GGGGGTTTTTTT

Geometry & Topology

Volume 27 (2023)

Derived equivalences of hyperkähler varieties

LENNY TAELMAN

msp

msp

Geometry & Topology **27**:7 (2023) 2649–2693
DOI: 10.2140/gt.2023.27.2649
Published: 19 September 2023

Derived equivalences of hyperkähler varieties

LENNY TAELMAN

We show that the Looijenga–Lunts–Verbitsky Lie algebra acting on the cohomology of a hyperkähler variety is a derived invariant, and obtain from this a number of consequences for the action on cohomology of derived equivalences between hyperkähler varieties.

This includes a proof that derived equivalent hyperkähler varieties have isomorphic $\mathbb{Q}$–Hodge structures, the construction of a rational "Mukai lattice" functorial for derived equivalences, and the computation (up to index 2) of the image of the group of auto-equivalences on the cohomology of certain Hilbert squares of K3 surfaces.

14F05, 14J32

1 Introduction

1.1 Background

We briefly recall the background to our results. We refer to Huybrechts [24] for more details. For a smooth projective complex variety X, we denote by $\mathcal{D}X$ the bounded derived category of coherent sheaves on X. By a theorem of Orlov [37] any (exact, $\mathbb{C}$–linear) equivalence $\Phi\colon \mathcal{D}X_1 \xrightarrow{\sim} \mathcal{D}X_2$ comes from a Fourier–Mukai kernel $\mathcal{P} \in \mathcal{D}(X_1 \times X_2)$, and convolution with the Mukai vector $v(\mathcal{P}) \in \mathrm{H}(X_1 \times X_2, \mathbb{Q})$ defines an isomorphism

$$\Phi^{\mathrm{H}}\colon \mathrm{H}(X_1, \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}(X_2, \mathbb{Q})$$

between the total cohomology of X_1 and X_2. This isomorphism is not graded, and respects the Hodge structures only up to Tate twists. Nonetheless, Orlov has conjectured [38] that if X_1 and X_2 are derived equivalent, then for every i there exist (noncanonical) isomorphisms $\mathrm{H}^i(X_1, \mathbb{Q}) \cong \mathrm{H}^i(X_2, \mathbb{Q})$ of $\mathbb{Q}$–Hodge structures.

For every X we have a representation

$$\rho_X : \mathrm{Aut}(\mathcal{D}X) \to \mathrm{GL}(\mathrm{H}(X, \mathbb{Q})), \quad \Phi \mapsto \Phi^{\mathrm{H}}.$$

Its image is known for varieties with ample or antiample canonical class (in which case $\mathrm{Aut}(\mathcal{D}X)$ is small and well understood; see Bondal and Orlov [9]), for abelian varieties — see Golyshev, Lunts and Orlov [18] — and for K3 surfaces. To place our results in context, we recall the description of the image for K3 surfaces.

Let X be a K3 surface. Consider the *Mukai lattice*

$$\widetilde{\mathrm{H}}(X,\mathbb{Z}) := \mathrm{H}^0(X,\mathbb{Z}) \oplus \mathrm{H}^2(X,\mathbb{Z}(1)) \oplus \mathrm{H}^4(X,\mathbb{Z}(2)).$$

This is a Hodge structure of weight 0, and it comes equipped with a perfect bilinear form b of signature $(4,20)$. For convenience, we denote by α and β the natural generators of $\mathrm{H}^0(X,\mathbb{Z})$ and $\mathrm{H}^4(X,\mathbb{Z}(2))$ respectively, so that $\widetilde{\mathrm{H}}(X,\mathbb{Z}) = \mathbb{Z}\alpha \oplus \mathrm{H}^2(X,\mathbb{Z}(1)) \oplus \mathbb{Z}\beta$. The pairing b is the orthogonal sum of the intersection pairing on $\mathrm{H}^2(X,\mathbb{Z}(1))$ and the pairing on $\mathbb{Z}\alpha \oplus \mathbb{Z}\beta$ given by $b(\alpha,\alpha) = b(\beta,\beta) = 0$ and $b(\alpha,\beta) = -1$.

It was observed by Mukai [35] that if $\Phi\colon \mathcal{D}X_1 \xrightarrow{\sim} \mathcal{D}X_2$ is a derived equivalence between K3 surfaces, then Φ^{H} restricts to an isomorphism $\Phi^{\widetilde{\mathrm{H}}}\colon \widetilde{\mathrm{H}}(X_1,\mathbb{Z}) \to \widetilde{\mathrm{H}}(X_2,\mathbb{Z})$ respecting the pairing and Hodge structures. Denote by $\mathrm{Aut}(\widetilde{\mathrm{H}}(X,\mathbb{Z}))$ the group of isometries of $\widetilde{\mathrm{H}}(X,\mathbb{Z})$ respecting the Hodge structure, and by $\mathrm{Aut}^+(\widetilde{\mathrm{H}}(X,\mathbb{Z}))$ the subgroup (of index 2) consisting of those isometries that respect the orientation on a four-dimensional positive definite subspace of $\widetilde{\mathrm{H}}(X,\mathbb{R})$.

Theorem 1.1 [22; 26; 35; 36; 39] *Let X be a* K3 *surface. Then the image of ρ_X is* $\mathrm{Aut}^+(\widetilde{\mathrm{H}}(X,\mathbb{Z}))$. □

In this paper, we prove Orlov's conjecture on $\mathbb{Q}$–Hodge structures for hyperkähler varieties, construct a rational version of the Mukai lattice for hyperkähler varieties, and compute (up to index 2) the image of ρ_X for certain Hilbert squares of K3 surfaces. The main tool in these results is the Looijenga–Lunts–Verbitsky Lie algebra.

1.2 The LLV Lie algebra and derived equivalences

Let X be a smooth projective complex variety. By the hard Lefschetz theorem, every ample class $\lambda \in \mathrm{NS}(X)$ determines a Lie algebra $\mathfrak{g}_\lambda \subset \mathrm{End}(\mathrm{H}(X,\mathbb{Q}))$ isomorphic to $\mathfrak{sl}_2$. More generally, this holds for every cohomology class $\lambda \in \mathrm{H}^2(X,\mathbb{Q})$ (algebraic or not) satisfying the conclusion of the hard Lefschetz theorem. Looijenga and Lunts [33] and Verbitsky [46] have studied the Lie algebra $\mathfrak{g}(X) \subset \mathrm{End}(\mathrm{H}(X,\mathbb{Q}))$ generated by the collection of the Lie algebras $\mathfrak{g}_\lambda$. We will refer to this as the LLV Lie algebra. See Section 2.1 for more details.

We say that X is *holomorphic symplectic* if it admits a nowhere degenerate holomorphic symplectic form $\sigma \in \mathrm{H}^0(X, \Omega_X^2)$.

Theorem A (Section 2.4) *Let X_1 and X_2 be holomorphic symplectic varieties. Then for every equivalence $\Phi\colon \mathcal{D}X_1 \xrightarrow{\sim} \mathcal{D}X_2$ there exists a canonical isomorphism of rational Lie algebras*

$$\Phi^{\mathfrak{g}}\colon \mathfrak{g}(X_1) \xrightarrow{\sim} \mathfrak{g}(X_2)$$

with the property that the map $\Phi^{\mathrm{H}}\colon \mathrm{H}(X_1, \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}(X_2, \mathbb{Q})$ is equivariant with respect to $\Phi^{\mathfrak{g}}$.

Note that $\mathfrak{g}(X)$ is defined in terms of the grading and the cup product on $\mathrm{H}(X, \mathbb{Q})$, neither of which are preserved under derived equivalences.

To prove Theorem A we introduce a complex Lie algebra $\mathfrak{g}'(X)$ whose definition is similar to the rational Lie algebra $\mathfrak{g}(X)$, but where the action of $\mathrm{H}^2(X, \mathbb{Q})$ on $\mathrm{H}(X, \mathbb{Q})$ is replaced with a natural action of the Hochschild cohomology group $\mathrm{HH}^2(X)$ on Hochschild homology $\mathrm{HH}_\bullet(X)$. Since Hochschild cohomology and its action on Hochschild homology is known to be invariant under derived equivalences, it follows that $\mathfrak{g}'(X)$ is a derived invariant. We show that if X is holomorphic symplectic, then the isomorphism $\mathrm{HH}_\bullet(X) \to \mathrm{H}(X, \mathbb{C})$ (coming from the Hochschild–Kostant–Rosenberg isomorphism) maps $\mathfrak{g}'(X)$ to $\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$. This is closely related to Verbitsky's "mirror symmetry" for hyperkähler varieties [46; 47]. From this we deduce that the rational Lie algebra $\mathfrak{g}(X)$ is a derived invariant.

1.3 A rational Mukai lattice for hyperkähler varieties

A *hyperkähler* (or irreducible holomorphic symplectic) variety is a simply connected smooth projective variety X for which $\mathrm{H}^0(X, \Omega_X^2)$ is spanned by a nowhere degenerate form.

Let X be a hyperkähler variety. Consider the $\mathbb{Q}$–vector space

$$\widetilde{\mathrm{H}}(X, \mathbb{Q}) := \mathbb{Q}\alpha \oplus \mathrm{H}^2(X, \mathbb{Q}) \oplus \mathbb{Q}\beta$$

equipped with the bilinear form b which is the orthogonal sum of the Beauville–Bogomolov form on $\mathrm{H}^2(X, \mathbb{Q})$ and a hyperbolic plane $\mathbb{Q}\alpha \oplus \mathbb{Q}\beta$ with α and β isotropic and $b(\alpha, \beta) = -1$. By analogy with the case of a K3 surface, we will call $\widetilde{\mathrm{H}}(X, \mathbb{Q})$ the (rational) *Mukai lattice* of X. Looijenga and Lunts [33] and Verbitsky [46] have shown that the Lie algebra $\mathfrak{g}(X)$ can be canonically identified with $\mathfrak{so}(\widetilde{\mathrm{H}}(X, \mathbb{Q}))$;

see Section 3.1 for a precise statement. Moreover, Verbitsky [46] has shown that the subalgebra $\mathrm{SH}(X,\mathbb{Q})$ of $\mathrm{H}(X,\mathbb{Q})$ generated by $\mathrm{H}^2(X,\mathbb{Q})$ forms an irreducible sub-$\mathfrak{g}(X)$–module. Using this, we show that Theorem A implies:

Theorem B (Section 4.2) *Let X_1 and X_2 be hyperkähler varieties and*

$$\Phi\colon \mathcal{D}X_1 \xrightarrow{\sim} \mathcal{D}X_2$$

an equivalence. Then the induced isomorphism Φ^{H} restricts to an isomorphism $\Phi^{\mathrm{SH}}\colon \mathrm{SH}(X_1,\mathbb{Q}) \xrightarrow{\sim} \mathrm{SH}(X_2,\mathbb{Q})$.

Taking $X_1 = X_2 = X$ in Theorem B we obtain a homomorphism

$$\rho_X^{\mathrm{SH}}\colon \mathrm{Aut}(\mathcal{D}X) \to \mathrm{GL}(\mathrm{SH}(X,\mathbb{Q})).$$

The complex structure on a hyperkähler variety X induces a Hodge structure of weight 0 on $\widetilde{\mathrm{H}}(X,\mathbb{Q})$ given by

$$\widetilde{\mathrm{H}}(X,\mathbb{Q}) = \mathbb{Q}\alpha \oplus \mathrm{H}^2(X,\mathbb{Q}(1)) \oplus \mathbb{Q}\beta.$$

Denote by $\mathrm{Aut}\,\widetilde{\mathrm{H}}(X,\mathbb{Q})$ the group of Hodge isometries of $\widetilde{\mathrm{H}}(X,\mathbb{Q})$.

Theorem C (Section 4.2) *Let X be a hyperkähler variety of dimension $2d$ and second Betti number b_2. Assume that b_2 is odd or d is odd. Then ρ_X^{SH} factors over a map $\rho_X^{\widetilde{\mathrm{H}}}\colon \mathrm{Aut}(\mathcal{D}(X)) \to \mathrm{Aut}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$.*

See Sections 3.2 and 4.2 for an explicit description of the implicit map

$$\mathrm{Aut}(\widetilde{\mathrm{H}}(X,\mathbb{Q})) \to \mathrm{GL}(\mathrm{SH}(X,\mathbb{Q})).$$

Note that all known hyperkähler varieties satisfy the parity conditions in the theorem: there are two infinite series of deformation classes with odd b_2 (generalized Kummers and Hilbert schemes of points), and three exceptional deformation classes with odd d (K3, OG6, OG10).

1.4 Hodge structures of derived equivalent hyperkähler varieties

Another application of Theorem A is the following:

Theorem D (Section 5) *Let X_1 and X_2 be derived equivalent hyperkähler varieties. Then for every i the $\mathbb{Q}$–Hodge structures $\mathrm{H}^i(X_1,\mathbb{Q})$ and $\mathrm{H}^i(X_2,\mathbb{Q})$ are isomorphic.*

This confirms Orlov's conjecture for hyperkähler varieties. The proof is inspired by Soldatenkov [43].

1.5 Auto-equivalences of the Hilbert square of a K3 surface

In the second half of the paper we consider the problem of determining the image of ρ_X for certain hyperkähler varieties. An important difference with the first half of the paper is that *integral* structures (lattices, arithmetic subgroups, ...) will play an important role here.

As a first approximation to determining the image of ρ_X, we consider a variation of this problem which is deformation invariant. Let X be a smooth projective complex variety. If X' and X'' are smooth deformations of X (parametrized by paths in the base), and if $\Phi\colon \mathcal{D}X' \xrightarrow{\sim} \mathcal{D}X''$ is an equivalence, then we obtain an isomorphism as the composition

$$\mathrm{H}(X,\mathbb{Q}) \to \mathrm{H}(X',\mathbb{Q}) \xrightarrow{\Phi^{\mathrm{H}}} \mathrm{H}(X'',\mathbb{Q}) \to \mathrm{H}(X,\mathbb{Q}).$$

We define the *derived monodromy group* of X to be the subgroup $\mathrm{DMon}(X)$ of $\mathrm{GL}(\mathrm{H}(X,\mathbb{Q}))$ generated by all these isomorphisms. This group contains both the usual monodromy group of X and the image of $\rho_X\colon \mathrm{Aut}(\mathcal{D}X) \to \mathrm{GL}(\mathrm{H}(X,\mathbb{Q}))$.

If S is a K3 surface, then the result of Huybrechts, Macrì and Stellari [26] implies $\mathrm{DMon}(S) = \mathrm{O}^+(\widetilde{\mathrm{H}}(S,\mathbb{Z}))$, and that the image of ρ_S consists of those elements of $\mathrm{DMon}(S)$ that respect the Hodge structure on $\widetilde{\mathrm{H}}(S,\mathbb{Z})$. Similarly, for an abelian variety A, the results of [18] imply $\mathrm{DMon}(A) = \mathrm{Spin}(\mathrm{H}^1(A,\mathbb{Z}) \oplus \mathrm{H}^1(A^\vee,\mathbb{Z}))$, and that the image of ρ_A consists of those elements of $\mathrm{DMon}(A)$ that respect the Hodge structure on $\mathrm{H}^1(A,\mathbb{Z}) \oplus \mathrm{H}^1(A^\vee,\mathbb{Z})$.

Now let X be a hyperkähler variety of type $\mathrm{K3}^{[2]}$. We have $\mathrm{H}(X,\mathbb{Q}) = \mathrm{SH}(X,\mathbb{Q})$ and hence by Theorem C the action of $\mathrm{Aut}(\mathcal{D}X)$ on $\mathrm{H}(X,\mathbb{Q})$ factors over a subgroup $\mathrm{O}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$ of $\mathrm{GL}(\mathrm{H}(X,\mathbb{Q}))$.

For an integral lattice $\Lambda \subset \widetilde{\mathrm{H}}(X,\mathbb{Q})$ we denote by $\mathrm{O}^+(\Lambda) \subset \mathrm{O}(\Lambda)$ the subgroup consisting of those Hodge isometries that respect the orientation of a positive 4–plane in $\Lambda_{\mathbb{R}}$.

Theorem E (Section 9.4) *Let X be a hyperkähler variety deformation equivalent to the Hilbert square of a* K3 *surface. There is an integral lattice $\Lambda \subset \widetilde{\mathrm{H}}(X,\mathbb{Q})$ such that*

$$\mathrm{O}^+(\Lambda) \subset \mathrm{DMon}(X) \subset \mathrm{O}(\Lambda)$$

inside $\mathrm{O}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$.

See Section 9.4 for a precise description of Λ. As an abstract lattice, Λ is isomorphic to $\mathrm{H}^2(X,\mathbb{Z}) \oplus U$, but its image in $\widetilde{\mathrm{H}}(X,\mathbb{Q})$ is *not* $\mathbb{Z}\alpha \oplus \mathrm{H}^2(X,\mathbb{Z}) \oplus \mathbb{Z}\beta$.

Crucial in the proof of Theorem E is the *derived McKay correspondence* due to Bridgeland, King and Reid [11] and Haiman [21]. It provides an ample supply of elements of $\mathrm{DMon}(X)$: every deformation of X to the Hilbert square $S^{[2]}$ of a K3 surface S induces an inclusion $\mathrm{DMon}(S) \to \mathrm{DMon}(X)$. As part of the proof, we explicitly compute this inclusion.

We denote by $\mathrm{Aut}(\Lambda)$ the group of isometries of $\Lambda \subset \widetilde{\mathrm{H}}(X, \mathbb{Q})$ that respect the Hodge structure on $\widetilde{\mathrm{H}}(X, \mathbb{Q})$. It follows from Theorem E that $\mathrm{im}(\rho_X)$ is contained in $\mathrm{Aut}(\Lambda)$ for every X which is deformation equivalent to the Hilbert square of a K3 surface. For some X we can show that the upper bound in the above corollary is close to being sharp. Denote by $\mathrm{Aut}^+(\Lambda) \subset \mathrm{Aut}(\Lambda)$ the subgroup consisting of those Hodge isometries that respect the orientation of a positive 4–plane in $\Lambda_{\mathbb{R}}$.

Theorem F (Section 10.2) *Let S be a complex* K3 *surface and $X = S^{[2]}$. Assume that* $\mathrm{NS}(X)$ *contains a hyperbolic plane. Then* $\mathrm{Aut}^+(\Lambda) \subset \mathrm{im}(\rho_X) \subset \mathrm{Aut}(\Lambda)$.

Remark 1.2 To determine $\mathrm{im}\,\rho_X$ up to index 2 for a general hyperkähler of type $\mathrm{K3}^{[2]}$ new constructions of derived equivalences will be needed.

Remark 1.3 Theorems E and F leave an ambiguity of index 2, related to orientations on a maximal positive subspace of $\widetilde{\mathrm{H}}(X, \mathbb{R})$. In the case of K3 surfaces, it was conjectured by Szendrői [44] that derived equivalences must respect such orientation, and this was proven by Huybrechts, Macrì, and Stellari [26]. Their method is based on deformation to generic (formal or analytic) K3 surfaces of Picard rank 0, and on a complete understanding of the space of stability conditions on those [25]. It is far from clear if such a strategy can be used to remove the index 2 ambiguity for hyperkähler varieties of type $\mathrm{K3}^{[2]}$.

