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In 2002, Polterovich established that on closed aspherical symplectic manifolds, Hamiltonian diffeomorphisms of finite order, also called Hamiltonian torsion, must be trivial. We prove the first higher-dimensional Hamiltonian no-torsion theorems beyond that of Polterovich, by considering the dynamical aspects of the problem. Our results are threefold.

First, we show that closed symplectic Calabi–Yau and negative monotone symplectic manifolds admit no Hamiltonian torsion. A key role is played by a new notion of a Hamiltonian diffeomorphism with nonisolated fixed points.

Second, going beyond topological constraints by means of Smith theory in filtered Floer homology, barcodes and quantum Steenrod powers, we prove that every closed positive monotone symplectic manifold admitting Hamiltonian torsion is geometrically uniruled by pseudoholomorphic spheres. In fact, we produce nontrivial homological counts of such curves, answering a close variant of Problem 24 from the introductory monograph of McDuff and Salamon. This provides additional no-torsion results and obstructions to Hamiltonian actions of compact Lie groups, related to a celebrated result of McDuff from 2009, and lattices such as $\mathrm{SL}(k,\mathbb{Z})$ for $k \geq 2$. We also prove that there is no Hamiltonian torsion diffeomorphism with noncontractible orbits.

Third, by defining a new invariant of a Hamiltonian diffeomorphism, we prove a first nontrivial symplectic analogue of Newman's 1931 theorem on finite groups of transformations. Namely, for each monotone symplectic manifold there exists a neighborhood of the identity in the Hamiltonian group endowed with Hofer's metric or Viterbo's spectral metric that contains no finite subgroups.

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1 Introduction and main results

1.1 Introduction

The question of the existence of finite group actions on manifolds has been of interest in topology for a long time. It was in order to study this question that P A Smith [99] developed in the 1930s what is now called Smith theory for cohomology with $\mathbb{F}_p$ coefficients in the context of continuous actions of finite p-groups. We refer the reader to Borel [4], Bredon [5], Floyd [25] and Hsiang [45] for references on Smith theory.

Quite a lot of progress regarding this question has been obtained in low-dimensional topology (see for example Morgan [62]) and in smooth topology in arbitrary dimension (see for example Mundet i Riera [78]). As a first easy example, we remark that it is not hard to classify finite group actions on closed surfaces. Further progress was made in low-dimensional symplectic topology (Chen and Kwasik [10]) ruling out symplectic finite group actions acting trivially on homology on certain symplectic Calabi–Yau 4–manifolds (see also Wu and Liu [109]) by means of tools such as Seiberg–Witten theory, which are available only in dimension four.

In higher-dimensional symplectic topology,[1] while the existence of general symplectic finite group actions has to the best of our knowledge not been ruled out in any given setting,[2] it was shown by Polterovich [72] that nontrivial *Hamiltonian* finite group actions, which we refer to as *Hamiltonian torsion*, on symplectically aspherical manifolds do not exist. Essentially, the only other constraints on symplectic and Hamiltonian finite group actions in higher dimensions were obtained by Mundet i Riera [77], showing, roughly speaking, that finite groups acting in a Hamiltonian way (or symplectically in the simply connected case) must be approximately abelian: specifically, they satisfy the Jordan property. In turn, abelian Hamiltonian finite group actions do exist on closed symplectic manifolds such as toric varieties, which tend to have a lot of pseudoholomorphic curves. These developments, as well as further results that we describe below, have motivated Problem 24 from the list of problems that are "appealing in their own right and central to symplectic topology" in the monograph [59] of McDuff and Salamon. This problem seeks obstructions to the existence of Hamiltonian torsion related to the scarcity of pseudoholomorphic curves in the manifold. One of the goals of this paper is to produce a solution to a close version of Problem 24, proving a result which is, in a way, stronger.

[1] That is, in dimension $2n \geq 6$.

[2] See Section 1.2.4, however.

Another goal is to study the metric rigidity properties of Hamiltonian torsion, also alluded to in the presentation of this problem. Finally, we prove a topological rigidity result: all periodic orbits of a Hamiltonian isotopy whose time-one map is torsion must be contractible.

To motivate Problem 24 further, and to introduce a few important notions, we add that Hamiltonian actions of cyclic groups on rational ruled symplectic 4–manifolds — that is, symplectic S^2–bundles over S^2 — were recently shown to be induced by S^1–actions; see Chen [9] and Chiang and Kessler [11]. However, this is false for general symplectic 4–manifolds; see Remark 7. The strongest restriction to date on manifolds admitting nontrivial Hamiltonian S^1–actions was obtained by McDuff [57], who showed that all such manifolds must be *uniruled*, in the sense that at least one genus-zero k–point Gromov–Witten invariant for $k \geq 3$ involving the point class must not vanish. Of course, rational ruled symplectic 4–manifolds satisfy this condition, with $k = 3$: they are *strongly uniruled.* Either condition implies that these manifolds are *geometrically uniruled*: for each ω–compatible almost complex structure J and each point $p \in M$, there is a J–holomorphic sphere[3] passing through p. Finally, in Shelukhin [93] a new notion of uniruledness, $\mathbb{F}_p$–Steenrod uniruledness, was introduced for $p = 2$, and was generalized to odd primes $p > 2$ by work in progress of Shelukhin and Wilkins [97]; the quantum Steenrod p^{th} power of the cohomology class Poincaré dual to the point class is defined and deformed in the sense of not coinciding with the classical Steenrod p^{th} power. This notion similarly implies geometric uniruledness. It is currently not known whether it implies uniruledness in the sense of McDuff, but it is expected to do so; see Seidel [91] and Seidel and Wilkins [92] for first steps in this direction.

This paper proves the first higher-dimensional Hamiltonian no-torsion results since that of Polterovich, which hold beyond the symplectically aspherical case. Firstly, we prove that, in addition to symplectically aspherical manifolds, symplectically Calabi–Yau and negative monotone symplectic manifolds do not admit Hamiltonian torsion. An elementary argument then shows that if a closed symplectic manifold M admits Hamiltonian torsion, then it has a spherical homology class A such that $\langle c_1(TM), A\rangle > 0$ and $\langle [\omega], A\rangle > 0$; see Corollary 2. Our results have a similar flavor to the result of McDuff for S^1–actions: indeed, negative monotone and Calabi–Yau manifolds are not geometrically uniruled, and neither are the symplectically aspherical ones.

[3]This is a smooth map $u\colon \mathbb{CP}^1 \to M$ satisfying $Du \circ j = J \circ Du$ for the standard complex structure j on $\mathbb{CP}^1$. Such spheres and their significance in symplectic topology were discovered by Gromov [38]. We refer to McDuff and Salamon [58] for a detailed modern description of this notion.

Going far beyond topological restrictions, we further study restrictions on Hamiltonian torsion in the (positive) monotone case. Using recently discovered techniques, we show that in this case the existence of nontrivial Hamiltonian torsion implies $\mathbb{F}_p$–Steenrod uniruledness for certain primes p, and hence geometric uniruledness. This again fits well with the result of McDuff and in fact provides a partial solution to Problem 24 from the monograph [59] of McDuff and Salamon. Studying the properties of the quantum Steenrod operations and their relation to Gromov–Witten invariants further — see Seidel and Wilkins [92] and Wilkins [106; 107] for first inroads in this direction — might show that our solution is in fact quite complete. Furthermore, we are tempted to conjecture the following analogue of the result of McDuff.

Conjecture 1 Each closed symplectic manifold with nontrivial Hamiltonian torsion must be uniruled.

Before addressing further results on the metric properties of Hamiltonian torsion diffeomorphisms when they exist in the monotone case, we comment on our methods of proof. The main general idea of the paper is to treat such a diffeomorphism as a Hamiltonian dynamical system, despite the fact that it exhibits very simple periodic dynamics. Indeed, quite paradoxically, studying its asymptotic behavior for large iterations is effective, as it yields new topological and Floer-theoretical properties of such diffeomorphisms.

Curiously enough, on a more technical level, our arguments involve a recently discovered analogue of Smith theory in filtered Hamiltonian Floer homology (see Seidel [90], Shelukhin [95] and Shelukhin and Zhao [98]), and related notions of quantum Steenrod powers (see Shelukhin and Wilkins [97] and Wilkins [106; 107]). Previously these methods were applied to questions of existence of infinitely many periodic points (see again Shelukhin and Wilkins [97] and Shelukhin [95]) and, more restrictively, of obstructions on manifolds to admit Hamiltonian pseudorotations (see Shelukhin [93; 94] and Çineli, Ginzburg and Gürel [7]). In fact, a more precise general theme of this paper is that a Hamiltonian diffeomorphism of finite order behaves in many senses like a counterexample to the Conley conjecture. For instance, the statement of Corollary 2 is analogous to that of [36, Theorem 1.1] that provides the most general setting wherein the Conley conjecture is known to hold.

Our third and last series of results studies the metric rigidity of Hamiltonian torsion and related maps. We start by proving that the spectral norm (see Oh [65], Schwarz [86] and Viterbo [104]) of a Hamiltonian torsion element ϕ of order k on a closed rational symplectic manifold (ie a manifold for which $\langle[\omega], \pi_2(M)\rangle = \rho\cdot\mathbb{Z}$ with $\rho > 0$) satisfies $\gamma(\phi) \geq \rho/k$, and as an immediate consequence, the same estimate applies for the Hofer norm (see Hofer [40] and Lalonde and McDuff [51]).

More importantly, in our final main result, we prove that in the monotone case, given $\phi \in \mathrm{Ham}(M,\omega) \setminus \{\mathrm{id}\}$ of order k, ie with $\phi^k = \mathrm{id}$, there exists $m \in \mathbb{Z}/k\mathbb{Z}$ such that

$$\gamma(\phi^m) \geq \frac{\rho}{3}. \tag{1}$$

This last result should be considered a Hamiltonian analogue of the celebrated result of Newman [63] (see also Dress [16] and Smith [100]), the C^0–distance having been replaced by the spectral distance. Moreover we prove the stronger statement that if k is prime, then $\gamma(\phi^m) \geq \rho\lfloor k/2\rfloor/k$ for a certain $m \in \mathbb{Z}/k\mathbb{Z}$, and provide a similar statement in the context of Hamiltonian pseudorotations.

The bound (1) can further be seen to imply Newman's result in a special case, as follows. By Shelukhin [96, Theorem C] (see also Kawamoto [47]), when $M = \mathbb{CP}^n$ is the complex projective space with the standard symplectic form normalized so that $\mathbb{CP}^1$ has area 1, there is a constant c_n, depending only on the dimension, such that for all $\phi \in \mathrm{Ham}(M,\omega)$, the usual C^0–distance of ϕ to the identity satisfies

$$d_{C^0}(\phi, \mathrm{id}) \geq c_n \gamma(\phi).$$

Hence, if ϕ is of finite order, then by (1) there exists $m \in \mathbb{Z}$ such that

$$d_{C^0}(\phi^m, \mathrm{id}) \geq \frac{c_n}{3}.$$

It would be very interesting to see if the results of this paper can be extended to the case of Hamiltonian homeomorphisms, as defined in Buhovsky, Humilière and Seyfaddini [6]. This generalization does not seem to be straightforward because we use the properties of the linearization of the Hamiltonian diffeomorphism at its fixed points, as well as Smith theory in filtered Floer homology, which is not in general stable in the C^0–topology.

We close the introduction by noting that we expect that our results in the monotone case should extend to the semipositive case, once the relevant results of [95] and [97] have been generalized to the requisite setting. Since these generalizations would not considerably differ, in a conceptual way, from the arguments presented in this paper, but would necessitate more lengthy technical proofs, we defer their investigation to further publications.

1.2 Main results

We start with the following theorem of Polterovich [72], originally stated in the case where $\pi_2(M)=0$. For the reader's convenience we include its proof in Section 5.4.

Theorem A (Polterovich) *Let (M,ω) be a closed symplectically aspherical symplectic manifold. If G is a finite group, then each homomorphism $G\to \mathrm{Ham}(M,\omega)$ is trivial.*

In this paper we prove a number of additional "no-torsion" theorems of this kind, going beyond the symplectically aspherical case, and study the metric properties of Hamiltonian diffeomorphisms of finite order when such obstructions do not hold. Our conditions on the manifold that imply the absence of Hamiltonian torsion are of two kinds: the first is purely topological, and the second, perhaps more surprisingly, is in terms of pseudoholomorphic curves.

1.2.1 Topological conditions The first set of results of this paper is as follows.

Theorem B *Let (M,ω) be a closed negative monotone or closed symplectically Calabi–Yau symplectic manifold. If G is a finite group, then each homomorphism $G\to \mathrm{Ham}(M,\omega)$ is trivial.*

A simple exercise in linear algebra shows that the class of manifolds, which we call *symplectically nonpositive*, covered by Theorems A and B can be described concisely as those closed symplectic manifolds (M,ω) for which

$$\langle[\omega],A\rangle\cdot\langle c_1(TM),A\rangle\le 0\quad\text{for all } A\in\pi_2(M).$$

In other words, the following holds.

Corollary 2 *If a closed symplectic manifold (M,ω) admits a nontrivial homomorphism $G\to \mathrm{Ham}(M,\omega)$ from a finite group, then there exists an $A\in\pi_2(M)$ such that $\langle[\omega],A\rangle>0$ and $\langle c_1(TM),A\rangle>0$.*

For details of this implication see [36, Proof of Theorem 4.1].

Theorem B follows directly from Theorems C and D below. These two steps essentially generalize the notion of a perfect Hamiltonian diffeomorphism, ie one that has a finite number of contractible periodic points of all periods, to the case of compact path-connected isolated sets of fixed points. We call such an isolated set of fixed points of $\phi \in \mathrm{Ham}(M,\omega)$ a *generalized fixed point* of ϕ. Recall that a fixed point x of a Hamiltonian diffeomorphism $\phi = \phi_H^1$ is called contractible whenever the homotopy class $\alpha(x,\phi)$ of the path $\alpha(x,H) = \{\phi_H^t(x)\}$ for a Hamiltonian H generating ϕ is trivial. This class does not depend on the choice of Hamiltonian, by a classical argument in Floer theory. We call a generalized fixed point $\mathcal{F}$ of ϕ contractible if all fixed points $x \in \mathcal{F}$ are contractible. We denote by $\mathscr{F}$ the generalized periodic orbit, consisting of all $\alpha(x,H)$ for $x \in \mathcal{F}$, corresponding to the generalized fixed point $\mathcal{F}$. This is a subset of the free loop space $\mathcal{L}M$ of M. If $\mathcal{F}$ is contractible, we show that there exists a *capping* $\overline{\mathscr{F}}$ of $\mathscr{F}$, which is a lift of $\mathscr{F}$ to a suitable cover of the connected component $\mathcal{L}_{\mathrm{pt}}M$ of the loop space consisting of contractible loops. Finally, and crucially, we introduce the following notion: we call a generalized fixed point $\mathscr{F}$ *index-constant* if the mean-index $\Delta(H,\overline{x})$ for $\overline{x} \in \overline{\mathscr{F}}$ is constant as a function of $\overline{x} \in \overline{\mathscr{F}}$ (which is in turn determined by $x \in \mathcal{F}$ and the capping $\overline{\mathscr{F}}$). We refer to Section 2.1.3 for the definition of the mean-index.

We call $\phi \in \mathrm{Ham}(M,\omega)$ a *generalized perfect* Hamiltonian diffeomorphism if there exists a sequence $k_j \to \infty$ of iterations satisfying the following two properties: first, it contains a subsequence $l_i = k_{j_i}$ with $l_i \mid l_{i+1}$ for all i; second, for all $j \in \mathbb{Z}_{>0}$ the diffeomorphism ϕ^{k_j} has only a finite set, which does not depend on j, of contractible generalized fixed points, *which are all index-constant*.

Finally, we call a diffeomorphism ϕ with a finite number of (contractible) generalized fixed points *weakly nondegenerate* if for each (contractible) fixed point x of ϕ, the spectrum of the differential $D(\phi)_x$ at x contains points different from $1 \in \mathbb{C}$. Using the existence of ω–compatible almost complex structures invariant under a Hamiltonian diffeomorphism of finite order, we prove the following structural result.

Theorem C *Let (M,ω) be a closed symplectic manifold. Then a torsion Hamiltonian diffeomorphism $\phi \in \mathrm{Ham}(M,\omega)$ is a weakly nondegenerate generalized perfect Hamiltonian diffeomorphism. In fact, it is Floer–Morse–Bott and its generalized fixed points are symplectic submanifolds.*

While we do not require this for Theorem C, for most of our applications it is sufficient to assume that ϕ is p–torsion for a prime p, that is, $\phi^p = \mathrm{id}$.

Following the index arguments of Salamon and Zehnder [84], and their generalization due to Ginzburg and Gürel [34], we prove the following obstruction to the existence of weakly nondegenerate generalized perfect Hamiltonian diffeomorphisms.

Theorem D *Let a closed symplectic manifold (M,ω) be negative monotone or symplectically Calabi–Yau. Then (M,ω) does not admit weakly nondegenerate generalized perfect Hamiltonian diffeomorphisms.*

Together with Theorem C, Theorem D immediately implies Theorem B. In fact, in view of Cauchy's theorem for finite groups, to rule out all Hamiltonian finite group actions it is sufficient to rule out all Hamiltonian torsion of prime order. One can say that, almost paradoxically, we use the *large-time asymptotic behavior* of our Hamiltonian system to study its *periodic* dynamics! This is the main general idea of this paper.

As easy examples show, generalized perfect Hamiltonian diffeomorphisms do indeed exist on the manifolds of Theorem D if one drops the weak nondegeneracy assumption. For example, one can take $T^2 = S^1 \times S^1$, where $S^1 = \mathbb{R}/\mathbb{Z}$, to be the standard torus with (x,y) denoting a general point, and $\omega_{\mathrm{st}} = dx \wedge dy$ the standard symplectic form, and pick $\phi \in \mathrm{Ham}(T^2,\omega_{\mathrm{st}})$ given by $\phi = \phi_H^t$ for $t>0$, with $H \in C^\infty(T^2,\mathbb{R})$ given by $H(x,y) = \cos(2\pi y)$. It is easy to see that the set of contractible periodic points of ϕ consists precisely of the two isolated sets $\{y=0\}$ and $\left\{y=\frac{1}{2}\right\}$.

1.2.2 Conditions in terms of pseudoholomorphic curves Our second set of results deals with the class of monotone symplectic manifolds. It is evident that far more than topological conditions is necessary to rule out Hamiltonian torsion in this case, since each Hamiltonian S^1–manifold, such as $\mathbb{CP}^n$ for example, admits Hamiltonian torsion. We formulate our restriction on the existence of Hamiltonian torsion geometrically as follows. For an ω–compatible almost complex structure J on a closed symplectic manifold (M,ω), we say that the manifold is *geometrically uniruled* if for each point $p \in M$, there exists a nonconstant J–holomorphic sphere $u\colon \mathbb{CP}^1 \to M$ such that $p \in \mathrm{im}(u)$.

Theorem E *Let (M,ω) be a closed monotone symplectic manifold that is not geometrically uniruled for some ω–compatible almost complex structure J. Then each homomorphism $G \to \mathrm{Ham}(M,\omega)$, where G is a finite group, is trivial.*

This is a corollary of the following more precise result involving the quantum Steenrod power operations.

Theorem F *Let (M,ω) be a closed monotone symplectic manifold that admits a Hamiltonian diffeomorphism of order $d>1$. Then the p^{th} quantum Steenrod power of the cohomology class $\mu \in H^{2n}(M;\mathbb{F}_p)$ Poincaré dual to the point class is deformed for all primes p coprime to d.*

Theorem F follows directly from Theorems G and I below.

Theorems E and F provide an obstruction to the existence of Hamiltonian diffeomorphisms of finite order in terms of pseudoholomorphic curves. The existence of an obstruction of this type was conjectured by McDuff and Salamon, and publicized as Problem 24 in their introductory monograph [59]. Therefore we provide a solution to a reasonable variant of this problem. Indeed, further investigations into the enumerative nature of quantum Steenrod operations might prove that our solution is in fact complete in the framework of monotone symplectic manifolds. Such investigations were initiated in Seidel and Wilkins [92] and Wilkins [106; 107].

In particular, in the special case where (M,ω) has minimal Chern number $N=n+1$, we deduce from Theorem F and the work of Seidel and Wilkins [92], as in Shelukhin [93], that nontrivial Hamiltonian torsion implies that the quantum product $[\mathrm{pt}]*[\mathrm{pt}]$ does not vanish. This means that the manifold is strongly rationally connected: it implies strong uniruledness, and moreover that for each pair of distinct points p_1, p_2 in M, and each ω–compatible almost complex structure J, there exists a J–holomorphic sphere in M passing through p_1 and p_2.

As mentioned above, the proof of Theorem F relies on two steps: Theorems G and I. These steps are aimed at showing that torsion Hamiltonian diffeomorphisms of closed monotone symplectic manifolds, which by Theorem C are generalized perfect and weakly nondegenerate, are moreover homologically minimal in the following sense. To formulate it precisely, we first discuss a useful technical notion.

Let $\mathbb{K}$ be a coefficient field. For a generalized fixed point $\mathcal{F}$ of a Hamiltonian diffeomorphism ψ, we define a generalized version $HF^{\mathrm{loc}}(\psi,\mathcal{F})$ of local Floer homology. Such notions date back to the original work of Floer [24; 23] and have been revisited a number of times: for example by Pozniak in [76]. It is naturally $\mathbb{Z}/2\mathbb{Z}$–graded.[4]

We call a Hamiltonian diffeomorphism a *generalized* $\mathbb{K}$ *pseudorotation* with the sequence k_j if it is generalized perfect with the sequence k_j and, further, $HF^{\mathrm{loc}}(\psi,\mathcal{F})\neq 0$ for all $\mathcal{F}\in\pi_0(\mathrm{Fix}(\psi))$ and the homological count

$$N(\psi,\mathbb{K}) := \sum_{\mathcal{F}\in\pi_0(\mathrm{Fix}(\psi))} \dim_{\mathbb{K}} HF^{\mathrm{loc}}(\psi,\mathcal{F})$$

of generalized fixed points of $\psi=\phi^{k_j}$ satisfies

$$N(\psi,\mathbb{K}) = \dim_{\mathbb{K}} H_*(M;\mathbb{K}) \quad \text{for all } j\in\mathbb{Z}_{>0}.$$

[4] We also define a $\mathbb{Z}$–graded version for a capped generalized 1–periodic point $\bar{\mathscr{F}}$ lifting $\mathcal{F}$.

