A higher-rank rigidity theorem for
convex real projective manifolds

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For convex real projective manifolds we prove an analogue of the higher-rank rigidity theorem of Ballmann and Burns and Spatzier.

1 Introduction

A real projective structure on a d–manifold M is an open cover $M = \bigcup_\alpha U_\alpha$ along with coordinate charts $\varphi_\alpha \colon U_\alpha \to \mathbb{P}(\mathbb{R}^{d+1})$ such that each transition function $\varphi_\alpha \circ \varphi_\beta^{-1}$ coincides with the restriction of an element in $\mathrm{PGL}_{d+1}(\mathbb{R})$. A *real projective manifold* is a manifold equipped with a real projective structure.

An important class of real projective manifolds is the convex real projective manifolds, which are defined as follows. First, a subset $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ is called a *properly convex domain* if there exists an affine chart which contains it as a bounded convex open set. In this case, the *automorphism group of* Ω is

$$\mathrm{Aut}(\Omega) := \{g \in \mathrm{PGL}_{d+1}(\mathbb{R}) : g\Omega = \Omega\}.$$

If $\Gamma \leq \mathrm{Aut}(\Omega)$ is a discrete subgroup that acts freely and properly discontinuously on Ω, then the quotient manifold $\Gamma \backslash \Omega$ is called a *convex real projective manifold*. Notice that local inverses to the covering map $\Omega \to \Gamma \backslash \Omega$ provide a real projective structure on the quotient. In the case when there exists a compact quotient, the domain Ω is called *divisible*. For more background see the expository papers by Benoist [7], Marquis [22] and Quint [25].

When $d \leq 3$, the structure of closed convex real projective d–manifolds is very well understood thanks to deep work of Benzécri [9], Goldman [16] and Benoist [6]. But, when $d \geq 4$, their general structure is mysterious.

We establish a dichotomy for convex real projective manifolds inspired by the theory of nonpositively curved Riemannian manifolds. In particular, a compact Riemannian manifold (M, g) with nonpositive curvature is said to have *higher rank* if every geodesic in the universal cover is contained in a totally geodesic subspace isometric to $\mathbb{R}^2$. Otherwise, (M, g) is said to have *rank one*. An important theorem of Ballmann [2] and Burns and Spatzier [11; 12] states that every compact irreducible Riemannian manifold with nonpositive curvature and higher rank is a locally symmetric space. This foundational result reduces many problems about nonpositively curved manifolds to the rank-one case. Further, rank-one manifolds possess very useful "weakly hyperbolic behavior" (see for instance Ballmann [1] and Knieper [20]).

In the context of convex real projective manifolds, the natural analogue of isometrically embedded copies of $\mathbb{R}^2$ are properly embedded simplices, see Section 2.6 below, which leads to a definition of higher rank:

Definition 1.1   (i)   A properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ has *higher rank* if for every $p, q \in \Omega$ there exists a properly embedded simplex $S \subset \Omega$ with $\dim(S) \geq 2$ and $[p, q] \subset S$.

(ii)   If a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ does not have higher rank, then we say that Ω has *rank one*.

There are two basic families of properly convex domains with higher rank: reducible domains (see Section 2.4) and symmetric domains with real rank at least two.

A properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is called *symmetric* if there exists a semisimple Lie group $G \leq \mathrm{PGL}_d(\mathbb{R})$ which preserves Ω and acts transitively. In this case, the *real rank of* Ω is defined to be the real rank of G. Koecher and Vinberg characterized the irreducible symmetric properly convex domains and proved that G must be locally isomorphic to either

(i) $\mathrm{SO}(1, m)$ with $d = m + 1$,

(ii) $\mathrm{SL}_m(\mathbb{R})$ with $d = \frac{1}{2}(m^2 + m)$,

(iii) $\mathrm{SL}_m(\mathbb{C})$ with $d = m^2$,

(iv) $\mathrm{SL}_m(\mathbb{H})$ with $d = 2m^2 - m$, or

(v) $E_{6(-26)}$ with $d = 27$.

For details see Faraut and Korányi [15], Koecher [21] and Vinberg [28; 29]. Borel [10] proved that every semisimple Lie group contains a cocompact lattice, which implies that every symmetric properly convex domain is divisible.

We prove that these two families of examples are the only divisible domains with higher rank. In fact, we show that being symmetric with real rank at least two is equivalent to a number of other "higher rank" conditions. Before stating the main result we need a few more definitions.

Definition 1.2   • Given $g \in \mathrm{PGL}_d(\mathbb{R})$, let

$$\lambda_1(g) \geq \lambda_2(g) \geq \cdots \geq \lambda_d(g)$$

denote the absolute values of the eigenvalues of some (hence any) lift of g to $\mathrm{SL}_d^{\pm}(\mathbb{R}) := \{h \in \mathrm{GL}_d(\mathbb{R}) : \det h = \pm 1\}$.

   • $g \in \mathrm{PGL}_d(\mathbb{R})$ is *proximal* if $\lambda_1(g) > \lambda_2(g)$. In this case, let $\ell_g^+ \in \mathbb{P}(\mathbb{R}^d)$ denote the eigenline of g corresponding to $\lambda_1(g)$.

   • $g \in \mathrm{PGL}_d(\mathbb{R})$ is *biproximal* if g and g^{-1} are both proximal. In this case, define $\ell_g^- := \ell_{g^{-1}}^+$.

Next we define a distance on the boundary using projective line segments:

Definition 1.3   Given a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ the (possibly infinite valued) *simplicial distance* on $\partial\Omega$ is defined by

$s_{\partial\Omega}(x, y)$
$= \inf\{k : \exists a_0, \ldots, a_k \text{ with } x = a_0, y = a_k \text{ and } [a_j, a_{j+1}] \subset \partial\Omega \text{ for } 0 \leq j \leq k-1\}.$

We will prove a characterization of higher rank in the context of convex real projective manifolds:

Theorem 1.4   (see Section 9)   *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is an irreducible properly convex domain and $\Gamma \leq \mathrm{Aut}(\Omega)$ is a discrete group acting cocompactly on Ω. Then the following are equivalent*:

   (i) Ω *is symmetric with real rank at least two.*

   (ii) Ω *has higher rank.*

   (iii)   *The extreme points of Ω form a closed proper subset of $\partial\Omega$.*

   (iv) $[x_1, x_2] \subset \partial\Omega$ *for every two extreme points $x_1, x_2 \in \partial\Omega$.*

   (v) $s_{\partial\Omega}(x, y) \leq 2$ *for all $x, y \in \partial\Omega$.*

   (vi) $s_{\partial\Omega}(x, y) < +\infty$ *for all $x, y \in \partial\Omega$.*

   (vii) Γ *has higher rank in the sense of Prasad and Raghunathan (see Section 8).*

(viii) *For every $g \in \Gamma$ with infinite order, the cyclic group $g^{\mathbb{Z}}$ has infinite index in the centralizer of g in Γ.*

(ix) *Every $g \in \Gamma$ with infinite order has at least three fixed points in $\partial\Omega$.*

(x) *$[\ell_g^+, \ell_g^-] \subset \partial\Omega$ for every biproximal element $g \in \Gamma$.*

(xi) *$s_{\partial\Omega}(\ell_g^+, \ell_g^-) < +\infty$ for every biproximal element $g \in \Gamma$.*

M Islam [18] has recently defined and studied rank-one isometries of a properly convex domain. These are analogous to the classical definition of rank-one isometries of CAT(0) spaces (see [1]) and are defined as follows:

Definition 1.5 (Islam [18]) Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain. An element $g \in \text{Aut}(\Omega)$ is a *rank-one isometry* if g is biproximal and $s_{\partial\Omega}(\ell_g^+, \ell_g^-) > 2$.

Remark 1.6 (1) When $g \in \text{Aut}(\Omega)$ is a rank-one isometry, the properly embedded line segment $(\ell_g^+, \ell_g^-) \subset \Omega$ is preserved by g. Further, g acts by translations on (ℓ_g^+, ℓ_g^-) in the following sense: if H_Ω is the Hilbert metric on Ω, then there exists $T > 0$ such that

$$H_\Omega(g^n(x), x) = nT$$

for all $n \geq 0$ and $x \in (\ell_g^+, \ell_g^-)$.

(2) Islam [18, Proposition 6.3] also proved a weaker characterization of rank-one isometries: $g \in \text{Aut}(\Omega)$ is a rank-one isometry if and only if g acts by translations on a properly embedded line segment $(a, b) \subset \Omega$ and $s_{\partial\Omega}(a, b) > 2$.

As an immediate consequence of Theorem 1.4:

Corollary 1.7 *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is an irreducible properly convex domain and $\Gamma \leq \text{Aut}(\Omega)$ is a discrete group acting cocompactly on Ω. Then the following are equivalent*:

(i) *Ω has rank one.*

(ii) *Γ contains a rank-one isometry.*

Islam has also established a number of remarkable results when the automorphism group contains a rank-one isometry; see [18] for details. For instance:

Corollary 1.8 (consequence of Theorem 1.4 and [18, Theorem 1.5]) *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is an irreducible properly convex domain and $\Gamma \leq \text{Aut}(\Omega)$ is a discrete group acting cocompactly on Ω. If $d \geq 3$ and Ω is not symmetric with real rank at least two, then Γ is an acylindrically hyperbolic group.*

1.1 Outline of the proof of Theorem 1.4

The difficult part is showing that any one of conditions (ii)–(xi) implies that the domain is symmetric with real rank at least two.

One key idea is to construct and study special semigroups in $\mathbb{P}(\mathrm{End}(\mathbb{R}^d))$ associated to each boundary face. This is accomplished as follows. First, motivated by a lemma of Benoist [5, Lemma 2.2], we consider a compactification of a subgroup of $\mathrm{PGL}_d(\mathbb{R})$:

Definition 1.9 Given a subgroup $G \leq \mathrm{PGL}_d(\mathbb{R})$ let

$$\overline{G}^{\mathrm{End}} \subset \mathbb{P}(\mathrm{End}(\mathbb{R}^d))$$

denote the closure of G in $\mathbb{P}(\mathrm{End}(\mathbb{R}^d))$.

Next, for a dividing group, we introduce subsets of this compactification:

Definition 1.10 Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $\Gamma \leq \mathrm{Aut}(\Omega)$ is a discrete group acting cocompactly on Ω. If $F \subset \partial\Omega$ is a boundary face and $V := \mathrm{Span}\, F \subset \mathbb{R}^d$, then define

$$\overline{\Gamma}_F^{\mathrm{End}} := \{T \in \overline{\Gamma}^{\mathrm{End}} : \mathrm{image}(T) \subset V\}$$

and

$$\overline{\Gamma}_{F,\star}^{\mathrm{End}} := \{T \in \overline{\Gamma}^{\mathrm{End}} : \mathrm{image}(T) = V \text{ and } \ker(T) \cap V = \{0\}\}.$$

We then prove:

Theorem 3.1 *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is an irreducible properly convex domain and $\Gamma \leq \mathrm{Aut}(\Omega)$ is a discrete group acting cocompactly on Ω. If Ω is nonsymmetric, $F \subset \partial\Omega$ is a boundary face, $V := \mathrm{Span}\, F \subset \mathbb{R}^d$, and $\dim(V) \geq 2$, then:*

(a) *If $T \in \overline{\Gamma}_F^{\mathrm{End}}$, then $T(\Omega) \subset \overline{F}$.*

(b) *If $T \in \overline{\Gamma}_{F,\star}^{\mathrm{End}}$, then $T(F)$ is an open subset of F.*

(c) *The set*

$$\{T|_V : T \in \overline{\Gamma}_{F,\star}^{\mathrm{End}}\}$$

is a nondiscrete Zariski-dense semigroup in $\mathbb{P}(\mathrm{End}(V))$.

Using Theorem 3.1 we will show that any one of Theorem 1.4(ii)–(xi) implies that the domain is symmetric with real rank at least two. Here is a sketch of the argument: First suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is an irreducible properly convex domain, $\Gamma \leq \mathrm{Aut}(\Omega)$ is a discrete group acting cocompactly on Ω, and any one of Theorem 1.4(ii)–(xi) is true.

Then let $\mathcal{E}_\Omega \subset \partial\Omega$ denote the extreme points of Ω. We will show that there exists a boundary face $F \subset \partial\Omega$ such that

$$\text{(1)} \qquad\qquad F \cap \bar{\mathcal{E}}_\Omega = \varnothing.$$

By choosing F minimally, we can also assume that $\bar{\mathcal{E}}_\Omega$ intersects every boundary face of strictly smaller dimension. As before, let $V := \operatorname{Span} F$. Then using (1) we show that $T|_V \in \operatorname{Aut}(F)$ for every $T \in \overline{\Gamma}_{F,\star}^{\operatorname{End}}$. Therefore Theorem 3.1 implies that either Ω is symmetric or $\operatorname{Aut}(F)$ is a nondiscrete Zariski-dense subgroup of $\operatorname{PGL}(V)$. In the latter case, it is fairly easy to deduce that $\operatorname{PSL}(V) \subset \operatorname{Aut}(F)$, see Lemma 4.5 below, which is impossible. So Ω must be symmetric.

