Geometry & Topology

Volume 27 (2023)

Weighted K–stability and coercivity with applications to extremal Kähler and Sasaki metrics

Vestislav Apostolov
Simon Jubert
Abdellah Lahdili

msp

Geometry & Topology **27**:8 (2023) 3229–3302
DOI: 10.2140/gt.2023.27.3229
Published: 9 November 2023

Weighted K–stability and coercivity with applications to extremal Kähler and Sasaki metrics

VESTISLAV APOSTOLOV
SIMON JUBERT
ABDELLAH LAHDILI

We show that a compact weighted extremal Kähler manifold, as defined by the third author (2019), has coercive weighted Mabuchi energy with respect to a maximal complex torus $\mathbb{T}^{\mathbb{C}}$ in the reduced group of complex automorphisms. This provides a vast extension and a unification of a number of results concerning Kähler metrics satisfying special curvature conditions, including Kähler metrics with constant scalar curvature, extremal Kähler metrics, Kähler–Ricci solitons, and their weighted extensions. Our result implies the strict positivity of the weighted Donaldson–Futaki invariant of any nonproduct $\mathbb{T}^{\mathbb{C}}$–equivariant smooth Kähler test configuration with reduced central fibre, a property known as $\mathbb{T}^{\mathbb{C}}$–equivariant weighted K–polystability on such test configurations. It also yields the $\mathbb{T}^{\mathbb{C}}$–uniform weighted K–stability on the class of smooth $\mathbb{T}^{\mathbb{C}}$–equivariant polarized test configurations with reduced central fibre. For a class of fibrations constructed from principal torus bundles over a product of Hodge cscK manifolds, we use our results in conjunction with results of Chen and Cheng (2021), He (2019) and Han and Li (2022) in order to characterize the existence of extremal Kähler metrics and Calabi–Yau cones associated to the total space, in terms of the coercivity of the weighted Mabuchi energy of the fibre. This yields a new existence result for Sasaki–Einstein metrics on certain Fano toric fibrations, extending the results of Futaki, Ono and Wang (2009) in the toric Fano case, and of Mabuchi and Nakagawa (2013) in the case of Fano $\mathbb{P}^1$–bundles.

32Q20, 53C25, 53C55, 58J60; 14J45, 32J27

Introduction

We are concerned with the existence and obstruction theory of a class of special Kähler metrics, called *weighted constant scalar curvature metrics*, which were introduced by the third author in [54; 55], giving a vast extension of the notion of Kähler metrics of constant scalar curvature (cscK for short), and providing the unification of a number of related notions of Kähler metrics satisfying special curvature conditions.

0.1 The weighted cscK problem

Let X be a smooth compact complex m–dimensional manifold with a given de Rham cohomology class $\alpha \in H^{1,1}(X,\mathbb{R})$ of Kähler metrics, and let $\mathbb{T} \subset \mathrm{Aut}_r(X)$ denote a fixed compact torus in the *reduced* group $\mathrm{Aut}_r(X)$ of automorphisms of X, ie the connected subgroup of automorphisms of X generated by the Lie algebra of real holomorphic vector fields with zeros; see eg Gauduchon [41]. It is well known that $\mathbb{T}$ acts in a hamiltonian way with respect to any $\mathbb{T}$–invariant Kähler metric $\omega \in \alpha$, and the corresponding momentum map μ_ω sends X onto a compact convex polytope $\Delta \subset \mathfrak{t}^*$ in the dual vector space $\mathfrak{t}^*$ of the Lie algebra $\mathfrak{t}$ of $\mathbb{T}$; see Atiyah [9] and Guillemin and Sternberg [44]. Furthermore, up to translations, Δ is independent of the choice of $\omega \in \alpha$. We shall further fix Δ, giving rise to a *normalization* of the corresponding momentum maps $\{\mu_\omega \mid \omega \in \alpha\}$.

Following [55], let $v(\mu) > 0$ and $w(\mu)$ be smooth functions defined on Δ. One can then consider the following condition for $\mathbb{T}$–invariant Kähler metrics ω in α (and fixed polytope Δ), called (v,w)*–cscK metrics*:

$$\mathrm{Scal}_v(\omega) = w(\mu_\omega). \tag{1}$$

Here the so-called v*–scalar curvature* of ω is defined by

$$\mathrm{Scal}_v(\omega) := v(\mu_\omega)\,\mathrm{Scal}(\omega) + 2\Delta_\omega v(\mu_\omega) + \langle g_\omega, \mu_\omega^*(\mathrm{Hess}(v))\rangle, \tag{2}$$

with $\mathrm{Scal}(\omega)$ being the usual scalar curvature of the riemannian metric g_ω associated to ω, Δ_ω the Laplace operator of g_ω, and where the contraction $\langle\cdot,\cdot\rangle$ is taken between the smooth $\mathfrak{t}^* \otimes \mathfrak{t}^*$–valued function g_ω on X (the restriction of the riemannian metric g_ω to $\mathfrak{t} \subset C^\infty(X, TX)$) and the smooth $\mathfrak{t} \otimes \mathfrak{t}$–valued function ${\mu_\omega}^*(\mathrm{Hess}(v))$ on X (given by the pullback by μ_ω of $\mathrm{Hess}(v) \in C^\infty(\Delta, \mathfrak{t} \otimes \mathfrak{t})$). The relevance of (1) to various geometric conditions is discussed in detail in [55], but we mention below a few special cases which partly motivate our study:

- $v = 1$ and w is a constant: this is the familiar cscK problem.
- $v = 1$ and $w = \ell$ with ℓ an affine-linear function on $\mathfrak{t}^*$: (1) then describes an extremal Kähler metric in the sense of Calabi [21].
- $v = e^\ell$ and $w = 2(\ell + a)e^\ell$, where ℓ is an affine-linear function on $\mathfrak{t}^*$ and a is a constant corresponding to the so-called μ–cscK (see Inoue [50]), extending the notion of Kähler–Ricci solitons (see Tian and Zhu [71]) defined when X is Fano and $\alpha = 2\pi c_1(X)$.

- $v=\ell^{-m-1}$, $w=a\ell^{-m-2}$ and $\alpha=c_1(L)$, where ℓ is a positive affine-linear function on Δ, m is the complex dimension of X, a is a constant, and L is a polarization of X: (1) then describes a scalar flat cone Kähler metric on the affine cone $(L^{-1})^{\times}$ polarized by the lift of $\xi=d\ell$ to L^{-1} via ℓ; see Apostolov, Calderbank and Legendre [2; 7].

In general, the problem of finding a $\mathbb{T}$–invariant Kähler metric $\omega\in\alpha$ solving (1) is obstructed in a similar way that the cscK problem is obstructed by the vanishing of the Futaki invariant: for any $\mathbb{T}$–invariant Kähler metric $\omega\in\alpha$ and any affine-linear function ℓ on $\mathfrak{t}^*$, one must have

$$\mathrm{Fut}_{v,w}(\ell):=\int_X(\mathrm{Scal}_v(\omega)-w(\mu_\omega))\ell(\mu_\omega)\omega^m=0, \tag{3}$$

should a solution to (1) exist. In [55], an unobstructed modification of (1) is proposed, extending Calabi's notion [21] of extremal Kähler metrics. To this end, suppose that $v, w_0>0$ are positive smooth functions on Δ. One can then find a unique affine-linear function $\ell^{\mathrm{ext}}_{v,w_0}(\mu)$ on $\mathfrak{t}^*$, called *the extremal function*, such that (3) holds for the weights $(v,w)=(v,\ell^{\mathrm{ext}}_{v,w_0}w_0)$. In this case, a solution of the (v,w)–cscK problem (1) is referred to as a (v,w_0)–*extremal Kähler metric*. We emphasize that (v,w_0)–extremal Kähler metrics are (v,w)–cscK metrics with a special property of the weight function w, namely, $w=\ell w_0$ with $w_0>0$ on Δ and ℓ affine-linear. In particular, (v,w)–cscK metrics with $w\neq 0$ on Δ are (v,w)–extremal with $\ell^{\mathrm{ext}}_{v,w}=\mathrm{sign}(w|_\Delta)$ and $(v,0)$–cscK metrics are (v,w)–extremal with $\ell^{\mathrm{ext}}_{v,w}=0$ for any $w>0$. It follows that all the above listed special cases are examples of (v,w)–extremal Kähler metrics, and thus the setup of (v,w)–extremal Kähler metrics allows one to study all these cases together.

0.2 Relation to v–solitons

Motivated by works of T Mabuchi [58; 59] and subsequent work by Berman and Nyström [15], Y Han and C Li [45] have recently introduced and studied the general notion of a *weighted v–soliton* on a smooth Fano variety X, as follows. In the setup explained above, we let $\alpha=2\pi c_1(X)$ and consider the natural action of $\mathbb{T}$ on K_X^{-1}, which fixes the momentum polytope Δ of $(X,\alpha,\mathbb{T})$ and normalizes the momentum map μ_ω for any $\mathbb{T}$–invariant Kähler metric $\omega\in\alpha$. For a (smooth) positive weight function $v(\mu)$ on Δ, one defines a v–soliton as a $\mathbb{T}$–invariant Kähler metric $\omega\in\alpha$, such that

$$\rho_\omega-\omega=\tfrac{1}{2}dd^c\log v(\mu_\omega), \tag{4}$$

where ρ_ω denotes the Ricci form of ω. Notice that when $v(\mu)=e^{\langle\mu,\xi\rangle}$ for some $\xi\in\mathfrak{t}$, one gets the well-studied class of *Kähler–Ricci solitons* [71] whereas the case when

$v(\mu)$ is a positive affine-linear function on Δ corresponds to the *Mabuchi solitons* studied in [58; 59]. As we shall see below, other choices for v are also geometrically meaningful. We make the following useful observation:

Proposition 1 *Let X be a smooth Fano manifold and $\mathbb{T} \subset \mathrm{Aut}(X)$ a compact torus. A $\mathbb{T}$–invariant Kähler metric $\omega \in 2\pi c_1(X)$ is a v–soliton if and only if ω is (v, w)–cscK with $w(\mu) := 2(m + \langle d \log v, \mu\rangle)v(\mu)$.*

We use the above result in order to make connection with the recent paper [45] (where the authors obtain a complete Yau–Tian–Donaldson type correspondence for the existence of v–Ricci solitons), which will play an important role in our present study of (v, w)–cscK metrics.

We also notice that v–solitons can be viewed as $(\bar{v}, \bar{w})$–cscK metrics for *different* choices of weights. This is for instance the case when $v(\mu) = \ell(\mu)^{-(m+2)}$, where $\ell(\mu) = \langle \xi, \mu\rangle + a$ is positive affine-linear on Δ. Whereas Proposition 1 identifies the v–soliton as a (v, w)–cscK metric with

$$v = \ell^{-(m+2)} \quad \text{and} \quad w = 2\ell^{-(m+3)}(-2\ell + (m+2)a),$$

we also observe:

Proposition 2 *Let $(X, \mathbb{T})$ be a smooth Fano variety and $\ell(\mu) = (\langle \xi, \mu\rangle + a)$ a positive affine-linear function on its canonical polytope Δ. A $\mathbb{T}$–invariant Kähler metric $\omega \in 2\pi c_1(X)$ is an $\ell^{-(m+2)}$–soliton if and only if the lift $\hat{\xi}$ of $\xi = d\ell$ to K_X via ℓ is the Reeb vector field of a Sasaki–Einstein structure defined on the unit circle bundle $N \subset K_X$ with respect to the hermitian metric on K_X with curvature $-\omega$. This condition is also equivalent to ω being an $(\ell^{-m-1}, 2ma\ell^{-m-2})$–cscK metric.*

0.3 Main results

Similarly to the usual cscK case, it is shown by Lahdili [55] that the solutions of (1) can be characterized as minimizers of a functional $\boldsymbol{M}_{v,w}$ defined on the space of $\mathbb{T}$–invariant Kähler metrics in α, extending the *Mabuchi energy* to the weighted setting (see Section 1 below for the precise definition). After the deep works of Berman, Darvas and Lu [14] and Chen and Cheng [23], it is now well understood that the coercivity of the Mabuchi energy is equivalent to the existence of a cscK metric in a given cohomology class. Noting that, by the results in [55], any (v, w)–extremal metric is invariant under a *maximal* compact torus in $\mathrm{Aut}_r(X)$, our first main result is an extension of one direction of the correspondence in the cscK case to the weighted setting.

Theorem 1 *Suppose $\mathbb{T} \subset \mathrm{Aut}_r(X)$ is a maximal torus in the reduced group of automorphisms of X, and $\omega_0 \in \alpha$ a $\mathbb{T}$–invariant (v, w_0)–extremal Kähler metric. Then the weighted Mabuchi energy $\boldsymbol{M}_{v,w}$ (with $w = \ell^{\mathrm{ext}}_{v,w_0} w_0$) is coercive relative to the complex torus $\mathbb{T}^{\mathbb{C}}$ in the sense of Darvas and Rubinstein [29], ie there exist positive real constants λ and δ such that for any $\mathbb{T}$–invariant Kähler metric $\omega \in \alpha$,*

$$\boldsymbol{M}_{v,w}(\omega) \geq \lambda \inf_{\sigma \in \mathbb{T}^{\mathbb{C}}} \boldsymbol{J}(\sigma^* \omega) - \delta,$$

where $\boldsymbol{J}$ denotes the Aubin functional on the space of Kähler metrics; *see Definition 3.1.*

Our proof of Theorem 1 adapts to the case when the torus $\mathbb{T} \subset \mathrm{Aut}_r(X)$ is not necessarily maximal. Instead of $\mathbb{T}^{\mathbb{C}}$ one takes the infimum of $\boldsymbol{J}(\sigma^*\omega)$ over $\widehat{\mathbb{G}} := \mathrm{Aut}^{\mathbb{T}}_r(X)$, the connected component of the identity of the centralizer of $\mathbb{T}$ in $\mathrm{Aut}_r(X)$ (which by [55] is a reductive group if X admits a (v, w_0)–extremal $\mathbb{T}$–invariant Kähler metric; see Remark 7.7 for more details). Furthermore, we can also consider *any* reductive connected subgroup group $\mathbb{G} = \mathbb{K}^{\mathbb{C}} \subset \widehat{\mathbb{G}}$ with a compact form $\mathbb{K}$ containing $\mathbb{T}$, and restrict $\boldsymbol{M}_{v,w}$ to the space of $\mathbb{K}$–invariant Kähler metrics in α as in Han and Li [45].[1]

As noticed by Berman, Darvas and Lu [14] (in the polarized case) and by Sjöström Dyrefelt [66] (in the more general Kähler case), the coercivity of the Mabuchi energy yields a sharp estimate of the sign of the Donaldson–Futaki invariant of a $\mathbb{T}$–equivariant test configuration. In our weighted setting, we consider $\mathbb{T}$–equivariant (compactified) Kähler test configurations $(\mathscr{X}, \mathscr{A})$ associated to $(X, \alpha, \mathbb{T})$, which have smooth total space. To any such test configuration one can associate a *weighted Donaldson–Futaki* invariant by the formula (see [55])

$$\mathscr{F}_{v,w}(\mathscr{X}, \mathscr{A}) := -\int_{\mathscr{X}} (\mathrm{Scal}_v(\Omega) - w(\mu_\Omega))\Omega^{[m+1]} + (8\pi) \int_X v(\mu_\omega)\omega^{[m]},$$

where $\Omega \in \mathscr{A}$ and $\omega \in \alpha$ are $\mathbb{T}$–invariant Kähler forms on $\mathscr{X}$ and X, respectively, with respective Δ–normalized momentum maps μ_Ω and μ_ω, and $\mathrm{Scal}_v(\Omega)$ is the v–scalar curvature of Ω defined by (2). In the above formula, for any 2–form ψ we use the convention $\psi^{[k]} := \psi^k / k!$. Thus, $\mathscr{F}_{v,w}(\mathscr{X}, \mathscr{A})$ extends to the weighted setting the expression of the Donaldson–Futaki invariant of $(\mathscr{X}, \mathscr{A})$ in terms of intersection numbers (see Odaka [62] and Wang [72]).

Corollary 1 *Under the hypotheses of Theorem 1, for any $\mathbb{T}$–equivariant smooth Kähler test configuration $(\mathscr{X}, \mathscr{A})$ of $(X, \alpha, \mathbb{T})$ which has a reduced central fibre,*

$$\mathscr{F}_{v,w}(\mathscr{X}, \mathscr{A}) \geq 0,$$

[1] We are grateful to Chi Li for pointing this out to us.

with equality if and only if $(\mathscr{X}, \mathscr{A})$ *is a product test configuration. Furthermore, if* $\alpha = 2\pi c_1(L)$ *corresponds to a polarization* L *of* X *and* $(\mathscr{X}, \mathscr{L}, \mathbb{T})$ *is a* $\mathbb{T}$*–equivariant smooth polarized test configuration of* (X, L) *as above,*

$$\mathscr{F}_{v,w}(\mathscr{X}, \mathscr{A}) \geq \lambda \boldsymbol{J}^{\mathrm{NA}}_{\mathbb{T}^{\mathbb{C}}}(\mathscr{X}, \mathscr{A}),$$

where $\mathscr{A} = 2\pi c_1(\mathscr{L})$, $\lambda > 0$ *is the constant appearing in Theorem 1, and* $\boldsymbol{J}^{\mathrm{NA}}_{\mathbb{T}^{\mathbb{C}}}(\mathscr{X}, \mathscr{A})$ *is the* $\mathbb{T}^{\mathbb{C}}$*–relative non-Archimedean* $\boldsymbol{J}$*–functional of the test configuration introduced in Hisamoto [49] and Li [57]; see (20).*

Corollary 1 improves the ($\mathbb{T}$–equivariant) (v, w)–K–semistability established in Lahdili [56, Theorem 2] to ($\mathbb{T}$–equivariant) (v, w)–K–polystability on the test configurations as above, and, in the projective case, further to $\mathbb{T}^{\mathbb{C}}$–uniform (v, w)–K–stability in the sense of [49; 57]. As we already mentioned, the fist part of Corollary 1 was proved by Berman, Darvas and Lu [14], Sjöström Dyrefelt [66] and Stoppa [67] in the cscK case ($v = 1$ and w is a constant), and by Dervan [33], [66] and Stoppa and Székelyhidi [68] in the unweighted extremal case ($v = 1 = w_0$). We however notice that in the extremal case our proof uses directly the coercivity of the *relative* Mabuchi energy (which follows from Theorem 1) whereas the proofs in [33; 68] and [66] are based on the Arezzo–Pacard existence results of extremal metrics on blow-ups (see Arezzo, Pacard and Singer [8]), and on the coercivity of the unweighted Mabuchi energy $\boldsymbol{M}_{1,c}$ established in [14; 66], respectively. The $\mathbb{T}^{\mathbb{C}}$–uniform (v, w)–K–stability statement in the second part of Corollary 1 is established in the cscK case in [49; 57], and in the case of a v–soliton by Han and Li [45]. Our proof of Corollary 1 in the general weighted case follows easily from Theorem 1 by the established techniques in the cscK case; see Section 4.

Another notable special case where our results apply is when $\alpha = c_1(L)$ for an ample line bundle L over X, and $v = \ell^{-m-1}$ and $w_0 = \ell^{-m-3}$ for a positive affine-linear function on Δ. It is observed by Apostolov and Calderbank [2] that in this case a (v, w_0)–extremal Kähler metric in α describes an *extremal Sasaki* metric on the total space N of the unit circle bundle in L^{-1} with respect to the hermitian metric with curvature $-\omega$, and Reeb vector field corresponding to the lift of $d\ell$ to L^{-1} via ℓ. In this special case, the first part of Corollary 1 above was obtained by Apostolov, Calderbank and Legendre [7] for *polarized* test configurations (see Theorem 1, Conjecture 5.8 and Remark 5.9 in [7]), by using the results in He and Li [48] which establish an analogue of Theorem 1 in the Sasaki case. Thus, our proofs of Theorem 1 and Corollary 1 allow one to recast and further generalize [7, Theorem 1] entirely within the framework of the weighted Kähler geometry of X.

0.4 Method of proof

We now discuss briefly the method of proof of Theorem 1 above. It is an application of the general coercivity principle of Darvas and Rubinstein [29, Theorem 3.4]; see Section 3. This principle is used in the cscK case by Berman, Darvas and Lu [14], and our approach is mainly inspired by these two references. Noting that in the weighted extremal case $\boldsymbol{M}_{v,w}$ is $\mathbb{G}$–invariant and $\mathbb{G} := \mathbb{T}^{\mathbb{C}}$ is reductive, by the results of [29], in order to obtain Theorem 1 one needs to

(i) extend $\boldsymbol{M}_{v,w}$ to the space $\mathcal{E}^1(X,\omega_0)$ of ω_0–relative plurisubharmonic functions of maximal mass and finite energy;

(ii) show that the extension is convex and continuous along weak d_1–geodesics in $\mathcal{E}^1(X,\omega_0)$;

(iii) establish a compactness result for the extension of $\boldsymbol{M}_{v,w}$; and

(iv) show the uniqueness modulo the action of $\mathbb{G}$ (and in particular the regularity) of the weak minimizers of $\boldsymbol{M}_{v,w}$, under the assumption that a (v,w_0)–extremal metric exists.

The steps (i), (ii) and (iii) in the unweighted cscK case are obtained by Berman, Darvas and Lu [13] and follow from the Chen–Tian formula of $\boldsymbol{M}_{1,1}$. The analogous formula for $\boldsymbol{M}_{v,w}$ is obtained by Lahdili [55], but the presence of weights does not allow for a straightforward generalization of the arguments in [13]. Similar difficulty arises in Berman and Nyström [15] in the framework of v–solitons on a Fano variety, where the authors were able to obtain a suitable extension of the *weighted Ding functional* to the space $\mathcal{E}^1(X,\omega_0)$. This functional has milder dependence on the weights than the weighted Mabuchi functional we consider. Indeed, the arguments of [15] yield the existence of a continuous extension to $\mathcal{E}^1(X,\omega_0)$ of one of the three terms in the Chen–Tian decomposition of $\boldsymbol{M}_{v,w}$, which depend on the weight w. Building on [15], Han and Li [45] proposed a new approach to the extension problem in the case of v–solitons, based on an idea going back to Donaldson [36] (see in particular the proof of Proposition 3 in [36]), which amounts to considering suitable fibre bundles Y over a cscK base B and fibre X, and showing that the weighted quantities on X correspond to the restrictions of unweighted quantities on the total space Y. This is the *semisimple principal* $(X,\mathbb{T})$*–fibration construction*, which we review in the next subsection. Going further than [45], we express in general the scalar curvature of a bundle-compatible Kähler metric on Y in terms of the weighted scalar curvature of X, and show that the usual (unweighted) Mabuchi energy on Y restricts to a suitably weighted Mabuchi energy on X. It thus follows that, at least for suitable polynomial weights v, the remaining terms of the Chen–Tian decomposition of $\boldsymbol{M}_{v,w}$ can be extended to $\mathcal{E}^1(X,\omega_0)$ simply by restricting to the fibres the corresponding (unweighted) extension of the Mabuchi energy of Y. The final crucial observation for obtaining the extension for *any* weights is that $\boldsymbol{M}_{v,w}$ depends *linearly* and *continuously* on (v,w), so that one can further use (as in [45]) the Stone–Weierstrass approximation theorem over $C^0(\Delta)$. With this in place, and using the weighted analogue of the uniqueness (see Berman and Berndtsson [11]) achieved by Lahdili [56], we can adapt the arguments from [14].

0.5 Applications to the semisimple principal fibration construction

We briefly review here the semisimple principal bundle construction, which is not only a key tool in our proof of Theorem 1, but also provides a framework for further geometric applications of our results, extending the setting of the generalized Calabi construction in Apostolov, Calderbank, Gauduchon and Tønnesen-Friedman [6].

We denote by $\mathbb{T}$ a compact r–dimensional torus with Lie algebra $\mathfrak{t}$ and lattice $\Lambda\subset\mathfrak{t}$ of generators of S^1–subgroups, ie $\mathbb{T}=\mathfrak{t}/2\pi\Lambda$. Let $B=B_1\times\cdots\times B_k$ be a $2n$–dimensional cscK manifold which is a product of compact cscK Hodge Kähler $2n_a$–manifolds (B_a,ω_{B_a}) for $a=1,\dots,k$. We then consider a principal $\mathbb{T}$–bundle $\pi\colon P\to B$ endowed with a connection 1–form $\theta\in\Omega^1(P,\mathfrak{t})$ with curvature

$$d\theta=\sum_{a=1}^{k}(\pi^*\omega_{B_a})\otimes p_a \quad \text{for } p_a\in\Lambda.$$

For any smooth compact Kähler $2m$–manifold $(X,\omega_X,\mathbb{T})$, endowed with a hamiltonian isometric action of the torus $\mathbb{T}$ as in the setup above, we can construct the *principal $(X,\mathbb{T})$–fibration*

$$Y:=(X\times P)/\mathbb{T}\to B,$$

where the $\mathbb{T}$–action on the product is $\sigma(x,p)=(\sigma^{-1}x,\sigma p)$ for $x\in X$, $p\in P$ and $\sigma\in\mathbb{T}$. Using the chosen connection on P, the almost complex structures on X and B lift to define a CR structure on the product $X\times P$, and thus endow Y with the structure of a $2(m+n)$–dimensional smooth complex manifold. Furthermore, Y comes equipped with an induced holomorphic fibration $\pi\colon Y\to B$, with smooth complex fibres X, and induced fibrewise $\mathbb{T}$–action. Fixing constants $c_a\in\mathbb{R}$ such that, for each $a=1,\dots,k$, the affine-linear function $\langle p_a, \mu\rangle + c_a$ on $\mathfrak{t}^*$ is strictly positive on the momentum image Δ of X, one can define a lifted Kähler metric ω_Y on Y which, pulled back to $X \times P$, has the form

$$\omega_Y := \omega_X + \sum_{a=1}^{k} (\langle p_a, \mu_\omega\rangle + c_a)\pi^*\omega_{B_a} + \langle d\mu_\omega \wedge \theta\rangle,$$

where $\langle \cdot\,,\cdot\rangle$ stands for the natural pairing of $\mathfrak{t}$ and $\mathfrak{t}^*$. Thus $\langle p_a, \mu_\omega\rangle$ is a smooth function and $\langle d\mu_\omega \wedge \theta\rangle$ is a 2–form on $X \times P$. As we show in Section 5, when ω_X varies in a given Kähler class of X, the corresponding Kähler metric ω_Y will vary in a fixed Kähler class on Y. We also notice that when $(X, \omega_X, \mathbb{T})$ is a smooth *toric* Kähler manifold, the setup above reduces to the theory of *semisimple rigid* toric fibrations studied by Apostolov, Calderbank, Gauduchon and Tønnesen-Friedman [3; 4; 6]. Inspired by these results, we show that the scalar curvature of ω_Y can be expressed in terms of the p–weighted scalar curvature of (X, ω_X), where the weight function $p(\mu)$ is a polynomial depending on the fixed data (p_a, c_a, n_a) of the construction. With this observation in mind, we show that (similarly to the case of semisimple rigid toric fibrations recently studied by Jubert [52]) the recent results Chen and Cheng [23] and He [47] can be used to obtain a converse of Theorem 1 in the case of semisimple principal fibrations.

Theorem 2 *Suppose Y is a semisimple principal $(X, \mathbb{T})$–fibration, with a Kähler metric ω_Y induced by a $\mathbb{T}$–invariant Kähler metric ω_X on X. Suppose, moreover, that $\mathbb{T}$ is a maximal torus in the reduced group of automorphisms* $\mathrm{Aut}_r(X)$. *Then, the following conditions are equivalent:*

(i) *Y admits an extremal Kähler metric in the Kähler class $[\omega_Y]$.*

(ii) *X admits a $\mathbb{T}$–invariant $(p, \tilde{w})$–cscK metric in the Kähler class $[\omega_X]$, with weights*

$$p(\mu) = \prod_{a=1}^{k} (\langle p_a, \mu\rangle + c_a)^{n_a}, \quad \tilde{w}(\mu) = p(\mu)\biggl(-\sum_{a=1}^{k} \frac{\mathrm{Scal}(\omega_{B_a})}{\langle p_a, \mu\rangle + c_a} + \ell^{\mathrm{ext}}(\mu)\biggr),$$

where ℓ^{ext} is an affine-linear function determined by the condition (3).

(iii) *The weighted Mabuchi energy $\boldsymbol{M}^X_{p,\tilde{w}}$ of $(X, [\omega_X], \mathbb{T})$ is coercive with respect to $\mathbb{T}^{\mathbb{C}}$, where p and $\tilde{w}$ are the weights defined in* (ii).

Compared to the general setting of Dervan and Sektnan [35], the semisimple principal $(X, \mathbb{T})$–fibration (trivially) satisfies the condition of optimal symplectic connection. Accordingly, one can conclude by [35] that $(Y, [\omega_Y])$ admits an extremal Kähler metric, provided that (X,ω_X) is cscK, and if we take large enough constants c_a. As a matter of fact, the conclusion also follows under the more general assumption that (X,ω_X) is extremal, by the proof of [4, Theorem 3]. The novelty of Theorem 2 is therefore in the fact that it gives a precise condition (in terms of X) for the existence of an extremal Kähler metric in a *given* Kähler class $[\omega_Y]$, also revealing that $(X,[\omega_X])$ need not be extremal in general. We finally note that in the case of toric fibre, [52] provides a further equivalence with a certain weighted notion of *uniform K–stability* of the corresponding Delzant polytope.

If all the factors (B_a,ω_{B_a}) of the base are positive Kähler–Einstein manifolds, and the fibre $(X,\mathbb{T})$ is a smooth Fano variety, the semisimple principal $(X,\mathbb{T})$–fibration construction can produce a smooth Fano variety Y for suitable choice of the principal $\mathbb{T}$–bundle over B; see Lemma 5.11. In this case, combining Han and Li [45, Theorem 3.5] with our results:

Theorem 3 *Suppose Y is a Fano semisimple principal $(X,\mathbb{T})$–fibration, obtained from the product of positive Kähler–Einstein Hodge manifolds (B_a,ω_{B_a}) and a smooth Fano fibre $(X,\mathbb{T})$ via Lemma 5.11. Suppose also that $\mathbb{T}$ is a maximal torus in the automorphism group* $\mathrm{Aut}(X)$*. Then Y admits a v–soliton in $2\pi c_1(Y)$, provided that the weighted Mabuchi functional $\boldsymbol{M}^X_{pv,\tilde{w}}$ of $(X,\mathbb{T},2\pi c_1(X))$ is coercive with respect to $\mathbb{T}^{\mathbb{C}}$, where p is the weight defined in Theorem 2*(ii) *and*

$$\tilde{w}=2pv(m+\langle d\log v,\mu\rangle+\langle d\log p,\mu\rangle).$$

If, furthermore, the fibre $(X,\mathbb{T})$ is a smooth toric Fano variety, then this equation is equivalent to the vanishing of the Futaki invariant (3) associated to the weights $(pv,\tilde{w})$ on X. In particular, any Fano semisimple principal $(X,\mathbb{T})$–fibration with smooth toric Fano fibre $(X,\mathbb{T})$ admits a Kähler–Ricci soliton, and the corresponding affine cone $(K_Y)^{\times}$ admits a Calabi–Yau cone metric, given by a Sasaki–Einstein structure on a unit circle bundle associated to the canonical bundle K_Y.

The existence of a Kähler–Ricci soliton in the above setting is essentially known even though we didn't find it explicitly stated in the literature. In the toric case (ie when $Y=X$ and B is a point) the result follows by Wang and Zhu [73] (see also Datar and Székelyhidi [30]), and for $\mathbb{P}^1$–bundles by Apostolov, Calderbank, Gauduchon and Tønnesen-Friedman [5], Dancer and Wang [26] and Koiso [53]. In the general case the result can be obtained from Podestà and Spiro [63], which in turn extends [73] to the framework of multiplicity-free manifolds, but the arguments can be also adapted to the case of semisimple principal $(X,\mathbb{T})$–fibrations; see Apostolov, Calderbank, Gauduchon and Tønnesen-Friedman [6, Remark 7] and Donaldson [37]. Our approach, however, builds on the idea of [30]. There are also related existence results for Kähler–Ricci solitons on spherical manifolds; see Delcroix [31] and Delgove [32]. On the other hand, the existence of Sasaki–Einstein metrics seems to be new in the above stated generality. Indeed, in the toric case the claim follows from Futaki, Ono and Wang [40], and there are known existence results (see Boyer and Tønnesen-Friedman [20], Gauntlett, Martelli, Sparks and Waldram [42] and Mabuchi and Nakagawa [60]) on $\mathbb{P}^1$–bundles. We expect our arguments to extend to spherical manifolds too.

