformation-equivalent hyperkähler varieties of dimension $2d$. Let $\Phi\colon\mathcal{D}X_1\xrightarrow{\sim}\mathcal{D}X_2$ be an* *equivalence. Then there is a unique Hodge isometry* $\Phi^{\widetilde{\mathrm{H}}}$ *making the square*

$$\begin{array}{ccc} \mathrm{SH}(X_1,\mathbb{Q}) & \xrightarrow{\Phi^{\mathrm{SH}}} & \mathrm{SH}(X_2,\mathbb{Q}) \\ \downarrow{\scriptstyle\Psi} & & \downarrow{\scriptstyle\Psi} \\ \mathrm{Sym}^d\,\widetilde{\mathrm{H}}(X_1,\mathbb{Q}) & \xrightarrow{\mathrm{Sym}^d(\Phi^{\widetilde{\mathrm{H}}})} & \mathrm{Sym}^d\,\widetilde{\mathrm{H}}(X_2,\mathbb{Q}) \end{array}$$

commute. The formation of $\Phi^{\widetilde{\mathrm{H}}}$ *is functorial in* Φ*.*

Proof Since X_1 and X_2 are deformation equivalent, we can choose an isometry $\varphi\colon \widetilde{\mathrm{H}}(X_1,\mathbb{Q}) \xrightarrow{\sim} \widetilde{\mathrm{H}}(X_2,\mathbb{Q})$. Moreover, X_1 and X_2 have the same Fujiki constant, so $\mathrm{Sym}^d\,\varphi$ restricts to an isometry between the images of Ψ. Then by Theorem 4.6 and Proposition 4.4, there is a unique isometry $\psi \in \mathrm{O}(\widetilde{\mathrm{H}}(X_2,\mathbb{Q}))$ such that $\Phi^{\widetilde{\mathrm{H}}} := \psi\varphi$ makes the square commute. Uniqueness forces its formation to be functorial.

That $\Phi^{\widetilde{\mathrm{H}}}$ respects the Hodge structures follows from the same argument as in the proof of Proposition 4.9. □

If d is even, then both existence and uniqueness of $\Phi^{\widetilde{\mathrm{H}}}$ in the statement of Theorem 4.10 fail. However, if we moreover assume that $b_2(X)$ is odd, then one can use the description of $G(V,d)$ from Proposition 4.4 to salvage this, at the cost of keeping track of a determinant character.

Define an *orientation* on X to be the choice of a generator of $\det \mathrm{H}^2(X,\mathbb{R})$, up to $\mathbb{R}^\times_{>0}$. Equivalently, an orientation is the choice of generator of $\det \widetilde{\mathrm{H}}(X,\mathbb{R})$ up to $\mathbb{R}^\times_{>0}$. Define the *sign* $\epsilon(\varphi)$ of a Hodge isometry $\varphi\colon \widetilde{\mathrm{H}}(X_1,\mathbb{Q}) \xrightarrow{\sim} \widetilde{\mathrm{H}}(X_2,\mathbb{Q})$ as $\epsilon(\varphi) = 1$ if φ respects the orientations and $\epsilon(\varphi) = -1$ otherwise. A derived equivalence between oriented hyperkähler varieties is a derived equivalence between the underlying unoriented hyperkähler varieties.

Theorem 4.11 (d even) *Assume that* d *is even, and that* $\Phi\colon \mathcal{D}X_1 \xrightarrow{\sim} \mathcal{D}X_2$ *is a derived equivalence between oriented hyperkähler varieties of dimension* $2d$*. Assume that* X_1 *and* X_2 *have odd* b_2*, and that the quadratic spaces* $\mathrm{H}^2(X_1,\mathbb{Q})$ *and* $\mathrm{H}^2(X_2,\mathbb{Q})$ *are isometric. Then there exists a unique Hodge isometry* $\Phi^{\widetilde{\mathrm{H}}}$ *making the square*

$$\begin{array}{ccc} \mathrm{SH}(X_1,\mathbb{Q}) & \xrightarrow{\epsilon(\Phi^{\widetilde{\mathrm{H}}})\Phi^{\mathrm{SH}}} & \mathrm{SH}(X_2,\mathbb{Q}) \\ \downarrow{\scriptstyle\Psi} & & \downarrow{\scriptstyle\Psi} \\ \mathrm{Sym}^d\,\widetilde{\mathrm{H}}(X_1,\mathbb{Q}) & \xrightarrow{\mathrm{Sym}^d(\Phi^{\widetilde{\mathrm{H}}})} & \mathrm{Sym}^d\,\widetilde{\mathrm{H}}(X_2,\mathbb{Q}) \end{array}$$

commute. Moreover, the formation of $\Phi^{\tilde{\mathrm{H}}}$ *is functorial for composition of derived equivalences between hyperkähler varieties equipped with orientations.*

Proof The argument is quite similar to the proof of Theorem 4.10. Choose an isometry $\varphi\colon \tilde{\mathrm{H}}(X_1,\mathbb{Q}) \xrightarrow{\sim} \tilde{\mathrm{H}}(X_2,\mathbb{Q})$. Because the dimension of $\tilde{\mathrm{H}}(X_i,\mathbb{Q})$ is odd, we may replace φ with $-\varphi$ if necessary to ensure that φ respects the orientations, and hence we may assume $\epsilon(\varphi)=1$. The map φ induces an isometry $\mathrm{Sym}^d\,\varphi$, which restricts to an isometry $\varphi^{\mathrm{SH}}\colon \mathrm{SH}(X_1,\mathbb{Q}) \to \mathrm{SH}(X_2,\mathbb{Q})$.

By Theorem 4.6, there is a $\psi \in G(\tilde{\mathrm{H}}(X_2,\mathbb{Q}),d)$ such that $\Phi^{\mathrm{SH}} = \psi\circ\varphi^{\mathrm{SH}}$, and by Proposition 4.4, we have that $\psi = \det(\psi_0)S_{[d]}(\psi_0)$ for a unique $\psi_0\in \mathrm{O}(\tilde{\mathrm{H}}(X_2,\mathbb{Q}))$. Now take $\Phi^{\tilde{\mathrm{H}}} := \psi_0\circ\varphi$. Then $\epsilon(\Phi^{\tilde{\mathrm{H}}}) = \det(\psi_0)$ and $\mathrm{Sym}^d(\Phi^{\tilde{\mathrm{H}}})$ lifts to the map $\det(\psi_0)^{-1}\psi\circ\varphi^{\mathrm{SH}} = \epsilon(\Phi^{\tilde{\mathrm{H}}})\Phi^{\mathrm{SH}}$ as claimed.

Proposition 4.4 forces $\Phi^{\tilde{\mathrm{H}}}$ to be unique, and this implies the functoriality for composition. Compatibility with Hodge structures follows from the same argument as in the proof of Proposition 4.9. □

Remark 4.12 If X_1 and X_2 are hyperkähler varieties belonging to one of the known families, and if $\Phi\colon \mathcal{D}X_1 \xrightarrow{\sim} \mathcal{D}X_2$ is an equivalence, then the hypotheses of either Theorem 4.10 or Theorem 4.11 are satisfied. Indeed, X_1 and X_2 will have the same dimension $2d$ and because they have isomorphic LLV Lie algebra, they have the same second Betti number b_2. Going through the list of known families, one sees that this implies that X_1 and X_2 are deformation equivalent. In particular, they have isometric H^2. Finally, all known hyperkähler varieties of dimension $2d$ with d even have odd b_2.

Taking $X_1 = X_2$ in Theorems 4.10 and 4.11 yields Theorem C from the introduction:

Theorem 4.13 *Let X be a hyperkähler variety of dimension $2d$. Assume that either d is odd or that d is even and $b_2(X)$ is odd. Then the representation*

$$\rho^{\mathrm{SH}}\colon \mathrm{Aut}\,\mathcal{D}(X) \to \mathrm{GL}(\mathrm{SH}(X,\mathbb{Q}))$$

factors over a map $\rho^{\tilde{\mathrm{H}}}\colon \mathrm{Aut}\,\mathcal{D}(X) \to \mathrm{O}(\tilde{\mathrm{H}}(X,\mathbb{Q}))$.

Remark 4.14 For d odd, the implicit map $\mathrm{O}(\tilde{\mathrm{H}}(X,\mathbb{Q})) \to \mathrm{GL}(\mathrm{SH}(X,\mathbb{Q}))$ is the natural map coming from the isomorphism $\mathrm{SH}(X,\mathbb{Q}) \cong S_{[d]}\tilde{\mathrm{H}}(X,\mathbb{Q}))$. For d even (and b_2 odd), it is the twist of the natural map with the determinant character

$$\mathrm{O}(\tilde{\mathrm{H}}(X,\mathbb{Q})) \to \{\pm 1\}.$$

5 Hodge structures

In this section we prove Theorem D from the introduction.

For a nondegenerate quadratic space V over $\mathbb{Q}$ we will make use of the algebraic groups $\mathbf{SO}(V)$, $\mathbf{Spin}(V)$, and $\mathbf{GSpin}(V)$ (sometimes denoted $\mathbf{CSpin}(V)$) over $\mathbb{Q}$. These groups sit in a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_2 & \longrightarrow & \mathbf{Spin}(V) & \longrightarrow & \mathbf{SO}(V) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbf{G}_m & \longrightarrow & \mathbf{GSpin}(V) & \longrightarrow & \mathbf{SO}(V) & \longrightarrow & 1 \end{array} \tag{6}$$

from which one deduces an exact sequence

$$1 \to \mu_2 \to \mathbf{G}_m \times \mathbf{Spin}(V) \to \mathbf{GSpin}(V) \to 1, \tag{7}$$

where the first map is the diagonal embedding $\epsilon \mapsto (\epsilon, \epsilon)$. Alternatively, one can use (7) as the definition of **GSpin**, and deduce the existence of the above commutative diagram.