1.2 Outline of the paper

In Section 2 we recall some preliminary material. In Section 3 we prove Theorem 3.1. In Section 4 we prove the rigidity result mentioned in the previous subsection.

The rest of the paper is devoted to the proof of the various equivalences in Theorem 1.4. In Sections 5, 6, and 7 we prove some new results about the action of the automorphism group. In Section 8 we consider the rank of a group in the sense of Prasad and Raghunathan. Finally, in Section 9 we prove Theorem 1.4.

Acknowledgments

2 Preliminaries

2.1 Notation

Given a linear subspace $V \subset \mathbb{R}^d$, we let $\mathbb{P}(V) \subset \mathbb{P}(\mathbb{R}^d)$ denote its projectivization. In all other cases, given some object o, we will let $[o]$ be the projective equivalence class of o. For instance:

(i) If $v \in \mathbb{R}^d \setminus \{0\}$, let $[v]$ denote the image of v in $\mathbb{P}(\mathbb{R}^d)$.

(ii) If $\phi \in \operatorname{GL}_d(\mathbb{R})$, let $[\phi]$ denote the image of ϕ in $\operatorname{PGL}_d(\mathbb{R})$.

(iii) If $T \in \operatorname{End}(\mathbb{R}^d) \setminus \{0\}$, let $[T]$ denote the image of T in $\mathbb{P}(\operatorname{End}(\mathbb{R}^d))$.

We also identify $\mathbb{P}(\mathbb{R}^d) = \mathrm{Gr}_1(\mathbb{R}^d)$, so for instance if $x \in \mathbb{P}(\mathbb{R}^d)$ and $V \subset \mathbb{R}^d$ is a linear subspace, then $x \in \mathbb{P}(V)$ if and only if $x \subset V$.

Finally, given a subset X of $\mathbb{R}^d$ (respectively $\mathbb{P}(\mathbb{R}^d)$), we will let Span $X \subset \mathbb{R}^d$ denote the smallest linear subspace containing X (respectively the preimage of X).

2.2 Convexity and line segments

A subset $C \subset \mathbb{P}(\mathbb{R}^d)$ is called *convex* if there exists an affine chart which contains it as a convex subset. A subset $C \subset \mathbb{P}(\mathbb{R}^d)$ is called *properly convex* if there exists an affine chart which contains it as a bounded convex subset. For convex subsets, we make some topological definitions:

Definition 2.1 Let $C \subset \mathbb{P}(\mathbb{R}^d)$ be a convex set. The *relative interior of C*, denoted by rel-int(C), is the interior of C in its span and the *boundary of C* is $\partial C := \overline{C} \setminus \mathrm{rel\text{-}int}(C)$.

A *line segment* in $\mathbb{P}(\mathbb{R}^d)$ is a connected subset of a projective line. Given two points $x, y \in \mathbb{P}(\mathbb{R}^d)$ there is no canonical line segment with endpoints x and y, but we will use the convention that if $C \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex set and $x, y \in \overline{C}$, then (when the context is clear) we will let $[x, y]$ denote the closed line segment joining x to y which is contained in $\overline{C}$. In this case, we will also let $(x, y) = [x, y] \setminus \{x, y\}$, $[x, y) = [x, y] \setminus \{y\}$, and $(x, y] = [x, y] \setminus \{x\}$.

2.3 Irreducibility

A subgroup $\Gamma \leq \mathrm{PGL}_d(\mathbb{R})$ is *irreducible* if $\{0\}$ and $\mathbb{R}^d$ are the only Γ–invariant linear subspaces of $\mathbb{R}^d$, and *strongly irreducible* if every finite-index subgroup is irreducible.

We will use the following observation several times:

Observation 2.2 *If $\Gamma \leq \mathrm{PGL}_d(\mathbb{R})$ is strongly irreducible, $x_1, \ldots, x_k \in \mathbb{P}(\mathbb{R}^d)$, and*

$$V_1, \ldots, V_k \subsetneq \mathbb{R}^d$$

are linear subspaces, then there exists $g \in \Gamma$ such that $g x_j \notin \mathbb{P}(V_j)$ for all $1 \leq j \leq k$.

Proof Let $G = \overline{\Gamma}^{\mathrm{Zar}}$ denote the Zariski closure of Γ in $\mathrm{PGL}_d(\mathbb{R})$ and let $G_0 \leq G$ denote the connected component of the identity of G (in the Zariski topology). Then $G_0 \cap \Gamma$ is a finite-index subgroup of Γ and hence G_0 is irreducible. So each set

$$\mathcal{O}_j = \{g \in G_0 : g x_j \notin \mathbb{P}(V_j)\}$$

is nonempty and Zariski open in G_0. Hence $\mathcal{O} = \bigcap_{j=1}^k \mathcal{O}_j$ is nonempty and Zariski open in G_0. Since $\Gamma \cap G_0$ is Zariski dense in G_0, there exists some $g \in \Gamma \cap \mathcal{O}$. $\qquad\square$

2.4 Zariski closures

An open convex cone $C \subset \mathbb{R}^d$ is *reducible* if there exists a nontrivial vector space decomposition $\mathbb{R}^d = V_1 \oplus V_2$ and convex cones $C_1 \subset V_1$ and $C_2 \subset V_2$ such that $C = C_1 + C_2$. Otherwise, C is said to be *irreducible*. The preimage in $\mathbb{R}^d$ of a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is the union of a cone and its negative; when this cone is reducible (respectively irreducible) we say that Ω is *reducible* (respectively *irreducible*).

Benoist determined the Zariski closures of discrete groups acting cocompactly on irreducible properly convex domains:

Theorem 2.3 (Benoist [5]) *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is an irreducible properly convex domain and $\Gamma \leq \mathrm{Aut}(\Omega)$ is a discrete group acting cocompactly on Ω. Then either*

(i) *Ω is symmetric, or*

(ii) *Γ is Zariski dense in $\mathrm{PGL}_d(\mathbb{R})$.*

2.5 The Hilbert distance

In this section we recall the definition of the Hilbert metric. But first some notation:

Given a projective line $L \subset \mathbb{P}(\mathbb{R}^d)$ and four distinct points $a, x, y, b \in L$ we define the *cross ratio* by
$$[a, x, y, b] = \frac{|x - b||y - a|}{|x - a||y - b|},$$
where $|\cdot|$ is some (any) norm in some (any) affine chart of $\mathbb{P}(\mathbb{R}^d)$ containing a, x, y and b.

Next, for $x, y \in \mathbb{P}(\mathbb{R}^d)$ distinct, let $L_{x,y} \subset \mathbb{P}(\mathbb{R}^d)$ denote the projective line containing x and y.

Definition 2.4 Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain. The *Hilbert distance* on Ω, denoted by H_Ω, is defined as follows: if $x, y \in \Omega$ are distinct, then
$$H_\Omega(x, y) = \tfrac{1}{2} \log[a, x, y, b],$$
where $\partial\Omega \cap L_{x,y} = \{a, b\}$ with the ordering a, x, y, b along $L_{x,y}$.

The following result is classical; see for instance [13, Section 28].

Proposition 2.5 *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain. Then H_Ω is a complete $\mathrm{Aut}(\Omega)$–invariant metric on Ω which generates the standard topology on Ω. Moreover, if $p, q \in \Omega$, then there exists a geodesic joining p and q whose image is the line segment $[p, q]$.*

2.6 Properly embedded simplices

In this subsection we recall the definition of properly embedded simplices.

Definition 2.6 A subset $S \subset \mathbb{P}(\mathbb{R}^d)$ is a *simplex* if there exists $g \in \mathrm{PGL}_d(\mathbb{R})$ and $k \geq 0$ such that

$$gS = \{[x_1 : \cdots : x_{k+1} : 0 : \cdots : 0] \in \mathbb{P}(\mathbb{R}^d) : x_1 > 0, \ldots, x_{k+1} > 0\}.$$

In this case, we write $\dim(S) = k$ (notice that S is homeomorphic to $\mathbb{R}^k$).

Definition 2.7 Suppose that $A \subset B \subset \mathbb{P}(\mathbb{R}^d)$. Then A is *properly embedded in B* if the inclusion map $A \hookrightarrow B$ is a proper map (relative to the subspace topology).

By [23, Proposition 1.7], [17], or [26] the Hilbert metric on a simplex is isometric to a normed space, and so:

Observation 2.8 *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $S \subset \Omega$ is a properly embedded simplex. Then (S, H_Ω) is quasi-isometric to $\mathbb{R}^{\dim S}$.*

2.7 Limits of linear maps

Every $T \in \mathbb{P}(\mathrm{End}(\mathbb{R}^d))$ induces a map

$$\mathbb{P}(\mathbb{R}^d) \setminus \mathbb{P}(\ker T) \to \mathbb{P}(\mathbb{R}^d)$$

defined by $x \to T(x)$. We will frequently use:

Observation 2.9 *If $(T_n)_{n \geq 1}$ converges in $\mathbb{P}(\mathrm{End}(\mathbb{R}^d))$ to $T \in \mathbb{P}(\mathrm{End}(\mathbb{R}^d))$, then*

$$T(x) = \lim_{n \to \infty} T_n(x)$$

for all $x \in \mathbb{P}(\mathbb{R}^d) \setminus \mathbb{P}(\ker T)$. Moreover, the convergence is uniform on compact subsets of $\mathbb{P}(\mathbb{R}^d) \setminus \mathbb{P}(\ker T)$.

2.8 The faces and extreme points of a properly convex domain

Definition 2.10 Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain. For $x \in \overline{\Omega}$ let $F_\Omega(x)$ denote the (*open*) *face* of x; that is,

$$F_\Omega(x) = \{x\} \cup \{y \in \overline{\Omega} : \exists \text{ an open line segment in } \overline{\Omega} \text{ containing } x \text{ and } y\}.$$

If $x \in \partial\Omega$ and $F_\Omega(x) = \{x\}$, then x is called an *extreme point of* Ω. Finally, let

$$\mathcal{E}_\Omega \subset \partial\Omega$$

denote the set of all extreme points.

These subsets have some basic properties:

Observation 2.11 *Suppose that* $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ *is a properly convex domain.*

(i) *If* $x \in \Omega$, *then* $F_\Omega(x) = \Omega$.

(ii) $F_\Omega(x)$ *is open in its span.*

(iii) $y \in F_\Omega(x)$ *if and only if* $x \in F_\Omega(y)$ *if and only if* $F_\Omega(x) = F_\Omega(y)$.

(iv) *If* $y \in \partial F_\Omega(x)$, *then* $F_\Omega(y) \subset \partial F_\Omega(x)$ *and* $F_\Omega(y) = F_{F_\Omega(x)}(y)$.

(v) *If* $x, y \in \overline{\Omega}$ *and* $z \in (x, y)$, *then*

$$(p, q) \subset F_\Omega(z)$$

for all $p \in F_\Omega(x)$ *and* $q \in F_\Omega(y)$.

Proof These are all simple consequences of convexity. □

We will also use results about the action of the automorphism group:

Proposition 2.12 [19, Proposition 5.6] *Suppose that* $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ *is a properly convex domain,* $p_0 \in \Omega$, *and* $(g_n)_{n \geq 1}$ *is a sequence in* $\mathrm{Aut}(\Omega)$ *such that*

(i) $g_n(p_0) \to x \in \partial\Omega$,

(ii) $g_n^{-1}(p_0) \to y \in \partial\Omega$, *and*

(iii) g_n *converges in* $\mathbb{P}(\mathrm{End}(\mathbb{R}^d))$ *to* $T \in \mathbb{P}(\mathrm{End}(\mathbb{R}^d))$.

Then image $T \subset \mathrm{Span}\, F_\Omega(x)$, $\mathbb{P}(\ker T) \cap \Omega = \varnothing$, *and* $y \in \mathbb{P}(\ker T)$.

In the case of "nontangential" convergence we can say more:

Proposition 2.13 [19, Proposition 5.7] *Suppose that* $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ *is a properly convex domain,* $p_0 \in \Omega$, $x \in \partial\Omega$, $(p_n)_{n \geq 1}$ *is a sequence in* $[p_0, x)$ *converging to* x, *and* $(g_n)_{n \geq 1}$ *is a sequence in* $\mathrm{Aut}(\Omega)$ *such that*

$$\sup_{n \geq 1} H_\Omega(g_n(p_0), p_n) < +\infty.$$

If g_n *converges in* $\mathbb{P}(\mathrm{End}(\mathbb{R}^d))$ *to* $T \in \mathbb{P}(\mathrm{End}(\mathbb{R}^d))$, *then*

$$T(\Omega) = F_\Omega(x),$$

and hence image $T = \mathrm{Span}\, F_\Omega(x)$.