0.6 Structure of the paper

In Section 1, we recall the setup of weighted cscK metrics and state the main results we shall need from Lahdili [55; 56]. In Section 2, we recall the notion of v–solitons from Han and Li [45] and Mabuchi [58], and establish the equivalences stated in Propositions 1 and 2. Sections 3 and 4 review and recast in the weighted setting, respectively, the coercivity principle of Darvas and Rubinstein [29] and its application to stability (see Berman, Darvas and Lu [14] and Sjöström Dyrefelt [66]), thus outlining the main steps needed for the proof of Theorem 1 and from it deriving Corollary 1. In Section 5, we introduce the semisimple principal $(X, \mathbb{T})$–fibration construction, and establish the main geometric properties allowing us to extend the results from Apostolov, Calderbank, Gauduchon and Tønnesen-Friedman [6]. In Section 6, we use an idea from [45] in order to define an extension of the weighted Mabuchi energy to the space $\mathcal{E}^1(X, \omega_0)$, and show its convexity and compactness properties. In Section 7, we extend the arguments of [14] to show that weak minimizers of the weighted Mabuchi energy are smooth. Here, we complete the proof of Theorem 1. In Section 8, we detail the proofs of Theorems 2 and 3. In the appendices, we present some technical computational results, detailing the linearization of the scalar and the twisted scalar curvature of a semisimple principal $(X, \mathbb{T})$–fibre and recasting the weighted Futaki invariant (3), which are needed for the proofs of Theorems 2 and 3.

Acknowledgements Apostolov was supported in part by an NSERC discovery grant and a connect talent grant of the Région Pays de la Loire. Jubert was supported by PhD fellowships of the UQAM and the University of Toulouse III – Paul Sabatier. Lahdili was supported by a postdoctoral fellowship of the BICMR in Beijing. We are very grateful to the referee for their careful reading of the manuscript and many valuable suggestions, which substantially improved our work. We thank D Calderbank, T Darvas, R Dervan, E Inoue, E Legendre, C Li and G Tian for their interest and comments on the manuscript. Apostolov thanks P Gauduchon for sharing with him his unpublished notes, and C Tønnesen-Friedman for bringing [42] to our attention.

1 Preliminaries on the weighted cscK problem

We recall the setup from [55]. Let X be a smooth compact, connected Kähler manifold of (real) dimension $2m$, and let

$$\mathcal{K}(X,\omega_0) = \{\varphi \in C^\infty(X) \mid \omega_\varphi := \omega_0 + dd^c\varphi > 0\}$$

be the space of ω_0–relative smooth Kähler potentials on X. We let $\mathbb{T} \subset \mathrm{Aut}_r(X)$ be a fixed compact torus in the *reduced* group of automorphisms of X, ie the connected closed subgroup $\mathrm{Aut}_r(X)$ of the group of complex automorphisms $\mathrm{Aut}(X)$, whose Lie algebra is the space of holomorphic vector fields of X with zeros; see eg [41]. Equivalently, $\mathrm{Aut}_r(X)$ is the connected component of the identity of the kernel of the natural group homomorphism from $\mathrm{Aut}(X)$ to the Albanese torus, and is known to be isomorphic to the linear algebraic group in the Chevalley-type decomposition of $\mathrm{Aut}(X)$; see [38]. We denote by $C^\infty_{\mathbb{T}}(X)$ the space of $\mathbb{T}$–invariant smooth functions on X and introduce the space

$$\mathcal{K}_{\mathbb{T}}(X,\omega_0) := \mathcal{K}(X,\omega_0) \cap C^\infty_{\mathbb{T}}(X),$$

of $\mathbb{T}$–invariant relative Kähler potentials, assuming also that ω_0 is $\mathbb{T}$–invariant.

It is well known that the action of $\mathbb{T}$ on (X,ω_0) is hamiltonian, and we let $\mu_0 : X \to \mathfrak{t}^*$ be a momentum map, where $\mathfrak{t}$ is the Lie algebra of $\mathbb{T}$ and $\mathfrak{t}^*$ the dual vector space. By the convexity theorem [9; 44], the image $\Delta := \mu_0(X) \subset \mathfrak{t}^*$ is a compact convex polytope. For any $\varphi \in \mathcal{K}_{\mathbb{T}}(X,\omega_0)$, the smooth $\mathfrak{t}^*$–valued function

$$\mu_\varphi = \mu_0 + d^c\varphi \tag{5}$$

is the $\mathbb{T}$–momentum map of (X,ω_φ), normalized by the condition $\mu_\varphi(X) = \Delta$. In the above formula, $d^c\varphi$ is viewed as a smooth $\mathfrak{t}^*$–valued function via the identity $\langle d^c\varphi, \xi\rangle := d^c\varphi(\xi)$ for any $\xi \in \mathfrak{t} \subset C^\infty(X, TX)$.

1.1 The (v, w)–constant scalar curvature Kähler metrics

Following [55], let $v(\mu) > 0$ and $w(\mu)$ be smooth functions on Δ. One can then consider the condition (1) for a $\mathbb{T}$–invariant Kähler metric ω_φ in α (and the fixed polytope Δ), called a (v, w)*–cscK metric*. We thus want to solve the PDE for $\varphi \in \mathcal{K}_{\mathbb{T}}(X,\omega_0)$

$$\mathrm{Scal}_v(\omega_\varphi) = w(\mu_\varphi), \tag{6}$$

where

$$\mathrm{Scal}_v(\omega_\varphi) := v(\mu_\varphi)\,\mathrm{Scal}(\omega_\varphi) + 2\Delta_{\omega_\varphi} v(\mu_\varphi) + \langle g_\varphi, \mu_\varphi^*(\mathrm{Hess}(v))\rangle.$$

As we explained in the introduction, the problem of finding $\omega_\varphi \in \alpha$ solving (6) is obstructed by the condition (3), and in the case when v and w_0 are positive weights, this can be resolved (similarly to the approach in [21]) by finding a unique affine-linear function $\ell^{\mathrm{ext}}_{v,w_0}(\mu)$ on $\mathfrak{t}^*$, called *the extremal function*, such that for any ω_φ,

$$\int_X (\mathrm{Scal}_v(\omega_\varphi) - \ell^{\mathrm{ext}}_{v,w_0}(\mu_\varphi) w_0(\mu_\varphi))\ell(\mu_\varphi)\omega_\varphi^{[m]} = 0 \quad \text{for all } \ell \in \mathrm{Aff}(\mathfrak{t}^*).$$

Geometrically, the above condition means that the weighted cscK problem with weights $(v, w) = (v, \ell^{\mathrm{ext}}_{v,w_0} w_0)$ is unobstructed in terms of (3), and a solution ω_φ of the $(v, \ell^{\mathrm{ext}}_{v,w_0} w_0)$–cscK problem is referred to as a (v, w_0)*–extremal metric*.

1.2 The weighted Mabuchi energy

Definition 1.1 [55] Let v and w be weight functions on Δ with $v(\mu) > 0$. The weighted Mabuchi energy $\boldsymbol{M}_{v,w}$ on $\mathcal{K}_{\mathbb{T}}(X, \omega_0)$ is defined by

$$(d_\varphi \boldsymbol{M}_{v,w})(\dot\varphi) = -\int_X (\mathrm{Scal}_v(\omega_\varphi) - w(\mu_\varphi))\dot\varphi\,\omega_\varphi^{[m]}, \quad \boldsymbol{M}_{v,w}(0) = 0.$$

Remark 1.2 It follows from the above definition and the results in [55] that for a constant c, $\boldsymbol{M}_{v,w}(\varphi + c) = \boldsymbol{M}_{v,w}(\varphi)$ if and only if v and w satisfy the integral relation

$$\int_X \mathrm{Scal}_v(\omega_0)\omega_0^{[m]} = \int_X w(\mu_0)\omega_0^{[m]}. \tag{7}$$

Furthermore, by the results in [55], (7) is a necessary condition for the existence of a solution of (6) and it is incorporated in the definition of $\boldsymbol{M}_{v,w}$ given in [55] via the constant $c_{v,w}(\alpha)$ in front of w, but we do not assume a priori this condition in the current article. It is however automatically satisfied if α admits a $\mathbb{T}$–invariant (v, w)–cscK metric, or if we consider the weights $(v, w) = (v, \ell^{\mathrm{ext}}_{v,w_0} w_0)$ corresponding to (v, w_0)–extremal Kähler metrics. In these cases, we shall write $\boldsymbol{M}_{v,w}(\omega_\varphi)$ to emphasize that the weighted Mabuchi functional acts on the space of $\mathbb{T}$–invariant Kähler metrics in $\alpha = [\omega_0]$.

The following result is established in [56], generalizing [11] to arbitrary weights $v > 0$ and w:

Theorem 1.3 *If ω is a $\mathbb{T}$–invariant (v, w)–cscK metric on $(X, \alpha, \mathbb{T}, \Delta)$, then for any $\varphi \in \mathcal{K}_{\mathbb{T}}(X, \omega_0)$ we have $\boldsymbol{M}_{v,w}(\omega_\varphi) \geq \boldsymbol{M}_{v,w}(\omega)$.*

1.3 The automorphism group of a (v, w_0)–extremal Kähler manifold

In what follows we will consider *connected* Lie groups. We recall that we have set $\mathrm{Aut}_r(X)$ to be the connected component of the identity of the kernel of the Albanese homomorphism and, similarly, we denote by $\mathrm{Aut}_r^{\mathbb{T}}(X)$ the connected component of the identity of the centralizer of the torus $\mathbb{T}$ in $\mathrm{Aut}_r(X)$. We shall use the following result, established in [55, Theorem B.1] (see also [39]) and [56, Remark 2]:

Proposition 1.4 *If $(X, \alpha, \mathbb{T})$ admits a (v, w_0)–extremal Kähler metric ω, then the connected component of the identity $\mathrm{Aut}_r^{\mathbb{T}}(X)$ of the subgroup of $\mathbb{T}$–commuting automorphisms in $\mathrm{Aut}_r(X)$ is reductive, and ω is invariant under the action of a maximal compact connected subgroup of $\mathrm{Aut}_r^{\mathbb{T}}(X)$. In particular, the isometry group of (X, ω) contains a maximal torus $\mathbb{T}_{\max} \subset \mathrm{Aut}_r(X)$ with $\mathbb{T} \subset \mathbb{T}_{\max}$. If, furthermore, $\mathbb{T} = \mathbb{T}_{\max}$, then $\mathrm{Aut}_r^{\mathbb{T}}(X) = \mathbb{T}^{\mathbb{C}}$.*

Because of this result, we shall often assume (without loss of generality for solving (6)) that $\mathbb{T} = \mathbb{T}_{\max} \subset \mathrm{Aut}_r(X)$ and thus $\mathrm{Aut}_r^{\mathbb{T}}(X) = \mathbb{T}^{\mathbb{C}}$.

1.4 Uniqueness of the (v, w_0)–extremal Kähler metrics

Another key result in the theory is the extension in [56] of the uniqueness results [11; 24] to the weighted setting.

Theorem 1.5 *Suppose ω and ω' are $\mathbb{T}$–invariant (v, w_0)–extremal Kähler metrics. Then there exists $\sigma \in \mathrm{Aut}_r^{\mathbb{T}}(X)$ such that $\sigma^*(\omega') = \omega$. In particular, if $\mathbb{T} \subset \mathrm{Aut}_r(X)$ is maximal, then the uniqueness holds modulo $\mathbb{T}^{\mathbb{C}}$.*

2 v–solitons as weighted cscK metrics

We review here the definition of v–solitons on a Fano manifold, following [15; 45], and discuss their link with (v, w)–cscK metrics.

We thus suppose throughout this section that X is a smooth Fano manifold $\alpha := 2\pi c_1(X)$ and $\mathbb{T} \subset \mathrm{Aut}(X)$ a fixed compact torus. (We recall here that on a Fano manifold $\mathrm{Aut}_r(X)$ coincides with the connected component of the identity of the full automorphism group.) We further consider the *natural* action of $\mathbb{T}$ on the anticanonical bundle K_X^{-1} of X, which normalizes the momentum map μ_ω of each $\mathbb{T}$–invariant Kähler metric $\omega \in \alpha$, and fixes the momentum image Δ. We shall sometimes refer to this normalization as the *canonical normalization* of Δ. In this setup, we recall:

Definition 2.1 Let $v > 0$ be a positive smooth weight function on Δ. A v–soliton on X is a $\mathbb{T}$–invariant Kähler metric $\omega \in 2\pi c_1(X)$ which satisfies the relation (4).

In the special case $v = e^{\langle \xi, \mu \rangle}$ we obtain a Kähler–Ricci soliton in the sense of [71].

Lemma 2.2 *A $\mathbb{T}$–invariant Kähler metric $\omega \in 2\pi c_1(X)$ is a v–soliton if and only if ω is a (v, w)–cscK metric with weight $w(\mu) = 2v(\mu)[m + \langle d \log v(\mu), \mu \rangle]$.*

Proof We start by showing that (4) implies that ω is (v, w)–cscK with the weight w specified in the lemma. Taking the trace in (4) with respect to ω gives

$$
\begin{aligned}
\text{(8)} \quad \mathrm{Scal}(\omega) - 2m &= -\Delta_\omega(\log v(\mu_\omega)) \\
&= -\frac{1}{v(\mu_\omega)} \Delta_\omega(v(\mu_\omega)) - \frac{1}{v(\mu_\omega)^2} g_\omega(dv(\mu_\omega), dv(\mu_\omega)) \\
&= -\sum_{i=1}^{m} \frac{v_{,i}(\mu_\omega)}{v(\mu_\omega)} (\Delta_\omega \mu_\omega^{\xi_i}) + \sum_{i,j=1}^{m} \frac{v_{,ij}(\mu_\omega)}{v(\mu_\omega)} g_\omega(\xi_i, \xi_j) \\
&\qquad - \sum_{i,j=1}^{m} \frac{v_{,i}(\mu_\omega) v_{,j}(\mu_\omega)}{v(\mu_\omega)^2} g_\omega(\xi_i, \xi_j),
\end{aligned}
$$

where $(\xi_i)_{i=1,\dots,r}$ is a basis of $\mathfrak{t}$ and $v_{,i}$ denotes the partial derivative in direction of ξ_i. On the other hand, by taking the interior product of (4) with ξ_i and using that ξ_i is Killing with respect to ω, we get

$$
-d\Delta_\omega \mu_\omega^{\xi_i} + 2d\mu_\omega^{\xi_i} = d\big(d^c(\log v(\mu_\omega))(\xi_i)\big) = d\Bigg(\sum_{j=1}^{m} \frac{v_{,j}(\mu_\omega)}{v(\mu_\omega)} g_\omega(\xi_i, \xi_j)_\omega\Bigg),
$$

where $\mu_\omega^{\xi} := \langle \mu_\omega, \xi \rangle$ is the momentum of ξ. It follows that

$$
\text{(9)} \qquad -\Delta_\omega \mu_\omega^{\xi_i} + 2\mu_\omega^{\xi_i} = \sum_{j=1}^{m} \frac{v_{,j}(\mu_\omega)}{v(\mu_\omega)} g_\omega(\xi_i, \xi_j) + c
$$

for some constant c. As we consider the canonical normalization of μ_ω (corresponding to the natural lifted $\mathbb{T}$–action on K_X^{-1}), one can see that $c = 0$. Indeed, the infinitesimal actions A_i of the elements of the basis $(\xi_i)_i$ on smooth sections of K_X^{-1} are given by $A_i(s) := \mathcal{L}_{\xi_i} s$. We denote by H_g the induced hermitian metric on K_X^{-1} through the riemannian metric g_ω of ω (so that H_g has curvature ρ_ω) and by $H = v(\mu_\omega) H_g$ the induced hermitian metric with curvature ω (by using (4)); comparing the actions of the corresponding Chern connections, $\nabla^g_{\xi_i}$ and $\nabla^H_{\xi_i} = \nabla^g_{\xi_i} - \frac{1}{2}\sqrt{-1} d^c \log v(\mu_\omega)(\xi_i) \,\mathrm{id}$ on smooth sections of K_X^{-1} with the infinitesimal actions A_i gives (see eg [41, Propositions 8.8.2 and 8.8.3])

$$A_i(s) = \nabla^g_{\xi_i} s + \tfrac{1}{2}\sqrt{-1}(\Delta_\omega \mu_\omega^{\xi_i})s \quad \text{and} \quad A_i(s) = \nabla^H_{\xi_i} s + \sqrt{-1}\mu_\omega^{\xi_i} s. \tag{10}$$

We thus deduce $\frac{1}{2}\Delta_\omega \mu_\omega^{\xi_i} = \mu_\omega^{\xi_i} - \frac{1}{2}d^c(\log v(\mu_\omega))(\xi_i)$, ie $c = 0$ in (9).

Now, letting $c = 0$ in (9), multiplying it by $v_{,i}(\mu_\omega)/v(\mu_\omega)$, and taking the sum over i gives

$$\sum_{i,j=1}^m \frac{v_{,i}(\mu_\omega)v_{,j}(\mu_\omega)}{v(\mu_\omega)^2} g_\omega(\xi_i,\xi_j) = \sum_{i=1}^r \frac{v_{,i}(\mu_\omega)}{v(\mu_\omega)}(\Delta_\omega \mu_\omega^{\xi_i}) - 2\sum_{i=1}^r \frac{v_{,i}(\mu_\omega)}{v(\mu_\omega)}\mu_\omega^{\xi_i},$$

which, substituted back into (8), yields

$$\begin{aligned}
\mathrm{Scal}(\omega) - 2m &= -2\sum_{i=1}^r \frac{v_{,i}(\mu_\omega)}{v(\mu_\omega)}\Delta_\omega \mu_\omega^{\xi_i} + \sum_{i,j=1}^m \frac{v_{,ij}(\mu_\omega)}{v(\mu_\omega)} g_\omega(\xi_i,\xi_j) + 2\sum_{i=1}^r \frac{v_{,i}(\mu_\omega)}{v(\mu_\omega)}\mu_\omega^{\xi_i} \\
&= -2\sum_{i=1}^r \frac{v_{,i}(\mu_\omega)}{v(\mu_\omega)}\Delta_\omega \mu_\omega^{\xi_i} + 2\sum_{i,j=1}^m \frac{v_{,ij}(\mu_\omega)}{v(\mu_\omega)} g_\omega(\xi_i,\xi_j) \\
&\qquad\qquad - \sum_{i,j=1}^m \frac{v_{,ij}(\mu_\omega)}{v(\mu_\omega)} g_\omega(\xi_i,\xi_j) + 2\langle d\log v, \mu_\omega\rangle \\
&= -2\Delta_\omega(v(\mu_\omega)) - \sum_{i,j=1}^m \frac{v_{,ij}(\mu_\omega)}{v(\mu_\omega)} g_\omega(\xi_i,\xi_j) + 2\langle d\log v, \mu_\omega\rangle.
\end{aligned}$$

Thus, $\mathrm{Scal}_v(\omega) = w(\mu_\omega)$.

Now we show the converse. To this end, let $\omega \in 2\pi c_1(X)$ be a $\mathbb{T}$–invariant Kähler metric, $v > 0$ a positive smooth function on the canonically normalized polytope Δ and $w = 2(m + \langle d\log, \mu\rangle)v$ the weight defined in Lemma 2.2. Let $h \in C^\infty_{\mathbb{T}}(X)$ be an ω–relative Ricci potential, ie

$$\rho_\omega - \omega = \tfrac{1}{2}dd^c h.$$

Taking the trace with respect to ω and the interior product with $\xi \in \mathfrak{t}$ in the above identity, we get

$$\mathrm{Scal}(\omega) = 2m - \Delta_\omega h \quad \text{and} \quad \Delta_\omega \mu_\omega^\xi + \mathcal{L}_{J\xi} h = 2\mu_\omega^\xi, \tag{11}$$

where we have used the canonical normalization of μ_ω to determine the additive constant in the second inequality (as we did for (9)). Similar computations as in the first part of the proof (using (11)) give

$$
\begin{aligned}
\text{(12)}\quad \mathrm{Scal}_v(\omega) - w(\mu_\omega)
&= -v(\mu_\omega)(\Delta_\omega h) + 2\sum_{i=1}^r v_{,i}(\mu_\omega)(\Delta_\omega \mu_\omega^{\xi_i}) - \sum_{i,j=1}^r v_{,ij}(\mu_\omega)(\xi_i,\xi_j)_\omega \\
&\qquad + 2\sum_{i=1}^r v_{,i}(\mu_\omega)\mu_\omega^{\xi_i} \\
&= -v(\mu_\omega)(\Delta_\omega h) + \sum_{i=1}^r v_{,i}(\mu_\omega) g_\omega(dh, d\mu_\omega^{\xi_i}) + \sum_{i=1}^r v_{,i}(\mu_\omega)(\Delta_\omega \mu_\omega^{\xi_i}) \\
&\qquad - \sum_{i,j=1}^r v_{,ij}(\mu_\omega) g_\omega(\xi_i,\xi_j) \\
&= -v(\mu_\omega)(\Delta_{\omega,v} h) + \sum_{i=1}^r v_{,i}(\mu_\omega)(\Delta_\omega \mu_\omega^{\xi_i}) - \sum_{i,j=1}^r v_{,ij}(\mu_\omega) g_\omega(\xi_i,\xi_j),
\end{aligned}
$$

where $\Delta_{\omega,v} := (1/v(\mu_\omega))\delta_\omega v(\mu_\omega) d$ is the weighted Laplacian; see Appendix A. Using the second equality in (8), we compute

$$
\begin{aligned}
v(\mu_\omega)\Delta_{\omega,v}(\log v(\mu_\omega)) &= v(\mu_\omega)(\Delta_\omega \log v(\mu_\omega)) - \sum_{i=1}^m v_i(\mu_\omega) g_\omega(d(\log v(\mu_\omega)), d\mu_\omega^{\xi_i}) \\
&= v(\mu_\omega)(\Delta_\omega \log v(\mu_\omega)) - \sum_{i,j=1}^m \frac{v_{,i}(\mu_\omega)v_{,j}(\mu_\omega)}{v(\mu_\omega)} g_\omega(\xi_i,\xi_j) \\
&= \sum_{i=1}^r v_{,i}(\mu_\omega)(\Delta_\omega \mu_\omega^{\xi_i}) - \sum_{i,j=1}^r v_{,ij}(\mu_\omega) g_\omega(\xi_i,\xi_j).
\end{aligned}
$$

Substituting back in (12),

$$
\mathrm{Scal}_v(\omega) - w(\mu_\omega) = v(\mu_\omega)\Delta_{\omega,v}(\log v(\mu_\omega) - h). \tag{13}
$$

It follows that if ω is (v,w)–cscK then $h = \log v(\mu_\omega) + c$ by the maximum principle, showing that ω satisfies (4). □

Remark 2.3 Using the second relation in (11) it follows that, under the canonical normalization of μ_ω,

$$
\int_X \mu_\omega^\xi e^h \omega^{[m]} = 0 \quad \text{for } \xi \in \mathfrak{t}. \tag{14}
$$

This is precisely the normalization of μ_ω used in [71, Section 2].

Lemma 2.4 *Define $v := \ell^{-(m+2)}$ for $\ell(\mu) = \langle \xi, \mu \rangle + a$ positive affine-linear on Δ. Then $\omega \in 2\pi c_1(X)$ is a v–soliton if and only if ω is an $(\ell^{-(m+1)}, 2ma\ell^{-(m+2)})$–cscK metric.*

Proof The proof is similar to that of Lemma 2.2.

If ω is a v–soliton with $v := \ell^{-(m+2)}$, specializing (8) and (9) to the specific choice of v and letting $f := \ell(\mu_\omega) = \mu_\omega^\xi + a$, we get the identities

$$\mathrm{Scal}(\omega) = 2m + (m+2)\Delta_\omega \log f \quad \text{and} \quad -\Delta_\omega f + 2f = \frac{m+2}{f} g_\omega(df, df) + 2a.$$

Multiplying the first equality by f^2 and taking the sum with the second equality multiplied by mf gives

$$f^2\,\mathrm{Scal}(\omega) - 2(m+1) f \Delta_\omega f - (m+1)(m+2) g_\omega(df, df) = 2maf. \tag{15}$$

The right side is the $(m+2, f)$–scalar curvature (see [2]) and it is straightforward to check that (15) is equivalent to the condition that ω is an $(\ell^{-(m+1)}, 2ma\ell^{-(m+2]})$–cscK metric.

In the other direction, for any $\mathbb{T}$–invariant Kähler metric $\omega \in 2\pi c_1(X)$, we let

$$f := \ell(\mu_\omega) = \mu_\omega^\xi + a > 0$$

be the corresponding Killing potential and let $h \in C^\infty_{\mathbb{T}}(X)$ be such that $\rho_\omega - \omega = \frac{1}{2} dd^c h$. From (11) we have

$$\mathrm{Scal}(\omega) = 2m - \Delta_\omega h \quad \text{and} \quad -\Delta_\omega f + 2f = -g_\omega(df, dh) + 2a.$$

Multiplying the first identity by f^2 and summing with the second identity multiplied by mf gives

$$\begin{aligned} f^2\,\mathrm{Scal}(\omega) - 2(m+1) f\Delta_\omega f - (m+1)(m+2) g_\omega(df, df) - 2maf \\ = -f^2\big(\Delta_\omega(h + (m+2)\log f) + m g_\omega(d\log f, dh + (m+2)\, d\log f)\big). \end{aligned} \tag{16}$$

If we suppose that (15) holds, we conclude, again by the maximum principle, that $(m+2)\log f + h$ must be constant. ∎

Remark 2.5 Lemmas 2.2 and 2.4 give two different realizations of the same $\ell^{-(m+2)}$–soliton as a weighted cscK metric, with weights $\big(\ell^{-(m+2)}, 2(-2\ell + (m+2)a)\ell^{-(m+3)}\big)$ and $(\ell^{-(m+1)}, 2am\ell^{-(m+2)})$, respectively.

We derive from Lemma 2.4 and the correspondence in [2] the following fact, which does not seem to have been noticed before:

Lemma 2.6 *On a Fano manifold $(X, \mathbb{T})$, a $\mathbb{T}$–invariant Kähler metric $\omega \in 2\pi c_1(X)$ is an $\ell^{-(m+2)}$–soliton with respect to a positive affine-linear function $\ell = \langle \xi, \mu \rangle + a$ if and only if the lift $\hat{\xi}$ of the vector field ξ to K_X, via the hermitian connection ∇^h with curvature $-\omega$ and the ω–momentum $\ell(\mu_\omega)$ of ξ, is a Reeb vector of a Sasaki–Einstein (transversal) structure of transversal scalar curvature $2am$, defined on the unit circle bundle N of (K_X, h).*

Proof By Lemma 2.4, we need to show that an $(\ell^{-(m+1)}, 2am\ell^{-(m+2)})$–cscK metric in $2\pi c_1(X)$ corresponds to a Sasaki–Einstein structure as defined in the statement. By [2, Theorem 1], the condition that ω is $(\ell^{-(m+1)}, 2am\ell^{-(m+2)})$–cscK is equivalent to the condition that the corresponding Sasaki structure has transversal scalar curvature equal to $2ma$ (notice that $a > 0$ by the positivity of ℓ over the canonical polytope Δ). Any Sasaki structure of constant transversal scalar curvature on $N \subset K_X$ is transversally Kähler–Einstein as $c_1(K_X^\times) = 0$, and therefore the first Chern class of the CR distribution of N vanishes; see eg [19, Corollary 5.3; 40, Proposition 4.3]. □

Remark 2.7 The correspondence in Lemma 2.6 is, in fact, local and can be deduced directly from the relation between the transversal Ricci tensors of the two Sasaki structures on the CR manifold $N \subset K_X$, defined by $\hat{\xi}$ and the regular Reeb vector field $\hat{\chi}$, respectively, according to [51] and Gauduchon (personal communication).

Proof of Propositions 1 and 2 Propositions 1 and 2 from the introduction follow directly from Lemmas 2.2, 2.4 and 2.6. □

3 The coercivity principle: the plan of the proof of Theorem 1

We consider the following general setup, based on the results of [27; 29; 70]. As before, we let $\mathbb{T} \subset \mathrm{Aut}_r(X)$ be a fixed connected compact torus in the reduced group of automorphisms of X, and denote by $\mathbb{G} = \mathbb{T}^{\mathbb{C}} \subset \mathrm{Aut}_r(X)$ the corresponding complex torus.

Following [27], we consider the L_1–length function on $\mathcal{K}(X, \omega_0)$, introduced on a smooth curve ψ_t for $t \in [0, 1]$ by

$$L_1(\psi_t) := \int_0^1 \left(\int_X |\dot{\psi}_t| \omega_{\psi_t}^{[m]} \right) ds,$$

and, for $\varphi_0, \varphi_1 \in \mathcal{K}(X, \omega_0)$, we let

$$d_1(\varphi_0, \varphi_1) := \inf\{L_1(\psi_t) \mid \psi_t \in \mathcal{K}(X, \omega_0) \text{ for } t \in [0, 1], \psi_0 = \varphi_0 \text{ and } \psi_1 = \varphi_1\}.$$

Similarly, we define d_1 on $\mathcal{K}_{\mathbb{T}}(M,\omega_0)$ by considering the infimum over smooth curves in $\mathcal{K}_{\mathbb{T}}(X,\omega_0)$. It is proved in [27] that $(\mathcal{K}(X,\omega_0),d_1)$ is a metric space, and it is observed in [29] that $(\mathcal{K}_{\mathbb{T}}(X,\omega_0),d_1)$ is a metric subspace of $(\mathcal{K}(X,\omega_0),d_1)$.

Recall the following well-known functionals on $\mathcal{K}(X,\omega_0)$:

Definition 3.1 Let $\boldsymbol{I}$ denote the functional on $\mathcal{K}(X,\omega_0)$ defined by

$$(d_\varphi \boldsymbol{I})(\dot{\varphi}) = \int_X \dot{\varphi}\omega_\varphi^{[m]}, \quad \boldsymbol{I}(0)=0,$$

and let $\boldsymbol{J}(\varphi) := \int_X \varphi\omega_0^{[m]} - \boldsymbol{I}(\varphi)$.

Remark 3.2 For any constant c, we have that $\boldsymbol{I}(\varphi+c)=\boldsymbol{I}(\varphi)+c\,\mathrm{Vol}(X,\omega_0)$ (where $\mathrm{Vol}(X,\omega_0)=\int_X {\omega_0}^{[m]}$ is the total volume of (X,ω_0)), whereas $\boldsymbol{J}(\varphi+c)=\boldsymbol{J}(\varphi)$, ie we can see $\boldsymbol{J}$ as a functional on the space of Kähler metrics in the Kähler class $\alpha=[\omega_0]$, which motivates the notation $\boldsymbol{J}(\omega_\varphi)$. One can further show that $\boldsymbol{J}(\omega_\varphi)\geq 0$ with equality if and only if $\omega_\varphi=\omega_0$.

By the above remark, for any Kähler metric ω_φ in the Kähler class $[\omega_0]$, there exists a uniquely determined ω_0–relative potential $\varphi\in\mathcal{K}(X,\omega_0)$ satisfying

$$\boldsymbol{I}(\varphi)=0.$$

We shall denote by $\mathring{\mathcal{K}}(X,\omega_0)$ (and $\mathring{\mathcal{K}}_{\mathbb{T}}(X,\omega_0)$) the subspaces of *normalized* ω_0–relative Kähler potentials satisfying the above equality. We notice that the group $\mathbb{G}=\mathbb{T}^{\mathbb{C}}$ naturally acts on the space of Kähler metrics in $[\omega_0]$, preserving the subspace of $\mathbb{T}$–invariant Kähler metrics. This induces an action $[\mathbb{G}]$ on the spaces $\mathring{\mathcal{K}}(X,\omega_0)$ and $\mathring{\mathcal{K}}_{\mathbb{T}}(X,\omega_0)$ such that

$$\omega_{\sigma[\varphi]}=\sigma^*(\omega_\varphi) \quad \text{for all } \sigma\in\mathbb{G} \text{ and } \varphi\in\mathring{\mathcal{K}}(X,\omega_0).$$

We introduce the $\mathbb{G}$–relative distance on $\mathring{\mathcal{K}}(X,\omega_0)$ and $\mathring{\mathcal{K}}_{\mathbb{T}}(X,\omega_0)$ by

$$d_1^{[\mathbb{G}]}(\varphi_0,\varphi_1)=\inf_{\sigma_0,\sigma_1\in\mathbb{G}} d_1(\sigma_0[\varphi_0],\sigma_1[\varphi_1]).$$

It is proved in [29] that $d_1^{[\mathbb{G}]}$ is $\mathbb{G}$–invariant, ie $d_1^{[\mathbb{G}]}(\sigma[\varphi_0],\sigma[\varphi_1])=d_1^{[\mathbb{G}]}(\varphi_0,\varphi_1)$, and thus

$$d_1^{[\mathbb{G}]}(\varphi_0,\varphi_1)=\inf_{\sigma\in\mathbb{G}} d_1(\varphi_0,\sigma[\varphi_1]).$$

Definition 3.3 Let $\boldsymbol{F}$ be a functional on $\mathcal{K}_{\mathbb{T}}(X,\omega_0)$. We say that $\boldsymbol{F}$ is $\mathbb{G}$–*coercive* if there exist uniform positive constants (λ,δ) such that

$$\boldsymbol{F}(\varphi)\geq\lambda d_1^{[\mathbb{G}]}(0,\varphi)-\delta \quad \text{for all } \varphi\in\mathring{\mathcal{K}}_{\mathbb{T}}(X,\omega_0). \tag{17}$$

It is sometimes more natural to introduce $\mathbb{G}$–coercivity in terms of the functional $\boldsymbol{J}$ via:

Proposition 3.4 [29] *$\boldsymbol{F}$ is $\mathbb{G}$–coercive if and only if there exist uniform positive constants (λ', δ') such that*

$$\boldsymbol{F}(\varphi) \geq \lambda' \inf_{\sigma \in \mathbb{G}} \boldsymbol{J}(\sigma^* \omega_\varphi) - \delta' \quad \text{for all } \varphi \in \mathcal{K}_{\mathbb{T}}(X, \omega_0). \tag{18}$$

Remark 3.5 If $\boldsymbol{F}$ is $\mathbb{G}$–coercive, then it is bounded below by (17).