Remark 1.4 That a lattice of signature $(4, b_2 - 2)$ should play a role in describing the image of ρ_X for hyperkähler varieties X was expected from the physics literature — see Dijkgraaf [16] — but it is not clear where the lattice should come from, nor what its precise description should be for general hyperkähler varieties. In the above results, the lattice Λ arises in a rather implicit way, and one may hope for a more concrete interpretation of its elements.

Remark 1.5 It is tempting to try to conjecture a description of the group $\mathrm{Aut}(\mathcal{D}X)$ in terms of an action on a space of stability conditions on X, generalizing Bridgeland's work on K3 surfaces [10]. However, there is a representation-theoretic obstruction against doing this naively. The central charge of a hypothetical stability condition on X

takes values in $\mathrm{H}(X,\mathbb{C})$, yet Theorems E and F suggest the central charge should take values in $\widetilde{\mathrm{H}}(X,\mathbb{C})$. If X is of type $\mathrm{K3}^{[2]}$, then $\mathrm{H}(X,\mathbb{C})$ and $\widetilde{\mathrm{H}}(X,\mathbb{C})$ are nonisomorphic irreducible $\mathrm{DMon}(X)$–modules, so this would require a modification of the notion of stability condition.

Acknowledgements

I am grateful to Nick Addington, Thorsten Beckmann, Dion Leijnse, Eyal Markman, Zoë Schroot, and the referees for many valuable comments on earlier versions of this paper. This project received funding from the European Research Council (ERC), grant 864145, and from the Dutch Research Council (NWO).

2 The LLV Lie algebra of a smooth projective variety

In this section we recall the construction of Looijenga and Lunts [33] and Verbitsky [46] of a Lie algebra acting naturally on the cohomology of algebraic varieties. For holomorphic symplectic varieties we show that this Lie algebra is a derived invariant.

2.1 The LLV Lie algebra

Let F be a field of characteristic zero and M be a $\mathbb{Z}$–graded F–vector space of finite F–dimension. Denote by h the endomorphism of M that is multiplication by n on M_n.

Let e be an endomorphism of M of degree 2. We say that e *has the hard Lefschetz property* if for every $n \geq 0$ the map $e^n\colon M_{-n} \to M_n$ is an isomorphism. This is equivalent to the existence of an $f \in \mathrm{End}(M)$ such that the relations

$$[h,e]=2e, \quad [h,f]=-2f, \quad [e,f]=h \tag{1}$$

hold in $\mathrm{End}(M)$. Thus, (e,h,f) forms an $\mathfrak{sl}_2$–triple and defines a Lie homomorphism $\mathfrak{sl}_2 \to \mathrm{End}(M)$.

Proposition 2.1 *Assume that e has the hard Lefschetz property. Then the element f satisfying (1) is unique, and if e and h lie in a semisimple sub-Lie algebra $\mathfrak{g} \subset \mathrm{End}(M)$, then so does f.*

Proof The action of $\mathrm{ad}\, e$ on $\mathrm{End}(M)$ has the hard Lefschetz property for the grading defined by $\mathrm{ad}\, h$. In particular,

$$(\mathrm{ad}\, e)^2\colon \mathrm{End}(M)_{-2} \xrightarrow{\sim} \mathrm{End}(M)_2$$

is an isomorphism. It sends f to $-2e$, so f is indeed uniquely determined.

If e and h lie in $\mathfrak{g}$, then $\mathfrak{g} \subset \operatorname{End}(M)$ is graded and the above map restricts to an injective map

$$(\operatorname{ad} e)^2 \colon \mathfrak{g}_{-2} \hookrightarrow \mathfrak{g}_2.$$

Since h is diagonalizable, it is contained in a Cartan subalgebra of $\mathfrak{g}$. The symmetry of the resulting root system implies that $\dim \mathfrak{g}_{-n} = \dim \mathfrak{g}_n$ for all n. In particular, the map $(\operatorname{ad} e)^2$ defines an isomorphism between $\mathfrak{g}_{-2}$ and $\mathfrak{g}_2$; thus f lies in $\mathfrak{g}$. □

Let $\mathfrak{a}$ be an abelian Lie algebra and $e \colon \mathfrak{a} \to \mathfrak{gl}(M)$, defined by $a \mapsto e_a$, a Lie homomorphism. We say that e *has the hard Lefschetz property* if $e(\mathfrak{a}) \subset \mathfrak{gl}(M)_2$ and if there exists some $a \in \mathfrak{a}$ such that e_a has the hard Lefschetz property. Note that this is a Zariski open condition on $a \in \mathfrak{a}$.

If $e \colon \mathfrak{a} \to \mathfrak{gl}(M)$ has the hard Lefschetz property, then we denote by $\mathfrak{g}(\mathfrak{a}, M)$ the Lie algebra generated by the $\mathfrak{sl}_2$–triples (e_a, h, f_a) for $a \in \mathfrak{a}$ such that e_a has the hard Lefschetz property. We say that $(\mathfrak{a}, M)$ is a *Lefschetz module* if $\mathfrak{g}(\mathfrak{a}, M)$ is semisimple.

Now let X be a smooth projective complex variety of dimension d. Denote by $M := \mathrm{H}(X, \mathbb{Q})[d]$ the shifted total cohomology of X (with middle cohomology in degree 0). For a class $\lambda \in \mathrm{H}^2(X, \mathbb{Q})$, consider the endomorphism $e_\lambda \in \operatorname{End}(M)$ given by cup product with λ. If λ is ample, then e_λ has the hard Lefschetz property, so the map $e \colon \mathrm{H}^2(X, \mathbb{Q}) \to \mathfrak{gl}(M)$ has the hard Lefschetz property. We denote the corresponding Lie algebra by $\mathfrak{g}(X) := \mathfrak{g}(\mathrm{H}^2(X, \mathbb{Q}), M)$.

Proposition 2.2 [33, 1.6, 1.9] *$(\mathrm{H}^2(X, \mathbb{Q}), M)$ is a Lefschetz module.* □

In other words, $\mathfrak{g}(X)$ is a semisimple Lie algebra over $\mathbb{Q}$.

2.2 Hochschild homology and cohomology

Let X be a smooth projective variety of dimension d with canonical bundle $\omega_X := \Omega_X^d$. Its *Hochschild cohomology* is defined as

$$\mathrm{HH}^n(X) := \operatorname{Ext}^n_{X \times X}(\Delta_* \mathcal{O}_X, \Delta_* \mathcal{O}_X)$$

and its *Hochschild homology* is defined as

$$\mathrm{HH}_n(X) := \operatorname{Ext}^{d-n}_{X \times X}(\Delta_* \mathcal{O}_X, \Delta_* \omega_X).$$

Composition of extensions defines maps

$$\mathrm{HH}^n \otimes \mathrm{HH}^m \to \mathrm{HH}^{n+m}, \quad \mathrm{HH}^n \otimes \mathrm{HH}_m \to \mathrm{HH}_{m-n},$$

making $\mathrm{HH}_\bullet(X)$ into a graded module over the graded ring $\mathrm{HH}^\bullet(X)$.

The Hochschild–Kostant–Rosenberg isomorphism (twisted by the square root of the Todd class as in [30; 15]) defines isomorphisms

$$I^n\colon \mathrm{HH}^n(X) \xrightarrow{\sim} \bigoplus_{i+j=n} \mathrm{H}^i\bigl(X, \textstyle\bigwedge^j T_X\bigr), \quad I_n\colon \mathrm{HH}_n(X) \xrightarrow{\sim} \displaystyle\bigoplus_{j-i=n} \mathrm{H}^i(X, \Omega_X^j).$$

Under these isomorphisms, multiplication in $\mathrm{HH}^\bullet(X)$ corresponds to the operation induced by the product in $\bigwedge^\bullet T_X$, and the action of $\mathrm{HH}^\bullet(X)$ on $\mathrm{HH}_\bullet(X)$ corresponds to the action induced by the contraction action of $\bigwedge^\bullet T_X$ on $\Omega_X^\bullet$; see [12; 13].

Together with the degeneration of the Hodge–de Rham spectral sequence, the isomorphism $I_\bullet$ defines an isomorphism

$$\mathrm{HH}_\bullet(X) \xrightarrow{\sim} \mathrm{H}(X, \mathbb{C}).$$

This map does not respect the grading; rather it maps HH_i to the i^{th} column of the Hodge diamond (normalized so that the 0^{th} column is the central column $\bigoplus_p \mathrm{H}^{p,p}$). Combining with the action of $\mathrm{HH}^\bullet$ on $\mathrm{HH}_\bullet$, we obtain an action of the ring $\mathrm{HH}^\bullet(X)$ on $\mathrm{H}(X, \mathbb{C})$.

Theorem 2.3 *Let $\Phi\colon \mathcal{D}X_1 \xrightarrow{\sim} \mathcal{D}X_2$ be a derived equivalence between smooth projective complex varieties. Then we have natural graded isomorphisms*

$$\Phi^{\mathrm{HH}^\bullet}\colon \mathrm{HH}^\bullet(X_1) \xrightarrow{\sim} \mathrm{HH}^\bullet(X_2), \quad \Phi^{\mathrm{HH}_\bullet}\colon \mathrm{HH}_\bullet(X_1) \xrightarrow{\sim} \mathrm{HH}_\bullet(X_2),$$

compatible with the ring structure on $\mathrm{HH}^\bullet$ and the module structure on $\mathrm{HH}_\bullet$, and such that the square

$$\begin{array}{ccc} \mathrm{HH}_\bullet(X_1) & \xrightarrow{I} & \mathrm{H}(X_1, \mathbb{C}) \\ \downarrow{\scriptstyle \Phi^{\mathrm{HH}_\bullet}} & & \downarrow{\scriptstyle \Phi^{\mathrm{H}}} \\ \mathrm{HH}_\bullet(X_2) & \xrightarrow{I} & \mathrm{H}(X_2, \mathbb{C}) \end{array}$$

commutes.

Proof See [13; 34]. □

2.3 The Hochschild Lie algebra of a holomorphic symplectic variety

Now assume that X is holomorphic symplectic of dimension $2d$. That is, we assume that there exists a symplectic form $\sigma \in \mathrm{H}^0(X, \Omega_X^2)$. Note that this implies that a Zariski-dense collection of $\sigma \in \mathrm{H}^0(X, \Omega_X^2)$ will be nowhere degenerate.

Through the isomorphism $I\colon \mathrm{HH}_\bullet(X) \to \mathrm{H}(X, \mathbb{C})$, the vector space $\mathrm{H}(X, \mathbb{C})$ becomes a module under the ring $\mathrm{HH}^\bullet(X)$.

Lemma 2.4 $\mathrm{HH}^\bullet(X) \cong \mathrm{H}^\bullet(X,\mathbb{C})$ *as graded rings, and* $\mathrm{H}(X,\mathbb{C})$ *is free of rank one as an* $\mathrm{HH}^\bullet(X)$*–module.*

Proof A symplectic form σ defines an isomorphism $\Omega^1_X \xrightarrow{\sim} T_X$, and hence an isomorphism of algebras $\bigwedge^\bullet \Omega^1_X \xrightarrow{\sim} \bigwedge^\bullet T_X$. Combining this with the Hochschild–Kostant–Rosenberg isomorphism I and the degeneration of the Hodge–de Rham spectral sequence, we obtain a chain of isomorphisms of graded rings

$$\mathrm{HH}^\bullet(X) \xrightarrow{\sim} \mathrm{H}^\bullet\big(X, \textstyle\bigwedge^\bullet T_X\big) \xrightarrow{\sim} \mathrm{H}^\bullet(X,\Omega^\bullet_X) \xrightarrow{\sim} \mathrm{H}^\bullet(X,\mathbb{C}).$$

This proves the first assertion. For the second it suffices to observe that the module $\mathrm{HH}_\bullet(X,\mathbb{C})$ is generated by $\sigma^d \in \mathrm{HH}_{2d}(X) = \mathrm{H}^0(X,\Omega^{2d}_X)$. □

Consider the endomorphisms $h_p, h_q \in \mathrm{End}(\mathrm{H}(X,\mathbb{C}))$ given by

$$h_p = p-d, \quad h_q = q-d \quad \text{on } \mathrm{H}^{p,q}.$$

These define the Hodge bigrading on $\mathrm{H}(X,\mathbb{C})$, normalized to be symmetric along the central part $\mathrm{H}^{d,d}$. Note that $h = h_p + h_q$. The action of $\mathrm{HH}^n(X)$ on $\mathrm{H}(X,\mathbb{C})$ has degree n for the grading defined by $h' = h_q - h_p$.

Lemma 2.4 and hard Lefschetz imply:

Corollary 2.5 *For a Zariski-dense collection of* $\mu \in \mathrm{HH}^2(X)$, *the action by* μ,

$$e'_\mu \colon \mathrm{H}(X,\mathbb{C}) \to \mathrm{H}(X,\mathbb{C}),$$

has the hard Lefschetz property with respect to the grading defined by h'. □

In particular, for every such μ we have a complex subalgebra $\mathfrak{g}_\mu \subset \mathrm{End}(\mathrm{H}(X,\mathbb{C}))$ isomorphic to $\mathfrak{sl}_2$, and the collection of such algebras generates a Lie algebra which we denote by $\mathfrak{g}'(X) \subset \mathrm{End}(\mathrm{H}(X,\mathbb{C}))$. From Lemma 2.4 we also obtain:

Corollary 2.6 *The complex Lie algebras* $\mathfrak{g}'(X)$ *and* $\mathfrak{g}(X) \otimes_\mathbb{Q} \mathbb{C}$ *are isomorphic.* □

In the next section, we will show something stronger: that $\mathfrak{g}'(X)$ and $\mathfrak{g}(X) \otimes_\mathbb{Q} \mathbb{C}$ coincide as sub-Lie algebras of $\mathrm{End}(\mathrm{H}(X,\mathbb{C}))$. Theorem A then follows by combining this with the following proposition:

Proposition 2.7 *Assume that* X_1 *and* X_2 *are holomorphic symplectic varieties. Then for every equivalence* $\Phi\colon \mathcal{D}X_1 \xrightarrow{\sim} \mathcal{D}X_2$ *there exists a canonical isomorphism of complex Lie algebras*

$$\Phi^{\mathfrak{g}'} \colon \mathfrak{g}'(X_1) \xrightarrow{\sim} \mathfrak{g}'(X_2).$$

It has the property that the map $\Phi^{\mathrm{H}}\colon \mathrm{H}(X_1,\mathbb{C}) \xrightarrow{\sim} \mathrm{H}(X_2,\mathbb{C})$ *is equivariant with respect to* $\Phi^{\mathfrak{g}'}$.

Proof This follows immediately from Theorem 2.3. □

2.4 Comparison of the two Lie algebras and proof of Theorem A

The remainder of this section is devoted to the proof of the following:

Proposition 2.8 *If* X *is holomorphic symplectic, then* $\mathfrak{g}(X)\otimes_{\mathbb{Q}}\mathbb{C} = \mathfrak{g}'(X)$ *as sub-Lie algebras of* $\mathrm{End}(\mathrm{H}(X,\mathbb{C}))$.

Let X be holomorphic symplectic. If $\mathcal{F}$ is a coherent $\mathcal{O}_X$–module then we will simply write $\mathrm{H}^i(\mathcal{F})$ for $\mathrm{H}^i(X,\mathcal{F})$. We have decompositions

$$\mathrm{H}^2(X,\mathbb{C}) = \mathrm{H}^2(\mathcal{O}_X)\oplus \mathrm{H}^1(\Omega^1_X)\oplus \mathrm{H}^0(\Omega^2_X)$$

and

$$\mathrm{HH}^2(X) = \mathrm{H}^2(\mathcal{O}_X)\oplus \mathrm{H}^1(T_X)\oplus \mathrm{H}^0\big(\textstyle\bigwedge^2 T_X\big).$$

We will use the same symbol λ to denote an element $\lambda\in \mathrm{H}^2(X,\mathbb{C})$ and the endomorphism of $\mathrm{End}(\mathrm{H}(X,\mathbb{C}))$ given by cup product with λ. Note that $\lambda \in \mathfrak{g}(X)\otimes_{\mathbb{Q}}\mathbb{C}$ by construction. Similarly, we will use the same symbol for $\mu\in\mathrm{HH}^2(X)$ and the resulting $\mu\in\mathrm{End}(\mathrm{H}(X,\mathbb{C}))$, given by contraction with μ. We have $\mu\in\mathfrak{g}'(X)$.

For a symplectic form $\sigma\in\mathrm{H}^0(\Omega^2_X)$, we denote by $\check{\sigma}\in\mathrm{H}^0\big(\bigwedge^2 T_X\big)$ the image of the form $\sigma\in\mathrm{H}^0(\Omega^2_X)$ under the isomorphism $\Omega^2_X\to\bigwedge^2 T_X$ defined by σ. In suitable local coordinates, we have

$$\sigma = \mathrm{d}u_1\wedge\mathrm{d}v_1+\cdots+\mathrm{d}u_d\wedge\mathrm{d}v_d$$

and

$$\check{\sigma} = \frac{\partial}{\partial u_1}\wedge\frac{\partial}{\partial v_1}+\cdots+\frac{\partial}{\partial u_d}\wedge\frac{\partial}{\partial v_d}.$$

Lemma 2.9 *If* σ *is a nowhere degenerate symplectic form then* $(\sigma,h_p,\check{\sigma})$ *is an* $\mathfrak{sl}_2$–*triple in* $\mathrm{End}(\mathrm{H}(X,\mathbb{C}))$.

Proof Clearly σ has degree 2 and $\check{\sigma}$ has degree -2 for the grading given by h_p, so $[h_p,\sigma]=2\sigma$ and $[h_p,\check{\sigma}]=-2\check{\sigma}$.

We need to show that $[\sigma,\check{\sigma}]=h_p$. This follows immediately from a local computation: in the above local coordinates, one verifies that on the standard basis of Ω^p the commutator $[\sigma,\check{\sigma}]$ acts as $p-d$. □

Note that the existence of one nowhere degenerate σ implies that a Zariski-dense collection of $\sigma \in \mathrm{H}^0(\Omega_X^2)$ is nowhere degenerate.

Lemma 2.10 *For a Zariski-dense collection $\alpha \in \mathrm{H}^2(X, \mathcal{O}_X)$, there is $\check{\alpha} \in \mathrm{End}(\mathrm{H}(X, \mathbb{C}))$ such that $(\alpha, h_q, \check{\alpha})$ is an $\mathfrak{sl}_2$–triple.*

Proof This follows from Lemma 2.9 and Hodge symmetry. □

Lemma 2.11 *For all $\tau \in \mathrm{H}^0\big(X, \bigwedge^2 T_X\big)$ the endomorphism τ lies in $\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$.*

Proof It suffices to show that this holds for a Zariski-dense collection of τ; hence we may assume without loss of generality that $\tau = \check{\sigma}$ with σ and $\check{\sigma}$ as in Lemma 2.9. Let α and $\check{\alpha}$ be as in Lemma 2.10. Because σ and h_p commute with both α and h_q, we have that every element of the $\mathfrak{sl}_2$–triple $(\sigma, h_p, \check{\sigma})$ commutes with every element of the $\mathfrak{sl}_2$–triple $(\alpha, h_q, \check{\alpha})$. From this, it follows that

$$(\alpha + \sigma, h, \check{\alpha} + \check{\sigma}) \quad \text{and} \quad (\alpha - \sigma, h, \check{\alpha} - \check{\sigma})$$

are $\mathfrak{sl}_2$–triples. Since the elements $\alpha \pm \sigma$ lie in $\mathrm{H}^2(X, \mathbb{C})$, and apparently have the hard Lefschetz property, we conclude that the endomorphisms $\check{\alpha} \pm \check{\sigma}$ lie in $\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$; hence also $\tau = \check{\sigma}$ lies in $\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$. □

Corollary 2.12 *h_p and h_q lie in $\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$.*

Proof By Lemma 2.9 we have $h_p = [\sigma, \check{\sigma}]$, which by Lemma 2.11 lies in $\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$. Since $h_q = h - h_p$ we also have that h_q lies in $\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$. □

Fix a $\tau \in \mathrm{H}^0\big(X, \bigwedge^2 T_X\big)$ that is nowhere degenerate as an alternating form on Ω_X^1. This defines isomorphisms $c_\tau \colon \Omega_X^1 \to T_X$ and $c_\tau \colon \mathrm{H}^1(\Omega_X^1) \to \mathrm{H}^1(T_X)$ given by contracting sections of Ω_X^1 with τ.