Proposition 5.7 in [19] is stated differently, so we provide the proof:

Proof Proposition 2.12 implies $T(\Omega) \subset F_\Omega(x)$, so we have to prove $T(\Omega) \supset F_\Omega(x)$.
Fix $y \in F_\Omega(x)$. Then we can pick a sequence $(y_n)_{n \geq 1}$ in $[p_0, y)$ such that

$$\sup_{n \geq 1} H_\Omega(y_n, p_n) < \infty.$$

Thus

$$\sup_{n \geq 1} H_\Omega(g_n^{-1}(y_n), p_0) < \infty.$$

So there exists $n_j \to \infty$ such that the limit

$$q := \lim_{j \to \infty} g_{n_j}^{-1}(y_{n_j})$$

exists in Ω. Notice that $q \notin \mathbb{P}(\ker T)$ by Proposition 2.12 and so the "moreover" part
of Observation 2.9 implies that

$$T(q) = \lim_{n \to \infty} g_n(q) = \lim_{j \to \infty} g_{n_j}(q) = \lim_{j \to \infty} g_{n_j}(g_{n_j}^{-1}(y_{n_j})) = \lim_{j \to \infty} y_{n_j} = y.$$

Since y was arbitrary, $F_\Omega(x) \subset T(\Omega)$. □

2.9 Proximal elements

In this section we recall some basic properties of proximal elements. For more background we refer the reader to [8].

Definition 2.14 Suppose that $F : M \to M$ is a C^1 map of a manifold M. Then a fixed point $x \in M$ of F is *attractive* if $|\lambda| < 1$ for every eigenvalue λ of $d(F)_x : T_x M \to T_x M$.

A straightforward calculation provides a characterization of proximality:

Observation 2.15 *Suppose that $g \in \mathrm{PGL}_d(\mathbb{R})$ and x is a fixed point of the g action on $\mathbb{P}(\mathbb{R}^d)$. Then the following are equivalent*:

  (i) *x is an attractive fixed point of g.*

  (ii) *g is proximal and $x = \ell_g^+$.*

Next we explain the global dynamics of a proximal element.

Definition 2.16 If $g \in \mathrm{PGL}_d(\mathbb{R})$ is proximal, then define $H_g^- \in \mathrm{Gr}_{d-1}(\mathbb{R}^d)$ to be the unique g–invariant linear hyperplane with

$$\ell_g^+ \oplus H_g^- = \mathbb{R}^d.$$

If g is biproximal, then also define $H_g^+ := H_{g^{-1}}^-$.

When $g \in \mathrm{PGL}_d(\mathbb{R})$ is proximal, H_g^- is usually called the repelling hyperplane of g. This is motivated by the following observation:

Observation 2.17 *If $g \in \mathrm{PGL}_d(\mathbb{R})$ is proximal, then*

$$T_g := \lim_{n \to \infty} g^n$$

exists in $\mathbb{P}(\mathrm{End}(\mathbb{R}^d))$. Moreover, image $T_g = \ell_g^+$, $\ker T_g = H_g^-$, and

$$\text{image } T_g \oplus \ker T_g = \mathbb{R}^d.$$

Hence

$$\ell_g^+ = \lim_{n \to \infty} g^n x$$

for all $x \in \mathbb{P}(\mathbb{R}^d) \setminus \mathbb{P}(H_g^-)$.

Observation 2.18 *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain. If $g \in \mathrm{Aut}(\Omega)$ is proximal, then ℓ_g^+ is an extreme point of $\partial \Omega$ and $\mathbb{P}(H_g^-) \cap \partial \Omega = \varnothing$.*

Proof Proposition 2.12 implies that $\ell_g^+ \in \partial \Omega$ and $\mathbb{P}(H_g^-) \cap \partial \Omega = \varnothing$. Let $F = F_\Omega(\ell_g^+)$ and $V = \mathrm{Span}\, F$. Then $g(V) = V$. Let $\bar{g} \in \mathrm{GL}_d(\mathbb{R})$ be a lift of $g \in \mathrm{PGL}_d(\mathbb{R})$ and let $h \in \mathrm{GL}(V)$ denote the element obtained by restricting $\bar{g}$ to V. Notice that h is proximal since $\ell_g^+ \subset V$. Further $[h] \in \mathrm{Aut}(F)$ and $h(\ell_g^+) = \ell_g^+$. Since $\mathrm{Aut}(F)$ acts properly on F and $\ell_g^+ \in F$, the cyclic group

$$[h]^{\mathbb{Z}} \leq \mathrm{Aut}(F) \leq \mathrm{PGL}(V)$$

must be relatively compact. This implies that every eigenvalue of h has the same absolute value. Then, since h is proximal, V must be one-dimensional and so $F = \{\ell_g^+\}$. Thus ℓ_g^+ is an extreme point. $\qquad \square$

The following result can be viewed as a converse to Observation 2.17 and will be used to construct proximal elements.

Proposition 2.19 *Suppose that $(g_n)_{n \geq 1}$ is a sequence in $\mathrm{PGL}_d(\mathbb{R})$ and*

$$T := \lim_{n \to \infty} g_n$$

exists in $\mathbb{P}(\mathrm{End}(\mathbb{R}^d))$. If $\dim(\mathrm{image}\, T) = 1$ and

$$\text{image } T \oplus \ker T = \mathbb{R}^d,$$

then, for n sufficiently large, g_n is proximal and

$$\text{image } T = \lim_{n \to \infty} \ell_{g_n}^+.$$

Proof Since $g_n \to T$ in $\mathbb{P}(\mathrm{End}(\mathbb{R}^d))$,

$$\lim_{n\to\infty} g_n(x) = T(x) = \mathrm{image}\, T \in \mathbb{P}(\mathbb{R}^d)$$

for all $x \in \mathbb{P}(\mathbb{R}^d) \setminus \mathbb{P}(\ker T)$. Moreover, the convergence is uniform on compact subsets of $\mathbb{P}(\mathbb{R}^d) \setminus \mathbb{P}(\ker T)$.

By assumption,

$$\mathrm{image}\, T \notin \mathbb{P}(\ker T),$$

so we can find a compact neighborhood U of image T in $\mathbb{P}(\mathbb{R}^d)$ such that U is homeomorphic to a closed ball and

$$U \cap \mathbb{P}(\ker T) = \varnothing.$$

Then, by passing to a tail, we can assume that $g_n(U) \subset U$ for all n. So, by the Brouwer fixed-point theorem, each g_n has a fixed point $x_n \in U$. Since U can be chosen arbitrarily small,

$$\mathrm{image}\, T = \lim_{n\to\infty} x_n.$$

We claim that, for n large, x_n is an attractive fixed point of g_n. By [Observation 2.15](#) this will finish the proof. Let $f_n \colon \mathbb{P}(\mathbb{R}^d) \to \mathbb{P}(\mathbb{R}^d)$ be the diffeomorphism induced by g_n, that is $f_n(x) = g_n(x)$ for all x. Then, since each g_n acts by projective linear transformations, we see that the f_n converge locally uniformly in the C^∞ topology on $\mathbb{P}(\mathbb{R}^d) \setminus \mathbb{P}(\ker T)$ to the constant map $f \equiv \mathrm{image}\, T$. So, fixing a Riemannian metric on $\mathbb{P}(\mathbb{R}^d)$, we have

$$\lim_{n\to\infty} \|d(f_n)_{x_n}\| = 0.$$

Hence, for n large, x_n is an attractive fixed point of g_n. $\square$

2.10 Rank-one isometries

In this section we state a characterization of rank-one isometries established in [18]:

Theorem 2.20 (Islam [18, Proposition 6.3]) *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $\gamma \in \mathrm{Aut}(\Omega)$. If*

$$\inf_{p\in\Omega} H_\Omega(\gamma(p), p) > 0$$

and γ fixes two points $x, y \in \partial\Omega$ with $s_{\partial\Omega}(x, y) > 2$, then:

(i) *γ is biproximal and $\{\ell_\gamma^+, \ell_\gamma^-\} = \{x, y\}$. In particular, γ is a rank-one isometry.*

(ii) *The only points fixed by γ in $\partial\Omega$ are ℓ_γ^+ and ℓ_γ^-.*

(iii) If $w \in \partial\Omega$, *then*
$$(\ell_\gamma^+, w) \cup (w, \ell_\gamma^-) \subset \Omega.$$

(iv) If $z \in \partial\Omega \setminus \{\ell_\gamma^\pm\}$, *then*
$$s_{\partial\Omega}(\ell_\gamma^\pm, z) = \infty.$$

Remark 2.21 Notice that (iv) is a consequence of (iii).

3 A semigroup associated to a boundary face

Theorem 3.1 *Suppose that* $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ *is an irreducible properly convex domain and* $\Gamma \leq \mathrm{Aut}(\Omega)$ *is a discrete group acting cocompactly on* Ω. *If* Ω *is nonsymmetric,* $F \subset \partial\Omega$ *is a boundary face,* $V := \mathrm{Span}\, F$, *and* $\dim(V) \geq 2$, *then:*

(a) *If* $T \in \overline{\Gamma}_F^{\mathrm{End}}$, *then* $T(\Omega) \subset \overline{F}$.

(b) *If* $T \in \overline{\Gamma}_{F,\star}^{\mathrm{End}}$, *then* $T(F)$ *is an open subset of* F.

(c) *The set*
$$\{T|_V : T \in \overline{\Gamma}_{F,\star}^{\mathrm{End}}\}$$

is a nondiscrete Zariski-dense semigroup in $\mathbb{P}(\mathrm{End}(V))$.

The proof of Theorem 3.1 will follow from a series of lemmas, many of which hold in greater generality.

For the rest of the section fix a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ and a subgroup $\Gamma \leq \mathrm{Aut}(\Omega)$. Notice that we are not (currently) assuming that Ω is irreducible, that Γ is discrete, or that Γ acts cocompactly on Ω.

Observation 3.2   (a) *If* $T \in \overline{\Gamma}^{\mathrm{End}}$, *then* $\mathbb{P}(\ker T) \cap \Omega = \varnothing$.

(b) *If* $S, T \in \overline{\Gamma}^{\mathrm{End}}$ *and* $\mathrm{image}\, T \setminus \ker S \neq \varnothing$, *then* $S \circ T \in \overline{\Gamma}^{\mathrm{End}}$.

Proof Part (a) follows immediately from Proposition 2.12.

For part (b), fix $S, T \in \overline{\Gamma}^{\mathrm{End}}$ with $\mathrm{image}\, T \setminus \ker S \neq \varnothing$. By hypothesis $S \circ T$ is a well-defined element of $\mathbb{P}(\mathrm{End}(\mathbb{R}^d))$. To show that $S \circ T \in \overline{\Gamma}^{\mathrm{End}}$, fix sequences $(g_n)_{n \geq 1}$ and $(h_n)_{n \geq 1}$ in Γ such that
$$S = \lim_{n\to\infty} g_n \quad \text{and} \quad T = \lim_{n\to\infty} h_n$$

in $\mathbb{P}(\mathrm{End}(\mathbb{R}^d))$. Then, since $S \circ T \neq 0$,
$$S \circ T = \lim_{n\to\infty} g_n h_n$$

in $\mathbb{P}(\mathrm{End}(\mathbb{R}^d))$. So $S \circ T \in \overline{\Gamma}^{\mathrm{End}}$. $\qquad\square$

Lemma 3.3 *If $F \subset \partial\Omega$ is a boundary face and $T \in \overline{\Gamma}_F^{\mathrm{End}}$, then $T(\Omega) \subset \overline{F}$.*

Proof Suppose $T \in \overline{\Gamma}_F^{\mathrm{End}}$. Then there exists a sequence $(g_n)_{n \geq 1}$ in Γ such that

$$T = \lim_{n \to \infty} g_n$$

in $\mathbb{P}(\mathrm{End}(\mathbb{R}^d))$. Since $\mathbb{P}(\ker T) \cap \Omega = \varnothing$,

$$T(p) = \lim_{n \to \infty} g_n(p) \in \overline{\Omega}$$

for all $p \in \Omega$. So $T(\Omega) \subset \overline{\Omega}$. Since $\mathrm{image}(T) \subset V$,

$$T(\Omega) \subset \mathbb{P}(V) \cap \overline{\Omega} = \overline{F}. \qquad \square$$

Lemma 3.4 *If $F \subset \partial\Omega$ is a boundary face and $T \in \overline{\Gamma}_{F,\star}^{\mathrm{End}}$, then $T(F)$ is an open subset of F.*

Proof By definition and [Observation 3.2]

$$(\Omega \cup F) \cap \mathbb{P}(\ker T) \subset (\Omega \cup \mathbb{P}(V)) \cap \mathbb{P}(\ker T) = \varnothing.$$

So T induces a continuous map on $\Omega \cup F$. Since $F \subset \overline{\Omega}$, the previous lemma implies that

$$T(F) \subset \overline{T(\Omega)} \subset \overline{F}.$$

Since $V \cap \ker T = \{0\}$, $T(F)$ is an open subset of $\mathbb{P}(V)$. So

$$T(F) \subset \mathrm{rel\text{-}int}(\overline{F}) = F. \qquad \square$$

Lemma 3.5 *If $F \subset \partial\Omega$ is a boundary face, then the set*

$$\{T|_V : T \in \overline{\Gamma}_{F,\star}^{\mathrm{End}}\}$$

is a semigroup in $\mathbb{P}(\mathrm{End}(V))$.