We will write $\mathrm{SO}(V)$, $\mathrm{Spin}(V)$, and $\mathrm{GSpin}(V)$ for the groups of $\mathbb{Q}$–points of these algebraic groups. Note that the above exact sequences of algebraic groups need not induce exact sequences of groups of $\mathbb{Q}$–points, and the obstruction can be described in terms of Galois cohomology. The sequence for the Spin–cover of $\mathbf{SO}(V)$ induces an exact sequence

$$1 \to \{\pm 1\} \to \mathrm{Spin}(V) \to \mathrm{SO}(V) \to \mathrm{H}^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \{\pm 1\}) = \mathbb{Q}^\times/(\mathbb{Q}^\times)^2,$$

where the connecting homomorphism $\mathrm{SO}(V) \to \mathbb{Q}^\times/(\mathbb{Q}^\times)^2$ is the spinor norm. By Hilbert 90, we have $\mathrm{H}^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \overline{\mathbb{Q}}^\times) = \{1\}$ and the analogous sequence for the GSpin–cover does induce a short exact sequence

$$1 \to \mathbb{Q}^\times \to \mathrm{GSpin}(V) \to \mathrm{SO}(V) \to 1. \tag{8}$$

This will be used crucially in the proof of Theorem D.

Lemma 5.1 *Let X be a hyperkähler variety of dimension $2d$. There exists a unique action of $\mathbf{GSpin}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$ on $\mathrm{H}(X,\mathbb{Q})$ such that*

(i) *the action of $\mathbf{Spin}(\widetilde{\mathrm{H}}(X,\mathbb{Q})) \subset \mathbf{GSpin}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$ integrates the action of $\mathfrak{g}(X) = \mathfrak{so}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$;*

(ii) *a section $\lambda \in \mathbf{G}_m \subset \mathbf{GSpin}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$ acts as λ^{i-2d} on $\mathrm{H}^i(X,\mathbb{Q})$.*

Proof The action of $\mathfrak{so}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$ integrates to an action of the simply connected algebraic group $\mathbf{Spin}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$. This commutes with the action of $\mathbf{G}_m$ for which λ acts as λ^{i-2d} on $\mathrm{H}^i(X,\mathbb{Q})$, and we obtain an action of $\mathbf{G}_m \times \mathbf{Spin}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$ on $\mathrm{H}(X,\mathbb{Q})$. The lemma claims that this descends to an action of the quotient group $\mathbf{GSpin}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$.

By (7) it suffices to verify that the kernel μ_2 acts trivially, ie that $-1 \in \mathrm{Spin}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$ acts as $(-1)^i$ on $\mathrm{H}^i(X,\mathbb{Q})$. Any $\mathfrak{sl}_2$–triple $(e_\lambda, h, f_\lambda)$ in $\mathfrak{g}(X)$ induces an algebraic subgroup $\mathbf{SL}_2 \subset \mathbf{Spin}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$ with the property that $\mathrm{diag}(\mu,\mu^{-1}) \in \mathrm{SL}_2(\mathbb{Q})$ acts as μ^i on $\mathrm{H}^{2d+i}(X,\mathbb{Q})$. It follows that $\mathrm{diag}(-1,-1)$ must be mapped to the nontrivial central element $-1 \in \mathrm{Spin}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$, and that -1 acts as $(-1)^i$ on $\mathrm{H}^i(X,\mathbb{Q})$. □

Recall from Definition 4.7 that we have equipped $\widetilde{\mathrm{H}}(X,\mathbb{Q})$ and $\mathrm{H}^{\mathrm{ev}}(X,\mathbb{Q})$ with Hodge structures of weight 0. Similarly, we equip the odd cohomology of X with a Hodge structure of weight 1,

$$\mathrm{H}^{\mathrm{odd}}(X,\mathbb{Q}) := \bigoplus_i \mathrm{H}^{2i+1}(X,\mathbb{Q}(i)).$$

Lemma 5.2 *Let $g \in \mathrm{GSpin}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$. If the action of g on $\widetilde{\mathrm{H}}(X,\mathbb{Q})$ respects the Hodge structure, then so does its action on $\mathrm{H}^{\mathrm{ev}}(X,\mathbb{Q})$ and on $\mathrm{H}^{\mathrm{odd}}(X,\mathbb{Q})$.*

Proof This follows immediately from the fact that the Hodge structure is determined by the action of $h' \in \mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$ (see Section 2.3), and from the faithfulness of the $\mathfrak{g}(X)$–module $\widetilde{\mathrm{H}}(X,\mathbb{Q})$. □

Theorem 5.3 *Let X_1 and X_2 be hyperkähler varieties, and let $\Phi\colon \mathcal{D}X_1 \xrightarrow{\sim} \mathcal{D}X_2$ be an equivalence. Then for every i the $\mathbb{Q}$–Hodge structures $\mathrm{H}^i(X_1,\mathbb{Q})$ and $\mathrm{H}^i(X_2,\mathbb{Q})$ are isomorphic.*

Proof Consider the Lie algebra isomorphism $\Phi^{\mathfrak{g}}\colon \mathfrak{g}(X_1) \xrightarrow{\sim} \mathfrak{g}(X_2)$ from Theorem A. By Proposition 4.9, there exists a Hodge similitude $\phi\colon \widetilde{\mathrm{H}}(X_1,\mathbb{Q}) \xrightarrow{\sim} \widetilde{\mathrm{H}}(X_2,\mathbb{Q})$ such that the square

$$\begin{array}{ccc} \mathfrak{so}(\widetilde{\mathrm{H}}(X_1,\mathbb{Q})) & \xrightarrow{\mathrm{Ad}(\phi)} & \mathfrak{so}(\widetilde{\mathrm{H}}(X_2,\mathbb{Q})) \\ \downarrow & & \downarrow \\ \mathfrak{g}(X_1) & \xrightarrow{\Phi^{\mathfrak{g}}} & \mathfrak{g}(X_2) \end{array}$$

commutes. Here the vertical maps are the isomorphisms from Theorem 3.1.

The K3–type Hodge structure $\widetilde{\mathrm{H}}(X_2,\mathbb{Q})$ decomposes as $N \oplus T$, with N and T its algebraic and transcendental parts, respectively. The Hodge similitude ϕ maps the distinguished elements α_1 and β_1 of $\widetilde{\mathrm{H}}(X_1,\mathbb{Q})$ to N. By Witt cancellation, there exists a $\psi_N \in \mathrm{SO}(N)$ and $\lambda,\mu \in \mathbb{Q}^\times$ such that $\psi_N\phi(\alpha_1) = \lambda\alpha_2$ and $\psi_N\phi(\beta_1) = \mu\beta_2$. Extending by the identity, we find a Hodge isometry $\psi \in \mathrm{SO}(\widetilde{\mathrm{H}}(X_2,\mathbb{Q}))$ such that $\psi\phi\colon \widetilde{\mathrm{H}}(X_1,\mathbb{Q}) \xrightarrow{\sim} \widetilde{\mathrm{H}}(X_2,\mathbb{Q})$ is a *graded* Hodge similitude. In particular, the induced map $\psi\phi\colon \mathfrak{g}(X_1) \xrightarrow{\sim} \mathfrak{g}(X_2)$ is graded, and $\psi\phi$ maps the grading element $h_1 \in \mathfrak{g}(X_1)$ to the grading element $h_2 \in \mathfrak{g}(X_2)$.

By (8) the element ψ lifts to an element $\tilde\psi \in \mathrm{GSpin}(\widetilde{\mathrm{H}}(X_2,\mathbb{Q}))$, which by Lemma 5.1 and Lemma 5.2 induces automorphisms of the Hodge structures $\mathrm{H}^{\mathrm{ev}}(X_2,\mathbb{Q})$ and $\mathrm{H}^{\mathrm{odd}}(X_2,\mathbb{Q})$. Now, by construction, the composition $\tilde\psi \circ \Phi^{\mathrm{H}}$ defines isomorphisms

$$\tilde\psi \circ \Phi^{\mathrm{H}}\colon \mathrm{H}^{\mathrm{ev}}(X_1,\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^{\mathrm{ev}}(X_2,\mathbb{Q}), \quad \tilde\psi \circ \Phi^{\mathrm{H}}\colon \mathrm{H}^{\mathrm{odd}}(X_1,\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^{\mathrm{odd}}(X_2,\mathbb{Q}),$$

which respect both the grading and the Hodge structure, so they induce isomorphisms of Hodge structures $\mathrm{H}^i(X_1,\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^i(X_2,\mathbb{Q})$, for all i. □

6 Topological K–theory

6.1 Topological K–theory and the Mukai vector

We now briefly recall some basic properties of topological K–theory of projective algebraic varieties. See [1; 3; 4] for more details.

For every smooth and projective X over $\mathbb{C}$ we have a $\mathbb{Z}/2\mathbb{Z}$–graded abelian group

$$\mathrm{K}_{\mathrm{top}}(X) := \mathrm{K}^0_{\mathrm{top}}(X) \oplus \mathrm{K}^1_{\mathrm{top}}(X).$$

This is functorial for pullback and proper pushforward, and carries a product structure. The group $\mathrm{K}^0_{\mathrm{top}}(X)$ is the Grothendieck group of topological vector bundles on the differentiable manifold X^{an}. Pullback agrees with pullback of vector bundles, and the product structure agrees with the tensor product of vector bundles.