Lemma 2.13 *For all $\eta \in \mathrm{H}^1(\Omega_X^1)$, we have $[\tau, \eta] = c_\tau(\eta)$ in $\mathrm{End}(\mathrm{H}(X, \mathbb{C}))$.*

Proof This is again a local computation. If η is a local section of Ω_X^1, then a computation on a local basis shows $[\tau, \eta] = c_\tau(\eta)$ as maps $\Omega_X^p \to \Omega_X^{p-1}$. □

Corollary 2.14 *Every element η' of $\mathrm{H}^1(X, T_X)$ lies in $\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$.*

Proof (See also [19, 4.5] for the case of a hyperkähler variety.) Every such η' is of the form $c_\tau(\eta)$ for a unique $\eta \in \mathrm{H}^1(\Omega_X^1)$, and hence the corollary follows from Lemmas 2.13 and 2.11 and the fact that η lies in $\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$. □

We can now finish the comparison of the two Lie algebras.

Proof of Proposition 2.8 By Corollary 2.6 it suffices to show that $\mathfrak{g}'(X)$ is contained in $\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$. By Proposition 2.1 it suffices to show that h' is contained in $\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$, and that for almost every $a \in \mathrm{HH}^2(X)$ we have that the action of a on $\mathrm{H}(X, \mathbb{C})$ is contained in $\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$. This follows from Lemma 2.11, Corollaries 2.12 and 2.14, and the fact that the action of any $\alpha \in \mathrm{H}^2(\mathcal{O}_X)$ lies in $\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$. □

Together with Proposition 2.7, this proves Theorem A.

3 Rational cohomology of hyperkähler varieties

3.1 The BBF form and the LLV Lie algebra

Let X be a complex hyperkähler variety of dimension $2d$. We denote by

$$b = b_X \colon \mathrm{H}^2(X, \mathbb{Q}) \times \mathrm{H}^2(X, \mathbb{Q}) \to \mathbb{Q}$$

its Beauville–Bogomolov–Fujiki, and by c_X its Fujiki constant. These are related by

$$\int_X \lambda^{2d} = \frac{(2d)!}{2^d\, d!} c_X b(\lambda, \lambda)^d \tag{2}$$

for $\lambda \in \mathrm{H}^2(X, \mathbb{Q})$; see eg [41].

We extend b to a bilinear form on

$$\widetilde{\mathrm{H}}(X, \mathbb{Q}) := \mathbb{Q}\alpha \oplus \mathrm{H}^2(X, \mathbb{Q}) \oplus \mathbb{Q}\beta,$$

by declaring α and β to be orthogonal to $\mathrm{H}^2(X, \mathbb{Q})$, and setting $b(\alpha, \beta) = -1$, $b(\alpha, \alpha) = 0$ and $b(\beta, \beta) = 0$. We equip $\widetilde{\mathrm{H}}(X, \mathbb{Q})$ with a grading satisfying $\deg \alpha = -2$ and $\deg \beta = 2$, and for which $\mathrm{H}^2(X, \mathbb{Q})$ sits in degree 0. This induces a grading on the Lie algebra $\mathfrak{so}(\widetilde{\mathrm{H}}(X, \mathbb{Q}))$.

For $\lambda \in \mathrm{H}^2(X, \mathbb{Q})$ we consider the endomorphism $e_\lambda \in \mathfrak{so}(\widetilde{\mathrm{H}}(X, \mathbb{Q}))$ given by $e_\lambda(\alpha) = \lambda$, $e_\lambda(\mu) = b(\lambda, \mu)\beta$ for all $\mu \in \mathrm{H}^2(X, \mathbb{Q})$, and $e_\lambda(\beta) = 0$.

Theorem 3.1 (Looijenga–Lunts, Verbitsky) *There is a unique isomorphism of graded Lie algebras*

$$\mathfrak{so}(\widetilde{\mathrm{H}}(X, \mathbb{Q})) \xrightarrow{\sim} \mathfrak{g}(X)$$

that maps e_λ to e_λ for every $\lambda \in \mathrm{H}^2(X, \mathbb{Q})$.

Proof See [33, Proposition 4.5] or [46, Theorem 1.4] for the theorem over the real numbers. This readily descends to $\mathbb{Q}$; see [43, Proposition 2.9] for more details. □

The representation of $\mathfrak{so}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$ on $\mathrm{H}(X,\mathbb{Q})$ integrates to a representation of the group $\mathrm{Spin}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$ on $\mathrm{H}(X,\mathbb{Q})$. Let $\lambda \in \mathrm{H}^2(X,\mathbb{Q})$. Then e_λ is nilpotent, and hence $B_\lambda := \exp e_\lambda$ is an element of $\mathrm{Spin}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$. It acts on $\widetilde{\mathrm{H}}(X,\mathbb{Q})$ by

$$B_\lambda(r\alpha+\mu+s\beta) = r\alpha + (\mu + r\lambda) + \bigl(s + b(\mu,\lambda) + r\tfrac{1}{2}b(\lambda,\lambda)\bigr)\beta \tag{3}$$

for all $r, s \in \mathbb{Q}$ and $\mu \in \mathrm{H}^2(X,\mathbb{Q})$. The action on the total cohomology of X is given by:

Proposition 3.2 *B_λ acts as multiplication by $\mathrm{ch}(\lambda)$ on $\mathrm{H}(X,\mathbb{Q})$.* □

In particular, if $\mathcal{L}$ is a line bundle on X and $\Phi\colon \mathcal{D}X \to \mathcal{D}X$ is the equivalence that maps $\mathcal{F}$ to $\mathcal{F}\otimes\mathcal{L}$, then $\Phi^{\mathrm{H}} = B_{c_1(\mathcal{L})}$.

3.2 The Verbitsky component of cohomology

Let X be a complex hyperkähler variety of dimension $2d$. We define the *even cohomology* of X as the graded $\mathbb{Q}$–algebra

$$\mathrm{H}^{\mathrm{ev}}(X,\mathbb{Q}) := \bigoplus_n \mathrm{H}^{2n}(X,\mathbb{Q}),$$

and the *Verbitsky component* of the cohomology of X as the sub-$\mathbb{Q}$–algebra $\mathrm{SH}(X,\mathbb{Q})$ of $\mathrm{H}^{\mathrm{ev}}(X,\mathbb{Q})$ generated by $\mathrm{H}^2(X,\mathbb{Q})$. Clearly, $\mathrm{SH}(X,\mathbb{Q})[2d]$ is a sub-Lefschetz module of $\mathrm{H}^{\mathrm{ev}}(X,\mathbb{Q})[2d]$ for $\mathrm{H}^2(X,\mathbb{Q})$.

Lemma 3.3 (Verbitsky [8; 45]) *The kernel of the $\mathbb{Q}$–algebra homomorphism*

$$\mathrm{Sym}^\bullet \mathrm{H}^2(X,\mathbb{Q}) \twoheadrightarrow \mathrm{SH}(X,\mathbb{Q})$$

is generated by the elements λ^{d+1} with $\lambda \in \mathrm{H}^2(X,\mathbb{Q})$ satisfying $b(\lambda,\lambda)=0$. □

Lemma 3.4 (Verbitsky) *$\mathrm{SH}(X,\mathbb{Q})[2d]$ is an irreducible Lefschetz module.*

Proof It is the smallest sub-Lefschetz module of $\mathrm{H}^{\mathrm{ev}}(X,\mathbb{Q})[2d]$ having a nontrivial component of degree $-2d$. □

Verbitsky also describes the space $\mathrm{SH}(X,\mathbb{Q})$ explicitly. Below we normalize this description, and use it to compute the Mukai pairing on $\mathrm{SH}(X,\mathbb{Q})$.

Proposition 3.5 *There is a unique map*

$$\Psi\colon \mathrm{SH}(X,\mathbb{Q})[2d] \to \mathrm{Sym}^d \widetilde{\mathrm{H}}(X,\mathbb{Q})$$

satisfying

(i) Ψ *is morphism of Lefschetz modules*,

(ii) $\Psi(1) = \alpha^d/d!$.

Note that the Lefschetz module structure on $\mathrm{Sym}^d \widetilde{\mathrm{H}}(X,\mathbb{Q})$ is given by the Leibniz rule

$$e_\lambda(x_1 \cdots x_d) := \sum_i x_1 \cdots e_\lambda(x_i) \cdots x_d.$$

Proof Uniqueness is clear. For existence, consider the map

$$\widetilde{\Psi}\colon \mathrm{Sym}^\bullet \mathrm{H}^2(X,\mathbb{Q}) \to \mathrm{Sym}^d \widetilde{\mathrm{H}}(X,\mathbb{Q}),$$

given by

$$\lambda_1 \cdots \lambda_n \mapsto e_{\lambda_1} \cdots e_{\lambda_n}(\alpha^d/d!).$$

This map is well defined since the e_{λ_i} commute. Moreover, the map is graded and satisfies $\widetilde{\Psi}(\lambda x) = e_\lambda \widetilde{\Psi}(x)$ for all $\lambda \in \mathrm{H}^2(X,\mathbb{Q})$ and $x \in \mathrm{Sym}^\bullet \mathrm{H}^2(X,\mathbb{Q})$. To show that $\widetilde{\Psi}$ induces a morphism of Lefschetz modules with the desired properties it now suffices to verify that it vanishes on the ideal generated by the λ^{d+1} for $\lambda \in \mathrm{H}^2(X,\mathbb{Q})$ satisfying $b(\lambda,\lambda) = 0$. Equivalently, it suffices to show that for every $x \in \mathrm{Sym}^d \widetilde{\mathrm{H}}(X,\mathbb{Q})$ and for every $\lambda \in \mathrm{H}^2(X,\mathbb{Q})$ with $b(\lambda,\lambda) = 0$ we have $e_\lambda^{d+1}(x) = 0$.

Without loss of generality, we may assume that x is a monomial of the form

$$x = \alpha^i \beta^j \lambda_1 \cdots \lambda_m, \quad i + j + m = d, \quad \lambda_i \in \mathrm{H}^2(X,\mathbb{Q}).$$

For degree reasons, we have $e_\lambda^k(\beta^j \lambda_1 \cdots \lambda_m) = 0$ for $k > m$. Moreover, it follows from $b(\lambda,\lambda) = 0$ that $e_\lambda^k(\alpha^i) = 0$ for $k > i$. Combining these, one concludes that $e_\lambda^{d+1}(x) = 0$, which is what we had to prove. □

Lemma 3.6 $$\Psi(\mathrm{pt}_X) = \beta^d/c_X.$$

Proof Choose $\lambda \in \mathrm{H}^2(X,\mathbb{Q})$ with $b(\lambda,\lambda) \neq 0$. Then we have

$$\Psi(\lambda^{2d}) = e_\lambda^{2d}\left(\frac{\alpha^d}{d!}\right) = \frac{(2d)!}{2^d d!} b(\lambda,\lambda)^d \beta^d. \tag{4}$$

Dividing by (2) gives the claimed identity. □

Consider the contraction (or Laplacian) operator

$$\Delta\colon \operatorname{Sym}^d \widetilde{\mathrm{H}}(X,\mathbb{Q}) \to \operatorname{Sym}^{d-2} \widetilde{\mathrm{H}}(X,\mathbb{Q}),$$

given by

$$x_1 \dots x_d \mapsto \sum_{i<j} b(x_i, x_j) x_1 \cdots \hat{x}_i \cdots \hat{x}_j \cdots x_d.$$

This is a morphism of Lefschetz modules, or equivalently of $\mathfrak{so}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$–modules.

Lemma 3.7 *The sequence of Lefschetz modules*

$$0 \to \mathrm{SH}(X,\mathbb{Q})[2d] \xrightarrow{\Psi} \operatorname{Sym}^d \widetilde{\mathrm{H}}(X,\mathbb{Q}) \xrightarrow{\Delta} \operatorname{Sym}^{d-2} \widetilde{\mathrm{H}}(X,\mathbb{Q}) \to 0$$

is exact.

Proof Since $\Delta\Psi(1) = 0$, we have $\Delta \circ \Psi = 0$. The map Δ is well known to be a surjective map of $\mathfrak{so}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$–modules with irreducible kernel. Since Ψ is nonzero and $\mathrm{SH}(X,\mathbb{Q})$ is irreducible, it follows that the sequence is exact. □

The *Mukai pairing* [14] on $\mathrm{H}^{\mathrm{ev}}(X,\mathbb{Q})$ restricts to a pairing b_{SH} on $\mathrm{SH}(X,\mathbb{Q})$. It pairs elements of degree m with elements of degree $2d-m$, according to the formula

$$b_{\mathrm{SH}}(\lambda_1 \cdots \lambda_m,\, \mu_1 \cdots \mu_{2d-m}) = (-1)^m \int_X \lambda_1 \cdots \lambda_m \mu_1 \cdots \mu_{2d-m}.$$

Note that $b_{\mathrm{SH}}(e_\lambda x, y) + b_{\mathrm{SH}}(x, e_\lambda y) = 0$ for all $x, y \in \mathrm{SH}(X,\mathbb{Q})$ and $\lambda \in \mathrm{H}^2(X,\mathbb{Q})$, so b_{SH} is $\mathfrak{so}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$–invariant.

The pairing on $\widetilde{\mathrm{H}}(X,\mathbb{Q})$ induces a pairing on $\operatorname{Sym}^d \widetilde{\mathrm{H}}(X,\mathbb{Q})$ defined by

$$b_{[d]}(x_1 \cdots x_d, y_1 \cdots y_d) := (-1)^d \sum_{\sigma \in \mathfrak{S}_d} \prod_i b(x_i, y_{\sigma i}).$$

By construction, $b_{[d]}$ is $\mathfrak{so}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$–invariant. The map Ψ is almost an isometry, in the following sense:

Proposition 3.8 *For all $x, y \in \mathrm{SH}(X,\mathbb{Q})$,*

$$c_X b_{[d]}(\Psi x, \Psi y) = b_{\mathrm{SH}}(x, y).$$

Proof Both the Mukai form on $\mathrm{SH}(X,\mathbb{Q})[2d]$ and the pairing on $\operatorname{Sym}^d \widetilde{\mathrm{H}}(X,\mathbb{Q})$ are $\mathfrak{so}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$–invariant. Since $\mathrm{SH}(X,\mathbb{Q})$ is an irreducible $\mathfrak{so}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$–module, it suffices to verify the identity for some $x, y \in \mathrm{SH}(X,\mathbb{Q})$ with $b_{\mathrm{SH}}(x, y) \neq 0$.

Let $\lambda \in \mathrm{H}^2(X, \mathbb{Q})$ with $b(\lambda, \lambda) \neq 0$. We have

$$b_{\mathrm{SH}}(1, \lambda^{2d}) = \int_X \lambda^{2d} = \frac{(2d)!}{2^d\, d!} c_X b(\lambda, \lambda)^d \neq 0.$$

By (4),

$$\Psi(\lambda^{2d}) = \frac{(2d)!}{2^d\, d!} b(\lambda, \lambda)^d \beta^d,$$

and hence

$$c_X b_{[d]}(\Psi(1), \Psi(\lambda^{2d})) = \frac{c_X (2d)!}{2^d\, (d!)^2} b_{[d]}(\alpha^d, \beta^d) = \frac{c_X (2d)!}{2^d\, d!} b(\lambda, \lambda)^d,$$

which agrees with the above expression for $b_{\mathrm{SH}}(1, \lambda^{2d})$. □

Remark 3.9 If X is of type $\mathrm{K3}^{[d]}$ then $c_X = 1$ and Ψ is an isometry.

4 Action of derived equivalences on the Verbitsky component

In this section we prove Theorems B and C from the introduction.

4.1 A representation-theoretical construction

Let K be a field of characteristic different from 2, and let $V = (V, b)$ be a nondegenerate quadratic space over K. Let d be a positive integer and consider the space

$$S_{[d]} V := \ker(\mathrm{Sym}^d\, V \xrightarrow{\Delta} \mathrm{Sym}^{d-2}\, V).$$

The Lie algebra $\mathfrak{so}(V)$ acts faithfully on $S_{[d]} V$, inducing an inclusion

$$\mathfrak{so}(V) \subset \mathrm{End}(S_d V).$$

Consider the normalizer of $\mathfrak{so}(V)$ in $\mathrm{GL}(S_{[d]} V)$, that is, the group

$$N(V, d) := \{ g \in \mathrm{GL}(S_{[d]} V) \mid g\, \mathfrak{so}(V) g^{-1} = \mathfrak{so}(V) \}.$$

Proposition 4.1 *Assume that K is separably closed. Then there is an exact sequence*

$$1 \to \{\pm 1\} \to \mathrm{O}(V) \times K^{\times} \to N(V, d) \to 1,$$

where the inclusion maps ϵ to (ϵ, ϵ^d) and the surjection maps (φ, λ) to $\lambda S_{[d]}(\varphi)$.

Proof The only nontrivial part is surjectivity of $\mathrm{O}(V) \times K^{\times} \to N(V, d)$. Denote by

$$\sigma\colon \mathrm{O}(V) \to N(V, d), \quad \varphi \mapsto S_{[d]}(\varphi),$$

the restriction of this map to the first component.

The representation $S_{[d]}V$ of $\mathfrak{so}(V)$ is irreducible, so by Schur's lemma the centralizer of $\mathfrak{so}(V)$ in $\mathrm{GL}(S_{[d]}V)$ is $K^\times$, and we have an exact sequence

$$1 \to K^\times \to N(V,d) \xrightarrow{\psi} \mathrm{Aut}(\mathfrak{so}(V)).$$

It therefore suffices to show that the image of ψ equals the image of $\psi \circ \sigma$.

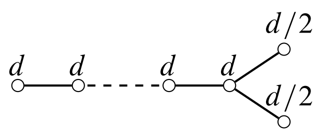
The adjoint group of $\mathfrak{so}(V)$ is $\mathrm{PSO}(V)$, and we have a short exact sequence

$$1 \to \mathrm{PSO}(V) \to \mathrm{Aut}(\mathfrak{so}(V)) \to \mathrm{Out}(\mathfrak{so}(V)) \to 1, \tag{5}$$

where $\mathrm{Out}(\mathfrak{so}(V))$ coincides with the group of symmetries of the Dynkin diagram.