Proof Fix $T_1, T_2 \in \overline{\Gamma}_{F,\star}^{\mathrm{End}}$. Then

$$\mathrm{image}\, T_2 \setminus \ker T_1 = V \setminus \ker T_1 = V \setminus \{0\} \neq \varnothing,$$

and so $T_1 \circ T_2 \in \overline{\Gamma}^{\mathrm{End}}$ by [Observation 3.2].

We first show $\ker(T_1 \circ T_2) \cap V = \{0\}$. Suppose $v \in \ker(T_1 \circ T_2) \cap V$. Then $T_2(v) \in \ker T_1$. But $\mathrm{image}\, T_2 = V$ and $\ker T_1 \cap V = \{0\}$, so $T_2(v) = 0$ and so $v \in \ker T_2 \cap V = \{0\}$. So $v = 0$, and thus

$$(2) \qquad\qquad \{0\} = \ker(T_1 \circ T_2) \cap V.$$

Next, by definition,

$$\mathrm{image}(T_1 \circ T_2) \subset \mathrm{image}\, T_1 = V.$$

So by (2) and dimension counting

$$\text{image}(T_1 \circ T_2) = V.$$

Thus $T_1 \circ T_2 \in \overline{\Gamma}_{F,\star}^{\text{End}}$.

Since image $T_2 = V$

$$T_1|_V \circ T_2|_V = (T_1 \circ T_2)|_V,$$

so

$$(T_1 \circ T_2)|_V \in \{T|_V : T \in \overline{\Gamma}_{F,\star}^{\text{End}}\}.$$

Then, since $T_1, T_2 \in \overline{\Gamma}_{F,\star}^{\text{End}}$ were arbitrary, we see that

$$\{T|_V : T \in \overline{\Gamma}_{F,\star}^{\text{End}}\}$$

is a semigroup in $\mathbb{P}(\text{End}(V))$. □

The next lemma requires a definition.

Definition 3.6 A point $x \in \partial\Omega$ is a *conical limit point of* Γ if there exist $p_0 \in \Omega$, a sequence $(p_n)_{n\geq 1}$ in $[p_0, x)$ with $p_n \to x$, and a sequence $(\gamma_n)_{n\geq 1}$ in Γ with

$$\sup_{n\geq 1} H_\Omega(\gamma_n(p_0), p_n) < +\infty.$$

Notice that if Γ acts cocompactly on Ω then every boundary point is a conical limit point.

Lemma 3.7 Suppose $x \in \partial\Omega$ is a conical limit point of Γ, $F = F_\Omega(x)$, $V = \text{Span } F$, and $\dim(V) = k$. If $k \geq 2$ and the image of $\Gamma \hookrightarrow \text{PGL}(\bigwedge^k \mathbb{R}^d)$ is strongly irreducible (eg Γ is Zariski dense in $\text{PGL}_d(\mathbb{R})$), then there exists a sequence $(g_n)_{n\geq 1}$ in Γ with:

(i) $g_n \to T$ in $\mathbb{P}(\text{End}(\mathbb{R}^d))$, where $T \in \overline{\Gamma}_{F,\star}^{\text{End}}$.

(ii) $g_1|_V, g_2|_V, \ldots$ are pairwise distinct elements of $\mathbb{P}(\text{Lin}(V, \mathbb{R}^d))$.

Proof By hypothesis there exist $p_0 \in \Omega$, a sequence $(p_n)_{n\geq 1}$ in $[p_0, x)$ with $p_n \to x$, and a sequence $(\gamma_n)_{n\geq 1}$ in Γ with

$$\sup_{n\geq 1} H_\Omega(\gamma_n(p_0), p_n) < +\infty.$$

After passing to a subsequence we can suppose that the limit

$$S = \lim_{n\to\infty} \gamma_n$$

exists in $\mathbb{P}(\text{End}(\mathbb{R}^d))$. Then, by Proposition 2.13,

$$\text{image } S = \text{Span } F = V,$$

and so $S \in \overline{\Gamma}_F^{\text{End}}$. By passing to another subsequence we can suppose that

$$V_\infty = \lim_{n \to \infty} \gamma_n^{-1} V$$

exists in $\text{Gr}_k(\mathbb{R}^d)$.

Let $V = \text{Span}\{v_1, \ldots, v_k\}$, $V_\infty = \text{Span}\{u_1, \ldots, u_k\}$, and $\ker S = \text{Span}\{s_1, \ldots, s_{d-k}\}$, and let $W_1 = [u_1 \wedge \cdots \wedge u_k]$ and

$$W_2 = \{\alpha \in \wedge^k \mathbb{R}^d : \alpha \wedge s_1 \wedge \cdots \wedge s_{d-k} = 0\}.$$

Since the image of $\Gamma \hookrightarrow \text{PGL}(\wedge^k \mathbb{R}^d)$ is strongly irreducible, [Observation 2.2](#) implies that there exists some $\phi \in \Gamma$ such that $\phi[v_1 \wedge \cdots \wedge v_k] \notin W_1 \cup W_2$. Equivalently, $\ker S \cap \phi V = \{0\}$ and $\phi V \neq V_\infty$.

Define $g_n := \gamma_n \phi$. Then

$$T := S \circ \phi = \lim_{n \to \infty} g_n$$

exists in $\mathbb{P}(\text{End}(\mathbb{R}^d))$. Further, $\text{image } T = \text{image } S = V$ and

$$\ker T \cap V = \phi^{-1}(\ker S \cap \phi V) = \{0\},$$

so $T \in \overline{\Gamma}_{F,\star}^{\text{End}}$. Also, since $T(V) = V$,

$$V = T(V) = \lim_{n \to \infty} g_n V.$$

Next we claim that $g_n V \neq V$ for n sufficiently large. Notice that $g_n V = V$ if and only if $g_n^{-1} V = V$ if and only if $\gamma_n^{-1} V = \phi V$. But $\gamma_n^{-1} V \to V_\infty$ and $\phi V \neq V_\infty$, so $g_n V \neq V$ for n sufficiently large.

Finally, since $g_n V \to V$ and $g_n V \neq V$ for n sufficiently large, we can pass to a subsequence so that $V, g_1 V, g_2 V, \ldots$ are pairwise distinct subspaces. Thus $g_1|_V, g_2|_V, \ldots$ must be pairwise distinct. □

Lemma 3.8 *Suppose $x \in \partial\Omega$ is a conical limit point of Γ, $F = F_\Omega(x)$, $V = \text{Span } F$, and $\dim(V) = k$. If $k \geq 2$ and the image of $\Gamma \hookrightarrow \text{PGL}(\wedge^k \mathbb{R}^d)$ is strongly irreducible (eg Γ is Zariski dense in $\text{PGL}_d(\mathbb{R})$), then the set*

$$\{T|_V : T \in \overline{\Gamma}_{F,\star}^{\text{End}}\}$$

is nondiscrete in $\mathbb{P}(\text{End}(V))$.

Proof Let $T \in \overline{\Gamma}_{F,\star}^{\text{End}}$ and $(g_n)_{n \geq 1}$ be as in the previous lemma. Since $g_1|_V, g_2|_V, \ldots$ are pairwise distinct and each $g_n|_V$ is determined by its values on any set of $\dim V + 1$ points in general position, after passing to a subsequence we can find a point $x_0 \in F$ such that $g_1(x_0), g_2(x_0), \ldots$ are pairwise distinct.

Since $x_0 \in F$ and $\mathbb{P}(\ker T) \cap F = \varnothing$,

$$T(x_0) = \lim_{n \to \infty} g_n(x_0).$$

Since $g_1(x_0), g_2(x_0), \ldots$ are pairwise distinct, by passing to another sequence we can assume that $g_n(x_0) \neq T(x_0)$ for all n. Then, for each n there exists a unique projective line L_n containing $T(x_0)$ and $g_n(x_0)$. By passing to a subsequence we can suppose that L_n converges to a projective line L. Then let $W \subset \mathbb{R}^d$ be the two-dimensional linear subspace with $L = \mathbb{P}(W)$.

Fix some $W' \in \mathrm{Gr}_k(\mathbb{R}^d)$ with $W \subset W'$ and suppose that $V = \mathrm{Span}\{v_1, \ldots, v_k\}$, $W' = \mathrm{Span}\{w_1, \ldots, w_k\}$, and $\ker T = \mathrm{Span}\{t_1, \ldots, t_{d-k}\}$. Let

$$U = \{\alpha \in \wedge^k \mathbb{R}^d : \alpha \wedge t_1 \wedge \cdots \wedge t_{d-k} = 0\}.$$

Since the image of $\Gamma \hookrightarrow \mathrm{PGL}(\wedge^k \mathbb{R}^d)$ is strongly irreducible, [Observation 2.2](#) implies that there exists $\varphi \in \Gamma$ such that $\varphi[v_1 \wedge \cdots \wedge v_k] \notin U$ and $\varphi[w_1 \wedge \cdots \wedge w_k] \notin U$. Hence $\ker T \cap \varphi V = \{0\}$ and $\ker T \cap \varphi W = \{0\}$.

Notice that $T\varphi T = \lim_{n \to \infty} g_n \varphi g_n$ is in $\overline{\Gamma}_{F,\star}^{\mathrm{End}}$. Then replacing $(g_n)_{n \geq 1}$ with a tail, we can assume that

$$S_n := T\varphi g_n \in \overline{\Gamma}_{F,\star}^{\mathrm{End}}$$

for all n.

We claim that the set

$$\{S_n(x_0) : n \geq 0\} \subset F$$

is infinite. For this calculation we fix an affine chart $\mathbb{A}$ of $\mathbb{P}(\mathbb{R}^d)$ which contains $\overline{\Omega}$. We then identify $\mathbb{A}$ with $\mathbb{R}^{d-1}$ so that $T(x_0) = 0$ and

$$\mathbb{A} \cap L = \{(t, 0, \ldots, 0) : t \in \mathbb{R}\}.$$

Since $\ker T \cap \varphi V = \{0\}$, in these coordinates the map $T\varphi$ is smooth in a neighborhood of $0 = T(x_0)$. Further, since $\ker T \cap \varphi W = \{0\}$, in these coordinates

$$d(T\varphi)_0(1, 0, \ldots, 0) \neq 0.$$

Now, since $L_n \to L$ and $g_n(x_0) \to T(x_0)$ in these coordinates,

$$g_n(x_0) = (t_n, 0, \ldots, 0) + \mathrm{o}(|t_n|)$$

for some sequence $(t_n)_{n \geq 1}$ converging to 0. Then, in these coordinates,

$$S_n(x_0) = T\varphi g_n(x_0) = T\varphi((t_n, 0, \ldots, 0) + \mathrm{o}(|t_n|))$$
$$= T\varphi T(x_0) + t_n d(T\varphi)_0(1, 0, \ldots, 0) + \mathrm{o}(|t_n|).$$

Since $d(T\varphi)_0(1, 0, \ldots, 0) \neq 0$ and $t_n \to 0$, we see that the set $\{S_n(x_0) : n \geq 0\}$ is infinite.

Finally, since $S_n|_V \to T\varphi T|_V$, this implies that

$$\{S_n|_V : n \geq 0\} \cup \{T\varphi T|_V\}$$

is nondiscrete in $\mathbb{P}(\mathrm{End}(V))$. □

Lemma 3.9 *Suppose $x \in \partial\Omega$ is a conical limit point of Γ, $F = F_\Omega(x)$, $V = \mathrm{Span}\, F$, and $\dim(V) = k$. If $k \geq 2$ and Γ is Zariski dense in $\mathrm{PGL}_d(\mathbb{R})$, then*

$$\{T|_V : T \in \overline{\Gamma}_{F,\star}^{\mathrm{End}}\}$$

is Zariski dense in $\mathbb{P}(\mathrm{End}(V))$.