By [3, Section 1.10], the Chern character can be extended to odd degree, inducing a $\mathbb{Z}/2\mathbb{Z}$–graded map

$$v_X^{\mathrm{top}}\colon \mathrm{K}_{\mathrm{top}}(X) \to \mathrm{H}(X,\mathbb{Q}),$$

given by $v_X^{\mathrm{top}}(\mathcal{F}) = \sqrt{\mathrm{Td}_X} \cdot \mathrm{ch}(\mathcal{F})$. The image of v_X^{top} is a $\mathbb{Z}$–lattice of full rank.

There is a “forgetful” map $\mathrm{K}^0(X) \to \mathrm{K}_{\mathrm{top}}(X)$ from the Grothendieck group of algebraic vector bundles (or equivalently of the triangulated category $\mathcal{D}X$). This is compatible with pullback, multiplication, and proper pushforward. The Mukai vector

$$v_X : \mathrm{K}^0(X) \to \mathrm{H}(X, \mathbb{Q})$$

factors over v_X^{top}.

If $\mathcal{P}$ is an object in $\mathcal{D}(X \times Y)$ then convolution with its class in $\mathrm{K}^0_{\mathrm{top}}(X \times Y)$ defines a map $\Phi^{\mathrm{K}}_{\mathcal{P}} \colon \mathrm{K}_{\mathrm{top}}(X) \to \mathrm{K}_{\mathrm{top}}(Y)$, in such a way that the diagram

$$\begin{array}{ccccc} \mathrm{K}^0(X) & \longrightarrow & \mathrm{K}_{\mathrm{top}}(X) & \xrightarrow{v_X^{\mathrm{top}}} & \mathrm{H}(X, \mathbb{Q}) \\ \big\downarrow \Phi_{\mathcal{P}} & & \big\downarrow \Phi^{\mathrm{K}}_{\mathcal{P}} & & \big\downarrow \Phi^{\mathrm{H}}_{\mathcal{P}} \\ \mathrm{K}^0(Y) & \longrightarrow & \mathrm{K}_{\mathrm{top}}(Y) & \xrightarrow{v_Y^{\mathrm{top}}} & \mathrm{H}(Y, \mathbb{Q}). \end{array}$$

commutes.

6.2 Equivariant topological K–theory

The above formalism largely generalizes to an equivariant setting. Again, we briefly recall the most important properties; see [5; 6; 28; 42] for more details.

If X is a smooth projective complex variety equipped with an action of a finite group G, we denote by $\mathrm{K}^0_G(X)$ the Grothendieck group of G–equivariant algebraic vector bundles on X, or equivalently the Grothendieck group of the bounded derived category $\mathcal{D}_G X$ of G–equivariant coherent $\mathcal{O}_X$–modules. This is functorial for pullback along G–equivariant maps and pushforward along G–equivariant proper maps.

Similarly, we have the G–equivariant topological K–theory

$$\mathrm{K}_{\mathrm{top},G}(X) := \mathrm{K}^0_{\mathrm{top},G}(X) \oplus \mathrm{K}^1_{\mathrm{top},G}(X),$$

where $\mathrm{K}^0_{\mathrm{top},G}(X)$ is the Grothendieck group of topological G–equivariant vector bundles.

There is a natural map $\mathrm{K}^0_G(X) \to \mathrm{K}^0_{\mathrm{top},G}(X)$ compatible with pullback and tensor product. If $f : X \to Y$ is *proper* and G–equivariant, then we have a pushforward map $f_* \colon \mathrm{K}_{\mathrm{top},G}(X) \to \mathrm{K}_{\mathrm{top},G}(Y)$. There is a Riemann–Roch theorem [5; 28], stating that the square

$$\begin{array}{ccc} \mathrm{K}^0_G(X) & \longrightarrow & \mathrm{K}_{\mathrm{top},G}(X) \\ \big\downarrow f_* & & \big\downarrow f_* \\ \mathrm{K}^0_G(Y) & \longrightarrow & \mathrm{K}_{\mathrm{top},G}(Y) \end{array}$$

commutes.

Now assume that we have a finite group G acting on X, and a finite group H acting on Y. If $\mathcal{P}$ is an object in $\mathcal{D}_{G\times H}(X\times Y)$, then convolution with $\mathcal{P}$ induces a functor $\Phi_{\mathcal{P}}\colon \mathcal{D}_G X \to \mathcal{D}_H Y$, see [40] for more details. Similarly, convolution with the class of $\mathcal{P}$ in $\mathrm{K}^0_{\mathrm{top},G\times H}(X\times Y)$ induces a map $\Phi^{\mathrm{K}}_{\mathcal{P}}\colon \mathrm{K}_{\mathrm{top},G}(X)\to \mathrm{K}_{\mathrm{top},H}(Y)$. These satisfy the usual Fourier–Mukai calculus, and moreover they are compatible in the sense that the square

$$\begin{array}{ccc} \mathrm{K}^0_G(X) & \longrightarrow & \mathrm{K}_{\mathrm{top},G}(X) \\ \downarrow{\scriptstyle \Phi_{\mathcal{P}}} & & \downarrow{\scriptstyle \Phi^{\mathrm{K}}_{\mathcal{P}}} \\ \mathrm{K}^0_H(Y) & \longrightarrow & \mathrm{K}_{\mathrm{top},H}(Y) \end{array}$$

commutes.

7 Cohomology of the Hilbert square of a K3 surface

Let S be a K3 surface and $X = S^{[2]}$ its Hilbert square. In the coming few paragraphs we recall the structure of the cohomology of X in terms of the cohomology of S. See [7; 17; 23] for more details.

7.1 Line bundles on the Hilbert square

Let $G=\{1,\sigma\}$ be the group of order two, acting on $S\times S$ by permuting the factors. The Hilbert square X sits in a diagram

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow^{p} & & \searrow^{q} & \\ S\times S & & & & X \end{array}$$

where $p\colon Z\to S\times S$ is the blow-up along the diagonal, and where $q\colon Z\to X$ is the quotient map for the natural action of G on Z. Denote by $R\subset Z$ the exceptional divisor of p. Then R equals the ramification locus of q. We have $q_*\mathcal{O}_Z = \mathcal{O}_X\oplus\mathcal{E}$ for some line bundle $\mathcal{E}$, and $q^*\mathcal{E}\cong\mathcal{O}_Z(-R)$.

If $\mathcal{L}$ is a line bundle on S then

$$\mathcal{L}_2 := (q_* p^*(\mathcal{L}\boxtimes\mathcal{L}))^G$$

is a line bundle on X. The map

$$\mathrm{Pic}(S)\oplus\mathbb{Z}\to\mathrm{Pic}(X),\quad (\mathcal{L},n)\mapsto \mathcal{L}_2\otimes\mathcal{E}^{\otimes n},$$

is an isomorphism.

7.2 Cohomology of the Hilbert square

There is an isomorphism

$$\mathrm{H}^2(S,\mathbb{Z})\oplus\mathbb{Z}\delta \xrightarrow{\sim} \mathrm{H}^2(X,\mathbb{Z})$$

with the property that $c_1(\mathcal{L})$ is mapped to $c_1(\mathcal{L}_2)$, and δ is mapped to $c_1(\mathcal{E})$. We will use this isomorphism to identify $\mathrm{H}^2(S,\mathbb{Z})\oplus\mathbb{Z}\delta$ with $\mathrm{H}^2(X,\mathbb{Z})$. The Beauville–Bogomolov form on $\mathrm{H}^2(X,\mathbb{Z})$ satisfies

$$b_X(\lambda,\lambda)=b_S(\lambda,\lambda),\quad b_X(\lambda,\delta)=0,\quad b_X(\delta,\delta)=-2$$

for all $\lambda\in\mathrm{H}^2(S,\mathbb{Z})$.

The cup product defines an isomorphism $\mathrm{Sym}^2\,\mathrm{H}^2(X,\mathbb{Q})\xrightarrow{\sim}\mathrm{H}^4(X,\mathbb{Q})$. By Poincaré duality, there is a unique $q_X\in\mathrm{H}^4(X,\mathbb{Q})$ representing the Beauville–Bogomolov form, in the sense that

$$\int_X q_X\lambda_1\lambda_2=b_X(\lambda_1,\lambda_2) \tag{9}$$

for all $\lambda_1,\lambda_2\in\mathrm{H}^2(X,\mathbb{Z})$. Multiplication by q_X defines an isomorphism $\mathrm{H}^2(X,\mathbb{Q})\to\mathrm{H}^6(X,\mathbb{Q})$, and, for all $\lambda_1,\lambda_2,\lambda_3\in\mathrm{H}^2(X,\mathbb{Q})$,

$$\lambda_1\lambda_2\lambda_3=b_X(\lambda_1,\lambda_2)q_X\lambda_3+b_X(\lambda_2,\lambda_3)q_X\lambda_1+b_X(\lambda_3,\lambda_1)q_X\lambda_2 \tag{10}$$

in $\mathrm{H}^6(X,\mathbb{Q})$. Finally, for all $\lambda\in\mathrm{H}^2(X,\mathbb{Q})$ the Fujiki relation

$$\int_X\lambda^4=3b_X(\lambda,\lambda)^2 \tag{11}$$

holds.