If $\dim V = 2n+1$, then we have $\mathrm{PSO}(V) = \mathrm{SO}(V)$. The Dynkin diagram (of type B_n) has no nontrivial automorphisms, so $\mathrm{Aut}(\mathfrak{so}(V,b)) = \mathrm{SO}(V)$. The composition $\psi \circ \sigma$ maps $\mathrm{SO}(V)$ identically to $\mathrm{SO}(V)$, and we conclude that the image of ψ is the image of $\psi \circ \sigma$.

Now assume $\dim V = 2n$. Since K is algebraically closed, $\mathrm{PSO}(V) = \mathrm{SO}(V)/\{\pm 1\}$. The larger group $\mathrm{O}(V)/\{\pm 1\}$ embeds in $\mathrm{Aut}\,\mathfrak{so}(V)$, with elements of determinant -1 in $\mathrm{O}(V)$ inducing the reflection in the horizontal axis in the Dynkin diagram (of type D_n). For $n \neq 4$, this inclusion is an equality, while for $n = 4$ "triality" gives extra automorphisms. However, expressed on simple roots the highest weight of the representation $S_{[d]}V$ of $\mathfrak{so}(V)$ is



such that for $n = 4$ the extra automorphisms of $\mathfrak{so}(V)$ do not lift to automorphisms of $S_{[d]}V$. We conclude that the image of ψ is contained in $\mathrm{O}(V)/\{\pm 1\}$ and that the composition $\psi \circ \sigma$ is the natural map $\mathrm{O}(V) \to \mathrm{O}(V)/\{\pm 1\}$, so also in this case the image of ψ coincides with the image of $\psi \circ \sigma$. □

Remark 4.2 The condition that K is algebraically closed is needed in the case of even $\dim V$. If K is not algebraically closed, then one still has the exact sequence (5), but one should be careful to define $\mathrm{PSO}(V)$ as the group of K–points of the algebraic group $\mathbf{PSO}(V)$ over K. In general, this group is bigger than $\mathrm{SO}(V)/\{\pm 1\}$). In particular, not every element of $N(V,d)$ can be lifted to $\mathrm{O}(V) \times K^\times$.

Proposition 4.3 *Let V_1 and V_2 be nondegenerate quadratic spaces over K. Assume that there is a linear isomorphism $f : S_{[d]}V_1 \to S_{[d]}V_2$ such that $f\,\mathfrak{so}(V_1)\,f^{-1} = \mathfrak{so}(V_2)$*

as subspaces of $\mathrm{End}(V_2)$. *Then there exists a* $\mu \in K^\times$ *and a similitude* $\varphi\colon V_1 \to V_2$ *such that* $f = \mu S_{[d]}(\varphi)$.

Proof Let $\overline{K}$ be a separable closure of K. Consider the $\mathrm{Gal}(\overline{K}/K)$–sets

$$S := \{\varphi\colon V_{1,\overline{K}} \to V_{2,\overline{K}} \mid \varphi \text{ is a similitude}\}$$

and

$$N := \{g\colon S_{[d]}V_{1,\overline{K}} \to S_{[d]}V_{2,\overline{K}} \mid g\,\mathfrak{so}(V_{1,\overline{K}})g^{-1} = \mathfrak{so}(V_{2,\overline{K}})\}$$

and the Galois-equivariant map

$$\xi\colon \overline{K}^\times \times S \to N, \quad (\mu,\varphi) \mapsto \mu S_{[d]}(\varphi).$$

The map ξ is surjective. Indeed, since over a separably closed field the quadratic spaces are isometric, we may assume without loss of generality that $V_1 = V_2$. Then $N = N(V_{1,\overline{K}}, d)$ and the surjectivity follows from Proposition 4.1 (it suffices even to consider isometries instead of similitudes).

The group $\overline{K}^\times$ acts on $\overline{K}^\times \times S$ by $\lambda(\mu,\varphi) := (\lambda^{-d}\mu, \lambda\varphi)$ and the fibers of ξ are principal homogenous spaces under this action.

The map f defines a Galois-invariant element $f \in N$, so its fiber $\xi^{-1}(f)$ carries a natural Galois action. By Hilbert 90, we have $\mathrm{H}^1(\mathrm{Gal}(\overline{K}/K, \overline{K}^\times) = \{1\}$, which implies that $\xi^{-1}(f)$ contains a Galois-invariant element (μ,φ). □

The bilinear form b on V induces a bilinear form $b_{[d]}$ on $S_{[d]}V$ defined as

$$b_{[d]}(x_1\cdots x_d, y_1\cdots y_d) := (-1)^d \sum_{\sigma\in S_n} \prod_i b(x_i, y_{\sigma i}),$$

Consider the group

$$G(V,d) := N(V,d) \cap \mathrm{O}(S_{[d]}, b_{[d]})$$

of isometries of $S_{[d]}V$ that preserve the subspace $\mathfrak{so}(V)$ of $\mathrm{End}\, S_{[d]}V$.

Proposition 4.4 *If* d *is odd, then the map*

$$\mathrm{O}(V) \to G(V,d), \quad \varphi \mapsto S_{[d]}(\varphi),$$

is an isomorphism. If d *is even and* $\dim V$ *is odd, then the map*

$$\mathrm{O}(V) \to G(V,d), \quad \varphi \mapsto \det(\varphi)S_{[d]}(\varphi),$$

is an isomorphism.

Proof Assume first that K is separably closed. The short exact sequence of Proposition 4.1 restricts to a short exact sequence

$$1 \to \{\pm 1\} \to \mathrm{O}(V) \times \{\pm 1\} \to G(V,d) \to 1,$$

from which one verifies directly that the given maps are isomorphisms. If K is not separably closed, then the result follows from taking Galois invariants. □

Remark 4.5 If both d and $\dim V$ are even, one obtains

$$G(V_{\overline{K}}, d) \cong \mathrm{O}(V_{\overline{K}})/\{\pm 1\} \times \{\pm 1\}.$$

Note, however, that in general there are more Galois-invariant elements than just those in $\mathrm{O}(V)/\{\pm 1\}$. See also Remark 4.2.

4.2 The Verbitsky component

Theorem 4.6 *Let X_1 and X_2 be hyperkähler varieties and $\Phi\colon \mathcal{D}X_1 \to \mathcal{D}X_2$ an equivalence. Then the induced isomorphism $\Phi^{\mathrm{H}}\colon \mathrm{H}(X_1,\mathbb{Q}) \to \mathrm{H}(X_2,\mathbb{Q})$ restricts to an isomorphism $\Phi^{\mathrm{SH}}\colon \mathrm{SH}(X_1,\mathbb{Q}) \to \mathrm{SH}(X_2,\mathbb{Q})$. Moreover:*

(i) *Φ^{SH} is an isometry with respect to the Mukai pairings.*

(ii) *$\Phi^{\mathrm{SH}}\mathfrak{g}(X_1)(\Phi^{\mathrm{SH}})^{-1} = \mathfrak{g}(X_2)$ in $\mathrm{End}(\mathrm{SH}(X_2,\mathbb{Q}))$.*

Proof Note that $\mathrm{SH}(X,\mathbb{Q})$ can be characterized as the minimal sub-$\mathfrak{g}(X)$–module of $\mathrm{H}(X,\mathbb{Q})$ whose Hodge structure attains the maximal possible level (width). It then follows from Theorem A and from Lemma 3.4 that Φ^{H} restricts to an isomorphism

$$\Phi^{\mathrm{SH}}\colon \mathrm{SH}(X_1,\mathbb{Q}) \xrightarrow{\sim} \mathrm{SH}(X_2,\mathbb{Q})$$

respecting the Lie algebras $\mathfrak{g}(X_1)$ and $\mathfrak{g}(X_2)$. By [14], the map Φ^{H} respects the Mukai pairings, and the theorem follows. □

Definition 4.7 For a complex hyperkähler variety we equip $\mathrm{SH}(X,\mathbb{Q})$ and $\widetilde{\mathrm{H}}(X,\mathbb{Q})$ with Hodge structures of weight 0, given by

$$\mathrm{SH}(X,\mathbb{Q}) \subset \mathrm{H}^{\mathrm{ev}}(X,\mathbb{Q}) = \bigoplus_n \mathrm{H}^{2n}(X,\mathbb{Q}(n))$$

and

$$\widetilde{\mathrm{H}}(X,\mathbb{Q}) = \mathbb{Q}\alpha \oplus \mathrm{H}^2(X,\mathbb{Q}(1)) \oplus \mathbb{Q}\beta.$$

Lemma 4.8 *Let X be a hyperkähler variety of dimension $2d$. Then the map*

$$\Psi\colon \mathrm{SH}(X,\mathbb{Q}) \to \mathrm{Sym}^d\, \widetilde{\mathrm{H}}(X,\mathbb{Q})$$

of Proposition 3.5 is a morphism of Hodge structures of weight 0.

Proof One verifies directly that the "action map"

$$\mathrm{H}^2(X,\mathbb{Q}(1))\otimes\widetilde{\mathrm{H}}(X,\mathbb{Q})\to\widetilde{\mathrm{H}}(X,\mathbb{Q}),$$

which maps (λ,x) to $e_\lambda(x)$ is a map of Hodge structures. From this it follows that the action map

$$\mathrm{H}^2(X,\mathbb{Q}(1))\otimes\mathrm{Sym}^d\,\widetilde{\mathrm{H}}(X,\mathbb{Q})\to\mathrm{Sym}^d\,\widetilde{\mathrm{H}}(X,\mathbb{Q})$$

is a map of Hodge structures, and that the map

$$\widetilde{\Psi}\colon\mathrm{Sym}^\bullet\,\mathrm{H}(X,\mathbb{Q}(1))\to\mathrm{Sym}^d\,\widetilde{\mathrm{H}}(X,\mathbb{Q})$$

from the proof of Proposition 3.5 is a morphism of Hodge structures.

Since multiplication in the cohomology of X preserves the Hodge structure, the quotient map $\mathrm{Sym}^\bullet\,\mathrm{H}(X,\mathbb{Q}(1))\to\mathrm{SH}(X,\mathbb{Q})$ is also a morphism of Hodge structures, and hence so is the map Ψ constructed in the proof of Proposition 3.5. □

Proposition 4.9 *Let X_1 and X_2 be derived equivalent hyperkähler varieties. Then there exists a Hodge similitude $\varphi\colon\widetilde{\mathrm{H}}(X_1,\mathbb{Q})\xrightarrow{\sim}\widetilde{\mathrm{H}}(X_2,\mathbb{Q})$ and a scalar $\lambda\in\mathbb{Q}^\times$ such that the square*

$$\begin{array}{ccc}\mathrm{SH}(X_1,\mathbb{Q}) & \xrightarrow{\Phi^{\mathrm{SH}}} & \mathrm{SH}(X_2,\mathbb{Q})\\ \downarrow{\Psi} & & \downarrow{\Psi}\\ \mathrm{Sym}^d\,\widetilde{\mathrm{H}}(X_1,\mathbb{Q}) & \xrightarrow{\lambda\,\mathrm{Sym}^d(\varphi)} & \mathrm{Sym}^d\,\widetilde{\mathrm{H}}(X_2,\mathbb{Q})\end{array}$$

commutes.

Proof Recall from Lemma 3.7 that the image of Ψ is precisely $S_{[d]}\widetilde{\mathrm{H}}\subset\mathrm{Sym}^d\,\widetilde{\mathrm{H}}$. It then follows from Theorem 4.6 and Proposition 4.3 that there exists a similitude φ and a scalar λ that make the square commute.

It remains to check that φ respects the Hodge structures. The Hodge structure on $\widetilde{\mathrm{H}}(X_i,\mathbb{Q})$ is given by a morphism $h_i\colon\mathbb{C}^\times\to\mathrm{O}(\widetilde{\mathrm{H}}(X_i,\mathbb{R}))$, and the preceding lemma implies that the Hodge structure on $\mathrm{SH}(X_i,\mathbb{Q})$ is given by composing h_i with the injective map $\mathrm{O}(\widetilde{\mathrm{H}}(X_i,\mathbb{R}))\to\mathrm{GL}(\mathrm{SH}(X_i,\mathbb{R}))$. Since φ maps the Hodge structure on $\mathrm{SH}(X_1,\mathbb{Q})$ to the Hodge structure on $\mathrm{SH}(X_2,\mathbb{Q})$, we conclude that φ maps h_1 to h_2. □

Theorem 4.10 (d odd) *Assume that d is odd, and that X_1 and X_2 are deformation-equivalent hyperkähler varieties of dimension $2d$. Let $\Phi\colon\mathcal{D}X_1\xrightarrow{\sim}\mathcal{D}X_2$ be an* *equivalence. Then there is a unique Hodge isometry* $\Phi^{\widetilde{\mathrm{H}}}$ *making the square*

$$\begin{array}{ccc} \mathrm{SH}(X_1,\mathbb{Q}) & \xrightarrow{\Phi^{\mathrm{SH}}} & \mathrm{SH}(X_2,\mathbb{Q}) \\ \downarrow{\scriptstyle\Psi} & & \downarrow{\scriptstyle\Psi} \\ \mathrm{Sym}^d\, \widetilde{\mathrm{H}}(X_1,\mathbb{Q}) & \xrightarrow{\mathrm{Sym}^d(\Phi^{\widetilde{\mathrm{H}}})} & \mathrm{Sym}^d\, \widetilde{\mathrm{H}}(X_2,\mathbb{Q}) \end{array}$$

commute. The formation of $\Phi^{\widetilde{\mathrm{H}}}$ *is functorial in* Φ.

Proof Since X_1 and X_2 are deformation equivalent, we can choose an isometry $\varphi\colon \widetilde{\mathrm{H}}(X_1,\mathbb{Q}) \xrightarrow{\sim} \widetilde{\mathrm{H}}(X_2,\mathbb{Q})$. Moreover, X_1 and X_2 have the same Fujiki constant, so $\mathrm{Sym}^d\,\varphi$ restricts to an isometry between the images of Ψ. Then by Theorem 4.6 and Proposition 4.4, there is a unique isometry $\psi \in \mathrm{O}(\widetilde{\mathrm{H}}(X_2,\mathbb{Q}))$ such that $\Phi^{\widetilde{\mathrm{H}}} := \psi\varphi$ makes the square commute. Uniqueness forces its formation to be functorial.

That $\Phi^{\widetilde{\mathrm{H}}}$ respects the Hodge structures follows from the same argument as in the proof of Proposition 4.9. □

If d is even, then both existence and uniqueness of $\Phi^{\widetilde{\mathrm{H}}}$ in the statement of Theorem 4.10 fail. However, if we moreover assume that $b_2(X)$ is odd, then one can use the description of $G(V,d)$ from Proposition 4.4 to salvage this, at the cost of keeping track of a determinant character.

Define an *orientation* on X to be the choice of a generator of $\det \mathrm{H}^2(X,\mathbb{R})$, up to $\mathbb{R}^\times_{>0}$. Equivalently, an orientation is the choice of generator of $\det \widetilde{\mathrm{H}}(X,\mathbb{R})$ up to $\mathbb{R}^\times_{>0}$. Define the *sign* $\epsilon(\varphi)$ of a Hodge isometry $\varphi\colon \widetilde{\mathrm{H}}(X_1,\mathbb{Q}) \xrightarrow{\sim} \widetilde{\mathrm{H}}(X_2,\mathbb{Q})$ as $\epsilon(\varphi)=1$ if φ respects the orientations and $\epsilon(\varphi)=-1$ otherwise. A derived equivalence between oriented hyperkähler varieties is a derived equivalence between the underlying unoriented hyperkähler varieties.

Theorem 4.11 (d even) *Assume that d is even, and that $\Phi\colon \mathcal{D}X_1 \xrightarrow{\sim} \mathcal{D}X_2$ is a derived equivalence between oriented hyperkähler varieties of dimension $2d$. Assume that X_1 and X_2 have odd b_2, and that the quadratic spaces $\mathrm{H}^2(X_1,\mathbb{Q})$ and $\mathrm{H}^2(X_2,\mathbb{Q})$ are isometric. Then there exists a unique Hodge isometry $\Phi^{\widetilde{\mathrm{H}}}$ making the square*

$$\begin{array}{ccc} \mathrm{SH}(X_1,\mathbb{Q}) & \xrightarrow{\epsilon(\Phi^{\widetilde{\mathrm{H}}})\Phi^{\mathrm{SH}}} & \mathrm{SH}(X_2,\mathbb{Q}) \\ \downarrow{\scriptstyle\Psi} & & \downarrow{\scriptstyle\Psi} \\ \mathrm{Sym}^d\, \widetilde{\mathrm{H}}(X_1,\mathbb{Q}) & \xrightarrow{\mathrm{Sym}^d(\Phi^{\widetilde{\mathrm{H}}})} & \mathrm{Sym}^d\, \widetilde{\mathrm{H}}(X_2,\mathbb{Q}) \end{array}$$

commute. Moreover, the formation of $\Phi^{\widetilde{\mathrm{H}}}$ *is functorial for composition of derived equivalences between hyperkähler varieties equipped with orientations.*

Proof The argument is quite similar to the proof of Theorem 4.10. Choose an isometry $\varphi\colon \widetilde{\mathrm{H}}(X_1,\mathbb{Q}) \xrightarrow{\sim} \widetilde{\mathrm{H}}(X_2,\mathbb{Q})$. Because the dimension of $\widetilde{\mathrm{H}}(X_i,\mathbb{Q})$ is odd, we may replace φ with $-\varphi$ if necessary to ensure that φ respects the orientations, and hence we may assume $\epsilon(\varphi)=1$. The map φ induces an isometry $\mathrm{Sym}^d\,\varphi$, which restricts to an isometry $\varphi^{\mathrm{SH}}\colon \mathrm{SH}(X_1,\mathbb{Q}) \to \mathrm{SH}(X_2,\mathbb{Q})$.

By Theorem 4.6, there is a $\psi \in G(\widetilde{\mathrm{H}}(X_2,\mathbb{Q}),d)$ such that $\Phi^{\mathrm{SH}} = \psi\circ\varphi^{\mathrm{SH}}$, and by Proposition 4.4, we have that $\psi = \det(\psi_0)S_{[d]}(\psi_0)$ for a unique $\psi_0 \in \mathrm{O}(\widetilde{\mathrm{H}}(X_2,\mathbb{Q}))$. Now take $\Phi^{\widetilde{\mathrm{H}}} := \psi_0\circ\varphi$. Then $\epsilon(\Phi^{\widetilde{\mathrm{H}}}) = \det(\psi_0)$ and $\mathrm{Sym}^d(\Phi^{\widetilde{\mathrm{H}}})$ lifts to the map $\det(\psi_0)^{-1}\psi\circ\varphi^{\mathrm{SH}} = \epsilon(\Phi^{\widetilde{\mathrm{H}}})\Phi^{\mathrm{SH}}$ as claimed.

Proposition 4.4 forces $\Phi^{\widetilde{\mathrm{H}}}$ to be unique, and this implies the functoriality for composition. Compatibility with Hodge structures follows from the same argument as in the proof of Proposition 4.9. □

Remark 4.12 If X_1 and X_2 are hyperkähler varieties belonging to one of the known families, and if $\Phi\colon \mathcal{D}X_1 \xrightarrow{\sim} \mathcal{D}X_2$ is an equivalence, then the hypotheses of either Theorem 4.10 or Theorem 4.11 are satisfied. Indeed, X_1 and X_2 will have the same dimension $2d$ and because they have isomorphic LLV Lie algebra, they have the same second Betti number b_2. Going through the list of known families, one sees that this implies that X_1 and X_2 are deformation equivalent. In particular, they have isometric H^2. Finally, all known hyperkähler varieties of dimension $2d$ with d even have odd b_2.