Proof Let Z_0 be the Zariski closure of

$$\{T|_V : T \in \overline{\Gamma}_{F,\star}^{\mathrm{End}}\}$$

in $\mathbb{P}(\mathrm{End}(V))$.

[Lemma 3.7](#) implies that $\overline{\Gamma}_{F,\star}^{\mathrm{End}}$ is nonempty, so fix $T \in \overline{\Gamma}_{F,\star}^{\mathrm{End}}$. Then define

$$Z_1 = \{g \in \mathrm{PGL}_d(\mathbb{R}) : \mathrm{rank}(T \circ g|_V) < \dim(V)\}.$$

Notice that Z_1 is a proper Zariski-closed set in $\mathrm{PGL}_d(\mathbb{R})$ since $\mathrm{rank}(T) = \dim(V)$. Also define

$$Z_2 = \{g \in \mathrm{PGL}_d(\mathbb{R}) : T \circ g|_V \in Z_0\}.$$

Notice that Z_2 is a Zariski-closed subset of $\mathrm{PGL}_d(\mathbb{R})$.

We claim that $\Gamma \subset Z_1 \cup Z_2$. If $g \in \Gamma \setminus Z_1$, then $\mathrm{rank}(T \circ g|_V) = \dim V$ and

$$\mathrm{image}(T \circ g|_V) \subset \mathrm{image}\, T = V.$$

So $(T \circ g)(V) = V$, which implies that $T \circ g \in \overline{\Gamma}_{F,\star}^{\mathrm{End}}$, and hence that $g \in Z_2$. So $\Gamma \subset Z_1 \cup Z_2$.

Then, since Z_1 is a proper Zariski closed subset of $\mathrm{PGL}_d(\mathbb{R})$ and Γ is Zariski dense in $\mathrm{PGL}_d(\mathbb{R})$, we see that $Z_2 = \mathrm{PGL}_d(\mathbb{R})$. Therefore

$$Z_0 \supset \{T \circ g|_V : g \in Z_2\} = \{T \circ g|_V : g \in \mathrm{PGL}_d(\mathbb{R})\} \supset \mathrm{PGL}(V),$$

since $\mathrm{image}\, T = V$. Thus $Z_0 = \mathbb{P}(\mathrm{End}(V))$. □

Proof of [Theorem 3.1](#) Parts (a) and (b) follow from Lemmas [3.3](#) and [3.4](#), respectively. Since Γ acts cocompactly on Ω, every point in $\partial\Omega$ is a conical limit point, and

Theorem 2.3 implies that Γ is Zariski dense in $\mathrm{PGL}_d(\mathbb{R})$. So part (c) follows from Lemmas 3.3, 3.8, and 3.9. □

4 The main rigidity theorem

Recall that $\mathcal{E}_\Omega \subset \partial\Omega$ denotes the set of extreme points of a properly convex domain Ω. In this section we prove the following rigidity result:

Theorem 4.1 *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is an irreducible properly convex divisible domain and there exists a boundary face $F \subset \partial\Omega$ such that*

$$F \cap \bar{\mathcal{E}}_\Omega = \varnothing.$$

Then Ω is symmetric with real rank at least two.

The rest of the section is devoted to the proof of the theorem, so suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ satisfies the hypothesis of the theorem. Then let $\Gamma \leq \mathrm{Aut}(\Omega)$ be a discrete group acting cocompactly on Ω.

We assume, for a contradiction, that Ω is not symmetric with real rank at least two.

Lemma 4.2 *It holds that Ω is not symmetric.*

Proof If Ω were symmetric, then by assumption it would have real rank one. Then, by the characterization of symmetric convex divisible domains, Ω coincides with the unit ball in some affine chart. Therefore $\mathcal{E}_\Omega = \partial\Omega$, which is impossible since there exists a boundary face $F \subset \partial\Omega$ such that

$$F \cap \bar{\mathcal{E}}_\Omega = \varnothing. \qquad \square$$

Now we fix a boundary face $F \subset \partial\Omega$, where

$$\bar{\mathcal{E}}_\Omega \cap F = \varnothing$$

and if $F' \subset \partial\Omega$ is a face with $\dim F' < \dim F$ then

$$\bar{\mathcal{E}}_\Omega \cap F' \neq \varnothing.$$

Then define $V := \mathrm{Span}\, F$.

Lemma 4.3 *If $T \in \bar{\Gamma}^{\mathrm{End}}_{F,\star}$, then the map*

$$F \to \mathbb{P}(V), \quad p \mapsto T(p),$$

is in $\mathrm{Aut}(F)$.

Proof Notice that $T|_V \in \mathrm{PGL}(V)$ since $T(V) \subset V$ and $\ker T \cap V = \{0\}$. So we just have to show that $T(F) = F$. Theorem 3.1(b) says that $T(F) \subset F$, and so we just have to show that $F \subset T(F)$.

Fix $y \in F$. Since the set $T(\overline{F}) \cap F$ is closed in F, there exists $x_0 \in T(\overline{F}) \cap F$ such that

$$H_F(y, x_0) = \min_{x \in T(\overline{F}) \cap F} H_F(y, x).$$

Since $T|_V \in \mathrm{PGL}(V)$, the set $T(F)$ is open in F. So we either have $y = x_0 \in T(F)$ or $x_0 \in T(\partial F)$. Suppose for a contradiction that $x_0 \in T(\partial F)$. Then let $x_0' \in \partial F$ be the point where $T(x_0') = x_0$. Next, let $F' \subset \partial F$ be the face of x_0'. Then $\dim F' < \dim F$, so

$$\overline{\mathcal{E}}_\Omega \cap F' \neq \varnothing.$$

Thus we can find $z \in F'$ and a sequence $(z_n)_{n \geq 1}$ in $\mathcal{E}_\Omega$ such that $z_n \to z$. Since $z \in F'$, there exists an open line segment L in $\overline{F}$ which contains z and x_0'. Then $T(L)$ is an open line segment in $\overline{F}$ since $T|_V \in \mathrm{PGL}(V)$. So, since $T(x_0') \in F$, we also have $T(z) \in F$, and since

$$T \in \overline{\Gamma}_{F,\star}^{\mathrm{End}} \subset \overline{\Gamma}^{\mathrm{End}},$$

there exists a sequence $(g_n)_{n \geq 1}$ in Γ such that $g_n \to T$ in $\mathbb{P}(\mathrm{End}(\mathbb{R}^d))$. Now note that $z \notin \mathbb{P}(\ker T)$ since $\ker T \cap V = \{0\}$. So by the "moreover" part of Observation 2.9,

$$T(z) = \lim_{n \to \infty} g_n(z_n) \in F.$$

However, $g_n(z_n) \in \mathcal{E}_\Omega$, and so

$$T(z) \in \overline{\mathcal{E}}_\Omega \cap F = \varnothing.$$

Thus we have a contradiction. Hence $y = x_0 \in T(F)$, and since $y \in F$ was arbitrary we have $F \subset T(F)$. ☐

Lemma 4.4 $\mathrm{Aut}(F)$ *is nondiscrete and Zariski dense in* $\mathrm{PGL}(V)$.

Proof This follows immediately from Lemma 4.3 and Theorem 3.1(c). ☐

Lemma 4.5 $\qquad\qquad\qquad \mathrm{PSL}(V) \subset \mathrm{Aut}(F)$.

Proof Let $\mathrm{Aut}_0(F)$ denote the connected component of the identity in $\mathrm{Aut}(F)$ and let $\mathfrak{g} \subset \mathfrak{sl}(V)$ denote the Lie algebra of $\mathrm{Aut}_0(F)$. Then $\mathfrak{g} \neq \{0\}$ since $\mathrm{Aut}(F)$ is closed and nondiscrete. Also $\mathrm{Aut}_0(F)$ is normalized by $\mathrm{Aut}(F)$, and so

$$\mathrm{Ad}(g)\mathfrak{g} = \mathfrak{g}$$

for all $g \in \mathrm{Aut}(F)$. Then, since $\mathrm{Aut}(F)$ is Zariski dense in $\mathrm{PGL}(V)$, we see that

$$\mathrm{Ad}(g)\mathfrak{g} = \mathfrak{g}$$

for all $g \in \mathrm{PGL}(V)$. Since the representation $\mathrm{Ad} : \mathrm{PGL}(V) \to \mathrm{GL}(\mathfrak{sl}(V))$ is irreducible, we must have $\mathfrak{g} = \mathfrak{sl}(V)$. Thus $\mathrm{Aut}_0(F) = \mathrm{PSL}(V)$. □

Proof of Theorem 4.1 The previous lemma immediately implies a contradiction: fix $x \in F$, then

$$\mathbb{P}(V) \supset F \supset \mathrm{Aut}(F) \cdot x \supset \mathrm{PSL}(V) \cdot x = \mathbb{P}(V).$$

So $F = \mathbb{P}(V)$, which contradicts the fact that Ω is properly convex. □

5 Density of biproximal elements

In this section we prove a density result for the attracting and repelling fixed points of biproximal elements. To state the result we need one definition: if $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $\Gamma \leq \mathrm{Aut}(\Omega)$, then the *limit set of* Γ is

$$\mathcal{L}_\Omega(\Gamma) = \bigcup_{p \in \Omega} \overline{\Gamma \cdot p} \cap \partial\Omega.$$

Equivalently, a point $x \in \partial\Omega$ is in $\mathcal{L}_\Omega(\Gamma)$ if and only if there exist $p \in \Omega$ and a sequence $(\gamma_n)_{n \geq 1}$ in Γ such that $\gamma_n(p) \to x$.

Theorem 5.1 *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $\Gamma \leq \mathrm{Aut}(\Omega)$ is a strongly irreducible group. If $x, y \in \mathcal{L}_\Omega(\Gamma)$ are extreme points of Ω and $(x, y) \subset \Omega$, then there exists a sequence of biproximal elements $(g_n)_{n \geq 1}$ in Γ such that*

$$\lim_{n \to \infty} \ell^+_{g_n} = x \quad \text{and} \quad \lim_{n \to \infty} \ell^-_{g_n} = y.$$

Before proving the theorem we state and prove one corollary:

Corollary 5.2 *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is an irreducible properly convex domain and $\Gamma \leq \mathrm{Aut}(\Omega)$ is a discrete group that acts cocompactly on Ω. If $x, y \in \partial\Omega$ are extreme points and $(x, y) \subset \Omega$, then there exists a sequence of biproximal elements $(g_n)_{n \geq 1}$ in Γ such that*

$$\lim_{n \to \infty} \ell^+_{g_n} = x \quad \text{and} \quad \lim_{n \to \infty} \ell^-_{g_n} = y.$$

Proof A result of Vey [27, Theorem 5] implies that Γ is strongly irreducible and Proposition 2.13 implies that $\partial\Omega = \mathcal{L}_\Omega(\Gamma)$, so Theorem 5.1 implies the corollary. □

Proof of Theorem 5.1 By definition there exist $p \in \Omega$ and a sequence $(\gamma_n)_{n \geq 1}$ in Γ such that $\gamma_n(p) \to x$. Passing to a subsequence, we can suppose the limits

$$T^+ = \lim_{n \to \infty} \gamma_n \quad \text{and} \quad T^- = \lim_{n \to \infty} \gamma_n^{-1}$$

exist in $\mathbb{P}(\mathrm{End}(\mathbb{R}^d))$. By Proposition 2.12

$$\mathrm{image}\, T^+ \subset \mathrm{Span}\, F_\Omega(x) = \mathrm{Span}\{x\} = x,$$

and so image $T^+ = x$. Proposition 2.12 also implies that $\mathbb{P}(\ker T^-) \cap \Omega = \varnothing$ and $x \in \mathbb{P}(\ker T^-)$. Notice that $y \notin \mathbb{P}(\ker T^-)$ since $(x, y) \subset \Omega$.

Similarly, we can find a sequence $(\phi_n)_{n \geq 1}$ in Γ such that the limits

$$S^+ = \lim_{n \to \infty} \phi_n \quad \text{and} \quad S^- = \lim_{n \to \infty} \phi_n^{-1}$$

exist in $\mathbb{P}(\mathrm{End}(\mathbb{R}^d))$, image $S^+ = y$, and $x \notin \mathbb{P}(\ker S^-)$.

Fix some $x' \in \mathrm{image}\, T^-$ and $y' \in \mathrm{image}\, S^-$. Since Γ is strongly irreducible, by Observation 2.2 there exists $h \in \Gamma$ such that:

(i) $h(y') \notin \mathbb{P}(\ker T^+)$; hence, $h(\mathrm{image}\, S^-) \not\subset \ker T^+$.
(ii) $hS^-(x) \notin \mathbb{P}(\ker T^+)$.
(iii) $h(x') \notin \mathbb{P}(\ker S^+)$; hence, $h(\mathrm{image}\, T^-) \not\subset \ker S^+$.
(iv) $hT^-(y) \notin \mathbb{P}(\ker S^+)$.