We recall that usually an $\mathbb{F}_p$ pseudorotation is defined analogously, with the sequence $k_j = p^{j-1}$, and with the additional hypothesis that each $\mathcal{F} \in \pi_0(\mathrm{Fix}(\psi))$ for $\psi = \phi^{k_j}$ consists of a single point. Unless otherwise stated, a *generalized* $\mathbb{F}_p$ *pseudorotation* will be considered with the sequence $k_j = p^{j-1}$.

In view of the discussion in Shelukhin [95; 96], this homological minimality for a Hamiltonian diffeomorphism ψ with a finite number of generalized fixed points is equivalent to the absence of finite bars in the barcode $\mathcal{B}(\psi)$ of ψ, a notion of recent interest in symplectic topology; see eg Kislev and Shelukhin [48], Polterovich and Shelukhin [74], Polterovich, Shelukhin and Stojisavljević [75] and Shelukhin [95; 96]. It also implies the equality

$$\mathrm{Spec}^{\mathrm{ess}}(F;\mathbb{K}) = \mathrm{Spec}^{\mathrm{vis}}(F;\mathbb{K})$$

between two homologically defined subsets of the spectrum associated to a Hamiltonian $F \in \mathcal{H}$ generating ψ. Recall that the spectrum $\mathrm{Spec}(F)$ of F is the set of critical values of the action functional of F. For a coefficient field $\mathbb{K}$, there is a nested sequence of subsets

$$\mathrm{Spec}^{\mathrm{ess}}(F;\mathbb{K}) \subset \mathrm{Spec}^{\mathrm{vis}}(F;\mathbb{K}) \subset \mathrm{Spec}(F).$$

Here the *essential spectrum* $\mathrm{Spec}^{\mathrm{ess}}(F;\mathbb{K})$ is the set of values of all spectral invariants associated to F, in other words the set of starting points of infinite bars in the barcode of F. The *visible spectrum* $\mathrm{Spec}^{\mathrm{vis}}(F;\mathbb{K})$ is the set of action values of capped (generalized) periodic orbits of F that have nonzero local Floer homology, in other words the set of endpoints of all bars in the barcode. It is not hard to modify the definitions of the two homological spectra to include multiplicities, in which case their equality would be equivalent to homological minimality.

The first step in the proof of Theorem F, which is nontrivial and uses Smith theory in filtered Floer homology (cf [95; 98]), is the following reduction.

Theorem G *Let (M,ω) be a closed monotone symplectic manifold. Suppose that $\phi \in \mathrm{Ham}(M,\omega)$ is a Hamiltonian diffeomorphism of prime order $q \geq 2$. Then*:

(i) *For each prime p different from q, the q–torsion diffeomorphism ϕ is a weakly nondegenerate generalized pseudorotation over $\mathbb{F}_p$, with the sequence k_j given by the monotone increasing ordering of the set*

$$\{k \in \mathbb{Z}_{>0} \mid k \neq 0 \ (\mathrm{mod}\ q)\}.$$

(ii) *Moreover, for each Hamiltonian H generating ϕ, and each coefficient field $\mathbb{K}$ of characteristic p coprime to q, we have*
$$\mathrm{Spec}^{\mathrm{ess}}(H;\mathbb{K}) = \mathrm{Spec}^{\mathrm{vis}}(H;\mathbb{K}),$$
and for all k coprime to q, we have
$$\mathrm{Spec}^{\mathrm{ess}}(H^{(k)};\mathbb{Q}) = k\cdot \mathrm{Spec}^{\mathrm{ess}}(H;\mathbb{Q}) + \rho\cdot\mathbb{Z}.$$

(iii) *Finally, part (i) holds also for $p = q$, and in part (ii), equalities*
$$\mathrm{Spec}^{\mathrm{ess}}(H;\mathbb{K}) = \mathrm{Spec}^{\mathrm{vis}}(H;\mathbb{K}),$$
$$\mathrm{Spec}^{\mathrm{ess}}(H^{(k)};\mathbb{K}) = k\cdot \mathrm{Spec}^{\mathrm{ess}}(H;\mathbb{K}) + \rho\cdot\mathbb{Z}$$
hold with arbitrary coefficient field $\mathbb{K}$, and moreover,
$$\mathrm{Spec}^{\mathrm{vis}}(H;\mathbb{K}) = \mathrm{Spec}(H).$$

The proof of Theorem G appears in Section 5.9. For the moment, we briefly explain the approach used to prove Theorem G(i). Following the main theme of the proof of Theorem C, we use information about large iterations of H to study the periodic Hamiltonian diffeomorphism $\phi = \phi_H^1$ that it generates. More precisely, let $\psi = \phi^k$ with k coprime to q. Combining the theory of barcodes of Hamiltonian diffeomorphisms (see Proposition 23), and Smith-type inequalities in filtered Floer homology (see Theorem N), we observe that for the *bar-lengths*
$$\beta_1(\psi,\mathbb{F}_p) \le \cdots \le \beta_{K(\psi,\mathbb{F}_p)}(\psi,\mathbb{F}_p)$$
of ψ, we have the following inequality. Set
$$\beta_{\mathrm{tot}}(\psi,\mathbb{F}_p) = \beta_1(\psi,\mathbb{F}_p) + \cdots + \beta_{K(\psi,\mathbb{F}_p)}(\psi,\mathbb{F}_p)$$
to be the *total bar-length* of ψ. Then
$$\beta_{\mathrm{tot}}(\psi^{p^m},\mathbb{F}_p) \ge p^m\cdot\beta_{\mathrm{tot}}(\psi,\mathbb{F}_p).$$
However, $\beta_{\mathrm{tot}}(\psi^{p^m},\mathbb{F}_p)$ is bounded, since it can take at most $q-1$ values. This implies
$$\beta_{\mathrm{tot}}(\psi,\mathbb{F}_p) = 0,$$
which in turn implies part (i), by the theory of barcodes; see Proposition 23.

Remark 3 We separate part (iii) of Theorem G because it requires a different proof, relying on Proposition 5 below. The first statement of part (iii) is obtained via Proposition 5 by classical Smith theory combined with the classical homological Arnol'd conjecture, outlined in Chiang and Kessler [11, Remark 7.1] with details for $p = 2$. One could also obtain this statement by a suitable generalization of Theorem N on Smith theory in filtered Floer homology, which is, however, out of the scope of this paper.

Remark 4 When the order q is not prime, a version of Theorem G still holds. We leave its somewhat lengthier formulation to the interested reader, since we do not require it for our arguments, only observing that part (i) holds under the assumption that p does not divide q, and the sequence of iterations is given by $\{k \in \mathbb{Z}_{>0} \mid \gcd(k,q)=1\}$ and part (ii) holds as stated.

The following statement is a key component of the proof of Theorem G(iii). It relies on the generalization of the Morse–Bott theory of Pozniak [76, Theorem 3.4.11] to the situation with signs and orientations, as in for example Schmaschke [85, Chapter 9], Fukaya, Oh, Ohta and Ono [28, Chapter 8], or Wehrheim and Woodward [105]. However, it is not entirely straightforward, because as classical examples show, it is false in the general Floer–Morse–Bott situation, while in our case it holds because of the existence of special ω–compatible almost complex structures adapted to the situation.

Proposition 5 *Let (M,ω) be a closed symplectic manifold, and $\phi \in \mathrm{Ham}(M,\omega)$ a Hamiltonian diffeomorphism of finite order $d \geq 2$. Let $\mathcal{F}$ be a path-connected component of the fixed-point set of ϕ. Finally, let R be a commutative unital ring. Then the local Floer homology of ϕ at $\mathcal{F}$ with coefficients in R satisfies*

$$HF^{\mathrm{loc}}(\phi,\mathcal{F}) \cong H(\mathcal{F};R).$$

The proof of Theorem G has the following by-product, which is a new analogue, for Hamiltonian torsion, of the classical consequence of Floer theory, whereby the map $\pi_1(\mathrm{Ham}(M,\omega)) \to \pi_1(M)$ is trivial.

Theorem H *Let (M,ω) be a closed monotone symplectic manifold, and let ϕ in $\mathrm{Ham}(M,\omega)$ be a Hamiltonian diffeomorphism of finite order. Then **all** the fixed points of ϕ are contractible.*

The second step in the argument proving Theorem F is the following statement. It essentially follows the arguments of Shelukhin [94] and Shelukhin and Wilkins [97].

Theorem I *Let (M,ω) be a closed monotone symplectic manifold that admits a weakly nondegenerate generalized $\mathbb{F}_p$ pseudorotation for a prime $p \geq 2$. Then the p^{th} quantum Steenrod power of the cohomology class $\mu \in H^{2n}(M;\mathbb{F}_p)$ Poincaré dual to the point class is deformed.*

Theorems G and I immediately imply Theorem F and therefore, by a Gromov compactness argument, Theorem E.

1.2.3 Applications to actions of Lie groups and lattices To conclude the discussion of our first two sets of results, we discuss their implications to the question of existence of Hamiltonian actions of possibly disconnected Lie groups, and lattices in Lie groups, on closed symplectic manifolds.

A well-known result of Delzant [15] (see [73] for an alternative argument) implies that a simple Lie group can only act nontrivially on a closed symplectic manifold if it is compact. A compact zero-dimensional Lie group is finite, whence Theorems B and E provide topological and geometrical obstructions to their actions. The identity component K_0 of a compact Lie group K of positive dimension is a compact connected Lie group of positive dimension, and as such admits a maximal torus $T \cong (S^1)^k$ of positive dimension, whose conjugates cover the whole group K_0. Therefore, the absence of Hamiltonian torsion, as in Theorems A, B, E and F, implies that a nontrivial K–action yields a nontrivial K_0–action, since otherwise it would factor through K/K_0, which is finite. This in turn yields a nontrivial T–action and a fortiori a nontrivial S^1–action. A celebrated result of McDuff [57] then shows that nontrivial S^1–actions imply uniruledness in the sense of k–point genus-zero Gromov–Witten invariants, and hence geometric uniruledness. We therefore obtain the following result.

Corollary 6 *Let (M,ω) be a closed positive monotone symplectic manifold that is not geometrically uniruled, or a negative monotone or symplectically Calabi–Yau symplectic manifold. Then each homomorphism $K \to \mathrm{Ham}(M,\omega)$ for a compact Lie group K must be trivial.*

Moreover, by a simple continuity argument, a nontrivial continuous S^1–action implies a nontrivial $\mathbb{Z}/p\mathbb{Z}$–action for each prime p. Therefore Theorems B and E imply the above corollary for symplectically aspherical, symplectically Calabi–Yau, negative monotone, or monotone symplectic manifolds directly, without relying on the result of McDuff. Moreover, Theorem F also implies that if a positive monotone symplectic manifold admits a nontrivial Hamiltonian S^1–action, it must be $\mathbb{F}_p$–Steenrod uniruled for all primes p. It is seen from examples due to Seidel and Wilkins [92] that there exist closed monotone symplectic manifolds that are uniruled in the sense of Gromov–Witten invariants, and yet not $\mathbb{F}_p$–Steenrod uniruled for certain primes p. More precisely, the monotone blowup M of $\mathbb{CP}^2$ at 6 points is not $\mathbb{F}_2$–Steenrod uniruled, but is evidently uniruled in the Gromov–Witten sense.

The following discussion shows that for a certain nonmonotone 6–point blowup of $\mathbb{CP}^2$ there exists a Hamiltonian involution that cannot be inscribed into an S^1–action.

Note that [92, Example 1.7] and Theorem F imply that the *monotone* blowup M admits no Hamiltonian torsion of order other than 2. It would be interesting to construct a nontrivial Hamiltonian involution of M or prove that it does not exist.

Remark 7 In [12], Chiang and Kessler gave an example of a symplectic involution, ie $\phi \in \mathrm{Symp}(M_0)$ such that $\phi^2 = \mathrm{id}$, of a certain nonmonotone 6–point blowup M_0 of the standard $\mathbb{CP}^2$, with blowup sizes $\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{3}{16}, \frac{1}{8}$. This involution belongs to the symplectic Torelli group $\mathrm{Symp}^h(M_0)$ of symplectomorphisms acting trivially on homology, and has the property that it does not belong to any S^1–subgroup of $\mathrm{Ham}(M_0)$. Li, Li and Wu [53] showed in particular that the mapping-class group $\pi_0 \,\mathrm{Symp}^h(M_0)$ is isomorphic to the quotient $G_6 = P_6(S^2)/Z$ of the spherical pure braid group $P_6(S^2)$ on 6 strands by its center $Z \cong \mathbb{Z}/2\mathbb{Z}$. It is well known that G_6 has no torsion; see González-Meneses [37] for a beautiful account of related subjects. This implies that $\phi \in \mathrm{Symp}_0(M_0) = \mathrm{Ham}(M_0)$, showing that ϕ is a Hamiltonian involution that does not belong to any S^1–subgroup.

We note that McDuff's theorem was proven by showing that certain loops of Hamiltonian diffeomorphisms in a blow-up of the manifold are nontrivial, and detectable by Seidel's representation [87]. It would be interesting to investigate the existence of nontrivial Hamiltonian loops associated to Hamiltonian diffeomorphisms of finite order. For a Hamiltonian H generating $\phi \in \mathrm{Ham}(M,\omega)$ of order d, the Hamiltonian $H^{(d)}$ generates a loop homotopic to $\{(\phi_H^t)^d\}$. The noncontractibility of this loop is not obvious since for a rotation $\phi_{2\pi/3}$ of S^2 by angle $2\pi/3$ about the z–axis, the loop $\{\phi^3_{t\cdot 2\pi/3}\}$ is not contractible in $\mathrm{Ham}(S^2,\omega_{\mathrm{st}})$, while the loop $\{\phi^3_{-t\cdot 4\pi/3}\}$ is contractible therein, yet $\phi_{-4\pi/3} = \phi_{2\pi/3}$.

Finally we can argue, following the work of Polterovich [72] on the Hamiltonian Zimmer conjecture, that $\mathrm{SL}(k,\mathbb{Z})$ for $k \geq 2$ has no nontrivial Hamiltonian actions on symplectically aspherical, symplectically Calabi–Yau, negative monotone, or monotone and not geometrically uniruled closed symplectic manifolds. Indeed, it is well known that $\mathrm{SL}(k,\mathbb{Z})$ for $k \geq 2$ is generated by elements of finite order. We remark, however, that the case of finite-index subgroups of $\mathrm{SL}(k,\mathbb{Z})$ with $k \geq 3$ is much more difficult and seems to be currently out of reach of our methods.

1.2.4 Symplectic actions It makes sense to study finite group actions by more general symplectic diffeomorphisms than Hamiltonian ones. In particular, a classical statement in the topology of hyperbolic surfaces is that diffeomorphisms of finite order cannot be isotopic to the identity. Further progress in this direction was made in low-dimensional symplectic topology; see Chen [9], Chen and Kwasik [10] and Wu and Liu [109]. In this section we collect remarks and results in the higher-dimensional setting.

Let us denote by $\mathrm{Symp}(M,\omega)$ the group of diffeomorphisms preserving the symplectic form, and by $\mathrm{Symp}_0(M,\omega)$ its identity component. Of course $\mathrm{Ham}(M,\omega)$ is a subgroup of $\mathrm{Symp}_0(M,\omega)$.

We first make the observation that if $\mathrm{Ham}(M,\omega)$ and $\mathrm{Symp}_0(M,\omega)$ coincide, Hamiltonian no-torsion theorems yield no-torsion theorems for elements of $\mathrm{Symp}_0(M,\omega)$. Let $\Gamma_\omega \subset H^1(M,\mathbb{R})$ be the well-known *flux group*, defined as the image of the map $\mathrm{Flux}\colon \pi_1(\mathrm{Symp}(M,\omega)) \to H^1(M,\mathbb{R})$ given by integrating ω over the two-cycle traced by a loop of symplectomorphisms applied to one-cycles. It is a finitely generated abelian group. The exact sequence

$$1 \to \mathrm{Ham}(M,\omega) \to \mathrm{Symp}_0(M,\omega) \to H^1(M,\mathbb{R})/\,\Gamma_\omega \to 1$$

therefore implies that $\mathrm{Ham}(M,\omega) = \mathrm{Symp}_0(M,\omega)$ if and only if $H^1(M,\mathbb{R}) = 0$. Second, following Polterovich [72, Example 1.3.C], by the same exact sequence we note that whenever $\Gamma_\omega = 0$, all torsion elements in $\mathrm{Symp}_0(M,\omega)$ must in fact be Hamiltonian. By a result of McDuff [56, Theorem 1], this happens for homologically monotone and negative monotone symplectic manifolds, ie when $[\omega] = \kappa \cdot c_1(TM)$ for some $\kappa \neq 0$ as elements of $H^2(M,\mathbb{R})$. By a result of Kędra [49], this also holds for closed symplectically aspherical manifolds (M,ω), ie when $[\omega] = 0$ on $\pi_2(M)$, of nonvanishing Euler characteristic or when the center of $\pi_1(M)$ is trivial; see also Kędra, Kotschick and Morita [50].

We expect that the methods developed in this paper will yield new results on torsion in symplectomorphism groups and plan to investigate this in a further publication.

1.2.5 Metric properties Our third and final set of results studies the metric properties of Hamiltonian torsion diffeomorphisms, in cases that are not ruled out by our previous arguments, for example on $\mathbb{CP}^n$.

Recall that the *spectral pseudonorm* of a Hamiltonian $H \in C^\infty(S^1 \times M,\mathbb{R})$ on a closed symplectic manifold (M,ω) is defined in terms of Hamiltonian spectral invariants as

$$\gamma(H) = c([M],H) + c([M],\overline{H}),$$

and the *spectral norm* of $\phi \in \mathrm{Ham}(M,\omega)$ is set as

$$\gamma(\phi) = \inf_{\phi_H^1 = \phi} \gamma(H).$$

We refer to Section 2 for a more in-depth discussion of this interesting notion, remarking for now that this is a conjugation-invariant and nondegenerate norm on $\mathrm{Ham}(M,\omega)$, yielding a bi-invariant metric

$$d_\gamma(f,g)=\gamma(gf^{-1}).$$

This was shown in large generality in Oh [65], Schwarz [86] and Viterbo [104].

Furthermore, whenever defined, $\gamma(\phi)$ provides a lower bound on the celebrated Hofer distance $d_{\mathrm{Hofer}}(\phi,\mathrm{id})$, defined as

$$d_{\mathrm{Hofer}}(\phi,\mathrm{id})=\inf_{\phi_H^1=\phi}\int_0^1 \max_M H(t,-)-\min_M H(t,-)\,dt;$$

see Hofer [40] and Lalonde and McDuff [51]. Finally in Buhovsky, Humilière and Seyfaddini [6], Kawamoto [47] and Shelukhin [96] it was shown, in various degrees of generality, that $\gamma(\phi)$ is bounded by the C^0–distance $d_{C^0}(\phi,\mathrm{id})$ of ϕ to the identity, at least in a small d_{C^0}–neighborhood of the identity.

These and numerous other recent results show that the spectral norm γ is an important measure of a Hamiltonian diffeomorphism. Here, we provide lower bounds on $\gamma(\phi)$, under the assumption that ϕ is of finite order. Our first result is relatively general and quite straightforward, and follows essentially from the homogeneity of the action functional under iteration. However, it underlines the fact that the finite order condition implies certain metric rigidity.

Theorem J *Let (M,ω) be a closed rational symplectic manifold, with rationality constant $\rho>0$, ie $\langle[\omega],\pi_2(M)\rangle=\rho\cdot\mathbb{Z}$. Suppose that $\phi\in\mathrm{Ham}(M,\omega)$ is a nontrivial Hamiltonian diffeomorphism of order d, ie $\phi^d=\mathrm{id}$. Then $\gamma(\phi)\geq\rho/d$.*

As a further consequence of Theorem G, which requires considerably more complex methods, we obtain the following analogue of Newman's theorem for the spectral norm of Hamiltonian torsion elements. This result is the first nontrivial result of its kind in symplectic topology, and is implicitly conjectured in the formulation of [59, Problem 24].

Theorem K *Let (M,ω) be a closed monotone symplectic manifold of rationality constant $\rho>0$. Consider a Hamiltonian diffeomorphism $\phi\in\mathrm{Ham}(M,\omega)$ of order $d>1$. Then there exists $m\in\mathbb{Z}/d\mathbb{Z}$ such that*

$$\gamma(\phi^m)\geq\frac{\rho}{3}.$$

Here the coefficients are in an arbitrary field $\mathbb{K}$. *In fact, if* $d = p$ *is prime, we prove the stronger statement that there exists* $m \in \mathbb{Z}/p\mathbb{Z}$ *such that*

$$\gamma(\phi^m) \geq \frac{\rho \cdot \lfloor p/2 \rfloor}{p}.$$

The key notion in the proof of this result is a new invariant of a Hamiltonian diffeomorphism $\phi \in \mathrm{Ham}(M,\omega)$, which we call the *spectral length* $l(\phi,\mathbb{K})$ of ϕ with coefficients in a field $\mathbb{K}$. It is defined as the minimal diameter of $\mathrm{Spec}^{\mathrm{ess}}(H;\mathbb{K}) \cap I$ over intervals $I = (a-\rho, a] \subset \mathbb{R}$ of length ρ, where H is a Hamiltonian with $\phi_H^1 = \phi$. In particular, we show that this minimum does not depend on the choice of the Hamiltonian H. We show the key property that $l(\phi,\mathbb{K}) \leq \gamma(\phi,\mathbb{K})$ and that, in our case, the spectral length behaves in a controlled way with respect to iterations. By a combinatorial analysis of our situation we consequently deduce Theorem K. We expect $l(\phi,\mathbb{K})$ to have additional applications in quantitative symplectic topology, which we plan to investigate.

Theorem K is generally speaking sharp, as can be seen from the rotation ϕ of S^2 by $2\pi/3$ about the z–axis. In this case $\phi^3 = \mathrm{id}$ and $\gamma(\phi) = \gamma(\phi^2) = \gamma(\phi^{-1}) = \rho/3$, where ρ is the area of the sphere. Observe moreover that the lower bound in Theorem K does not depend on the order of ϕ. In particular if $d = 2$, then Theorem J gives the stronger lower bound $\gamma(\phi) \geq \rho/2$, which is again sharp for the π–rotation of S^2 about the z–axis. We recall that Newman's theorem is a directly analogous assertion, but for the C^0–distance to the identity, in the setting of homeomorphisms of smooth manifolds. In contrast to our result, the constant in Newman's theorem is not explicit.