Taking $X_1 = X_2$ in Theorems 4.10 and 4.11 yields Theorem C from the introduction:

Theorem 4.13 *Let X be a hyperkähler variety of dimension $2d$. Assume that either d is odd or that d is even and $b_2(X)$ is odd. Then the representation*

$$\rho^{\mathrm{SH}}\colon \mathrm{Aut}\,\mathcal{D}(X) \to \mathrm{GL}(\mathrm{SH}(X,\mathbb{Q}))$$

factors over a map $\rho^{\widetilde{\mathrm{H}}}\colon \mathrm{Aut}\,\mathcal{D}(X) \to \mathrm{O}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$.

Remark 4.14 For d odd, the implicit map $\mathrm{O}(\widetilde{\mathrm{H}}(X,\mathbb{Q})) \to \mathrm{GL}(\mathrm{SH}(X,\mathbb{Q}))$ is the natural map coming from the isomorphism $\mathrm{SH}(X,\mathbb{Q}) \cong S_{[d]}\widetilde{\mathrm{H}}(X,\mathbb{Q}))$. For d even (and b_2 odd), it is the twist of the natural map with the determinant character

$$\mathrm{O}(\widetilde{\mathrm{H}}(X,\mathbb{Q})) \to \{\pm 1\}.$$

5 Hodge structures

In this section we prove Theorem D from the introduction.

For a nondegenerate quadratic space V over $\mathbb{Q}$ we will make use of the algebraic groups $\mathbf{SO}(V)$, $\mathbf{Spin}(V)$, and $\mathbf{GSpin}(V)$ (sometimes denoted $\mathbf{CSpin}(V)$) over $\mathbb{Q}$. These groups sit in a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_2 & \longrightarrow & \mathbf{Spin}(V) & \longrightarrow & \mathbf{SO}(V) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbf{G}_m & \longrightarrow & \mathbf{GSpin}(V) & \longrightarrow & \mathbf{SO}(V) & \longrightarrow & 1 \end{array} \tag{6}$$

from which one deduces an exact sequence

$$1 \to \mu_2 \to \mathbf{G}_m \times \mathbf{Spin}(V) \to \mathbf{GSpin}(V) \to 1, \tag{7}$$

where the first map is the diagonal embedding $\epsilon \mapsto (\epsilon, \epsilon)$. Alternatively, one can use (7) as the definition of $\mathbf{GSpin}$, and deduce the existence of the above commutative diagram.

We will write $\mathrm{SO}(V)$, $\mathrm{Spin}(V)$, and $\mathrm{GSpin}(V)$ for the groups of $\mathbb{Q}$–points of these algebraic groups. Note that the above exact sequences of algebraic groups need not induce exact sequences of groups of $\mathbb{Q}$–points, and the obstruction can be described in terms of Galois cohomology. The sequence for the Spin–cover of $\mathbf{SO}(V)$ induces an exact sequence

$$1 \to \{\pm 1\} \to \mathrm{Spin}(V) \to \mathrm{SO}(V) \to \mathrm{H}^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \{\pm 1\}) = \mathbb{Q}^\times/(\mathbb{Q}^\times)^2,$$

where the connecting homomorphism $\mathrm{SO}(V) \to \mathbb{Q}^\times/(\mathbb{Q}^\times)^2$ is the spinor norm. By Hilbert 90, we have $\mathrm{H}^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \overline{\mathbb{Q}}^\times) = \{1\}$ and the analogous sequence for the GSpin–cover does induce a short exact sequence

$$1 \to \mathbb{Q}^\times \to \mathrm{GSpin}(V) \to \mathrm{SO}(V) \to 1. \tag{8}$$

This will be used crucially in the proof of Theorem D.

Lemma 5.1 *Let X be a hyperkähler variety of dimension $2d$. There exists a unique action of $\mathbf{GSpin}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$ on $\mathrm{H}(X,\mathbb{Q})$ such that*

- (i) *the action of $\mathbf{Spin}(\widetilde{\mathrm{H}}(X,\mathbb{Q})) \subset \mathbf{GSpin}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$ integrates the action of $\mathfrak{g}(X) = \mathfrak{so}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$;*
- (ii) *a section $\lambda \in \mathbf{G}_m \subset \mathbf{GSpin}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$ acts as λ^{i-2d} on $\mathrm{H}^i(X,\mathbb{Q})$.*

Proof The action of $\mathfrak{so}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$ integrates to an action of the simply connected algebraic group $\mathbf{Spin}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$. This commutes with the action of $\mathbf{G}_m$ for which λ acts as λ^{i-2d} on $\mathrm{H}^i(X,\mathbb{Q})$, and we obtain an action of $\mathbf{G}_m \times \mathbf{Spin}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$ on $\mathrm{H}(X,\mathbb{Q})$. The lemma claims that this descends to an action of the quotient group $\mathbf{GSpin}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$.

By (7) it suffices to verify that the kernel μ_2 acts trivially, ie that $-1 \in \mathrm{Spin}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$ acts as $(-1)^i$ on $\mathrm{H}^i(X,\mathbb{Q})$. Any $\mathfrak{sl}_2$–triple $(e_\lambda, h, f_\lambda)$ in $\mathfrak{g}(X)$ induces an algebraic subgroup $\mathbf{SL}_2 \subset \mathbf{Spin}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$ with the property that $\mathrm{diag}(\mu,\mu^{-1}) \in \mathrm{SL}_2(\mathbb{Q})$ acts as μ^i on $\mathrm{H}^{2d+i}(X,\mathbb{Q})$. It follows that $\mathrm{diag}(-1,-1)$ must be mapped to the nontrivial central element $-1 \in \mathrm{Spin}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$, and that -1 acts as $(-1)^i$ on $\mathrm{H}^i(X,\mathbb{Q})$. □

Recall from Definition 4.7 that we have equipped $\widetilde{\mathrm{H}}(X,\mathbb{Q})$ and $\mathrm{H}^{\mathrm{ev}}(X,\mathbb{Q})$ with Hodge structures of weight 0. Similarly, we equip the odd cohomology of X with a Hodge structure of weight 1,

$$\mathrm{H}^{\mathrm{odd}}(X,\mathbb{Q}) := \bigoplus_i \mathrm{H}^{2i+1}(X,\mathbb{Q}(i)).$$

Lemma 5.2 *Let* $g \in \mathrm{GSpin}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$. *If the action of* g *on* $\widetilde{\mathrm{H}}(X,\mathbb{Q})$ *respects the Hodge structure, then so does its action on* $\mathrm{H}^{\mathrm{ev}}(X,\mathbb{Q})$ *and on* $\mathrm{H}^{\mathrm{odd}}(X,\mathbb{Q})$.

Proof This follows immediately from the fact that the Hodge structure is determined by the action of $h' \in \mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$ (see Section 2.3), and from the faithfulness of the $\mathfrak{g}(X)$–module $\widetilde{\mathrm{H}}(X,\mathbb{Q})$. □

Theorem 5.3 *Let* X_1 *and* X_2 *be hyperkähler varieties, and let* $\Phi\colon \mathcal{D}X_1 \xrightarrow{\sim} \mathcal{D}X_2$ *be an equivalence. Then for every* i *the* $\mathbb{Q}$*–Hodge structures* $\mathrm{H}^i(X_1,\mathbb{Q})$ *and* $\mathrm{H}^i(X_2,\mathbb{Q})$ *are isomorphic.*

Proof Consider the Lie algebra isomorphism $\Phi^{\mathfrak{g}}\colon \mathfrak{g}(X_1) \xrightarrow{\sim} \mathfrak{g}(X_2)$ from Theorem A. By Proposition 4.9, there exists a Hodge similitude $\phi\colon \widetilde{\mathrm{H}}(X_1,\mathbb{Q}) \xrightarrow{\sim} \widetilde{\mathrm{H}}(X_2,\mathbb{Q})$ such that the square

$$\begin{array}{ccc} \mathfrak{so}(\widetilde{\mathrm{H}}(X_1,\mathbb{Q})) & \xrightarrow{\mathrm{Ad}(\phi)} & \mathfrak{so}(\widetilde{\mathrm{H}}(X_2,\mathbb{Q})) \\ \downarrow & & \downarrow \\ \mathfrak{g}(X_1) & \xrightarrow{\Phi^{\mathfrak{g}}} & \mathfrak{g}(X_2) \end{array}$$

commutes. Here the vertical maps are the isomorphisms from Theorem 3.1.

The K3–type Hodge structure $\widetilde{\mathrm{H}}(X_2, \mathbb{Q})$ decomposes as $N \oplus T$, with N and T its algebraic and transcendental parts, respectively. The Hodge similitude ϕ maps the distinguished elements α_1 and β_1 of $\widetilde{\mathrm{H}}(X_1, \mathbb{Q})$ to N. By Witt cancellation, there exists a $\psi_N \in \mathrm{SO}(N)$ and $\lambda, \mu \in \mathbb{Q}^\times$ such that $\psi_N \phi(\alpha_1) = \lambda\alpha_2$ and $\psi_N \phi(\beta_1) = \mu\beta_2$. Extending by the identity, we find a Hodge isometry $\psi \in \mathrm{SO}(\widetilde{\mathrm{H}}(X_2, \mathbb{Q}))$ such that $\psi\phi \colon \widetilde{\mathrm{H}}(X_1, \mathbb{Q}) \xrightarrow{\sim} \widetilde{\mathrm{H}}(X_2, \mathbb{Q})$ is a *graded* Hodge similitude. In particular, the induced map $\psi\phi \colon \mathfrak{g}(X_1) \xrightarrow{\sim} \mathfrak{g}(X_2)$ is graded, and $\psi\phi$ maps the grading element $h_1 \in \mathfrak{g}(X_1)$ to the grading element $h_2 \in \mathfrak{g}(X_2)$.

By (8) the element ψ lifts to an element $\widetilde{\psi} \in \mathrm{GSpin}(\widetilde{\mathrm{H}}(X_2, \mathbb{Q}))$, which by Lemma 5.1 and Lemma 5.2 induces automorphisms of the Hodge structures $\mathrm{H}^{\mathrm{ev}}(X_2, \mathbb{Q})$ and $\mathrm{H}^{\mathrm{odd}}(X_2, \mathbb{Q})$. Now, by construction, the composition $\widetilde{\psi} \circ \Phi^{\mathrm{H}}$ defines isomorphisms

$$\widetilde{\psi} \circ \Phi^{\mathrm{H}} \colon \mathrm{H}^{\mathrm{ev}}(X_1, \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^{\mathrm{ev}}(X_2, \mathbb{Q}), \quad \widetilde{\psi} \circ \Phi^{\mathrm{H}} \colon \mathrm{H}^{\mathrm{odd}}(X_1, \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^{\mathrm{odd}}(X_2, \mathbb{Q}),$$

which respect both the grading and the Hodge structure, so they induce isomorphisms of Hodge structures $\mathrm{H}^i(X_1, \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^i(X_2, \mathbb{Q})$, for all i. □

6 Topological K–theory

6.1 Topological K–theory and the Mukai vector

We now briefly recall some basic properties of topological K–theory of projective algebraic varieties. See [1; 3; 4] for more details.

For every smooth and projective X over $\mathbb{C}$ we have a $\mathbb{Z}/2\mathbb{Z}$–graded abelian group

$$\mathrm{K}_{\mathrm{top}}(X) := \mathrm{K}^0_{\mathrm{top}}(X) \oplus \mathrm{K}^1_{\mathrm{top}}(X).$$

This is functorial for pullback and proper pushforward, and carries a product structure. The group $\mathrm{K}^0_{\mathrm{top}}(X)$ is the Grothendieck group of topological vector bundles on the differentiable manifold X^{an}. Pullback agrees with pullback of vector bundles, and the product structure agrees with the tensor product of vector bundles.

By [3, Section 1.10], the Chern character can be extended to odd degree, inducing a $\mathbb{Z}/2\mathbb{Z}$–graded map

$$v_X^{\mathrm{top}} \colon \mathrm{K}_{\mathrm{top}}(X) \to \mathrm{H}(X, \mathbb{Q}),$$

given by $v_X^{\mathrm{top}}(\mathcal{F}) = \sqrt{\mathrm{Td}_X} \cdot \mathrm{ch}(\mathcal{F})$. The image of v_X^{top} is a $\mathbb{Z}$–lattice of full rank.

There is a "forgetful" map $\mathrm{K}^0(X) \to \mathrm{K}_{\mathrm{top}}(X)$ from the Grothendieck group of algebraic vector bundles (or equivalently of the triangulated category $\mathcal{D}X$). This is compatible with pullback, multiplication, and proper pushforward. The Mukai vector

$$v_X : \mathrm{K}^0(X) \to \mathrm{H}(X, \mathbb{Q})$$

factors over v_X^{top}.

If $\mathcal{P}$ is an object in $\mathcal{D}(X \times Y)$ then convolution with its class in $\mathrm{K}^0_{\mathrm{top}}(X \times Y)$ defines a map $\Phi^{\mathrm{K}}_{\mathcal{P}} \colon \mathrm{K}_{\mathrm{top}}(X) \to \mathrm{K}_{\mathrm{top}}(Y)$, in such a way that the diagram

$$\begin{array}{ccccc} \mathrm{K}^0(X) & \longrightarrow & \mathrm{K}_{\mathrm{top}}(X) & \xrightarrow{v_X^{\mathrm{top}}} & \mathrm{H}(X, \mathbb{Q}) \\ \downarrow{\scriptstyle \Phi_{\mathcal{P}}} & & \downarrow{\scriptstyle \Phi^{\mathrm{K}}_{\mathcal{P}}} & & \downarrow{\scriptstyle \Phi^{\mathrm{H}}_{\mathcal{P}}} \\ \mathrm{K}^0(Y) & \longrightarrow & \mathrm{K}_{\mathrm{top}}(Y) & \xrightarrow{v_Y^{\mathrm{top}}} & \mathrm{H}(Y, \mathbb{Q}). \end{array}$$

commutes.

6.2 Equivariant topological *K*–theory

The above formalism largely generalizes to an equivariant setting. Again, we briefly recall the most important properties; see [5; 6; 28; 42] for more details.

If X is a smooth projective complex variety equipped with an action of a finite group G, we denote by $\mathrm{K}^0_G(X)$ the Grothendieck group of G–equivariant algebraic vector bundles on X, or equivalently the Grothendieck group of the bounded derived category $\mathcal{D}_G X$ of G–equivariant coherent $\mathcal{O}_X$–modules. This is functorial for pullback along G–equivariant maps and pushforward along G–equivariant proper maps.

Similarly, we have the G–equivariant topological K–theory

$$\mathrm{K}_{\mathrm{top},G}(X) := \mathrm{K}^0_{\mathrm{top},G}(X) \oplus \mathrm{K}^1_{\mathrm{top},G}(X),$$

where $\mathrm{K}^0_{\mathrm{top},G}(X)$ is the Grothendieck group of topological G–equivariant vector bundles.

There is a natural map $\mathrm{K}^0_G(X) \to \mathrm{K}^0_{\mathrm{top},G}(X)$ compatible with pullback and tensor product. If $f : X \to Y$ is *proper* and G–equivariant, then we have a pushforward map $f_* \colon \mathrm{K}_{\mathrm{top},G}(X) \to \mathrm{K}_{\mathrm{top},G}(Y)$. There is a Riemann–Roch theorem [5; 28], stating that the square

$$\begin{array}{ccc} \mathrm{K}^0_G(X) & \longrightarrow & \mathrm{K}_{\mathrm{top},G}(X) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ \mathrm{K}^0_G(Y) & \longrightarrow & \mathrm{K}_{\mathrm{top},G}(Y) \end{array}$$

commutes.

Now assume that we have a finite group G acting on X, and a finite group H acting on Y. If $\mathcal{P}$ is an object in $\mathcal{D}_{G\times H}(X\times Y)$, then convolution with $\mathcal{P}$ induces a functor $\Phi_{\mathcal{P}}\colon \mathcal{D}_G X \to \mathcal{D}_H Y$, see [40] for more details. Similarly, convolution with the class of $\mathcal{P}$ in $\mathrm{K}^0_{\mathrm{top},G\times H}(X\times Y)$ induces a map $\Phi^{\mathrm{K}}_{\mathcal{P}}\colon \mathrm{K}_{\mathrm{top},G}(X)\to \mathrm{K}_{\mathrm{top},H}(Y)$. These satisfy the usual Fourier–Mukai calculus, and moreover they are compatible in the sense that the square

$$\begin{array}{ccc} \mathrm{K}^0_G(X) & \longrightarrow & \mathrm{K}_{\mathrm{top},G}(X) \\ \downarrow{\scriptstyle \Phi_{\mathcal{P}}} & & \downarrow{\scriptstyle \Phi^{\mathrm{K}}_{\mathcal{P}}} \\ \mathrm{K}^0_H(Y) & \longrightarrow & \mathrm{K}_{\mathrm{top},H}(Y) \end{array}$$

commutes.

7 Cohomology of the Hilbert square of a K3 surface

Let S be a K3 surface and $X = S^{[2]}$ its Hilbert square. In the coming few paragraphs we recall the structure of the cohomology of X in terms of the cohomology of S. See [7; 17; 23] for more details.

7.1 Line bundles on the Hilbert square

Let $G=\{1,\sigma\}$ be the group of order two, acting on $S\times S$ by permuting the factors. The Hilbert square X sits in a diagram

$$\begin{array}{ccccc} & & Z & & \\ & {\scriptstyle p}\swarrow & & \searrow{\scriptstyle q} & \\ S\times S & & & & X \end{array}$$

where $p\colon Z\to S\times S$ is the blow-up along the diagonal, and where $q\colon Z\to X$ is the quotient map for the natural action of G on Z. Denote by $R\subset Z$ the exceptional divisor of p. Then R equals the ramification locus of q. We have $q_*\mathcal{O}_Z=\mathcal{O}_X\oplus\mathcal{E}$ for some line bundle $\mathcal{E}$, and $q^*\mathcal{E}\cong\mathcal{O}_Z(-R)$.

If $\mathcal{L}$ is a line bundle on S then

$$\mathcal{L}_2 := (q_*p^*(\mathcal{L}\boxtimes\mathcal{L}))^G$$

is a line bundle on X. The map

$$\mathrm{Pic}(S)\oplus\mathbb{Z}\to\mathrm{Pic}(X),\quad (\mathcal{L},n)\mapsto\mathcal{L}_2\otimes\mathcal{E}^{\otimes n},$$

is an isomorphism.

7.2 Cohomology of the Hilbert square

There is an isomorphism

$$\mathrm{H}^2(S,\mathbb{Z})\oplus\mathbb{Z}\delta \xrightarrow{\sim} \mathrm{H}^2(X,\mathbb{Z})$$

with the property that $c_1(\mathcal{L})$ is mapped to $c_1(\mathcal{L}_2)$, and δ is mapped to $c_1(\mathcal{E})$. We will use this isomorphism to identify $\mathrm{H}^2(S,\mathbb{Z})\oplus\mathbb{Z}\delta$ with $\mathrm{H}^2(X,\mathbb{Z})$. The Beauville–Bogomolov form on $\mathrm{H}^2(X,\mathbb{Z})$ satisfies

$$b_X(\lambda,\lambda)=b_S(\lambda,\lambda),\quad b_X(\lambda,\delta)=0,\quad b_X(\delta,\delta)=-2$$

for all $\lambda\in\mathrm{H}^2(S,\mathbb{Z})$.