Then consider $g_n = \gamma_n \circ h \circ \phi_n^{-1}$. By our choice of h, we have $T^+ \circ h \circ S^- \neq 0$ and hence

$$T^+ \circ h \circ S^- = \lim_{n \to \infty} g_n$$

in $\mathbb{P}(\mathrm{End}(\mathbb{R}^d))$. Notice that $\mathrm{image}(T^+ \circ h \circ S^-) = \mathrm{image}\, T^+ = x$ and, by our choice of h,

$$x \notin \mathbb{P}(\ker(T^+ \circ h \circ S^-)).$$

So

$$\mathrm{image}(T^+ \circ h \circ S^-) + \ker(T^+ \circ h \circ S^-) = x + \ker(T^+ \circ h \circ S^-) = \mathbb{R}^d,$$

and hence, by Proposition 2.19, g_n is proximal for n sufficiently large and $\ell_{g_n}^+ \to x$. By similar reasoning g_n^{-1} is proximal for n sufficiently large and $\ell_{g_n}^- = \ell_{g_n^{-1}}^+ \to y$. $\square$

6 North–south dynamics

In this section we prove a stronger version of Theorem 5.1 for pairs of extreme points in the limit set whose simplicial distance is greater than two.

Theorem 6.1 *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $\Gamma \leq \mathrm{Aut}(\Omega)$ is strongly irreducible. Assume $x, y \in \mathcal{L}_\Omega(\Gamma)$ are extreme points of Ω and $s_{\partial\Omega}(x, y) > 2$. If $A, B \subset \overline{\Omega}$ are neighborhoods of x and y, then there exists $g \in \Gamma$ with*

$$g(\overline{\Omega} \setminus B) \subset A \quad and \quad g^{-1}(\overline{\Omega} \setminus A) \subset B.$$

Remark 6.2 Theorem 6.1 is an analogue of a result for CAT(0) spaces; see Chapter 3 and specifically Theorem 3.4 of [3].

Before proving the theorem we state and prove one corollary:

Corollary 6.3 *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is an irreducible properly convex domain and $\Gamma \leq \mathrm{Aut}(\Omega)$ is a discrete group acting cocompactly on Ω. Assume $x, y \in \partial\Omega$ are extreme points and $s_{\partial\Omega}(x, y) > 2$. If $A, B \subset \overline{\Omega}$ are neighborhoods of x and y, then there exists $g \in \Gamma$ with*

$$g(\overline{\Omega} \setminus B) \subset A \quad and \quad g^{-1}(\overline{\Omega} \setminus A) \subset B.$$

Proof A result of Vey [27, Theorem 5] implies that Γ is strongly irreducible and Proposition 2.13 implies that $\partial\Omega = \mathcal{L}_\Omega(\Gamma)$, so Theorem 6.1 implies the corollary. □

Lemma 6.4 *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $\gamma \in \mathrm{Aut}(\Omega)$ is biproximal, and $s_{\partial\Omega}(\ell_\gamma^+, \ell_\gamma^-) > 2$. If $A, B \subset \overline{\Omega}$ are neighborhoods of ℓ_γ^+ and ℓ_γ^-, then there exists $N \geq 0$ such that*

$$\gamma^n(\overline{\Omega} \setminus B) \subset A \quad and \quad \gamma^{-n}(\overline{\Omega} \setminus A) \subset B$$

for all $n \geq N$.

Proof Observation 2.17 implies that

$$(3) \qquad\qquad \ell_\gamma^+ = \lim_{n\to\infty} \gamma^n(x)$$

for all $x \in \mathbb{P}(\mathbb{R}^d) - \mathbb{P}(H_g^-)$ and the convergence is locally uniform.

We claim that

$$\mathbb{P}(H_g^-) \cap \overline{\Omega} = \{\ell_g^-\}.$$

Proposition 2.12 implies that $\{\ell_g^-\} \subset \mathbb{P}(H_g^-) \cap \overline{\Omega}$ and that $\Omega \cap \mathbb{P}(H_g^-) = \varnothing$. So if $y \in \mathbb{P}(H_g^-) \cap \overline{\Omega}$ then $[y, \ell_g^-] \subset \mathbb{P}(H_g^-) \cap \overline{\Omega}$, and hence $[y, \ell_g^-] \subset \partial\Omega$. Then, by Theorem 2.20(ii), we have $y = \ell_g^+$. So $\mathbb{P}(H_g^-) \cap \overline{\Omega} \subset \{\ell_g^-\}$ and the claim is established.

Then, by the locally uniform convergence in (3), there exists $N_1 > 0$ such that

$$\gamma^n(\overline{\Omega} \setminus B) \subset A$$

for all $n \geq N_1$.

Repeating the same argument with γ^{-1} shows that there exists $N_2 > 0$ such that

$$\gamma^{-n}(\overline{\Omega} \setminus A) \subset B$$

for all $n \geq N_2$.

Then $N = \max\{N_1, N_2\}$ satisfies the conclusion of the lemma. $\qquad\square$

Proof of Theorem 6.1 By Theorem 5.1 there exists a sequence of biproximal elements $(g_n)_{n \geq 1}$ in Γ such that

$$\lim_{n \to \infty} \ell_{g_n}^+ = x \quad \text{and} \quad \lim_{n \to \infty} \ell_{g_n}^- = y.$$

Since $s_{\partial\Omega}(x, y) > 2$ we may pass to a tail of $(g_n)_{n \geq 1}$ and assume that

$$s_{\partial\Omega}(\ell_{g_n}^+, \ell_{g_n}^-) > 2$$

for all n.

Next, fix n sufficiently large that $\ell_{g_n}^+ \in A$ and $\ell_{g_n}^- \in B$. Then, by Lemma 6.4, there exists $m \geq 0$ such that

$$g_n^m(\overline{\Omega} \setminus B) \subset A \quad \text{and} \quad g_n^{-m}(\overline{\Omega} \setminus A) \subset B,$$

so $g = g_n^m$ satisfies the theorem. $\qquad\square$

7 Fixed points and centralizers

In this section we prove the following result, connecting the number of boundary fixed points of an element with the size of its centralizer:

Theorem 7.1 *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is an irreducible properly convex domain and $\Gamma \leq \mathrm{Aut}(\Omega)$ is a discrete group that acts cocompactly on Ω. If $g \in \Gamma$ has infinite order then the following are equivalent:*

(i) *There exist two distinct points $x, y \in \partial\Omega$ fixed by g with $s_{\partial\Omega}(x, y) < +\infty$.*

(ii) *g fixes at least three points in $\partial\Omega$.*

(iii) *The cyclic group $g^{\mathbb{Z}}$ has infinite index in its centralizer.*

Corollary 7.2 *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is an irreducible properly convex domain and $\Gamma \le \mathrm{Aut}(\Omega)$ is a discrete group that acts cocompactly on Ω. If $g \in \Gamma$ is biproximal, then the following are equivalent:*

(i) $[\ell_g^+, \ell_g^-] \subset \partial\Omega$.

(ii) $s_{\partial\Omega}(\ell_g^+, \ell_g^-) < +\infty$.

(iii) *g has at least three fixed points in $\partial\Omega$.*

(iv) *The cyclic group $g^{\mathbb{Z}}$ has infinite index in its centralizer.*

We will first recall some results established in [19], then prove the theorem and corollary.

7.1 Maximal abelian subgroups and minimal translation sets

Theorem 7.3 (Islam and Zimmer [19, Theorem 1.6]) *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $\Gamma \le \mathrm{Aut}(\Omega)$ is a discrete group that acts cocompactly on Ω. If $A \le \Gamma$ is a maximal abelian subgroup of Γ then there exists a properly embedded simplex $S \subset \Omega$ such that*

(i) *S is A–invariant,*

(ii) *A acts cocompactly on S, and*

(iii) *A fixes each vertex of S.*

Moreover, A has a finite-index subgroup isomorphic to $\mathbb{Z}^{\dim(S)}$.

Remark 7.4 The above result is a special case of [19, Theorem 1.6], which holds in the more general case when $\Gamma \le \mathrm{Aut}(\Omega)$ is a naive convex cocompact subgroup.

Definition 7.5 Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $g \in \mathrm{Aut}(\Omega)$. Define the *minimal translation length of g* to be

$$\tau_\Omega(g) := \inf_{x \in \Omega} H_\Omega(x, g(x))$$

and the *minimal translation set of g* to be

$$\mathrm{Min}_\Omega(g) = \{x \in \Omega : H_\Omega(g(x), x) = \tau_\Omega(g)\}.$$

Cooper, Long and Tillmann [14] showed that the minimal translation length of an element can be determined from its eigenvalues:

Proposition 7.6 [14, Proposition 2.1] *If $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $g \in \mathrm{Aut}(\Omega)$, then*

$$\tau_\Omega(g) = \tfrac{1}{2} \log \frac{\lambda_1(g)}{\lambda_d(g)}.$$

Remark 7.7 Recall that

$$\lambda_1(g) \geq \lambda_2(g) \geq \cdots \geq \lambda_d(g)$$

denote the absolute values of the eigenvalues of some (and hence any) lift of g to $\mathrm{SL}_d^{\pm}(\mathbb{R}) := \{h \in \mathrm{GL}_d(\mathbb{R}) : \det h = \pm 1\}$.

As a consequence of Proposition 7.6, we observe the following:

Observation 7.8 *If $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $p_0 \in \Omega$, and $g \in \mathrm{Aut}(\Omega)$, then*

$$\lim_{n \to \infty} \frac{1}{n} H_\Omega(g^n(p_0), p_0) = \tau_\Omega(g).$$

Proof Proposition 7.6 implies that $\tau_\Omega(g^n) = n\tau_\Omega(g)$, and hence

$$\liminf_{n \to \infty} \frac{1}{n} H_\Omega(g^n(p_0), p_0) \geq \tau_\Omega(g).$$

For the other inequality, fix $\epsilon > 0$ and $q \in \Omega$ with $H_\Omega(g(q), q) < \tau_\Omega(g) + \epsilon$. Then

$$
\begin{aligned}
\limsup_{n \to \infty} \frac{H_\Omega(g^n(p_0), p_0)}{n} \\
&\leq \limsup_{n \to \infty} \frac{H_\Omega(g^n(q), q) + 2H_\Omega(p_0, q)}{n} \\
&\leq \limsup_{n \to \infty} \frac{H_\Omega(g^n(q), g^{n-1}(q)) + \cdots + H_\Omega(g(q), q) + 2H_\Omega(p_0, q)}{n} \\
&= \limsup_{n \to \infty} H_\Omega(g(q), q) + \frac{2H_\Omega(p_0, q)}{n} < \tau_\Omega(g) + \epsilon.
\end{aligned}
$$

Since $\epsilon > 0$ was arbitrary, the proof is complete. $\square$

Next, given a group G and an element $g \in G$, let $C_G(g)$ denote the centralizer of g in G. Then given a subset $X \subset G$, define

$$C_G(X) = \bigcap_{x \in X} C_G(x).$$

Theorem 7.9 (Islam and Zimmer [19, Theorem 1.10]) *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $\Gamma \leq \mathrm{Aut}(\Omega)$ is a discrete group that acts cocompactly on Ω, and $A \leq \Gamma$ is an abelian subgroup. Then*

$$\mathrm{Min}_\Omega(A) := \bigcap_{a \in A} \mathrm{Min}_\Omega(a)$$

is nonempty and $C_\Gamma(A)$ acts cocompactly on the convex hull of $\mathrm{Min}_\Omega(A)$ in Ω.

Remark 7.10 The above result is a special case of [19, Theorem 1.9], which holds in the more general case when $\Gamma \leq \mathrm{Aut}(\Omega)$ is a naive convex cocompact subgroup.

Proposition 7.11 *Suppose that $S \subset \mathbb{P}(\mathbb{R}^d)$ is a simplex. If $g \in \mathrm{Aut}(S)$ fixes every vertex of S, then $\mathrm{Min}_S(g) = S$.*

Proof See for instance [19, Proposition 7.3]. □

Observation 7.12 *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $\Gamma \leq \mathrm{Aut}(\Omega)$ is a discrete group. If $g \in \Gamma$ is biproximal and $(\ell_g^+, \ell_g^-) \subset \Omega$, then $g^{\mathbb{Z}}$ has finite index in $C_\Gamma(g)$.*

Proof First notice that $C_\Gamma(g)$ preserves (ℓ_g^+, ℓ_g^-). Since $\mathrm{Aut}(\Omega)$ acts properly on Ω and $\Gamma \leq \mathrm{Aut}(\Omega)$ is discrete, we see that $C_\Gamma(g)$ acts properly on (ℓ_g^+, ℓ_g^-). Then $g^{\mathbb{Z}}$ has finite index in $C_\Gamma(g)$ since $g^{\mathbb{Z}}$ acts cocompactly on (ℓ_g^+, ℓ_g^-). □

7.2 Proof of Theorem 7.1

Fix a maximal abelian subgroup $A \leq \Gamma$ which contains g. Then, by Theorem 7.3, there exists $S \subset \Omega$ such that

- S is a properly embedded simplex,
- A acts cocompactly on S,
- A fixes every vertex of S, and
- A has a finite-index subgroup isomorphic to $\mathbb{Z}^{\dim(S)}$.