As in [27], one can consider the metric completion $(\mathcal{E}^1(X, \omega_0), d_1)$ of $(\mathcal{K}(X, \omega_0), d_1)$, which can be characterized by a suitable continuously embedded subspace in $L^1(X, \omega_0)$; similarly we let $(\mathcal{E}^1_{\mathbb{T}}(X, \omega_0), d_1)$ be the metric completion of $(\mathcal{K}_{\mathbb{T}}(X, \omega_0), d_1)$, which, again by the results in [29], can be viewed as the closed subspace of $\mathbb{T}$–invariant elements of $\mathcal{E}^1(X, \omega_0)$. It will be important for us that $(\mathcal{E}^1_{\mathbb{T}}(X, \omega_0), d_1)$ is a *geodesic space*, ie each two elements $\psi_0, \psi_1 \in \mathcal{E}^1_{\mathbb{T}}(X, \omega_0)$ can be connected with a curve ψ_t for $t \in [0, 1]$ in $(\mathcal{E}^1_{\mathbb{T}}(X, \omega_0), d_1)$, called a *weak geodesic*, obtained as the limit of $C^{1,\bar{1}}$–geodesics between elements of $\mathcal{K}_{\mathbb{T}}(X, \omega_0)$; see [22; 27]. This object is a curve $\varphi_t \in \mathcal{E}^1_{\mathbb{T}}(X, \omega_0)$, of regularity $C^{1,1}([0, 1] \times X)$, which is uniquely associated to each $\varphi_0, \varphi_1 \in \mathcal{K}_{\mathbb{T}}(X, \omega_0)$; see [16; 22; 25] and the proof of Proposition 5.8 for more details about the weak $C^{1,\bar{1}}$–geodesics.

In [29, Theorem 3.4], the following general principle is established:

Theorem 3.6 (coercivity principle) *Let $\boldsymbol{F} : \mathcal{K}_{\mathbb{T}}(X, \omega_0) \to \mathbb{R}$ be a lower semicontinuous (lsc) functional with respect to d_1, and $\boldsymbol{F} : \mathcal{E}^1_{\mathbb{T}}(X, \omega_0) \to \mathbb{R} \cup \{+\infty\}$ be its largest lsc extension. Suppose, furthermore, that $\boldsymbol{F}(\varphi + c) = \boldsymbol{F}(\varphi) =: \boldsymbol{F}(\omega_\varphi)$ and $\boldsymbol{F}(\sigma^*\omega_\varphi) = \boldsymbol{F}(\omega_\varphi)$ for any $\varphi \in \mathcal{K}_{\mathbb{T}}(X, \omega_0)$ and $\sigma \in \mathbb{G}$, and that $\boldsymbol{F}$ satisfies:*

- (i) ***Convexity*** *For each $\varphi_0, \varphi_1 \in \mathcal{K}_{\mathbb{T}}(X, \omega_0)$ and the $C^{1,\bar{1}}$–geodesic φ_t joining φ_0 and φ_1, $t \to \boldsymbol{F}(\varphi_t)$ is continuous and convex.*
- (ii) ***Regularity*** *If $\psi \in \mathcal{E}^1_{\mathbb{T}}(X, \omega_0)$ is a minimizer of $\boldsymbol{F}$, then $\psi \in \mathcal{K}_{\mathbb{T}}(X, \omega_0)$.*
- (iii) ***Uniqueness*** *$\mathbb{G}$ acts transitively on the set of minimizers of $\boldsymbol{F}$.*
- (iv) ***Compactness*** *If $\{\psi_j\}_j \in \mathcal{E}^1_{\mathbb{T}}(X, \omega_0)$ satisfies $\lim_{j\to\infty} \boldsymbol{F}(\psi_j) = \inf_{\mathcal{E}^1_{\mathbb{T}}(X,\omega_0)} \boldsymbol{F}$ and, for some $C > 0$, $d_1(0, \psi_j) \leq C$, then there exists a $\psi \in \mathcal{E}^1_{\mathbb{T}}(X, \omega_0)$ and a subsequence $\{\psi_{j_k}\}_k$ with $\psi_{j_k} \to \psi$ in $(\mathcal{E}^1_{\mathbb{T}}(X, \omega_0), d_1)$.*

Then, the following two conditions are equivalent:

- *$\boldsymbol{F}$ has minimizer in $\mathcal{K}_{\mathbb{T}}(X, \omega_0)$.*
- *$\boldsymbol{F}$ is $\mathbb{G}$**–coercive**.*

The above result provides a clear framework for achieving the proof of Theorem 1: we need to find a suitable largest lsc extension of the weighted Mabuchi functional $\boldsymbol{M}_{v,w}$ to the space $\mathcal{E}^1_{\mathbb{T}}(X,\omega_0)$ and show it satisfies the properties (i)–(iv). Notice that the invariance of $\boldsymbol{M}_{v,w}$ under the action of $\mathbb{G}=\mathbb{T}^{\mathbb{C}}$ is equivalent to the necessary condition (3) for the existence of a (v,w)–cscK metric, whereas (iii) will follow from Theorem 1.5 once the regularity condition (ii) is established. Furthermore, the property (i) is proved in [56, Theorem 1], so the core of our argument is to define the extension of $\boldsymbol{M}_{v,w}$ to $\mathcal{E}^1_{\mathbb{T}}(X,\omega_0)$ and establish the properties (ii) and (iv). These steps will be detailed in Theorems 6.1, 7.1 and 6.17, respectively.

4 K–stability via coercivity: deriving Corollary 1 from Theorem 1

We use the following general setup, based on the results of [10; 14; 18; 49; 57; 66; 70] which deal with the K–polystability and uniform K–stability in the unweighted cscK case. Let $\mathbb{T}\subset \mathrm{Aut}_r(X)$ be a connected compact torus in the reduced group of automorphisms of X.

Definition 4.1 A $\mathbb{T}$–equivariant Kähler test configuration $(\mathscr{X},\mathscr{A})$ associated to $(X,\alpha,\mathbb{T})$ is a *normal* compact Kähler space $\mathscr{X}$ endowed with

- a flat morphism $\pi\colon \mathscr{X}\to\mathbb{P}^1$;
- a $\mathbb{C}^*$–action ρ covering the standard $\mathbb{C}^*$–action on $\mathbb{P}^1$, and a $\mathbb{T}$–action commuting with ρ and preserving π;
- a $\mathbb{T}\times\mathbb{C}^*$–equivariant biholomorphism $\Pi_0\colon(\mathscr{X},\backslash\pi^{-1}(0))\cong X\times(\mathbb{P}^1\setminus\{0\})$;
- a Kähler class $\mathscr{A}\in H^{1,1}(\mathscr{X},\mathbb{R})$ such that $(\Pi_0^{-1})^*(\mathscr{A})|_{X\times\{\tau\}}=\alpha$.

We say that $(\mathscr{X},\mathscr{A})$ is *smooth* if $\mathscr{X}$ is smooth and *dominating* if Π_0 extends to a $\mathbb{T}\times\mathbb{C}^*$–equivariant morphism

$$\Pi\colon \mathscr{X}\to X\times\mathbb{P}^1. \tag{19}$$

$(\mathscr{X},\mathscr{A})$ is called *trivial* if it is dominating and Π is an isomorphism; $(\mathscr{X},\mathscr{A})$ is called *product* if $\pi^{-1}(0)\cong X$. If (X,L) is a smooth polarized variety and $\alpha=2\pi c_1(L)$, a *polarized* test configuration is a normal polarized variety $(\mathscr{X},\mathscr{L})$ such that, for some $r\in\mathbb{N}^*$, $(\mathscr{X},(1/r)2\pi c_1(\mathscr{L}))$ defines a Kähler test configuration of (X,α) and, under Π_0, $(X,\mathscr{L}|_{X\times\{\tau\}})\cong(X,L^r)$.

4.1 Non-Archimedean functionals

We recall that any $\mathbb{T}\times\mathbb{S}^1$–invariant Kähler metric $\Omega\in\mathscr{A}$ on $\mathscr{X}$ gives rise to a smooth ray of $\mathbb{T}$–invariant Kähler metrics $\omega_t\in\alpha$ on X defined by

$$\omega_t := \rho(e^{-t+is})^*(\Omega)|_{X\times\{1\}}.$$

Definition 4.2 Let $\boldsymbol{F}$ be a functional defined on the space of $\mathbb{T}$–invariant Kähler metrics on X in the class α. We say that $\boldsymbol{F}$ admits a *non-Archimedean version* $\boldsymbol{F}^{\mathrm{NA}}$, defined on a subclass C of $\mathbb{T}$–equivariant Kähler test configurations $(\mathscr{X},\mathscr{A})$ associated to $(X,\alpha,\mathbb{T})$ if, for any $(\mathscr{X},\mathscr{A})\in C$ and any induced smooth ray of $\mathbb{T}$–invariant Kähler metrics $\omega_t\in\alpha$ on X, the slope $\lim_{t\to\infty}\boldsymbol{F}(\omega_t)/t$ is well defined and given by a quantity $\boldsymbol{F}^{\mathrm{NA}}(\mathscr{X},\mathscr{A})$ which is independent of the choice of the $\mathbb{T}\times\mathbb{S}^1$–invariant Kähler form $\Omega\in\mathscr{A}$.

We give below two key examples of non-Archimedean versions of known functionals. The first one is established in the polarized case in [18] and in the generality we consider in [34; 65]:

Example 4.3 The functional $\boldsymbol{J}$ introduced in Definition 3.1 admits a non-Archimedean version defined, up to a positive-dimensional multiplicative constant, on the class of smooth $\mathbb{T}$–equivariant dominating Kähler test configurations $(\mathscr{X},\mathscr{L})$ by

$$\boldsymbol{J}^{\mathrm{NA}}(\mathscr{X},\mathscr{A}) = \frac{((\Pi^*\alpha)^m\cdot\mathscr{A})_{\mathscr{X}}}{(\alpha^m)_X} - \frac{1}{m+1}\frac{(\mathscr{A}^{m+1})_{\mathscr{X}}}{(\alpha^m)_X},$$

where Π is the morphism (19) and α denotes both the Kähler class on X and its pullback to $X\times\mathbb{P}^1$.

The above expression generalizes to dominating smooth test configurations which are only *relatively nef* (in the terminology of [66]), thus also providing a non-Archimedean version of $\boldsymbol{J}$ for any Kähler test configuration. Indeed, by the equivariant Hironaka resolution, any $\mathbb{T}$–equivariant test configuration can be dominated by a smooth relatively nef Kähler dominating test configuration, and the computation of $\boldsymbol{J}^{\mathrm{NA}}$ on the latter does not depend on the choice made.

The non-Archimedean functional $\boldsymbol{J}^{\mathrm{NA}}$ defined above is always nonnegative and equals zero precisely when $(\mathscr{X},\mathscr{A})$ is the *trivial* test configuration. This statement is proved in [18, Theorem 7.9] in the polarized case, and follows from the results in [66] in the Kähler case; see in particular [66, Lemma 4.8] with G trivial and recall that the $\boldsymbol{J}$–norm is Lipschitz equivalent to the d_1–distance, so that the unique weak geodesic ray associated to a test configuration with vanishing $\boldsymbol{J}^{\mathrm{NA}}$–norm must be constant, and hence the test configuration must be trivial by [66, Corollary 3.12]. Thus, $\boldsymbol{J}^{\mathrm{NA}}$ can be thought of as a "norm" on the space of Kähler test configurations.

In order to obtain a norm which is zero for more general *product* test configurations, in [33; 49; 57] the authors consider smooth rays $\tilde{\omega}_t \in \alpha$ of $\mathbb{T}$–invariant Kähler metrics on X which are obtained by composing an induced ray ω_t from a $\mathbb{T}\times\mathbb{S}^1$–invariant Kähler metric $\Omega \in \mathscr{A}$ on $\mathscr{X}$ with the flow of a vector field $J\xi$, where $\xi \in \mathfrak{t}$, ie $\tilde{\omega}_t = \exp(tJ\xi)^*(\omega_t)$. They show that the slope

$$\lim_{t\to\infty} \frac{\boldsymbol{J}(\tilde{\omega}_t)}{t} =: \boldsymbol{J}^{\mathrm{NA}}(\mathscr{X}_\xi, \mathscr{A}_\xi)$$

is well defined and independent of the choice of induced ray ω_t. We notice that when $\xi \in 2\pi\Lambda$ is a lattice element (or more generally is rational), ξ induces a $\mathbb{C}^*$–action ρ_ξ on $\mathscr{X}$ and $\tilde{\omega}_t$ is an induced smooth ray from another Kähler test configuration $(\mathscr{X}_\xi, \mathscr{A}_\xi)$, called the ξ*–twist* of $(\mathscr{X}, \mathscr{A})$, obtained from $\mathscr{X}$ by composing the initial $\mathbb{C}^*$–action ρ with ρ_ξ and compactifying trivially at infinity. (For instance, the product test configurations are precisely the ξ–twists of the trivial test configuration.) In this case, $\boldsymbol{J}^{\mathrm{NA}}(\mathscr{X}_\xi, \mathscr{A}_\xi)$ is just the non-Archimedean $\boldsymbol{J}$–functional computed as in Example 4.3 on $(\mathscr{X}_\xi, \mathscr{A}_\xi)$. For a general ξ, the quantity $(\mathscr{X}_\xi, \mathscr{A}_\xi)$ in this notation is not a test configuration in the usual sense (it is sometimes refereed to as an $\mathbb{R}$*–test configuration*) but the value $\boldsymbol{J}^{\mathrm{NA}}(\mathscr{X}_\xi, \mathscr{A}_\xi)$ can be obtained as a continuous extension of the corresponding quantity for rational ξ's. Following [49; 57], we let

$$\boldsymbol{J}^{\mathrm{NA}}_{\mathbb{T}^{\mathbb{C}}}(\mathscr{X}, \mathscr{A}) := \inf_{\xi\in\mathfrak{t}} \boldsymbol{J}^{\mathrm{NA}}(\mathscr{X}_\xi, \mathscr{A}_\xi) \geq 0. \tag{20}$$

A key observation [18; 49; 57] in the polarized case is that the equality in (20) holds if and only if $(\mathscr{X}, \mathscr{L})$ is a *product* test configuration. Furthermore, according to [49, Theorem B; 57, Theorem 3.14]:

Example 4.3 (continued) In the polarized case, the quantity $\boldsymbol{J}^{\mathrm{NA}}_{\mathbb{T}^{\mathbb{C}}}(\mathscr{X}, \mathscr{A})$ introduced in (20) defines a non-Archimedean version of the functional

$$\boldsymbol{J}_{\mathbb{T}^{\mathbb{C}}}(\omega) := \inf_{\sigma\in\mathbb{T}^{\mathbb{C}}} \boldsymbol{J}(\sigma^*(\omega))$$

on the class of $\mathbb{T}$–equivariant polarized test configuration of $(X, L, \mathbb{T})$.

Our second example is established in [55, Theorem 7]:

Example 4.4 Consider the weighted Mabuchi functional $\boldsymbol{M}_{v,w}$ from Definition 1.1 and assume that the relation (7) holds; see Remark 1.2. Then $\boldsymbol{M}_{v,w}$ admits a non-Archimedean version defined on *smooth* $\mathbb{T}$–equivariant Kähler test configurations with *reduced central fibre*, given by

$$\mathscr{F}_{v,w}(\mathscr{X},\mathscr{A}) := -\int_{\mathscr{X}} (\mathrm{Scal}_v(\Omega) - w(\mu_\Omega))\Omega^{[m+1]} + 8\pi \int_X v(\mu_\omega)\omega^{[m]}, \tag{21}$$

where $\Omega \in \mathscr{A}$ is any $\mathbb{T}$–invariant Kähler metric on $\mathscr{X}$ with Δ–normalized $\mathbb{T}$–momentum map $\mu_\Omega \colon \mathscr{X} \to \Delta$ and v–scalar curvature $\mathrm{Scal}_v(\Omega)$, and $\omega \in \alpha$ is any $\mathbb{T}$–invariant Kähler metric on X with Δ–normalized $\mathbb{T}$–momentum map $\mu_\omega \colon X \to \Delta$.

Definition 4.5 The right side of (21) is independent of $\Omega \in \mathscr{A}$ and $\omega \in \alpha$ (see [55]) and is referred to as the (v,w)*–weighted Donaldson–Futaki invariant* of a smooth $\mathbb{T}$–equivariant Kähler test configuration $(\mathscr{X},\mathscr{A})$.

Remark 4.6 In the unweighted case (ie $v = 1$ and $w = 4m\pi c_1(X)\cdot\alpha^{m-1}/\alpha^m$), $\mathscr{F}_{v,w}(\mathscr{X},\mathscr{A})$ admits an equivalent expression in terms of intersection cohomology numbers on $\mathscr{X}$; see [62; 72]. This allows one to extend the definition of the (unweighted) Donaldson–Futaki invariant to *any* normal Kähler test configuration. For arbitrary weight functions $v > 0$ and w, we don't have as yet a general definition for $\mathscr{F}_{v,w}$, but (21) can be readily extended to orbifold test configurations. We also notice that the assumption on the central fibre in Example 4.4 is necessary in order to ensure the equality $\mathscr{F}_{v,w} = \boldsymbol{M}^{\mathrm{NA}}_{v,w}$; see [65] for a general formula of the non-Archimedean version of the unweighted Mabuchi energy. It will be interesting to obtain a non-Archimedean version of $\boldsymbol{M}_{v,w}$ for any orbifold $\mathbb{T}$–equivariant Kähler test configuration.

4.2 $\boldsymbol{F}^{\mathrm{NA}}$–stability

Definition 4.7 Let $\boldsymbol{F}$ be a functional defined on the space of $\mathbb{T}$–invariant Kähler metrics on X in the Kähler class α, and suppose $\boldsymbol{F}$ admits a non-Archimedean version $\boldsymbol{F}^{\mathrm{NA}}(\mathscr{X},\mathscr{A})$ (see Definition 4.2), defined on a class C of $\mathbb{T}$–equivariant Kähler test configurations $(\mathscr{X},\mathscr{A})$ associated to $(X,\alpha,\mathbb{T})$. We suppose that C contains the product test configurations. We say that:

(i) $(X,\alpha,\mathbb{T})$ is $\mathbb{T}$*–equivariant* $\boldsymbol{F}^{\mathrm{NA}}$*–semistable* (on test configurations of C) if for any $(\mathscr{X},\mathscr{A}) \in C$ we have $\boldsymbol{F}^{\mathrm{NA}}(\mathscr{X},\mathscr{A}) \geq 0$.

(ii) $(X,\alpha,\mathbb{T})$ is $\mathbb{T}$*–equivariant* $\boldsymbol{F}^{\mathrm{NA}}$*–polystable* (on test configurations of C) if it is $\mathbb{T}$–equivariant $\boldsymbol{F}^{\mathrm{NA}}$–semistable and, further, $\boldsymbol{F}^{\mathrm{NA}}(\mathscr{X},\mathscr{A}) = 0$ if and only if $(\mathscr{X},\mathscr{A})$ is a product test configuration.

(iii) $(X,\alpha,\mathbb{T})$ is $\mathbb{T}^{\mathbb{C}}$–*uniform* $\boldsymbol{F}^{\mathrm{NA}}$–*stable* (on test configurations of C) if there exists a uniform positive constant $\lambda > 0$ such that, for any test configuration $(\mathscr{X},\mathscr{A}) \in C$,
$$\boldsymbol{F}^{\mathrm{NA}}(\mathscr{X},\mathscr{A}) \geq \lambda \boldsymbol{J}^{\mathrm{NA}}_{\mathbb{T}^{\mathbb{C}}}(\mathscr{X},\mathscr{A}),$$
where $\boldsymbol{J}^{\mathrm{NA}}_{\mathbb{T}^{\mathbb{C}}}(\mathscr{X},\mathscr{L})$ is as introduced in (20).

Remark 4.8 If $\boldsymbol{F}$ is bounded below, $(X,\alpha,\mathbb{T})$ is $\mathbb{T}$–equivariant $\boldsymbol{F}^{\mathrm{NA}}$–semistable. Furthermore both (ii) and (iii) imply (i) and, in the polarized case, (iii) implies (ii) by the results in [18; 49; 57].

Theorem 4.9 [14; 49; 57; 66] *Suppose $\boldsymbol{F}$ is a functional defined on the space of $\mathbb{T}$–invariant Kähler metrics in α, which is $\mathbb{T}$–relatively $\mathbb{T}^{\mathbb{C}}$–proper. Suppose, furthermore, that $\boldsymbol{F}$ admits a non-Archimedean version $\boldsymbol{F}^{\mathrm{NA}}$ defined for a class C of $\mathbb{T}$–equivariant Kähler test configurations of $(X,\alpha,\mathbb{T})$. Then $(X,\alpha,\mathbb{T})$ is $\mathbb{T}$–equivariant $\boldsymbol{F}^{\mathrm{NA}}$–polystable on C. If, moreover, (X,L) is a polarized variety and $\alpha = 2\pi c_1(L)$, then $(X,\alpha,\mathbb{T})$ is $\mathbb{T}^{\mathbb{C}}$–uniform $\boldsymbol{F}^{\mathrm{NA}}$–stable on polarized test configurations in C.*

Proof For the first part, we follow [66] with some minor modifications. We want to show that if $\boldsymbol{F}^{\mathrm{NA}}(\mathscr{X},\mathscr{A}) = 0$, then $(\mathscr{X},\mathscr{A})$ is a product test configuration.

Fix a $\mathbb{T}\times\mathbb{S}^1$–invariant Kähler form $\Omega \in \mathscr{A}$ and let ω_t be the corresponding ray of smooth $\mathbb{T}$–invariant Kähler forms in α, and $\psi_t \in \mathcal{K}_{\mathbb{T}}(X,\omega_0)$ the *normalized* smooth ray of Kähler potentials satisfying $\boldsymbol{I}(\psi_t) = 0$. According to [65], the Kähler test configuration $(\mathscr{X},\mathscr{A})$ also determines a unique $C^{1,\bar{1}}$ weak geodesic ray φ_t in $\mathcal{K}^{1,\bar{1}}(X,\omega_0)$, emanating from ψ_0. Furthermore, φ_t is invariant under $\mathbb{T}$ (by its uniqueness) provided that we have $\psi_0 \in \mathcal{K}_{\mathbb{T}}(X,\omega_0)$. According to [66, Proposition 4.2], we can consider instead of $\mathscr{A}$ the *relative Kähler class* $\mathscr{A}_c = \mathscr{A} - c[X_0] = \mathscr{A} - c\pi^*(\mathcal{O}_{\mathbb{P}^1}(1))$ (for a constant c determined from $\mathscr{A}$ and where $[X_0]$ denotes the divisor corresponding to the central fibre X_0 of $\mathscr{X}$) such that the $C^{1,\bar{1}}$ weak geodesic ray φ^c_t corresponding to $(\mathscr{X},\mathscr{A}_c)$ is the projection of φ_t to the slice $\mathcal{K}^{1,\bar{1}}_{\mathbb{T}}(X,\omega_0) \cap \boldsymbol{I}^{-1}(0)$. Notice that the smooth $(1,1)$–form $\Omega - c\pi^*\omega_{\mathrm{FS}} \in \mathscr{A}_c$ defines the same smooth ray ω_t of $\mathbb{T}$–invariant Kähler metrics, and thus the same ray of smooth potentials ψ_t is in $\mathcal{K}_{\mathbb{T}}(X,\omega_0) \cap \boldsymbol{I}^{-1}(0)$ and $\boldsymbol{F}^{\mathrm{NA}}(\mathscr{X},\mathscr{A}_c) = \boldsymbol{F}^{\mathrm{NA}}(\mathscr{X},\mathscr{A}) = 0$. The key point is that (17) and
$$\lim_{t\to\infty} \frac{\boldsymbol{F}(\omega_{\psi_t})}{t} = \boldsymbol{F}^{\mathrm{NA}}(\mathscr{X},\mathscr{A}_c) = 0$$
yield an estimate $0 \leq d_1^{[\mathbb{G}]}(0,\psi_t) \leq o(t)$, which is shown in [66, Lemma 4.8] to be equivalent to $0 \leq d_1^{[\mathbb{G}]}(0,\varphi^c_t) \leq o(t)$. We can now apply the arguments in the proof of

the implication "(2) $\Longrightarrow$ (5)" of [66, Theorem 4.4] by replacing the Mabuchi energy with the abstract functional $\boldsymbol{F}$ and the group $\mathrm{Aut}_0(X)$ with $\mathbb{T}^{\mathbb{C}}$, and noting that in our $\mathbb{T}$–relative situation instead of the cscK potential ψ_0 in [66, Proposition 4.10] we can take *any* Kähler potential in $\mathcal{K}_{\mathbb{T}}(X,\omega_0)$ (as ω_{ψ_0} is $\mathbb{T}$–invariant and $\mathbb{T}^{\mathbb{C}}$ is reductive). We thus deduce the implication (5) of [66], namely, that the geodesic ray φ_t^c associated to $(\mathscr{X},\mathscr{A}_c)$ is given by the ω_0–relative Kähler potentials of $\exp(tJ\xi)^*(\omega_{\psi_0})$ in $\boldsymbol{I}^{-1}(0)$, where ξ is a vector field in the Lie algebra of $\mathbb{T}$; it follows from [66, Theorem A.6] that $(\mathscr{X},\mathscr{A}_c)$, and hence also $(\mathscr{X},\mathscr{A})$, is a product test configuration.

The second part follows immediately from (18) and Example 4.3 (continued). □

We next apply Theorem 4.9 to $\boldsymbol{F}=\boldsymbol{M}_{v,w}$ and $\boldsymbol{F}^{\mathrm{NA}}=\mathscr{F}_{v,w}$.

Definition 4.10 Let $\boldsymbol{F}^{\mathrm{NA}}=\mathscr{F}_{v,w}$, where $\mathscr{F}_{v,w}$ is defined on any smooth $\mathbb{T}$–equivariant test configuration via the formula (21); see Definition 4.5. We then refer to the $\boldsymbol{F}^{\mathrm{NA}}$–stability notions introduced in Definition 4.7(i)–(iii) as $\mathbb{T}$–equivariant (v,w)–K–semistability, $\mathbb{T}$–equivariant (v,w)–K–polystability, and $\mathbb{T}^{\mathbb{C}}$–uniform (v,w)–K–stability, respectively, on $\mathbb{T}$–invariant dominating smooth Kähler test configurations with reduced central fibre.

Proof of Corollary 1 modulo Theorem 1 By definition of $\boldsymbol{M}_{v,w}$ (see Definition 1.1),

$$\boldsymbol{M}_{v,w}(\varphi+c)=\boldsymbol{M}_{v,w}(\varphi)+c\int_X(\mathrm{Scal}_v(\omega_\varphi)-w(\mu_\varphi))\omega_\varphi^{[m]},$$

showing that if $\boldsymbol{M}_{v,w}$ is bounded below on $\mathcal{K}_{\mathbb{T}}(X,\omega_0)$ (in particular if $\boldsymbol{M}_{v,w}$ is $\mathbb{T}$–relatively $\mathbb{T}^{\mathbb{C}}$–proper), then the relation (7) holds and $\boldsymbol{M}_{v,w}$ defines a functional on the space of $\mathbb{T}$–invariant Kähler metrics in α (see Remark 1.2). In this case, Example 4.4 tells us that $\mathscr{F}_{v,w}(\mathscr{X},\mathscr{A})$ defines a non-Archimedean version of $\boldsymbol{M}_{v,w}$. We can now apply Theorem 4.9. □

5 Semisimple principal fibrations

Let (X,ω) be a compact Kähler $2m$–manifold, endowed with a hamiltonian isometric action of an r–dimensional torus $\mathbb{T}$. As $\mathbb{T}$ will act on various spaces, we shall use at times super- and subscripts to emphasize the space on which $\mathbb{T}$ acts. For instance, $\mathbb{T}_X$ will denote the $\mathbb{T}$–action on X. Let $\mathfrak{t}$ be the Lie algebra of $\mathbb{T}$ and $\Lambda\subset\mathfrak{t}$ the lattice of generators of circle groups in $\mathbb{T}$ (ie $\mathbb{T}=\mathfrak{t}/2\pi\Lambda$). We denote by $\mu_\omega\colon X\to\Delta\subset\mathfrak{t}^*$ the normalized $\mathbb{T}_X$–momentum map of ω (the map whose image is a fixed compact convex polytope $\Delta\subset\mathfrak{t}^*$).

Let $B = B_1 \times \cdots \times B_k$ be a $2n$–dimensional cscK manifold where each (B_a, ω_{B_a}), for $a = 1, \ldots, k$, is a compact cscK Hodge Kähler $2n_a$–manifold (ie $\frac{1}{2\pi}[\omega_{B_a}]$ is in $H^2(B_a, \mathbb{Z})$), and $\pi_B \colon P \to B$ a principal $\mathbb{T}$–bundle endowed with a connection 1–form $\theta \in \Omega^1(P, \mathfrak{t})$ with curvature

$$d\theta = \sum_{a=1}^{k} (\pi_B^* \omega_{B_a}) \otimes p_a \quad \text{for } \ p_a \in \Lambda. \tag{22}$$

Remark 5.1 The principal $\mathbb{T}$–bundle P above can be described in terms of r complex line bundles over B as follows. Fixing a lattice basis $\{\xi_1, \ldots, \xi_r\}$ of $\mathfrak{t}$ and writing $p_a = \sum_{i=1}^r p_{ai}\xi_i$ for $p_{ai} \in \mathbb{Z}$ with $a = 1, \ldots, k$ (22) yields that P is the (fiberwise) product of r principal U(1)–bundles $P_i \to B$, where each P_i is associated to a complex line bundle L_i^* on B with Chern class $2\pi c_1(L_i^*) = -\sum_{a=1}^r p_{ai}\pi_B^*[\omega_{B_a}]$:

$$2\pi c_1(P) := -2\pi \sum_{i=1}^{r} c_1(L_i^*) \otimes \xi_i = \sum_{a=1}^{k} \pi_B^*[\omega_{B_a}] \otimes p_a.$$

Fixing a connection 1–form θ on P as in (22) amounts to introducing a hermitian metric h_i^* on each L_i^*, with curvature $-\sum_{a=1}^r p_{ai}\pi_B^*(\omega_{B_a})$, and identifying $P_i \subset L_i^*$ with the corresponding unitary $\mathbb{S}^1$–bundle.

Let $\mathcal{D} = \mathrm{ann}(\theta) \subset TP$ be the horizontal distribution defined by θ, leading to a splitting

$$TP = \mathcal{D} \oplus \mathfrak{t}_P,$$

where $\mathfrak{t}_P$ denotes the Lie algebra of $\mathbb{T}_P$ inside $C^\infty(P, TP)$, corresponding to the $\mathbb{T}$–action $\mathbb{T}_P$ on P. The lift J_B of the integrable almost complex structure of B to $\mathcal{D}$ gives rise to a CR structure $(\mathcal{D}, J_B)$ on P (of codimension r).

We further let $Z := X \times P$ and consider the induced $\mathbb{T}$–action, denoted by $\mathbb{T}_Z$, generated by $(-\xi_i^X + \xi_i^P)$ for any basis of Λ as above. We thus define

$$Y := Z/\mathbb{T}_Z.$$

It follows that Y is a $2(m+n)$–dimensional smooth manifold, and $\pi_Y \colon Z = X \times P \to Y$ is a principal $\mathbb{T}$–bundle over Y, whereas $\pi_B \colon P \to B$ defines a fibration $\pi_B \colon Y \to B$ with smooth fibres X, as summarized in the diagram

$$\begin{tikzcd}
 & Z = X \times P \arrow[dl, "/\mathbb{T}_P"'] \arrow[d, "\pi_B"] \arrow[dr, "/\mathbb{T}_Z"] & \\
X \times B \arrow[dr, "\pi_B"'] & B & Y \arrow[dl, "\pi_B"]
\end{tikzcd}$$

The $\mathbb{T}_X$–action on the factor X in $Z = X \times P$ descends to a $\mathbb{T}$–action on Y, denoted by $\mathbb{T}_Y$, which preserves each fibre (and thus coincides with the action of $\mathbb{T}_X$). Notice that the 1–form θ also defines a connection 1–form on Z with horizontal distribution $\mathscr{H}$:

$$T(X \times P) = \mathscr{H} \oplus \mathfrak{t}_Z, \quad \mathscr{H} = TX \oplus \mathcal{D} = \operatorname{ann}(\theta). \tag{23}$$

This gives rise to an induced CR structure $(\mathscr{H}, J = J_X \oplus J_B)$ of codimension r on Z, which is clearly invariant under the $\mathbb{T}_Z$–action, and therefore defines a $\mathbb{T}_Y$–invariant complex structure J_Y on Y.