The cup product defines an isomorphism $\mathrm{Sym}^2\,\mathrm{H}^2(X,\mathbb{Q})\xrightarrow{\sim}\mathrm{H}^4(X,\mathbb{Q})$. By Poincaré duality, there is a unique $q_X\in\mathrm{H}^4(X,\mathbb{Q})$ representing the Beauville–Bogomolov form, in the sense that

$$\int_X q_X\lambda_1\lambda_2=b_X(\lambda_1,\lambda_2) \tag{9}$$

for all $\lambda_1,\lambda_2\in\mathrm{H}^2(X,\mathbb{Z})$. Multiplication by q_X defines an isomorphism $\mathrm{H}^2(X,\mathbb{Q})\to\mathrm{H}^6(X,\mathbb{Q})$, and, for all $\lambda_1,\lambda_2,\lambda_3\in\mathrm{H}^2(X,\mathbb{Q})$,

$$\lambda_1\lambda_2\lambda_3=b_X(\lambda_1,\lambda_2)q_X\lambda_3+b_X(\lambda_2,\lambda_3)q_X\lambda_1+b_X(\lambda_3,\lambda_1)q_X\lambda_2 \tag{10}$$

in $\mathrm{H}^6(X,\mathbb{Q})$. Finally, for all $\lambda\in\mathrm{H}^2(X,\mathbb{Q})$ the Fujiki relation

$$\int_X\lambda^4=3b_X(\lambda,\lambda)^2 \tag{11}$$

holds.

7.3 Todd class of the Hilbert square

Proposition 7.1 $\mathrm{Td}_X=1+\frac{5}{2}q_X+3[\mathrm{pt}].$

Proof See also [23, Section 23.4]. Since the Todd class is invariant under the monodromy group of X, we necessarily have

$$\mathrm{Td}_X=1+sq_X+t[\mathrm{pt}]$$

for some $s,t\in\mathbb{Q}$. By Hirzebruch–Riemann–Roch, for every line bundle L on S with $c_1(L)=\lambda$,

$$\chi(X,L_2)=\int_X\mathrm{ch}(\lambda)\mathrm{Td}_X=\frac{1}{24}\int_X\lambda^4+\frac{s}{2}\int_X\lambda^2q_X+t.$$

By the relations (11) and (9), the right-hand side reduces to

$$\tfrac{1}{8}b(\lambda,\lambda)^2 + \tfrac{1}{2}sb(\lambda,\lambda) + t.$$

By [23, Section 23.4] or [17, 5.1], the left-hand side computes to

$$\chi(X, L_2) = \tfrac{1}{8}b(\lambda,\lambda)^2 + \tfrac{5}{4}b(\lambda,\lambda) + 3.$$

Comparing the two expressions yields the result. □

8 Derived McKay correspondence

8.1 The derived McKay correspondence

As in Section 7.1, we consider a K3 surface S, its Hilbert square $X = S^{[2]}$, the maps $p\colon Z \to S \times S$ and $q\colon Z \to X$, and the group $G = \{1, \sigma\}$ acting on $S \times S$ and Z.

The *derived McKay correspondence* [11] is the triangulated functor

$$\mathrm{BKR}\colon \mathcal{D}^b(X) \to \mathcal{D}^b_G(S \times S)$$

given as the composition

$$\mathrm{BKR}\colon \mathcal{D}X \xrightarrow{q^*} \mathcal{D}_G(Z) \xrightarrow{p_*} \mathcal{D}_G(S \times S),$$

where the first functor maps $\mathcal{F}$ to $q^*\mathcal{F}$ equipped with the trivial G–linearization. By [11, Theorem 1.1; 21, Theorem 5.1], the functor BKR is an equivalence of categories.

Its inverse has been described in [31, Section 4]. Denote by $j : Z \to S \times S \times X$ the G–equivariant closed immersion induced by p and q. The exceptional divisor $R \subset Z$ is G–invariant and hence defines a G–equivariant sheaf $\mathcal{O}(R)$, and a G–equivariant sheaf $\mathcal{Q} := j_*\mathcal{O}_Z(R)$ in $\mathcal{D}_G(S \times S \times X)$.

Proposition 8.1 *The inverse equivalence of* BKR *is given by the equivariant Fourier–Mukai transform with respect to $\mathcal{Q}$. It maps $\mathcal{F} \in \mathcal{D}_G(S \times S)$ to the object*

$$(q_* p^* \mathcal{F})^{\sigma=-1} \otimes \mathcal{E}^{-1}$$

of $\mathcal{D}(X)$.

Proof The first statement is [31, 4.1]. By the adjunction formula for $j : Z \to S\times S \to X$, this implies that $\mathcal{F}$ is mapped to $\big(q_*(p^*\mathcal{F} \otimes \mathcal{O}_Z(R))\big)^G \in \mathcal{D}(X)$. If we upgrade the line bundle $\mathcal{E}$ on X to a G–equivariant (for the trivial action on X) line bundle $\mathcal{E}_-$

by making σ act as -1, then $q^*\mathcal{E}_- \cong \mathcal{O}_Z(-R)$ as G–equivariant line bundles on Z. Applying the projection formula once more for the equivariant map q, we find

$$(q_*(p^*\mathcal{F}\otimes\mathcal{O}_Z(R)))^G \cong (q_*p^*\mathcal{F}\otimes\mathcal{E}_-^{-1})^G \cong (q_*p^*\mathcal{F})^{\sigma=-1}\otimes\mathcal{E}^{-1}. \qquad \square$$

Now let S_1 and S_2 be K3 surfaces with Hilbert squares X_1 and X_2. As was observed by Ploog [39], any equivalence $\Phi\colon \mathcal{D}S_1 \xrightarrow{\sim} \mathcal{D}S_2$ induces an equivalence

$$\mathcal{D}_G(S_1\times S_2) \xrightarrow{\sim} \mathcal{D}_G(S_2\times S_2),$$

and hence, via the derived McKay correspondence, an equivalence $\Phi^{[2]}\colon \mathcal{D}X_1 \xrightarrow{\sim} \mathcal{D}X_2$.

8.2 Topological K–theory of the Hilbert square

Theorem 8.2 *The composition*

$$\mathrm{BKR}_{\mathrm{top}}\colon \mathrm{K}_{\mathrm{top}}(X) \xrightarrow{q^*} \mathrm{K}_{\mathrm{top},G}(Z) \xrightarrow{p_*} \mathrm{K}_{\mathrm{top},G}(S\times S)$$

is an isomorphism.

Proof (See also [11, Section 10].) This is a purely formal consequence of the calculus of equivariant Fourier–Mukai transforms sketched in Section 6.2. The functor BKR and its inverse are given by kernels $\mathcal{P}\in\mathcal{D}_G(X\times S\times S)$ and $\mathcal{Q}\in\mathcal{D}_G(S\times S\times X)$. The map $\mathrm{BKR}_{\mathrm{top}}$ is given by convolution with the class of $\mathcal{P}$ in $\mathrm{K}^0_{\mathrm{top},G}(X\times S\times S)$. The identities in $\mathrm{K}^0(X\times X)$ and $\mathrm{K}^0_{G\times G}(S\times S\times S\times S)$ witnessing that $\mathcal{P}$ and $\mathcal{Q}$ are mutually inverse equivalences induce analogous identities in $\mathrm{K}^0_{\mathrm{top}}$. These show that convolution with the class of $\mathcal{Q}$ defines a two-sided inverse to $\mathrm{BKR}_{\mathrm{top}}$. $\square$

Consider the map

$$\psi^{\mathrm{K}}\colon \mathrm{K}^0_{\mathrm{top}}(X)\to \mathrm{K}^0_{\mathrm{top}}(S\times S)^G$$

obtained as the composition of $\mathrm{BKR}_{\mathrm{top}}$ and the forgetful map from $\mathrm{K}^0_{\mathrm{top},G}(S\times S)$ to $\mathrm{K}^0_{\mathrm{top}}(S\times S)$. Also, consider the map

$$\theta^{\mathrm{K}}\colon \mathrm{K}^0_{\mathrm{top}}(S)\to \mathrm{K}^0_{\mathrm{top}}(X), \quad [\mathcal{F}]\mapsto \mathrm{BKR}^{-1}_{\mathrm{top}}([\mathcal{F}\boxtimes\mathcal{F},1]-[\mathcal{F}\boxtimes\mathcal{F},-1]),$$

where $[\mathcal{F}\boxtimes\mathcal{F},\pm1]$ denotes the class of the topological vector bundle $\mathcal{F}\boxtimes\mathcal{F}$ equipped with $\pm$ the natural G–linearization.

By construction, these maps are "functorial" in $\mathcal{D}S$, in the following sense:

Proposition 8.3 *If $\Phi\colon \mathcal{D}S_1 \xrightarrow{\sim} \mathcal{D}S_2$ is a derived equivalence between* K3 *surfaces, and $\Phi^{[2]}\colon \mathcal{D}X_1 \xrightarrow{\sim} \mathcal{D}X_2$ is the induced equivalence between their Hilbert squares, then the squares*

$$\begin{array}{ccc} \mathrm{K}^0_{\mathrm{top}}(X_1) & \xrightarrow{\psi^{\mathrm{K}}} & \mathrm{K}^0_{\mathrm{top}}(S_1 \times S_1)^G \\ \downarrow{\scriptstyle \Phi^{[2],\mathrm{K}}} & & \downarrow{\scriptstyle \Phi^{\mathrm{K}}\otimes\Phi^{\mathrm{K}}} \\ \mathrm{K}^0_{\mathrm{top}}(X_2) & \xrightarrow{\psi^{\mathrm{K}}} & \mathrm{K}^0_{\mathrm{top}}(S_2 \times S_2)^G \end{array} \qquad \begin{array}{ccc} \mathrm{K}^0_{\mathrm{top}}(S_1) & \xrightarrow{\theta^{\mathrm{K}}} & \mathrm{K}^0_{\mathrm{top}}(X_1) \\ \downarrow{\scriptstyle \Phi^{\mathrm{K}}} & & \downarrow{\scriptstyle \Phi^{[2],\mathrm{K}}} \\ \mathrm{K}^0_{\mathrm{top}}(S_2) & \xrightarrow{\theta^{\mathrm{K}}} & \mathrm{K}^0_{\mathrm{top}}(X_2) \end{array}$$

commute. □

Proposition 8.4 *The sequence*

$$0 \to \mathrm{K}^0_{\mathrm{top}}(S) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\theta^{\mathrm{K}}} \mathrm{K}^0_{\mathrm{top}}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\psi^{\mathrm{K}}} \mathrm{K}^0_{\mathrm{top}}(S \times S)^G \otimes_{\mathbb{Z}} \mathbb{Q} \to 0$$

is exact.

Proof In the proof, we will implicitly identify $\mathrm{K}_{\mathrm{top},G}(S \times S)$ and $\mathrm{K}_{\mathrm{top}}(X)$.

Note that the map θ^{K} is additive. Indeed, let $\mathcal{F}_1$ and $\mathcal{F}_2$ be (topological) vector bundles on S. Then the cross term $\theta^{\mathrm{K}}[\mathcal{F}_1 \oplus \mathcal{F}_2] - \theta^{\mathrm{K}}[\mathcal{F}_1] - \theta^{\mathrm{K}}[\mathcal{F}_2]$ computes to

$$\bigl[\mathcal{F}_1 \boxtimes \mathcal{F}_2 \oplus \mathcal{F}_2 \boxtimes \mathcal{F}_1,\ \bigl(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\bigr)\bigr] - \bigl[\mathcal{F}_1 \boxtimes \mathcal{F}_2 \oplus \mathcal{F}_2 \boxtimes \mathcal{F}_1,\ \bigl(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\bigr)\bigr],$$

which vanishes because the matrices $\bigl(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\bigr)$ and $\bigl(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\bigr)$ are conjugated over $\mathbb{Z}$.

Next we observe that $\psi^{\mathrm{K}}\colon \mathrm{K}^0_{\mathrm{top}}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathrm{K}^0_{\mathrm{top}}(S \times S)^G \otimes_{\mathbb{Z}} \mathbb{Q}$ is surjective. Indeed, by the Künneth formula [2], the group $\mathrm{K}^0_{\mathrm{top}}(S \times S)^G \otimes_{\mathbb{Z}} \mathbb{Q}$ is generated by classes of the form $[\mathcal{F}_1 \boxtimes \mathcal{F}_2 \oplus \mathcal{F}_2 \boxtimes \mathcal{F}_1]$, and these lie in the image of ψ^{K}.

Also, the composition $\psi^{\mathrm{K}}\theta^{\mathrm{K}}$ vanishes. Computing the $\mathbb{Q}$–dimensions one sees that it suffices to show that θ^{K} is injective to conclude that the sequence is exact.

Pulling back to the diagonal and taking invariants defines a map

$$\mathrm{K}^0_{\mathrm{top}}(S) \xrightarrow{\theta^{\mathrm{K}}} \mathrm{K}^0_{\mathrm{top},G}(S \times S) \xrightarrow{\Delta^*} \mathrm{K}^0_{\mathrm{top},G}(S) \xrightarrow{(-)^G} \mathrm{K}^0_{\mathrm{top}}(S).$$

This composition computes to

$$[\mathcal{F}] \mapsto [\mathrm{Sym}^2 \mathcal{F}] - \bigl[\textstyle\bigwedge^2 \mathcal{F}\bigr].$$

This coincides with the second Adams operation, which is injective on $\mathrm{K}^0_{\mathrm{top}}(S) \otimes_{\mathbb{Z}} \mathbb{Q}$, since it has eigenvalues 1, 2, and 4. We conclude that θ^{K} is injective, and the proposition follows. □

8.3 A computation in the cohomology of the Hilbert square

We now come to the technical heart of our computation of the derived monodromy of the Hilbert square of a K3 surface.

Consider the map $\theta^{\mathrm{H}} \colon \mathrm{H}(S,\mathbb{Q}) \to \mathrm{H}(X,\mathbb{Q})$ given by

$$\theta^{\mathrm{H}}(s+\lambda+t\,\mathrm{pt}_S) = (s\delta+\lambda\delta+tq_X\delta)\cdot e^{-\delta/2}, \tag{12}$$

for all $s,t\in\mathbb{Q}$ and $\lambda\in\mathrm{H}^2(S,\mathbb{Q})$. See Section 7.2 for the definition of $\delta\in\mathrm{H}^2(X,\mathbb{Q})$ and $q_X\in\mathrm{H}^4(X,\mathbb{Q})$.

Proposition 8.5 *The square*

$$\begin{array}{ccc} \mathrm{K}^0_{\mathrm{top}}(S) & \xrightarrow{\theta^{\mathrm{K}}} & \mathrm{K}^0_{\mathrm{top}}(X) \\ \downarrow{\scriptstyle v_S^{\mathrm{top}}} & & \downarrow{\scriptstyle v_X^{\mathrm{top}}} \\ \mathrm{H}(S,\mathbb{Q}) & \xrightarrow{\theta^{\mathrm{H}}} & \mathrm{H}(X,\mathbb{Q}) \end{array}$$

commutes.

Proof Since $\mathrm{K}^0_{\mathrm{top}}(S)\otimes_{\mathbb{Z}}\mathbb{Q}$ is additively generated by line bundles, it suffices to show

$$v_X^{\mathrm{top}}(\theta^{\mathrm{K}}(\mathcal{L})) = \big(\delta+\lambda\delta+\big(\tfrac{1}{2}b(\lambda,\lambda)+1\big)q_X\delta\big)\cdot e^{-\delta/2} \tag{13}$$

for a topological line bundle $\mathcal{L}$ with $\lambda=c_1(\mathcal{L})$. Deforming S if necessary, we may assume that $\mathcal{L}$ is algebraic.

Using Proposition 8.1 and the fact that the natural map

$$\mathcal{L}_2\otimes q_*\mathcal{O}_Z \to q_*p^*(\mathcal{L}\boxtimes\mathcal{L})$$

is an isomorphism of $\mathcal{O}_X$–modules, we find

$$\mathrm{BKR}^{-1}[\mathcal{L}\boxtimes\mathcal{L},1]=\mathcal{L}_2,\quad \mathrm{BKR}^{-1}[\mathcal{L}\boxtimes\mathcal{L},-1]=\mathcal{E}^{-1}\otimes\mathcal{L}_2.$$

We conclude that θ^{K} maps $\mathcal{L}$ to $[\mathcal{L}_2](1-[\mathcal{E}^{-1}])$ in $\mathrm{K}^0(X)$.

We compute its image under v_X. Using the formula for the Todd class from Proposition 7.1, we find

$$v_X(\theta^{\mathrm{K}}(\mathcal{L})) = \big(1+\tfrac{5}{4}q_X+\cdots\big)\exp(\lambda)(1-e^{-\delta}).$$

Since $1-e^{-\delta}$ has no term in degree 0, the degree 8 part of the square root of the Todd class is irrelevant, so we have

$$v_X(\theta^{\mathrm{K}}(\mathcal{L})) = \big(1+\tfrac{5}{4}q_X\big)\exp(\lambda)(1-e^{-\delta}).$$

By the Fujiki relation (11) from Section 7.2, we have $\lambda^3\delta = 0$, so the above can be rewritten as

$$v_X(\theta^{\mathrm{K}}(\mathcal{L})) = \left(1+\tfrac{5}{4}q_X\right)\cdot\left(\delta+\lambda\delta+\tfrac{1}{2}\lambda^2\delta\right)\cdot\frac{1-e^{-\delta}}{\delta}.$$

Since $q_X\delta\lambda = b(\delta,\lambda) = 0$, we can rewrite this further as

$$v_X(\theta^{\mathrm{K}}(\mathcal{L})) = \left(1+\tfrac{1}{4}q_X\right)\cdot\left(\delta+\lambda\delta+\left(\tfrac{1}{2}b(\lambda,\lambda)+1\right)q_X\delta\right)\cdot\frac{1-e^{-\delta}}{\delta}.$$

Comparing this with the right-hand side of (13), we see that it suffices to show

$$\left(1+\tfrac{1}{4}q_X\right)\cdot(1-e^{-\delta}) = \delta e^{-\delta/2}$$

in $\mathrm{H}(X,\mathbb{Q})$. This boils down to the identities

$$\tfrac{1}{6}\delta^3+\tfrac{1}{4}\delta q_X = \tfrac{1}{8}\delta^3, \quad \tfrac{1}{24}\delta^4+\tfrac{1}{8}\delta^2 q_X = \tfrac{1}{48}\delta^4$$

in $\mathrm{H}^6(X,\mathbb{Q})$ and $\mathrm{H}^8(X,\mathbb{Q})$, respectively. These follow easily from the relations (9), (10), and (11) in Section 7.2. □

9 Derived monodromy group of the Hilbert square of a K3 surface

9.1 Derived monodromy groups

Let X be a smooth projective complex variety. We call a *deformation* of X the data of a smooth projective variety X', a proper smooth family $\mathcal{X}\to B$, a path $\gamma\colon[0,1]\to\mathcal{X}$, and isomorphisms $X \xrightarrow{\sim} \mathcal{X}_{\gamma(0)}$ and $X' \xrightarrow{\sim} \mathcal{X}_{\gamma(1)}$. We will informally say that X' is a deformation of X, the other data being implicitly understood. Parallel transport along γ defines an isomorphism $\mathrm{H}(X,\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}(X',\mathbb{Q})$.