Finally, we remark that analogous statements hold for generalized $\mathbb{F}_p$ pseudorotations ϕ with sufficiently large admissible sequences. For example, for the sequence $k_j = p^{j-1}$, we get the lower bound $\gamma(\phi^{k_j}) \geq \rho/(p+1)$ for some $j \in \mathbb{Z}_{>0}$, which is saturated by the rotation of S^2 by $2\pi/(p+1)$ about the z–axis. For the sequence $k_j = j$, we obtain the following lower bound, which is saturated by any $2\pi\theta$–rotation on S^2 about the z–axis, where $\theta \notin \mathbb{Q}$.

Theorem L *Let* $\phi \in \mathrm{Ham}(M,\omega)$ *be a generalized* $\mathbb{K}$ *pseudorotation with sequence* $k_j = j$ *on a closed monotone symplectic manifold* (M,ω) *with rationality constant* ρ. *Then*

$$\sup_{j \in \mathbb{Z}_{>0}} \gamma(\phi^{k_j}) \geq \frac{\rho}{2},$$

the coefficients being taken in $\mathbb{K}$.

This result is new in this generality even for strongly nondegenerate pseudorotations. Moreover, Theorem L applies to irrational elements of effective Hamiltonian S^1–actions, and Theorem K applies to rational elements. In particular, by considering the element $\left[\frac{1}{2}\right] \in S^1 = \mathbb{R}/\mathbb{Z}$, we obtain that the Hofer length of such a Hamiltonian S^1–action is at least ρ. In the case of semifree S^1–actions, this lower bound can be deduced from McDuff and Slimowitz [60], where it is also proven that the S^1–action is Hofer length-minimizing among Hamiltonian loops in the same free homotopy class. Our results do not prove such homotopical minimality. However, they do apply in the case where the action is not semifree, where no such results are known. In fact such Hamiltonian loops may well be nullhomotopic; see also Karshon and Pearl [46] for more general shortening results in this case. Finally, we observe that in the special case where (M,ω) is a complex projective space, a similar result to Theorem L can be obtained in a different way by following the methods of Ginzburg and Gürel [35].

2 Preliminary material

2.1 Basic setup

In this section, we recall established aspects of the theory of Hamiltonian diffeomorphisms on symplectic manifolds. Throughout the article, (M,ω) denotes a $2n$–dimensional closed symplectic manifold.

Definition 8 (monotone, negative monotone and symplectically Calabi–Yau) Suppose that the cohomology class of the symplectic form ω is proportional to the first Chern class, ie
$$[\omega] = \kappa \cdot c_1(TM)$$
for some $\kappa \neq 0$, on the image $H_2^S(M;\mathbb{Z})$ of the Hurewicz map $\pi_2(M) \to H_2(M;\mathbb{Z})$. If $\kappa < 0$ we call (M,ω) *negative monotone*, and if $\kappa > 0$ we call it (*positive*) *monotone*. If the first Chern class $c_1(TM)$ vanishes on the image of the Hurewicz map, we say that (M,ω) is *symplectically Calabi–Yau*.

The symplectic manifold (M,ω) is called *rational* whenever $\mathcal{P}_\omega = \langle [\omega], H_2^S(M;\mathbb{Z})\rangle$ is a discrete subgroup of $\mathbb{R}$. If $\mathcal{P}_\omega \neq 0$, then $\mathcal{P}_\omega = \rho \cdot \mathbb{Z}$ for $\rho > 0$, which we call the *rationality constant* of (M,ω). If $\mathcal{P}_\omega = 0$ we call (M,ω) *symplectically aspherical*.[5]

[5]In the literature the additional condition $\langle c_1(TM), H_2^S(M;\mathbb{Z})\rangle = 0$, which we do not require, is often imposed. This condition allows one to introduce a $\mathbb{Z}$–grading on the Floer complex, which we do not require once $\mathcal{P}_\omega = 0$.

Finally we recall that the minimal Chern number of (M,ω) is the index

$$N = N_M = [\mathbb{Z} : I]$$

in $\mathbb{Z}$ of the subgroup $I = \operatorname{im}\big(c_1(TM) \colon \pi_2(M) \to \mathbb{Z}\big)$. Namely, $[\mathbb{Z} : I] = |\mathbb{Z}/I|$ is the cardinality of the quotient group $\mathbb{Z}/I$.

2.1.1 Hamiltonian isotopies and diffeomorphisms We next consider normalized 1–periodic Hamiltonian functions $H \in \mathcal{H} \subset C^\infty(\mathrm{S}^1 \times M, \mathbb{R})$, where $\mathcal{H}$ is the space of Hamiltonians normalized so that $H(t,-)$ has zero ω^n–mean for all $t \in [0,1]$. For each $H \in \mathcal{H}$ we have the corresponding time-dependent vector field X_H^t defined by the relation $\omega(X_H^t, \cdot) = -dH_t$. In particular, to each Hamiltonian function we can associate a Hamiltonian isotopy $\{\phi_H^t\}$ induced by X_H^t and its time-one map $\phi_H = \phi_H^1$. We omit the H from this notation whenever it is clear from context. Such maps ϕ_H are called Hamiltonian diffeomorphisms and they form a group denoted by $\mathrm{Ham}(M,\omega)$.

For a Hamiltonian diffeomorphism $\phi \in \mathrm{Ham}(M,\omega)$, we denote the set of its *contractible* fixed points by $\mathrm{Fix}(\phi)$. Contractible means the homotopy class $\alpha(x,\phi)$ of the path $\alpha(x,H) = \{\phi_H^t(x)\}$ for a Hamiltonian $H \in \mathcal{H}$ generating ϕ is trivial. This class does not depend on the choice of Hamiltonian, by a classical argument in Floer theory. We write $x^{(k)}$ for the image of $x \in \mathrm{Fix}(\phi)$ under the inclusion $\mathrm{Fix}(\phi) \subset \mathrm{Fix}(\phi^k)$.

We denote by $H^{(k)} \in C^\infty(\mathrm{S}^1 \times M, \mathbb{R})$ the k^{th} iteration of a Hamiltonian function H, given by $H^{(k)}(t,x) = kH(kt,x)$. Note that $\phi_{H^{(k)}} = \phi_H^k$. There is a bijective correspondence between $\mathrm{Fix}(\phi_H)$ and contractible 1–periodic orbits of the isotopy $\{\phi_H^t\}$, thus for $x \in \mathrm{Fix}(\phi_H)$, we denote by $x(t)$ the 1–periodic orbit given by $x(t) = \phi_H^t(x)$ and, similarly, by $x^{(k)}(t)$ the 1–periodic orbit given by $x^{(k)}(t) = \phi_{H^{(k)}}^t(x^{(k)})$.

2.1.2 The Hamiltonian action functional Let $x \colon \mathrm{S}^1 \to M$ be a contractible loop. It is then possible to extend this map to a *capping* of x, namely, a map $\bar{x} \colon \mathrm{D}^2 \to M$ such that $\bar{x}|_{\mathrm{S}^1} = x$. Let $\mathcal{L}_{\mathrm{pt}}M$ denote the space of contractible loops in M and consider the equivalence relation on capped loops given by

$$(x,\bar{x}) \sim (y,\bar{y}) \qquad \iff \qquad x = y \quad \text{and} \quad \bar{x} \,\#\, (-\bar{y}) \in \ker([\omega]) \cap \ker(c_1),$$

where $\bar{x} \,\#\, (-\bar{y})$ stands for gluing the disks along their boundaries with the orientation of $\bar{y}$ reversed. Here $\ker([\omega])$ and $\ker(c_1)$ denote the kernels of the homomorphisms $H_2^S(M;\mathbb{Z}) \to \mathbb{R}$ induced by $[\omega]$ and $c_1(TM)$. The quotient space $\widetilde{\mathcal{L}}_{\mathrm{pt}}M$ of capped loops by the above equivalence relation is a covering of $\mathcal{L}_{\mathrm{pt}}M$ with the group of deck transformations isomorphic to $\Gamma = H_2^S(M;\mathbb{Z})/\big(\ker([\omega]) \cap \ker(c_1)\big)$. Note that if (M,ω) is positive or negative monotone or symplectically Calabi–Yau, then $\ker([\omega]) \cap \ker(c_1) = \ker([\omega])$, whence $\Gamma = H_2^S(M;\mathbb{Z})/\ker([\omega])$. Note also that $\Gamma \cong \pi_2(M)/\big(\ker([\omega]') \cap \ker(c_1')\big)$, where the maps $[\omega]', c_1' \colon \pi_2(M) \to \mathbb{R}$ are the compositions of $[\omega]$ and c_1 with the Hurewicz homomorphism $\pi_2(M) \to H_2^S(M;\mathbb{Z})$.

We write $(x,\overline{x})$, or simply $\overline{x}$, for the equivalence class of the capped loop. With this notation, to each $A \in \Gamma$ we associate the deck transformation sending a capped loop $\overline{x}$ to $\overline{x} \# A$. We define the Hamiltonian action functional $\mathcal{A}_H \colon \widetilde{\mathcal{L}}_{\mathrm{pt}} M \to \mathbb{R}$ of a 1–periodic Hamiltonian H by

$$\mathcal{A}_H(\overline{x}) = \int_0^1 H(t,x(t))\,dt - \int_{\overline{x}} \omega.$$

Observe that the critical points of the Hamiltonian action functional are exactly $(x,\overline{x})$ for x a contractible 1–periodic orbit satisfying the equation $x'(t) = X_H^t(x(t))$. We denote by $\mathcal{O}(H)$ the set of such orbits, and by $\widetilde{\mathcal{O}}(H)$ the set of critical points of $\mathcal{A}_H$. The *action spectrum* of H is defined as $\operatorname{Spec}(H) = \mathcal{A}_H(\widetilde{\mathcal{O}}(H))$. We remark, following [86], that in the rational case the action spectrum is a closed nowhere-dense subset of $\mathbb{R}$. In addition, if $A \in \Gamma$ then

$$\mathcal{A}_H(\overline{x} \# A) = \mathcal{A}_H(\overline{x}) - \int_A \omega,$$

and for $\overline{x}^{(k)}$, the k^{th} iteration of $\overline{x}$ with the naturally inherited capping, we have

$$\mathcal{A}_{H^{(k)}}(\overline{x}^{(k)}) = k\mathcal{A}_H(\overline{x}).$$

Definition 9 (nondegenerate and weakly nondegenerate orbits) A 1–periodic orbit x of H is called *nondegenerate* if 1 is not an eigenvalue of the linearized time-one map $D(\phi_H^1)_{x(0)}$ at $x(0)$. We call x *weakly nondegenerate* if there exists at least one eigenvalue of $D(\phi_H^1)_{x(0)}$ different from 1. We say that a Hamiltonian H is *nondegenerate* (resp. *weakly nondegenerate*) if all its 1–periodic orbits are nondegenerate (resp. weakly nondegenerate).

The nondegeneracy of an orbit x of H is equivalent to

$$\operatorname{graph}(\phi_H) = \{(x,\phi_H(x)) \mid x \in M\}$$

intersecting the diagonal $\Delta_M \subset M \times M$ transversely at $(x(0),x(0))$. Following [84], for any Hamiltonian H and $\epsilon > 0$, there exists a nondegenerate Hamiltonian H' satisfying $\|H - H'\|_{C^2} < \epsilon$. This fact is key in the definition of filtered Floer homology of degenerate Hamiltonians and for local Floer homology.

2.1.3 Mean-index and the Conley–Zehnder index Following [84; 34], the *mean-index* $\Delta(H,\overline{x})$ of a capped orbit $\overline{x}$ of a possibly degenerate Hamiltonian H is a real number measuring the sum of the angles swept by certain eigenvalues of $\{D(\phi_H^t)_{x(t)}\}$ lying on the unit circle. Here a trivialization induced by the capping is used in order to view $\{D(\phi_H^t)_{x(t)}\}$ as a path in $\mathrm{Sp}(2n,\mathbb{R})$. One can show that for the time-one map $\phi=\phi_H$ generated by the Hamiltonian H, the mean-index depends only on the class $\widetilde{\phi}$ of $\{\phi_H^t\}_{t\in[0,1]}$ in the universal cover $\widetilde{\mathrm{Ham}}(M,\omega)$, making the notation $\Delta(\widetilde{\phi},\overline{x})$ suitable. In addition, the mean-index depends continuously on $\widetilde{\phi}$ in the C^1–topology and on the capped orbit, and it behaves well with iterations,

$$\Delta(\widetilde{\phi}^k,\overline{x}^{(k)})=k\cdot\Delta(\widetilde{\phi},\overline{x}).$$

Meanwhile, the *Conley–Zehnder index* $\mathrm{CZ}(H,\overline{x})$ of a nondegenerate capped 1–periodic orbit $\overline{x}$ is integer-valued, and roughly measures the winding number of the abovementioned eigenvalues. Once again, the index only depends on $\widetilde{\phi}$, so we can also write $\mathrm{CZ}(H,\overline{x})=\mathrm{CZ}(\widetilde{\phi},\overline{x})$. We shall use the same normalization as in [34], namely, $\mathrm{CZ}(H,\overline{x})=n$ if x is a nondegenerate maximum of an autonomous Hamiltonian H with small Hessian and $\overline{x}$ is the constant capping. We shall omit the H or $\widetilde{\phi}$ in the notation when it is clear from the context. We remark that for an element $A\in\Gamma$,

$$\Delta(\overline{x}\,\#\,A)=\Delta(\overline{x})-2\langle c_1(TM),A\rangle\quad\text{and}\quad\mathrm{CZ}(\overline{x}\,\#\,A)=\mathrm{CZ}(\overline{x})-2\langle c_1(TM),A\rangle.$$

Also, in the case that $\overline{x}$ is nondegenerate, we have

$$|\Delta(\overline{x})-\mathrm{CZ}(\overline{x})|<n. \tag{2}$$

Following [79; 73; 18], we observe that a version of the Conley–Zehnder index can be defined even in the case where the capped orbit is degenerate. It is called the Robbin–Salamon index, and it coincides with the usual Conley–Zehnder index in the nondegenerate case. Furthermore, we note that the mean-index can be equivalently defined by

$$\Delta(\widetilde{\phi}_H,\overline{x})=\lim_{k\to\infty}\frac{1}{k}\,\mathrm{CZ}(\widetilde{\phi}_H^k,\overline{x}^{(k)}), \tag{3}$$

where we are slightly abusing notation in the sense that CZ here means the Robbin–Salamon index so as to include the degenerate case. The limit in (3) exists, as the Robbin–Salamon index is a quasimorphism $\mathrm{CZ}\colon\widetilde{\mathrm{Sp}}(2n,\mathbb{R})\to\mathbb{R}$; see eg [14] and [18, Section 3.3.4]. In particular, as can be seen directly from its definition in [84], the mean-index is induced by a homogeneous quasimorphism $\Delta\colon\widetilde{\mathrm{Sp}}(2n,\mathbb{R})\to\mathbb{R}$.

Moreover, this map is continuous, and satisfies the additivity property

$$\Delta(\Phi\Psi) = \Delta(\Phi) + \Delta(\Psi)$$

for all $\Phi \in \pi_1(\mathrm{Sp}(2n,\mathbb{R})) \subset \widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ and all $\Psi \in \widetilde{\mathrm{Sp}}(2n,\mathbb{R})$.

2.2 Floer theory

Floer theory was first introduced by A Floer [21; 22; 23] as a generalization of the Morse–Novikov homology for the Hamiltonian action functional defined above. We refer to [67] for details on the constructions described in this subsection, and to [1; 88; 110] for a discussion of canonical orientations.

2.2.1 Filtered and total Floer homology We review the construction of *filtered Floer homology* in order to recall some basic properties and set notation.

Let H be a nondegenerate 1–periodic Hamiltonian on a rational symplectic manifold (M,ω) and $\mathbb{K}$ a fixed base field. For $a \in \mathbb{R}\backslash\mathrm{Spec}(H)$ and $\{J_t \in \mathcal{J}(M,\omega)\}_{t\in S^1}$ a generic loop of ω–compatible almost complex structures, set

$$CF_k(H;J)^{<a} = \Big\{\sum \lambda_{\bar{x}}\cdot\bar{x} \;\Big|\; \bar{x}\in\widetilde{\mathcal{O}}(H),\, \mathrm{CZ}(\bar{x}) = k,\, \lambda_{\bar{x}}\in\mathbb{K},\, \mathcal{A}_H(\bar{x}) < a\Big\},$$

where $\#\{\lambda_{\bar{x}} \neq 0 \mid \mathcal{A}_H(\bar{x}) > c\} < \infty$ for every $c \in \mathbb{R}$. Intuitively, it is the vector space over $\mathbb{K}$ generated by the critical points of the Hamiltonian action functional of filtration level $< a$. The graded $\mathbb{K}$–vector space $CF_*(H,J)^{<a}$ is endowed with the Floer differential $d_{H;J}$, which is defined as the signed count of isolated solutions (quotiented out by the $\mathbb{R}$–action) of the asymptotic boundary value problem on maps $u\colon \mathbb{R}\times S^1 \to M$ defined by the negative gradient of $\mathcal{A}_H$; see [83; 84]. In other words, the boundary operator counts the finite-energy solutions to the Floer equation,

$$\frac{\partial u}{\partial s} + J_t(u)\left(\frac{\partial u}{\partial t} - X_H^t(u)\right) = 0 \quad \text{such that } E(u) = \int_{\mathbb{R}}\int_{S^1}\left\|\frac{\partial u}{\partial s}\right\|^2 dt\,ds < \infty,$$

which converge as s tends to $\pm\infty$ to periodic orbits x_- and x_+ such that the capping $\bar{x}_-\# u$ is equivalent to $\bar{x}_+$ and $\mathrm{CZ}(\bar{x}_-) - \mathrm{CZ}(\bar{x}_+) = 1$. In this case the Floer trajectory u satisfies $E(u) = \mathcal{A}_H(\bar{x}_-) - \mathcal{A}_H(\bar{x}_+)$. We thus obtain the *filtered Floer chain complex* $(CF_*(H;J)^{<a}, d_{H;J})$, which is a subcomplex of the *total Floer chain complex* (corresponding to $a = +\infty$). Furthermore, for an interval $I = (a,b)$ with $a < b$ and $a,b \in (\mathbb{R}\setminus\mathrm{Spec}(H))\cup\{\pm\infty\}$, we define the Floer complex in the action window I as the quotient complex

$$CF_*(H;J)^I = CF_*(H;J)^{<b}/CF_*(H;J)^{<a},$$

where $CF_*(H;J)^{<-\infty} = 0$. The resulting homology of this complex $HF_*(H)^I$ is the *Floer homology* of H *in the action window* I and it is independent of the generic choice of almost complex structure J. So the (total) Floer homology of H can be obtained by setting $a = -\infty$ and $b = +\infty$. We note that in the positive and negative monotone case $CF_*(H;J)$ is naturally a module over the Novikov field $\Lambda_{\mathbb{K}} = \mathbb{K}[\![q^{-1}, q]$ with q a variable of degree $2N$. Indeed we define $q^{-1} \cdot \overline{x} = \overline{x} \# A_0$ for A_0 the generator of Γ with $\langle c_1(TM), A_0\rangle = N$, and extend it to a module structure in the natural way. In the Calabi–Yau case, $CF_*(H;J)$ is a module over the Novikov field

$$\Lambda_{\mathbb{K},\omega} = \Big\{\sum a_i T^{\lambda_i} \;\Big|\; a_i \in \mathbb{K},\, \lambda_i \in \mathcal{P}_\omega,\, \lambda_i \to \infty\Big\}.$$

While we shall not use it in the paper, we remark that in the general case, it is a module over the Novikov field

$$\Lambda_{\mathbb{K},\omega,c_1} = \Big\{\sum a_i T^{A_i} \;\Big|\; a_i \in \mathbb{K},\, A_i \in \Gamma,\, \omega(A_i) \to \infty\Big\}.$$

Observe that by interpolating between distinct Hamiltonians through generic families and writing the Floer continuation map, where the Hamiltonian perturbation term and the almost complex structure depend on the $\mathbb{R}$–coordinate, one can show that $HF_*(H)$ does not depend on the Hamiltonian, and $HF_*(H)^I$ depends only on the homotopy class of $\{\phi_H^t\}_{t\in[0,1]}$ in the universal cover $\widetilde{\mathrm{Ham}}(M,\omega)$ of the Hamiltonian group $\mathrm{Ham}(M,\omega)$. Also, when M is rational the above construction extends by a standard continuity argument to degenerate Hamiltonians.

Remark 10 Theorems B, D and J partially deal with negative monotone or general spherically rational symplectic manifolds. It is important to emphasize that for our arguments to apply to this case in full generality, we must make use of the machinery of virtual cycles to guarantee that the Floer differential is well defined. In this case, the ground field $\mathbb{K}$ should be of characteristic zero. Our arguments are not sensitive to the specific approach to questions of transversality. We refer to [33; 55; 28; 80] for early works on the subject, subsequently augmented in [32, Chapters 15–20; 31, Section 9; 30, Section 8; 29, Section 19]. We refer to [32, Chapter 1.4] for an overview of other approaches to virtual fundamental cycles. This includes the theory of polyfolds initiated in [42; 43; 44]; see [20] for a recent survey. We also note that [69] provides foundations of Hamiltonian Floer theory in full generality. Furthermore, we mention the following cases where classical transversality techniques are applicable. First, if (M,ω) is a semipositive[6] symplectic manifold — that is, if (M,ω) is symplectically

[6]The terminology "weakly monotone" also appears in the literature for the same notion.

aspherical, symplectically Calabi–Yau, positive monotone, or if the minimal Chern number of (M,ω) is $N \geq n-2$ — then classical transversality applies by [41]. Second, if the manifold is homologically rational, ie the symplectic form can be scaled so that all of its periods are integers, then classical transversality applies by [8] following [13].

2.2.2 The irrational case In this paper we also consider the case in which the manifold M is symplectically Calabi–Yau, which includes the possibility of it being irrational. In this case we have to work a little harder if H is degenerate since the continuation argument above does not work as before, as nonspectral a, b for H do not necessarily remain nonspectral even for arbitrarily small perturbations H_1 of H. Moreover, the resulting homology groups depend on the choice of nondegenerate perturbation H_1. We shall follow [39] to work around this issue.