We now consider Kähler metrics on Y, compatible with the fibre bundle construction of the above form. To simplify the notation, we denote by $\omega_a := \omega_{B_a}$ the (fixed) cscK metric on each factor B_a, by ω a $\mathbb{T}$–invariant Kähler structure in the class α on X, and by $\tilde{\omega}$ the resulting Kähler structure on Y, which is defined in terms of a basic 2–form on $Z = X \times P$, depending on k real constants $c_a \in \mathbb{R}$ (which will be fixed) such that, for each $a = 1, \dots, k$, the affine-linear function $\langle p_a, \mu \rangle + c_a$ on $\mathfrak{t}^*$ is strictly positive on the momentum image Δ:

$$\begin{aligned} \tilde{\omega} &:= \omega + \sum_{a=1}^{k} (\langle p_a, \mu_\omega \rangle + c_a) \pi_B^* \omega_a + \langle d\mu_\omega \wedge \theta \rangle \\ &= \omega + \sum_{a=1}^{k} c_a (\pi_B^* \omega_a) + d(\langle \mu_\omega, \theta \rangle). \end{aligned} \tag{24}$$

In the above expression, $\langle \cdot, \cdot \rangle$ stands for the natural pairing between $\mathfrak{t}$ and $\mathfrak{t}^*$. Thus $\langle p_a, \mu_\omega \rangle$ is a smooth function, $\langle \mu_\omega, \theta \rangle$ is a 1–form, and $\langle d\mu_\omega \wedge \theta \rangle$ is a 2–form on Z. One can directly check from the above expression that $\tilde{\omega}$ is closed and $\mathbb{T}_Z$–basic, and is positive definite on $(\mathscr{H}, J_X \oplus J_B)$, so it is the pullback of a Kähler form on Y. We shall tacitly identify in the sequel the Kähler form on Y with its pullback (24) on $Z = X \times P$. Notice that $\tilde{\omega}$ is $\mathbb{T}_Y$–invariant and μ_ω, seen as a smooth $\mathbb{T}_Z$–invariant function on Z, is the Δ–normalized momentum map.

Remark 5.2 The horizontal part $\tilde{\omega}_h := \tilde{\omega}|_{\mathscr{H}}$ of the 2–form $\tilde{\omega}$ on $Z = X \times P \xrightarrow{\pi_B} X \times B$ is invariant and basic with respect to the action $\mathbb{T}_P$ on the factor P, and thus induces a hermitian (non-Kähler in general) metric on $X \times B = X \times \prod_{a=1}^{k} B_a$, given by

$$\tilde{\omega}_h = \omega + \sum_{a=1}^{k} (\langle p_a, \mu_\omega \rangle + c_a) \omega_a,$$

which is an instance of warped geometry. On can thus think of $(X \times B, \tilde{\omega}_h)$ and $(Y, \tilde{\omega})$ as being related by the twist construction of [69] applied to $(Z, \tilde{\omega}, \mathbb{T}_Z)$ and $(Z, \tilde{\omega}, \mathbb{T}_P)$.

Definition 5.3 The Kähler manifold $(Y, \mathbb{T}_Y)$ constructed as above will be called a *semisimple* $(X, \mathbb{T})$*–principal fibration* associated to the Kähler manifold $(X, \mathbb{T})$ and the product cscK manifold $B = B_1 \times \cdots \times B_k$. The $\mathbb{T}_Y$–invariant Kähler metric $\widetilde{\omega}$ on Y constructed from a $\mathbb{T}_X$–invariant Kähler metric ω on X (and fixed cscK metrics ω_a on B_a) will be called *bundle-compatible*.

Remark 5.4 In the case when $(X, \mathbb{T}, \omega)$ is a toric Kähler manifold, a semisimple $(X, \mathbb{T})$–principal fibration endowed with a bundle-compatible Kähler metric is an example of a semisimple rigid toric fibration in the sense of [6], and is thus described by the *generalized Calabi construction* with a global product structure on the base and no blow-downs.

5.1 The space of functions

The above bundle construction gives rise to a natural embedding of the space $C^\infty_{\mathbb{T}}(X)$ of $\mathbb{T}_X$–invariant smooth functions on X to the space $C^\infty_{\mathbb{T}}(Y)$ of $\mathbb{T}_Y$–invariant smooth functions on Y: for any $\varphi \in C^\infty_{\mathbb{T}}(X)$ we consider the induced function on $Z = X \times P$, which is clearly $\mathbb{T}_Z$–invariant, and thus descends to a smooth $\mathbb{T}_Y$–invariant function on Y. We shall tacitly identify φ and its image in $C^\infty_{\mathbb{T}}(Y)$, ie we shall consider

$$C^\infty_{\mathbb{T}}(X) \subset C^\infty_{\mathbb{T}}(Y).$$

Notice that the above embedding is closed in the Fréchet topology, as we can identify a smooth $\mathbb{T}_X$–invariant function on X with a smooth $\mathbb{T}_Y$–invariant function φ on Y, which has the property

$$d_P(\pi_Y^* \varphi) = 0$$

on $Z = X \times P$.

More generally, for any $\mathbb{T}_Y$–invariant smooth function $\psi \in C^\infty_{\mathbb{T}}(Y)$, its lift $\pi_Y^*\psi$ to $Z = X \times P$ is a smooth function which is both $\mathbb{T}_Z$– and $\mathbb{T}_X$–invariant, or equivalently $\mathbb{T}_X$– and $\mathbb{T}_P$–invariant. It thus follows that $\pi_Y^*\psi$ can be equivalently viewed as a $\mathbb{T}_X$–invariant smooth function on $X \times B$, ie we have an identification

$$C^\infty_{\mathbb{T}}(Y) \cong C^\infty_{\mathbb{T}}(X \times B). \tag{25}$$

In particular, for any fixed point $x \in X$, we shall denote by $\psi_x \in C^\infty(B)$ the induced smooth function on B, and for any fixed point $b \in B$ by $\psi_b \in C^\infty_{\mathbb{T}}(X)$ the induced function on X. We thus have the identification

$$C^\infty_{\mathbb{T}}(X) \cong \{\psi \in C^\infty_{\mathbb{T}}(Y) \mid d_B \psi_x = 0 \text{ for all } x \in X\}.$$

5.2 The space of bundle-compatible Kähler metrics

We shall next use the construction of (24) in order to identify the space $\mathcal{K}_{\mathbb{T}}(X,\omega_0)$ of $\mathbb{T}_X$–invariant ω_0–relative Kähler potentials on X as a subset of the space $\mathcal{K}_{\mathbb{T}}(Y,\widetilde{\omega}_0)$ of $\mathbb{T}_Y$–invariant $\widetilde{\omega}_0$–relative Kähler potentials on Y.

Lemma 5.5 *Let $\omega_\varphi=\omega_0+d_Xd_X^c\varphi$ be a $\mathbb{T}_X$–invariant Kähler form on X in the Kähler class $\alpha=[\omega_0]$, where $\varphi\in\mathcal{K}_{\mathbb{T}}(X,\omega_0)$ is a $\mathbb{T}_X$–invariant smooth function on X. Denote by μ_φ the momentum map of $\mathbb{T}_X$ with respect to ω_φ, satisfying the normalization $\mu_\varphi(X)=\Delta$, and by $\widetilde{\omega}_\varphi$ the induced Kähler metric on Y, via (24). Then*

$$\widetilde{\omega}_\varphi=\widetilde{\omega}_0+d_Yd_Y^c\varphi,$$

where φ stands for the induced smooth function on Y.

Proof Recall that $\mu_\varphi=\mu_0+d_X^c\varphi$; see (5). By (24), the pullback of $\widetilde{\omega}_\varphi$ to $Z=X\times P$ is

$$\begin{aligned}\widetilde{\omega}_\varphi&=\omega_\varphi+\sum_{a=1}^k c_a(\pi_B^*\omega_a)+d_Z\langle\mu_\varphi,\theta\rangle=\omega_0+\sum_{a=1}^k c_a(\pi_B^*\omega_a)+d_Xd_X^c\varphi+d_Z\langle\mu_\varphi,\theta\rangle\\&=\widetilde{\omega}_0+d_Zd_X^c\varphi+d_Z(\langle d_X^c\varphi,\theta\rangle),\end{aligned}$$

so it is enough to check that

$$d_Y^c\varphi=d_X^c\varphi+\langle d_X^c\varphi,\theta\rangle \tag{26}$$

for any $\mathbb{T}_X$–invariant smooth function φ on X. To this end, let us choose a basis $\{\xi_1,\dots,\xi_r\}$ of $\mathfrak{t}$, with dual basis $\{\xi^1,\dots,\xi^r\}$ of $\mathfrak{t}^*$, and write

$$d_X^c\varphi=\sum_{j=1}^r(d_X^c\varphi)(\xi_j^X)\xi^j\quad\text{and}\quad\theta=\sum_{j=1}^r\theta_j\xi_j$$

for 1–forms θ_j on Z such that θ_j is zero on $\mathscr{H}$ and $\theta_j(\xi_i^P)=\theta_j(-\xi_i^X+\xi_i^P)=\delta_{ij}$. Thus, (26) is equivalent to

$$d_Y^c\varphi=d_X^c\varphi+\sum_{j=1}^r(d_X^c\varphi)(\xi_j^X)\theta_j.$$

Evaluating the right side of the above equality on the generators $(-\xi_j^X+\xi_j^P)$ of $\mathfrak{t}_Z$, we see that it is a π_Y–basic 1–form on Z, and thus is the pullback of a 1–form on Y via π_Y. The claim follows easily. □

Thus, Lemma 5.5 defines an embedding $\mathcal{K}_{\mathbb{T}}(X,\omega_0)\subset\mathcal{K}_{\mathbb{T}}(Y,\widetilde{\omega}_0)$ and we have also identified in Section 5.1 a natural embedding of the space of $\mathbb{T}_X$–invariant functions on X into the space of $\mathbb{T}_Y$–invariant functions on Y, through their pullbacks to $Z=X\times P$.

Letting $\theta := \sum_{j=1}^r \theta_j \otimes \xi_j^P$ be the decomposition of the connection 1–form θ on P in a basis $\{\xi_1, \dots, \xi_r\}$ of the lattice $\Lambda \subset \mathfrak{t}$, and $\theta^{\wedge r} := \theta_1 \wedge \cdots \wedge \theta_r$, it follows from (24) and Lemma 5.5 that for any $\varphi \in \mathcal{K}_{\mathbb{T}}(X, \omega_0) \subset \mathcal{K}_{\mathbb{T}}(Y, \tilde{\omega}_0)$, the measure $\tilde{\omega}_\varphi^{[m+n]}$ on Y is the pushforward of the measure on Z

$$\frac{1}{(2\pi)^r}\tilde{\omega}_\varphi^{[m+n]} \wedge \theta^{\wedge r} = \frac{1}{(2\pi)^r}\big(p(\mu_\varphi)\omega_\varphi^{[m]} \wedge \textstyle\bigwedge_{a=1}^k \pi_B^*\omega_a^{[n_a]}\big) \wedge \theta^{\wedge r}, \tag{27}$$

where

$$p(\mu) := \prod_{a=1}^k (\langle p_a, \mu\rangle + c_a)^{n_a} \quad \text{for } n_a = \dim_{\mathbb{C}}(B_a) \tag{28}$$

is a positive polynomial on Δ, determined by the semisimple $(X, \mathbb{T})$–principal fibration Y and the given bundle-compatible Kähler class on it. It thus follows that any $\mathbb{T}_X$–invariant integrable function f on X defines an integrable $\mathbb{T}_Y$–invariant function on Y and, for any $\varphi \in \mathcal{K}_{\mathbb{T}}(X, \omega_0) \subset \mathcal{K}_{\mathbb{T}}(Y, \tilde{\omega}_0)$,

$$\int_Y f\tilde{\omega}_\varphi^{[n+m]} = \mathrm{Vol}(B, \omega_B)\int_X p(\mu_\varphi) f\omega_\varphi^{[m]}. \tag{29}$$

Corollary 5.6 *There exists an embedding $\mathcal{K}_{\mathbb{T}}(X, \omega_0) \subset \mathcal{K}_{\mathbb{T}}(Y, \tilde{\omega}_0)$ such that, for any smooth curve $\psi_t \in \mathcal{K}_{\mathbb{T}}(X, \omega_0) \subset \mathcal{K}_{\mathbb{T}}(Y, \tilde{\omega}_0)$,*

$$L_1^Y(\psi_t) = \mathrm{Vol}(B, \omega_B)L_{1,p}^X(\psi_t),$$

where $p(\mu)$ is the positive weight function on Δ defined in (28), $L_{1,p}^X$ is the $p(\mu)$–weighted length function on $\mathcal{K}_{\mathbb{T}}(X, \omega_0)$ given by

$$L_{1,p}^X(\psi_t) := \int_0^1\Big(\int_X |\dot{\psi}_t| p(\mu_{\psi_t})\omega_{\psi_t}^{[m]}\Big)\,dt,$$

and L_1^Y is the length function on $\mathcal{K}_{\mathbb{T}}(Y, \tilde{\omega}_0)$ corresponding to the weight $p = 1$. In particular, $d_1^Y(\varphi_0, \varphi_1) = \mathrm{Vol}(B, \omega_B)d_{1,p}^X(\varphi_0, \varphi_1)$ for any $\varphi_0, \varphi_1 \in \mathcal{K}_{\mathbb{T}}(X, \omega_0) \subset \mathcal{K}_{\mathbb{T}}(Y, \tilde{\omega}_0)$, where $d_{1,p}^X$ is the induced distance via the length functional $L_{1,p}^X$.

Proof This is a direct consequence of (29). □

Lemma 5.7 *Let φ be a smooth $\mathbb{T}_X$–invariant function on X, also considered as a smooth $\mathbb{T}_Y$–invariant function on Y, and ω be a $\mathbb{T}_X$–invariant Kähler metric on X with $\tilde{\omega}$ the corresponding $\mathbb{T}_Y$–invariant Kähler metric on Y given by (24). Then*

$$\|d\varphi\|_\omega^2 = \|d\varphi\|_{\tilde{\omega}}^2.$$

Proof We use that

$$\|d\varphi\|^2_{\omega} = \frac{d_X\varphi \wedge d^c_X\varphi \wedge \omega^{[m-1]}}{\omega^{[m]}} = \frac{d_X\varphi \wedge d^c_X\varphi \wedge \omega^{[m-1]} \wedge p(\mu_\omega)(\pi^*_B\omega_B)^{[n]} \wedge \theta^{\wedge r}}{\omega^{[m]} \wedge p(\mu_\omega)(\pi^*_B\omega_B)^{[n]} \wedge \theta^{\wedge r}},$$

$$\|d\varphi\|^2_{\tilde\omega} = \frac{d_Y\varphi \wedge d^c_Y\varphi \wedge \tilde\omega^{[m+n-1]}}{\tilde\omega^{[m+n]}} = \frac{d_Y\varphi \wedge d^c_Y\varphi \wedge \tilde\omega^{[m+n-1]} \wedge \theta^{\wedge r}}{\tilde\omega^{[m+n]} \wedge \theta^{\wedge r}}$$

(where the right sides are written on $X \times P$) together with $d_X\varphi = d_Y\varphi$ and (27). □

Proposition 5.8 *The embedding in Corollary 5.6 is totally geodesic with respect to the weak $C^{1,\bar 1}$ geodesics.*

Proof Let $\varphi_0, \varphi_1 \in \mathcal{K}_{\mathbb{T}}(X, \omega_0)$. If φ_0 and φ_1 can be connected by a *smooth* geodesic φ_t, ie with a smooth curve in $\mathcal{K}_{\mathbb{T}}(X, \omega_0)$ such that

$$\ddot\varphi = \|d\dot\varphi\|^2_{\omega_\varphi}, \tag{30}$$

then, by Lemma 5.7, φ_t is also a smooth geodesic in $\mathcal{K}_{\mathbb{T}}(Y, \tilde\omega_0)$ connecting φ_0 and φ_1 in $\mathcal{K}_{\mathbb{T}}(Y, \tilde\omega_0)$.

In general, by the results in [22], φ_0 and φ_1 can be connected only with a weak $C^{1,\bar 1}$–geodesic in $\mathcal{K}^{1,\bar 1}_{\mathbb{T}}(X, \omega_0)$, where $\mathcal{K}^{1,\bar 1}(X, \omega_0)$ stand for the space of $C^1(X)$ functions φ on X such that $\omega_0 + dd^c\varphi \geq 0$ and has bounded coefficients as a $(1,1)$–current. More precisely, letting $\Sigma := \{1 < z < e\} \subset \mathbb{C}$, it is shown in [22] that there exists a unique weak solution (ie a positive $(1,1)$–current in the sense of Bedford and Taylor) of the homogeneous Monge–Ampère equation

$$\begin{gathered}(\pi^*_X\omega_0 + d_Xd^c_X\Phi)^{m+1} = 0, \quad \pi^*_X\omega_0 + d_Xd^c_X\Phi \geq 0 \quad \text{for } \Phi \in C^{1,\alpha}(X \times \bar\Sigma),\\ \Phi(x,1) = \varphi_0(x), \quad \Phi(x,e) = \varphi_1(x).\end{gathered} \tag{31}$$

It was later shown in [25] that Φ is actually of regularity $C^{1,1}(X \times \bar\Sigma)$. Note that, by uniqueness, Φ is $\mathbb{T}$–invariant as soon as φ_0 and φ_1 are. The link with (30) is (see [64]) that if Φ were actually smooth, we could recover the smooth geodesic φ_t joining φ_0 and φ_1 by letting $t := \log|z|$ and $\varphi_t(x) := \Phi(x, \log|z|)$. In the general case, the curve φ_t of (weak) ω_0–relative plurisubharmonic potentials (of regularity $C^{1,1}(X \times [0,1])$) is referred to as the *weak* $C^{1,\bar 1}$–geodesic joining φ_0 and φ_1.

We are thus going to check that any weak $C^{1,\bar 1}$–geodesic on X (invariant under $\mathbb{T}_X$) defines, via Lemma 5.5, a $C^{1,\bar 1}$–geodesic on Y. To this end, we need to show that Φ satisfies

$$(\pi^*_Y\tilde\omega_0 + d_Yd^c_Y\Phi)^{m+n+1} = 0 \quad \text{and} \quad \pi^*_Y\tilde\omega_0 + d_Yd^c_Y\Phi \geq 0, \tag{32}$$

the regularity statements being automatically satisfied on Y.

By the results in [16; 22], Φ can be approximated as $\varepsilon \to 0$, both in the weak sense of currents and in $C^{1,\alpha}(X \times \overline{\Sigma})$ (for a fixed $\alpha \in (0,1)$), by smooth functions $\Psi^\varepsilon(x,z)$ on $X \times \overline{\Sigma}$ for $\varepsilon > 0$ which solve

$$\begin{aligned}(\pi_X^*\omega_0 + d_X d_X^c \Psi^\varepsilon)^{[m+1]} &= \varepsilon((\pi_X^*\omega_0)^{[m]} \wedge (dx \wedge dy)),\\ \pi_X^*\omega_0 + d_X d_X^c \Psi^\varepsilon > 0, \quad \Psi^\varepsilon(x,1) &= \varphi_0, \quad \Psi^\varepsilon(x,e) = \varphi_1.\end{aligned} \tag{33}$$

By the uniqueness of the smooth solution of (33) (and using that both φ_0 and φ_1 are $\mathbb{T}_X$–invariant), we have that $\Psi^\varepsilon(x,z)$ is a $\mathbb{T}_X$–invariant smooth function on X for any $z \in \overline{\Sigma}$. Furthermore, the positivity condition on the second line yields that $\Psi^\varepsilon(x,z) \in \mathcal{K}_{\mathbb{T}}(X,\omega_0)$ for any $z \in \overline{\Sigma}$. We can then also see $\Psi^\varepsilon(x,z)$, via its pullback to $X \times P \times \overline{\Sigma}$, as a $\mathbb{T}_Y$–invariant smooth function on $Y \times \overline{\Sigma}$; the arguments in the proof of Lemma 5.5 yield that $\pi_Y^*\tilde{\omega}_0 + d_Y d_Y^c \Psi^\varepsilon > 0$ on $Y \times \overline{\Sigma}$. Also, by the same proof, we have the equality of volume forms, on $X \times P \times \overline{\Sigma}$,

$$\begin{aligned}(\pi_Y^*\tilde{\omega}_0 + d_Y d_Y^c \Psi^\varepsilon)^{[m+n+1]} \wedge \theta^{\wedge r} &= p(\mu_{\Psi^\varepsilon})(\pi_X^*\omega_0 + dd^c\Psi^\varepsilon)^{[m+1]} \wedge (\pi_B^*\omega_B)^{[n]} \wedge \theta^{\wedge r}\\ &= \varepsilon p(\mu_{\Psi^\varepsilon})(\pi_X^*\omega_0)^{[m+1]} \wedge (\pi_B^*\omega_B)^{[n]} \wedge \theta^{\wedge r},\end{aligned} \tag{34}$$

where, we recall, $p(\mu) := \prod_{a=1}^k (\langle p_a, \mu\rangle + c_a)^{n_a}$ and $\theta^{\wedge r} := \theta_1 \wedge \cdots \wedge \theta_r$ (for $\theta = \sum_{i=1}^r \theta_i \otimes \xi_i^P$ with respect to a basis $\{\xi_1, \dots, \xi_r\}$ of $\Lambda \subset \mathfrak{t}$), and, for any fixed $z \in \overline{\Sigma}$, μ_{Ψ^ε} denotes the normalized $\mathbb{T}_X$–momentum map (5) of $\omega_0 + d_X d_X^c \Psi^\varepsilon$. Notice that, as p is uniformly bounded on Δ by positive constants, it follows by (34) that

$$\lim_{\varepsilon \to 0}((\pi_Y^*\tilde{\omega}_0 + d_Y d_Y^c \Psi^\varepsilon)^{[m+n+1]} \wedge \theta^{\wedge r}) = 0$$

weakly (as measures on $Z \times \overline{\Sigma}$). The measure $(\pi_Y^*\tilde{\omega}_0 + d_Y d_Y^c \Psi^\varepsilon)^{[m+n+1]}$ is the pushforward measure of $(\pi_Y^*\tilde{\omega}_0 + d_Y d_Y^c \Psi^\varepsilon)^{[m+n+1]} \wedge \theta^{\wedge r}$ to Y, so we obtain, on Y,

$$\lim_{\varepsilon \to 0}((\pi_Y^*\tilde{\omega}_0 + d_Y d_Y^c \Psi^\varepsilon)^{[m+n+1]}) = 0.$$

Furthermore, using the $C^{1,\alpha}$–convergence of Ψ^ε to Φ, we get the weak convergences (of positive $(1,1)$–currents)

$$\begin{aligned}\lim_{\varepsilon \to 0}(\pi_Y^*\tilde{\omega}_0 + d_Y d_Y^c \Psi^\varepsilon) &= \pi_Y^*\tilde{\omega}_0 + d_Y d_Y^c \Phi \geq 0,\\ 0 = \lim_{\varepsilon \to 0}(\pi_Y^*\tilde{\omega}_0 + d_Y d_Y^c \Psi^\varepsilon)^{[m+n+1]} &= (\pi_Y^*\tilde{\omega}_0 + d_Y d_Y^c \Phi)^{[m+n+1]}.\end{aligned}$$

Thus, (32) follows. ∎

Lemma 5.9 *Let v be a smooth positive weight function on Δ, let ω and $\tilde{\omega}$ be $\mathbb{T}$–invariant Kähler metrics on X and Y, respectively, given by (24), and suppose (B_a,ω_a) has constant scalar curvature $\mathrm{Scal}(\omega_a)=s_a$. Then the v–scalar curvature $\mathrm{Scal}(\tilde{\omega})$, considered as smooth function on $X\times P$, is given by*

$$\mathrm{Scal}_v(\tilde{\omega})=\frac{1}{p(\mu_\omega)}\,\mathrm{Scal}_{pv}(\omega)+v(\mu_\omega)q(\mu_\omega), \tag{35}$$

with $p(\mu)=\prod_{a=1}^k(\langle p_a,\mu\rangle+c_a)^{n_a}$ and $q(\mu)=\sum_{a=1}^k s_a/(\langle p_a,\mu\rangle+c_a)$. In particular, ω is a $(pv,\tilde{w})$–cscK metric on X if and only if $\tilde{\omega}$ is a (v,w)–cscK metric on Y, with

$$\tilde{w}(\mu)=p(\mu)(w(\mu)-v(\mu)q(\mu)).$$

Proof We apply the arguments in the proof of [3, Proposition 7] to both $(X,\mathbb{T}_X)$ and $(Y,\mathbb{T}_Y)$ to compute the corresponding scalar curvatures, and compare the results.

On X, we consider the open dense subset $\mathring{X}\subset X$ of stable points of the $\mathbb{T}_X$–action, and take the quotient $\pi_S\colon\mathring{X}\to S:=\mathring{X}/\mathbb{T}_X^{\mathbb{C}}$ under the induced complexified action $\mathbb{T}_X^{\mathbb{C}}\cong(\mathbb{C}^*)^r$ (thus S is a complex $2(m-r)$–dimensional orbifold).

Consider the pointwise ω–orthogonal and $\mathbb{T}$–invariant decomposition

$$T\mathring{X}=\mathcal{H}\oplus\mathfrak{t}_X\oplus J\mathfrak{t}_X,$$

and write the Kähler structure (g,J,ω) on X as

$$g=g_{\mathcal{H}}+\sum_{i,j=1}^r H_{ij}(\eta_i\otimes\eta_j+J\eta_i\otimes J\eta_j),\quad \omega=\omega_{\mathcal{H}}+\sum_{i,j=1}^r H_{ij}\eta_i\wedge J\eta_j,$$

where, for a fixed basis $\{\xi_1,\dots,\xi_r\}$ of $\mathfrak{t}$, the 1–forms η_j on $\mathring{X}$ are defined by $(\eta_j)|_{\mathcal{H}}=0$, $\eta_j(\xi_i^X)=\delta_{ij}$; $\eta_j(J\xi_i^X)=0$ and $H_{ij}=g(\xi_i^X,\xi_j^X)$.

We next fix a local volume form Vol_S on S in some holomorphic coordinates, and pointwise write

$$\omega_{\mathcal{H}}^{[m-r]}=Q\pi_S^*(\mathrm{Vol}_S) \tag{36}$$

for some positive (locally defined) smooth function Q on $\mathring{X}$ (where both $\omega_H^{[m-1]}$ and $\pi_S^*(\mathrm{Vol}_S)$ are seen as sections of $\bigwedge^{m-1}\mathcal{H}^*$). According to [3, Proposition 7],

$$\kappa:=-\tfrac{1}{2}(\log Q+\log\det(H_{ij})) \tag{37}$$

is a (local) Ricci potential of ω, ie $\rho_\omega=d_Xd_X^c\kappa$, and thus

$$\mathrm{Scal}(\omega)=-2\frac{d_Xd_X^c\kappa\wedge\omega^{[m-1]}}{\omega^{[m]}}.$$

We can now make a similar argument on Y, noting that the Kähler reduction of $\mathring{Y}$ by the induced $\mathbb{T}_Y$–action is $S \times B$; taking a local volume form in holomorphic coordinates on $S \times B$ of the form $\mathrm{Vol}_S \wedge \mathrm{Vol}_{B_1} \wedge \cdots \wedge \mathrm{Vol}_{B_k}$, and using (24), we see that a Ricci potential on Y (when pulled back to $X \times P$) is written as

$$\tilde{\kappa} = \sum_{a=1}^{k} \kappa_a - \tfrac{1}{2}(\log \widetilde{Q} + \log\det(H_{ij})),$$

where $\kappa_a := -\frac{1}{2}\log(\omega_a^{[n_a]}/\mathrm{Vol}_{B_a})$ is a Ricci potential of (B_a, ω_a) and

$$\widetilde{Q} = p(\mu_\omega)Q.$$

Thus,

$$\tilde{\kappa} = \sum_{a=1}^{k} \kappa_a + \kappa - \tfrac{1}{2}\log p(\mu_\omega), \tag{38}$$

as functions on $X \times P$. Introducing a basis $(\xi_i)_i$ of Λ and writing the connection 1–form $\theta \in \Omega^1(P, \mathfrak{t})$ as $\theta = \sum_{j=1}^{r} \theta_j \otimes \xi_j^P$ (where the 1–forms θ_j on P are such that θ_j is zero on $\mathcal{D}$ and $\theta_j(\xi_i^P) = \delta_{ij}$), we compute the scalar curvature of $\tilde{\omega}$ to be

$$\mathrm{Scal}(\tilde{\omega}) = \begin{cases} -(d_Y d_Y^c \tilde{\kappa} \wedge \tilde{\omega}^{[m+n-1]})/(\tilde{\omega}^{[m+n]}) & \text{on } Y, \\ -(d_Y d_Y^c \tilde{\kappa} \wedge \tilde{\omega}^{[m+n-1]} \wedge \theta^{\wedge r})/(\tilde{\omega}^{[m+n]} \wedge \theta^{\wedge r}) & \text{on } X \times P. \end{cases} \tag{39}$$

By (26) and (38), the pullback of $d_Y d_Y^c \tilde{\kappa}$ to $X \times P$ is given by

$$\begin{aligned} d_Y d_Y^c \tilde{\kappa} &= d_Y d_Y^c\big(\kappa - \tfrac{1}{2}\log p(\mu_\omega)\big) + \sum_{a=1}^{k} d_Y d_Y^c \kappa_a \\ &= d_X d_X^c\big(\kappa - \tfrac{1}{2}\log p(\mu_\omega)\big) + \sum_{j=1}^{r} d_X\big(d_X^c\big(\kappa - \tfrac{1}{2}\log p(\mu_\omega)\big)(\xi_j^X)\big) \wedge \theta_j \\ &\qquad + \sum_{j=1}^{r} d_X^c\big(\kappa - \tfrac{1}{2}\log p(\mu_\omega)\big)(\xi_j) d_P\theta_j + \sum_{a=1}^{k} dd_{B_a}^c \kappa_a \\ &= d_X d_X^c \kappa - \tfrac{1}{2} d_X d_X^c(\log p(\mu_\omega)) \\ &\qquad + \sum_{j=1}^{r} d_X\big(d_X^c\big(\kappa - \tfrac{1}{2}\log p(\mu_\omega)\big)(\xi_j^X)\big) \wedge \theta_j \\ &\qquad + \sum_{a=1}^{k} d_X^c\big(\kappa - \tfrac{1}{2}\log p(\mu_\omega)\big)(p_a)(\pi_B^*\omega_a) + \sum_{a=1}^{k} dd_{B_a}^c \kappa_a, \end{aligned} \tag{40}$$

where in the last equality we used (22) and we have denoted by p_a the induced vector field on X by the element $p_a \in \mathfrak{t}$. We shall compute the term $d_X^c \kappa(p_a)$ on $\mathring{X}$. Using (37), we get

$$d^c_X\kappa(p_a)=\frac{1}{2}\left(\frac{\mathcal{L}_{Jp_a}Q}{Q}+\mathrm{tr}(H^{-1}_{ij}(\mathcal{L}_{Jp_a}H_{ij}))\right). \tag{41}$$

Taking the wedge product of both sides of (36) with

$$\left(\sum_{i,j=1}^r H_{ij}\eta_i\wedge J\eta_j\right)^{[r]}=\det(H_{ij})\bigwedge_{j=1}^r(\eta_j\wedge J\eta_j)$$

gives

$$\omega^{[m]}=Q\pi_S^*\,\mathrm{Vol}_S\wedge\det(H_{ij})\bigwedge_{j=1}^r(\eta_j\wedge J\eta_j).$$

Applying the Lie derivative $\mathcal{L}_{Jp_a}$ to the above equality yields

$$\begin{aligned}(\Delta_\omega\mu_\omega^{p_a})\omega^{[m]}=(\mathcal{L}_{J\xi_a}Q)\pi_S^*\,\mathrm{Vol}_S\wedge\det(H_{ij})\textstyle\bigwedge_{j=1}^r(\eta_j\wedge J\eta_j)&\\ +Q\pi_S^*\,\mathrm{Vol}_S\wedge\mathcal{L}_{Jp_a}\big(\det(H_{ij})\textstyle\bigwedge_{j=1}^r\eta_j\wedge J\eta_j\big)&\end{aligned}$$

and

$$\begin{aligned}&Q\pi_S^*\,\mathrm{Vol}_S\wedge\mathcal{L}_{Jp_a}\big(\det(H_{ij})\textstyle\bigwedge_{j=1}^r(\eta_j\wedge J\eta_j)\big)\\ &\qquad=\big(\mathrm{tr}(H^{-1}_{ij}(\mathcal{L}_{Jp_a}H_{ij}))\big)Q\pi_S^*\,\mathrm{Vol}_S\wedge(\det(H_{ij}))\textstyle\bigwedge_{j=1}^r\eta_j\wedge J\eta_j,\end{aligned}$$

where we used that $\mathcal{L}_{Jp_a}\eta_j$ is a basic form (since $(\mathcal{L}_{Jp_a}\eta_j)(\xi_i)=-\eta_j([Jp_a,\xi_i])=0$). We thus get $\Delta_\omega\mu_\omega^{p_a}=\mathcal{L}_{Jp_a}Q/Q+\mathrm{tr}(H^{-1}_{ij}(\mathcal{L}_{Jp_a}H_{ij}))$ or equivalently, in terms of (41),

$$d^c_X\kappa(p_a)=\tfrac{1}{2}(\Delta_\omega\mu_\omega^{p_a}). \tag{42}$$

Using the above equation in (40), we continue the computation:

$$\begin{aligned}d_Yd^c_Y\tilde{\kappa}=d_Xd^c_X\kappa-\tfrac{1}{2}dd^c_X(\log p(\mu_\omega))+\sum_{j=1}^r d_X\big(d^c_X\big(\kappa-\tfrac{1}{2}\log p(\mu_\omega)\big)(\xi_j^X)\big)\wedge\theta_j&\\ +\frac{1}{2}\sum_{a=1}^k\left(\Delta_\omega\mu_\omega^{p_a}+\frac{\mathcal{L}_{Jp_a}(p(\mu_\omega))}{p(\mu_\omega)}\right)(\pi_B^*\omega_a)+\sum_{a=1}^k d_{B_a}d^c_{B_a}\kappa_a.&\end{aligned} \tag{43}$$