If X' and X'' are deformations of X, and if $\phi\colon X'\to X''$ is an isomorphism of projective varieties, then we obtain a composite isomorphism

$$\mathrm{H}(X,\mathbb{Q}) \xrightarrow[\sim]{} \mathrm{H}(X',\mathbb{Q}) \xrightarrow[\sim]{\phi} \mathrm{H}(X'',\mathbb{Q}) \xrightarrow[\sim]{} \mathrm{H}(X,\mathbb{Q}).$$

We call such an isomorphism a *monodromy operator* for X, and denote by $\mathrm{Mon}(X)$ the subgroup of $\mathrm{GL}(\mathrm{H}(X,\mathbb{Q}))$ generated by all monodromy operators.

If X' and X'' are deformations of X, and if $\Phi\colon \mathcal{D}X' \xrightarrow{\sim} \mathcal{D}X''$ is an equivalence, then we obtain an isomorphism

$$\mathrm{H}(X,\mathbb{Q}) \xrightarrow[\sim]{} \mathrm{H}(X',\mathbb{Q}) \xrightarrow[\sim]{\Phi^{\mathrm{H}}} \mathrm{H}(X'',\mathbb{Q}) \xrightarrow[\sim]{} \mathrm{H}(X,\mathbb{Q}).$$

We call such an isomorphism a *derived monodromy operator* for X, and denote by $\mathrm{DMon}(X)$ the subgroup of $\mathrm{GL}(\mathrm{H}(X,\mathbb{Q}))$ generated by all derived monodromy operators.

By construction, the derived monodromy group is deformation invariant. It contains the usual monodromy group, and the image of ρ_X, and we have a commutative square of groups

$$\begin{array}{ccc} \mathrm{Aut}(X) & \hookrightarrow & \mathrm{Aut}(\mathcal{D}X) \\ \downarrow & & \downarrow \rho_X \\ \mathrm{Mon}(X) & \hookrightarrow & \mathrm{DMon}(X) \end{array}$$

Remark 9.1 The above definition is somewhat ad hoc, and should be considered a poor man's derived monodromy group. This is sufficient for our purposes. A more mature definition should involve all noncommutative deformations of X.

Proposition 9.2 *If S is a* K3 *surface, then* $\mathrm{DMon}(S) = \mathrm{O}^+(\widetilde{\mathrm{H}}(S,\mathbb{Z}))$.

Proof Indeed, if $\Phi\colon \mathcal{D}S_1 \to \mathcal{D}S_2$ is an equivalence, then

$$\Phi^{\mathrm{H}}\colon \widetilde{\mathrm{H}}(S_1,\mathbb{Z}) \to \widetilde{\mathrm{H}}(S_2,\mathbb{Z})$$

preserves the Mukai form, as well as a natural orientation on four-dimensional positive subspaces; see [26, Section 4.5]. Also any deformation preserves the Mukai form and the natural orientation, so any derived monodromy operator will land in $\mathrm{O}^+(\widetilde{\mathrm{H}}(S,\mathbb{Z}))$.

The converse inclusion can be easily obtained from the Torelli theorem, together with the results of [22; 39] on derived auto-equivalences of K3 surfaces. Alternatively, one can use that the group $\mathrm{O}^+(\widetilde{\mathrm{H}}(S,\mathbb{Z}))$ is generated by reflections in -2–vectors δ. By the Torelli theorem, any such -2–vector will become algebraic on a suitable deformation S' of S, and by [32] there exists a spherical object $\mathcal{E}$ on S' with Mukai vector $v(\mathcal{E}) = \delta$. The spherical twist in $\mathcal{E}$ then shows that reflection in δ is indeed a derived monodromy operator. □

9.2 Action of $\mathrm{DMon}(S)$ on $\mathrm{H}(X,\mathbb{Q})$

By the derived McKay correspondence, any derived equivalence $\Phi_S : \mathcal{D}S_1 \xrightarrow{\sim} \mathcal{D}S_2$ between K3 surfaces induces a derived equivalence $\Phi_X : \mathcal{D}X_1 \xrightarrow{\sim} \mathcal{D}X_2$ between the corresponding Hilbert squares. By Propositions 8.3 and 8.4, the induced map Φ_X^H only depends on Φ_S^H. Since any deformation of a K3 surface S induces a deformation of $X = S^{[2]}$, we conclude that we have a natural homomorphism

$$\mathrm{DMon}(S) \to \mathrm{DMon}(X),$$

and hence an action of $\mathrm{DMon}(S)$ on $\mathrm{H}(X,\mathbb{Q})$. In this subsection, we will explicitly compute this action. As a first approximation, we determine the $\mathrm{DMon}(S)$–module structure of $\mathrm{H}(X,\mathbb{Q})$, up to isomorphism.

Proposition 9.3 *We have* $\mathrm{H}(X,\mathbb{Q}) \cong \widetilde{\mathrm{H}}(S,\mathbb{Q}) \oplus \mathrm{Sym}^2 \widetilde{\mathrm{H}}(S,\mathbb{Q})$ *as representations of* $\mathrm{DMon}(S) = \mathrm{O}^+(\widetilde{\mathrm{H}}(S,\mathbb{Z}))$.

Proof This follows from Propositions 8.3 and 8.4. □

Since $\mathfrak{g}(X)$ is a purely topological invariant, it is preserved under deformations. In particular, Theorem 4.13 implies that we have an inclusion $\mathrm{DMon}(X) \subset \mathrm{O}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$. We conclude there exists a unique map of algebraic groups h making the square

$$\begin{array}{ccc} \mathrm{DMon}(S) & \longrightarrow & \mathrm{DMon}(X) \\ \downarrow & & \downarrow \\ \mathrm{O}(\widetilde{\mathrm{H}}(S,\mathbb{Q})) & \xrightarrow{h} & \mathrm{O}(\widetilde{\mathrm{H}}(X,\mathbb{Q})) \end{array} \tag{14}$$

commute.

Recall that in (3) we defined an isometry B_λ of $\widetilde{\mathrm{H}}(X,\mathbb{Q})$ for every $\lambda \in \mathrm{H}^2(X,\mathbb{Q})$.

Theorem 9.4 *The map h in the square (14) is given by*

$$g \mapsto \det(g) \cdot (B_{-\delta/2} \circ \iota(g) \circ B_{\delta/2}),$$

with $\iota\colon \mathrm{O}(\widetilde{\mathrm{H}}(S,\mathbb{Q})) \to \mathrm{O}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$ *the natural inclusion.*

The proof of this theorem will occupy the remainder of this section.

Consider the unique homomorphism of Lie algebras $\iota\colon \mathfrak{g}(S) \to \mathfrak{g}(X)$ that respects the grading and maps e_λ to e_λ for all $\lambda \in \mathrm{H}^2(S,\mathbb{Q}) \subset \mathrm{H}^2(X,\mathbb{Q})$. Under the isomorphism of Theorem 3.1 this corresponds to the map $\mathfrak{so}(\widetilde{\mathrm{H}}(S,\mathbb{Q})) \to \mathfrak{so}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$ induced by the inclusion of quadratic spaces $\widetilde{\mathrm{H}}(S,\mathbb{Q}) \subset \widetilde{\mathrm{H}}(X,\mathbb{Q})$.

Recall from Section 8.3 the map $\theta^{\mathrm{H}}\colon \mathrm{H}(S,\mathbb{Q}) \to \mathrm{H}(X,\mathbb{Q})$.

Lemma 9.5 *The map* $\theta^{\mathrm{H}}\colon \mathrm{H}(S,\mathbb{Q}) \to \mathrm{H}(X,\mathbb{Q})$ *is equivariant with respect to*

$$\theta^{\mathfrak{g}}\colon \mathfrak{g}(S) \to \mathfrak{g}(X), \quad x \mapsto B_{-\delta/2} \circ \iota(x) \circ B_{\delta/2}.$$

Proof We have $\theta^{\mathrm{H}} = e^{-\delta/2} \cdot \theta_0^{\mathrm{H}}$, with

$$\theta_0^{\mathrm{H}}(s + \lambda + t\mathrm{pt}_S) = s\delta + \lambda\delta + tq_X\delta.$$

The map θ_0^{H} respects the grading, and we claim that for every $\mu \in \mathrm{H}^2(S,\mathbb{Q})$ the diagram

$$\begin{array}{ccccc} \mathrm{H}(S,\mathbb{Q}) & \xrightarrow{\theta_0^{\mathrm{H}}} & \mathrm{H}(X,\mathbb{Q}) & \xrightarrow{e^{-\delta/2}} & \mathrm{H}(X,\mathbb{Q}) \\ \downarrow e_\mu & & \downarrow e_\mu & & \downarrow e^{-\delta/2}e_\mu e^{\delta/2} \\ \mathrm{H}(S,\mathbb{Q}) & \xrightarrow{\theta_0^{\mathrm{H}}} & \mathrm{H}(X,\mathbb{Q}) & \xrightarrow{e^{-\delta/2}} & \mathrm{H}(X,\mathbb{Q}) \end{array}$$

commutes. Indeed, we have

$$e_\mu(\theta_0^{\mathrm{H}}(s + \lambda + t\mathrm{pt}_S)) = s\delta\mu + \lambda\delta\mu + tq_X\delta\mu,$$
$$\theta_0^{\mathrm{H}}(e_\mu(s + \lambda + t\mathrm{pt}_S)) = s\delta\mu + b(\lambda,\mu)q_X\delta.$$

One verifies easily that these agree, using the identities (10) and (9) from Section 7.2 and the fact that $b(\lambda,\delta) = b(\mu,\delta) = 0$. This shows that the left-hand square commutes. The right-hand square commutes trivially, so the outer rectangle commutes, which shows that $\theta^{\mathrm{H}} = e^{-\delta/2} \cdot \theta_0^{\mathrm{H}}$ is indeed equivariant with respect to $\theta^{\mathfrak{g}}$. □

Lemma 9.6 *There is an isomorphism*

$$\det(\widetilde{\mathrm{H}}(X,\mathbb{Q})) \otimes \mathrm{Sym}^2(\widetilde{\mathrm{H}}(X,\mathbb{Q})) \cong \mathrm{H}(X,\mathbb{Q}) \oplus \det(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$$

of representations of $G = \mathrm{O}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$.

Proof This follows from Lemma 3.7, Theorem 4.13 and Remark 4.14. □

We are now ready to prove the main result of this subsection.

Proof of Theorem 9.4 By Proposition 8.5, the map θ^{H} is equivariant for the action of $\mathrm{DMon}(S)$. Lemma 9.5 then implies that

$$h(g) = B_{-\delta/2} \circ \iota(g) \circ B_{\delta/2}$$

for all $g \in \mathrm{SO}(\widetilde{\mathrm{H}}(S,\mathbb{Q}))$. We have an orthogonal decomposition

$$\widetilde{\mathrm{H}}(X,\mathbb{Q}) = B_{-\delta/2}(\widetilde{\mathrm{H}}(S,\mathbb{Q})) \oplus C$$

with C of rank 1. Since $\mathrm{SO}(\widetilde{\mathrm{H}}(S,\mathbb{Q}))$ is normal in $\mathrm{O}(\widetilde{\mathrm{H}}(S,\mathbb{Q}))$, the action of $\mathrm{O}(\widetilde{\mathrm{H}}(S,\mathbb{Q}))$ (via h) must preserve this decomposition. With respect to this decomposition h must then be given by

$$h(g) = (B_{-\delta/2} \circ g\epsilon_1(g) \circ B_{\delta/2}) \oplus \epsilon_2(g),$$

where the $\epsilon_i(g)\colon \mathrm{O}(\widetilde{\mathrm{H}}(S,\mathbb{Q})) \to \{\pm 1\}$ are quadratic characters. This leaves four possibilities for h. One verifies that $\epsilon_1 = \epsilon_2 = \det g$ is the only possibility compatible with Proposition 9.3 and Lemma 9.6, and the theorem follows. □

9.3 A transitivity lemma

In this section we prove a lattice-theoretical lemma that will play an important role in the proofs of Theorems E and F.

Let $b\colon L \times L \to \mathbb{Z}$ be an even nondegenerate lattice. Let U be a hyperbolic plane with basis consisting of isotropic vectors α and β satisfying $b(\alpha,\beta) = -1$.

As before, to a $\lambda \in L$ we associate the isometry $B_\lambda \in \mathrm{O}(U \oplus L)$ defined as

$$B_\lambda(r\alpha + \mu + s\beta) = r\alpha + (\mu + r\lambda) + \big(s + b(\mu,\lambda) + r\tfrac{1}{2}b(\lambda,\lambda)\big)\beta$$

for all $r, s \in \mathbb{Z}$ and $\mu \in L$. Let γ be the isometry of $U \oplus L$ given by $\gamma(\alpha) = \beta$, $\gamma(\beta) = \alpha$, and $\gamma(\lambda) = -\lambda$ for all $\lambda \in L$.

Lemma 9.7 *Let L be an even lattice containing a hyperbolic plane. Let $G \subset \mathrm{O}(U \oplus L)$ be the subgroup generated by γ and by B_λ for all $\lambda \in L$. Then, for all $\delta \in U \oplus L$ with $\delta^2 = -2$ and for all $g \in \mathrm{O}(U \oplus L)$, there exists a $g' \in G$ such that $g'g$ fixes δ.*

Proof This follows from classical results of Eichler. A convenient modern source is [20, Section 3], whose notation we adopt. The isometry B_λ coincides with the Eichler transvection $t(\beta, -\lambda)$. The conjugate $\gamma B_\lambda \gamma^{-1}$ is the Eicher transvection $t(\alpha,\lambda)$. Hence G contains the subgroup $E_U(L) \subset \mathrm{O}(U \oplus L)$ of unimodular transvections with respect to U. By [20, Proposition 3.3], there exists a $g' \in E_U(L)$ mapping $g\delta$ to δ. □

9.4 Proof of Theorem E

Let X be a hyperkähler variety of type $\mathrm{K3}^{[2]}$. Let $\delta \in \mathrm{H}^2(X,\mathbb{Z})$ be any class satisfying $\delta^2 = -2$ and $b(\delta,\lambda) \in 2\mathbb{Z}$ for all $\lambda \in \mathrm{H}^2(X,\mathbb{Z})$. For example, if $X = S^{[2]}$, we may take $\delta = c_1(\mathcal{E})$ as in Section 7.2. Consider the integral lattice

$$\Lambda := B_{\delta/2}(\mathbb{Z}\alpha \oplus \mathrm{H}^2(X,\mathbb{Z}) \oplus \mathbb{Z}\beta) \subset \widetilde{\mathrm{H}}(X,\mathbb{Q}).$$

The subgroup $\Lambda \subset \widetilde{\mathrm{H}}(X,\mathbb{Q})$ does not depend on the choice of δ. In this section, we will prove Theorem E. More precisely, we will show:

Theorem 9.8 $$\mathrm{O}^+(\Lambda) \subset \mathrm{DMon}(X) \subset \mathrm{O}(\Lambda).$$

We start with the lower bound.

Proposition 9.9 $\mathrm{O}^+(\Lambda) \subset \mathrm{DMon}(X)$ *as subgroups of* $\mathrm{O}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$.

Proof Since the derived monodromy group is invariant under deformation, we may assume without loss of generality that $X = S^{[2]}$ for a K3 surface S and $\delta = c_1(\mathcal{E})$ as in Section 7.2.

The shift functor [1] on $\mathcal{D}X$ acts as -1 on $\mathrm{H}(X,\mathbb{Q})$, which coincides with the action of $-1 \in \mathrm{O}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$. In particular, $-1 \in \mathrm{O}^+(\Lambda)$ lies in $\mathrm{DMon}(X)$, so it suffices to show that $\mathrm{SO}^+(\Lambda)$ is contained in $\mathrm{DMon}(X)$.

Consider the isometry $\gamma \in \mathrm{O}^+(\widetilde{\mathrm{H}}(S,\mathbb{Q}))$ given by $\gamma(\alpha) = -\beta$, $\gamma(\beta) = -\alpha$, and $\gamma(\lambda) = \lambda$ for all $\lambda \in \mathrm{H}^2(S,\mathbb{Q})$. Then $\det(\gamma) = -1$ and by Theorem 9.4 its image $h(\gamma)$ interchanges $B_{\delta/2}\alpha$ and $B_{\delta/2}\beta$ and acts by -1 on $B_{\delta/2}\mathrm{H}^2(X,\mathbb{Z})$. Since γ lies in $\mathrm{DMon}(S) \subset \mathrm{O}(\widetilde{\mathrm{H}}(S,\mathbb{Q}))$, we have that $h(\gamma)$ lies in $\mathrm{DMon}(X) \subset \mathrm{O}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$.

Let $G \subset \mathrm{O}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$ be the subgroup generated by $h(\gamma)$ and the isometries B_λ for $\lambda \in \mathrm{H}^2(X,\mathbb{Z})$. Clearly G is contained in $\mathrm{DMon}(X)$.

Let g be an element of $\mathrm{SO}^+(\Lambda)$, and consider the image $gB_{\delta/2}\delta$ of $B_{\delta/2}\delta$. By Lemma 9.7 there exists a $g' \in G \subset \mathrm{DMon}(X)$ such that $g'g$ fixes $B_{\delta/2}\delta$. But then $g'g$ acts on
$$(B_{\delta/2}\delta)^\perp = B_{\delta/2}(\mathbb{Z}\alpha \oplus \mathrm{H}^2(S,\mathbb{Z}) \oplus \mathbb{Z}\beta)$$
with determinant 1 and preserving the orientation of a maximal positive subspace. In particular, $g'g$ lies in the image of $\mathrm{DMon}(S) \to \mathrm{DMon}(X)$, and we conclude that g lies in $\mathrm{DMon}(X)$. □

The proof of the upper bound is now almost purely group-theoretical. Denote by $\mathrm{SO}^+(\Lambda)$ the intersection $\mathrm{O}^+(\Lambda) \cap \mathrm{SO}(\Lambda)$. This group coincides with the kernel of the spinor norm on $\mathrm{SO}(\Lambda)$.

Proposition 9.10 $\mathrm{SO}(\Lambda)$ *is the unique maximal arithmetic subgroup of* $\mathrm{SO}(\Lambda \otimes_{\mathbb{Z}} \mathbb{Q})$ *containing* $\mathrm{SO}^+(\Lambda)$.

Proof More generally, this holds for any even lattice Λ with the property that the quadratic form $q(x) = b(x,x)/2$ on the $\mathbb{Z}$–module Λ is semiregular [29, Section IV.3].