7.3 Todd class of the Hilbert square

Proposition 7.1 $$\mathrm{Td}_X=1+\tfrac{5}{2}q_X+3[\mathrm{pt}].$$

Proof See also [23, Section 23.4]. Since the Todd class is invariant under the monodromy group of X, we necessarily have

$$\mathrm{Td}_X=1+sq_X+t[\mathrm{pt}]$$

for some $s,t\in\mathbb{Q}$. By Hirzebruch–Riemann–Roch, for every line bundle L on S with $c_1(L)=\lambda$,

$$\chi(X,L_2)=\int_X\mathrm{ch}(\lambda)\mathrm{Td}_X=\frac{1}{24}\int_X\lambda^4+\frac{s}{2}\int_X\lambda^2q_X+t.$$

By the relations (11) and (9), the right-hand side reduces to

$$\tfrac{1}{8}b(\lambda,\lambda)^2 + \tfrac{1}{2}sb(\lambda,\lambda) + t.$$

By [23, Section 23.4] or [17, 5.1], the left-hand side computes to

$$\chi(X, L_2) = \tfrac{1}{8}b(\lambda,\lambda)^2 + \tfrac{5}{4}b(\lambda,\lambda) + 3.$$

Comparing the two expressions yields the result. □

8 Derived McKay correspondence

8.1 The derived McKay correspondence

As in Section 7.1, we consider a K3 surface S, its Hilbert square $X = S^{[2]}$, the maps $p\colon Z \to S \times S$ and $q\colon Z \to X$, and the group $G = \{1, \sigma\}$ acting on $S \times S$ and Z.

The *derived McKay correspondence* [11] is the triangulated functor

$$\mathrm{BKR}\colon \mathcal{D}^b(X) \to \mathcal{D}^b_G(S \times S)$$

given as the composition

$$\mathrm{BKR}\colon \mathcal{D}X \xrightarrow{q^*} \mathcal{D}_G(Z) \xrightarrow{p_*} \mathcal{D}_G(S \times S),$$

where the first functor maps $\mathcal{F}$ to $q^*\mathcal{F}$ equipped with the trivial G–linearization. By [11, Theorem 1.1; 21, Theorem 5.1], the functor BKR is an equivalence of categories.

Its inverse has been described in [31, Section 4]. Denote by $j : Z \to S \times S \times X$ the G–equivariant closed immersion induced by p and q. The exceptional divisor $R \subset Z$ is G–invariant and hence defines a G–equivariant sheaf $\mathcal{O}(R)$, and a G–equivariant sheaf $\mathcal{Q} := j_*\mathcal{O}_Z(R)$ in $\mathcal{D}_G(S \times S \times X)$.

Proposition 8.1 *The inverse equivalence of* BKR *is given by the equivariant Fourier–Mukai transform with respect to $\mathcal{Q}$. It maps $\mathcal{F} \in \mathcal{D}_G(S \times S)$ to the object*

$$(q_* p^* \mathcal{F})^{\sigma=-1} \otimes \mathcal{E}^{-1}$$

of $\mathcal{D}(X)$.

Proof The first statement is [31, 4.1]. By the adjunction formula for $j : Z \to S \times S \to X$, this implies that $\mathcal{F}$ is mapped to $\big(q_*(p^*\mathcal{F} \otimes \mathcal{O}_Z(R))\big)^G \in \mathcal{D}(X)$. If we upgrade the line bundle $\mathcal{E}$ on X to a G–equivariant (for the trivial action on X) line bundle $\mathcal{E}_-$

by making σ act as -1, then $q^*\mathcal{E}_- \cong \mathcal{O}_Z(-R)$ as G–equivariant line bundles on Z. Applying the projection formula once more for the equivariant map q, we find

$$(q_*(p^*\mathcal{F} \otimes \mathcal{O}_Z(R)))^G \cong (q_* p^*\mathcal{F} \otimes \mathcal{E}_-^{-1})^G \cong (q_* p^*\mathcal{F})^{\sigma=-1} \otimes \mathcal{E}^{-1}. \qquad \square$$

Now let S_1 and S_2 be K3 surfaces with Hilbert squares X_1 and X_2. As was observed by Ploog [39], any equivalence $\Phi\colon \mathcal{D}S_1 \xrightarrow{\sim} \mathcal{D}S_2$ induces an equivalence

$$\mathcal{D}_G(S_1 \times S_2) \xrightarrow{\sim} \mathcal{D}_G(S_2 \times S_2),$$

and hence, via the derived McKay correspondence, an equivalence $\Phi^{[2]}\colon \mathcal{D}X_1 \xrightarrow{\sim} \mathcal{D}X_2$.

8.2 Topological K–theory of the Hilbert square

Theorem 8.2 *The composition*

$$\mathrm{BKR}_{\mathrm{top}}\colon \mathrm{K}_{\mathrm{top}}(X) \xrightarrow{q^*} \mathrm{K}_{\mathrm{top},G}(Z) \xrightarrow{p_*} \mathrm{K}_{\mathrm{top},G}(S \times S)$$

is an isomorphism.

Proof (See also [11, Section 10].) This is a purely formal consequence of the calculus of equivariant Fourier–Mukai transforms sketched in Section 6.2. The functor BKR and its inverse are given by kernels $\mathcal{P} \in \mathcal{D}_G(X \times S \times S)$ and $\mathcal{Q} \in \mathcal{D}_G(S \times S \times X)$. The map $\mathrm{BKR}_{\mathrm{top}}$ is given by convolution with the class of $\mathcal{P}$ in $\mathrm{K}^0_{\mathrm{top},G}(X \times S \times S)$. The identities in $\mathrm{K}^0(X \times X)$ and $\mathrm{K}^0_{G\times G}(S \times S \times S \times S)$ witnessing that $\mathcal{P}$ and $\mathcal{Q}$ are mutually inverse equivalences induce analogous identities in $\mathrm{K}^0_{\mathrm{top}}$. These show that convolution with the class of $\mathcal{Q}$ defines a two-sided inverse to $\mathrm{BKR}_{\mathrm{top}}$. $\square$

Consider the map

$$\psi^{\mathrm{K}}\colon \mathrm{K}^0_{\mathrm{top}}(X) \to \mathrm{K}^0_{\mathrm{top}}(S \times S)^G$$

obtained as the composition of $\mathrm{BKR}_{\mathrm{top}}$ and the forgetful map from $\mathrm{K}^0_{\mathrm{top},G}(S \times S)$ to $\mathrm{K}^0_{\mathrm{top}}(S \times S)$. Also, consider the map

$$\theta^{\mathrm{K}}\colon \mathrm{K}^0_{\mathrm{top}}(S) \to \mathrm{K}^0_{\mathrm{top}}(X), \quad [\mathcal{F}] \mapsto \mathrm{BKR}^{-1}_{\mathrm{top}}([\mathcal{F} \boxtimes \mathcal{F}, 1] - [\mathcal{F} \boxtimes \mathcal{F}, -1]),$$

where $[\mathcal{F} \boxtimes \mathcal{F}, \pm 1]$ denotes the class of the topological vector bundle $\mathcal{F} \boxtimes \mathcal{F}$ equipped with $\pm$ the natural G–linearization.

By construction, these maps are "functorial" in $\mathcal{D}S$, in the following sense:

Proposition 8.3 *If $\Phi\colon \mathcal{D}S_1 \xrightarrow{\sim} \mathcal{D}S_2$ is a derived equivalence between* K3 *surfaces, and $\Phi^{[2]}\colon \mathcal{D}X_1 \xrightarrow{\sim} \mathcal{D}X_2$ is the induced equivalence between their Hilbert squares, then the squares*

$$\begin{array}{ccc} \mathrm{K}^0_{\mathrm{top}}(X_1) & \xrightarrow{\psi^{\mathrm{K}}} & \mathrm{K}^0_{\mathrm{top}}(S_1\times S_1)^G \\ \downarrow{\Phi^{[2],\mathrm{K}}} & & \downarrow{\Phi^{\mathrm{K}}\otimes\Phi^{\mathrm{K}}} \\ \mathrm{K}^0_{\mathrm{top}}(X_2) & \xrightarrow{\psi^{\mathrm{K}}} & \mathrm{K}^0_{\mathrm{top}}(S_2\times S_2)^G \end{array} \qquad \begin{array}{ccc} \mathrm{K}^0_{\mathrm{top}}(S_1) & \xrightarrow{\theta^{\mathrm{K}}} & \mathrm{K}^0_{\mathrm{top}}(X_1) \\ \downarrow{\Phi^{\mathrm{K}}} & & \downarrow{\Phi^{[2],\mathrm{K}}} \\ \mathrm{K}^0_{\mathrm{top}}(S_2) & \xrightarrow{\theta^{\mathrm{K}}} & \mathrm{K}^0_{\mathrm{top}}(X_2) \end{array}$$

commute. □

Proposition 8.4 *The sequence*

$$0 \to \mathrm{K}^0_{\mathrm{top}}(S)\otimes_{\mathbb{Z}}\mathbb{Q} \xrightarrow{\theta^{\mathrm{K}}} \mathrm{K}^0_{\mathrm{top}}(X)\otimes_{\mathbb{Z}}\mathbb{Q} \xrightarrow{\psi^{\mathrm{K}}} \mathrm{K}^0_{\mathrm{top}}(S\times S)^G\otimes_{\mathbb{Z}}\mathbb{Q} \to 0$$

is exact.

Proof In the proof, we will implicitly identify $\mathrm{K}_{\mathrm{top},G}(S\times S)$ and $\mathrm{K}_{\mathrm{top}}(X)$.