Recall that, by (27), on Z we have $\tilde{\omega}^{[m+n]}\wedge\theta^{\wedge r}=p(\mu_\omega)\omega^{[m]}\wedge\bigwedge_{a=1}^k\pi_B^*\omega_a^{[n_a]}\wedge\theta^{\wedge r}$. Similarly, by (24),

$$\begin{aligned}&\tilde{\omega}^{[m+n-1]}\wedge\theta^{\wedge r}\\ &\quad=\sum_{b=1}^k\left(\frac{p(\mu_\omega)}{\langle\mu_\omega,p_b\rangle+c_b}\omega^{[m]}\wedge(\pi_B^*\omega_b)^{[n_b-1]}\wedge\textstyle\bigwedge_{a=1,a\neq b}^k(\pi_B^*\omega_a)^{[n_a]}\wedge\theta^{\wedge r}\right)\\ &\qquad+p(\mu_\omega)\omega^{[m-1]}\wedge\textstyle\bigwedge_{a=1}^k(\pi_B^*\omega_a)^{[n_a]}\wedge\theta^{\wedge r}.\end{aligned} \tag{44}$$

Using (39), (43), (27) and (44), we obtain

$$
\begin{aligned}
(45)\quad \mathrm{Scal}(\tilde\omega)
&= \mathrm{Scal}(\omega)+\Delta_\omega(\log p(\mu_\omega)) \\
&\qquad +\sum_{a=1}^{k}\left(\frac{n_a}{\langle\mu_\omega,p_a\rangle+c_a}\left[\Delta_\omega\mu_\omega^{p_a}+\frac{\mathcal{L}_{Jp_a}p(\mu_\omega)}{p(\mu_\omega)}\right]+\frac{s_a}{\langle\mu_\omega,p_a\rangle+c_a}\right) \\
&= \mathrm{Scal}(\omega)+\sum_{a=1}^{k} n_a\Delta_\omega(\log(\langle\mu_\omega,p_a\rangle+c_a)) \\
&\qquad +\sum_{a=1}^{k}\left(\frac{n_a}{\langle\mu_\omega,p_a\rangle+c_a}\left[\Delta_\omega(\langle\mu_\omega,p_a\rangle)+\frac{\mathcal{L}_{Jp_a}(p(\mu_\omega))}{p(\mu_\omega)}\right]+\frac{s_a}{\langle\mu_\omega,p_a\rangle+c_a}\right) \\
&= \mathrm{Scal}(\omega)-\sum_{a,b=1}^{k}\frac{n_an_bg(p_a,p_b)}{(\langle\mu_\omega,p_a\rangle+c_a)(\langle\mu_\omega,p_b\rangle+c_b)} \\
&\qquad +\sum_{a=1}^{k}\left(\frac{2n_a\Delta_\omega(\langle\mu_\omega,p_a\rangle)}{\langle\mu_\omega,p_a\rangle+c_a}+\frac{n_a|\xi_a|^2_{g_\omega}}{(\langle\mu_\omega,p_a\rangle+c_a)^2}+\frac{s_a}{\langle\mu_\omega,p_a\rangle+c_a}\right).
\end{aligned}
$$

On the other hand, using a basis (ξ_i) of $\mathfrak{t}$ with a dual basis (ξ^i) of $\mathfrak{t}^*$,

$$
\begin{aligned}
\mathrm{Scal}_p(\omega) &:= p(\mu_\omega)\,\mathrm{Scal}(\omega)+2\sum_{i=1}^{r}p_{,i}(\mu_\omega)\Delta_\omega(\langle\mu_\omega,\xi_i\rangle)-\sum_{i,j=1}^{r}p_{,ij}(\mu_\omega)g_\omega(\xi_i,\xi_j) \\
&= p(\mu_\omega)\,\mathrm{Scal}(\omega)+2\sum_{i=1}^{r}\Delta_\omega(\langle\mu_\omega,\xi_i\rangle)\sum_{a=1}^{k}\frac{n_a\xi^i(p_a)p(\mu_\omega)}{\langle\mu_\omega,p_a\rangle+c_a} \\
&\qquad +\sum_{i,j=1}^{r}g_\omega(\xi_i,\xi_j)\left[\sum_{a=1}^{k}\frac{n_a\xi^i(p_a)\xi^j(p_a)p(\mu_\omega)}{(\langle\mu_\omega,p_a\rangle+c_a)^2}\right. \\
&\qquad\qquad \left.-\sum_{a,b=1}^{k}\frac{n_an_b\xi^i(p_a)\xi^j(p_a)p(\mu_\omega)}{(\langle\mu_\omega,p_a\rangle+c_a)(\langle\mu_\omega,p_b\rangle+c_b)}\right] \\
&= p(\mu_\omega)\,\mathrm{Scal}(\omega)+2\Delta_\omega(\langle\mu_\omega,p_a\rangle)\sum_{a=1}^{k}\frac{n_ap(\mu_\omega)}{\langle\mu_\omega,p_a\rangle+c_a} \\
&\qquad +\left[\sum_{a=1}^{k}\frac{n_a|p_a|^2_{g_\omega}p(\mu_\omega)}{(\langle\mu_\omega,p_a\rangle+c_a)^2}-\sum_{a,b=1}^{k}\frac{n_an_bg_\omega(p_a,p_b)p(\mu_\omega)}{(\langle\mu_\omega,p_a\rangle+c_a)(\langle\mu_\omega,p_b\rangle+c_b)}\right].
\end{aligned}
$$

Comparing the above expression with (45),

$$\mathrm{Scal}(\tilde{\omega}) = \frac{1}{p(\mu_\omega)} \mathrm{Scal}_p(\omega) + \left(\sum_{a=1}^{k} \frac{s_a}{\langle \mu_\omega, p_a \rangle + c_a} \right). \tag{46}$$

Using that (as functions on $X \times P$) $\mu_{\tilde{\omega}} = \mu_\omega$ and $g_\omega(\xi_i, \xi_j) = g_{\tilde{\omega}}(\xi_i, \xi_j)$ (see the proof of Lemma 5.7), we further compute, from (46),

$$\begin{aligned}
\mathrm{Scal}_v(\tilde{\omega}) &= v(\mu_{\tilde{\omega}}) \,\mathrm{Scal}(\tilde{\omega}) + 2 \sum_{i=1}^{r} v_{,i}(\mu_{\tilde{\omega}}) \Delta^Y_{\tilde{\omega}}(\langle \mu_{\tilde{\omega}}, \xi_i \rangle) - \sum_{i,j=1}^{r} v_{,ij}(\mu_{\tilde{\omega}}) g_{\tilde{\omega}}(\xi_i, \xi_j) \\
&= \frac{v(\mu_\omega)}{p(\mu_\omega)} \mathrm{Scal}_p(\omega) + v(\mu_\omega) q(\mu_\omega) + 2 \sum_{i=1}^{r} v_{,i}(\mu_\omega) \Delta^Y_{\tilde{\omega}}(\langle \mu_\omega, \xi_i \rangle) \\
&\qquad - \sum_{i,j=1}^{r} v_{,ij}(\mu_\omega) g_{\tilde{\omega}}(\xi_i, \xi_j) \\
&= \frac{v(\mu_\omega)}{p(\mu_\omega)} \mathrm{Scal}_p(\omega) + v(\mu_\omega) q(\mu_\omega) + 2 \sum_{i=1}^{r} v_{,i}(\mu_\omega) \Delta^X_{\omega,p}(\langle \mu_\omega, \xi_i \rangle) \\
&\qquad - \sum_{i,j=1}^{r} v_{,ij}(\mu_\omega) g_\omega(\xi_i, \xi_j),
\end{aligned}$$

where, for passing to the last line, we used the identity $\Delta^Y_{\tilde{\omega}} = \Delta^X_{\omega,p}$ established in Lemma A.3. As

$$\Delta^X_{\omega,p}(\psi) := \frac{1}{p(\mu_\omega)} \delta_\omega(p(\mu_\omega) d\psi) = \Delta^X_\omega(\psi) - \sum_{j=1}^{r} \frac{p_{,j}(\mu_\omega)}{p(\mu_\omega)} g_\omega(d\mu^{\xi_j}_\omega, d\psi), \tag{47}$$

we further get

$$\begin{aligned}
&\frac{v(\mu_\omega)}{p(\mu_\omega)} \mathrm{Scal}_p(\omega) + 2 \sum_{i=1}^{r} v_{,i}(\mu_\omega) \Delta^X_{\omega,p}(\langle \mu_\omega, \xi_i \rangle) - \sum_{i,j=1}^{r} v_{,ij}(\mu_\omega) g_\omega(\xi_i, \xi_j) \\
&\quad = v(\mu_\omega) \,\mathrm{Scal}(\omega) + 2 \sum_{i=1}^{r} \frac{v(\mu_\omega) p_{,i}(\mu_\omega)}{p(\mu_\omega)} \Delta^X_\omega(\langle \mu_\omega, \xi_i \rangle) \\
&\qquad - \sum_{i,j=1}^{r} \frac{v(\mu_\omega) p_{,ij}(\mu_\omega)}{p(\mu_\omega)} g_\omega(\xi_i, \xi_j) + 2 \sum_{i=1}^{r} v_{,i}(\mu_\omega) \Delta^X_\omega(\langle \mu_\omega, \xi_i \rangle) \\
&\qquad - 2 \sum_{i,j=1}^{r} \frac{v_{,i}(\mu_\omega) p_{,j}(\mu_\omega)}{p(\mu_\omega)} g^X(\xi_i, \xi_j) - \sum_{i,j=1}^{r} v_{,ij}(\mu_\omega) g_\omega(\xi_i, \xi_j)
\end{aligned}$$

$$= \frac{1}{p(\mu_\omega)}\Big((pv)(\mu_\omega)\,\mathrm{Scal}_\omega + 2\sum_{i=1}^{r}(pv)_{,i}(\mu_\omega)\Delta^X_\omega(\langle \mu_\omega, \xi_i\rangle) - \sum_{i,j=1}^{r}(pv)_{,ij}(\mu_\omega)g_\omega(\xi_i,\xi_j)\Big)$$

$$= \frac{1}{p(\mu_\omega)}\,\mathrm{Scal}_{pv}(\omega).$$

The expression (35) follows from the above formulas. □

Lemma 5.10 *The restriction of the weighted Mabuchi energy $\boldsymbol{M}^Y_{v,w}$ on Y to the subspace $\mathcal{K}_{\mathbb{T}}(X,\omega_0) \subset \mathcal{K}_{\mathbb{T}}(Y,\tilde{\omega}_0)$ is equal to $C\boldsymbol{M}^X_{pv,\tilde{w}}$, where p, w and $\tilde{w}$ are as given in Lemma 5.9 and $C = \mathrm{Vol}(B,\omega_B)$.*

Proof This is a direct corollary of Lemma 5.9 and Definition 1.1. □

We now specialize to the case when each (B_a,ω_a) is a Hodge Kähler–Einstein manifold with positive scalar curvature $s_a = 2n_a k_a$, where $k_a \in \mathbb{N}$. Equivalently, $2\pi c_1(B_a) = k_a[\omega_a]$ for a positive integer k_a and an integral Kähler class $\frac{1}{2\pi}[\omega_a]$. Notice that k_a must be a positive divisor of the Fano index $\mathrm{Ind}(B_a)$ of B_a, which yields the a priori bound $1 \le k_a \le \mathrm{Ind}(B_a)$. We also assume that $(X,\mathbb{T})$ is Fano, with canonically normalized momentum polytope Δ. We then have:

Lemma 5.11 *In the setting above, if the affine-linear functions $\langle p_a,\mu\rangle + k_a$ are positive on Δ, then the bundle-compatible Kähler metric $\tilde{\omega}$ on Y corresponding to the constants $c_a = k_a$ belongs to de Rham class $2\pi c_1(Y)$. Furthermore, $\tilde{\omega}$ is a v–soliton if and only if ω is a pv–soliton.*

Proof By using (38) and rearranging the terms in (40), we have the relation (written on Z)

$$\rho_{\tilde{\omega}} = \rho_\omega + \sum_{a=1}^{k}(\langle p_a, \mu_{\rho_\omega}\rangle + c_a)\omega_a + \langle d_X\mu_{\rho_\omega} \wedge \theta\rangle + \sum_{a=1}^{k}(\rho_a - c_a\omega_a) - \tfrac{1}{2}d_Y d^c_Y \log p(\mu_\omega), \tag{48}$$

where $\rho_{\tilde{\omega}}$, ρ_ω and ρ_a denote the Ricci forms of $(Y,\tilde{\omega})$, (X,ω) and (B_a,ω_a), respectively, pulled back to Z, and $\mu_{\rho_\omega} := d^c_X\kappa$ is the "momentum map" with respect to the Ricci form ρ_ω. As in (42), we have $\mu_{\rho_\omega} = \frac{1}{2}\Delta_\omega\mu_\omega$. Suppose $\rho_\omega - \omega = \frac{1}{2}d_X d^c_X h$ for some $\mathbb{T}$–invariant smooth function on X; by using that the momentum polytope Δ is canonically normalized, we have (see (11)) $\mu_{\rho_\omega} - \mu_\omega = d^c h$. A closer look at the proof of Lemma 5.5 and the relation (48) (with $c_a = s_a/(2n_a) = k_a$) show that

$$\rho_{\tilde{\omega}} - \tilde{\omega} = \tfrac{1}{2} d_Y d_Y^c \tilde{h} \quad \text{with } \tilde{h} := h - \log p(\mu_\omega).$$

The claim follows from the above. □

Remark 5.12 Lemma 5.11 provides a useful way to *construct* semisimple $(X, \mathbb{T})$–principal Fano fibrations. Indeed, for given positive Hodge Kähler–Einstein manifolds (B_a, ω_a) as above with corresponding integer constants k_a, and a given Fano manifold $(X, \mathbb{T})$ with associated canonical polytope Δ, one can try to find the possible principal $\mathbb{T}$–bundles P over $B = \prod_{a=1}^k B_a$ for which the corresponding semisimple $(X, \mathbb{T})$–principal fibration is Fano. Such principal $\mathbb{T}$–bundles P are in correspondence with the choice of lattice elements $p_a \in \Lambda \subset \mathfrak{t}$ and Lemma 5.11 tells us that for a set of elements p_a, to determine a Fano semisimple $(X, \mathbb{T})$–principal fibration Y it is sufficient to check that, for all a,

$$\langle p_a, \mu \rangle + k_a > 0 \quad \text{on } \Delta.$$

For instance, if we take $B = B_1 = \mathbb{P}^1$ with a Fubini–Study metric ω_1 of scalar curvature 4 (so that $k_1 = 2$ and ω_1 is primitive) and $(X, \mathbb{T}) = (\mathbb{P}^1, \mathbb{S}^1)$ with canonical polytope $\Delta = [-1, 1]$, then the possible Fano $(\mathbb{P}^1, \mathbb{S}^1)$–principal fibrations will correspond to $p_1 \in \mathbb{Z}$ such that $p_1 \mu + 2 > 0$ on $[-1, 1]$, ie $p_1 = \pm 1, 0$ are the only possible values. This gives rise to the Fano surfaces $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-1)) \cong \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1))$ and $\mathbb{P}^1 \times \mathbb{P}^1$. In general, the isomorphism class of the principal $\mathbb{T}$–bundle P over B, and hence also the semisimple $(X, \mathbb{T})$–principal Fano fibration constructed as above, is encoded by the Hodge classes $\frac{1}{2\pi}[\omega_a] \otimes p_a = 1/k_a c_1(B_a) \otimes p_a \in H^2(B, \mathbb{Z})^r$. The a priori bounds $1 \le k_a \le \mathrm{Ind}(B_a)$ for k_a show that, for given base $B = \prod_{a=1}^k B_a$ and fibre $(X, \mathbb{T})$, there are only a finite number of semisimple $(X, \mathbb{T})$–principal Fano fibrations constructed this way.

Remark 5.13 The relationship between the Ricci potentials $\tilde{h}$ and h established in the proof of Lemma 5.11 and (29) yield, via Remark 2.3, that, if the momentum map μ_ω of $(X, \omega, \mathbb{T}_X)$ is canonically normalized, then the momentum map $\mu_{\tilde{\omega}} = \mu_\omega$ of the corresponding bundle-compatible Kähler metric $\tilde{\omega}$ on $(Y, \mathbb{T}_Y)$ is also canonically normalized.

We finish this section with a straightforward extension of [6, Lemma 5]:

Lemma 5.14 *Suppose Y is a semisimple principal $(X,\mathbb{T})$–fibration over B, such that $\mathbb{T}$ is a maximal torus in the reduced group of automorphisms $\mathrm{Aut}_r(X)$. Let $\tilde{\omega}$ be a bundle-compatible Kähler metric on Y corresponding to a $\mathbb{T}$–invariant Kähler metric ω on X, and $\mathbb{K}_B\subset \mathrm{Aut}_r(B)$ be a maximal compact torus in the reduced group of automorphisms of B which (without loss of generality by the Lichnerowicz–Matsushima theorem) belongs to the isometry group of ω_B. Then $\tilde{\omega}$ is invariant under the action of a maximal torus $\mathbb{K}_Y\subset \mathrm{Aut}_r(Y)$, and we have an exact sequence*

$$\{0\}\to \mathrm{Lie}(\mathbb{T}_Y)\to \mathrm{Lie}(\mathbb{K}_Y)\to \mathrm{Lie}(\mathbb{K}_B)\to \{0\}.$$

Furthermore, for any positive weight functions v and w_0 defined on $\Delta\subset\mathfrak{t}^*$, there exists a unique affine-linear function $\ell^{\mathrm{ext}}_{v,w_0}$ on $\mathfrak{t}^*$ such that, when pulled back to the dual Lie algebra $\mathfrak{k}_Y^*$ of $\mathbb{K}_Y$, $(v, w_0\ell^{\mathrm{ext}}_{v,w_0})$ satisfies (3) with respect to $\tilde{\omega}$ on Y for any affine-linear function ℓ on $\mathfrak{k}_Y^*$.

Proof This proof is not materially different than the proof of [6, Lemma 5] (which is made in the case when $(X,\mathbb{T})$ is toric and $v=w_0=1$). We only give a sketch. A Killing potential f for a Killing vector field $K\in\mathfrak{k}_B:=\mathrm{Lie}(\mathbb{K}_B)$ is of the form $f=\sum_{a=1}^k f_a$, where f_a is a Killing potential of (B_a,ω_a). Letting $\tilde{K}$ be the horizontal lift of K to P (using the $\mathfrak{t}_P$–valued connection 1–form θ), one can check that the vector field on P

$$\hat{K}=\tilde{K}+\sum_{a=1}^k f_a\xi^P_{p_a}$$

is a CR vector field on $(P,\mathcal{D},J_B)$, hence also on $(Z,\mathscr{H},J_B\oplus J_X)$. Furthermore, a direct verification in (24) reveals that

$$\iota_{\hat{K}}\tilde{\omega}=-d\Big(\sum_{a=1}^k(\langle p_a,\mu_\omega\rangle+c_a)f_a\Big), \tag{49}$$

so $\hat{K}$ also preserves $\tilde{\omega}$. We thus obtain a lift $\hat{\mathfrak{k}}_B$ of the Lie algebra $\mathfrak{k}_B=\mathrm{Lie}(T_B)$ to Z, which clearly commutes with the action $\mathbb{T}_Z$ and preserves both the CR structure of $(Z,\mathscr{H})$ and the 2–form $\tilde{\omega}$. The Lie algebra $\mathfrak{k}_Y$ of $\mathbb{K}_Y$ is then induced by $\mathfrak{t}_X\oplus\hat{\mathfrak{k}}_B\subset TZ$, which descend to an abelian Lie algebra of Killing fields on Y. The maximality of $\mathbb{K}_Y\subset\mathrm{Aut}_r(Y)$ and the exactness of the sequence follow from the maximality of each $\mathbb{K}_B\subset\mathrm{Aut}_r(B)$ and $\mathbb{T}\subset\mathrm{Aut}_r(X)$, and the fact that (recall that Y is a locally trivial X–fibre bundle and therefore the fibres have trivial normal bundle) any holomorphic vector field on Y projects under π_B to a holomorphic vector field on B. For the final claim in Lemma 5.14, notice that by (49) the Killing potentials of all lifted Killing vector fields $\widehat{K}$ from B are of the form $\sum_{a=1}^{k}(\langle p_a,\mu_\omega\rangle + c_a)f_a$. Thus, by Lemma 5.9 and using (27), the integral condition (3) on $(Y,\tilde{\omega})$ will be zero for any such Killing potential as soon as we normalize $\int_{B_a} f_a\omega_a^{n_a} = 0$ and assume $\ell^{\mathrm{ext}}_{v,w_0} \in \mathrm{Aff}(\mathfrak{t}^*_X)$. On the other hand, examining (3) on $(Y,\tilde{\omega})$ for the Killing potentials $\ell(\mu_{\tilde{\omega}})$ for $\ell \in \mathrm{Aff}(\mathfrak{t}^*)$ reduces (again by Lemma 5.9 and (29)) to an integral relation on (X,ω) which defines a unique element $\ell^{\mathrm{ext}}_{v,w_0} \in \mathrm{Aff}(\mathfrak{t}^*)$. □

6 Weighted functionals and distances and their extensions

Let ω_0 be a $\mathbb{T}$–invariant Kähler metric in the Kähler class α, denote by $\mathrm{PSH}_{\mathbb{T}}(X,\omega_0)$ the space of $\mathbb{T}$–invariant ω_0–plurisubharmonic functions in $L^1(X,\omega_0)$, and define the class of potentials of full volume by

$$\mathcal{E}_{\mathbb{T}}(X,\omega_0) := \left\{\varphi \in \mathrm{PSH}_{\mathbb{T}}(X,\omega_0) \;\middle|\; \int_X \mathrm{MA}(\varphi) = \int_X \omega_0^{[n]}\right\}.$$

According to [27], the d_1–completion of $\mathcal{K}_{\mathbb{T}}(X,\omega_0)$ can be identified with the subspace of potentials of finite energy:

$$\mathcal{E}^1_{\mathbb{T}}(X,\omega_0) = \left\{\varphi \in \mathcal{E}_{\mathbb{T}}(X,\omega_0) \;\middle|\; \int_X |\varphi|\,\mathrm{MA}(\varphi) < \infty\right\}.$$

Our main result in this section will be the existence of an lsc extension of the weighted Mabuchi functional (defined in Definition 1.1 on the space $\mathcal{K}_{\mathbb{T}}(X,\omega_0)$) to a functional on $\mathcal{E}^1_{\mathbb{T}}(X,\omega_0)$. Our starting point is that the weighted Mabuchi energy $\boldsymbol{M}_{v,w}$ admits a weighted Chen–Tian decomposition [55, Theorem 5] into energy and entropy parts

$$(50)\quad \boldsymbol{M}_{v,w}(\varphi) = \int_X \log\left(\frac{v(\mu_\varphi)\omega_\varphi^m}{\omega_0^m}\right) v(\mu_\varphi)\omega_\varphi^{[m]} - 2\boldsymbol{I}_v^{\rho_{\omega_0}}(\varphi) + \boldsymbol{I}_w(\varphi) - \int_X \log(v(\mu_0))v(\mu_0)\omega_0^{[m]},$$

where ρ_{ω_0} is the Ricci form of ω_0 and the functionals $\boldsymbol{I}_w$ and $\boldsymbol{I}_v^{\rho_{\omega_0}}$ are introduced in Definition 6.2. We want to show:

Theorem 6.1 *For smooth weight functions $v(\mu)$ and $w(\mu)$ such that $v(\mu) > 0$ on Δ, the weighted Mabuchi energy $\boldsymbol{M}_{v,w}\colon \mathcal{K}_{\mathbb{T}}(X,\omega_0) \to \mathbb{R}$ extends using (50) to the largest d_1–lsc functional $\boldsymbol{M}_{v,w}\colon \mathcal{E}^1_{\mathbb{T}}(X,\omega_0) \to \mathbb{R}\cup\{\infty\}$ which is convex along the finite-energy geodesics of $\mathcal{E}_{\mathbb{T}}(X,\omega_0)$. Additionally, the extended weighted Mabuchi energy $\boldsymbol{M}_{v,w}$ is linear in v and w, uniformly continuous in w in the $C^0(\Delta)$ topology, and continuous with respect to v in the $C^1(\Delta)$ topology.*

The above result is well known for the unweighted case, by [13], and we will follow a similar path to get an extension in the weighted case. The proof of Theorem 6.1 will be given at the end of the section, and we detail below the definition and extension of each component of (50).

6.1 The weighted Aubin–Mabuchi functionals

Definition 6.2 [55] For a smooth weight function $v(\mu)$ on Δ we let $\boldsymbol{I}_v$ denote the functional on $\mathcal{K}_{\mathbb{T}}(X,\omega_0)$ defined by

$$(d_\varphi \boldsymbol{I}_v)(\dot\varphi) = \int_X \dot\varphi v(\mu_\varphi)\omega_\varphi^{[n]}, \quad \boldsymbol{I}_v(0)=0,$$

and let $\boldsymbol{J}_v := \int_X \varphi v(\mu_0)\omega_0^{[m]} - \boldsymbol{I}_v(\varphi)$. Furthermore, for a fixed $\mathbb{T}$–invariant closed $(1,1)$–form ρ on X with momentum $\mu_\rho\colon X \to \mathfrak{t}^*$, we define the *$\rho$–twisted Aubin–Mabuchi functional* $\boldsymbol{I}_v^\rho\colon \mathcal{K}_{\mathbb{T}}(X,\omega_0)\to\mathbb{R}$ by

$$(d_\varphi \boldsymbol{I}_v^\rho)(\dot\varphi) := \int_X \dot\varphi\big(v(\mu_\varphi)\rho\wedge\omega_\varphi^{[m-1]} + \langle (dv)(\mu_\varphi),\mu_\rho\rangle\omega_\varphi^{[m]}\big), \quad \boldsymbol{I}_v^\rho(0)=0.$$

For $v\equiv 1$, we let $\boldsymbol{I}_1=\boldsymbol{I}$, $\boldsymbol{J}_1=\boldsymbol{J}$ and $\boldsymbol{I}_v^\rho=\boldsymbol{I}^\rho$, and notice that $\boldsymbol{I}$ and $\boldsymbol{J}$ are the functionals introduced in Definition 3.1

Remark 6.3 It follows from the above definition and the results in [55] that for any weight $v(x)$ and a constant c, $\boldsymbol{J}_v(\varphi+c)=\boldsymbol{J}_v(\varphi)$, allowing one to see $\boldsymbol{J}_v$ as a functional on the space of $\mathbb{T}$–invariant Kähler metrics in the Kähler class $\alpha=[\omega_0]$, and motivating the notation $\boldsymbol{J}_v(\omega_\varphi)$. Notice also that $\boldsymbol{I}_v$, $\boldsymbol{J}_v$ and $\boldsymbol{I}_v^\rho$ are linear in v. In the case when $v>0$, $\boldsymbol{J}_v$ is nonnegative (see Lemma 6.4), whereas $\boldsymbol{I}_v$ is monotone in the sense that, for any $\varphi_0,\varphi_1\in\mathcal{K}_{\mathbb{T}}(X,\omega_0)$ with $\varphi_1(x)\ge\varphi_0(x)$,

$$\boldsymbol{I}_v(\varphi_1)-\boldsymbol{I}_v(\varphi_0)\ge \inf_\Delta(v)\int_X(\varphi_1-\varphi_0)\omega_{\varphi_0}^{[m]}.$$

The above inequality follows by Definition 6.2, integrating the derivative of $\boldsymbol{I}_v$ along the path $t\varphi_1+(1-t)\varphi_0\in\mathcal{K}_{\mathbb{T}}(X,\omega_0)$, and integrating by parts.

The following is established in [45, (2.37)]:

Lemma 6.4 *Let $v>0$. There exists a uniform constant $C=C(X,\omega_0,v)>0$ such that*

$$\frac{1}{C}\boldsymbol{J}(\varphi)\le\boldsymbol{J}_v(\varphi)\le C\boldsymbol{J}(\varphi).$$

Proof Let $\varphi_t := \varphi_0 + t\varphi$ with $\varphi := \varphi_1 - \varphi_0$ and $\omega_{\varphi_t} = \omega_{\varphi_0} + t\,dd^c\varphi$ for $t \in [0,1]$. We compute

$$\begin{aligned}
\boldsymbol{J}_v(\varphi) = \boldsymbol{J}_v(\varphi_1) - \boldsymbol{J}_v(\varphi_0) &= \int_0^1 \int_X \varphi(v(\mu_\omega)\omega^{[m]} - v(\mu_{\varphi_s})\omega_{\varphi_s}^{[m]})\,ds \\
&= -\int_0^1 \int_X \varphi\Big(\int_0^s \frac{d}{dt}[v(\mu_{\varphi_t})\omega_{\varphi_t}^{[m]}]\,dt\Big)\,ds \\
&= -\int_0^1 \int_X \varphi\Big(\int_0^s \big(g_{\varphi_t}(d[\log\circ v(\mu_{\varphi_t})], d\varphi) - \Delta_{\varphi_t}(\varphi)\big)v(\mu_{\varphi_t})\omega_{\varphi_t}^{[m]}\,dt\Big)\,ds \\
&= -\int_0^1 \int_0^s \Big(\int_X \varphi\, d[v(\mu_{\varphi_t})] \wedge d^c\varphi \wedge \omega_{\varphi_t}^{[m-1]} + \varphi\, dd^c\varphi \wedge v(\mu_{\varphi_t})\omega_{\varphi_t}^{[m-1]}\Big)\,dt\,ds \\
&= \int_0^1 \int_0^s \Big(\int_X v(\mu_{\varphi_t})\,d\varphi \wedge d^c\varphi \wedge \omega_{\varphi_t}^{[m-1]}\Big)\,dt\,ds \\
&= \int_0^1 \int_0^s \Big(\int_X v(\mu_{\varphi_t})\,d\varphi \wedge d^c\varphi \wedge (t\omega_\varphi + (1-t)\omega)^{[m-1]}\Big)\,dt\,ds \\
&= \sum_{j=0}^{m-1} \int_0^1 \int_0^s \Big(\int_X t^j(1-t)^{m-j-1} v(\mu_{\varphi_t})\,d\varphi \wedge d^c\varphi \wedge \omega^{[j]} \wedge \omega_\varphi^{[m-j-1]}\Big)\,dt\,ds,
\end{aligned}$$

where, in the fourth equality, we have used that

$$\frac{d}{dt}[v(\mu_{\varphi_t})] = \sum_{i=1}^{r} v_{,i}(\mu_{\varphi_t})(d^c\varphi)(\xi_i) = g_{\varphi_t}(d[v(\mu_{\varphi_t})], d\varphi) \tag{51}$$

for any basis $(\xi_i)_{i=1,\dots,r}$ of $\mathfrak{t}$. It follows that

$$\frac{1}{C}\boldsymbol{J}(\varphi) \le \boldsymbol{J}_v(\varphi) \le C\,\boldsymbol{J}(\varphi),$$

where $C = C(X,\alpha,v)$ is a constant such that $1/C \leqslant v \leqslant C$ on Δ_α. □

Lemma 6.5 *Suppose v and w are smooth functions on Δ. Then*

$$\begin{aligned}
|\boldsymbol{J}_v(\varphi) - \boldsymbol{J}_w(\varphi)| &\le \|v-w\|_{C^0(\Delta)}\boldsymbol{J}_1(\varphi), \\
|\boldsymbol{I}_v(\varphi) - \boldsymbol{I}_w(\varphi)| &\le \|v-w\|_{C^0(\Delta)}(\|\varphi\|_{L^1(X,\omega_0)} + \boldsymbol{J}_1(\varphi)).
\end{aligned}$$

In particular, for a fixed $\varphi \in \mathcal{K}_{\mathbb{T}}(X,\omega_0)$, $\boldsymbol{I}_v(\varphi)$ and $\boldsymbol{J}_v(\varphi)$ are uniformly continuous in v.