For a fixed Hamiltonian H and action window $I=(a,b)$ with $a<b$, where $a,b \in (\mathbb{R} \setminus \mathrm{Spec}(H)) \cup \{\pm\infty\}$, consider the set of nondegenerate perturbations $\widetilde{H}$ whose action spectra do not include a and b and $H \leq \widetilde{H}$, ie $H(t,x) \leq \widetilde{H}(t,x)$ for all $x \in M$ and $t \in S^1$. Note that such perturbations $\widetilde{H}$ of a mean-normalized H will in general not be mean-normalized. However, this does not present an issue for our purposes. Observe that $\leq$ induces a partial order in the set of perturbations. In addition, by considering a monotone decreasing homotopy $\widetilde{H}^s$ from $\widetilde{H}^0$ to $\widetilde{H}^1$, one obtains an induced homomorphism between the Floer homology groups. These give rise to continuation maps $HF_*(H')^I \to HF_*(H'')^I$ whenever $H'' \leq H'$. Therefore, we can define the filtered Floer homology of H by taking the direct limit

$$HF_*(H)^I = \varinjlim HF_*(H')^I$$

over the homology groups of the perturbations satisfying the aforementioned conditions. We remark that in the case where H is nondegenerate or M is rational, this definition coincides with the usual filtered Floer homology groups.

2.2.3 Local Floer homology In this section we shall follow [34] in order to briefly review the construction of the *local Floer homology* of a Hamiltonian H at a capping $\bar{x}$ of an isolated 1–periodic orbit x.

Since $\bar{x}$ is isolated we can find an isolating neighborhood U of x in the extended phase-space $S^1 \times M$ whose closure does not intersect the image $\{(t, y(t))\}_{t\in[0,1]}$ of any other orbit y of H. For a sufficiently C^2–small nondegenerate perturbation H' of H, the orbit x splits into finitely many 1–periodic orbits $\mathcal{O}(H',x)$ of H', which are contained in U and whose cappings are inherited from $\bar{x}$. We denote by $\mathcal{O}(H',\bar{x})$

the capped 1–periodic orbits $\bar{x}$ splits into. Moreover, we can also guarantee that any Floer trajectory and any broken trajectory between capped orbits in $\mathcal{O}(H', \bar{x})$ are contained in U. For a base field $\mathbb{K}$ we consider the vector space $CF_*(H, \bar{x})$ generated by $\mathcal{O}(H', \bar{x})$, which by the above observation naturally inherits a Floer differential and a grading by the Conley–Zehnder index. The homology of this chain complex is independent of the choice of the perturbation H' once it is close enough to H, and it is called the local Floer homology of H at $\bar{x}$; it is denoted by $HF_*^{\mathrm{loc}}(H, \bar{x})$. This group depends only on the class $\tilde{\phi}$ of $\{\phi_H^t\}$ in the universal cover $\widetilde{\mathrm{Ham}}(M, \omega)$, and the capped orbit $\bar{x}$. Namely, homotopic paths have choices of cappings of orbits corresponding to a fixed point $x \in \mathrm{Fix}(\phi)$ in bijection, and the corresponding groups are canonically isomorphic. Hence we write $HF_*^{\mathrm{loc}}(H, \bar{x}) = HF_*^{\mathrm{loc}}(\tilde{\phi}, \bar{x})$. If we ignore the $\mathbb{Z}$–grading, then the group depends only on $\phi = \phi_H^1$ and $x \in \mathrm{Fix}(\phi)$. In this case, we write $HF^{\mathrm{loc}}(\phi, x)$ for the corresponding local homology group, which is naturally only $\mathbb{Z}/(2)$–graded.

Let $\bar{x}$ be a capped 1–periodic orbit of a Hamiltonian H. We define the *support* of $HF_*^{\mathrm{loc}}(H, \bar{x})$ to be the collection of integers k such that $HF_k^{\mathrm{loc}}(H, \bar{x}) \neq 0$. By the continuity of the mean-index and by equation (2), it follows that $HF_*^{\mathrm{loc}}(H, \bar{x})$ is supported in the interval $[\Delta(\bar{x}) - n, \Delta(\bar{x}) + n]$. One can show that if x is weakly nondegenerate then $HF_*^{\mathrm{loc}}(H, \bar{x})$ is actually supported in $(\Delta(\bar{x}) - n, \Delta(\bar{x}) + n)$. We shall explore the idea behind the proof of this second fact later as we use the same argument to prove a similar claim in slightly greater generality. Namely, we extend it to an isolated compact path-connected family of contractible fixed points.

2.3 Quantum homology and PSS isomorphism

In the present section we describe the quantum homology of a symplectic manifold. It might be helpful to think of it as the Hamiltonian Floer homology in the case where the Hamiltonian is a C^2–small time-independent Morse function. Alternatively, one may consider it as the cascade approach [26] to Morse homology for the unperturbed symplectic area functional on the space $\widetilde{\mathcal{L}}_{\mathrm{pt}} M$. For a more detailed exposition of these subjects we refer to [52; 67; 89].

2.3.1 Quantum homology Fix a ground field $\mathbb{K}$. Consider the Novikov field $\Lambda = \Lambda_{\mathbb{K}} = \mathbb{K}[\![q^{-1}, q]$ of (M, ω) in the positive and negative monotone cases, where $\deg(q) = 2N$ and $\Lambda = \Lambda_{\mathbb{K},\omega}$ in the Calabi–Yau case. We set

$$QH(M) = QH(M, \mathbb{K}) = H_*(M; \Lambda)$$

as a Λ–module. This module has the structure of a graded-commutative unital algebra over $\Lambda_{\mathbb{K}}$ whose product, denoted by $*$, is defined in terms of 3–point genus-zero Gromov–Witten invariants [54; 58; 81; 82; 108]. It can be thought of as a deformation of the usual intersection product on homology. As in the classical homology algebra, the unit for this quantum product is the fundamental class $[M]$ of M.

2.3.2 Piunikhin–Salamon–Schwarz isomorphism Under our conventions for the Conley–Zehnder index, one obtains a map

$$\mathrm{PSS}\colon QH_*(M) \to HF_{*-n}(H)$$

by counting (for generic auxiliary data) certain isolated configurations. More precisely, the configurations considered consist of negative gradient trajectories $\gamma\colon(-\infty,0]\to M$ of a generic Morse–Smale pair[7] incident at $\gamma(0)$ with the asymptotic $\lim_{s\to-\infty} u(s,\cdot)$ of a map $u\colon\mathbb{R}\times S^1 \to M$ of finite energy, satisfying the Floer equation

$$\frac{\partial u}{\partial s} + J_t(u)\left(\frac{\partial u}{\partial t} - X_K^t(u)\right) = 0$$

for $(s,t)\in\mathbb{R}\times S^1$ and $K(s,t)\in C^\infty(M,\mathbb{R})$ a small perturbation of $\beta(s)H_t$ such that $K(s,t)=\beta(s)H_t$ for $s\ll -1$ and for $s\gg +1$. Here $\beta\colon\mathbb{R}\to[0,1]$ is a smooth function satisfying $\beta(s)=0$ for $s\ll -1$ and $\beta(s)=1$ for $s\gg +1$. This map produces an isomorphism of $\Lambda_{\mathbb{K}}$–modules, which intertwines the quantum product on $QH(M)$ with the pair-of-pants product on Hamiltonian Floer homology. This map is called the *Piunikhin–Salamon–Schwarz isomorphism*.

2.4 Spectral invariants in Floer theory

We review the theory of spectral invariants following the works of [73; 34; 67], which contain a more exhaustive list of properties and finer details of the construction.

Let (M,ω) be a closed symplectic manifold, H a generic Hamiltonian and $\{J_t\}_{t\in S^1}$ a loop of ω–compatible almost complex structures. For $a\in\mathbb{R}\setminus\mathrm{Spec}(H)$, the inclusion of the filtered Floer complex into the total complex induces a homomorphism

$$i_a\colon HF(H)^{<a}\to HF(H).$$

[7] That is, a Morse function $f\in C^\infty(M,\mathbb{R})$ and Riemannian metric g on M, satisfying the Morse–Smale condition.

For each $\alpha_M \in QH_*(M) \setminus \{0\}$, using the PSS isomorphism $QH_*(M) \cong HF_{*-n}(H)$ we then define

$$c(\alpha_M, H) = \inf\{a \in \mathbb{R} \mid \mathrm{PSS}(\alpha_M) \in \mathrm{im}(i_a)\}.$$

It is not hard to see that the spectral invariants do not depend on the choice of an almost complex structure. In addition, for $H \in \mathcal{H}$ the spectral invariant $c(\alpha_M, H)$ depends only on the class $\widetilde{\phi}_H$ of $\{\phi_H^t\}$ in the universal cover $\widetilde{\mathrm{Ham}}(M, \omega)$; consequently, we also denote it by $c(\alpha_M, \widetilde{\phi}_H) = c(\alpha_M, H)$.

Definition 11 (non-Archimedean valuation) Let Λ be a field. A non-Archimedean valuation on Λ is a function $\nu\colon \Lambda \to \mathbb{R} \cup \{+\infty\}$ such that

(i) $\nu(x) = +\infty$ if and only if $x = 0$,

(ii) $\nu(xy) = \nu(x) + \nu(y)$ for all $x, y \in \Lambda$,

(iii) $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$ for all $x, y \in \Lambda$.

The Novikov field $\Lambda_{\mathbb{K}} = \mathbb{K}[\![q^{-1}, q]$ possesses a non-Archimedean valuation

$$\nu\colon \Lambda_{\mathbb{K}} \to \mathbb{R} \cup \{+\infty\}$$

given by setting $\nu(0) = +\infty$ and

$$\nu\Big(\sum a_j q^j\Big) = -\max\{j \mid a_j \neq 0\}. \tag{4}$$

Spectral invariants enjoy a wealth of useful properties, established by Schwarz [86], Viterbo [104], Oh [58; 64; 66] and generalized by Usher [101; 102], all of which hold for closed rational symplectic manifolds, using the machinery of virtual cycles as discussed in Remark 10 if necessary. We summarize some of the relevant properties for our purposes:

(i) **Spectrality** For each $\alpha_M \in QH(M) \setminus \{0\}$ and $H \in \mathcal{H}$,

$$c(\alpha_M, \widetilde{\phi}_H) \in \mathrm{Spec}(H).$$

(ii) **Stability** For any $H, G \in \mathcal{H}$ and $\alpha_M \in QH(M) \setminus \{0\}$,

$$\int_0^1 \min_M (H_t - G_t)\, dt \leq c(\alpha_M, \widetilde{\phi}_H) - c(\alpha_M, \widetilde{\phi}_G) \leq \int_0^1 \max_M (H_t - G_t)\, dt.$$

(iii) **Triangle inequality** For any $H, G \in \mathcal{H}$ and $\alpha_M, \alpha'_M \in QH(M) \setminus \{0\}$,

$$c(\alpha_M * \alpha'_M, \widetilde{\phi}_H \widetilde{\phi}_G) \leq c(\alpha_M, \widetilde{\phi}_H) + c(\alpha'_M, \widetilde{\phi}_G).$$

(iv) **Value at identity** For every $\alpha_M \in QH(M) \setminus \{0\}$,

$$c(\alpha_M, \widetilde{\mathrm{id}}) = -\rho \cdot \nu(\alpha_M),$$

where ρ is the rationality constant of (M, ω) and ν is as in (4).

(v) **Novikov action** For all $H \in \mathcal{H}$, $\alpha_M \in QH(M)$ and $\lambda \in \Lambda_{\mathbb{K}}$,

$$c(\lambda\alpha_M, H) = c(\alpha_M, H) - \rho \cdot \nu(\lambda).$$

(vi) **Non-Archimedean property** For all $\alpha_M, \alpha'_M \in QH(M)$,

$$c(\alpha_M + \alpha'_M, H) \leq \max\{c(\alpha_M, H), c(\alpha'_M, H)\}.$$

By the continuity property, the spectral invariants are defined for all $H \in \mathcal{H}$, and all the above listed properties apply in this generality. Further, we observe that for $\alpha_M \in QH(M)$ satisfying $\alpha_M * \alpha_M = \alpha_M$, the triangle inequality for the spectral invariants implies

$$c(\alpha_M, \widetilde{\phi}_{H^{(k)}}) = c(\alpha_M, \widetilde{\phi}^k_H) \leq k \cdot c(\alpha_M, \widetilde{\phi}_H).$$

2.4.1 Spectral norm For a Hamiltonian $H \in \mathcal{H}$, we define its spectral pseudonorm by

$$\gamma(H) = c([M], \widetilde{\phi}_H) + c([M], \widetilde{\phi}_{\overline{H}}), \tag{5}$$

where $\overline{H}$ is the Hamiltonian function $\overline{H}(t,x) = -H(1-t,x)$. A result of [65; 86; 104] shows that

$$\gamma(\phi) = \inf_{\phi^1_H = \phi} \gamma(H)$$

defines a nondegenerate norm $\gamma\colon \mathrm{Ham}(M,\omega) \to \mathbb{R}_{\geq 0}$ and yields a bi-invariant distance $\gamma(\phi,\phi') = \gamma(\phi'\phi^{-1})$. We call $\gamma(\phi)$ the *spectral norm* of ϕ and $\gamma(\phi,\phi')$ the *spectral distance* between ϕ and ϕ'.

2.4.2 Carrier of the spectral invariant In this section we review the definition of *carriers of the spectral invariant*, mainly following [34]. We observe that while we are going to introduce the notion of carriers specifically for the fundamental class $[M] \in QH(M)$, it can be done so for any nontrivial quantum homology class μ.

First, we fix $\alpha_M = [M]$ and write $c(H) = c(\widetilde{\phi}_H) = c([M], \widetilde{\phi}_H)$. Observe that in the case of a nondegenerate Hamiltonian H, we have

$$c(\widetilde{\phi}_H) = \inf\{\mathcal{A}_H(\sigma) \mid \sigma \in CF_n(H),\ \mathrm{PSS}([M]) = [\sigma]\},$$

where $\mathcal{A}_H(\sigma) = \max\{\mathcal{A}_H(\overline{x}) \mid \lambda_{\overline{x}} \neq 0\}$ for $\sigma = \sum \lambda_{\overline{x}}\overline{x}$. That is, it is the maximum action of a capped orbit $\overline{x}$ entering $\sigma \in CF_n(H)$. By the spectrality property of spectral invariants, the infimum is obtained. Consequently, there exists a cycle σ satisfying $[\sigma] = \mathrm{PSS}([M])$ such that $\mathcal{A}_H(\bar{x}) = c(\widetilde{\phi}_H)$ for an orbit $\bar{x}$ entering σ. We call $\bar{x}$ the *carrier of the spectral invariant* and observe that in order to guarantee its uniqueness, all the 1–periodic orbits of H need to have distinct action values. In order to generalize the notion of carriers to the case where H is degenerate and has isolated orbits, we first recall that for each C^2–small nondegenerate perturbation H', every capped 1–periodic orbit $\bar{x}$ splits into several nondegenerate 1–periodic orbits $\mathcal{O}(H', \bar{x})$, with their capping inherited from $\bar{x}$.

Definition 12 (carrier of degenerate H with isolated orbits) A capped 1–periodic orbit $\bar{x}$ is said to be a carrier of the spectral invariant if there exists a sequence $\{H'_k\}$ of nondegenerate perturbations C^2–converging to H such that for each k, one of the orbits in $\mathcal{O}(H'_k, \bar{x})$ is a carrier of the spectral invariant for H'_k. A uncapped orbit is said to be a carrier if it becomes one for a suitable capping.

As in the nondegenerate case, the uniqueness of the carrier follows from all the 1–periodic orbits having distinct action values. In this case, the carrier becomes independent of the choice of sequence $\{H'_k\}$. In addition, the definition of a carrier and the continuity of the action functional and of the mean-index readily yield

$$c(\widetilde{\phi}_H) = \mathcal{A}_H(\bar{x}) \quad \text{and} \quad 0 \le \Delta(\widetilde{\phi}_H, \bar{x}) \le 2n,$$

where the inequalities can be made strict in the case where the orbit x is weakly nondegenerate. In [34] the following result was obtained.

Lemma 13 *Suppose H only has isolated 1–periodic orbits, and let $\bar{x}$ be a carrier of the spectral invariant of the fundamental class. Then $HF_n^{\mathrm{loc}}(H, \bar{x}) \neq 0$.*

In Section 3.1.2 below, we generalize this statement to the case of isolated path-connected sets of periodic orbits, and also to arbitrary quantum homology classes.

3 Isolated connected sets of periodic points

3.1 Generalized perfect Hamiltonians

Recall that a Hamiltonian H is called perfect if it has a finite number of contractible periodic points of all periods. We consider the more general condition where H has finitely many isolated path-connected families of periodic orbits, which in turn implies that $\mathrm{Fix}(\phi_H)$ is composed of finitely many isolated path-connected sets.

Definition 14 A Hamiltonian diffeomorphism $\phi \in \mathrm{Ham}(M,\omega)$ is *generalized perfect* whenever the following conditions are met:

(i) $\mathrm{Fix}(\phi)$ has finitely many isolated compact path-connected components.

(ii) There exists a sequence of integers $k_i \to \infty$ which contains a subsequence $l_i = k_{j_i}$ with $l_i \mid l_{i+1}$ for all i, and for which $\mathrm{Fix}(\phi^{k_i}) = \mathrm{Fix}(\phi)$ for all i.

(iii) For each isolated path-connected component $\mathcal{F}$ of $\mathrm{Fix}(\phi)$, and for all i, the mean-index $\Delta(H^{(k_i)}, \bar{x}^{(k_i)})$, where $x \in \mathcal{F}$ and $\bar{x} \in \bar{\mathscr{F}}$, is a constant function of $x \in \mathcal{F}$. We denote this constant by $\Delta(H^{(k_i)}, \bar{\mathscr{F}}^{(k_i)})$.

An isolated path-connected component $\mathcal{F} \subset \mathrm{Fix}(\phi)$ can be thought of as, and is indeed called in this paper, a *generalized fixed point*. In this section we slightly generalize some of the theory discussed in Section 2, allowing us to treat generalized perfect Hamiltonians. We observe that the third condition in Definition 14 is not vacuous: indeed, one can construct an example of a generalized fixed point $\mathcal{F}$ where the mean-index is not a constant function of $x \in \mathcal{F}$, by means of the Hamiltonian suspension construction [71, Section 3.1] applied to an appropriate contractible Hamiltonian loop of S^2. However, as stated in Theorem C, a p–torsion Hamiltonian diffeomorphism is weakly nondegenerate generalized perfect: in particular, the mean-index is constant on each generalized fixed point.

3.1.1 Lifts of generalized 1–periodic orbits Let (M,ω) be a closed symplectic manifold and H a Hamiltonian function generating a Hamiltonian diffeomorphism ϕ_H on M whose set of contractible fixed points consists of a finite number of path-connected components. Denote the path-connected components of $\mathrm{Fix}(\phi_H)$ by $\mathcal{F}_1, \dots, \mathcal{F}_m$. For each j and each $x \in \mathcal{F}_j$, there is a corresponding contractible loop $x(t) = \phi_H^t(x)$, thus to each isolated fixed-point set $\mathcal{F}_j$ we can associate a subset $\mathscr{F}_j$ of the space $\mathcal{L}_{\mathrm{pt}}M$ of all contractible loops in M. It is natural to ask whether the generalized orbits $\mathscr{F}_j$ lift to the Γ–cover $\widetilde{\mathcal{L}}_{\mathrm{pt}}M$ in a suitable manner, namely, if the preimage under the projection $\mathrm{Pr}\colon \widetilde{\mathcal{L}}_{\mathrm{pt}}M \to \mathcal{L}_{\mathrm{pt}}M$ is composed of isolated path-connected "copies" of $\mathscr{F}_j$. We show that the lift exists, and we denote by $\bar{\mathscr{F}}_j$ a particular lift of $\mathscr{F}_j$. This is analogous to a capping of an orbit in the case of a usual Hamiltonian.

Consider the set $\mathscr{F} \subset \mathcal{L}_{\mathrm{pt}}M$ associated to $\mathcal{F} \in \pi_0(\mathrm{Fix}(\phi_H))$ and let $i : \mathscr{F} \to \mathcal{L}_{\mathrm{pt}}M$ be the natural inclusion map. Formally, we are asking when, given a loop $x_0 \in \mathscr{F}$ and $\bar{x}_0 \in \mathrm{Pr}^{-1}(\{x_0\})$, a lift of i exists: namely, a unique map $f : \mathscr{F} \to \widetilde{\mathcal{L}}_{\mathrm{pt}}M$ such that $f(x_0) = \bar{x}_0$ and $\mathrm{Pr} \circ f = i$. From the theory of covering spaces, the existence of the lift is equivalent to $i_*(\pi_1(\mathscr{F}, x_0)) \subset \mathrm{Pr}_*(\pi_1(\widetilde{\mathcal{L}}_{\mathrm{pt}}M, \bar{x}_0))$.

Proposition 15 *Let (M,ω) be a symplectic manifold in one of the three classes considered in this paper, and ϕ_H a generalized perfect Hamiltonian diffeomorphism. Then each generalized orbit $\mathscr{F}$ can be lifted to $\overline{\overline{\mathscr{F}}}$ in a unique manner specified by a loop $x_0 \in \mathscr{F}$ and an element in its fiber $\overline{x}_0 \in \mathrm{Pr}^{-1}(x_0)$.*

Proof Let γ be a loop in $\mathscr{F}$ such that $\gamma_0 = x_0$. We show that we can find a loop $\widetilde{\gamma}$ in $\widetilde{\mathcal{L}}_{\mathrm{pt}} M$ such that $i \circ \gamma = \mathrm{Pr} \circ \widetilde{\gamma}$, which implies the claim of the theorem. We build $\widetilde{\gamma}$ in a natural way by defining the capping at γ_s to be given by gluing the "cylinder" given by traversing the loop γ from 0 to s to the capping $\overline{x}_0$. To see that the capped orbits $\widetilde{\gamma}_0$ and $\widetilde{\gamma}_1$ are equivalent in $\widetilde{\mathcal{L}}_{\mathrm{pt}} M$, we show that

$$\int_{\mathrm{T}^2} \gamma^* \omega = 0 \tag{6}$$

for every loop γ in $\mathscr{F}$. We can then guarantee the existence of a lift. Equation (6) follows from the continuity of $\mathcal{A}_H$ and the fact that $\mathrm{Spec}(H)$ has zero measure in $\mathbb{R}$. Indeed, $\mathcal{A}_H(\widetilde{\gamma}_s) = \mathcal{A}_H(\widetilde{\gamma}_0)$ for every s, otherwise, the fact that $\widetilde{\gamma}_s$ is a critical point for each s would imply that $\mathcal{A}_H\big(\bigcup_{0 \le t \le s} \widetilde{\gamma}_t\big)$ is a positive measure subset of $\mathrm{Spec}(H)$. Finally, $\mathcal{A}_H(\widetilde{\gamma}_1) = \mathcal{A}_H(\widetilde{\gamma}_0)$ amounts to fulfilling the sufficient condition given by equation (6). Therefore for the three classes we consider, the proof is complete since in this case $\Gamma \cong \pi_2(M)/\ker([\omega])$ and hence it is only necessary to verify (6). Alternatively, one can prove that $\langle [T^2], \gamma^*(c_1) \rangle = 0$ directly, by replacing γ with a map $\gamma_1 : S^2 \to M$ with $\langle [T^2], \gamma^*(c_1) \rangle = \langle [S^2], \gamma_1^*(c_1) \rangle$, which vanishes by our assumption on the manifold and (6). □

3.1.2 Generalized local Floer homology In this section, we define a version of local Floer homology for a generalized capped orbit $\overline{\overline{\mathscr{F}}} \subset \widetilde{\mathcal{L}}_{\mathrm{pt}} M$ of a 1–periodic Hamiltonian H in a way that is closely related to what was done in [61; 34]. The only differences are that we are beyond the symplectically aspherical case and we are dealing with path-connected components of $\mathrm{Fix}(\phi_H)$ instead of isolated points. The proofs of [61] used to define the local Floer homology are valid in this case with nearly no modifications. The notion of local Floer homology in a more general setting goes back to the original work of Floer [24; 23] and has been revisited a number of times, for example in the work of Pozniak [76]. The main ingredients of the construction are as follows.