Since g has infinite order, $\dim(S) \geq 1$.

We consider a number of cases and prove that in each case (i), (ii), and (iii) are either all true or all false.

Case 1 Assume $\dim(S) \geq 2$. Then clearly (i), (ii), and (iii) are all true.

Case 2 Assume $\dim(S) = 1$. Let v^+ and v^- be the vertices of S and fix some $p_0 \in S$. Then, after possibly relabeling, we can assume that

$$\lim_{n \to \pm\infty} g^n(p_0) = v^\pm.$$

Case 2(a) Assume $s_{\partial\Omega}(v^+, v^-) > 2$. Then Theorem 2.20 implies that g is a rank-one isometry and $v^\pm = \ell_g^\pm$. Theorem 2.20 also implies that v^+ and v^- are the only fixed points of g in $\partial\Omega$ and $s_{\partial\Omega}(v^+, v^-) = \infty$. Hence (i) and (ii) are false. Observation 7.12 implies that $g^{\mathbb{Z}}$ has finite index in $C_\Gamma(g)$ and hence (iii) is false.

Case 2(b) Assume $s_{\partial\Omega}(v^+, v^-) = 2$. Then, by definition, (i) is true. Fix $y_0 \in \partial\Omega$ such that $[v^+, y_0] \cup [y_0, v^-]$.

Pick a sequence $n_j \to \infty$ such that the limits

$$T^{\pm} := \lim_{j\to\infty} g^{\pm n_j}$$

exist in $\mathbb{P}(\mathrm{End}(\mathbb{R}^d))$. Then [Proposition 2.12](#) implies that $v^{\mp} \in \mathbb{P}(\ker T^{\pm})$ and $\mathbb{P}(\ker T^{\pm}) \cap \Omega = \varnothing$. This implies that $v^{\pm} \notin \mathbb{P}(\ker T^{\pm})$ since $(v^+, v^-) \subset \Omega$. Also, g commutes with $T^{\pm}$ and hence $g\mathbb{P}(\ker T^{\pm}) = \mathbb{P}(\ker T^{\pm})$.

Passing to a further sequence, we can suppose that $g^{\pm n_j}(y_0) \to y^{\pm}$. Then

$$[v^+, y^{\pm}] \cup [y^{\pm}, v^-] \subset \partial\Omega$$

and so, since $(v^+, v^-) \subset \Omega$, $y^{\pm}$ must be distinct from v^+ and v^-. Since $g^{\pm n_j}(x) \to v^{\pm}$ for all $x \in \mathbb{P}(\mathbb{R}^d) \setminus \mathbb{P}(\ker T^{\pm})$, we must have $y \in \mathbb{P}(\ker T^+ \cap \ker T^-)$. Thus the set

$$C := \partial\Omega \cap \mathbb{P}(\ker T^+ \cap \ker T^-)$$

is nonempty. Then g has a fixed point $y \in C$ since C is g–invariant, closed, and convex, so g has at least three fixed points in $\partial\Omega$ and (ii) is true.

Recall that $v^{\mp} \in \mathbb{P}(\ker T^{\pm})$ and $\mathbb{P}(\ker T^{\pm}) \cap \Omega = \varnothing$; hence,

$$[v^+, y] \cup [y, v^-] \subset \partial\Omega.$$

Let S' be the open simplex with vertices v^+, v^- and y. Since $(v^+, v^-) \subset \Omega$ we have $S' \subset \Omega$. In particular,

$$(4) \qquad\qquad H_{S'}(p, q) \geq H_{\Omega}(p, q)$$

for all $p, q \in S'$. Since $p_0 \in (v^-, v^+) \subset S' \subset \Omega$, [Observation 7.8](#) implies that

$$\tau_{\Omega}(g) = \lim_{n\to\infty} \frac{H_{\Omega}(g^n(p_0), p_0)}{n} = \lim_{n\to\infty} \frac{H_{S'}(g^n(p_0), p_0)}{n} = \tau_{S'}(g).$$

Then, by [(4)](#) and [Proposition 7.11](#),

$$S' = \mathrm{Min}_{S'}(g) \subset \mathrm{Min}_{\Omega}(g).$$

Now we claim that $g^{\mathbb{Z}}$ has infinite index in $C_{\Gamma}(g)$. [Theorem 7.9](#) implies that there is a compact set $K \subset \Omega$ such that

$$S' \cup (v^+, v^-) \subset C_{\Gamma}(g) \cdot K.$$

Further, $g^{\mathbb{Z}}$ preserves (v^+, v^-), so it is enough to show that

$$\sup_{p \in S'} H_{\Omega}(p, (v^+, v^-)) = \infty.$$

Fix $(p_n)_{n \geq 1}$ in S' converging to y. Since $(v^+, v^-) \subset \Omega$ and $[v^+, y] \cup [y, v^-] \subset \partial\Omega$, Observation 2.11 implies that the faces $F_\Omega(v^+)$, $F_\Omega(v^-)$, and $F_\Omega(y)$ are all distinct. Then, by the definition of the Hilbert metric,

$$\lim_{n \to \infty} H_\Omega(p_n, (v^+, v^-)) = \infty.$$

Thus $g^{\mathbb{Z}}$ has infinite index in $C_\Gamma(g)$ and so (iii) is true. $\qed$

7.3 Proof of Corollary 7.2

Theorem 7.1 implies that (ii) $\implies$ (iii) $\iff$ (iv), and by definition (i) $\implies$ (ii). Finally, by Observation 7.12, (iv) $\implies$ (i). $\qed$

8 Rank in the sense of Prasad and Raghunathan

In this section we consider the rank of a group in the sense of [24].

Definition 8.1 (Prasad and Raghunathan) Suppose that Γ is an abstract group. For $i \geq 0$ let $A_i(\Gamma) \subset \Gamma$ be the subset of elements whose centralizer contains a free abelian group of rank at most i as a subgroup of finite index. Next define $r(\Gamma)$ to be the minimal $i \in \{0, 1, 2, \dots\} \cup \{\infty\}$ such that there exist $\gamma_1, \dots, \gamma_m \in \Gamma$ with

$$\Gamma \subset \bigcup_{j=1}^{m} \gamma_j A_i(\Gamma).$$

Then the *Prasad–Raghunathan rank of* Γ is defined to be

$$\mathrm{rank}_{\mathrm{PR}}(\Gamma) := \sup\{r(\Gamma^*) : \Gamma^* \text{ is a finite-index subgroup of } \Gamma\}.$$

Prasad and Raghunathan computed the rank of lattices in semisimple Lie groups, which implies:

Theorem 8.2 [24, Theorem 3.9] *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is an irreducible properly convex domain. If Ω is symmetric with real rank r and $\Gamma \leq \mathrm{Aut}(\Omega)$ is a discrete group acting cocompactly on Ω, then $\mathrm{rank}_{\mathrm{PR}}(\Gamma) = r$.*

As a corollary to Selberg's lemma we get a lower bound on the Prasad–Raghunathan rank:

Corollary 8.3 *If $\Gamma \leq \mathrm{PGL}_d(\mathbb{R})$ is a finitely generated infinite group, $\mathrm{rank}_{\mathrm{PR}}(\Gamma) \geq 1$.*

Proof By Selberg's lemma, there exists a finite-index torsion-free subgroup $\Gamma^* \leq \Gamma$. Notice that every element of $A_0(\Gamma^*)$ has finite order and hence $A_0(\Gamma^*) = \{\mathrm{id}\}$. Then, since Γ^* is infinite,

$$\mathrm{rank}_{\mathrm{PR}}(\Gamma) \geq r(\Gamma^*) \geq 1. \qquad \square$$

In this section we will show that the existence of a rank-one isometry implies that the Prasad–Raghunathan rank is one.

Proposition 8.4 *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $\Gamma \leq \mathrm{Aut}(\Omega)$ is a finitely generated strongly irreducible discrete group. If there exists a biproximal element $g \in \Gamma$ with $(\ell_g^+, \ell_g^-) \subset \Omega$, then*

$$\mathrm{rank}_{\mathrm{PR}}(\Gamma) = 1.$$

Remark 8.5 The proof of Proposition 8.4 is a simple modification of Ballmann and Eberlein's proof [4] of the analogous statement for CAT(0) groups.

The rest of the section is devoted to the proof of Proposition 8.4, so suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$, $\Gamma \leq \mathrm{Aut}(\Omega)$, and $g \in \Gamma$ satisfy the hypothesis of the proposition. By Corollary 8.3 it is enough to fix a finite-index subgroup $\Gamma^* \subset \Gamma$ and show that $r(\Gamma^*) \leq 1$. Also, by replacing g with a sufficiently large power, we may assume that $g \in \Gamma^*$.

Lemma 8.6 *Suppose that $x_1, x_2 \in \partial\Omega$ and $(x_1, x_2) \subset \Omega$. If $A, B \subset \partial\Omega$ are open sets with $\bar{A} \cap \bar{B} = \varnothing$, then we can find disjoint neighborhoods V_1 and V_2 of x_1 and x_2 such that for each $\varphi \in \mathrm{Aut}(\Omega)$ at least one of the following occurs:*

(i) $\varphi(V_1) \cap A = \varnothing$.

(ii) $\varphi(V_1) \cap B = \varnothing$.

(iii) $\varphi(V_2) \cap A = \varnothing$.

(iv) $\varphi(V_2) \cap B = \varnothing$.

Proof The following argument is essentially the proof of Lemma 3.10 in [4].

Fix a distance $d_{\mathbb{P}}$ on $\mathbb{P}(\mathbb{R}^d)$ induced by a Riemannian metric. Then, for each n and $j = 1, 2$, let $V_{j,n}$ be a neighborhood of x_j whose diameter with respect to $d_{\mathbb{P}}$ is less than $1/n$.

Suppose for a contradiction that the lemma is false. Then, for each n, there exists $\varphi_n \in \mathrm{Aut}(\Omega)$ such that

(5) $$\varphi_n(V_{j,n}) \cap A \neq \varnothing \quad \text{and} \quad \varphi_n(V_{j,n}) \cap B \neq \varnothing$$

for $j = 1, 2$. By passing to a subsequence, we can suppose that

$$T := \lim_{n \to \infty} \varphi_n$$

exists in $\mathbb{P}(\mathrm{End}(\mathbb{R}^d))$. Then

$$T(u) = \lim_{n \to \infty} \varphi_n(u)$$

for all $u \in \mathbb{P}(\mathbb{R}^d) \setminus \mathbb{P}(\ker T)$. Moreover, the convergence is uniform on compact subsets of $\mathbb{P}(\mathbb{R}^d) \setminus \mathbb{P}(\ker T)$.

Proposition 2.12 implies that $\mathbb{P}(\ker T) \cap \Omega = \varnothing$. Then, since $(x_1, x_2) \subset \Omega$, it is impossible for both x_1 and x_2 to be contained in $\mathbb{P}(\ker T)$. So, after possibly relabelling, we may assume that $x_1 \notin \mathbb{P}(\ker T)$.

By (5) there exist sequences $a_n, b_n \in \partial\Omega$ converging to x_1 such that $\varphi_n(a_n) \in A$ and $\varphi_n(b_n) \in B$. Then, since $x_1 \notin \mathbb{P}(\ker T)$,

$$T(x_1) = \lim_{n \to \infty} \varphi_n(a_n) \in \bar{A} \quad \text{and} \quad T(x_1) = \lim_{n \to \infty} \varphi_n(b_n) \in \bar{B}.$$

So $T(x_1) \in \bar{A} \cap \bar{B} = \varnothing$, which is a contradiction. $\qquad\square$

Lemma 8.7 $\hspace{3cm} r(\Gamma^*) \leq 1.$

Proof The following argument is essentially the proof of Theorem 3.1 in [4].

Since Γ is strongly irreducible Γ^* is also strongly irreducible, so, by Observation 2.2, there exists $\phi \in \Gamma^*$ such that

$$\phi \ell_g^+, \phi \ell_g^-, \ell_g^+ \text{ and } \ell_g^-$$

are all distinct. Then $h := \phi g \phi^{-1}$ is biproximal, $\ell_h^\pm = \phi \ell_g^\pm$, and

$$(\ell_h^+, \ell_h^-) = \phi(\ell_g^+, \ell_g^-) \subset \Omega.$$

Fix open neighborhoods $A, B \subset \partial\Omega$ of ℓ_h^+ and ℓ_h^- such that $\bar{A} \cap \bar{B} = \varnothing$. Then let $V_1, V_2 \subset \partial\Omega$ be neighborhoods of ℓ_g^+ and ℓ_g^- such that A, B, V_1 and V_2 satisfy Lemma 8.6.