Proof The first relation follows from Lemma 6.4 whereas the second inequality follows from the first and Definition 6.2. □

Lemma 6.6 *The restrictions of $\boldsymbol{I}_1^Y$ and $\boldsymbol{J}_1^Y$ to the subspace $\mathcal{K}_{\mathbb{T}}(X,\omega_0) \subset \mathcal{K}_{\mathbb{T}}(Y,\widetilde{\omega}_0)$ are equal to $C\,\boldsymbol{I}_p^X$ and $C\,\boldsymbol{J}_p^X$, respectively, where $p(\mu)$ is the weight function defined* *in Lemma 5.9 and $C = \mathrm{Vol}(B, \omega_B)$. Furthermore, if $\tilde{\rho}$ is a Kähler form on Y induced by a Kähler form ρ on X using (24), then the restriction of $(\boldsymbol{I}_1^{\tilde{\rho}})^Y$ to the subspace $\mathcal{K}_{\mathbb{T}}(X, \omega_0)$ equals $C(\boldsymbol{I}_p^{\rho})^X$.*

Proof The first part follows from the definition of $\boldsymbol{I}_1^Y$, using that

$$\tilde{\omega}_\varphi^{[n+m]} \wedge \theta^{\wedge r} = p(\mu_\varphi)\omega_\varphi^{[m]} \wedge \omega_B^{[n]} \wedge \theta^{\wedge r}$$

on Z.

Similarly, if $\tilde{\rho}$ is a $(1,1)$–form Y whose pullback to Z is

$$\tilde{\rho} := \rho + \sum_{a=1}^{k} (\langle p_a, \mu_\rho\rangle + c_a))\pi_B^*\omega_a + \langle d\mu_\rho \wedge \theta\rangle, \tag{52}$$

we compute

$$\begin{aligned}(d_\varphi \boldsymbol{I}_p^{\rho})^X(\dot{\varphi}) &= \int_X \dot{\varphi}[p(\mu_\varphi)\rho \wedge \omega_\varphi^{[m-1]} + \langle (dp)(\mu_\varphi), \mu_\rho\rangle \omega_\varphi^{[m]}] \\ &= \frac{1}{\mathrm{Vol}(B, \omega_B)} \int_Y \dot{\varphi}\tilde{\rho} \wedge \tilde{\omega}_\varphi^{[n+m-1]} = \frac{1}{\mathrm{Vol}(B, \omega_B)} (d_\varphi \boldsymbol{I}^{\tilde{\rho}})^Y(\dot{\varphi}).\end{aligned}$$

The claim follows, as $(\boldsymbol{I}_p^{\rho})^X(0) = 0 = (\boldsymbol{I}^{\tilde{\rho}})^Y(0)$. □

6.2 The weighted d_1–distance

Definition 6.7 Let $v > 0$ be a positive function on Δ. For $\varphi_0, \varphi_1 \in \mathcal{K}_{\mathbb{T}}(X, \omega_0)$ we let

$$d_{1,v}(\varphi_0, \varphi_1) := \inf_{\psi(t)} \{L_{1,v}(\psi(t)) \mid \psi(t,x) \in C_{\mathbb{T}}^\infty([0,1] \times X) \text{ and } \psi(t) \in \mathcal{K}_{\mathbb{T}}(X, \omega_0)\},$$

where

$$L_{1,v}(\psi(t)) := \int_0^1 \left(\int_X |\dot{\psi}(t)| v(\mu_{\psi(t)}) \omega_{\psi(t)}^{[m]} \right) dt.$$

For $v \equiv 1$, we have $d_{1,1} = d_1$, where d_1 is the distance introduced in Section 3.

Lemma 6.8 *For any weight $v > 0$, there exists uniform constant $C = C(X, \omega_0, v) > 0$ such that*

$$\frac{1}{C} d_1(\varphi_0, \varphi_1) \le d_{1,v}(\varphi_0, \varphi_1) \le C d_1(\varphi_0, \varphi_1) \quad \text{for all } \varphi_0, \varphi_1 \in \mathcal{K}_{\mathbb{T}}(X, \omega_0), \tag{53}$$

where $d_1 := d_{1,1}$ is the distance introduced in [27]. In particular, $d_{1,v}$ is a distance on $\mathcal{K}_{\mathbb{T}}(X, \omega_0)$ which is quasiisometric with d_1.

Proof The relation (53) follows from the fact that $v(\mu)$ is positive and uniformly bounded on Δ. This yields that $d_{1,v}$ is a distance, as d_1 is a distance according to [27]. □

Lemma 6.9 *For any smooth weight $v > 0$,*

$$|\boldsymbol{I}_v(\varphi_0) - \boldsymbol{I}_v(\varphi_1)| \leq d_{1,v}(\varphi_0, \varphi_1) \leq C d_1(\varphi_0, \varphi_1) \quad \text{for all } \varphi_0, \varphi_1 \in \mathcal{K}_{\mathbb{T}}(X, \omega_0).$$

Proof For any smooth curve φ_t between φ_1 and φ_2, using Definition 6.2,

$$|\boldsymbol{I}_v(\varphi_0) - \boldsymbol{I}_v(\varphi_1)| = \left| \int_0^1 (d_{\varphi_t} \boldsymbol{I}_v)(\dot{\varphi}_t)\, dt \right| \leq L_{1,v}(\varphi_t).$$

The claim follows from the above and Lemma 6.8. □

6.3 Extensions to $\mathcal{E}^1_{\mathbb{T}}(X, \omega_0)$

Lemma 6.10 *For any smooth weight v, the functionals $\boldsymbol{I}_v$ and $\boldsymbol{J}_v$ continuously extend to the space $\mathcal{E}^1_{\mathbb{T}}(X, \omega_0)$. Furthermore, for any $\psi \in \mathcal{E}^1_{\mathbb{T}}(X, \omega_0)$, the extended functionals are linear and uniformly continuous in v, in the topology $C^0(\Delta)$.*

Proof $\boldsymbol{I}_v$ is d_1–Lipschitz by Lemma 6.9; for $\boldsymbol{J}_v$, we get from Definition 6.2 that

$$|\boldsymbol{J}_v(\varphi_0) - \boldsymbol{J}_v(\varphi_1)| \leq \int_X |\varphi_0 - \varphi_1| \omega_0^{[m]} + |\boldsymbol{I}_v(\varphi_0) - \boldsymbol{I}_v(\varphi_1)|.$$

Combining the above inequality with Lemma 6.9 and [27, Corollary 5.7], there exists a uniform positive constant $C = C(X, \omega_0, v)$ and, for any fixed positive real number $R > 0$, an increasing continuous function $F_R \colon \mathbb{R}_+ \to \mathbb{R}_+$, $F(0) = 0$, defined in terms of (X, ω_0, R), such that, for any $\varphi_0, \varphi_1 \in \mathcal{K}_{\mathbb{T}}(X, \omega_0)$ with $d_1(0, \varphi_i) \leq R$,

$$|\boldsymbol{J}_v(\varphi_0) - \boldsymbol{J}_v(\varphi_1)| \leq C d_1(\varphi_0, \varphi_1) + F_R(d_1(\varphi_0, \varphi_1)),$$

showing that $\boldsymbol{J}_v$ is locally uniformly continuous on $(\mathcal{K}_{\mathbb{T}}(X, \omega_0), d_1)$ and thus extends continuously to $(\mathcal{E}^1_{\mathbb{T}}(X, \omega_0), d_1)$.

The v–linearity of $\boldsymbol{I}_v$ and $\boldsymbol{J}_v$ is clear by continuity; see Remark 6.3. The continuity with respect to v follows from the continuous extensions of the inequalities in Lemma 6.5, noting that we have already shown that $\boldsymbol{J}_v$, $\boldsymbol{J}_w$, $\boldsymbol{J}$, $\boldsymbol{I}_v$ and $\boldsymbol{I}_w$ all extend continuously, whereas $\|\cdot\|_{L^1(X,\omega_0)}$ extends continuously by [27, Theorem 5.8]. □

Corollary 6.11 *The metric completion of $(\mathcal{K}_{\mathbb{T}}(X, \omega_0) \cap \boldsymbol{I}_v^{-1}(0), d_1)$ is the complete geodesic metric space $(\mathcal{E}^1_{\mathbb{T}}(X, \omega_0) \cap \boldsymbol{I}_v^{-1}(0), d_1)$.*

Proof Similarly to [29, Lemma 5.2], one can show that $\boldsymbol{I}_v$ is linear along finite-energy geodesics. As $\boldsymbol{I}_v \colon \mathcal{E}^1_{\mathbb{T}}(X, \omega_0) \to \mathbb{R}$ is d_1–continuous, it follows that $\mathcal{E}^1_{\mathbb{T}}(X, \omega_0) \cap \boldsymbol{I}_v^{-1}(0)$ is a d_1–closed subspace. □

Lemma 6.12 *Let v be a smooth weight function and ρ a $\mathbb{T}$–invariant closed $(1,1)$–form. The functional $\boldsymbol{I}_v^{\rho}\colon \mathcal{K}_{\mathbb{T}}(X,\omega_0)\to\mathbb{R}$ extends to a d_1–continuous functional on $\mathcal{E}^1_{\mathbb{T}}(X,\omega_0)$ which is bounded on d_1–bounded subsets of $\mathcal{E}^1_{\mathbb{T}}(X,\omega_0)$. Furthermore, the extended functional is linear and uniformly continuous in v, in the $C^1(\Delta)$ topology.*

Proof Following the proof of [14, Proposition 4.4], we show that $\boldsymbol{I}_v^{\rho}$ is locally uniformly d_1–continuous and bounded on d_1–bounded subsets of $\mathcal{K}_{\mathbb{T}}(X,\omega_0)$. Letting $\varphi_0,\varphi_1\in\mathcal{K}_{\mathbb{T}}(X,\omega_0)$, we put $\varphi_s:=s\varphi_1+(1-s)\varphi_0$ for $s\in[0,1]$ and compute

$$
\begin{aligned}
(54)\quad \boldsymbol{I}_v^{\rho}(\varphi_1)-\boldsymbol{I}_v^{\rho}(\varphi_0)&=\int_0^1\frac{d}{ds}\boldsymbol{I}_v^{\rho}(\varphi_s)\,ds\\
&=\int_0^1\int_X(\varphi_1-\varphi_0)\big(v(\mu_{\varphi_s})\rho\wedge\omega_{\varphi_s}^{[m-1]}+\langle(dv)(\mu_{\varphi_s}),\mu_\rho\rangle\omega_{\varphi_s}^{[m]}\big)\,ds\\
&=\int_X(\varphi_1-\varphi_0)\sum_{j=0}^{m-1}v_{j,m-1}(\mu_{\varphi_0},\mu_{\varphi_1})\rho\wedge\omega_{\varphi_1}^{[j]}\wedge\omega_{\varphi_0}^{[m-j-1]}\\
&\quad+\int_X(\varphi_1-\varphi_0)\sum_{j=0}^{m}\langle(dv)_{j,m}(\mu_{\varphi_0},\mu_{\varphi_1}),\mu_\rho\rangle\omega_{\varphi_1}^{[j]}\wedge\omega_{\varphi_0}^{[m-j]},
\end{aligned}
$$

where $v_{j,k}(\mu_0,\mu_1)$ and $(dv)_{j,k}(\mu_0,\mu_1)$ are defined on $\Delta\times\Delta$ by

$$
\begin{aligned}
v_{j,k}(\mu_0,\mu_1)&:=\int_0^1 s^j(1-s)^{k-j}v(s\mu_1+(1-s)\mu_0),\\
(dv)_{j,k}(\mu_0,\mu_1)&=\int_0^1 s^j(1-s)^{k-j}(dv)(s\mu_1+(1-s)\mu_0).
\end{aligned}
$$

Using the computation (54),

$$
\begin{aligned}
(55)\quad |\boldsymbol{I}_v^{\rho}(\varphi_1)-\boldsymbol{I}_v^{\rho}(\varphi_0)|&\leqslant C\int_X|\varphi_1-\varphi_0|\sum_{j=0}^{m-1}\omega_0\wedge\omega_{\varphi_1}^{[j]}\wedge\omega_{\varphi_0}^{[m-j-1]}\\
&\quad+C\int_X|\varphi_1-\varphi_0|\sum_{j=0}^{m}\omega_{\varphi_1}^{[j]}\wedge\omega_{\varphi_0}^{[m-j]}\\
&\leqslant C\int_X|\varphi_1-\varphi_0|\omega_{(\varphi_0+\varphi_1)/4}^{[m]},
\end{aligned}
$$

where in the first inequality we use that the functions $\langle(dv)_{j,k}(\mu_{\varphi_0},\mu_{\varphi_1}),\mu_\rho\rangle$ and $v_{j,k}(\mu_{\varphi_0},\mu_{\varphi_1})$ are bounded on $\Delta\times\Delta$ and $-C\omega_0<\rho<C\omega_0$ for some constant $C>1$, and in the second inequality we use the observation $\omega_{(\varphi_0+\varphi_1)/4}=\frac{1}{2}\omega_0+\frac{1}{4}\omega_{\varphi_0}+\frac{1}{4}\omega_{\varphi_1}$. Using the estimate (55) we can show, similarly to [14, Proposition 4.4], that for any $R>0$ there is an increasing continuous function $F_R\colon\mathbb{R}\to\mathbb{R}$ with $F_R(0)=0$ such that

$$
|\boldsymbol{I}_v^{\rho}(\varphi_1)-\boldsymbol{I}_v^{\rho}(\varphi_0)|\leqslant F_R(d_1(\varphi_0,\varphi_1))
$$

for any $\varphi_0, \varphi_1 \in \mathcal{K}_{\mathbb{T}}(X, \omega_0) \cap \{\varphi \mid d_1(0, \varphi) < R\}$. It follows that $\boldsymbol{I}_v^{\rho}$ extends to a d_1–continuous functional on $\mathcal{E}^1_{\mathbb{T}}(X, \omega_0)$ which is bounded on d_1–bounded subsets of $\mathcal{E}^1_{\mathbb{T}}(X, \omega_0)$.

For the last statement, let v and w be two (smooth) positive weight functions and $\varphi \in \mathcal{K}_{\mathbb{T}}(X, \omega_0)$. Taking $\varphi_1 = \varphi$ and $\varphi_0 = 0$ in the computation (54),

$$\boldsymbol{I}_v^{\rho}(\varphi) = \int_X \varphi \sum_{j=0}^{m-1} v_{j,m-1}(\mu_0, \mu_\varphi)\rho \wedge \omega_\varphi^{[j]} \wedge \omega_0^{[m-j-1]} + \int_X \varphi \sum_{j=0}^{m} \langle (dv)_{j,m}(\mu_0, \mu_\varphi), \mu_\rho\rangle \omega_\varphi^{[j]} \wedge \omega_0^{[m-j]}.$$

Let $C > 1$ such that $-C\omega_0 < \rho < C\omega_0$. Using the above formula,

$$\begin{aligned}
|\boldsymbol{I}_v^{\rho}(\varphi) - \boldsymbol{I}_w^{\rho}(\varphi)| &= |\boldsymbol{I}_{v-w}^{\rho}(\varphi)| \\
&\leqslant C\int_X |\varphi| \sum_{j=0}^{m-1} |(v-w)_{j,m-1}(\mu_0, \mu_\varphi)| \omega_\varphi^{[j]} \wedge \omega_0^{[m-j]} \\
&\quad + C\int_X |\varphi| \sum_{j=0}^{m} |\langle (d(v-w))_{j,m}(\mu_0, \mu_\varphi), \mu_\rho\rangle| \omega_\varphi^{[j]} \wedge \omega_0^{[m-j]} \\
&\leqslant C\|v-w\|_{C^1(\Delta)} \int_X \sum_{j=0}^{m} |\varphi| \omega_\varphi^{[j]} \wedge \omega_0^{[m-j]} \\
&\leqslant C\|v-w\|_{C^1(\Delta)} \int_X |\varphi| (2\omega_0 + dd^c\varphi)^{[m]} \\
&\leqslant C\|v-w\|_{C^1(\Delta)} \int_X |\varphi| \omega_\varphi^{[m]}.
\end{aligned}$$

Using approximation by decreasing sequences in $\mathcal{K}_{\mathbb{T}}(X, \omega_0)$, the above estimate holds for $\mathcal{E}^1_{\mathbb{T}}(X, \omega_0)$. □

Following Berman and Nyström [15] and the recent work of Han and Li [45], we now define the extension of weighted Monge–Ampère measures to the space $\mathcal{E}_{\mathbb{T}}(X, \omega_0)$.

Proposition 6.13 *Let $v > 0$ be a smooth weight function. For any $\varphi \in \mathcal{K}_{\mathbb{T}}(X, \omega_0)$, let*

$$\mathrm{MA}_v(\varphi) := v(\mu_\varphi)\omega_\varphi^{[m]}.$$

Then $\mathrm{MA}_v(\varphi)$ extends to a well-defined Radon measure defined for any $\varphi \in \mathcal{E}_{\mathbb{T}}(X, \omega_0)$ such that, for any decreasing sequence $(\varphi_j)_j$ of elements in $\mathcal{K}_{\mathbb{T}}(X, \omega_0)$ converging to φ (which exists by [17]), *we have $\lim_{j\to\infty} \mathrm{MA}_v(\varphi_j) = \mathrm{MA}_v(\varphi)$.*

Proof The result is established in [15; 45] for $\omega_0 \in \alpha = c_1(L)$ a Kähler Hodge class on a projective variety X. The method of Han and Li [45, Proposition 2.2], which uses the semisimple principal fibration construction and polynomial approximations, extends to the case of an arbitrary Kähler class $\alpha = [\omega_0]$. Below we give details of this construction, for the reader's convenience.

Let $\varphi \in \mathcal{E}_{\mathbb{T}}(X, \omega_0)$. Following the proof of [45, Proposition 2.2], we first define $\mathrm{MA}_p(\varphi)$ for a positive polynomial weight of the form $p(\mu) := \prod_{a=1}^{k}(\langle p_a, \mu\rangle + c_a)^{n_a}$, and extend the definition linearly on p for finite sums of such polynomials. We can then use the Bernstein approximation theorem of an arbitrary positive v with polynomials of the above form in order to obtain $\mathrm{MA}_v(\varphi)$.

We start with a semisimple principal $(X, \mathbb{T})$–fibration Y (see Section 5) with corresponding polynomial weight $p(\mu) := \prod_{a=1}^{k}(\langle p_a, \mu\rangle + c_a)^{n_a}$; see (28). As the choice of the base $B = B_1 \times \cdots \times B_k$ does not matter, we can simply take (as in [45]) B to be the product of projective spaces $(B_a, \omega_a) = (\mathbb{P}^{n_a}, \omega_a)$ endowed with Fubini–Study metrics of scalar curvatures $2n_a(n_a+1)$, and P to be the principal $\mathrm{U}(1)^r$–bundle over B, obtained from the tensor products P_i of (the pullbacks to B of) the natural principal $\mathrm{U}(1)$–bundles of degrees p_{ai} over $\mathbb{P}^{n_a}$; see Remark 5.1.

Using [17, Theorem 1], there is a decreasing sequence

$$\varphi_j \in \mathrm{PSH}_{\mathbb{T}}(X, \omega_0) \cap C^{\infty}(X) = \mathcal{K}_{\mathbb{T}}(X, \omega_0)$$

converging towards φ. By Lemma 5.5 we have $\varphi_j \in \mathcal{K}_{\mathbb{T}}(Y, \tilde{\omega}_0)$ and, by (29), for any $\mathbb{T}_X$–invariant continuous function f on X,

$$\int_X f p(\mu_{\varphi_j})(\omega_0 + d_X d_X^c \varphi_j)^{[m]} = \frac{1}{\mathrm{Vol}(B, \omega_B)} \int_Y f(\tilde{\omega}_0 + d_Y d_Y^c \varphi_j)^{[m+n]}.$$

Passing to the limit in both sides of the above equation, we can define $\mathrm{MA}_p^X(\varphi)$ on $\mathbb{T}$–invariant continuous functions f by

$$\int_X f\, \mathrm{MA}_p^X(\varphi) := \lim_{j\to\infty} \frac{1}{\mathrm{Vol}(B, \omega_B)} \int_Y f(\tilde{\omega}_0 + d_Y d_Y^c \varphi_j)^{[m+n]}. \tag{56}$$

Notice that by [43, Theorem 1.9] the limit exists and is well defined on Y (independent of the chosen sequence).

For a continuous function f on X which is not necessarily $\mathbb{T}_X$–invariant, we define

$$\int_X f\, \mathrm{MA}_p^X(\varphi) := \int_X f^{\mathbb{T}}\, \mathrm{MA}_p^X(\varphi),$$

where $f^{\mathbb{T}}$ is the $\mathbb{T}_X$–invariant function given by the average of f over the $\mathbb{T}_X$–action. It follows that $\mathrm{MA}_p^X(\varphi)$ is a well-defined Radon measure by the Riesz representation theorem.

We can extend the above definition by linearity in p on polynomials which are linear combinations with positive coefficients of polynomials of the above special form. Thus, for $\varphi \in \mathrm{PSH}_{\mathbb{T}}(X, \omega_0)$ and for two polynomials p and q on Δ,

$$\left|\int_X f\,\mathrm{MA}_p^X(\varphi) - \int_X f\,\mathrm{MA}_q^X(\varphi)\right| \leqslant \|p-q\|_{C^0(\Delta)} \int_X |f|\,\mathrm{MA}^X(\varphi) \tag{57}$$

for any $f \in C^0(X)$.

For an arbitrary smooth positive function v on Δ we can approximate v in $C^0(\Delta)$ by polynomials p_i as above (eg by using Bernstein's approximation theorem), and thus, for any continuous function f, the limit

$$\lim_{i\to\infty}\lim_{j\to\infty}\int_X f\,\mathrm{MA}_{p_i}^X(\varphi_j)$$

exists independently of the chosen approximation. We then define

$$\int_X f\,\mathrm{MA}_v^X(\varphi) := \lim_{i\to\infty}\lim_{j\to\infty}\int_X f\,\mathrm{MA}_{p_i}^X(\varphi_j).$$

By the Riesz representation theorem, $\mathrm{MA}_v^X(\varphi)$ is a well-defined Radon measure. □

Remark 6.14 For any $\varphi \in \mathcal{E}_{\mathbb{T}}(X, \omega_0)$, the measure $\mathrm{MA}_v(\varphi)$ is absolutely continuous with respect to $\mathrm{MA}(\varphi)$ since v is bounded on Δ. In particular, for any positive weight v,

$$\mathcal{E}^1_{\mathbb{T}}(X, \omega_0) = \left\{\varphi \in \mathcal{E}_{\mathbb{T}}(X, \omega_0) \;\middle|\; \int_X |\varphi|\,\mathrm{MA}_v(\varphi) < \infty\right\}.$$

Lemma 6.15 *Let v be a positive weight function and $\varphi_j, \varphi \in \mathcal{E}^1_{\mathbb{T}}(X, \omega_0)$ such that $d_1(\varphi_j, \varphi) \to 0$. Then $\mathrm{MA}_v(\varphi_j) \to \mathrm{MA}_v(\varphi)$ weakly.*

Proof Let $v(\mu)$ be a polynomial of the form $p(\mu) := \prod_{a=1}^k (\langle p_a, \mu\rangle + c_a)^{n_a}$ for $\varphi_j \in \mathcal{K}_{\mathbb{T}}(X, \omega_0)$, and f any continuous $\mathbb{T}$–invariant function on X. We then have, by the construction in Section 5,

$$\int_X f p(\mu_{\varphi_j})(\omega_0 + d_X d_X^c \varphi_j)^{[m]} = \frac{1}{\mathrm{Vol}(B, \omega_B)} \int_Y f(\tilde{\omega}_0 + d_Y d_Y^c \varphi_j)^{[m+n]}.$$

It follows that, for each $\varphi_j \in \mathcal{E}^1_{\mathbb{T}}(X, \omega_0)$ (using an approximation with a decreasing sequence of smooth potentials [17]),

$$\int_X f\,\mathrm{MA}_p^X(\varphi_j) = \frac{1}{\mathrm{Vol}(B, \omega_B)} \int_Y f\,\mathrm{MA}^Y(\varphi_j).$$

By [27, Theorem 5], $\mathrm{MA}^Y(\varphi_j) \to \mathrm{MA}^Y(\varphi)$ weakly as $j \to \infty$. It follows that

$$\lim_{j\to\infty} \int_X f\,\mathrm{MA}_p^X(\varphi_j) = \frac{1}{\mathrm{Vol}(B,\omega_B)} \int_Y f\,\mathrm{MA}^Y(\varphi) = \int_X f\,\mathrm{MA}_p^X(\varphi).$$

Using (56), we conclude that $\mathrm{MA}_p^X(\varphi_j) \to \mathrm{MA}_p^X(\varphi)$ weakly as $j \to \infty$.

For an arbitrary weight function $v \in C^0(\Delta)$, we take a sequence of polynomials p_i of the above form converging to v in $C^0(\Delta)$. For any continuous function f on X, using (57),

$$\begin{aligned}
&\left|\int_X f\,\mathrm{MA}_v(\varphi_j) - \int_X f\,\mathrm{MA}_v(\varphi)\right| \\
&\quad\leqslant \left|\int_X f\,\mathrm{MA}_v(\varphi_j) - \int_X f\,\mathrm{MA}_{p_i}(\varphi_j)\right| + \left|\int_X f\,\mathrm{MA}_{p_i}(\varphi_j) - \int_X f\,\mathrm{MA}_{p_i}(\varphi)\right| \\
&\qquad\qquad + \left|\int_X f\,\mathrm{MA}_{p_i}(\varphi) - \int_X f\,\mathrm{MA}_v(\varphi)\right| \\
&\quad\leqslant \left|\int_X f\,\mathrm{MA}_{p_i}(\varphi_j) - \int_X f\,\mathrm{MA}_{p_i}(\varphi)\right| \\
&\qquad\qquad + \|p_i - v\|_{C^0(\Delta)}\left(\int_X |f|\,\mathrm{MA}(\varphi_j) + \int_X |f|\,\mathrm{MA}(\varphi)\right).
\end{aligned}$$

Letting $j \to \infty$,

$$\lim_{j\to\infty}\left|\int_X f\,\mathrm{MA}_v(\varphi_j) - \int_X f\,\mathrm{MA}_v(\varphi)\right| \leqslant 2\|p_i - v\|_{C^0(\Delta)} \int_X |f|\,\mathrm{MA}(\varphi),$$

using the existence of the weak limits $\mathrm{MA}_{p_i}(\varphi_j) \to \mathrm{MA}_{p_i}(\varphi)$ and $\mathrm{MA}(\varphi_j) \to \mathrm{MA}(\varphi)$ as $j \to \infty$ (by [27, Theorem 5]). Taking the limit $i \to \infty$ in the above inequality,

$$\lim_{j\to\infty}\left|\int_X f\,\mathrm{MA}_v(\varphi_j) - \int_X f\,\mathrm{MA}_v(\varphi)\right| = 0.$$

It follows that $\mathrm{MA}_v(\varphi_j) \to \mathrm{MA}_v(\varphi)$ weakly as $j \to \infty$. □

For a finite measure χ on X we define the entropy of χ with respect to $\omega^{[m]}$ by

$$\mathrm{Ent}(\omega^{[m]}, \chi) := \int_X \log\left(\frac{\chi}{\omega^{[m]}}\right)\chi.$$

In the following lemma we show that the elements of $\mathcal{E}^1_{\mathbb{T}}(X,\omega_0)$ can be approximated in the d_1 distance by smooth potentials with converging entropy of the corresponding weighted Monge–Ampère measures. This is the weighted analogue of [14, Lemma 3.1].

Lemma 6.16 *If $v > 0$, then $\mathcal{E}^1_{\mathbb{T}}(X,\omega_0) \ni \varphi \mapsto \mathrm{Ent}(\omega_0^{[m]}, \mathrm{MA}_v(\varphi))$ is d_1–lsc. Further, for any $\varphi \in \mathcal{E}^1_{\mathbb{T}}(X,\omega_0)$, there exists a sequence of smooth potentials $\varphi_j \in \mathcal{K}_{\mathbb{T}}(X,\omega_0)$ such that $d_1(\varphi_j,\varphi) \to 0$ and $\mathrm{Ent}(\omega_0^{[m]}, \mathrm{MA}_v(\varphi_j)) \to \mathrm{Ent}(\omega_0^{[m]}, \mathrm{MA}_v(\varphi))$ as $j \to \infty$.*

Proof The proof follows closely the arguments of [14, Lemma 3.1]. By Lemma 6.15 and the fact that the entropy $\chi \mapsto \mathrm{Ent}(\omega_0^m, \chi)$ is lsc on the space of finite measures, with respect to the weak convergence of measures (see [11, Proposition 3.1]), it follows that the entropy $\varphi \mapsto \mathrm{Ent}(\omega_0^{[m]}, \mathrm{MA}_v(\varphi))$ is d_1–lsc. Let $\varphi \in \mathcal{E}^1_{\mathbb{T}}(X,\omega_0)$. If $\mathrm{Ent}(\omega_0^{[m]}, \mathrm{MA}_v(\varphi)) = \infty$ then any sequence $\varphi_j \in \mathcal{K}_{\mathbb{T}}(X,\omega_0)$ such that $d_1(\varphi_j,\varphi) \to 0$ satisfies $\mathrm{Ent}(\omega_0^{[m]}, \mathrm{MA}_v(\varphi)) \to \infty$ as $j \to \infty$. We suppose $\mathrm{Ent}(\omega_0^{[m]}, \mathrm{MA}_v(\varphi)) < \infty$ and we put $g := \mathrm{MA}_v(\varphi)/\omega_0^{[m]} \geqslant 0$, the density function of the measure $\mathrm{MA}_v(\varphi)$. From the proof of [14, Lemma 3.1], there exist a sequence of positive functions $g_j \in C^\infty_{\mathbb{T}}(X)$ such that $\|g - g_j\|_{L^1} \to 0$ and

$$\int_X g_j \log g_j \,\omega_0^{[m]} \to \mathrm{Ent}(\omega_0^{[m]}, \mathrm{MA}_v(\varphi)).$$

Using [45, Proposition 3.7], we can find a smooth potential $\varphi_j \in \mathcal{K}_{\mathbb{T}}(X,\omega_0)$ (which is unique up to adding a constant) such that $\mathrm{MA}_v(\varphi_j) = \bigl(\int_X v(\mu_0)\omega_0^m / \int_X g_j \omega_0^m\bigr) g_j \omega_0^{[m]}$. By [45, Lemma 2.16], up to passing to a subsequence of φ_j, there exists a $\psi \in \mathcal{E}^1_{\mathbb{T}}(X,\omega_0)$ such that $d_1(\psi,\varphi_j) \to 0$. Lemma 6.15 together with $\|g - g_j\|_{L^1} \to 0$ gives

$$\mathrm{MA}_v(\psi) = \lim_{j\to\infty} \mathrm{MA}_v(\varphi_j) = \mathrm{MA}_v(\varphi).$$

It follows that $\varphi = \psi$ (up to a constant) by [15, Theorem 2.18]. Thus, $d_1(\varphi,\varphi_j) \to 0$ and $\mathrm{Ent}(\omega_0^{[m]}, \mathrm{MA}_v(\varphi_j)) \to \mathrm{Ent}(\omega_0^{[m]}, \mathrm{MA}_v(\varphi))$ as $j \to \infty$. □

Now we are in position to prove Theorem 6.1.

Proof of Theorem 6.1 By Lemmas 6.10 and 6.12, the functionals $\boldsymbol{I}_w$ and $\boldsymbol{I}_v^{\rho\omega_0}$ extend as continuous functionals on $\mathcal{E}^1_{\mathbb{T}}(X,\omega_0)$. On the other hand, the entropy $\varphi \mapsto \mathrm{Ent}(\omega_0^m, \mathrm{MA}_v(\varphi))$ is d_1–lsc by Lemma 6.16. Thus, the weighted Chen–Tian decomposition (50) gives rise to an extension of the (v,w)–Mabuchi energy to a d_1–lsc functional $\boldsymbol{M}_{v,w}\colon \mathcal{E}^1_{\mathbb{T}}(X,\omega_0) \to \mathbb{R} \cup \{\infty\}$. Notice that (using the continuity of $\boldsymbol{I}_w$ and $\boldsymbol{I}_v^{\rho\omega_0}$) the restriction of $\boldsymbol{M}_{v,w}\colon \mathcal{E}^1_{\mathbb{T}}(X,\omega_0) \to \mathbb{R}\cup\{\infty\}$ on the subspace $\mathcal{K}^{1,\bar{1}}_{\mathbb{T}}(X,\omega_0)$ is equal to the weighted (v,w)–Mabuchi energy on that space defined in [56, Corollary 3]. By Lemma 6.16, for $\varphi \in \mathcal{E}^1_{\mathbb{T}}(X,\omega_0)$, we can find a sequence $\varphi_j \in \mathcal{K}_{\mathbb{T}}(X,\omega_0)$ such that $d_1(\varphi_j,\varphi) \to 0$ and

$$\lim_{j\to\infty} \boldsymbol{M}_{v,w}(\varphi_j) = \boldsymbol{M}_{v,w}(\varphi).$$

It follows that the extension $\boldsymbol{M}_{v,w}\colon \mathcal{E}^1_{\mathbb{T}}(X,\omega_0)\to\mathbb{R}\cup\{\infty\}$ using (50) is the largest d_1–lsc extension of $\boldsymbol{M}_{v,w}\colon \mathcal{K}_{\mathbb{T}}(X,\omega_0)\to\mathbb{R}$.