Note that the map θ^{K} is additive. Indeed, let $\mathcal{F}_1$ and $\mathcal{F}_2$ be (topological) vector bundles on S. Then the cross term $\theta^{\mathrm{K}}[\mathcal{F}_1\oplus\mathcal{F}_2]-\theta^{\mathrm{K}}[\mathcal{F}_1]-\theta^{\mathrm{K}}[\mathcal{F}_2]$ computes to

$$\left[\mathcal{F}_1\boxtimes\mathcal{F}_2\oplus\mathcal{F}_2\boxtimes\mathcal{F}_1,\ \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\right]-\left[\mathcal{F}_1\boxtimes\mathcal{F}_2\oplus\mathcal{F}_2\boxtimes\mathcal{F}_1,\ \left(\begin{smallmatrix}0&-1\\-1&0\end{smallmatrix}\right)\right],$$

which vanishes because the matrices $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0&-1\\-1&0\end{smallmatrix}\right)$ are conjugated over $\mathbb{Z}$.

Next we observe that $\psi^{\mathrm{K}}\colon \mathrm{K}^0_{\mathrm{top}}(X)\otimes_{\mathbb{Z}}\mathbb{Q}\to\mathrm{K}^0_{\mathrm{top}}(S\times S)^G\otimes_{\mathbb{Z}}\mathbb{Q}$ is surjective. Indeed, by the Künneth formula [2], the group $\mathrm{K}^0_{\mathrm{top}}(S\times S)^G\otimes_{\mathbb{Z}}\mathbb{Q}$ is generated by classes of the form $[\mathcal{F}_1\boxtimes\mathcal{F}_2\oplus\mathcal{F}_2\boxtimes\mathcal{F}_1]$, and these lie in the image of ψ^{K}.

Also, the composition $\psi^{\mathrm{K}}\theta^{\mathrm{K}}$ vanishes. Computing the $\mathbb{Q}$–dimensions one sees that it suffices to show that θ^{K} is injective to conclude that the sequence is exact.

Pulling back to the diagonal and taking invariants defines a map

$$\mathrm{K}^0_{\mathrm{top}}(S)\xrightarrow{\theta^{\mathrm{K}}}\mathrm{K}^0_{\mathrm{top},G}(S\times S)\xrightarrow{\Delta^*}\mathrm{K}^0_{\mathrm{top},G}(S)\xrightarrow{(-)^G}\mathrm{K}^0_{\mathrm{top}}(S).$$

This composition computes to

$$[\mathcal{F}]\mapsto[\mathrm{Sym}^2\mathcal{F}]-\left[\textstyle\bigwedge^2\mathcal{F}\right].$$

This coincides with the second Adams operation, which is injective on $\mathrm{K}^0_{\mathrm{top}}(S)\otimes_{\mathbb{Z}}\mathbb{Q}$, since it has eigenvalues 1, 2, and 4. We conclude that θ^{K} is injective, and the proposition follows. □

8.3 A computation in the cohomology of the Hilbert square

We now come to the technical heart of our computation of the derived monodromy of the Hilbert square of a K3 surface.

Consider the map $\theta^{\mathrm{H}}\colon \mathrm{H}(S,\mathbb{Q}) \to \mathrm{H}(X,\mathbb{Q})$ given by

$$\theta^{\mathrm{H}}(s+\lambda+t\,\mathrm{pt}_S) = (s\delta+\lambda\delta+tq_X\delta)\cdot e^{-\delta/2}, \tag{12}$$

for all $s,t\in\mathbb{Q}$ and $\lambda\in\mathrm{H}^2(S,\mathbb{Q})$. See Section 7.2 for the definition of $\delta\in\mathrm{H}^2(X,\mathbb{Q})$ and $q_X\in\mathrm{H}^4(X,\mathbb{Q})$.

Proposition 8.5 *The square*

$$\begin{array}{ccc} \mathrm{K}^0_{\mathrm{top}}(S) & \xrightarrow{\theta^{\mathrm{K}}} & \mathrm{K}^0_{\mathrm{top}}(X) \\ \big\downarrow v_S^{\mathrm{top}} & & \big\downarrow v_X^{\mathrm{top}} \\ \mathrm{H}(S,\mathbb{Q}) & \xrightarrow{\theta^{\mathrm{H}}} & \mathrm{H}(X,\mathbb{Q}) \end{array}$$

commutes.

Proof Since $\mathrm{K}^0_{\mathrm{top}}(S)\otimes_{\mathbb{Z}}\mathbb{Q}$ is additively generated by line bundles, it suffices to show

$$v_X^{\mathrm{top}}(\theta^{\mathrm{K}}(\mathcal{L})) = \big(\delta+\lambda\delta+\big(\tfrac{1}{2}b(\lambda,\lambda)+1\big)q_X\delta\big)\cdot e^{-\delta/2} \tag{13}$$

for a topological line bundle $\mathcal{L}$ with $\lambda=c_1(\mathcal{L})$. Deforming S if necessary, we may assume that $\mathcal{L}$ is algebraic.

Using Proposition 8.1 and the fact that the natural map

$$\mathcal{L}_2\otimes q_*\mathcal{O}_Z \to q_*p^*(\mathcal{L}\boxtimes\mathcal{L})$$

is an isomorphism of $\mathcal{O}_X$–modules, we find

$$\mathrm{BKR}^{-1}[\mathcal{L}\boxtimes\mathcal{L},1]=\mathcal{L}_2,\quad \mathrm{BKR}^{-1}[\mathcal{L}\boxtimes\mathcal{L},-1]=\mathcal{E}^{-1}\otimes\mathcal{L}_2.$$

We conclude that θ^{K} maps $\mathcal{L}$ to $[\mathcal{L}_2](1-[\mathcal{E}^{-1}])$ in $\mathrm{K}^0(X)$.

We compute its image under v_X. Using the formula for the Todd class from Proposition 7.1, we find

$$v_X(\theta^{\mathrm{K}}(\mathcal{L})) = \big(1+\tfrac{5}{4}q_X+\cdots\big)\exp(\lambda)(1-e^{-\delta}).$$

Since $1-e^{-\delta}$ has no term in degree 0, the degree 8 part of the square root of the Todd class is irrelevant, so we have

$$v_X(\theta^{\mathrm{K}}(\mathcal{L})) = \big(1+\tfrac{5}{4}q_X\big)\exp(\lambda)(1-e^{-\delta}).$$

By the Fujiki relation (11) from Section 7.2, we have $\lambda^3\delta = 0$, so the above can be rewritten as

$$v_X(\theta^{\mathrm{K}}(\mathcal{L})) = \bigl(1 + \tfrac{5}{4}q_X\bigr) \cdot \bigl(\delta + \lambda\delta + \tfrac{1}{2}\lambda^2\delta\bigr) \cdot \frac{1-e^{-\delta}}{\delta}.$$

Since $q_X\delta\lambda = b(\delta,\lambda) = 0$, we can rewrite this further as

$$v_X(\theta^{\mathrm{K}}(\mathcal{L})) = \bigl(1 + \tfrac{1}{4}q_X\bigr) \cdot \bigl(\delta + \lambda\delta + \bigl(\tfrac{1}{2}b(\lambda,\lambda) + 1\bigr)q_X\delta\bigr) \cdot \frac{1-e^{-\delta}}{\delta}.$$

Comparing this with the right-hand side of (13), we see that it suffices to show

$$\bigl(1 + \tfrac{1}{4}q_X\bigr) \cdot (1 - e^{-\delta}) = \delta e^{-\delta/2}$$

in $\mathrm{H}(X,\mathbb{Q})$. This boils down to the identities

$$\tfrac{1}{6}\delta^3 + \tfrac{1}{4}\delta q_X = \tfrac{1}{8}\delta^3, \quad \tfrac{1}{24}\delta^4 + \tfrac{1}{8}\delta^2 q_X = \tfrac{1}{48}\delta^4$$

in $\mathrm{H}^6(X,\mathbb{Q})$ and $\mathrm{H}^8(X,\mathbb{Q})$, respectively. These follow easily from the relations (9), (10), and (11) in Section 7.2. □

9 Derived monodromy group of the Hilbert square of a K3 surface

9.1 Derived monodromy groups

Let X be a smooth projective complex variety. We call a *deformation* of X the data of a smooth projective variety X', a proper smooth family $\mathcal{X} \to B$, a path $\gamma\colon [0,1] \to \mathcal{X}$, and isomorphisms $X \xrightarrow{\sim} \mathcal{X}_{\gamma(0)}$ and $X' \xrightarrow{\sim} \mathcal{X}_{\gamma(1)}$. We will informally say that X' is a deformation of X, the other data being implicitly understood. Parallel transport along γ defines an isomorphism $\mathrm{H}(X,\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}(X',\mathbb{Q})$.

If X' and X'' are deformations of X, and if $\phi\colon X' \to X''$ is an isomorphism of projective varieties, then we obtain a composite isomorphism

$$\mathrm{H}(X,\mathbb{Q}) \xrightarrow[\sim]{} \mathrm{H}(X',\mathbb{Q}) \xrightarrow[\sim]{\phi} \mathrm{H}(X'',\mathbb{Q}) \xrightarrow[\sim]{} \mathrm{H}(X,\mathbb{Q}).$$

We call such an isomorphism a *monodromy operator* for X, and denote by $\mathrm{Mon}(X)$ the subgroup of $\mathrm{GL}(\mathrm{H}(X,\mathbb{Q}))$ generated by all monodromy operators.