For each $\mathcal{F} \in \pi_0(\mathrm{Fix}(\phi_H))$, we can find an isolating neighborhood $U_{\mathcal{F}}$ of the corresponding generalized 1–periodic orbit $\mathscr{F}$ in the extended phase-space $\mathrm{S}^1 \times M$, ie

$$\{(t, \phi_H^t(x)) \mid t \in \mathrm{S}^1,\ x \in \mathcal{F}\} \subset U_{\mathcal{F}}.$$

Moreover, we can choose this collection of neighborhoods to be pairwise disjoint: $U_{\mathcal{F}}$ is disjoint from $U_{\mathcal{F}'}$ for each pair of distinct generalized fixed points $\mathcal{F}$ and $\mathcal{F}'$. Such an open set $U_{\mathcal{F}}$ in the extended phase-space can be constructed, using the isotopy ϕ_H^t, from an open neighborhood of $\mathcal{F}$ in M. Hence by a slight abuse of notation we think of $U_{\mathcal{F}}$ as a neighborhood of $\mathcal{F}$ in M.

Now there exists an $\epsilon > 0$ small enough that for any nondegenerate Hamiltonian perturbation H' satisfying $\|H - H'\|_{C^2} < \epsilon$, the set of orbits $\mathcal{O}(H', \mathscr{F})$ which $\mathscr{F}$ splits into is contained in $U_{\mathcal{F}}$, and so is every (broken) Floer trajectory connecting any such two orbits; see Lemma 21. We can now consider the complex $CF_*(H', \bar{\mathscr{F}})$ over a ground field $\mathbb{K}$ generated by the capped 1–periodic orbits $\mathcal{O}(H', \bar{\mathscr{F}})$ which $\bar{\mathscr{F}}$ splits into, where the cappings are naturally produced from the specific lift $\bar{\mathscr{F}}$. One can see that this complex is graded by the Conley–Zehnder index and has a well-defined differential. By a standard continuation argument, one can show that the homology of this complex is independent of the nondegenerate perturbation (once it is small enough) and of the choice of almost complex structure. We refer to the resulting homology as the local Floer homology of H at $\bar{\mathscr{F}}$, and denote it by $HF_*^{\mathrm{loc}}(H, \bar{\mathscr{F}})$. Write

$$\Delta^{\min}(H, \mathscr{F}) = \min_{\bar{x} \in \bar{\mathscr{F}}} \Delta(H, \bar{x}) \quad \text{and} \quad \Delta^{\max}(H, \mathscr{F}) = \max_{\bar{x} \in \bar{\mathscr{F}}} \Delta(H, \bar{x})$$

for the minimum and maximum of the mean-indices $\Delta(H, \bar{x})$ for $\bar{x} \in \bar{\mathscr{F}}$.

We claim that if $\mathscr{F}$ is a family of weakly nondegenerate orbits, then the support of $HF_*^{\mathrm{loc}}(H, \bar{\mathscr{F}})$ satisfies

$$\operatorname{Supp}(HF_*^{\mathrm{loc}}(H, \bar{\mathscr{F}})) \subset (\Delta^{\min}(H, \bar{\mathscr{F}}) - n, \Delta^{\max}(H, \bar{\mathscr{F}}) + n). \tag{7}$$

In fact, by a simple argument following from the continuity of the mean-index and inequality (2), one obtains that $\operatorname{Supp}(HF_*^{\mathrm{loc}}(H, \bar{\mathscr{F}}))$ satisfies the nonstrict version of (7). In order to obtain the strict inequalities, we use the assumption that $\mathcal{F}$ is weakly nondegenerate, and its compactness, to argue as in [84]. In the situation where the Hamiltonian is generalized perfect, we obtain the following.

Lemma 16 *Suppose H is a weakly nondegenerate generalized perfect Hamiltonian and let $\bar{\mathscr{F}}$ be a generalized capped orbit of H. Then $HF_*^{\mathrm{loc}}(H, \bar{\mathscr{F}})$ is supported in the open interval $(\Delta(H, \bar{\mathscr{F}}) - n, \Delta(H, \bar{\mathscr{F}}) + n)$.*

Furthermore, the notion of action carriers discussed in Section 2.4 remains valid in this generalized setting by altering isolated fixed points to generalized fixed points in

Definition 12. Thus, the spectral invariant $c([M], H)$ is carried by a capped generalized periodic orbit $\bar{\mathscr{F}}$ of H. In this case, we have the following generalization of Lemma 13, whose proof, once Lemma 21 below is taken into account, follows just as in [34].

Lemma 17 *Suppose H has only a finite number of generalized fixed points, and let $\bar{\mathscr{F}}$ be a carrier of the spectral invariant of the fundamental class. In this case $HF_n^{\mathrm{loc}}(H, \bar{\mathscr{F}}) \neq 0$.*

Remark 18 Consider $\mathcal{F} \in \pi_0(\mathrm{Fix}(\phi))$ and $\mathscr{F} \subset \mathcal{L}_{\mathrm{pt}} M$ the associated generalized 1–periodic orbit. We remark that different choices of lifts $\bar{\mathscr{F}}$ result in isomorphic local Floer homology groups, with a shift in index given by an integer multiple of $2N$. In particular, if $A \in \Gamma$, then

$$HF_*^{\mathrm{loc}}(H, \bar{\mathscr{F}} \# A) \cong HF_{*+2\langle c_1(TM), A\rangle}^{\mathrm{loc}}(H, \bar{\mathscr{F}}),$$

where $\bar{\mathscr{F}} \# A$ denotes the unique choice of lift containing the capped orbit $\bar{x} \# A$ for $x \in \mathcal{F}$ and $\bar{x} \in \bar{\mathscr{F}}$. From this discussion, we conclude that $\dim_{\mathbb{K}} HF_*^{\mathrm{loc}}(H, \bar{\mathscr{F}})$ does not depend on the capping of $\mathscr{F}$. Hence, the notation $\dim_{\mathbb{K}} HF_*^{\mathrm{loc}}(H, \mathcal{F})$ is justified in this case. Furthermore, when (M, ω) is symplectically Calabi–Yau the local Floer homology does not depend on the choice of lift, thus we denote it by $HF_*^{\mathrm{loc}}(H, \mathcal{F})$. This is analogous to the effect of recapping on local Floer homology in the case of isolated fixed points.

We shall require a slightly more general statement about carriers of quantum homology classes. The definition of a carrier $\bar{\mathscr{F}}$ of a quantum homology class $\alpha_M \in QH(M)$ is the same as for the fundamental class, with $[M]$ replaced by α_M. We then have the following result.

Lemma 19 *Let $\alpha_M \in QH_k(M) \setminus \{0\}$ be a homogeneous element of degree k. Suppose H has only finitely many (contractible) generalized fixed points and let $\bar{\mathscr{F}}$ be a carrier of the spectral invariant of α_M. Then $HF_{k-n}^{\mathrm{loc}}(H, \bar{\mathscr{F}}) \neq 0$.*

In fact a stronger result is true, of which this statement is a direct consequence. It was proven as [93, Theorem D] in the context of ϕ_H^1 with isolated fixed points, but its proof adapts essentially immediately to the context of a finite number of (contractible) generalized fixed points. Indeed, our case differs from the one in [93] by replacing fixed points by generalized fixed points everywhere, hence the only technical difference consists in establishing Lemma 21. We recall that the proof relies on homological perturbation techniques, starting from the decomposition of Section 3.1.4. It constitutes a Novikov-field version of the canonical Λ^0–complexes from [95]. The goal of these arguments is to introduce a new complex which calculates the same total homology but replaces each local Floer complex $CF_*^{\mathrm{loc}}(H, \bar{\mathscr{F}})$, which depends on a sufficiently C^2–small perturbation H_1 of H, by its homology $HF_*^{\mathrm{loc}}(H, \bar{\mathscr{F}})$. This is the local Floer homology of H at $\bar{\mathscr{F}}$, which does not depend on H_1. Note that since we work over a field, $CF_*^{\mathrm{loc}}(H, \bar{\mathscr{F}})$ is chain-homotopy equivalent to $HF_*^{\mathrm{loc}}(H, \bar{\mathscr{F}})$ with the zero differential. The complex obtained from the Floer complex of H_1 in this way computes the same total homology, as desired, but is also strict in the sense of strictly decreasing a natural filtration. Furthermore, it allows us to compute directly the filtered Floer homology of H.

Theorem M *Let (M, ω) be a closed symplectic manifold which is positive or negative monotone. Consider the class $\tilde{\phi} \in \widetilde{\mathrm{Ham}}(M, \omega)$ of the Hamiltonian flow $\{\phi_H^t\}_{t \in [0,1]}$ of $H \in \mathcal{H}$, with $\mathrm{Fix}(\phi_H^1)$ consisting of a finite number of generalized fixed points. Let $\mathbb{K}$ be a ground field which is arbitrary in the positive monotone case and of characteristic zero in the negative monotone case. Then there is a filtered homotopy-canonical complex $(C(H), d_H)$ over the Novikov field $\Lambda_{\mathbb{K}}$ on the action-completion of*

$$\bigoplus HF_*^{\mathrm{loc}}(\tilde{\phi}, \bar{\mathscr{F}}),$$

the sum running over all capped generalized 1–periodic orbits $\bar{\mathscr{F}} \in \tilde{\mathcal{O}}(H)$. Specifically, $C(H)$ consists of infinite sums $x = \sum y_i$ where $y_i \in HF_*^{\mathrm{loc}}(\tilde{\phi}, \bar{\mathscr{F}}_i)$ with $\mathcal{A}_H(\bar{\mathscr{F}}_i) \xrightarrow{i \to \infty} -\infty$. The complex $(C(H), d_H)$ is free and graded over $\Lambda_{\mathbb{K}}$, and is strict, ie $\mathcal{A}_H(d_H(y)) < \mathcal{A}_H(y)$ for all $y \in C(H)$, with respect to the non-Archimedean action-filtration $\mathcal{A}_H$ on $C(H)$, defined by

$$\mathcal{A}_H\Big(\sum \lambda_j y_j\Big) = \max\{-\nu(\lambda_j) + \mathcal{A}_H(y_j)\}, \quad \mathcal{A}_H(y_j) = \mathcal{A}_H(\bar{\mathscr{F}}_{i(j)}). \tag{8}$$

Here $\{y_j\}$ is a $\Lambda_{\mathbb{K}}$–basis of $C(H)$ that is determined by $\{y_j \mid i(j) = i\}$ being a basis of $HF_*^{\mathrm{loc}}(\tilde{\phi}, \bar{\mathscr{F}}_i)$, where $\mathrm{Fix}(\phi) = \{\mathcal{F}_i\}$ and for each i, $\bar{\mathscr{F}}_i$ is a choice of a lift of the generalized 1–periodic orbit $\mathscr{F}_i$ corresponding to $\mathcal{F}_i$ to a capped generalized periodic orbit in $\tilde{\mathcal{O}}(H)$. Furthermore, for all $a \in \mathbb{R} \setminus \mathrm{Spec}(H)$, the filtered homology $HF(H)^{<a}$ is given by $HF(C(H)^{<a})$, where $C(H)^{<a} = (\mathcal{A}_H)^{-1}(-\infty, a)$. In particular,

$$HF(H) = H(C(H), d_H) \cong QH(M; \Lambda_{\mathbb{K}}).$$

Moreover, for all $a \leq b$ with $a, b \in (\mathbb{R} \setminus \mathrm{Spec}(H)) \cup \{\infty\}$, the comparison map $HF(H)^{<a} \to HF(H)^{<b}$ is induced by the inclusion $C(H)^{<a} \to C(H)^{<b}$.

Definition 20 (visible spectrum) We define the visible spectrum of a Hamiltonian function H as

$$\mathrm{Spec}^{\mathrm{vis}}(H) = \{\mathcal{A}_H(\bar{\mathscr{F}}) \mid HF_*^{\mathrm{loc}}(H, \bar{\mathscr{F}}) \neq 0\},$$

where $\mathcal{A}_H(\bar{\mathscr{F}})$ denotes the action of any capped orbit $\bar{x} \in \bar{\mathscr{F}}$ for a lift $\bar{\mathscr{F}}$ associated to a generalized fixed point $\mathcal{F} \subset \mathrm{Fix}(\phi_H)$. Indeed, an argument similar to the proof of Proposition 15 shows that the restriction $\mathcal{A}_H|_{\bar{\mathscr{F}}}$ is constant. It is clear that $\mathrm{Spec}^{\mathrm{vis}}(H) \subset \mathrm{Spec}(H)$. In the context of barcodes (see Section 3.1.5), the visible spectrum corresponds to the endpoints of all bars of the barcode $\mathcal{B}(H)$ associated to the filtered Floer homology of H.

3.1.3 Crossing energy We show that for a C^2–small perturbation H' of a generalized perfect Hamiltonian H on a closed symplectic manifold (M, ω), every Floer trajectory u connecting orbits of H' contained in distinct isolating neighborhoods has energy bounded below by a constant independent of the perturbation. This is an important technical step.

Lemma 21 *There exist $\delta > 0$ and $\epsilon > 0$ such that for every nondegenerate perturbation H' of H satisfying $\|H - H'\|_{C^2} < \epsilon$, every orbit in $\mathcal{O}(H', \mathscr{F}_j)$ is contained in $U_{\mathcal{F}_j}$ for $j = 1, \dots, m$, every Floer trajectory u connecting capped orbits in distinct isolating neighborhoods satisfies $E(u) > \delta$, and every Floer trajectory connecting capped orbits in the same $\mathcal{O}(H', \bar{\mathscr{F}}_j)$ is contained in $U_{\mathcal{F}_j}$. Finally, if (M, ω) is rational, every Floer trajectory u connecting capped orbits in $\mathcal{O}(H', \bar{\mathscr{F}}_j)$, $\mathcal{O}(H', \bar{\mathscr{F}}'_j)$, for different cappings $\bar{\mathscr{F}}_j, \bar{\mathscr{F}}'_j$ of the same $\mathscr{F}_j$, has energy $E(u) \geq \rho/2$.*

Proof Suppose there exists a sequence of nondegenerate Hamiltonians $\{H'_k\}$ that C^2–converges to H, and a sequence of Floer trajectories u_k of H'_k connecting orbits in distinct isolating neighborhoods such that $E(u_k) \to 0$. Since H has finitely many generalized fixed points, we may suppose without loss of generality that all the Floer trajectories u_k connect orbits in $U_{\mathcal{F}}$ to orbits in $U_{\mathcal{F}'}$, where $\mathcal{F}, \mathcal{F}' \in \pi_0(\mathrm{Fix}(\phi_H))$ are distinct.

By a compactness result of [19], and arguing as in [61], we obtain the existence of a Floer trajectory u of H connecting an orbit in $U_{\mathcal{F}}$ to an orbit in $U_{\mathcal{F}'}$ such that $E(u) = 0$. Thus,

$$\frac{\partial u}{\partial s} = 0 \quad \text{and} \quad \frac{\partial u}{\partial t} = X_H^t,$$

which, in turn, implies that for each s, the loop $u_s = u(s, \cdot)$ is a 1–periodic orbit of H. This contradicts the fact that the generalized fixed points of H are isolated.

Note that in the above argument, if for each k, u_k connects orbits in the same $U_{\mathcal{F}_j}$ but is not contained in $U_{\mathcal{F}_j}$, then for k sufficiently large, $E(u_k) > \delta$ again. Indeed, otherwise we would again reach a contradiction by a compactness argument. However, if now u_k connects orbits in the same $\mathcal{O}(H'_k, \bar{\mathscr{F}}_j)$, then its energy, given by the difference of actions of its two asymptotic capped orbits, tends to zero as $k \to \infty$. We conclude that u_k must be contained in $U_{\mathcal{F}_j}$ for all k sufficiently large.

Finally, we remark that if u_k connects orbits in $\mathcal{O}(H'_k, \bar{\mathscr{F}}_j)$ and in $\mathcal{O}(H'_k, \bar{\mathscr{F}}'_j)$, where $\bar{\mathscr{F}}_j$ and $\bar{\mathscr{F}}'_j$ are different cappings of $\mathscr{F}_j$, then, if the symplectic manifold is rational, $E(u_k) \ge \rho - o(1)$ as $k \to \infty$. $\square$

3.1.4 Decomposition of Floer differential An important feature related to local Floer homology concerns the decomposition of the full differential defined on the complex $CF_*(H')$ into the sum of local differentials of complexes $CF_*^{\mathrm{loc}}(H, \bar{\mathscr{F}})$ — for all the different lifts of the finitely many generalized fixed points — and into an additional component we shall call D. Note that here, H' is a nondegenerate Hamiltonian C^2–close enough to H in the aforementioned sense. Namely, for each chain $\sigma \in CF_*(H')$, we have

$$\partial\sigma = \sum \tilde{\partial}_{\bar{\mathscr{F}}}\sigma + D\sigma, \tag{9}$$

where $\tilde{\partial}_{\bar{\mathscr{F}}}$ represents an extension of the local differential of the complex $CF_*^{\mathrm{loc}}(H, \bar{\mathscr{F}})$ obtained by setting $\tilde{\partial}_{\bar{\mathscr{F}}}\bar{x} = 0$ for every capped orbit which does not belong $\mathcal{O}(H', \bar{\mathscr{F}})$. Loosely speaking, D only "counts" Floer trajectories connecting orbits contained in disjoint isolating open sets $U_{\mathcal{F}}$.

Remark 22 Suppose that σ is a chain in the complex $CF_*(H')$ and $\bar{z}$ is an orbit entering $D\sigma$. Naturally, there exists $0 \le k \le m$ such that $\bar{z} \in CF_*(H, \bar{\mathscr{F}}_k)$ for a particular lift of $\mathscr{F}_k$, and a Floer trajectory u connecting an orbit $\bar{y} \in CF_*(H, \bar{\mathscr{F}}_l)$ to $\bar{z}$ for $l \ne k$ (and a particular lift of $\mathscr{F}_l$). We then obtain

$$\mathcal{A}_{H'}(\bar{z}) = \mathcal{A}_{H'}(\bar{y}) - E(u) < \mathcal{A}'_H(\bar{y}) - \delta, \tag{10}$$

where the first equality comes from the fact that the energy of a Floer trajectory connecting two capped orbits is equal to their action difference, and the δ comes from the uniform lower bound for the crossing energy from Lemma 21. In other words,

$$\mathcal{A}_{H'}(Dx) < \mathcal{A}_{H'}(x) - \delta$$

for all $x \ne 0$ in $CF_*(H')$.

3.1.5 Barcodes of Hamiltonian diffeomorphisms The proof of Theorem G uses notions and results regarding barcodes of Hamiltonian diffeomorphisms, in the case where they have a finite number of contractible generalized fixed points. Hitherto, this theory was developed mostly for the case where the generalized fixed points are in fact points, yet given Lemma 21, all relevant results generalize to our situation. In the next section we describe the main Smith-type inequality regarding the behavior of barcodes under iteration.

We will summarize the properties necessary for us, and refer to [74; 75; 103; 48; 96; 95] for further discussion of this notion, in the context of continuity in the Hofer distance and the spectral distance in particular. For convenience, we work in the setting of monotone symplectic manifolds, yet natural analogues of various statements exist in the semipositive, rational and general settings.

Proposition 23 *Let (M,ω) be a monotone symplectic manifold with $\mathcal{P}_\omega = \rho\cdot\mathbb{Z}$, and consider $\phi \in \mathrm{Ham}(M,\omega)$ with $\mathrm{Fix}(\phi)$ consisting of a finite number of generalized fixed points. Let $\mathbb{K}$ be a coefficient field. Let H be a Hamiltonian generating ϕ. Then $\mathrm{Spec}(H) \subset \mathbb{R}$ is a discrete subset, and there exists a countable collection*

$$\mathcal{B}(H) = \mathcal{B}(H;\mathbb{K}) = \{(I_i, m_i)\}_{i\in\mathcal{I}},$$

called the **barcode** *of H with coefficients in $\mathbb{K}$, of intervals I_i in $\mathbb{R}$ of the form $I_i = (a_i, b_i]$ or $I_i = (a_i,\infty)$, called* **bars** *with multiplicities $m_i \in \mathbb{Z}_{>0}$, such that the following properties hold:*

(i) *The group $\rho\cdot\mathbb{Z}$ acts on $\mathcal{B}(H)$ in the sense that for all $k\in\mathbb{Z}$ and all $(I,m)\in\mathcal{B}$, we have $(I+\rho k, m)\in\mathcal{B}$.*

(ii) *For each window $J=(a,b)$ in $\mathbb{R}$ with $a,b\notin\mathrm{Spec}(H)$, only a finite number of intervals I with $(I,m)\in\mathcal{B}$ have endpoints in J. Furthermore,*

$$\dim_{\mathbb{K}} HF(H)^J = \sum_{\substack{(I,m)\in\mathcal{B}(H)\\ \#\partial I\cap J=1}} m,$$

where for an interval $I=(a,b]$, we set $\partial I=\{a,b\}$, and for $I=(a,\infty)$, we set $\partial I=\{a\}$.