We now show that $t\mapsto \boldsymbol{M}_{v,w}(\varphi_t)$ for $t\in[0,1]$ is convex and continuous along the finite-energy geodesics $\varphi_t\in\mathcal{E}_{\mathbb{T}}(X,\omega_0)$. We will follow closely the arguments of [14, Theorem 4.7]. Let $\varphi_t\in\mathcal{E}^1_{\mathbb{T}}(X,\omega_0)$ for $t\in[0,1]$ be a finite-energy geodesic. Suppose that $t_0,t_1\in[0,1]$ with $t_0\leqslant t_1$. Using Lemma 6.16, we can find sequences $\varphi^j_{t_0},\varphi^j_{t_1}\in\mathcal{K}_{\mathbb{T}}(X,\omega_0)$ such that $d_1(\varphi^j_{t_0},\varphi_{t_0})\to 0$ and $d_1(\varphi^j_{t_1},\varphi_{t_1})\to 0$, and

$$\lim_{j\to\infty}\boldsymbol{M}_{v,w}(\varphi^j_{t_0})=\boldsymbol{M}_{v,w}(\varphi_{t_0})\quad\text{and}\quad \lim_{j\to\infty}\boldsymbol{M}_{v,w}(\varphi^j_{t_1})=\boldsymbol{M}_{v,w}(\varphi_{t_1}).$$

Let $t\mapsto\varphi^j_t\in\mathcal{K}^{1,\bar{1}}_{\mathbb{T}}(X,\omega_0)$ for $t\in[t_0,t_1]$ be the $C^{1,\bar{1}}$–weak geodesic segment connecting $\varphi^j_{t_0}$ and $\varphi^j_{t_1}$. By [56, Theorem 5], the function $[t_0,t_1]\ni t\mapsto\boldsymbol{M}_{v,w}(\varphi^j_t)$ is convex. Since $\boldsymbol{M}_{v,w}\colon\mathcal{E}^1_{\mathbb{T}}(X,\omega_0)\to\mathbb{R}\cup\{\infty\}$ is d_1 lsc,

$$\begin{aligned}\boldsymbol{M}_{v,w}(\varphi_t)&\leqslant\liminf_{j\to\infty}\boldsymbol{M}_{v,w}(\varphi^j_t)\\&\leqslant\left(\frac{t-t_0}{t_1-t_0}\right)\lim_{j\to\infty}\boldsymbol{M}_{v,w}(\varphi^j_{t_0})+\left(\frac{t_1-t}{t_1-t_0}\right)\lim_{j\to\infty}\boldsymbol{M}_{v,w}(\varphi^j_{t_1})\\&\leqslant\left(\frac{t-t_0}{t_1-t_0}\right)\boldsymbol{M}_{v,w}(\varphi_{t_0})+\left(\frac{t_1-t}{t_1-t_0}\right)\lim_{j\to\infty}\boldsymbol{M}_{v,w}(\varphi_{t_1}),\end{aligned}$$

where the second inequality uses the convexity of $t\mapsto\boldsymbol{M}_{v,w}(\varphi^j_t)$. Thus, $t\mapsto\boldsymbol{M}_{v,w}(\varphi_t)$ is convex and continuous up to the boundary of $[t_0,t_1]$ since it is d_1–lsc.

It remains to show that $\boldsymbol{M}_{v,w}\colon\mathcal{E}^1_{\mathbb{T}}(X,\omega_0)\to\mathbb{R}\cup\{\infty\}$ is linear and continuous in v and w. For smooth potentials $\varphi\in\mathcal{K}_{\mathbb{T}}(X,\omega_0)$,

$$\operatorname{Ent}(\omega_0^{[m]},\mathrm{MA}_v(\varphi))-\int_X\log(v(\mu_0))v(\mu_0)\omega_0^{[m]}=\int_X\log\left(\frac{\mathrm{MA}(\varphi)}{\omega_0^m}\right)\mathrm{MA}_v(\varphi),\tag{58}$$

which is manifestly linear in v. For $\varphi\in\mathcal{E}^1_{\mathbb{T}}(X,\omega_0)$ the above expression is still linear in v by Proposition 6.13. Substituting back in (50), and using Lemmas 6.10 and 6.12, it follows that $\boldsymbol{M}_{v,w}\colon\mathcal{E}^1_{\mathbb{T}}(X,\omega_0)\to\mathbb{R}\cup\{\infty\}$ is linear in v and w. From these two lemmas we know that $\boldsymbol{I}^\rho_v\colon\mathcal{E}^1_{\mathbb{T}}(X,\omega_0)\to\mathbb{R}$ and $\boldsymbol{I}_w\colon\mathcal{E}^1_{\mathbb{T}}(X,\omega_0)\to\mathbb{R}$ are uniformly continuous in v and w. For the remaining entropy part, we notice that, if $\varphi\in\mathcal{E}^1_{\mathbb{T}}(X,\omega_0)$, $v,v'\in C^\infty(\Delta)$ and $f\in C^0(X)$, then

$$\left|\int_X f\,\mathrm{MA}_v(\varphi)-\int_X f\,\mathrm{MA}_{v'}(\varphi)\right|\leqslant\|v-v'\|_{C^0(\Delta)}\int_X|f|\,\mathrm{MA}(\varphi),$$

which can be obtained again by approximating φ with a monotone sequence of smooth relative potentials and Proposition 6.13. So $C^\infty(\Delta)\times\mathcal{E}^1_{\mathbb{T}}(X,\omega_0)\ni(v,\varphi)\mapsto\mathrm{MA}_v(\varphi)$ is uniformly continuous with respect to v for the weak topology on the space of measures. Since the entropy $\chi \mapsto \mathrm{Ent}(\omega_0^m, \chi)$ is lsc on the space of finite measures with respect to the weak convergence of measures [11, Proposition 3.1], the term $\mathrm{Ent}(\omega_0^{[m]}, \mathrm{MA}_v(\varphi))$ is lsc with respect to v. The linearity with respect to v in the right side of (58) shows that $\mathrm{Ent}(\omega_0^{[m]}, \mathrm{MA}_v(\varphi))$ is in fact continuous with respect to v. □

We derive the following weighted version of the key compactness result from [12; 13]:

Theorem 6.17 *Any sequence $\varphi_j \in \mathcal{E}^1_{\mathbb{T}}(X, \omega_0)$ such that*

$$d_1(0, \varphi_j) \leqslant C \quad \text{and} \quad \boldsymbol{M}_{v,w}(\varphi_j) \leqslant C$$

admits a d_1–convergent subsequence.

Proof From (50) and Lemmas 6.9 and 6.12, $\mathrm{Ent}(\omega_0^{[m]}, \mathrm{MA}_v(\varphi))$ is uniformly bounded under the hypotheses. We conclude using [45, Lemma 2.16]. □

7 Regularity of the weak minimizers of the weighted Mabuchi energy

In this section, we establish the regularity of the weak minimizers of $\boldsymbol{M}_{v,w}$.

Theorem 7.1 *Suppose $\mathbb{T} \subset \mathrm{Aut}_r(X)$ is a maximal torus and $(X, \alpha, \mathbb{T})$ admits a (v, w)–cscK metric ω with $w = \ell^{\mathrm{ext}}_{v,w_0} w_0$, where $v, w_0 > 0$ are two positive smooth weight functions on Δ. If $\psi \in \mathcal{E}^1_{\mathbb{T}}(X, \omega_0)$ is a minimizer of the extended (v, w)–Mabuchi energy $\boldsymbol{M}_{v,w} \colon \mathcal{E}^1_{\mathbb{T}}(X, \omega_0) \to \mathbb{R} \cup \{\infty\}$, then $\psi \in \mathcal{K}_{\mathbb{T}}(X, \omega_0)$ is a smooth potential.*

The proof of this result, which is an adaptation of the arguments in [14], will occupy the reminder of the section.

Definition 7.2 Let $v(\mu) > 0$ and $w(\mu)$ be smooth weight functions on Δ and $\rho > 0$ a $\mathbb{T}$–invariant Kähler form on X. We let

$$\mathcal{M}_{v,w} := \bigl\{\psi \in \mathcal{E}^1_{\mathbb{T}}(X, \omega_0) \cap \boldsymbol{I}^{-1}(0) \mid \boldsymbol{M}_{v,w}(\psi) = \inf_{\varphi \in \mathcal{E}^1_{\mathbb{T}}} \boldsymbol{M}_{v,w}(\varphi)\bigr\} \tag{59}$$

and $\boldsymbol{M}^{\rho}_{v,w} := \boldsymbol{M}_{v,w} + \boldsymbol{I}^{\rho}$, where $\boldsymbol{I}^{\rho}$ is introduced via Lemma 6.12 and $v = 1$.

By [29, Lemma 5.2] and Theorem 6.1, the set $\mathcal{M}_{v,w}$ (when nonempty) is totally geodesic with respect to the finite-energy geodesics of $\mathcal{E}^1_{\mathbb{T}}(X, \omega_0)$. Furthermore, if there exists a $\psi_\rho \in \mathcal{M}_{v,w}$ such that $\boldsymbol{I}^{\rho}(\psi_\rho) = \inf_{\psi \in \mathcal{M}_{v,w}} \boldsymbol{I}^{\rho}(\psi)$, then ψ_ρ is unique by the strict convexity of $\boldsymbol{I}^\rho$ established in [14, Proposition 4.5]. Furthermore, by Theorem 6.1, the functional $\boldsymbol{M}^\rho_{v,w}: \mathcal{E}^1_{\mathbb{T}}(X,\omega_0) \to \mathbb{R}\cup\{\infty\}$ will also be strictly convex along finite-energy geodesics, showing the uniqueness of an element $\psi \in \mathcal{E}^1_{\mathbb{T}}(X,\omega_0) \cap \boldsymbol{I}^{-1}(0)$ such that $\boldsymbol{M}^\rho_{v,w}(\psi) = \inf_{\varphi\in\mathcal{E}^1_{\mathbb{T}}} \boldsymbol{M}^\rho_{v,w}(\varphi)$ (assuming that such minimizer ψ exists).

We then have a weighted version of the continuity method of [14, Proposition 3.1]:

Proposition 7.3 *Let $v>0$ and w be smooth weight functions on Δ. Suppose that $\mathcal{M}_{v,w}$ is nonempty and $\varphi \in \mathcal{K}_{\mathbb{T}}(X,\omega_0)\cap \boldsymbol{I}^{-1}(0)$. Then, for any $\lambda>0$, there exists a unique minimizer $\psi_\lambda \in \mathcal{E}^1_{\mathbb{T}}(X,\omega_0)\cap \boldsymbol{I}^{-1}(0)$ of $\boldsymbol{M}^{\lambda\omega_\varphi}_{v,w} := \boldsymbol{M}_{v,w} + \boldsymbol{I}^{\lambda\omega_\varphi}$. The curve $[0,\infty)\ni\lambda\mapsto\psi_\lambda \in \mathcal{E}^1_{\mathbb{T}}(X,\omega_0)\cap \boldsymbol{I}^{-1}(0)$ is d_1–continuous and d_1–bounded, and $\psi_0 := \lim_{\lambda\to 0}\psi_\lambda$ is the unique minimizer of $\boldsymbol{I}^{\omega_\varphi}$ on $\mathcal{M}_{v,w}$. Furthermore, for any $\psi\in\mathcal{M}_{v,w}$ and $\lambda>0$,*

$$I(\varphi,\psi_\lambda) \leqslant m(m+1) I(\varphi,\psi), \tag{60}$$

where $I(\varphi,\psi) := \int_X (\varphi-\psi)(\omega^m_\psi - \omega^m_\varphi)$.

Proof The proof is a straightforward adaptation of that of [14, Proposition 3.1]. □

We next need a weighted analogue of [14, Lemma 3.3]:

Lemma 7.4 *Let $v>0$ and w be smooth weight functions on Δ, and $\rho>0$ a smooth $\mathbb{T}$–invariant Kähler form on X. Let $\varphi_0\in\mathcal{K}_{\mathbb{T}}(X,\omega_0)$ and $\varphi_1\in\mathcal{E}^1_{\mathbb{T}}(X,\omega_0)$, and $[0,1]\ni t\mapsto \varphi_t\in\mathcal{E}^1_{\mathbb{T}}(X,\omega_0)$ be a finite-energy geodesic connecting φ_0 and φ_1. Then*

$$\lim_{t\to 0^+}\frac{\boldsymbol{M}^\rho_{v,w}(\varphi_t)-\boldsymbol{M}^\rho_{v,w}(\varphi_0)}{t} \geqslant \int_X (w(\mu_{\varphi_0}) - \mathrm{Scal}_v(\varphi_0))\dot{\varphi}_0\omega^{[m]}_{\varphi_0} + \int_X \dot{\varphi}_0\rho\wedge\omega^{[m-1]}_{\varphi_0},$$

where $\boldsymbol{M}^\rho_{v,w} := \boldsymbol{M}_{v,w} + \boldsymbol{I}^\rho$.

Proof By Theorem 6.1 and the fact that $\boldsymbol{I}^\rho$ is d_1–continuous (see [14] or Lemma 6.12), for any $t\in[0,1]$ there exists a sequence $(\varphi^k_t)_k \in \mathcal{K}_{\mathbb{T}}(X,\omega_0)$ such that

$$\lim_{k\to\infty} d_1(\varphi^k_t,\varphi_t) = 0 \quad\text{and}\quad \boldsymbol{M}^\rho_{v,w}(\varphi^k_t)\to \boldsymbol{M}^\rho_{v,w}(\varphi_t).$$

We let $[0,t]\ni s\mapsto\psi^k_s$ be the weak $C^{1,\bar{1}}$–geodesic joining $\varphi^k_0=\varphi_0$ with φ^k_t. By the proof of [56, Corollary 1],

$$\lim_{t\to 0^+}\frac{\boldsymbol{M}^\rho_{v,w}(\varphi^k_t)-\boldsymbol{M}^\rho_{v,w}(\varphi_0)}{t} \geqslant \int_X (w(\mu_{\varphi_0}) - \mathrm{Scal}_v(\varphi_0))\dot{\psi}^k_0\omega^{[m]}_{\varphi_0} + \int_X \dot{\psi}^k_0\rho\wedge\omega^{[m-1]}_{\varphi_0}.$$

According to [14, Lemma 3.4], we can use the dominated convergence theorem on the right side of the above inequality to conclude. □

The last step is to establish a weighted version of [14, Proposition 3.2.]:

Proposition 7.5 *Suppose $\mathbb{T} \subset \mathrm{Aut}_r(X)$ is a maximal torus, and let $v(\mu), w_0(\mu) > 0$ and $w = \ell^{\mathrm{ext}}_{v,w_0} w_0$. Suppose that $\varphi^* \in \mathcal{K}_{\mathbb{T}}(X, \omega_0) \cap \boldsymbol{I}^{-1}(0)$ is a (v,w)–cscK potential. Then, for any fixed Kähler form ω_φ with $\varphi \in \mathcal{K}_{\mathbb{T}}(X, \omega_0)$, there exists a $\sigma \in \mathbb{G} := \mathbb{T}^{\mathbb{C}}$ such that*

$$\inf_{\psi \in \mathcal{M}_{v,w}} \boldsymbol{I}^{\omega_\varphi}(\psi) = \boldsymbol{I}^{\omega_\varphi}(\sigma[\varphi^*]).$$

Proof As $\mathbb{G}$ is reductive, there exists a unique $\sigma \in \mathbb{G}$ such that

$$\boldsymbol{I}^{\omega_\varphi}(\sigma[\varphi^*]) = \inf_{\tau \in \mathbb{G}} \boldsymbol{I}^{\omega_\varphi}(\tau[\varphi^*]) \tag{61}$$

(see eg [29, Section 6] or [56, Lemma 11]), where, we recall, the $\mathbb{G}$ action on potentials is introduced via the slice $\boldsymbol{I}^{-1}(0)$. Let $\varphi_0 := \sigma[\varphi^*] \in \mathcal{K}_{\mathbb{T}}(X, \omega_0) \cap \boldsymbol{I}^{-1}(0)$, and $\psi_0 \in \mathcal{M}_{v,w}$ be the unique minimizer of $\boldsymbol{I}^{\omega_\varphi}$. We want to show that $\varphi_0 = \psi_0$.

For $\lambda > 0$ let the unique minimizer of $\boldsymbol{M}^{\lambda\omega_\varphi}_{v,w} = \boldsymbol{M}_{v,w} + \lambda \boldsymbol{I}^{\omega_\varphi}$ on $\mathcal{E}^1_{\mathbb{T}}(X, \omega_0) \cap \boldsymbol{I}^{-1}(0)$ be ψ_λ, as given by Proposition 7.3. By this proposition, $\lim_{\lambda \to 0} d_1(\psi_\lambda, \psi_0) = 0$. We denote by $\boldsymbol{V}_\lambda$ and $\boldsymbol{W}$ the differentials of $\boldsymbol{M}^{\lambda\omega_\varphi}_{v,w}$ and $\boldsymbol{I}^{\omega_\varphi}$, respectively, viewed as 1–forms on the Fréchet space $\mathcal{K}(X, \omega_0)$. We thus have, for all $\psi \in \mathcal{K}_{\mathbb{T}}(X, \omega_0)$ and for all $\dot\psi \in C^\infty_{\mathbb{T}}(X)$,

$$\begin{aligned} (\boldsymbol{V}_0)_\psi(\dot\psi) &= -\int_X (\mathrm{Scal}_v(\omega_\psi) - w(\mu_\psi)) \dot\psi \omega_\psi^{[m]}, \\ \boldsymbol{W}_\psi(\dot\psi) &= \int_X \dot\psi \omega_\varphi \wedge \omega_\psi^{[m-1]}, \\ (\boldsymbol{V}_\lambda)_\psi(\dot\psi) &= (\boldsymbol{V}_0)_\psi(\dot\psi) + \lambda \boldsymbol{W}_\psi(\dot\psi). \end{aligned} \tag{62}$$

Recall that the Mabuchi connection $\mathcal{D}$ on the Fréchet space $\mathcal{K}_{\mathbb{T}}(X, \omega_0)$ is introduced by

$$(\mathcal{D}_{\dot\varphi_t} \dot\psi_t)_{\varphi_t} := \ddot\psi_t - \langle d\dot\psi_t, d\dot\varphi_t \rangle_{\omega_{\varphi_t}},$$

where φ_t and ψ_t are smooth paths in $\mathcal{K}_{\mathbb{T}}(X, \omega_0)$. Using [55, Lemma B.1], we compute the covariant derivative of $\boldsymbol{V}_0$ with respect to the Mabuchi connection to be

$$\begin{aligned} ((\mathcal{D}_{\dot\psi_2} \boldsymbol{V}_0)(\dot\psi_1))_\psi = \int_X \big[& 2v(\mu_\psi)((\nabla^{\omega_\psi} d\dot\psi_1)^-, (\nabla^{\omega_\psi} d\dot\psi_2)^-)_{\omega_\psi} \\ & + (\mathrm{Scal}_v(\omega_\psi) - w(\mu_\psi))(d\dot\psi_1, d\dot\psi_2)_{\omega_\psi} \big] \omega_\psi^{[m]}, \end{aligned}$$

where $(\nabla^{\omega_\psi} d\dot\psi)^-$ denotes the $(2,0) + (0,2)$ part of the Hessian of $\dot\psi$ with respect to the Levi-Civita connection ∇^{ω_ψ} of ω_ψ. Taking $\psi = \varphi_0$ to be the (v,w)–cscK potential,

$$\begin{aligned} ((\mathcal{D}_{\dot\psi_2} \boldsymbol{V}_0)(\dot\psi_1))_{\varphi_0} &= 2 \int_X ((\nabla^{\omega_{\varphi_0}} d\dot\psi_1)^-, (\nabla^{\omega_{\varphi_0}} d\dot\psi_2)^-)_{\omega_{\varphi_0}} v(\mu_{\varphi_0}) \omega_{\varphi_0}^{[m]} \\ &= 2 \int_X \mathbb{L}_{\omega_{\varphi_0}, v}(\dot\psi_1) \dot\psi_2 \omega_{\varphi_0}^{[m]} = 2 \int_X \mathbb{L}_{\omega_{\varphi_0}, v}(\dot\psi_2) \dot\psi_1 \omega_{\varphi_0}^{[m]}, \end{aligned}$$

where the operator $\mathbb{L}_{\omega_\psi,v}(\dot\psi) := \delta_{\omega_\psi}\delta_{\omega_\psi}(v(\mu_\psi)(\nabla^{\omega_\psi} d\dot\psi)^-)$ is a fourth-order elliptic self-adjoint operator on (X,ω_ψ), with kernel given by the space of Killing potentials in $C^\infty_{\mathbb{T}}(X)$; see Appendix A.

As φ_0 is a (v,w)–cscK potential which satisfies (61), we have by [56, Lemma 10] that $\boldsymbol{W}_{\varphi_0}(\dot\psi)=0$ for any $\mathbb{T}$–invariant Killing potential $\dot\psi$ with respect to ω_{φ_0}. It follows that we can solve the linear equation (for a function $\dot\psi \in C^\infty_{\mathbb{T}}(X)$)

$$\mathbb{L}_{\omega_{\varphi_0},v}(\dot\psi) = \frac{\omega_\varphi \wedge \omega_{\varphi_0}^{[m-1]}}{\omega_{\varphi_0}^{[m]}},$$

as the right side is L^2–orthogonal (with respect to the measure $\omega_{\varphi_0}^{[m]}$) to the kernel of $\mathbb{L}_{\omega_{\varphi_0},v}$. Equivalently, there exists a $\dot\psi_0 \in C^\infty_{\mathbb{T}}(X)$ such that we have equality of 1–forms on $\mathcal{K}_{\mathbb{T}}(X,\omega_0)$:

$$(\mathcal{D}_{\dot\psi_0}\boldsymbol{V}_0)_{\varphi_0} = -\boldsymbol{W}_{\varphi_0}. \tag{63}$$

Let $\lambda \to \dot\phi_\lambda \in C^\infty_{\mathbb{T}}(X)$ be a smooth curve in the tangent space to $(\varphi_0+\lambda\dot\psi_0) \in \mathcal{K}_{\mathbb{T}}(X,\omega_0)$, defined for $\lambda>0$ small enough. We compute

$$\begin{aligned}\frac{d}{d\lambda}\Big|_{\lambda=0}(\boldsymbol{V}_\lambda)_{\varphi_0+\lambda\dot\psi_0}(\dot\phi_\lambda) &= \boldsymbol{W}_{\varphi_0}(\dot\phi_0)+((\mathcal{D}_{\dot\psi_0}\boldsymbol{V}_0)(\dot\phi_0))_{\varphi_0}+(\boldsymbol{V}_0)_{\varphi_0}\Big(\frac{d}{d\lambda}\Big|_{\lambda=0}\dot\phi_\lambda\Big)\\ &=0,\end{aligned} \tag{64}$$

where we have used (63) and that $(\boldsymbol{V}_0)_{\varphi_0}=0$ since φ_0 is a (v,w)–cscK potential; see (62). On the other hand, letting

$$f_\lambda := -\mathrm{Scal}_v(\omega_{\varphi_0+\lambda\dot\psi_0}) + w(\mu_{\varphi_0+\lambda\dot\psi_0}) + \langle \omega_{\varphi_0+\lambda\dot\psi_0}, \omega_\varphi\rangle_{\omega_\varphi},$$

it follows from (62) that, for any $\dot\phi \in C^\infty_{\mathbb{T}}(X)$,

$$(\boldsymbol{V}_\lambda)_{\varphi_0+\lambda\dot\psi_0}(\dot\phi) = \int_X \dot\phi\, f_\lambda\, \omega^{[m]}_{\varphi_0+\lambda\dot\psi_0}.$$

Thus (64) implies that $f_\lambda = O(\lambda^2)$ and

$$|(\boldsymbol{V}_\lambda)_{\varphi_0+\lambda\dot\psi_0}(\dot\phi)| \leqslant C\lambda^2 \sup_X |\dot\phi|.$$

Let $\psi_\lambda(t) \in \mathcal{E}^1_{\mathbb{T}}(X,\omega_0)$ be a finite-energy geodesic connecting $\psi_\lambda(0) := \psi_\lambda \in \mathcal{E}^1_{\mathbb{T}}(X,\omega_0)$ with $\psi_\lambda(1) := \varphi_0+\lambda\dot\psi_0 \in \mathcal{K}_{\mathbb{T}}(X,\omega_0)$ for $\lambda>0$ small enough. By Lemma 7.4,

$$\frac{d}{dt}\Big|_{t=1^-}\boldsymbol{M}^{\lambda\omega_\varphi}_{v,w}(\psi_\lambda(t)) \leqslant \int_X \dot\psi_\lambda(1)\, f_\lambda\, \omega^{[m]}_{\varphi_0+\lambda\dot\psi_0}.$$

By Proposition 7.3, $d_1(0,\psi_\lambda(0))$ is uniformly bounded. Also, $d_1(0,\psi_\lambda(1))$ is uniformly bounded for λ small enough since $\psi_\lambda(1) := \varphi_0+\lambda\dot\psi_0 \in \mathcal{K}_{\mathbb{T}}(X,\omega_0)$. We thus have that both $d_1(0,\psi_\lambda(0))$ and $d_1(0,\psi_\lambda(1))$ are uniformly bounded and, by [14, Lemma 3.4(ii)], we get

$$\int_X |\dot\psi_\lambda(1)|\omega^{[m]}_{\varphi_0+\lambda\dot\psi_0} = d_1(\psi_\lambda(0),\psi_\lambda(0)) \leqslant d_1(0,\psi_\lambda(0)) + d_1(0,\psi_\lambda(1)) \leqslant C.$$

From $f_\lambda = O(\lambda^2)$, we obtain

$$\frac{d}{dt}\Big|_{t=1^-} \boldsymbol{M}^{\lambda\omega_\varphi}_{v,w}(\psi_\lambda(t)) \leqslant O(\lambda^2).$$

As the unique minimizer of the strictly convex functional $\boldsymbol{M}^{\lambda\omega_\varphi}_{v,w}$ on $\mathcal{E}^1_{\mathbb{T}}(X,\omega_0)\cap \boldsymbol{I}^{-1}(0)$ is $\psi_\lambda(0)=\psi_\lambda$,

$$\frac{d}{dt}\Big|_{t=1^-} \boldsymbol{M}^{\lambda\omega_\varphi}_{v,w}(\psi_\lambda(t)) \geqslant \frac{d}{dt}\Big|_{t=0^+} \boldsymbol{M}^{\lambda\omega_\varphi}_{v,w}(\psi_\lambda(t)) \geqslant 0.$$

Using that the functions $t\mapsto \boldsymbol{I}^{\omega_\varphi}(\psi_\lambda(t))$ and $t\mapsto \boldsymbol{M}_{v,w}(\psi_\lambda(t))$ are both convex (this follows from [14, Proposition 4.5] and Theorem 6.1),

$$\begin{aligned} 0 \leqslant \lambda\Big(\frac{d}{dt}\Big|_{t=1^-} - \frac{d}{dt}\Big|_{t=0^+}\Big)\boldsymbol{I}^{\omega_\varphi}(\psi_\lambda(t)) &\leqslant \Big(\frac{d}{dt}\Big|_{t=1^-} - \frac{d}{dt}\Big|_{t=0^+}\Big)\boldsymbol{M}^{\lambda\omega_\varphi}_{v,w}(\psi_\lambda(t)) \\ &\leqslant O(\lambda^2). \end{aligned}$$

By the convexity of $t\mapsto \boldsymbol{I}^{\omega_\varphi}(\psi_\lambda(t))$, the last estimate also gives

$$\begin{aligned} 0 &\leqslant t\boldsymbol{I}^{\omega_\varphi}(\psi_\lambda(1)) + (1-t)\boldsymbol{I}^{\omega_\varphi}(\psi_\lambda(0)) - \boldsymbol{I}^{\omega_\varphi}(\psi_\lambda(t)) \\ &= t(1-t)\bigg(\frac{\boldsymbol{I}^{\omega_\varphi}(\psi_\lambda(1)) - \boldsymbol{I}^{\omega_\varphi}(\psi_\lambda(t))}{1-t}\bigg) - t(1-t)\bigg(\frac{-\boldsymbol{I}^{\omega_\varphi}(\psi_\lambda(0)) + \boldsymbol{I}^{\omega_\varphi}(\psi_\lambda(t))}{t}\bigg) \\ &\leqslant t(1-t)O(\lambda). \end{aligned}$$

Letting $\lambda\to 0$ and using the endpoint stability of the finite-energy geodesic segments (see [14, Proposition 4.3]) together with the d_1–continuity of $\boldsymbol{I}^{\omega_\varphi}$ [14, Proposition 4.4], $t\mapsto \boldsymbol{I}^{\omega_\varphi}(\psi(t))$ is linear along the finite-energy geodesic $\psi(t)=\lim_{\lambda\to 0^+}\psi_\lambda(t)$ connecting $\psi_0(0)=\psi_0$ and $\psi_0(1)=\varphi_0$. The strict convexity of $\boldsymbol{I}^{\omega_\varphi}$ along finite-energy geodesics [14, Proposition 4.5] then yields $\psi_0=\varphi_0=\sigma[\varphi^*]$. □

Now we are in position to prove Theorem 7.1 by the arguments in [14, Theorem 1.4].

Proof of Theorem 7.1 Without loss of generality, we can assume that the (v,w)–extremal metric $\omega^*=\omega_0$ is the initial metric, and we suppose $\psi_0\in\mathcal{E}^1_{\mathbb{T}}(X,\omega_0)\cap\boldsymbol{I}^{-1}(0)$ is a weak minimizer of $\boldsymbol{M}_{v,w}:\mathcal{E}^1_{\mathbb{T}}(X,\omega_0)\to\mathbb{R}\cup\{\infty\}$. We want to show that $\psi_0=\sigma[0]$ for some $\sigma\in\mathbb{G}=\mathbb{T}^{\mathbb{C}}$. It is well known (see [28] or Corollary 6.11) that there exists a sequence $\varphi_j\in\mathcal{K}_{\mathbb{T}}(X,\omega_0)\cap\boldsymbol{I}^{-1}(0)$ such that $d_1(\varphi_j,\psi_0)\to 0$. We set $\rho_j=\omega_0+dd^c\varphi_j$, which is a $\mathbb{T}$–invariant Kähler form.

Since ω_0 is a (v,w)–extremal metric, $\mathcal{M}_{v,w}$ is nonempty. By Proposition 7.3, the functional $\boldsymbol{M}_{v,w}^{\lambda\rho_j} = \boldsymbol{M}_{v,w} + \lambda \boldsymbol{I}^{\rho_j}$ has a unique minimizer $\psi_{j,\lambda} \in \mathcal{E}^1_{\mathbb{T}}(X,\omega_0) \cap \boldsymbol{I}^{-1}(0)$ such that

$$I(\varphi_j, \psi_{j,\lambda}) \leqslant m(m+1) I(\varphi_j, \psi_0).$$

By the quasitriangle identity [14, (2.16)],

$$I(\psi_0, \psi_{j,\lambda}) \leqslant C(I(\psi_0, \varphi_j) + I(\varphi_j, \psi_{j,\lambda})) \leqslant C(m^2+m+1) I(\varphi_j, \psi_0), \tag{65}$$

where $C > 0$ is a uniform constant depending only on m.

Let $j > 0$ be fixed. According to Proposition 7.3, $\psi_{j,0} := \lim_{\lambda\to 0} \psi_{j,\lambda}$ is the unique minimizer of $\boldsymbol{I}^{\lambda\rho_j}$ on $\mathcal{M}_{v,w}$, whereas Proposition 7.5 yields that there exists a $\sigma_j \in \mathbb{G}$ such that $\psi_{j,0} = \sigma_j[0]$. Letting $\lambda \to 0^+$ in (65) (and using the d_1–continuity of I; see eg [13] or Lemma 6.10),

$$I(\psi_0, \sigma_j[0]) \leqslant C(m^2+m+1) I(\varphi_j, \psi_0).$$

When $j \to \infty$ (using $d_1(\varphi_j, \psi_0) \to 0$), we get $I(\psi_0, \sigma_j[0]) \to 0$. By [12, Proposition 2.3; 27, Proposition 5.9], the latter limit is equivalent to $d_1(\sigma_j[0], \psi_0) \to 0$. Using [14, Lemma 3.7], there exists a $\sigma \in \mathbb{G}$ such that $\sigma[0] = \psi_0$. □

Remark 7.6 The arguments in the proofs of Proposition 7.5 and Theorem 7.1 extend if we remove the maximality assumption for $\mathbb{T} \subset \mathrm{Aut}_r(X)$, and replace $\mathbb{G} = \mathbb{T}^{\mathbb{C}}$ with the connected component of the identity $\widehat{\mathbb{G}} = \mathrm{Aut}_r^{\mathbb{T}}(X)$ of the centralizer of $\mathbb{T}$ in $\mathrm{Aut}_r(X)$. The key points are that $\widehat{\mathbb{G}}$ is reductive (see Proposition 1.4) and $\widehat{\mathbb{G}}$ acts transitively on the space of $\mathbb{T}$–invariant (v,w_0)–extremal Kähler metrics (see Theorem 1.5).