For such Λ, the group schemes $\mathbf{Spin}(\Lambda)$ and $\mathbf{SO}(\Lambda)$ are smooth over $\operatorname{Spec}\mathbb{Z}$; see eg [27]. In particular, for every prime p the subgroups $\operatorname{Spin}(\Lambda\otimes\mathbb{Z}_p)$ and $\operatorname{SO}(\Lambda\otimes\mathbb{Z}_p)$ of $\operatorname{Spin}(\Lambda\otimes\mathbb{Q}_p)$ and $\operatorname{SO}(\Lambda\otimes\mathbb{Q}_p)$, respectively, are maximal compact subgroups. It follows that the groups

$$\operatorname{Spin}(\Lambda) = \operatorname{Spin}(\Lambda\otimes\mathbb{Q}) \cap \prod_p \operatorname{Spin}(\Lambda\otimes\mathbb{Z}_p)$$

and

$$\operatorname{SO}(\Lambda) = \operatorname{SO}(\Lambda\otimes\mathbb{Q}) \cap \prod_p \operatorname{SO}(\Lambda\otimes\mathbb{Z}_p)$$

are maximal arithmetic subgroups of $\operatorname{Spin}(\Lambda\otimes\mathbb{Q})$ and $\operatorname{SO}(\Lambda\otimes\mathbb{Q})$, respectively.

The subgroup $\operatorname{SO}^+(\Lambda) \subset \operatorname{SO}(\Lambda)$ is the kernel of the spinor norm, and the short exact sequence $1 \to \mu_2 \to \mathbf{Spin} \to \mathbf{SO} \to 1$ of fppf sheaves on $\operatorname{Spec}\mathbb{Z}$ induces an exact sequence of groups

$$1 \to \{\pm 1\} \to \operatorname{Spin}(\Lambda) \to \operatorname{SO}^+(\Lambda) \to 1.$$

Let $\Gamma \subset \operatorname{SO}(\Lambda\otimes\mathbb{Q})$ be a maximal arithmetic subgroup containing $\operatorname{SO}^+(\Lambda)$. Let $\widetilde{\Gamma}$ be its inverse image in $\operatorname{Spin}(\Lambda\otimes\mathbb{Q})$, so that we have an exact sequence

$$1 \to \{\pm 1\} \to \widetilde{\Gamma} \to \Gamma \to \mathbb{Q}^\times/2.$$

Since the group $\widetilde{\Gamma}$ is arithmetic and contains $\operatorname{Spin}(\Lambda)$, we have $\widetilde{\Gamma} = \operatorname{Spin}(\Lambda)$. Moreover, Γ normalizes $\operatorname{SO}^+(\Lambda) = \ker(\Gamma \to \mathbb{Q}^\times/2)$, and, as the normalizer of an arithmetic subgroup of $\operatorname{SO}(\Lambda\otimes\mathbb{Q})$ is again arithmetic, Γ must equal the normalizer of $\operatorname{SO}^+(\Lambda)$. But then Γ contains $\operatorname{SO}(\Lambda)$, and we conclude $\Gamma = \operatorname{SO}(\Lambda)$. □

Corollary 9.11 $\operatorname{DMon}(X) \subset \operatorname{O}(\Lambda)$.

Proof $\operatorname{DMon}(X)$ preserves the integral lattice $\mathrm{K}_{\mathrm{top}}(X)$ in the representation $\mathrm{H}(X,\mathbb{Q})$ of $\operatorname{O}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$, and hence is contained in an arithmetic subgroup of

$$\operatorname{O}(\widetilde{\mathrm{H}}(X,\mathbb{Q})) = \operatorname{SO}(\widetilde{\mathrm{H}}(X,\mathbb{Q})) \times \{\pm 1\}.$$

By Proposition 9.9 it contains $\operatorname{SO}^+(\Lambda) \times \{\pm 1\}$, so we conclude from the preceding proposition that $\operatorname{DMon}(X)$ must be contained in $\operatorname{O}(\Lambda)$. □

Together with Proposition 9.9 this proves Theorem 9.8.

10 The image of $\mathrm{Aut}(\mathcal{D}X)$ on $\mathrm{H}(X,\mathbb{Q})$

10.1 Upper bound for the image of ρ_X

We continue with the notation of the previous section. In particular, we denote by X a hyperkähler variety of type $\mathrm{K3}^{[2]}$, and by $\Lambda \subset \widetilde{\mathrm{H}}(X,\mathbb{Q})$ the lattice defined in Section 9.4. We equip $\widetilde{\mathrm{H}}(X,\mathbb{Q})$ with the weight 0 Hodge structure

$$\widetilde{\mathrm{H}}(X,\mathbb{Q}) = \mathbb{Q}\alpha \oplus \mathrm{H}^2(X,\mathbb{Q}(1)) \oplus \mathbb{Q}\beta.$$

We denote by $\mathrm{Aut}(\Lambda) \subset \mathrm{O}(\Lambda)$ the group of isometries of Λ that preserve this Hodge structure.

Proposition 10.1 $\mathrm{im}(\rho_X) \subset \mathrm{Aut}(\Lambda)$.

Proof By Theorem 9.8 we have $\mathrm{im}(\rho_X) \subset \mathrm{O}(\Lambda)$. The Hodge structure on

$$\mathrm{H}(X,\mathbb{Q}) = \bigoplus_{n=0}^{4} \mathrm{H}^{2n}(X,\mathbb{Q}(n))$$

induces a Hodge structure on $\mathfrak{g}(X) \subset \mathrm{End}(\mathrm{H}(X,\mathbb{Q}))$, which agrees with the Hodge structure on $\mathfrak{so}(\widetilde{\mathrm{H}}(X,\mathbb{Q}))$ induced by the Hodge structure on $\widetilde{\mathrm{H}}(X,\mathbb{Q})$. If

$$\Phi\colon \mathcal{D}X \xrightarrow{\sim} \mathcal{D}X$$

is an equivalence, then $\Phi^{\mathrm{H}}\colon \mathrm{H}(X,\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}(X,\mathbb{Q})$ and $\Phi^{\mathfrak{g}}\colon \mathfrak{g}(X) \xrightarrow{\sim} \mathfrak{g}(X)$ are isomorphisms of $\mathbb{Q}$–Hodge structures, from which it follows that Φ^{H} must land in $\mathrm{Aut}(\Lambda) \subset \mathrm{O}(\Lambda)$. □

10.2 Lower bound for the image of ρ_X

We write $\mathrm{Aut}^+(\Lambda)$ for the index 2 subgroup $\mathrm{Aut}(\Lambda) \cap \mathrm{O}^+(\Lambda)$ of $\mathrm{Aut}(\Lambda)$.

Theorem 10.2 *Let S be a* K3 *surface and let X be the Hilbert square of S. Assume that* $\mathrm{NS}(X)$ *contains a hyperbolic plane. Then* $\mathrm{Aut}^+(\Lambda) \subset \mathrm{im}\,\rho_X \subset \mathrm{Aut}(\Lambda)$.

Proof In view of Proposition 10.1 we only need to show the lower bound. The argument for this is entirely parallel to the proof of Proposition 9.9. Recall that

$$\Lambda = B_{\delta/2}(\mathbb{Z}\alpha \oplus \mathrm{H}^2(S,\mathbb{Z}(1)) \oplus \mathbb{Z}\delta \oplus \mathbb{Z}\beta).$$

The shift functor $[1] \in \mathrm{Aut}(\mathcal{D}X)$ maps to $-1 \in \mathrm{Aut}^+(\Lambda)$, so it suffices to show that $\mathrm{Aut}^+(\Lambda) \cap \mathrm{SO}(\Lambda)$ is contained in $\mathrm{im}\,\rho_X$.

Let $\gamma_S \in \mathrm{Aut}(\mathcal{D}S)$ be the composition of the spherical twist in $\mathcal{O}_S$ with the shift [1]. On the Mukai lattice $\widetilde{\mathrm{H}}(S,\mathbb{Z}) = \mathbb{Z}\alpha \oplus \mathrm{H}^2(X,\mathbb{Z}(1)) \oplus \mathbb{Z}\beta$ this equivalence maps α to $-\beta$ and β to $-\alpha$ and is the identity on $\mathrm{H}^2(S,\mathbb{Z})$. Under the derived McKay correspondence this induces an autoequivalence $\gamma_X \in \mathrm{Aut}\,\mathcal{D}X$. By Theorem 9.4, the automorphism $\rho_X(\gamma_X) \in \mathrm{Aut}(\Lambda)$ interchanges $B_{\delta/2}\alpha$ and $B_{\delta/2}\beta$ and acts by -1 on $B_{\delta/2}\mathrm{H}^2(X,\mathbb{Z})$.

Denote by $G \subset \mathrm{Aut}(\Lambda)$ the subgroup generated by $\rho_X(\gamma_X)$ and the isometries $B_\lambda = \rho_X(- \otimes \mathcal{L})$ with $\mathcal{L}$ a line bundle of class $\lambda \in \mathrm{NS}(X)$. Clearly G is contained in the image of ρ_X. Note that G acts on the lattice

$$\Lambda_{\mathrm{alg}} := B_{\delta/2}(\mathbb{Z}\alpha \oplus \mathrm{NS}(X) \oplus \mathbb{Z}\beta)$$

and that by our assumption $\mathrm{NS}(X)$ contains a hyperbolic plane.

Let $g \in \mathrm{Aut}^+(\Lambda)$. By Lemma 9.7 applied to $L = \mathrm{NS}(X)$, there exists a $g' \in G$ such that $g'g$ fixes $B_{\delta/2}\delta$. But then $g'g$ acts on

$$(B_{\delta/2}\delta)^\perp = B_{\delta/2}(\mathbb{Z}\alpha \oplus \mathrm{H}^2(S,\mathbb{Z}) \oplus \mathbb{Z}\beta)$$

with determinant 1 and preserving the Hodge structure and the orientation of a maximal positive subspace. In particular, $g'g$ lies in the image of $\mathrm{Aut}(\mathcal{D}S)$, and we conclude that g lies in $\mathrm{im}\,\rho_X$. □

References

[1] **N Addington**, **R Thomas**, *Hodge theory and derived categories of cubic fourfolds*, Duke Math. J. 163 (2014) 1885–1927 MR Zbl

[2] **M F Atiyah**, *Vector bundles and the Künneth formula*, Topology 1 (1962) 245–248 MR Zbl

[3] **M F Atiyah**, **F Hirzebruch**, *Vector bundles and homogeneous spaces*, from "Differential geometry" (C B Allendoerfer, editor), Proc. Sympos. Pure Math. 3, Amer. Math. Soc., Providence, RI (1961) 7–38 MR Zbl

[4] **M F Atiyah**, **F Hirzebruch**, *The Riemann–Roch theorem for analytic embeddings*, Topology 1 (1962) 151–166 MR Zbl

[5] **M F Atiyah**, **G B Segal**, *The index of elliptic operators, II*, Ann. of Math. 87 (1968) 531–545 MR Zbl

[6] **P Baum**, **W Fulton**, **R MacPherson**, *Riemann–Roch and topological K–theory for singular varieties*, Acta Math. 143 (1979) 155–192 MR Zbl

[7] **A Beauville**, *Variétés Kähleriennes dont la première classe de Chern est nulle*, J. Differential Geom. 18 (1983) 755–782 MR Zbl

[8] **F A Bogomolov**, *On the cohomology ring of a simple hyperkähler manifold (on the results of Verbitsky)*, Geom. Funct. Anal. 6 (1996) 612–618 MR Zbl

[9] **A Bondal**, **D Orlov**, *Reconstruction of a variety from the derived category and groups of autoequivalences*, Compositio Math. 125 (2001) 327–344 MR Zbl

[10] **T Bridgeland**, *Stability conditions on $K3$ surfaces*, Duke Math. J. 141 (2008) 241–291 MR Zbl

[11] **T Bridgeland**, **A King**, **M Reid**, *The McKay correspondence as an equivalence of derived categories*, J. Amer. Math. Soc. 14 (2001) 535–554 MR Zbl

[12] **D Calaque**, **M Van den Bergh**, *Hochschild cohomology and Atiyah classes*, Adv. Math. 224 (2010) 1839–1889 MR Zbl

[13] **D Calaque**, **C A Rossi**, **M Van den Bergh**, *Căldăraru's conjecture and Tsygan's formality*, Ann. of Math. 176 (2012) 865–923 MR Zbl

[14] **A Caldararu**, *The Mukai pairing*, *I*: *The Hochschild structure*, preprint (2003) arXiv math/0308079

[15] **A Căldăraru**, *The Mukai pairing, II: The Hochschild–Kostant–Rosenberg isomorphism*, Adv. Math. 194 (2005) 34–66 MR Zbl

[16] **R Dijkgraaf**, *Instanton strings and hyper-Kähler geometry*, Nuclear Phys. B 543 (1999) 545–571 MR Zbl

[17] **G Ellingsrud**, **L Göttsche**, **M Lehn**, *On the cobordism class of the Hilbert scheme of a surface*, J. Algebraic Geom. 10 (2001) 81–100 MR Zbl

[18] **V Golyshev**, **V Lunts**, **D Orlov**, *Mirror symmetry for abelian varieties*, J. Algebraic Geom. 10 (2001) 433–496 MR Zbl

[19] **M Green**, **Y-J Kim**, **R Laza**, **C Robles**, *The LLV decomposition of hyper-Kähler cohomology*, preprint (2019) arXiv 1906.03432

[20] **V Gritsenko**, **K Hulek**, **G K Sankaran**, *Abelianisation of orthogonal groups and the fundamental group of modular varieties*, J. Algebra 322 (2009) 463–478 MR Zbl

[21] **M Haiman**, *Hilbert schemes, polygraphs and the Macdonald positivity conjecture*, J. Amer. Math. Soc. 14 (2001) 941–1006 MR Zbl

[22] **S Hosono**, **B H Lian**, **K Oguiso**, **S-T Yau**, *Autoequivalences of derived category of a $K3$ surface and monodromy transformations*, J. Algebraic Geom. 13 (2004) 513–545 MR Zbl

[23] **D Huybrechts**, *Compact hyperkähler manifolds*, from "Calabi–Yau manifolds and related geometries" (M Gross, D Huybrechts, D Joyce, editors), Springer (2003) 161–225 MR Zbl

[24] **D Huybrechts**, *Fourier–Mukai transforms in algebraic geometry*, Oxford Univ. Press (2006) MR Zbl

[25] **D Huybrechts**, **E Macrì**, **P Stellari**, *Stability conditions for generic $K3$ categories*, Compos. Math. 144 (2008) 134–162 MR Zbl

[26] **D Huybrechts**, **E Macrì**, **P Stellari**, *Derived equivalences of $K3$ surfaces and orientation*, Duke Math. J. 149 (2009) 461–507 MR Zbl

[27] **H Ikai**, *Spin groups over a commutative ring and the associated root data (odd rank case)*, Beiträge Algebra Geom. 46 (2005) 377–395 MR Zbl

[28] **R Joshua**, *Equivariant Riemann–Roch for G–quasi-projective varieties, I*, K–Theory 17 (1999) 1–35 MR Zbl

[29] **M-A Knus**, *Quadratic and Hermitian forms over rings*, Grundl. Math. Wissen. 294, Springer (1991) MR Zbl

[30] **M Kontsevich**, *Deformation quantization of Poisson manifolds*, Lett. Math. Phys. 66 (2003) 157–216 MR Zbl

[31] **A Krug**, *On derived autoequivalences of Hilbert schemes and generalized Kummer varieties*, Int. Math. Res. Not. 2015 (2015) 10680–10701 MR Zbl

[32] **S A Kuleshov**, *A theorem on the existence of exceptional bundles on surfaces of type $K3$*, Izv. Akad. Nauk SSSR Ser. Mat. 53 (1989) 363–378 MR Zbl In Russian; translated in Math. USSR-Izv. 34 (1990) 373–388

[33] **E Looijenga**, **V A Lunts**, *A Lie algebra attached to a projective variety*, Invent. Math. 129 (1997) 361–412 MR Zbl

[34] **E Macrì**, **P Stellari**, *Infinitesimal derived Torelli theorem for $K3$ surfaces*, Int. Math. Res. Not. 2009 (2009) 3190–3220 MR Zbl

[35] **S Mukai**, *On the moduli space of bundles on $K3$ surfaces, I*, from "Vector bundles on algebraic varieties", Tata Inst. Fund. Res. Stud. Math. 11, Tata Inst. Fund. Res., Bombay (1987) 341–413 MR Zbl

[36] **D O Orlov**, *Equivalences of derived categories and $K3$ surfaces*, J. Math. Sci. 84 (1997) 1361–1381 MR Zbl

[37] **D O Orlov**, *Derived categories of coherent sheaves and equivalences between them*, Uspekhi Mat. Nauk 58 (2003) 89–172 MR Zbl In Russian; translated in Russian Math. Surveys 58 (2003) 511–591

[38] **D O Orlov**, *Derived categories of coherent sheaves, and motives*, Uspekhi Mat. Nauk 60 (2005) 231–232 MR Zbl In Russian; translated in Russian Math. Surveys 60 (2005) 1242–1244

[39] **D Ploog**, *Groups of autoequivalences of derived categories of smooth projective varieties*, PhD thesis, Freie Universität Berlin (2005) Available at `https://www.mathematik.hu-berlin.de/~ploog/PAPERS/Ploog_phd.pdf`

[40] **D Ploog**, *Equivariant autoequivalences for finite group actions*, Adv. Math. 216 (2007) 62–74 MR Zbl

[41] **A Rapagnetta**, *On the Beauville form of the known irreducible symplectic varieties*, Math. Ann. 340 (2008) 77–95 MR Zbl

[42] **G Segal**, *Equivariant K–theory*, Inst. Hautes Études Sci. Publ. Math. 34 (1968) 129–151 MR Zbl

[43] **A Soldatenkov**, *On the Hodge structures of compact hyperkähler manifolds*, Math. Res. Lett. 28 (2021) 623–635 MR Zbl

[44] **B Szendrői**, *Diffeomorphisms and families of Fourier–Mukai transforms in mirror symmetry*, from "Applications of algebraic geometry to coding theory, physics and computation" (C Ciliberto, F Hirzebruch, R Miranda, M Teicher, editors), NATO Sci. Ser. II Math. Phys. Chem. 36, Kluwer, Dordrecht (2001) 317–337 MR Zbl

[45] **M S Verbitsky**, *Cohomology of compact hyperkaehler manifolds*, PhD thesis, Harvard University (1995) MR Available at `https://www.proquest.com/docview/304181888`

[46] **M Verbitsky**, *Cohomology of compact hyperkähler manifolds and its applications*, Geom. Funct. Anal. 6 (1996) 601–611 MR Zbl

[47] **M Verbitsky**, *Mirror symmetry for hyper-Kähler manifolds*, from "Mirror symmetry, III" (D H Phong, L Vinet, S-T Yau, editors), AMS/IP Stud. Adv. Math. 10, Amer. Math. Soc., Providence, RI (1999) 115–156 MR Zbl

Korteweg-de Vries Institute for Mathematics, University of Amsterdam
Amsterdam, Netherlands

`l.d.j.taelman@uva.nl`

Proposed: Richard P Thomas — Received: 17 March 2020
Seconded: Jim Bryan, Dan Abramovich — Revised: 2 November 2021

GEOMETRY & TOPOLOGY

msp.org/gt

See inside back cover or msp.org/gt for submission instructions.

The subscription price for 2023 is US $740/year for the electronic version, and $1030/year (+$70, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Geometry & Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Geometry & Topology (ISSN 1465-3060 printed, 1364-0380 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

GT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Geometry & Topology

Volume 27 Issue 7 (pages 2497–2936) 2023