(iii) *For $a\in\mathrm{Spec}(H)$ and $\epsilon>0$ sufficiently small that $(a-\epsilon,a+\epsilon)\cap\mathrm{Spec}(H)=\{a\}$, we have*

$$\dim_{\mathbb{K}} HF(H)^{(a-\epsilon,a+\epsilon)} = \sum_{\substack{(I,m)\in\mathcal{B}(H)\\ a\in\partial I}} m,$$

$$\dim_{\mathbb{K}} HF(H)^{(a-\epsilon,a+\epsilon)} = \sum_{\mathcal{A}(\bar{\mathscr{F}})=a} \dim_{\mathbb{K}} HF^{\mathrm{loc}}(H,\bar{\mathscr{F}}).$$

(iv) *There are $K(\phi,\mathbb{K})$ orbits of finite bars counted with multiplicity, and $B(\mathbb{K})$ orbits of infinite bars counted with multiplicity, under the $\rho\cdot\mathbb{Z}$ action on $\mathcal{B}(H)$. These numbers satisfy*

$$B(\mathbb{K}) = \dim_{\mathbb{K}} H_*(M;\mathbb{K}) \quad \textit{and} \quad N(\phi,\mathbb{K}) = 2K(\phi,\mathbb{K}) + B(\mathbb{K}),$$

where

$$N(\phi,\mathbb{K}) = \sum \dim_{\mathbb{K}} HF^{\mathrm{loc}}(\phi,\mathcal{F})$$

*is the **homological count of the fixed points** of ϕ, the sum running over all the set $\pi_0(\mathrm{Fix}(\phi))$ of its generalized fixed points.*

(v) *To each orbit $[(I,m)]$, with $I=(a,b]$, of finite bars, there corresponds a bar-length $|I| = b-a$, counted with multiplicity m. There are hence $K(\phi,\mathbb{K})$ **bar-lengths** corresponding to the orbits of finite bars,*

$$0 < \beta_1(\phi,\mathbb{K}) \le \cdots \le \beta_{K(\phi,\mathbb{K})}(\phi,\mathbb{K}),$$

which depend only on ϕ. We call

$$\beta(\phi,\mathbb{K}) = \beta_{K(\phi,\mathbb{K})}(\phi,\mathbb{K})$$

*the **boundary-depth** of ϕ, and*

$$\beta_{\mathrm{tot}}(\phi,\mathbb{K}) = \sum_{1\le j\le K(\phi,\mathbb{K})} \beta_j(\phi,\mathbb{K})$$

*its **total bar-length**.*

(vi) *Each spectral invariant $c(\alpha,H)\in \mathrm{Spec}(H)$ for $\alpha\in QH_*(M)\setminus\{0\}$ is a starting point of an infinite bar in $\mathcal{B}(H)$, and each such starting point is given by a spectral invariant.*[8]

(vii) *If H' is another Hamiltonian generating ϕ, then $\mathcal{B}(H') = \mathcal{B}(H)[c]$ for a certain constant $c\in\mathbb{R}$, where $\mathcal{B}(H)[c] = \{(I_i - c, m_i)\}_{i\in\mathcal{I}}$.*

(viii) *If $\mathbb{K}$ is a field extension of $\mathbb{F}$ and H is a Hamiltonian, then $\mathcal{B}(H;\mathbb{K}) = \mathcal{B}(H;\mathbb{F})$. In particular, $\mathcal{B}(H;\mathbb{K}) = \mathcal{B}(H;\mathbb{F}_p)$ if $\mathrm{char}(\mathbb{K}) = p$, and $\mathcal{B}(H;\mathbb{K}) = \mathcal{B}(H;\mathbb{Q})$ if $\mathrm{char}(\mathbb{K}) = 0$.*

[8] In fact, representatives for the set of orbits of infinite bars counting with multiplicity, can be obtained as spectral invariants of an orthogonal basis of $QH_*(M)$ over the Novikov field $\Lambda_{\mathbb{K}}$, with respect to the non-Archimedean filtration $l_H(-) = c(-,H)$. As we shall not require this stronger statement, we refer to [95; 96] for a discussion of the relevant notions.

3.1.6 Smith theory in filtered Floer homology One of the fundamental results of [95] is the following Smith-type inequality, that readily adapts to our setting by Lemma 21 and its generalization to the situation of branched covers of the cylinder as in [98, Proposition 9]. We refer to [95, Theorem D] for a detailed argument in the case of isolated fixed points, and observe that our generalization below is formulated in such a way that the same proof applies verbatim, by replacing fixed points by generalized fixed points everywhere.

Theorem N *Let (M,ω) be a monotone symplectic manifold, p a prime number, and $\phi \in \mathrm{Ham}(M,\omega)$ with $\mathrm{Fix}(\phi)$ and $\mathrm{Fix}(\phi^p)$ each consisting of a finite number of generalized fixed points, and such that the natural inclusion $\mathrm{Fix}(\phi) \to \mathrm{Fix}(\phi^p)$ restricts to a homeomorphism from each generalized fixed point $\mathcal{F}$ of ϕ to a generalized fixed point of ϕ^p, which we denote by $\mathcal{F}^{(p)}$. Then*

$$\beta_{\mathrm{tot}}(\phi^p,\mathbb{F}_p) \geq p\cdot\beta_{\mathrm{tot}}(\phi,\mathbb{F}_p).$$

This inequality will be the key component in the proof of Theorem G.

A somewhat simpler statement than Theorem N is the Smith inequality in generalized local Floer homology, whose proof is precisely as in [98] together with the crossing energy argument of Lemma 21.

Proposition 24 *Let (M,ω) be a closed symplectic manifold, p a prime number and $\phi \in \mathrm{Ham}(M,\omega)$. Suppose that $\mathrm{Fix}(\phi)$ and $\mathrm{Fix}(\phi^p)$ each consist of a finite number of generalized fixed points. Let $\mathcal{F}$ be a generalized fixed point of ϕ, such that the natural inclusion $\mathrm{Fix}(\phi) \to \mathrm{Fix}(\phi^p)$ restricted to $\mathcal{F}$ is a homeomorphism onto $\mathcal{F}^{(p)}$. Then*

$$\dim_{\mathbb{F}_p} HF^{\mathrm{loc}}(\phi,\mathcal{F}) \leq \dim_{\mathbb{F}_p} HF^{\mathrm{loc}}(\phi^p,\mathcal{F}^{(p)}).$$

3.1.7 Quantum Steenrod operations Quantum Steenrod operations are remarkable algebraic maps

$$Q\mathrm{St}_p\colon QH^*(M;\mathbb{F}_p) \to QH^*(M;\mathbb{F}_p)[\![u]\!]\langle\theta\rangle$$

for p a prime number, u a formal variable of degree 2, and θ a formal variable of degree 1. As is the usual quantum product, $Q\mathrm{St}_p$ is essentially defined by certain counts of configurations consisting of holomorphic curves in M incident with negative gradient trajectories of Morse functions in M. The main difference is that $Q\mathrm{St}_p$ uses p input and 1 output trajectories, and the counts are carried out in families parametrized by the classifying space $B(\mathbb{Z}/p\mathbb{Z})$ of $\mathbb{Z}/p\mathbb{Z}$. The investigation of the enumerative significance of these counts, in terms of various Gromov–Witten invariants, and its implications for mirror symmetry was started in [91; 92].

These operations were first proposed by Fukaya [27], and were formally introduced for $p = 2$ by Wilkins in [106]. They were then studied in [107] in relation to the equivariant pair-of-pants product of Seidel [90]. For a definition for $p > 2$ odd, see [91; 97]. The significance of quantum Steenrod operations in Hamiltonian dynamics was first observed in [93], and was further investigated in [7; 94; 97]. While for the moment these operations are defined in the setting of monotone symplectic manifolds, it is expected that they will be generalized to the semipositive (also called weakly monotone) setting.

One particular property of quantum Steenrod operations that we use in this paper, which was first observed in [93] for $p = 2$, and proved in [97] for $p > 2$, is that whenever

$$QSt_p(\mu) \neq u^{(p-1)n}\mu, \tag{11}$$

where $\mu \in H^{2n}(M;\mathbb{F}_p)$ is the cohomology class Poincaré dual to the point class, the symplectic manifold (M,ω) is geometrically uniruled: for each ω–compatible almost complex structure J on M, and each point $x \in M$, there exists a J–holomorphic sphere $u\colon \mathbb{CP}^1 \to M$ such that $x \in \operatorname{im}(u)$. Hence, we call a (monotone) symplectic manifold $\mathbb{F}_p$*–Steenrod uniruled* if condition (11) holds. The algebraic significance of this condition is that $u^{(p-1)n}\mu = \mathrm{St}_p(\mu)$, where St_p is the (slightly reformulated) total Steenrod p^{th} power of the class μ, and in general,

$$QSt_p = \mathrm{St}_p + O(q),$$

where $O(q)$ is a collection of terms involving the quantum variable q to power at least 1. These terms correspond to configurations involving J–holomorphic spheres of positive symplectic area, hence condition (11) means that the quantum Steenrod power of the point cohomology class is *deformed* by holomorphic spheres.

3.2 Floer cohomology

At times it shall be convenient to work with Floer cohomology and quantum cohomology of closed symplectic manifolds, instead of homology. All the preliminary results above adapt naturally to this setting. In fact, we may define

$$CF^*(H,J) = CF_{n-*}(\overline{H},\bar{J}),$$

where $\overline{H}(t,x) = -H(1-t,x)$, $\bar{J}_t(x) = J_{1-t}(x)$. The usual action functional $\mathcal{A}_H$ on the left-hand side takes the form $(-\mathcal{A}_{\overline{H}})$ on the right-hand side. Note that hereby the cohomological differential increases the filtration, the triangle inequality for spectral invariants has the opposite inequality, and infinite bars in the barcode are of the form $(-\infty, b)$. Local Floer cohomology is defined in the same way as for homology. Action carriers, and contribution to local Floer cohomology hold similarly: $c(\mu, H)$ for $\mu \in QH^{2n}(M)$ is carried by a capped generalized periodic orbit $\bar{\mathscr{F}}$ of H if, in the same sense as for homology, $\bar{\mathscr{F}}$ is a *lowest* action term in a *highest* minimal action representative of the image $\mathrm{PSS}_H(\mu)$ of μ under the PSS isomorphism [70] from the quantum cohomology $QH^*(M) \to HF^{*-n}(H)$ to the filtered Floer cohomology of the Hamiltonian H. For (M,ω) rational, in particular monotone, for each nonzero class $\mu \in QH^*(M)$, and for $H \in \mathcal{H}$ with $\#\pi_0(\mathrm{Fix}(\phi_H^1)) < \infty$, we have that $c(\mu, H)$ is carried by at least one generalized capped 1–periodic orbit $\bar{\mathscr{F}}$ of H. Furthermore, if μ is a homogeneous class of degree k, and $\bar{\mathscr{F}}$ carries $c(\mu, H)$, then $HF^{k-n}_{\mathrm{loc}}(H, \bar{\mathscr{F}}) \neq 0$.

We refer to [52] for further discussion of the comparison between Floer homology and Floer cohomology.

4 Cluster structure of the essential spectrum

Definition 25 (essential spectrum) We define the essential spectrum of a Hamiltonian function H as

$$\mathrm{Spec}^{\mathrm{ess}}(H) = \{c(\alpha, H) \mid \alpha \in QH_*(M) \setminus \{0\}\}.$$

Observe that the spectrality property of the spectral invariants is equivalent to the inclusion $\mathrm{Spec}^{\mathrm{ess}}(H) \subset \mathrm{Spec}(H)$. In fact, Lemma 19 implies that $\mathrm{Spec}^{\mathrm{ess}}(H) \subset \mathrm{Spec}^{\mathrm{vis}}(H)$ for Hamiltonian diffeomorphisms with a finite number of (contractible) generalized fixed points. In the context of barcodes (see Section 3.1.5), the essential spectrum corresponds to the endpoints of infinite bars of the barcode $\mathcal{B}(H)$ associated to the filtered Floer homology of H.

We next show that whenever $\gamma(H) < \rho$, the essential spectrum has a cluster structure determined by the subset produced by quantum homology classes of valuation 0.

Proposition 26 *Suppose M is a monotone symplectic manifold and H a Hamiltonian function on M. Then*

$$0 \leq c([M], \tilde{\phi}_H) - c(\alpha, \tilde{\phi}_H) \leq \gamma(H)$$

for all $\alpha \in QH(M)$ such that $\nu(\alpha) = 0$, including all $\alpha \in H_*(M) \subset QH(M)$.

Proof By the triangle inequality and the value at identity properties of the spectral invariant,

$$c(\alpha,\widetilde{\phi}_H)=c(\alpha*[M],\widetilde{\mathrm{id}}\,\widetilde{\phi}_H)\le c(\alpha,\widetilde{\mathrm{id}})+c([M],\widetilde{\phi}_H)=c([M],\widetilde{\phi}_H)$$

for all $\alpha\in QH(M)$ such that $\nu(\alpha)=0$. In addition,

$$0=c(\alpha,\widetilde{\mathrm{id}})=c(\alpha*[M],\widetilde{\phi}_H\widetilde{\phi}_{\overline{H}})\le c(\alpha,\widetilde{\phi}_H)+c([M],\widetilde{\phi}_{\overline{H}}).$$

Combining both inequalities we obtain

$$0\le c([M],\widetilde{\phi}_H)-c(\alpha,\widetilde{\phi}_H)\le c([M],\widetilde{\phi}_H)+c([M],\widetilde{\phi}_{\overline{H}})=\gamma(H),$$

which concludes the proof of the proposition. □

Proposition 27 *Let M be a monotone symplectic manifold with rationality constant $\rho>0$, let H be a Hamiltonian function on M with $\gamma(H)<\rho$ and let $\alpha\in QH(M)$. Then*

$$c([M],\widetilde{\phi}_H)-\rho<c(\alpha,\widetilde{\phi}_H)\le c([M],\widetilde{\phi}_H)$$

if, and only if, $\nu(\alpha)=0$.

Proof If $\nu(\alpha)=0$, then Proposition 26 and the hypothesis that $\gamma(H)<\rho$ imply that

$$c([M],\widetilde{\phi}_H)-\rho<c(\alpha,\widetilde{\phi}_H)\le c([M],\widetilde{\phi}_H).$$

Conversely, let $x_1,\dots,x_B$ be a homogeneous basis of $H_*(M)\subset QH(M)$ and write $c=c([M],\widetilde{\phi}_H)$. Then, by Proposition 26, we have $c(x_k,\widetilde{\phi}_H)\in(c-\rho,c]$ for all $1\le k\le B$. Also, for $q^j\in\Lambda_{\mathbb{K}}$, the equality $c(q^jx_k,\widetilde{\phi}_H)=c(x_k,\widetilde{\phi}_H)+j\rho$ implies that $c(q^jx_k,\widetilde{\phi}_H)\notin(c-\rho,c]$ for all $j\neq 0$. Thus, if $c(\alpha,\widetilde{\phi}_H)\in(c-\rho,c]$ for

$$\alpha=\lambda x_k=\sum a_jq^jx_k,$$

where $\lambda\in\Lambda$, the non-Archimedean property of the spectral invariant imposes that

$$\alpha=a_0x_k+\sum_{j<0}a_jq^jx_k,$$

which in turn implies $\nu(\alpha)=0$. In general, $\alpha\in QH(M)$ is of the form $\sum\lambda_kx_k$, where $\lambda_k=\sum a_j^{(k)}q^j$. Consequently, if $c(\alpha,\widetilde{\phi}_H)\in(c-\rho,c]$, we may argue as before to conclude

$$\alpha=\sum_k\biggl(a_0^{(k)}x_k+\sum_{j<0}a_j^{(k)}q^jx_k\biggr). \tag{12}$$

Thus, $\nu(\alpha)=0$, which concludes the proof of the claim. □

Remark 28 The above propositions are valid, albeit with minor modifications to the proofs, in the more general case where M is only assumed to be rational. If M is negative monotone, then the base field $\mathbb{K}$ is required to be of characteristic zero; see Remark 10.

Let $\phi \in \mathrm{Ham}(M,\omega)$, and suppose $\gamma(\phi) < \rho$. We can, therefore, find a Hamiltonian function H generating ϕ such that $\gamma(H) < \rho$. Our goal is to extract information from the cluster structure of H in order to bound $\gamma(\phi)$ from below. First we set notation. Put $\mathrm{S}^1_\rho = \mathbb{R}/\rho\cdot\mathbb{Z}$ and, for $a \in \mathbb{R}$, let $[a] \in \mathrm{S}^1_\rho$ be its equivalence class. For $\theta \in \mathrm{S}^1_\rho$, define

$$\Gamma_\theta = \{(a-\rho, a] \mid a \in \mathbb{R},\ [a] = \theta\}.$$

Note that the intervals in Γ_θ are disjoint and their union covers the real line. In addition, observe that, modulo $\rho\cdot\mathbb{Z}$, the set $\mathrm{Spec}^{\mathrm{ess}}(H) \cap \mathrm{I}$ does not depend on the interval $\mathrm{I} \in \Gamma_\theta$.

Definition 29 (spectral length) We define the *θ–parsed spectral length of H* as

$$l(H,\Gamma_\theta) = \mathrm{diam}(\mathrm{Spec}^{\mathrm{ess}}(H) \cap \mathrm{I}) = \sup\{|a-b| \mid a, b \in \mathrm{Spec}^{\mathrm{ess}}(H) \cap \mathrm{I}\},$$

where $\mathrm{I} \in \Gamma_\theta$ is arbitrary. For $\Gamma_H = \Gamma_{[c([M],H)]}$ we call $l(H,\Gamma_H)$ the *fundamental length of H*. Finally, we define the *spectral length* of $\phi \in \mathrm{Ham}(M,\omega)$ as

$$l(\phi) = \inf\{l(H,\Gamma_\theta) \mid \theta \in \mathrm{S}^1_\rho\}, \tag{13}$$

where H is any Hamiltonian function generating ϕ. The right-hand side of (13) does not depend on the choice of Hamiltonian: indeed, if H' is another Hamiltonian generating ϕ, then $\mathrm{Spec}^{\mathrm{ess}}(H') = \mathrm{Spec}^{\mathrm{ess}}(H) + c$ for a certain $c \in \mathbb{R}$ by Proposition 23(vii). (Another proof using Seidel elements is also possible.)

Remark 30 The following alternative definition of $l(\phi)$ helps calculate it in examples. Set $\pi : \mathbb{R} \to \mathrm{S}^1_\rho$ for the natural projection: $\pi(a) = [a]$. The image $\pi(\mathrm{Spec}^{\mathrm{ess}}(H)) \subset \mathrm{S}^1_\rho$ is then a finite set. Hence its complement consists of a finite number of open intervals $\{K_j\}_{j=1}^m$. In terms of these intervals,

$$l(\phi) = \rho - \max_j |K_j|,$$

where for an interval K in S^1_ρ we denote by $|K|$ the length of K with respect to the standard metric. Yet again, we may reformulate $l(\phi)$ more intuitively as the smallest length of an interval containing $\pi(\mathrm{Spec}^{\mathrm{ess}}(H))$, that is,

$$l(\phi) = \inf\{|L| \mid L \supset \pi(\mathrm{Spec}^{\mathrm{ess}}(H))\},$$

the infimum running over intervals L in S^1_ρ.

Remark 31 We can also define an a priori larger invariant $l'(\phi) \geq l(\phi)$ of ϕ by

$$l'(\phi) = \inf_{\phi_H^1 = \phi} l(H, \Gamma_H).$$

However, we find $l(\phi)$ more convenient for this paper.

Lemma 32 *The fundamental length of a Hamiltonian H satisfies*

$$l(H, \Gamma_H) \leq \gamma(H).$$

If, in addition, $\gamma(H) < \rho$, then we have equality, ie

$$l(H, \Gamma_H) = \gamma(H).$$

Proof By definition the θ–parsed spectral length of H is bounded above by ρ for any choice of $\theta \in \mathrm{S}^1$; in particular, $l(H, \Gamma_H) \leq \rho$. Thus, we need only to consider the case where $\gamma(H) < \rho$. Equation (12) in the proof of Proposition 27, or alternatively Proposition 23(vi), implies that $\#\{\mathrm{Spec}^{\mathrm{ess}}(H) \cap \mathrm{I}\} < \infty$ for $\mathrm{I} \in \Gamma_H$ and hence for $\mathrm{I} \in \Gamma_\theta$ for any $\theta \in \mathrm{S}^1_\rho$. Thus by Proposition 27 the fundamental length of H is given by

$$l(H, \Gamma_H) = c([M], H) - c(\alpha_{\min,H}, H),$$

where $\alpha_{\min,H} \in QH(M)$ has zero valuation. Consequently, Proposition 26 implies that $l(H, \Gamma_H) \leq \gamma(H)$. To prove equality, we observe that by the Poincaré duality property of spectral invariants (see [68; 17]) and the fact that the set $\mathrm{Spec}^{\mathrm{ess}}(H) \cap \mathrm{I}$ is finite, there exists $\beta \in QH(M) \setminus \{0\}$ such that $c(\beta, H) = -c([M], \overline{H})$. By adding $\gamma(H)$ to both sides of the equality we obtain $c(\beta, H) + \gamma(H) = c([M], H)$, which implies

$$c([M], H) - \rho < c(\beta, H) \leq c([M], H).$$

Therefore, $\gamma(H) \leq l(H, \Gamma_H)$, which gives us the claimed equality. □

Lemma 33 *Let ϕ be a Hamiltonian diffeomorphism. Then, $l(\phi) \leq \gamma(\phi)$.*

Proof Let H be any Hamiltonian function that generates ϕ. By definition $l(H, \Gamma_\theta) < \rho$ for every $\theta \in \mathrm{S}^1_\rho$; in particular, we have $l(\phi) < \rho$. Hence, if $\gamma(\phi) \geq \rho$, the desired inequality holds trivially. Therefore, we may suppose that $\gamma(\phi) < \rho$, in which case we may take H such that $\gamma(H) < \rho$. Consequently, Lemma 32 implies $l(H, \Gamma_H) = \gamma(H)$; in particular, we have that $l(\phi) \leq \gamma(H)$. If H' is any other Hamiltonian function generating ϕ, with $\gamma(H') \leq \gamma(H)$, the same argument implies $l(\phi) \leq \gamma(H')$. Thus, we conclude that $l(\phi) \leq \gamma(\phi)$. □

Remark 34 Lemma 32 immediately implies that if $\gamma(\phi) < \rho$, then $l'(\phi) = \gamma(\phi)$. It is not clear that the same holds for $l(\phi)$. However, we can prove that if $\gamma(\phi) < \rho/2$, then $l(\phi) = \gamma(\phi)$. Indeed, if $\gamma(H) < \gamma(\phi) + \epsilon < \rho/2$, by Lemma 32 we have $l(H, \Gamma_H) = \gamma(H) < \rho/2$. However, this implies that for arbitrary $\theta \in \mathrm{S}^1_\rho$, either $l(H, \Gamma_\theta) = l(H, \Gamma_H)$, if the partitions of $\mathrm{Spec}^{\mathrm{ess}}(H)$ into clusters corresponding to Γ_H and Γ_θ coincide, or $l(H, \Gamma_\theta) \geq \rho - l(H, \Gamma_H) > \rho/2 > l(H, \Gamma_H)$, if they do not. Hence, by taking the infima, $l(\phi) = \gamma(\phi)$.