If X' and X'' are deformations of X, and if $\Phi\colon \mathcal{D}X' \xrightarrow{\sim} \mathcal{D}X''$ is an equivalence, then we obtain an isomorphism

$$\mathrm{H}(X,\mathbb{Q}) \xrightarrow[\sim]{} \mathrm{H}(X',\mathbb{Q}) \xrightarrow[\sim]{\Phi^{\mathrm{H}}} \mathrm{H}(X'',\mathbb{Q}) \xrightarrow[\sim]{} \mathrm{H}(X,\mathbb{Q}).$$

We call such an isomorphism a *derived monodromy operator* for X, and denote by $\mathrm{DMon}(X)$ the subgroup of $\mathrm{GL}(\mathrm{H}(X,\mathbb{Q}))$ generated by all derived monodromy operators.

By construction, the derived monodromy group is deformation invariant. It contains the usual monodromy group, and the image of ρ_X, and we have a commutative square of groups

$$\begin{array}{ccc} \mathrm{Aut}(X) & \hookrightarrow & \mathrm{Aut}(\mathcal{D}X) \\ \downarrow & & \downarrow \rho_X \\ \mathrm{Mon}(X) & \hookrightarrow & \mathrm{DMon}(X) \end{array}$$

Remark 9.1 The above definition is somewhat ad hoc, and should be considered a poor man's derived monodromy group. This is sufficient for our purposes. A more mature definition should involve all noncommutative deformations of X.

Proposition 9.2 *If S is a* K3 *surface, then* $\mathrm{DMon}(S) = \mathrm{O}^+(\widetilde{\mathrm{H}}(S,\mathbb{Z}))$.

Proof Indeed, if $\Phi\colon \mathcal{D}S_1 \to \mathcal{D}S_2$ is an equivalence, then

$$\Phi^{\mathrm{H}}\colon \widetilde{\mathrm{H}}(S_1,\mathbb{Z}) \to \widetilde{\mathrm{H}}(S_2,\mathbb{Z})$$

preserves the Mukai form, as well as a natural orientation on four-dimensional positive subspaces; see [26, Section 4.5]. Also any deformation preserves the Mukai form and the natural orientation, so any derived monodromy operator will land in $\mathrm{O}^+(\widetilde{\mathrm{H}}(S,\mathbb{Z}))$.

The converse inclusion can be easily obtained from the Torelli theorem, together with the results of [22; 39] on derived auto-equivalences of K3 surfaces. Alternatively, one can use that the group $\mathrm{O}^+(\widetilde{\mathrm{H}}(S,\mathbb{Z}))$ is generated by reflections in -2–vectors δ. By the Torelli theorem, any such -2–vector will become algebraic on a suitable deformation S' of S, and by [32] there exists a spherical object $\mathcal{E}$ on S' with Mukai vector $v(\mathcal{E}) = \delta$. The spherical twist in $\mathcal{E}$ then shows that reflection in δ is indeed a derived monodromy operator. □

9.2 Action of $\mathrm{DMon}(S)$ on $\mathrm{H}(X,\mathbb{Q})$

By the derived McKay correspondence, any derived equivalence $\Phi_S\colon \mathcal{D}S_1 \xrightarrow{\sim} \mathcal{D}S_2$ between K3 surfaces induces a derived equivalence $\Phi_X\colon \mathcal{D}X_1 \xrightarrow{\sim} \mathcal{D}X_2$ between the corresponding Hilbert squares. By Propositions 8.3 and 8.4, the induced map Φ_X^H only depends on Φ_S^H. Since any deformation of a K3 surface S induces a deformation of $X = S^{[2]}$, we conclude that we have a natural homomorphism

$$\mathrm{DMon}(S) \to \mathrm{DMon}(X),$$

and hence an action of $\mathrm{DMon}(S)$ on $\mathrm{H}(X, \mathbb{Q})$. In this subsection, we will explicitly compute this action. As a first approximation, we determine the $\mathrm{DMon}(S)$–module structure of $\mathrm{H}(X, \mathbb{Q})$, up to isomorphism.

Proposition 9.3 *We have* $\mathrm{H}(X, \mathbb{Q}) \cong \widetilde{\mathrm{H}}(S, \mathbb{Q}) \oplus \mathrm{Sym}^2 \widetilde{\mathrm{H}}(S, \mathbb{Q})$ *as representations of* $\mathrm{DMon}(S) = \mathrm{O}^+(\widetilde{\mathrm{H}}(S, \mathbb{Z}))$.

Proof This follows from Propositions 8.3 and 8.4. □

Since $\mathfrak{g}(X)$ is a purely topological invariant, it is preserved under deformations. In particular, Theorem 4.13 implies that we have an inclusion $\mathrm{DMon}(X) \subset \mathrm{O}(\widetilde{\mathrm{H}}(X, \mathbb{Q}))$. We conclude there exists a unique map of algebraic groups h making the square

$$\begin{array}{ccc} \mathrm{DMon}(S) & \longrightarrow & \mathrm{DMon}(X) \\ \downarrow & & \downarrow \\ \mathrm{O}(\widetilde{\mathrm{H}}(S, \mathbb{Q})) & \xrightarrow{h} & \mathrm{O}(\widetilde{\mathrm{H}}(X, \mathbb{Q})) \end{array} \tag{14}$$

commute.

Recall that in (3) we defined an isometry B_λ of $\widetilde{\mathrm{H}}(X, \mathbb{Q})$ for every $\lambda \in \mathrm{H}^2(X, \mathbb{Q})$.

Theorem 9.4 *The map h in the square* (14) *is given by*

$$g \mapsto \det(g) \cdot (B_{-\delta/2} \circ \iota(g) \circ B_{\delta/2}),$$

with $\iota\colon \mathrm{O}(\widetilde{\mathrm{H}}(S, \mathbb{Q})) \to \mathrm{O}(\widetilde{\mathrm{H}}(X, \mathbb{Q}))$ *the natural inclusion.*

The proof of this theorem will occupy the remainder of this section.

Consider the unique homomorphism of Lie algebras $\iota\colon \mathfrak{g}(S) \to \mathfrak{g}(X)$ that respects the grading and maps e_λ to e_λ for all $\lambda \in \mathrm{H}^2(S, \mathbb{Q}) \subset \mathrm{H}^2(X, \mathbb{Q})$. Under the isomorphism of Theorem 3.1 this corresponds to the map $\mathfrak{so}(\widetilde{\mathrm{H}}(S, \mathbb{Q})) \to \mathfrak{so}(\widetilde{\mathrm{H}}(X, \mathbb{Q}))$ induced by the inclusion of quadratic spaces $\widetilde{\mathrm{H}}(S, \mathbb{Q}) \subset \widetilde{\mathrm{H}}(X, \mathbb{Q})$.

Recall from Section 8.3 the map $\theta^{\mathrm{H}}\colon \mathrm{H}(S, \mathbb{Q}) \to \mathrm{H}(X, \mathbb{Q})$.

Lemma 9.5 *The map* $\theta^{\mathrm{H}}\colon \mathrm{H}(S,\mathbb{Q}) \to \mathrm{H}(X,\mathbb{Q})$ *is equivariant with respect to*

$$\theta^{\mathfrak{g}}\colon \mathfrak{g}(S) \to \mathfrak{g}(X), \quad x \mapsto B_{-\delta/2} \circ \iota(x) \circ B_{\delta/2}.$$

Proof We have $\theta^{\mathrm{H}} = e^{-\delta/2} \cdot \theta_0^{\mathrm{H}}$, with

$$\theta_0^{\mathrm{H}}(s + \lambda + t\,\mathrm{pt}_S) = s\delta + \lambda\delta + tq_X\delta.$$

The map θ_0^{H} respects the grading, and we claim that for every $\mu \in \mathrm{H}^2(S,\mathbb{Q})$ the diagram

$$\begin{array}{ccccc} \mathrm{H}(S,\mathbb{Q}) & \xrightarrow{\theta_0^{\mathrm{H}}} & \mathrm{H}(X,\mathbb{Q}) & \xrightarrow{e^{-\delta/2}} & \mathrm{H}(X,\mathbb{Q}) \\ \downarrow{\scriptstyle e_\mu} & & \downarrow{\scriptstyle e_\mu} & & \downarrow{\scriptstyle e^{-\delta/2}e_\mu e^{\delta/2}} \\ \mathrm{H}(S,\mathbb{Q}) & \xrightarrow{\theta_0^{\mathrm{H}}} & \mathrm{H}(X,\mathbb{Q}) & \xrightarrow{e^{-\delta/2}} & \mathrm{H}(X,\mathbb{Q}) \end{array}$$

commutes. Indeed, we have

$$\begin{aligned} e_\mu(\theta_0^{\mathrm{H}}(s + \lambda + t\,\mathrm{pt}_S)) &= s\delta\mu + \lambda\delta\mu + tq_X\delta\mu, \\ \theta_0^{\mathrm{H}}(e_\mu(s + \lambda + t\,\mathrm{pt}_S)) &= s\delta\mu + b(\lambda,\mu)q_X\delta. \end{aligned}$$

One verifies easily that these agree, using the identities (10) and (9) from Section 7.2 and the fact that $b(\lambda,\delta) = b(\mu,\delta) = 0$. This shows that the left-hand square commutes. The right-hand square commutes trivially, so the outer rectangle commutes, which shows that $\theta^{\mathrm{H}} = e^{-\delta/2} \cdot \theta_0^{\mathrm{H}}$ is indeed equivariant with respect to $\theta^{\mathfrak{g}}$. □

Lemma 9.6 *There is an isomorphism*

$$\det(\widetilde{\mathrm{H}}(X,\mathbb{Q})) \otimes \mathrm{Sym}^2(\widetilde{\mathrm{H}}(X,\mathbb{Q})) \cong \mathrm{H}(X,\mathbb{Q}) \oplus \det(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$$

of representations of $G = \mathrm{O}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$.