By further shrinking each V_j, we can assume that each $\partial\Omega \setminus V_j$ is homeomorphic to a closed ball.

Next, let $U_1 \subset V_1$ be a closed neighborhood of ℓ_g^+ such that, if $x \in U_1$ and $y \in \partial\Omega \setminus V_1$, then $s_{\partial\Omega}(x, y) > 2$. Such a choice is possible by Theorem 2.20(ii). In a similar fashion,

let $U_2 \subset V_2$ be a closed neighborhood of ℓ_g^- such that, if $x \in U_2$ and $y \in \partial\Omega \setminus V_2$, then $s_{\partial\Omega}(x, y) > 2$.

By further shrinking each U_j, we can assume that each U_j is homeomorphic to a closed ball.

By [Observation 2.18](), each $\ell_g^\pm$ and $\ell_h^\pm$ is an extreme point of Ω. Furthermore, by [Theorem 2.20](iii),

$$s_{\partial\Omega}(\ell_g^\pm, \ell_h^\pm) = \infty = s_{\partial\Omega}(\ell_g^\pm, \ell_h^\mp).$$

So, by [Theorem 6.1](), there exist $\varphi_1, \varphi_2, \psi_1, \psi_2 \in \Gamma^*$ such that

  (i) $\varphi_1(\partial\Omega \setminus A) \subset U_1$ and $\varphi_1^{-1}(\partial\Omega \setminus U_1) \subset A$,

 (ii) $\psi_1(\partial\Omega \setminus A) \subset U_2$ and $\psi_1^{-1}(\partial\Omega \setminus U_2) \subset A$,

(iii) $\varphi_2(\partial\Omega \setminus B) \subset U_1$ and $\varphi_2^{-1}(\partial\Omega \setminus U_1) \subset B$,

 (iv) $\psi_2(\partial\Omega \setminus B) \subset U_2$ and $\psi_2^{-1}(\partial\Omega \setminus U_2) \subset B$.

We claim that

$$\Gamma^* = \varphi_1^{-1} A_1(\Gamma^*) \cup \psi_1^{-1} A_1(\Gamma^*) \cup \varphi_2^{-1} A_1(\Gamma^*) \cup \psi_2^{-1} A_1(\Gamma^*).$$

Fix $\gamma \in \Gamma^*$. By construction, at least one of the four possibilities in [Lemma 8.6]() must occur.

Case 1 Assume $\gamma(V_1) \cap A = \varnothing$. Then

$$(6) \qquad\qquad \varphi_1\gamma(U_1) \subsetneq \varphi_1\gamma(V_1) \subset \varphi_1(\partial\Omega \setminus A) \subset U_1,$$

so, by the Brouwer fixed-point theorem, $\varphi_1\gamma$ has a fixed point in $x \in U_1$ (recall that U_1 is homeomorphic to a closed ball). Further,

$$(\varphi_1\gamma)^{-1}(\partial\Omega \setminus V_1) \subset (\varphi_1\gamma)^{-1}(\partial\Omega \setminus U_1) \subset \gamma^{-1}(A) \subset \partial\Omega \setminus V_1,$$

so $\varphi_1\gamma$ also has a fixed point in $y \in \partial\Omega \setminus V_1$. Now, by construction, $s_{\partial\Omega}(x, y) > 2$. So, by [Theorem 2.20](i), either

$$\inf_{p \in \Omega} H_\Omega(\varphi_1\gamma(p), p) = 0$$

or $\varphi_1\gamma$ is biproximal with

$$\{x, y\} = \{\ell_{\varphi_1\gamma}^+, \ell_{\varphi_1\gamma}^-\}.$$

In the latter case, $(\ell_{\varphi_1\gamma}^+, \ell_{\varphi_1\gamma}^-) \subset \Omega$, and so $\varphi_1\gamma \in A_1(\Gamma)$ by [Observation 7.12](). Thus we have reduced to showing that

$$\inf_{p \in \Omega} H_\Omega(\varphi_1\gamma(p), p) > 0.$$

Assume for a contradiction that

$$\inf_{p \in \Omega} H_\Omega(\varphi_1 \gamma(p), p) = 0.$$

Then, by Proposition 7.6, we have

$$\lambda_1(\varphi_1 \gamma) = \lambda_2(\varphi_1 \gamma) = \cdots = \lambda_d(\varphi_1 \gamma).$$

Since x and y are eigenlines of $\varphi_1 \gamma$, this implies that $\varphi_1 \gamma$ fixes every point of the line (x, y). Then, since $\mathrm{Aut}(\Omega)$ acts properly on Ω and Γ^* is discrete, the group

$$K = \{(\varphi_1 \gamma)^n : n \in \mathbb{Z}\}$$

is finite. So $(\varphi_1 \gamma)^N = \mathrm{id}$ for some large N. Then (6) implies that

$$U_1 = (\varphi_1 \gamma)^N(U_1) \subsetneq U_1.$$

So we have a contradiction, and hence

$$\inf_{p \in \Omega} H_\Omega(\varphi_1 \gamma(p), p) > 0$$

and so $\varphi_1 \gamma \in A_1(\Gamma^*)$.

Case 2 Assume $\gamma(V_1) \cap B = \varnothing$. Then arguing as in Case 1 shows that $\varphi_2 \gamma \in A_1(\Gamma^*)$.

Case 3 Assume $\gamma(V_2) \cap A = \varnothing$. Then arguing as in Case 1 shows that $\psi_1 \gamma \in A_1(\Gamma^*)$.

Case 4 Assume $\gamma(V_2) \cap B = \varnothing$. Then arguing as in Case 1 shows that $\psi_2 \gamma \in A_1(\Gamma^*)$.

Since $\gamma \in \Gamma^*$ was arbitrary,

$$\Gamma^* = \varphi_1^{-1} A_1(\Gamma^*) \cup \psi_1^{-1} A_1(\Gamma^*) \cup \varphi_2^{-1} A_1(\Gamma^*) \cup \psi_2^{-1} A_1(\Gamma^*).$$

Hence $r(\Gamma^*) \leq 1$. □

9 Proof of Theorem 1.4

Suppose for the rest of the section that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is an irreducible properly convex domain and $\Gamma \leq \mathrm{Aut}(\Omega)$ is a discrete group that acts cocompactly on Ω. We will show that the following conditions are equivalent:

(i) Ω is symmetric with real rank at least two.

(ii) Ω has higher rank.

(iii) The extreme points of Ω form a closed proper subset of $\partial\Omega$.
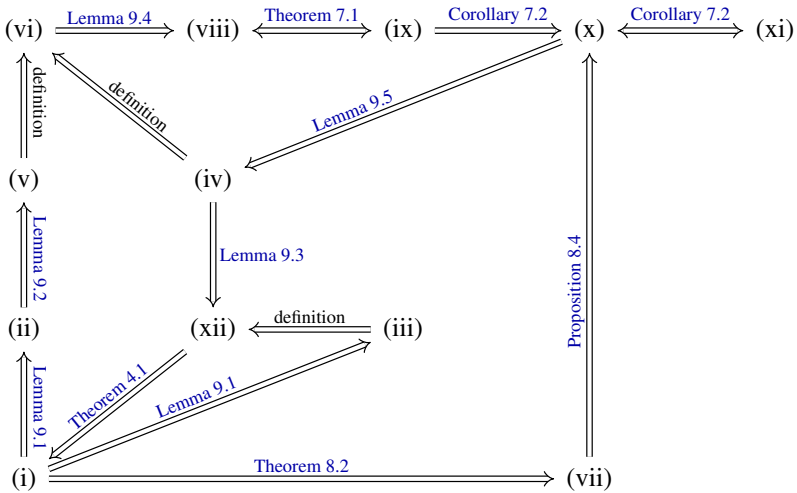
Figure 1: The proof of Theorem 1.4.

(iv) $[x_1, x_2] \subset \partial\Omega$ for every two extreme points $x_1, x_2 \in \partial\Omega$.

(v) $s_{\partial\Omega}(x, y) \leq 2$ for all $x, y \in \partial\Omega$.

(vi) $s_{\partial\Omega}(x, y) < +\infty$ for all $x, y \in \partial\Omega$.

(vii) Γ has higher rank in the sense of Prasad and Raghunathan.

(viii) For every $g \in \Gamma$ with infinite order, the cyclic group $g^{\mathbb{Z}}$ has infinite index in the centralizer $C_\Gamma(g)$ of g in Γ.

(ix) Every $g \in \Gamma$ with infinite order has at least three fixed points in $\partial\Omega$.

(x) $[\ell_g^+, \ell_g^-] \subset \partial\Omega$ for every biproximal element $g \in \Gamma$.

(xi) $s_{\partial\Omega}(\ell_g^+, \ell_g^-) < +\infty$ for every biproximal element $g \in \Gamma$.

(xii) There exists a boundary face $F \subset \partial\Omega$ such that

$$F \cap \bar{\mathcal{E}}_\Omega = \varnothing.$$

We verify all the implications shown in Figure 1. First notice that (iii) $\implies$ (xii), (iv) $\implies$ (vi), and (v) $\implies$ (vi) are by definition. The implication (i) $\implies$ (vii) is due to Prasad and Raghunathan; see Theorem 8.2 above. Proposition 8.4 implies that (vii) $\implies$ (x). Theorem 7.1 implies that (viii) $\iff$ (ix). Corollary 7.2 implies that (ix) $\implies$ (x) and (x) $\iff$ (xi). Theorem 4.1 implies that (xii) $\implies$ (i). The remaining implications in Figure 1 are given as lemmas below.

Lemma 9.1 (i) $\implies$ (ii) *and* (i) $\implies$ (iii).

Proof These implications follow from direct inspection of the short list of irreducible symmetric properly convex domains. □

Lemma 9.2 (ii) $\Longrightarrow$ (v).

Proof Suppose $x, y \in \partial\Omega$. If $[x, y] \subset \partial\Omega$, then $s_{\partial\Omega}(x, y) \leq 1$. If $(x, y) \subset \Omega$, then there exists a properly embedded simplex $S \subset \Omega$ with $\dim(S) \geq 2$ and $(x, y) \subset S$. Then
$$s_{\partial\Omega}(x, y) \leq s_{\partial S}(x, y) \leq 2.$$
Since $x, y \in \partial\Omega$ were arbitrary, we see that (v) holds. □

Lemma 9.3 (iv) $\Longrightarrow$ (xii).

Proof Fix a boundary face $F \subset \partial\Omega$ of maximal dimension. We claim that
$$\bar{\mathcal{E}}_\Omega \cap F = \varnothing.$$
Otherwise, there exists $x \in F$ and a sequence $x_n \in \mathcal{E}_\Omega$ such that $x_n \to x \in F$. Now fix an extreme point $y \in \partial\Omega \setminus \bar{F}$. Then, by hypothesis, $[x_n, y] \subset \partial\Omega$ for all n, so $[x, y] \subset \partial\Omega$.

Fix $z \in (x, y) \subset \partial\Omega$ and let C denote the convex hull of y and F. By [Observation 2.11](#),
$$\partial\Omega \supset F_\Omega(z) \supset \text{rel-int}(C).$$
Then
$$\dim F_\Omega(z) > \dim F,$$
which is a contradiction. So we must have $\bar{\mathcal{E}}_\Omega \cap F = \varnothing$, and hence (xii) holds. □

Lemma 9.4 (vi) $\Longrightarrow$ (viii).

Proof By [Theorem 7.3](#) every infinite-order element $g \in \Gamma$ preserves a properly embedded simplex $S \subset \Omega$ with $\dim(S) \geq 1$. Hence g fixes the vertices $v_1, \ldots, v_k$ of S. By hypothesis $s_{\partial\Omega}(v_1, v_2) < +\infty$ and hence, by [Theorem 7.1](#), $g^{\mathbb{Z}}$ has infinite index in the centralizer $C_\Gamma(g)$. □

Lemma 9.5 (x) $\Longrightarrow$ (iv).

Proof We prove the contrapositive: if there exist extreme points $x, y \in \partial\Omega$ with $(x, y) \subset \Omega$, then there exists a biproximal element $g \in \Gamma$ with $(\ell_g^+, \ell_g^-) \subset \Omega$. If such x and y exist, then by [Theorem 5.1](#) there exist biproximal elements $g_n \in \Gamma$ with $\ell_{g_n}^+ \to x$ and $\ell_{g_n}^- \to y$. Then, for n large, we must have $(\ell_{g_n}^+, \ell_{g_n}^-) \subset \Omega$. □

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