Lemma 35 *Let ϕ be a generalized Hamiltonian $\mathbb{K}$ pseudorotation with sequence $k_j = j$ and take a Hamiltonian H generating ϕ. Suppose that all the distances between pairs of points in $\mathrm{Spec}^{\mathrm{ess}}(H)$ are rational multiples of ρ. Then there exists a positive integer m such that $\gamma(\phi^m) \geq \rho$.*

Proof Fix the base coefficient field $\mathbb{K}$ for all homological notions in the proof. We can suppose $\gamma(\phi) < \rho$, otherwise the implication of the theorem would be true for $m = 1$. Furthermore, we note that the hypothesis of the theorem is independent of the choice of Hamiltonian function; thus, we may suppose that $\gamma(H) < \rho$, which by Lemma 32, implies $l(H, \Gamma_H) = \gamma(H)$. Hence, we have a cluster structure determined by finitely many values of the essential spectrum of H belonging to the interval

$$\mathrm{I}_H = (c([M], H) - \rho, c([M], H)].$$

Thus, setting

$$\mathrm{Spec}^{\mathrm{ess}}(H) \cap \mathrm{I}_H = \{c_1, \dots, c_B\},$$

by the hypothesis of the proposition we have

$$c_i - c_j = \frac{a_{ij}}{b_{ij}} \rho \in \rho \cdot \mathbb{Q} \cap (-\rho, \rho)$$

for all $i \neq j$. Note that any pair of points $\alpha, \beta \in \mathrm{Spec}^{\mathrm{ess}}(H)$ are of the form $\alpha = c_i + k\rho$ and $\beta = c_j + l\rho$ for integers l and k. Thus their difference is of the form

$$\alpha - \beta = \left(\frac{a_{ij}}{b_{ij}} + (k - l) \right) \rho. \tag{14}$$

Now, let m be the integer given by $\prod_{i<j} b_{ij}$. The facts that

$$\mathrm{Fix}(\phi^m) = \mathrm{Fix}(\phi), \quad \mathrm{Spec}^{\mathrm{ess}}(H) = \mathrm{Spec}^{\mathrm{vis}}(H) \quad \text{and} \quad HF^{\mathrm{loc}}(\phi^{k_j}, \mathcal{F}^{(k_j)}) \neq 0$$

for all generalized fixed points $\mathcal{F}$ of ϕ imply

$$\mathrm{Spec}^{\mathrm{ess}}(H^{(m)}) = m \cdot \mathrm{Spec}^{\mathrm{ess}}(H) + \rho \cdot \mathbb{Z}. \tag{15}$$

As a consequence of equations (14) and (15) and the definition of m, we have that $\mathrm{Spec}^{\mathrm{ess}}(H^{(m)}) = \rho \cdot \mathbb{Z} + c$ for a suitable constant $c \in \mathbb{R}$. Hence, $l(F, \Gamma_\theta) = 0$ for any Hamiltonian F generating ϕ^m and $\theta \in \mathrm{S}^1_\rho$. If $\gamma(F) < \rho$, then by Lemma 32 $\gamma(F) = l(F, \Gamma_F) = 0$, which is absurd since this would imply $\phi^m = \mathrm{id}$. Hence $\gamma(\phi^m) \geq \rho$. □

5 Proofs

5.1 Proof of Theorem C

Let (M, ω) be a closed symplectic manifold and consider a nontrivial $\phi \in \mathrm{Ham}(M, \omega)$ such that $\phi^p = \mathrm{id}$ for an integer[9] p. We can construct a Riemannian metric $\langle \cdot, \cdot \rangle$ which is invariant under the action of the group

$$G = \{\mathrm{id}, \phi, \dots, \phi^{p-1}\},$$

a fact that is true for any compact Lie group G. In other words, ϕ is an isometry with respect to this metric. We first show that $\mathrm{Fix}(\phi)$ is composed of finitely many isolated path-connected components.

Let $x \in \mathcal{F} \subset \mathrm{Fix}(\phi)$, where $\mathcal{F}$ is the path-connected component of x. We claim that there exists a neighborhood of x which does not intersect any other connected component of $\mathrm{Fix}(\phi)$. Suppose the contrary. Then x would be a limit point of $\mathrm{Fix}(\phi) \setminus \mathcal{F}$. In particular, if $B_\epsilon(x)$ is a normal ball of radius ϵ around x, then there exists a point $y \in B_\epsilon(x) \cap (\mathrm{Fix}(\phi) \setminus \mathcal{F})$ and we can consider the unique minimizing geodesic γ given by the exponential map, satisfying $\gamma(0) = x$ and $\gamma(1) = y$. However, ϕ is an isometry so we have that $\tilde{\gamma} = \phi \circ \gamma$ is also a minimizing geodesic satisfying $\tilde{\gamma}(0) = x$ and $\tilde{\gamma}(1) = y$, hence by uniqueness we must have $\mathrm{Image}(\gamma) \subset \mathrm{Fix}(\mathcal{F})$, contradicting the fact that y was in a distinct path-connected component. Since $\mathcal{F}$ is compact we can choose the radius ϵ of the normal ball uniformly so that $\mathcal{F}$ is in fact isolated, which by the compactness of M implies that there are only finitely many path-connected components.

Furthermore, if k is coprime to p then we have $\mathrm{Fix}(\phi^k) = \mathrm{Fix}(\phi)$. In fact, since p and k are relatively prime there exist integers a_k and b_k such that $a_k k + b_k p = 1$. Thus,

$$\phi = \phi^{a_k k + b_k p} = \phi^{a_k k} \phi^{b_k p} = \phi^{a_k k}.$$

[9] While we do not use it in this proof, it might help the reader to first assume that p is a prime.

So if x is a fixed point of ϕ^k then the above equality shows that x is also a fixed point of ϕ. Conversely, if x is a fixed point of ϕ it is clearly a fixed point for any of its iterations. Finally, the same argument shows that if x is contractible as a fixed point of ϕ^k it is also contractible as a fixed point of ϕ, and vice versa. Therefore $\mathrm{Fix}(\phi^k) = \mathrm{Fix}(\phi)$.

To show that ϕ is weakly nondegenerate we utilize the fact if M is connected and $f \in \mathrm{Iso}(M, \langle \cdot, \cdot \rangle)$ is such that $f(x) = x$ and $D(f)_x = \mathrm{id}_{T_x M}$ for a point $x \in M$, then $f = \mathrm{id}_M$. This can be proven by considering the nonempty closed set

$$S = \{y \in M \mid f(y) = y, D(f)_y = \mathrm{id}_{T_y M}\},$$

and noting that the existence of normal balls implies that S is also open. Applied to our context, we must then show that for every $x \in \mathrm{Fix}(\phi)$, $D(\phi)_x$ must have at least one eigenvalue different from 1, otherwise ϕ would have to be trivial. One way to see this is by noting that as $D(\phi)_x \in \mathrm{Sp}_{2n}(T_x M)$ is an element of finite order, its Jordan form is diagonal, hence it is trivial if and only if all its eigenvalues are equal to 1.

A slight modification of the above arguments, which amounts to the slice theorem [2, Theorem I.2.1], shows first that each connected component $\mathcal{F}$ of the fixed-point set of ϕ is a closed connected submanifold of M (and hence is path-connected). Moreover, for each $\mathcal{F}$ and $x \in \mathcal{F}$, $\ker(D(\phi)_x - \mathrm{id}_{T_x M}) = T_x \mathcal{F}$, which is to say that the graph of ϕ intersects the diagonal $\Delta \subset M \times M^-$ *cleanly*. In other words, ϕ is a Floer–Morse–Bott Hamiltonian diffeomorphism.

Finally, to prove that for a generalized fixed point $\mathcal{F}$ of ϕ, and capping $\overline{\mathscr{F}}$ of its corresponding generalized periodic orbit $\mathscr{F}$, the mean-index $\Delta(H, \overline{x})$ is constant as a function of $x \in \mathcal{F}$, we argue as follows. We shall prove that for a fixed $x_0 \in \mathcal{F}$, the function $f : \mathcal{F} \to \mathbb{R}$, given by $f(x) = \Delta(H, \overline{x}) - \Delta(H, \overline{x}_0)$, has integer values. By continuity of the mean-index this implies that f is identically constant, and as $f(x_0) = 0$, it is identically zero. This shows the required statement.

First we prove that f has integer values. Similarly to the case of a Riemannian metric, by [59, Proposition 2.5.6] we can find an ω–compatible almost complex structure J on M that is preserved by ϕ. This allows us to consider $D(\phi)_x \in \mathrm{Sp}_{2n}(T_x M)$ for all $x \in \mathcal{F}$ a unitary matrix, which has diagonal Jordan form, and is determined up to conjugation by its spectrum with geometric multiplicities. Furthermore its spectrum lies in the finite set $\mu_p \subset \mathbb{C}$ of p^{th} roots of unity. Therefore by continuity of the spectrum in the operator norm, which holds for normal and hence for unitary matrices in particular, the spectrum of $D(\phi)_x$ does not depend on $x \in \mathcal{F}$, and all $D(\phi)_x$ for $x \in \mathcal{F}$ are conjugate by appropriate unitary isomorphisms. Therefore $D(\phi)_x$ and $D(\phi)_{x_0}$ can be connected to the identity by conjugate paths, which therefore have equal mean-indices. Now, as the paths obtained from $D(\phi_H^t)_x$ and $D(\phi_H^t)_{x_0}$ by means of the cappings differ from these conjugate paths by suitable loops Φ and Φ_0 in the symplectic group, we obtain that $f(x) = \Delta(H, \overline{x}) - \Delta(H, \overline{x}_0) = \Delta(\Phi) - \Delta(\Phi_0) \in \mathbb{Z}$.

Finally, observe that with $D(\phi)_x$ being (ω_x, J_x)–unitary, $T_x\mathcal{F}$ is J_x–invariant, and the tangent space T_xM splits as a symplectic direct sum $T_x\mathcal{F} \oplus \mathcal{N}_x$, where $\mathcal{N}_x$ is the normal bundle to $\mathcal{F}$ at x (in fact this splitting can be obtained by taking $\mathcal{N}_x$ to be the Hermitian orthogonal complement of $T_x\mathcal{F}$). In particular, $\mathcal{F}$ is a symplectic submanifold of (M, ω).

Hence, the above discussion shows that ϕ is generalized perfect with sequence k_j being the monotone-increasing ordering of the set

$$\{k \in \mathbb{Z}_{>0} \mid \gcd(k, p) = 1\}. \qquad \square$$

Remark 36 We have just seen that a p–torsion Hamiltonian diffeomorphism ϕ is weakly nondegenerate generalized perfect. In our setting it is enough to consider the case where ϕ has prime order. In fact, if ϕ has order $d \geq 2$ and l is a prime that divides d, ie there is an integer m such that $d = lm$, we consider the Hamiltonian diffeomorphism $\psi = \phi^m$, which, in turn, has prime order. Equivalently, by Cauchy's theorem, if G is a finite group then for every prime p dividing its order there exists an element of order p.

5.2 Proof of Proposition 5

We first observe that by the universal coefficient formula, it is sufficient to prove the statement for $R = \mathbb{Z}$.

Now from [85, Chapter 9 and the proof of Theorem 2.3.2] as well as the translation of [76, Theorem 3.4.11] from the setting of Lagrangian clean intersections to the Floer–Morse–Bott Hamiltonian setting [3, Theorem 5.2.2] by means of the graph construction, it is direct to see that there is an isomorphism

$$HF^{\mathrm{loc}}(\phi, \mathcal{F}) \cong H(\mathcal{F}; \mathcal{O} \times_{\pm 1} \mathbb{Z})$$

of the local Floer homology and the homology of $\mathcal{F}$ with coefficients in a $\mathbb{Z}$–local system $\mathcal{O} \times_{\pm 1} \mathbb{Z}$, with structure group $\{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$, associated to a double cover $\mathcal{O}$

of $\mathcal{F}$ that we describe below. It is the goal of the proof to show that *in our case* this local system is trivial.

The local system $\mathcal{O}$ is defined as follows. For $x, y \in \mathcal{F}$, consider the space $P_{x,y}(\mathcal{F})$ of smooth maps $\gamma\colon \mathbb{R} \to \mathcal{F}$ such that

$$\lim_{s\to-\infty} \gamma(s) = x \quad \text{and} \quad \lim_{s\to\infty} \gamma(s) = y$$

for which the convergence is exponential with derivatives. Let $u_\gamma\colon \mathbb{R}\times \mathrm{S}^1 \to M$ denote the cylinder $u_\gamma(s,t) = \phi_H^t(\gamma(s))$. Look at the bundle $E_\gamma \to \mathbb{R}\times\mathrm{S}^1$ given by $E_\gamma = (u_\gamma)^*TM$. Now for each sufficiently small positive number $\epsilon \in (0,\epsilon_0)$, where ϵ_0 depends only on H and $\mathcal{F}$, consider real Cauchy–Riemann differential operators

$$D_\gamma\colon W^{1,p,\epsilon}(E_\gamma) \to L^{p,\epsilon}(E_\gamma)$$

between Sobolev spaces of sections of E_γ with ϵ–exponential decay as $|s|\to\infty$, that over $(-\infty,-C)$ and (C,∞), for a large $C>0$, coincide with real Cauchy–Riemann operators determined by a choice of an ω–compatible almost complex structure $\{J_t\}\in\mathcal{J}_M$ and connections whose parallel transport over the curve $\{(s,t)\}_{t\in[0,1]}$ (with s fixed) is determined by the linearization of ϕ_H^t at $\gamma(s)$. For $\epsilon>0$ sufficiently small, all these operators are Fredholm. Moreover, with the auxiliary data of connections and complex structures forming a contractible space, all these operators are furthermore homotopic to each other in the space of Fredholm operators. It is shown in [85] and [28, Chapter 8] that for $\gamma,\gamma' \in P_{x,y}(\mathcal{F})$, the orientation torsors $|D_\gamma|$ and $|D_{\gamma'}|$ of the determinant spaces $\det(D_\gamma)$ and $\det(D_{\gamma'})$ are canonically isomorphic.[10] We can therefore fix $x\in\mathcal{F}$, and set our local system $\mathcal{O}\to\mathcal{F}$ to be induced from the sets $|D_\gamma|$ for $\gamma\in P_{x,y}(\mathcal{F})$ with $y\in\mathcal{F}$, with the natural identifications provided by this isomorphism.

Now we prove that $\mathcal{O}$ is trivial in our case. Suppose $\gamma\in P_{x,y}(\mathcal{F})$. It is sufficient to show that $\det(D_\gamma)$ is canonically oriented. Now, as in the proof of Theorem C, in our case there exists an ω–compatible almost complex structure J on M which is invariant under ϕ. In particular, $D(\phi)_x\colon T_xM\to T_xM$ is (J_x,ω_x)–unitary for all $x\in\mathcal{F}$. This, together with the fact that the universal cover $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ deformation-retracts to the universal cover $\widetilde{U}(n)$ of its unitary subgroup, implies that D_γ is homotopic in the space of Fredholm operators, canonically up to a contractible choice of auxiliary

[10] Recall that the determinant line of a Fredholm operator D is the real vector space of dimension one defined as $\det(D) = \det(\mathrm{coker}(D))^\vee \otimes \det(\ker(D))$, where for a real finite-dimensional vector space V of dimension d, $\det(V) = \Lambda^d(V)$, and for a real vector space l of dimension one, its orientation torsor over the group ±1 is $|l| = (l\setminus\{0\})/(\mathbb{R}_{>0})$.

data, to a real Cauchy–Riemann operator $D\colon W^{1,p,\epsilon}(E_\gamma) \to L^{p,\epsilon}(E_\gamma)$ corresponding to a J–unitary connection. Call the homotopy[11] $\{D^r\}_{r\in[0,1]}$, where $D^0 = D_\gamma$ and $D^1 = D$. But such operators D are in fact complex Cauchy–Riemann operators, their kernels and cokernels are complex vector spaces, and hence their determinants are canonically oriented. Hence $|D|$ and $|D_\gamma|$ admit canonical elements o and o_γ. By a similar argument, following the definition of the isomorphisms $\psi_{\gamma,\gamma'}\colon |D_\gamma| \xrightarrow{\sim} |D_{\gamma'}|$ from [85; 28, Chapter 8], with the key point being that orientation gluing is natural with respect to homotopies [85, Lemma 9.4.1], we see that $\psi_{\gamma,\gamma'}(o_\gamma) = o_{\gamma'}$. Therefore $\mathcal{O}$ admits a continuous section, and hence is trivial. This finishes the proof. □

5.3 Proof of Theorem J

Consider $\phi \in \mathrm{Ham}(M,\omega)\setminus\{\mathrm{id}\}$ such that $\phi^d = \mathrm{id}$, and let H be a Hamiltonian function generating ϕ. Then $\gamma(H) > 0$ by the nondegeneracy of the spectral norm. Since ϕ has finite order d we have that $\{\phi^t_{H^{(d)}}\}_{t\in[0,1]}$ is a Hamiltonian loop, which, in addition to the fact that M has rationality constant $\rho > 0$, implies that $\mathrm{Spec}(H^{(d)}) = a + \rho\cdot\mathbb{Z}$ for a real constant a. One can show by a quick calculation that $\mathrm{Spec}(\overline{H}^{(d)}) = -a + \rho\cdot\mathbb{Z}$.

Furthermore, observe that $\mathrm{Spec}(H) \subset \mathrm{Spec}(H^{(d)})/d$. In fact, if $c \in \mathrm{Spec}(H)$ then there exists a 1–periodic capped orbit $\overline{x} \in \widetilde{\mathcal{O}}(H)$ such that $\mathcal{A}_H(\overline{x}) = c$. Consequently, $\mathcal{A}_{H^{(d)}}(\overline{x}^{(d)}) = d\cdot\mathcal{A}_H(\overline{x}) = d\cdot c$, which implies the claim when added to the fact that $\overline{x}^d$ is a critical point of $\mathcal{A}_{H^{(d)}}$.

Finally, the above observations imply that $\gamma(H) \in (\rho/d)\cdot\mathbb{Z}$. In particular, the fact $\gamma(H) \geq 0$ implies $\gamma(H) \geq \rho/d$. Since H was an arbitrary Hamiltonian generating ϕ, it is clear that $\gamma(\phi) \geq \rho/d$. □

5.4 Proof of Theorem A

Similarly to the case of Theorem J, $\phi^d = \mathrm{id}$ implies, in the symplectically aspherical setting, that for a Hamiltonian H generating ϕ, we have $\mathrm{Spec}(H^{(d)}) = \{a\}$ and $\mathrm{Spec}(\overline{H}^{(d)}) = \{-a\}$ for a constant $a \in \mathbb{R}$, so $\mathrm{Spec}(H) \subset \mathrm{Spec}(H^{(d)})/d = \{a/d\}$ consists of at most one point. Since $\mathrm{Spec}(H)$ contains $c([M],H)$, we obtain that

[11] In fact we apply a homotopy depending smoothly on $x_0 \in \mathcal{F}$ from the symplectic connections on $x^*(TM) \to S^1$ for $x(t) = \phi^t_H(x_0)$ given by the linearized flow of ϕ^t_H to unitary connections, while at all times preserving their monodromies $D(\phi^1_H)_{x_0}$ over S^1 for all $x_0 \in \mathcal{F}$. This means in particular that the kernels of the asymptotic operators for fixed x_0 do not depend on the homotopy parameter r, up to natural identification. This and the compactness of $\mathcal{F}$ imply that the $\epsilon > 0$ above can be chosen sufficiently small that all the operators D^r along the homotopy are indeed Fredholm as operators $W^{1,p,\epsilon}(E_\gamma) \to L^{p,\epsilon}(E_\gamma)$.

$c([M],H)=a/d$. Similarly, $c([M],\overline{H})=-a/d$. This means that $\gamma(H)=0$ and hence $\gamma(\phi)=0$, which implies by nondegeneracy of γ that $\phi=\mathrm{id}$. This finishes the proof. □

5.5 Proof of Theorem K

Consider $\phi\in\mathrm{Ham}(M,\omega)\setminus\{\mathrm{id}\}$ such that $\phi^p=\mathrm{id}$ for a prime number p. Fix a coefficient field $\mathbb{K}$. We show that there exists a positive integer m such that

$$\gamma(\phi^m)\geq\frac{\lfloor p/2\rfloor}{p}\cdot\rho, \tag{16}$$

where $\lfloor p/2\rfloor$ denotes the floor of $p/2$. We may suppose $p\geq 3$, since the case $p=2$ is settled by Theorem J for $m=1$. In this case, note that $\lfloor p/2\rfloor=(p-1)/2$. Supposing that $\gamma(\phi)<\rho(p-1)/2p$, we can find a Hamiltonian H generating ϕ satisfying $\gamma(H)<\rho(p-1)/2p$. In the proof of Theorem J we saw that $\gamma(H)$ must be a positive integer multiple of ρ/p. Therefore, by Lemma 32 we can find a positive integer $r\leq(p-3)/2$ such that

$$l(H,\Gamma_H)=\gamma(H)=\frac{r\rho}{p}. \tag{17}$$

In particular, we have that $2r<p$, which combined with the fact that $p>2$ implies that there exist integers a,b such that $a(2r)+bp=1$. Observe that b must be an odd integer, since $a(2r)$ is even while p and 1 are odd. Let k be the integer such that $-b=2k+1$. Furthermore, note that $a\neq 0$, and set $m=|a|$. There are two cases to be considered, depending on the sign of the integer a:

- If $a>0$, we have that $m(2r)-(2k+1)p=1$, which implies

$$\frac{mr}{p}-\frac{p+1}{2p}=k, \tag{18}$$

where $(p+1)/2=\lceil p/2\rceil$. Furthermore, since m and p are coprime, Theorem G implies that

$$\mathrm{Spec}^{\mathrm{ess}}(H^{(m)})=m\cdot\mathrm{Spec}^{\mathrm{ess}}(H)+\rho\mathbb{Z}. \tag{19}$$

Combining (17), (18) and (19) we obtain that there exist $c_0,c_1\in\mathrm{Spec}^{\mathrm{ess}}(H)$ such that

$$mc_1-mc_0=\frac{mr\rho}{p}=\frac{(p+1)\rho}{2p}+k\rho.$$

In addition, $mc_1 + j\rho$ and $mc_0 + j\rho$ belong to the essential spectrum of $H^{(m)}$ for every integer j. We conclude that for each $\theta \in \mathrm{S}^1_\rho$ and $\mathrm{I} \in \Gamma_\theta$, there exists an integer l such that either

$$mc_1 + l\rho, mc_0 + (k+l)\rho \in \mathrm{I} \quad \text{or} \quad mc_1 + l\rho, mc_0 + (k+l+1)\rho \in \mathrm{I}.$$

Consequently,

$$\begin{aligned} l(H^{(m)}, \Gamma_\theta) &\geq \min\{mc_1 - mc_0 - k\rho, mc_0 - mc_1 + (k+1)\rho\} \\ &= \min\left\{\frac{(p+1)\rho}{2p}, \frac{(p-1)\rho}{2p}\right\} = \frac{(p-1)\rho}{2p}. \end{aligned}$$

Since θ was arbitrary, we conclude

$$l(\phi^m) \geq \frac{\lfloor p/2 \rfloor}{p} \cdot \rho. \tag{20}$$

- If $a < 0$, an analogous argument can be made to show that once again (20) is valid.