Proof This follows from Lemma 3.7, Theorem 4.13 and Remark 4.14. □

We are now ready to prove the main result of this subsection.

Proof of Theorem 9.4 By Proposition 8.5, the map θ^{H} is equivariant for the action of $\mathrm{DMon}(S)$. Lemma 9.5 then implies that

$$h(g) = B_{-\delta/2} \circ \iota(g) \circ B_{\delta/2}$$

for all $g \in \mathrm{SO}(\widetilde{\mathrm{H}}(S,\mathbb{Q}))$. We have an orthogonal decomposition

$$\widetilde{\mathrm{H}}(X,\mathbb{Q}) = B_{-\delta/2}(\widetilde{\mathrm{H}}(S,\mathbb{Q})) \oplus C$$

with C of rank 1. Since $\mathrm{SO}(\widetilde{\mathrm{H}}(S,\mathbb{Q}))$ is normal in $\mathrm{O}(\widetilde{\mathrm{H}}(S,\mathbb{Q}))$, the action of $\mathrm{O}(\widetilde{\mathrm{H}}(S,\mathbb{Q}))$ (via h) must preserve this decomposition. With respect to this decomposition h must then be given by

$$h(g) = (B_{-\delta/2} \circ g\epsilon_1(g) \circ B_{\delta/2}) \oplus \epsilon_2(g),$$

where the $\epsilon_i(g)\colon \mathrm{O}(\widetilde{\mathrm{H}}(S,\mathbb{Q})) \to \{\pm 1\}$ are quadratic characters. This leaves four possibilities for h. One verifies that $\epsilon_1 = \epsilon_2 = \det g$ is the only possibility compatible with Proposition 9.3 and Lemma 9.6, and the theorem follows. □

9.3 A transitivity lemma

In this section we prove a lattice-theoretical lemma that will play an important role in the proofs of Theorems E and F.

Let $b\colon L \times L \to \mathbb{Z}$ be an even nondegenerate lattice. Let U be a hyperbolic plane with basis consisting of isotropic vectors α and β satisfying $b(\alpha,\beta) = -1$.

As before, to a $\lambda \in L$ we associate the isometry $B_\lambda \in \mathrm{O}(U \oplus L)$ defined as

$$B_\lambda(r\alpha + \mu + s\beta) = r\alpha + (\mu + r\lambda) + \big(s + b(\mu,\lambda) + r\tfrac{1}{2}b(\lambda,\lambda)\big)\beta$$

for all $r, s \in \mathbb{Z}$ and $\mu \in L$. Let γ be the isometry of $U \oplus L$ given by $\gamma(\alpha) = \beta$, $\gamma(\beta) = \alpha$, and $\gamma(\lambda) = -\lambda$ for all $\lambda \in L$.

Lemma 9.7 *Let L be an even lattice containing a hyperbolic plane. Let $G \subset \mathrm{O}(U \oplus L)$ be the subgroup generated by γ and by B_λ for all $\lambda \in L$. Then, for all $\delta \in U \oplus L$ with $\delta^2 = -2$ and for all $g \in \mathrm{O}(U \oplus L)$, there exists a $g' \in G$ such that $g'g$ fixes δ.*

Proof This follows from classical results of Eichler. A convenient modern source is [20, Section 3], whose notation we adopt. The isometry B_λ coincides with the Eichler transvection $t(\beta, -\lambda)$. The conjugate $\gamma B_\lambda \gamma^{-1}$ is the Eicher transvection $t(\alpha,\lambda)$. Hence G contains the subgroup $E_U(L) \subset \mathrm{O}(U \oplus L)$ of unimodular transvections with respect to U. By [20, Proposition 3.3], there exists a $g' \in E_U(L)$ mapping $g\delta$ to δ. □

9.4 Proof of Theorem E

Let X be a hyperkähler variety of type $\mathrm{K3}^{[2]}$. Let $\delta \in \mathrm{H}^2(X,\mathbb{Z})$ be any class satisfying $\delta^2 = -2$ and $b(\delta,\lambda) \in 2\mathbb{Z}$ for all $\lambda \in \mathrm{H}^2(X,\mathbb{Z})$. For example, if $X = S^{[2]}$, we may take $\delta = c_1(\mathcal{E})$ as in Section 7.2. Consider the integral lattice

$$\Lambda := B_{\delta/2}(\mathbb{Z}\alpha \oplus \mathrm{H}^2(X,\mathbb{Z}) \oplus \mathbb{Z}\beta) \subset \widetilde{\mathrm{H}}(X,\mathbb{Q}).$$

The subgroup $\Lambda \subset \widetilde{\mathrm{H}}(X, \mathbb{Q})$ does not depend on the choice of δ. In this section, we will prove Theorem E. More precisely, we will show:

Theorem 9.8 $\mathrm{O}^+(\Lambda) \subset \mathrm{DMon}(X) \subset \mathrm{O}(\Lambda)$.

We start with the lower bound.

Proposition 9.9 $\mathrm{O}^+(\Lambda) \subset \mathrm{DMon}(X)$ *as subgroups of* $\mathrm{O}(\widetilde{\mathrm{H}}(X, \mathbb{Q}))$.

Proof Since the derived monodromy group is invariant under deformation, we may assume without loss of generality that $X = S^{[2]}$ for a K3 surface S and $\delta = c_1(\mathcal{E})$ as in Section 7.2.

The shift functor [1] on $\mathcal{D}X$ acts as -1 on $\mathrm{H}(X, \mathbb{Q})$, which coincides with the action of $-1 \in \mathrm{O}(\widetilde{\mathrm{H}}(X, \mathbb{Q}))$. In particular, $-1 \in \mathrm{O}^+(\Lambda)$ lies in $\mathrm{DMon}(X)$, so it suffices to show that $\mathrm{SO}^+(\Lambda)$ is contained in $\mathrm{DMon}(X)$.

Consider the isometry $\gamma \in \mathrm{O}^+(\widetilde{\mathrm{H}}(S, \mathbb{Q}))$ given by $\gamma(\alpha) = -\beta$, $\gamma(\beta) = -\alpha$, and $\gamma(\lambda) = \lambda$ for all $\lambda \in \mathrm{H}^2(S, \mathbb{Q})$. Then $\det(\gamma) = -1$ and by Theorem 9.4 its image $h(\gamma)$ interchanges $B_{\delta/2}\alpha$ and $B_{\delta/2}\beta$ and acts by -1 on $B_{\delta/2}\mathrm{H}^2(X, \mathbb{Z})$. Since γ lies in $\mathrm{DMon}(S) \subset \mathrm{O}(\widetilde{\mathrm{H}}(S, \mathbb{Q}))$, we have that $h(\gamma)$ lies in $\mathrm{DMon}(X) \subset \mathrm{O}(\widetilde{\mathrm{H}}(X, \mathbb{Q}))$.

Let $G \subset \mathrm{O}(\widetilde{\mathrm{H}}(X, \mathbb{Q}))$ be the subgroup generated by $h(\gamma)$ and the isometries B_λ for $\lambda \in \mathrm{H}^2(X, \mathbb{Z})$. Clearly G is contained in $\mathrm{DMon}(X)$.

Let g be an element of $\mathrm{SO}^+(\Lambda)$, and consider the image $gB_{\delta/2}\delta$ of $B_{\delta/2}\delta$. By Lemma 9.7 there exists a $g' \in G \subset \mathrm{DMon}(X)$ such that $g'g$ fixes $B_{\delta/2}\delta$. But then $g'g$ acts on

$$(B_{\delta/2}\delta)^\perp = B_{\delta/2}(\mathbb{Z}\alpha \oplus \mathrm{H}^2(S, \mathbb{Z}) \oplus \mathbb{Z}\beta)$$

with determinant 1 and preserving the orientation of a maximal positive subspace. In particular, $g'g$ lies in the image of $\mathrm{DMon}(S) \to \mathrm{DMon}(X)$, and we conclude that g lies in $\mathrm{DMon}(X)$. □

The proof of the upper bound is now almost purely group-theoretical. Denote by $\mathrm{SO}^+(\Lambda)$ the intersection $\mathrm{O}^+(\Lambda) \cap \mathrm{SO}(\Lambda)$. This group coincides with the kernel of the spinor norm on $\mathrm{SO}(\Lambda)$.

Proposition 9.10 $\mathrm{SO}(\Lambda)$ *is the unique maximal arithmetic subgroup of* $\mathrm{SO}(\Lambda \otimes_{\mathbb{Z}} \mathbb{Q})$ *containing* $\mathrm{SO}^+(\Lambda)$.

Proof More generally, this holds for any even lattice Λ with the property that the quadratic form $q(x) = b(x,x)/2$ on the $\mathbb{Z}$–module Λ is semiregular [29, Section IV.3].

For such Λ, the group schemes $\mathbf{Spin}(\Lambda)$ and $\mathbf{SO}(\Lambda)$ are smooth over $\operatorname{Spec}\mathbb{Z}$; see eg [27]. In particular, for every prime p the subgroups $\mathrm{Spin}(\Lambda\otimes\mathbb{Z}_p)$ and $\mathrm{SO}(\Lambda\otimes\mathbb{Z}_p)$ of $\mathrm{Spin}(\Lambda\otimes\mathbb{Q}_p)$ and $\mathrm{SO}(\Lambda\otimes\mathbb{Q}_p)$, respectively, are maximal compact subgroups. It follows that the groups

$$\mathrm{Spin}(\Lambda) = \mathrm{Spin}(\Lambda\otimes\mathbb{Q}) \cap \prod_p \mathrm{Spin}(\Lambda\otimes\mathbb{Z}_p)$$

and

$$\mathrm{SO}(\Lambda) = \mathrm{SO}(\Lambda\otimes\mathbb{Q}) \cap \prod_p \mathrm{SO}(\Lambda\otimes\mathbb{Z}_p)$$

are maximal arithmetic subgroups of $\mathrm{Spin}(\Lambda\otimes\mathbb{Q})$ and $\mathrm{SO}(\Lambda\otimes\mathbb{Q})$, respectively.

The subgroup $\mathrm{SO}^+(\Lambda)\subset\mathrm{SO}(\Lambda)$ is the kernel of the spinor norm, and the short exact sequence $1\to\mu_2\to\mathbf{Spin}\to\mathbf{SO}\to 1$ of fppf sheaves on $\operatorname{Spec}\mathbb{Z}$ induces an exact sequence of groups

$$1\to\{\pm1\}\to\mathrm{Spin}(\Lambda)\to\mathrm{SO}^+(\Lambda)\to 1.$$

Let $\Gamma\subset\mathrm{SO}(\Lambda\otimes\mathbb{Q})$ be a maximal arithmetic subgroup containing $\mathrm{SO}^+(\Lambda)$. Let $\widetilde{\Gamma}$ be its inverse image in $\mathrm{Spin}(\Lambda\otimes\mathbb{Q})$, so that we have an exact sequence

$$1\to\{\pm1\}\to\widetilde{\Gamma}\to\Gamma\to\mathbb{Q}^\times/2.$$

Since the group $\widetilde{\Gamma}$ is arithmetic and contains $\mathrm{Spin}(\Lambda)$, we have $\widetilde{\Gamma}=\mathrm{Spin}(\Lambda)$. Moreover, Γ normalizes $\mathrm{SO}^+(\Lambda)=\ker(\Gamma\to\mathbb{Q}^\times/2)$, and, as the normalizer of an arithmetic subgroup of $\mathrm{SO}(\Lambda\otimes\mathbb{Q})$ is again arithmetic, Γ must equal the normalizer of $\mathrm{SO}^+(\Lambda)$. But then Γ contains $\mathrm{SO}(\Lambda)$, and we conclude $\Gamma=\mathrm{SO}(\Lambda)$. □

Corollary 9.11 $\mathrm{DMon}(X)\subset\mathrm{O}(\Lambda)$.

Proof $\mathrm{DMon}(X)$ preserves the integral lattice $\mathrm{K}_{\mathrm{top}}(X)$ in the representation $\mathrm{H}(X,\mathbb{Q})$ of $\mathrm{O}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$, and hence is contained in an arithmetic subgroup of

$$\mathrm{O}(\widetilde{\mathrm{H}}(X,\mathbb{Q})) = \mathrm{SO}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))\times\{\pm1\}.$$

By Proposition 9.9 it contains $\mathrm{SO}^+(\Lambda)\times\{\pm1\}$, so we conclude from the preceding proposition that $\mathrm{DMon}(X)$ must be contained in $\mathrm{O}(\Lambda)$. □

Together with Proposition 9.9 this proves Theorem 9.8.

10 The image of $\mathrm{Aut}(\mathcal{D}X)$ on $\mathrm{H}(X,\mathbb{Q})$

10.1 Upper bound for the image of ρ_X

We continue with the notation of the previous section. In particular, we denote by X a hyperkähler variety of type $\mathrm{K3}^{[2]}$, and by $\Lambda \subset \widetilde{\mathrm{H}}(X,\mathbb{Q})$ the lattice defined in Section 9.4. We equip $\widetilde{\mathrm{H}}(X,\mathbb{Q})$ with the weight 0 Hodge structure

$$\widetilde{\mathrm{H}}(X,\mathbb{Q}) = \mathbb{Q}\alpha \oplus \mathrm{H}^2(X,\mathbb{Q}(1)) \oplus \mathbb{Q}\beta.$$

We denote by $\mathrm{Aut}(\Lambda) \subset \mathrm{O}(\Lambda)$ the group of isometries of Λ that preserve this Hodge structure.

Proposition 10.1
$$\mathrm{im}(\rho_X) \subset \mathrm{Aut}(\Lambda).$$

Proof By Theorem 9.8 we have $\mathrm{im}(\rho_X) \subset \mathrm{O}(\Lambda)$. The Hodge structure on

$$\mathrm{H}(X,\mathbb{Q}) = \bigoplus_{n=0}^{4} \mathrm{H}^{2n}(X,\mathbb{Q}(n))$$

induces a Hodge structure on $\mathfrak{g}(X) \subset \mathrm{End}(\mathrm{H}(X,\mathbb{Q}))$, which agrees with the Hodge structure on $\mathfrak{so}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$ induced by the Hodge structure on $\widetilde{\mathrm{H}}(X,\mathbb{Q})$. If

$$\Phi\colon \mathcal{D}X \xrightarrow{\sim} \mathcal{D}X$$

is an equivalence, then $\Phi^{\mathrm{H}}\colon \mathrm{H}(X,\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}(X,\mathbb{Q})$ and $\Phi^{\mathfrak{g}}\colon \mathfrak{g}(X) \xrightarrow{\sim} \mathfrak{g}(X)$ are isomorphisms of $\mathbb{Q}$–Hodge structures, from which it follows that Φ^{H} must land in $\mathrm{Aut}(\Lambda) \subset \mathrm{O}(\Lambda)$. □

10.2 Lower bound for the image of ρ_X

We write $\mathrm{Aut}^+(\Lambda)$ for the index 2 subgroup $\mathrm{Aut}(\Lambda) \cap \mathrm{O}^+(\Lambda)$ of $\mathrm{Aut}(\Lambda)$.

Theorem 10.2 *Let S be a* K3 *surface and let X be the Hilbert square of S. Assume that* $\mathrm{NS}(X)$ *contains a hyperbolic plane. Then* $\mathrm{Aut}^+(\Lambda) \subset \mathrm{im}\,\rho_X \subset \mathrm{Aut}(\Lambda)$.

Proof In view of Proposition 10.1 we only need to show the lower bound. The argument for this is entirely parallel to the proof of Proposition 9.9. Recall that

$$\Lambda = B_{\delta/2}(\mathbb{Z}\alpha \oplus \mathrm{H}^2(S,\mathbb{Z}(1)) \oplus \mathbb{Z}\delta \oplus \mathbb{Z}\beta).$$

The shift functor $[1] \in \mathrm{Aut}(\mathcal{D}X)$ maps to $-1 \in \mathrm{Aut}^+(\Lambda)$, so it suffices to show that $\mathrm{Aut}^+(\Lambda) \cap \mathrm{SO}(\Lambda)$ is contained in $\mathrm{im}\,\rho_X$.

Let $\gamma_S \in \mathrm{Aut}(\mathcal{D}S)$ be the composition of the spherical twist in $\mathcal{O}_S$ with the shift [1]. On the Mukai lattice $\widetilde{\mathrm{H}}(S,\mathbb{Z}) = \mathbb{Z}\alpha \oplus \mathrm{H}^2(X,\mathbb{Z}(1)) \oplus \mathbb{Z}\beta$ this equivalence maps α to $-\beta$ and β to $-\alpha$ and is the identity on $\mathrm{H}^2(S,\mathbb{Z})$. Under the derived McKay correspondence this induces an autoequivalence $\gamma_X \in \mathrm{Aut}\,\mathcal{D}X$. By Theorem 9.4, the automorphism $\rho_X(\gamma_X) \in \mathrm{Aut}(\Lambda)$ interchanges $B_{\delta/2}\alpha$ and $B_{\delta/2}\beta$ and acts by -1 on $B_{\delta/2}\mathrm{H}^2(X,\mathbb{Z})$.

Denote by $G \subset \mathrm{Aut}(\Lambda)$ the subgroup generated by $\rho_X(\gamma_X)$ and the isometries $B_\lambda = \rho_X(-\otimes\mathcal{L})$ with $\mathcal{L}$ a line bundle of class $\lambda \in \mathrm{NS}(X)$. Clearly G is contained in the image of ρ_X. Note that G acts on the lattice

$$\Lambda_{\mathrm{alg}} := B_{\delta/2}(\mathbb{Z}\alpha \oplus \mathrm{NS}(X) \oplus \mathbb{Z}\beta)$$

and that by our assumption $\mathrm{NS}(X)$ contains a hyperbolic plane.

Let $g \in \mathrm{Aut}^+(\Lambda)$. By Lemma 9.7 applied to $L = \mathrm{NS}(X)$, there exists a $g' \in G$ such that $g'g$ fixes $B_{\delta/2}\delta$. But then $g'g$ acts on

$$(B_{\delta/2}\delta)^\perp = B_{\delta/2}(\mathbb{Z}\alpha \oplus \mathrm{H}^2(S,\mathbb{Z}) \oplus \mathbb{Z}\beta)$$

with determinant 1 and preserving the Hodge structure and the orientation of a maximal positive subspace. In particular, $g'g$ lies in the image of $\mathrm{Aut}(\mathcal{D}S)$, and we conclude that g lies in $\mathrm{im}\,\rho_X$. □

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