Hence, by Lemma 33 we obtain the inequality (16).

5.6 Proof of Theorem L

Consider a generalized pseudorotation ϕ as in Lemma 35. As a consequence of this lemma, we may suppose that there exist $c_1, c_2 \in \mathrm{Spec}^{\mathrm{ess}}(H)$ such that $c_1 - c_2 \in \rho \cdot (\mathbb{R} \setminus \mathbb{Q})$, otherwise $\gamma(\phi^m) \geq \rho$ for some positive integer m. Since the orbit of any irrational rotation in S^1 is dense, for every $\epsilon > 0$ there exists an integer m_ϵ such that

$$\frac{\rho}{2} - \epsilon < d_{S^1_\rho}([c_1], [m_\epsilon \cdot c_2]) \leq \frac{\rho}{2},$$

where for $x \in \mathbb{R}$ we denote by $[x] \in S^1_\rho = \mathbb{R}/\rho\mathbb{Z}$ its equivalence class, and $d_{S^1_\rho}$ is the distance function on S^1_ρ coming from the standard flat metric on $\mathbb{R}$. Therefore, arguing as in the proof of Theorem K we conclude

$$\sup_{k \in \mathbb{Z}_{>0}} \gamma(\phi^k) \geq \frac{\rho}{2}. \qquad \square$$

The proofs of Theorems D and I rely on the following observations regarding the mean-index. First, let $\widetilde{\phi}$ be a lift of ϕ to the universal cover $\widetilde{\mathrm{Ham}}(M, \omega)$ of $\mathrm{Ham}(M, \omega)$. As our path-connected isolated fixed-point sets are weakly nondegenerate, if the capping $\overline{\mathscr{F}}$ of the generalized 1–periodic orbit $\mathscr{F}$ corresponding to an isolated fixed-point set

$\mathcal{F} \subset \mathrm{Fix}(\phi)$ carries a cohomology class μ of Conley–Zehnder index n in $HF^n(\tilde{\phi}) \cong QH^{2n}(M, \Lambda_{\mathbb{K}})$, for a coefficient field $\mathbb{K}$, then its mean-index $\Delta = \Delta(\tilde{\phi}, \bar{\mathscr{F}})$ satisfies $\Delta - n < n < \Delta + n$. Hence,

$$\Delta(\tilde{\phi}, \bar{\mathscr{F}}) \in (0, 2n). \tag{21}$$

Similarly, if $\bar{\mathscr{F}}$ carries a homology class $u \in HF_n(\tilde{\phi}) \cong QH_{2n}(M, \Lambda_{\mathbb{K}})$, then (21) holds. Both of these implications follow from Lemma 19, equation (7) and Section 3.2. We will specifically use the case $u = [M]$, which follows from Lemma 17.

5.7 Proof of Theorem D

We first treat the negative monotone case. Choose $H \in \mathcal{H}$ so that the path $\{\phi_H^t\}_{t\in[0,1]}$ represents the class $\tilde{\phi}$ lifting ϕ. Let k_i be the sequence associated to ϕ as a generalized perfect Hamiltonian diffeomorphism. By the pigeonhole principle applied to the subsequence l_i with $l_i \,|\, l_{i+1}$ for all i, there exists an isolated fixed-point set $\mathcal{F} \subset \mathrm{Fix}(\phi)$, and an increasing subsequence of k_i, which we renumber and denote by r_i, such that $c([M], H^{(r_i)})$ is carried by a capping $\bar{\mathscr{G}}_i$ of the isolated set of 1–periodic orbits of the r_i–iterated Hamiltonian $H^{(r_i)}$ corresponding to $\mathcal{F}^{(r_i)}$. Set $\bar{\mathscr{G}} = \bar{\mathscr{G}}_1$. Since r_1 divides all r_i, by taking a power of ϕ we can assume that $r_1 = 1$.

Write $\bar{\mathscr{G}}_i$ as a recapped iteration of $\bar{\mathscr{G}}$, ie

$$\bar{\mathscr{G}}_i = \bar{\mathscr{G}}^{(r_i)} \# A_i. \tag{22}$$

We claim that for r_i large, $\omega(A_i) \geq 0$ and $c_1(A_i) > 0$, contradicting negative monotonicity. Indeed, write $\mathcal{A}_i$ for the action functional of $H^{(r_i)}$, and $\mathcal{A} := \mathcal{A}_1$. Then by (22) and the triangle inequality for spectral invariants,

$$r_i \mathcal{A}(\bar{\mathscr{G}}) - \omega(A_i) = \mathcal{A}_i(\bar{\mathscr{G}}_i) = c([M], H^{(r_i)}) \leq r_i c([M], H) = r_i \mathcal{A}(\bar{\mathscr{G}}). \tag{23}$$

Hence,

$$\omega(A_i) \geq 0.$$

However, as $\bar{\mathscr{G}}_i$ carries $c([M], H^{(r_i)})$, by (21) we have $\Delta(H^{(r_i)}, \bar{\mathscr{G}}_i) \in (0, 2n)$ and also $\Delta(H, \bar{\mathscr{G}}) \in (0, 2n)$. Hence $r_i \Delta(H, \bar{\mathscr{G}}) > 2n$ for r_i large enough, and

$$2n > \Delta(H^{(r_i)}, \bar{\mathscr{G}}_i) = r_i \Delta(H, \bar{\mathscr{G}}) - 2c_1(A_i). \tag{24}$$

Therefore

$$c_1(A_i) > 0,$$

which finishes the proof.

We now prove the symplectic Calabi–Yau case of the theorem. In this case, the mean-index of each capped fixed-point set $\overline{\mathscr{F}}$ does not depend on the capping. Hence we write $\Delta(H, \mathcal{F})$ for each generalized fixed point $\mathcal{F}$ for this mean-index. Then for each positive sequence $k_i \to \infty$ of iterations with ϕ^{k_i} having a fixed finite number of weakly nondegenerate generalized fixed points, we argue as follows. For each $\mathcal{F} \in \pi_0(\mathrm{Fix}(\phi))$,

$$\Delta(H^{(k_i)}, \mathcal{F}^{(k_i)}) = (k_i/k_1)\Delta(H^{(k_1)}, \mathcal{F}^{(k_1)}).$$

Hence, if $\Delta(H^{(k_1)}, \mathcal{F}^{(k_1)}) > 0$ then $\Delta(H^{(k_i)}, \mathcal{F}^{(k_i)}) > 2n$ for all k_i sufficiently large, and if $\Delta(H^{(k_1)}, \mathcal{F}^{(k_1)}) \leq 0$ then $\Delta(H^{(k_i)}, \mathcal{F}^{(k_i)}) \leq 0$ for all k_i. Now, as each $\mathcal{F}$ is weakly nondegenerate, we obtain by the same argument as for the proof of the support property of local Floer homology, Lemma 16, that for all k_i sufficiently large, $H^{(k_i)}$ admits a C^2–small nondegenerate Hamiltonian perturbation H_i without capped periodic orbits of Conley–Zehnder index n. However, this is in contradiction to the existence of the PSS isomorphism. Specifically, in this case $HF_n(H_i) = 0$ by definition of Floer homology, and by the PSS isomorphism $HF_n(H_i) \cong QH_{2n}(M) \neq 0$. Indeed $[M] \in QH_{2n}(M)$ is nonzero. □

The following result was first proven in [93] in the setting of a *pseudorotation* assuming that the quantum Steenrod square of the point cohomology class is undeformed, or in other words that (M, ω) is not $\mathbb{F}_2$–Steenrod uniruled. We observe that the same statement holds for generalized pseudorotations, with essentially the same proof, and with a small modification following [97], for all primes p. Here $\mu \in QH^{2n}(M, \Lambda_{\mathbb{F}_p})$ denotes the cohomology class Poincaré dual to the point.

Theorem O *Let ψ be a generalized $\mathbb{F}_p$ pseudorotation with sequence $k_j = p^{j-1}$ of a closed monotone symplectic manifold (M, ω) that is not $\mathbb{F}_p$–Steenrod uniruled. Then*

$$c(\mu, \tilde{\psi}^p) \geq p \cdot c(\mu, \tilde{\psi}) \tag{25}$$

for each $\tilde{\psi} \in \widetilde{\mathrm{Ham}}(M, \omega)$ covering ψ.

We proceed to the proof of Theorem I.

5.8 Proof of Theorem I

Choose $H \in \mathcal{H}$ so that the path $\{\phi_H^t\}_{t \in [0,1]}$ represents the class $\tilde{\phi}$ lifting ϕ. By the pigeonhole principle, there exists an isolated fixed-point set $\mathcal{F} \subset \mathrm{Fix}(\phi)$, and an increasing sequence k_i such that $c(\mu, H^{(r_i)})$ for $r_i = p^{k_i}$ is carried by a capping $\overline{\mathscr{G}}_i$ of the isolated set of 1–periodic orbits of the r_i–iterated Hamiltonian $H^{(r_i)}$ corresponding to $\mathcal{F}^{(r_i)}$. By taking a power of ϕ, we can assume that $r_1 = 1$, and set $\overline{\mathscr{G}} = \overline{\mathscr{G}}_1$. Write $\overline{\mathscr{G}}_i$ as a recapped iteration of $\overline{\mathscr{G}}$, ie

$$\overline{\mathscr{G}}_i = \overline{\mathscr{G}}^{(r_i)} \# A_i. \tag{26}$$

We claim that for r_i large, we get $\omega(A_i) \le 0$ and $c_1(A_i) > 0$, contradicting monotonicity. Indeed, write $\mathcal{A}_i$ for the action functional of $H^{(r_i)}$, and set $\mathcal{A} := \mathcal{A}_1$. Then by (26) and Theorem O,

$$r_i \mathcal{A}(\overline{\mathscr{G}}) - \omega(A_i) = \mathcal{A}_i(\overline{\mathscr{G}}_i) = c(\mu, H^{(r_i)}) \ge r_i c(\mu, H) = r_i \mathcal{A}(\overline{\mathscr{G}}).$$

Hence,

$$\omega(A_i) \le 0.$$

However, as $\overline{\mathscr{G}}_i$ carries $c(\mu, H^{(r_i)})$, by (21) we have $\Delta(H^{(r_i)}, \overline{\mathscr{G}}_i) \in (0, 2n)$ and also $\Delta(H, \overline{\mathscr{G}}) \in (0, 2n)$. Hence $r_i \Delta(H, \overline{\mathscr{G}}) > 2n$ for r_i large enough, and

$$2n > \Delta(H^{(r_i)}, \overline{\mathscr{G}}_i) = r_i \Delta(H, \overline{\mathscr{G}}) - 2c_1(A_i).$$

Therefore,

$$c_1(A_i) > 0.$$

5.9 Proof of Theorem G

Suppose that $\phi \in \mathrm{Ham}(M, \omega) \setminus \{\mathrm{id}\}$ is of prime order $q \ge 2$. Let $p \ge 2$ be a prime different from q. In particular, $\phi^{j \cdot p^k} \ne \mathrm{id}$ for all $k \in \mathbb{Z}$ and $1 \le j \le q-1$.

Write

$$B(\phi, \mathbb{F}_p) = \max_{1 \le j \le q-1} \beta_{\mathrm{tot}}(\phi^j, \mathbb{F}_p).$$

By Theorem N we obtain for $1 \le j \le q-1$ that

$$B(\phi, \mathbb{F}_p) \ge \beta_{\mathrm{tot}}(\phi^{j \cdot p^k}, \mathbb{F}_p) \ge p^k \beta_{\mathrm{tot}}(\phi^j, \mathbb{F}_p).$$

Choosing a sufficiently large positive k, this implies that for all $1 \le j \le q-1$,

$$\beta_{\mathrm{tot}}(\phi^j, \mathbb{F}_p) = 0,$$

whence by Proposition 23 all such ϕ^j are generalized $\mathbb{F}_p$ pseudorotations. They are weakly nondegenerate by Theorem C. In other words, the equality

$$\mathrm{Spec}^{\mathrm{vis}}(H; \mathbb{F}_p) = \mathrm{Spec}^{\mathrm{ess}}(H; \mathbb{F}_p)$$

follows directly from the fact that $\beta_{\mathrm{tot}}(\phi, \mathbb{F}_p) = 0$. This finishes the proof of part (i).

Let us prove that $\mathrm{Spec}^{\mathrm{vis}}(H^{(k)};\mathbb{Q}) = k\cdot\mathrm{Spec}^{\mathrm{vis}}(H;\mathbb{Q}) + \rho\cdot\mathbb{Z}$ for all $k\in\mathbb{Z}$ coprime with q. By the universal coefficient formula in local Floer homology, it is sufficient to prove the identity $\mathrm{Spec}^{\mathrm{vis}}(H^{(k)};\mathbb{F}_p) = k\cdot\mathrm{Spec}^{\mathrm{vis}}(H;\mathbb{F}_p) + \rho\cdot\mathbb{Z}$ for coefficients in $\mathbb{F}_p$ for an infinite sequence of primes p. Consider the primes p for which $p = a \pmod q$, where $a\in(\mathbb{F}_q)^*$ is a cyclic generator of the multiplicative group $(\mathbb{F}_q)^* = \mathrm{GL}(1,\mathbb{F}_q)$ of $\mathbb{F}_q$. In this case the set $\{\phi^{p^j} \mid j\in\mathbb{Z}_{\geq 0}\}$ coincides with

$$\{\phi^k \mid 1\leq k\leq q-1\} = \{\phi^k \mid k\neq 0 \pmod q\}.$$

Let $\bar{\mathscr{F}}$ be a capped generalized periodic orbit of H. It is enough to prove that

$$\dim_{\mathbb{F}_p} HF^{\mathrm{loc}}(H^{(p^j)},\bar{\mathscr{F}}^{(p^j)}) = \dim_{\mathbb{F}_p} HF^{\mathrm{loc}}(H,\bar{\mathscr{F}})$$

for all $j\in\mathbb{Z}_{\geq 0}$. Indeed, as explained above, each capped generalized fixed point of ϕ^k is a recapping of a p^j–iterated capped generalized fixed point of ϕ.

We know by the Smith inequality in generalized local Floer homology, Proposition 24, that $\dim_{\mathbb{F}_p} HF^{\mathrm{loc}}(H^{(p^j)},\bar{\mathscr{F}}^{(p^j)})$ is an increasing function of j. However, by the finite-order condition it takes only a finite number of values. Therefore it must be identically constant. This finishes the proof of part (ii).

Now we prove part (iii) relying on Proposition 5. First let $p=q$. Then for $\psi=\phi^k$, with k coprime to p,

$$N(\psi,\mathbb{F}_p) = \sum \dim_{\mathbb{F}_p} HF^{\mathrm{loc}}(\psi,\mathcal{F}) = \sum \dim_{\mathbb{F}_p} H(\mathcal{F};\mathbb{F}_p), \tag{27}$$

the sum running over all contractible generalized fixed points $\mathcal{F}$ of ψ, since by Proposition 5,

$$HF^{\mathrm{loc}}(\psi,\mathcal{F}) \cong H(\mathcal{F};\mathbb{F}_p)$$

for all generalized fixed points $\mathcal{F}$. We remark that $H(\mathcal{F};\mathbb{F}_p)\neq 0$. By Proposition 23, we know that

$$N(\psi,\mathbb{F}_p) \geq \dim_{\mathbb{F}_p} H(M;\mathbb{F}_p).$$

On the other hand, by the classical Smith inequality [99; 25; 4], we have

$$\sum \dim_{\mathbb{F}_p} H(\mathcal{F};\mathbb{F}_p) \leq \dim_{\mathbb{F}_p} H(M;\mathbb{F}_p), \tag{28}$$

the sum running over all the generalized fixed points of ψ. This yields

$$N(\psi,\mathbb{F}_p) = \dim_{\mathbb{F}_p} H(M;\mathbb{F}_p).$$

This finishes the proof of the first statement of part (iii).

To prove the second statement of part (iii), we first note that for $\mathrm{char}(\mathbb{K}) = p$,

$$\mathrm{Spec}^{\mathrm{ess}}(H,\mathbb{K}) = \mathrm{Spec}^{\mathrm{ess}}(H,\mathbb{F}_p) \quad \text{and} \quad \mathrm{Spec}^{\mathrm{vis}}(H,\mathbb{K}) = \mathrm{Spec}^{\mathrm{vis}}(H,\mathbb{F}_p)$$

by Proposition 23, and the equality

$$\mathrm{Spec}^{\mathrm{ess}}(H,\mathbb{F}_p) = \mathrm{Spec}^{\mathrm{vis}}(H,\mathbb{F}_p)$$

follows by the first statement of part (iii) and Proposition 23. For the last part, we note that by Proposition 5, $\mathrm{Spec}^{\mathrm{vis}}(H,\mathbb{K}) = \mathrm{Spec}(H)$ because

$$\dim HF^{\mathrm{loc}}(H,\bar{\mathscr{F}}) = \dim HF^{\mathrm{loc}}(\phi,\mathcal{F}) = \dim H(\mathcal{F};\mathbb{K}) > 0$$

for all capped contractible generalized 1–periodic orbits $\bar{\mathscr{F}}$ of H. Now for k coprime to q, $\mathrm{Spec}(H^{(k)}) = \{\mathcal{A}_{H^{(k)}}(\bar{\mathscr{F}}^{(k)} \# A)\}$, where the set runs over all $A \in \Gamma$, and $\bar{\mathscr{F}}$ runs over all capped contractible generalized 1–periodic orbits $\bar{\mathscr{F}}$ of H. Indeed, all the contractible generalized fixed points of ϕ^k are of the form $\mathcal{F}^{(k)}$ for $\mathcal{F}$ a contractible generalized fixed point of ϕ, and the identity quickly follows. Now using the homogeneity and the recapping properties of the action functional, we obtain

$$\mathrm{Spec}(H^{(k)}) = k\cdot\mathrm{Spec}(H) + \rho\cdot\mathbb{Z}.$$

Combined with the identities $\mathrm{Spec}^{\mathrm{ess}}(H^{(k)};\mathbb{K}) = \mathrm{Spec}^{\mathrm{vis}}(H^{(k)};\mathbb{K}) = \mathrm{Spec}(H^{(k)})$ and $\mathrm{Spec}^{\mathrm{ess}}(H;\mathbb{K}) = \mathrm{Spec}^{\mathrm{vis}}(H;\mathbb{K}) = \mathrm{Spec}(H)$, this finishes the proof.

5.10 Proof of Theorem H

First assume that ψ is of prime order p. Then the proof follows from equations (27) and (28). Indeed, the upper bound holds for *all* the generalized fixed points of ψ, and the lower bound $N(\psi,\mathbb{F}_p) \geq \dim_{\mathbb{F}_p} H(M;\mathbb{F}_p)$ takes into account *only contractible* generalized fixed points. If ψ had a noncontractible generalized fixed point, it would contribute $\dim_{\mathbb{F}_p} H(\mathcal{F};\mathbb{F}_p) > 0$ to the sum, making the equality impossible. Alternatively, one can argue by means of a suitable generalization of Theorem N with $p \neq q$.

Now suppose that ψ is of order p^k, with $k \geq 1$. As in Section 5.9, by Proposition 5

$$HF^{\mathrm{loc}}(\psi,\mathcal{F}) \cong H(\mathcal{F};\mathbb{F}_p)$$

for all generalized fixed points $\mathcal{F}$; and

$$N(\psi,\mathbb{F}_p) = \sum \dim_{\mathbb{F}_p} HF^{\mathrm{loc}}(\psi,\mathcal{F}) = \sum \dim_{\mathbb{F}_p} H(\mathcal{F};\mathbb{F}_p), \tag{29}$$

the sum running over all contractible generalized fixed points $\mathcal{F}$ of ψ. Moreover, by Proposition 23, we have

$$N(\psi, \mathbb{F}_p) \geq \dim_{\mathbb{F}_p} H(M; \mathbb{F}_p).$$

Finally, by the Smith inequality for finite p–groups [99; 25; 4] we again have

$$\sum \dim_{\mathbb{F}_p} H(\mathcal{F}; \mathbb{F}_p) \leq \dim_{\mathbb{F}_p} H(M; \mathbb{F}_p), \tag{30}$$

the sum running over all the generalized fixed points of ψ. Now, as in the case of order p, if ψ had a noncontractible generalized fixed point, it would contribute $\dim_{\mathbb{F}_p} H(\mathcal{F}; \mathbb{F}_p) > 0$ to the sum, making it impossible for (30) and (29) to hold simultaneously.

For ψ of arbitrary integer order $d = p_1^{k_1} \cdots p_m^{k_m}$, we proceed by induction. We have already shown the base of induction. Now we suppose that the result is true for all orders having at most $m-1$ distinct prime divisors, and prove it for ψ of order d as above. Then $\psi_1 = \psi^{p_1^{k_1}}$ is of order $d/p_1^{k_1}$, which has exactly $m-1$ prime divisors, and hence by induction all the fixed points of ψ_1 are contractible. This implies that the order of the homotopy class of each fixed point of ψ divides $p_1^{k_1}$. In the same way, we obtain that this order also divides $p_j^{k_j}$ for all $1 \leq j \leq m$, and therefore it divides $\gcd(p_1^{k_1}, \dots, p_m^{k_m}) = 1$. Therefore each fixed point of ψ is contractible. This finishes the proof.

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