Proof of Theorem 1 We apply the coercivity principle of [29]; see Theorem 3.6. By Theorem 6.1, the extension of the weighted Mabuchi energy $\boldsymbol{M}_{v,w}$ to the space $\mathcal{E}^1_{\mathbb{T}}(X,\omega_0)$ satisfies the hypotheses of Theorem 3.6 (the invariance of $\boldsymbol{M}_{v,w}$ under the action of $\mathbb{G} = \mathbb{T}^{\mathbb{C}}$ is equivalent to the necessary condition (3) for the existence of a (v,w)–cscK metric). We thus need to ensure that $\boldsymbol{M}_{v,w}$ further satisfies properties (i)–(iv) of Theorem 3.6. Theorem 6.1 also yields the convexity property (i), whereas the regularity property (ii) is established in Theorem 7.1. This last result also yields the uniqueness property (iii) via Theorem 1.5. Finally, the compactness property (iv) is established in Theorem 6.17. □

Remark 7.7 By virtue of Theorem 1.5 and Remark 7.6, the conclusion of Theorem 1 holds true if one drops the assumption that $\mathbb{T} \subset \mathrm{Aut}_r(X)$ is a maximal torus, but instead of $\mathbb{T}^{\mathbb{C}}$ one considers the larger reductive group $\widehat{\mathbb{G}} = \mathrm{Aut}_r^{\mathbb{T}}(X)$; see Proposition 1.4.

8 Proofs of Theorems 2 and 3

Proof of Theorem 2 The implication (ii) $\Longrightarrow$ (i) follows from Lemma 5.9, whereas (ii) $\Longrightarrow$ (iii) is established in Theorem 1. We shall prove (iii) $\Longrightarrow$ (ii) and (i) $\Longrightarrow$ (ii). The arguments are very similar to the ones in the proof of [52, Theorem 1], where the case when $(X,\mathbb{T})$ is toric is studied. The main idea is to show that, on a semisimple principal $(X,\mathbb{T})$–fibration, the continuity path used by Chen and Cheng [23] in the cscK case and its modification by He [47] to the extremal case can be adapted to bundle-compatible construction. We sketch the proof below for the reader's convenience.

(iii) $\Longrightarrow$ (ii) We shall work on Y. Let $\widetilde{\omega}_0$ be a bundle-compatible Kähler metric on Y, corresponding to a $\mathbb{T}_X$–invariant Kähler metric ω_0 on X. By Lemma 5.14, $\widetilde{\omega}_0$ is invariant under a maximal torus $\mathbb{K}_Y \subset \mathrm{Aut}_r(Y)$ (containing $\mathbb{T}_Y$), and, by this lemma and Lemma 5.10, the extremal affine-linear function corresponding to $\mathbb{K}_Y$ is the pullback to the vector space $\mathfrak{k}_Y^* = (\mathrm{Lie}(\mathbb{K}_Y))^*$ of the extremal affine-linear function $\ell^{\mathrm{ext}}(\mu)$ on $\mathfrak{t}$ defined in Theorem 2(ii). Furthermore, by Lemma 5.10, the restriction of $\boldsymbol{M}^Y_{1,\ell^{\mathrm{ext}}}$ to the subspace $\mathcal{K}_{\mathbb{T}}(X,\omega_0) \subset \mathcal{K}_{\mathbb{K}_Y}(Y,\widetilde{\omega}_0)$ (see Corollary 5.6 and Lemma 5.14) is a positive multiple of $\boldsymbol{M}^X_{p,\widetilde{w}}$, where the weights are those defined in Theorem 2(ii). In this setup, the main ingredients of the proof are as follows.

Step 1 Following [23; 46; 47], one considers the continuity path $\varphi_t \in \mathcal{K}_{\mathbb{K}_Y}(Y,\widetilde{\omega}_0)$ determined by the solution of the PDE

$$t(\mathrm{Scal}(\tilde{\omega}_{\varphi_t}) - \ell^{\mathrm{ext}}(\mu_{\tilde{\omega}_{\varphi_t}})) = (1-t)(\mathrm{tr}_{\tilde{\omega}_{\varphi_t}}(\tilde{\rho}) - (n+m)) \quad \text{for } t \in (0,1), \tag{66}$$

where $\tilde{\rho}$ is a suitable (fixed) $\mathbb{K}_Y$–invariant Kähler metric on Y in the class $[\widetilde{\omega}_0]$. By [23; 47] there exists $\tilde{\rho} \in [\widetilde{\omega}_0]$ and a $t_0 \in (0,1)$ such that a solution φ_t of (66) exists for t in the interval $[t_0,1)$. Furthermore, the solution $\varphi_t(y)$ is smooth as a function on $[t_0,1)\times Y$. The main observation of [52] is that, with a suitable choice for $\tilde{\rho}$, the path (66) can in fact be reduced to a continuity path on X. To see this we observe that, by [47, Proposition 3.1], one can take $\tilde{\rho}$ in (66) to be of the form $\tilde{\rho} = \widetilde{\omega}_0 + (1/r_0)dd^c f$ with r_0 large enough, where f is the smooth function on Y with zero mean with respect to $\widetilde{\omega}_0$ which solves the Laplace equation

$$\Delta_{\tilde{\omega}_0} f = \mathrm{Scal}(\tilde{\omega}_0) - \ell^{\mathrm{ext}}(\mu_{\tilde{\omega}_0}).$$

By Lemmas 5.9 and A.3, $f \in C^\infty_{\mathbb{T}}(X)$, whereas by Lemma 5.5 $\tilde{\rho}$ is bundle-compatible, ie

$$\tilde{\rho} = \rho + \sum_{a=1}^{k} (\langle p_a, \mu_\rho\rangle + c_a)\pi_B^*\omega_a + \langle d\mu_\rho \wedge \theta\rangle,$$

where $\rho = \omega_0 + (1/r_0)dd^c f$ is a $\mathbb{T}$–invariant Kähler metric on X; see (24). Using Lemma 5.9 and that both $\tilde{\omega}_\varphi$ and $\tilde{\rho}$ are of the form (24), we get a path of PDEs on X of the form

$$t(\mathrm{Scal}_p(\omega_{\varphi_t}) - \tilde{w}(\mu_{\omega_{\varphi_t}})) = (1-t)H(\varphi_t) \quad \text{for } t \in (t_0, 1), \tag{67}$$

where $\varphi_t \in \mathcal{K}_{\mathbb{T}}(X, \omega_0)$ and $H(\varphi_t) := \mathrm{tr}_{\tilde{\omega}_{\varphi_t}}(\tilde{\rho}) - (n+m)$ is manifestly a second-order differential operator on X for $\varphi_t \in \mathcal{K}_{\mathbb{T}}(X, \omega_0) \subset \mathcal{K}_{\mathbb{K}_Y}(Y, \tilde{\omega}_0)$. Then the solution φ_t for $t \in [t_0, 1)$ of (66) will actually belong to $\mathcal{K}_{\mathbb{T}}(X, \omega_0) \subset \mathcal{K}_{\mathbb{K}_Y}(Y, \tilde{\omega}_0)$. This last point is a consequence of the implicit function theorem (used in [46; 47] to establish the openness), which can be applied directly to (67); to find the linearization of (67), we use [46] that the linearization of $H(\varphi)$ on Y is the operator $\mathbb{H}^{\tilde{\rho}}_{\tilde{\omega}_\varphi, 1}$ (see Definition A.1), so that, by virtue of Lemma A.3, the linearization of $H(\varphi)$ when restricted to $\mathcal{K}_{\mathbb{T}}(X, \omega_0) \subset \mathcal{K}_{\mathbb{K}_Y}(Y, \tilde{\omega}_0)$ is given by the p–weighted operator $\mathbb{H}^{\rho}_{\omega_\varphi, p}$ introduced in Appendix A. Similar argument allows us to identify the linearization of $\mathrm{Scal}_p(\omega_\varphi)$; see also [55, Lemma B1]. We refer the reader to [52, Section 6] for further details.

Step 2 The next ingredient is a deep result from [23] with a complement in [47], showing that if $\boldsymbol{M}^Y_{1,\ell^{\mathrm{ext}}}$ is G–coercive along the continuity path φ_t with respect to a reductive subgroup $G \subset \mathrm{Aut}_r(Y)$ containing the torus generated by the extremal vector field $\xi^Y_{\mathrm{ext}} = d\ell^{\mathrm{ext}} \in \mathfrak{t}_Y$ in its centre, then there exists a subsequence of times $j \to 1$ and elements $\sigma_j \in G$ such that $\sigma_j^*(\tilde{\omega}_{\varphi_j})$ converges in $C^\infty(Y)$ to an extremal Kähler metric $\tilde{\omega}_1$. In our case, assuming (iii), we have that $\boldsymbol{M}^Y_{1,\ell^{\mathrm{ext}}}(\varphi_t) = \mathrm{Vol}(B, \omega_B)\boldsymbol{M}^X_{p,\tilde{w}}(\varphi_t)$ (see Lemma 5.10) is $\mathbb{G} = \mathbb{T}^{\mathbb{C}}_Y$–coercive (see Lemmas 6.6 and 6.4 and Proposition 3.4). We can thus find $\sigma_j \in \mathbb{T}^{\mathbb{C}}_Y$ and φ_j as above. The Kähler metrics $\sigma_j^*(\tilde{\omega}_{\varphi_j})$ are bundle-compatible in the sense of Definition 5.3, and thus are of the form

$$\sigma_j^*(\tilde{\omega}_{\varphi_j}) = \tilde{\omega}_0 + d_Y d^c_Y \sigma_j[\varphi_j] \quad \text{for } \sigma_j[\varphi_j] \in \mathcal{K}_{\mathbb{T}}(X, \omega_0) \subset \mathcal{K}_{\mathbb{K}_Y}(Y, \tilde{\omega}_0).$$

It follows that $\tilde{\omega}_1$ is a bundle-compatible extremal Kähler metric on Y (as $\mathcal{K}_{\mathbb{T}}(X, \omega_0)$ is $C^\infty(Y)$–closed in $\mathcal{K}_{\mathbb{K}_Y}(Y, \tilde{\omega}_0)$). By Lemma 5.9, the corresponding Kähler metric ω_1 on X is then $(p, \tilde{w})$–cscK.

(i) $\Longrightarrow$ (ii) The proof is very similar to the proof of (iii) $\Longrightarrow$ (ii). As in Step 1 above, we consider the continuity path (66), which defines potentials

$$\varphi_t \in \mathcal{K}_{\mathbb{T}}(X, \omega_0) \subset \mathcal{K}_{\mathbb{K}_Y}(Y, \tilde{\omega}_0) \quad \text{for } t \in [t_0, 1).$$

We can assume without loss of generality [21] that Y admits a $\mathbb{K}_Y$–invariant extremal Kähler metric in $[\tilde{\omega}_0]$, where $\mathbb{K}_Y \subset \mathrm{Aut}_r(Y)$ is the maximal torus given by Lemma 5.14.

This implies that $\boldsymbol{M}^Y_{1,\ell^{\mathrm{ext}}}$ is G–coercive for $G=\mathbb{K}_Y^{\mathbb{C}}$. Indeed, this can be justified, for instance, by applying Theorem 1 and Proposition 3.4 in the case $(v,w)=(1,\ell^{\mathrm{ext}})$. As in Step 2 of the proof of (iii) $\Longrightarrow$ (ii), we use [23; 47] and the G–coercivity of $\boldsymbol{M}^Y_{1,\ell^{\mathrm{ext}}}$ along the path in order to find a subsequence of times $j\to 1$ and elements $\sigma_j\in G$ such that $\sigma_j^*(\widetilde{\omega}_{\varphi_j})$ converges in $C^\infty(Y)$ to a $\mathbb{K}_Y$–invariant extremal Kähler metric $\widetilde{\omega}_1\in[\widetilde{\omega}_0]$. However, unlike the proof of (iii) $\Longrightarrow$ (ii), in general $\sigma_j^*(\widetilde{\omega}_{\varphi_j})$ and hence $\widetilde{\omega}_1$ are not bundle-compatible, as σ_j can act nontrivially on B (see the proof of Lemma 5.14). We thus need to slightly modify the argument in order to show that $\widetilde{\omega}_1$ still induces a $(p,\widetilde{w})$–cscK metric on any given fibre $X_b=\pi_B^{-1}(b)\subset Y$. We denote by $\omega_j(b):=(\widetilde{\omega}_{\varphi_j})|_{X_b}$ and $\overline{\omega}_j(b):=(\sigma_j^*(\widetilde{\omega}_{\varphi_j}))|_{X_b}$ the induced $\mathbb{T}$–invariant metrics on X_b. As $\widetilde{\omega}_{\varphi_j}$ is bundle-compatible, Lemma 5.9 yields

$$\mathrm{Scal}_p(\omega_j(b))=[p(\mu_{\widetilde{\omega}_{\varphi_j}})\,\mathrm{Scal}(\widetilde{\omega}_{\varphi_j})-p(\mu_{\widetilde{\omega}_{\varphi_j}})q(\mu_{\widetilde{\omega}_\varphi})]|_{X_b}.$$

Using that $\sigma_j\in\mathbb{K}_Y^{\mathbb{C}}$ sends the fibre X_b to the fibre $X_{\sigma_j(b)}$ (this follows from the construction of $\mathbb{K}_Y$ in the proof of Lemma 5.14), the above equality holds true for the metrics $\overline{\omega}_j(b)$, where in the right side we replace the metric $\widetilde{\omega}_{\varphi_j}$ on Y with $\overline{\omega}_j:=\sigma_j^*(\widetilde{\omega}_{\varphi_j})$. It thus follows by the smooth convergence of $\overline{\omega}_j(b)$ to $\omega_1(b)$ that

$$\begin{aligned}\mathrm{Scal}_p(\omega_1(b))&=[p(\mu_{\widetilde{\omega}_1})\,\mathrm{Scal}(\widetilde{\omega}_1)-p(\mu_{\widetilde{\omega}_1})q(\mu_{\widetilde{\omega}_1})]|_{X_b}\\&=[p(\mu_{\widetilde{\omega}_1})(\ell^{\mathrm{ext}}(\mu_{\widetilde{\omega}_1})-q(\mu_{\widetilde{\omega}_1})]|_{X_b}=\widetilde{w}(\mu_{\omega_1(b)}),\end{aligned}$$

where for the equalities on the second line we have used that the $\mathbb{K}_Y$–extremal function ℓ^{ext} is in $\mathrm{Aff}(\mathfrak{t}_X^*)$; see Lemma 5.14. Thus $\omega_1(b)$ is a $(v,\widetilde{w})$–cscK metric on X. □

Proof of Theorem 3 Han and Li introduced a functional $\boldsymbol{M}_v^{\mathrm{HL}}\colon\mathcal{K}_{\mathbb{T}}(X,\omega_0)\to\mathbb{R}$ whose critical points are the v–solitons; see [45, Lemma 4.4]. A careful inspection using (50) shows that $\boldsymbol{M}_v^{\mathrm{HL}}(\omega)=\boldsymbol{M}_{v,w}(\omega)-\int_X\log(v(\mu_\omega))v(\mu_\omega)\omega^{[m]}$, where w is the weight function defined in Proposition 1. Thus, the difference of the two functionals is a constant independent of the choice of a $\mathbb{T}$–invariant Kähler metric $\omega\in 2\pi c_1(X)$; see eg [55]. Thus, by [45, Theorem 3.5] applied to $(X,2\pi c_1(X),\mathbb{T})$ (and weights $pv,\widetilde{w}$), the $\mathbb{T}^{\mathbb{C}}$–coercivity of $\boldsymbol{M}^X_{pv,\widetilde{w}}$ is equivalent to the existence of a vp–soliton on X. By Lemma 5.11, this implies that Y admits a (bundle-compatible) v–soliton.

By [45, Theorem 1.7], the $\mathbb{T}^{\mathbb{C}}$–coercivity of $\boldsymbol{M}^X_{pv,\widetilde{w}}$ is also equivalent to the uniform vp–K–stability on $\mathbb{T}$–equivariant *special* test configurations. When $(X,\mathbb{T})$ is a toric Fano variety, the only such test configurations are the product test configurations, and thus, by [55, Proposition3], the condition is reduced to verifying (3) on X with respect to the weights $(pv,\widetilde{w})$.

Therefore, in order to show the existence of a Kähler–Ricci soliton, it is sufficient to find $\xi_0 \in \mathfrak{t}$ such that (3) is satisfied for the weight functions $v(\mu) = e^{\langle \xi_0, \mu \rangle} p(\xi)$ and $\tilde{w}(\mu) = 2p(\mu) e^{\langle \xi_0, \mu \rangle}(m + \langle \xi_0, \mu \rangle + \langle d \log p, \mu \rangle)$. We detail the proof of this fact below.

Let $\omega \in 2\pi c_1(X)$ be any $\mathbb{T}$–invariant Kähler metric with canonically normalized momentum map $\mu_\omega \colon X \to \Delta$. We then consider the following p–weighted version of a functional on $\mathfrak{t}$, defined originally by Tian and Zhu [71, Lemma 2.2]:

$$\xi \mapsto \int_X e^{\langle \xi, \mu_\omega \rangle} p(\mu_\omega) \omega^{[m]} \quad \text{for } \xi \in \mathfrak{t}. \tag{68}$$

The convexity and properness of the above functional follow by the arguments in [71, Lemma 2.2], but under our toric assumption these can also be seen directly by rewriting the right side in (68) as an integral over the Delzant polytope:

$$\xi \mapsto (2\pi)^m \int_\Delta e^{\langle \xi, \mu \rangle} p(\mu)\, d\mu.$$

Properness of this functional follows by the fact that the origin is in the interior of Δ (by the canonical normalization condition of Δ; see Remark 2.3). Let $\xi_0 \in \mathfrak{t}$ be the unique critical point of (68). We have that

$$\int_X \langle \zeta, \mu_\omega \rangle e^{\langle \xi_0, \mu_\omega \rangle} p(\mu_\omega) \omega^{[m]} = 0,$$

which is precisely the condition $\mathrm{Fut}_{v, \tilde{w}} = 0$ according to Lemma B.1.

The existence of a Sasaki–Einstein structure follows similarly. By Proposition 2, Lemma 5.11 and Proposition 1, in that order, we want to find $\xi_0 \in \mathfrak{t}$ such that (3) holds true for the weights given as in Proposition 1, with $v(\mu) = p(\mu)(\langle \xi_0, \mu \rangle + a)^{-(m+n+2)}$. (This will show the existence of a pv–soliton on the toric Fano manifold $(X, \mathbb{T})$ and hence a v–soliton on Y by the general arguments evoked above.) We argue based on [61], which introduced the *volume functional* on the space of *normalized* positive affine-linear functions on Δ. Strictly speaking, the functional in [61, Section 3] is introduced on the principal $\mathbb{S}^1$–bundle N over (X, ω) (which admits a natural strictly pseudoconvex CR structure $(\mathcal{D}, J)$ coming from X), and is then defined as the Sasaki volume of a $(\mathcal{D}, J)$–compatible normalized Sasaki–Reeb vector field $\hat{\xi}$ on N; using the point of view of [7] (see in particular Lemma 1.4), the volume functional can also be written on X, noting that positive affine-linear functions $\ell_\xi = \langle \xi, \mu \rangle + a$ over Δ are in bijection with Sasaki–Reeb vector fields $\hat{\xi}$ on $(N, \mathcal{D}, J)$, and the normalization condition used in [61] is equivalent to requiring $\ell_\xi(0) = a = 1$. Specifically, in our toric weighted setting, we let

$$\xi \mapsto \int_X (\langle \xi, \mu_\omega \rangle + 1)^{-(m+n+1)} p(\mu_\omega) \omega^{[m]} = (2\pi)^m \int_\Delta (\langle \xi, \mu \rangle + 1)^{-(m+n+1)} p(\mu)\, d\mu,$$

which is defined for $\xi \in \mathfrak{t}$ such that $(\langle \xi, \mu \rangle + 1) > 0$ on Δ. The properness of the functional follows by the fact that a canonically normalized Delzant polytope of a Fano toric manifold is determined by $\Delta = \{\mu : L_j(\mu) \geq 0\}$, where the affine-linear functions $L_j(\mu)$ satisfy $L_j(0) = 1$; see eg [1, Section 7.4]. The unique critical point $\xi_0 \in \mathfrak{t}$ of the above convex functional then satisfies

$$\int_X \langle \zeta, \mu_\omega \rangle (\langle \xi_0, \mu_\omega \rangle + 1)^{-(m+n+2)} p(\mu_\omega) \omega^{[m]} = 0 \quad \text{for } \zeta \in \mathfrak{t},$$

which, by Lemma B.1, is precisely the condition (3) for the weight functions considered. This concludes the proof of Theorem 3. □

Appendix A Weighted differential operators

Let $(X, \omega, \mathbb{T})$ be as in Section 1 and $v > 0$ be a positive smooth weight function defined over the polytope Δ. We denote by ∇^ω the Levi-Civita connection of the riemannian metric g_ω, and by δ_ω the formal adjoint of ∇^ω. We define the following weighted differential operators, which are self-adjoint with respect to the volume form $v(\mu_\omega)\omega^{[m]}$ on X:

Definition A.1 The *v–weighted Laplacian* of ψ is the second-order operator acting on smooth functions, defined by

$$\Delta_{\omega,v}(\psi) = \frac{1}{v(\mu_\omega)} \delta_\omega (v(\mu_\omega) d\psi). \tag{69}$$

The *v–weighted linear Lichnerowicz* operator is the fourth-order operator given by

$$\mathbb{L}_{\omega,v}(\psi) := \frac{\delta_\omega \delta_\omega (v(\mu_\omega)(\nabla^\omega d\psi)^-)}{v(\mu_\omega)}, \tag{70}$$

where $(\nabla^\omega d\phi)^-$ stands for the $(0,2)$–symmetric tensor of type $(2,0) + (0,2)$ with respect to the complex structure of X. For any $\mathbb{T}$–invariant Kähler form ρ on X, we define the second-order operator, given by

$$\mathbb{H}^\rho_{\omega,v}(\psi) := \langle \rho, dd^c\psi \rangle_\omega + \langle d\, \mathrm{tr}_\omega(\rho), d\psi \rangle_\omega + \frac{1}{v(\mu_\omega)} \langle \rho, dv(\mu_\omega) \wedge d^c\psi \rangle_\omega, \tag{71}$$

where $\mathrm{tr}_\omega(\rho) := (\rho \wedge \omega^{[m-1]})/\omega^{[m]} = \langle \rho, \omega \rangle_\omega$. The operator $\mathbb{H}^\rho_{\omega,v}$ is a v–weighted version of the linear operator used in [46].

A straightforward computation shows:

Lemma A.2 *The v–weighted Lichnerowicz operator can be written as*

$$\mathbb{L}_{\omega,v}(\psi) = \tfrac{1}{2}(\Delta_{\omega,v})^2(\psi) + \delta_{\omega,v}\big(\rho_{\omega,v}((d^c\psi)^\sharp)\big),$$

where $\delta_{\omega,v} := (1/v(\mu_\omega))\delta_\omega v(\mu_\omega)$ is the formal adjoint of the exterior derivative d on functions with respect to the weighted volume form $v(\mu_\omega)\omega^{[m]}$,

$$\rho_{\omega,v} := \rho_\omega - \tfrac{1}{2} dd^c(\log v(\mu_\omega))$$

is the Ricci form of the weighted volume form $v(\mu_\omega)\omega^{[m]}$, and $\sharp = g_\omega^{-1}$ stands for the riemannian duality between TM and T^*M by using the Kähler metric ω.

We now specialize to the case when $(Y, \tilde{\omega}, \mathbb{T}_Y)$ is a semisimple principal $(X, \omega, \mathbb{T}_X)$–fibration over B, as in Section 5. We then denote by $\Delta^Y_{\tilde{\omega}}$, $\mathbb{L}^Y_{\tilde{\omega}}$ and $(\mathbb{H}^{\tilde{\rho}}_{\tilde{\omega}})^Y$ the corresponding unweighted operators on $(Y, \tilde{\omega})$, where the Kähler form $\tilde{\rho}$ in the definition of $\mathbb{H}^{\tilde{\rho}}_{\tilde{\omega}}$ is bundle-compatible, ie given by (24) for a $\mathbb{T}_X$–invariant Kähler form ρ on X. We further let $\Delta^{B_a}_{\omega_a}$ denote the Laplacian on (B_a, ω_a), and Δ^B_x and $\mathbb{L}^B_x$ the Laplacian and Lichnerowicz operators on B, respectively, with respect to the Kähler metric $\omega_B(x) := \sum_{a=1}^k (\langle p_a, \mu_\omega(x)\rangle + c_a)\omega_a$. We thus have:

Lemma A.3 *Let ψ be a $\mathbb{T}_Y$–invariant smooth function on Y, seen as a $\mathbb{T}_X$–invariant function on $X \times B$ via (25), and $\tilde{\omega}$ a bundle-compatible $\mathbb{T}_Y$–invariant Kähler metric on Y associated to a $\mathbb{T}_X$–invariant Kähler metric ω on X. We then have*

$$\Delta^Y_{\tilde{\omega}}\psi = \Delta^X_{\omega,p}\psi_b + \Delta^B_x\psi_x,$$

$$\mathbb{L}^Y_{\tilde{\omega}}\psi = \mathbb{L}^X_{\omega,p}\psi_b + \mathbb{L}^B_x\psi_x + \Delta^B_x(\Delta^X_{\omega,p}\psi_b)_x + \Delta^X_{\omega,v}(\Delta^B_x\psi_x)_b + \sum_{a=1}^k Q_a(x)\Delta^{B_a}_{\omega_a}\psi_x$$

and

$$(\mathbb{H}^{\tilde{\rho}}_{\tilde{\omega},1})^Y\psi = (\mathbb{H}^{\rho}_{\omega,p})^X\psi_b + \sum_{a=1}^k P_a(x)\Delta^{B_a}_{\omega_a}\psi_x,$$

where $P_a(x)$ and $Q_a(x)$ are smooth $\mathbb{T}$–invariant functions on X, and ψ_x and ψ_b are the induced smooth functions on B and X, respectively, via (25).

Proof This first two equalities are established in [6] (see the proof of Lemma 8) in the special case when $(X, \omega, \mathbb{T}_X)$ is a toric variety, whereas the third identity is proved in [52] (also in the case when $(X, \mathbb{T}_X)$ is toric). These computations extend to the general setting with no substantial additional difficulty (by using Lemma A.2 for the second identity), but we include them below for the sake of self-containedness.

In the notation of Section 5,

$$(72)\quad \Delta^Y_{\tilde{\omega}}(\psi) = \begin{cases} -(d_Y d^c_Y\psi \wedge \tilde{\omega}^{[n+m-1]})/\tilde{\omega}^{[n+m]} & \text{on } Y, \\ -(d_Y d^c_Y\psi \wedge \tilde{\omega}^{[n+m-1]} \wedge \theta^{\wedge r})/(\tilde{\omega}^{[n+m]} \wedge \theta^{\wedge r}) & \text{on } Z = X \times P, \end{cases}$$

where $\theta^{\wedge r} := \bigwedge_{i=1}^{r} \theta_i$ with respect to any lattice basis $(\xi_i)_i$ of $\Lambda \subset \mathfrak{t}$. Viewing $d^c_{X\times B}\psi$ as a 1–form on Z, it admits the decomposition, with respect to (23),

$$(73)\qquad d^c_{X\times B}\psi = (d^c_{X\times B}\psi)_{\mathscr{H}} + \sum_{i=1}^{r}(d^c_{X\times B}\psi)(\xi_i^P - \xi_i^X)\theta_i = d^c_Y\psi - \langle d^c_X\psi, \theta\rangle.$$

We thus compute, on Z,

$$\begin{aligned}(74)\quad (d_Y d^c_Y\psi)_{(x,b)} &= d_Z\Big(d^c_X\psi + \sum_{j=1}^{r} d^c_X\psi(\xi_j^X)\theta_j + d^c_B\psi\Big)\\ &= d_Z d^c_X\psi + \sum_{j=1}^{r} d_Z(d^c_X\psi(\xi_j^X))\theta_j\\ &\qquad + \sum_{j=1}^{r} d^c_X\psi_b(\xi_j^X)\Big(\sum_{a=1}^{k}\xi^j(p_a)\pi_B^*\omega_a\Big) + d_Z d^c_B\psi\\ &= d_X d^c_X\psi_b + d_B d^c_B\psi_x + \sum_{j=1}^{r} d_Z(d^c_X\psi(\xi_j^X))\wedge\theta_j\\ &\qquad + \sum_{a=1}^{k} d^c_X\psi(p_a^X)\pi_B^*\omega_a + d_B d^c_X\psi + d_X d^c_B\psi,\end{aligned}$$

where for the third equality we used (22), as well as the identities $d_P d^c_X\psi = d_B d^c_X\psi$ and $d_P d^c_B\psi = d_B d^c_B\psi$ (which follow from the identification (25)). Using (27) and (44), we derive, from (72) and (74),

$$\Delta^Y_{\tilde\omega}(\psi)(x,b) = (\Delta^X_\omega\psi_b)(x) + (\Delta^B_{\omega_B(x)}\psi_x)(b) - \sum_{a=1}^{k}\frac{n_a}{\langle\mu_\omega, p_a\rangle + c_a}(d^c_X\psi_b)(p_a^X),$$

where, we recall, for a fixed $x \in X$ we have set $\omega_B(x) := \sum_{a=1}^{k}(\langle p_a, \mu_\omega\rangle + c_a)\omega_a$, and p_a^X denotes the vector field on X corresponding to $p_a \in \mathfrak{t}$. The first equality in the lemma follows from the identity (47), keeping in mind that, for any smooth function u on Δ and any $\mathbb{T}$–invariant smooth function ϕ on X,

$$g_\omega(d(u(\mu_\omega)), d\phi) = \sum_{i=1}^{r} u_{,i}(\mu_\omega)d^c\phi(\xi_i).$$

Now we establish the expression of the corresponding Lichnerowicz operators. Recall that (see eg [41])

$$(75)\qquad \mathbb{L}^Y_{\tilde\omega}\psi := \tfrac{1}{2}(\Delta^Y_{\tilde\omega})^2(\psi) + \delta_{\tilde\omega}(\rho_{\tilde\omega}(d^c_Y\psi)).$$

Using the decomposition of $\Delta^Y_{\tilde\omega}$ we have just established,

$$(\Delta^Y_{\tilde\omega})^2(\psi) = (\Delta^X_{\omega,p})^2(\psi_b) + (\Delta^B_x)^2(\psi_x) + \Delta^X_{\omega,p}(\Delta^B_x(\psi_x)) + \Delta^B_x(\Delta^X_{\omega,p}(\psi_b)). \tag{76}$$

It remains to compute the Ricci term in (75). From (43),

$$\rho_{\tilde\omega} = \rho_{\omega,p} + \pi_B^*\rho_{\omega_B} + \frac{1}{2}\sum_{a=1}^{k} \Delta^X_{\omega,p}(\langle\mu_\omega, p_a^X\rangle)\pi_B^*\omega_a \\ + \sum_{j=1}^{r} d_X\big(d^c_X\big(\kappa - \tfrac{1}{2}\log p(\mu_\omega)\big)(\xi^X_j)\big)\wedge\theta_j, \tag{77}$$

where $\rho_{\omega,p} := \rho_\omega - \frac{1}{2}d_X d^c_X \log p(\mu_\omega)$ is the Ricci form of the weighted volume form $p(\mu_\omega)\omega^{[m]}$. Using integration by parts, for any $\mathbb{T}_Y$–invariant smooth test function ϕ on Y, seen as a $\mathbb{T}_X$ and $\mathbb{T}_P$–invariant function on $Z = X\times P$ via (25),

$$\begin{aligned}
&\int_Z \phi\,\delta_{\tilde\omega}(\rho_{\tilde\omega}(d^c_Y\psi))\tilde\omega^{[n+m]}\wedge\theta^{\wedge r} \\
&\quad= -\int_Z \rho_{\tilde\omega}(d_Y\phi, d^c_Y\psi)\tilde\omega^{[n+m]}\wedge\theta^{\wedge r} \\
&\quad= \int_Z \rho_{\tilde\omega}\wedge d_Y\phi\wedge d^c_Y\psi\wedge\tilde\omega^{[n+m-2]}\wedge\theta^{\wedge r} \\
&\qquad\qquad - \frac{1}{2}\int_Z \mathrm{Scal}(\tilde\omega)\tilde g_{\tilde\omega}(d_Y\phi, d_Y\psi)\tilde\omega^{[n+m]}\wedge\theta^{\wedge r} \\
&\quad= \int_Z \rho_{\tilde\omega}\wedge d_Y\phi\wedge d^c_Y\psi\wedge\tilde\omega^{[n+m-2]}\wedge\theta^{\wedge r} \\
&\qquad\qquad - \frac{1}{2}\int_Z \Big(\frac{\mathrm{Scal}_p(\omega)}{p(\mu_\omega)} + q(\mu_\omega)\Big) d_Y\phi\wedge d^c_Y\psi\wedge\tilde\omega^{[n+m-1]}\wedge\theta^{\wedge r}.
\end{aligned} \tag{78}$$

From the above formula, using (44), (73) and (77), we compute (after some straightforward but long algebraic manipulations and integration by parts over X and B)

$$\begin{aligned}
&\delta^Y_{\tilde\omega}(\rho_{\tilde\omega}(d^c_Y\psi)) \\
&\quad= \delta^X_{\omega,p}(\rho_{\omega,p}(d^c_X\psi)) + \delta^B_{\omega_B(x)}(\rho_{\omega_B}(d^c_B\psi)) \\
&\qquad + \frac{1}{2}\sum_{a=1}^{k} \frac{q(\mu_\omega)}{\langle\mu_\omega, p_a\rangle + c_a}\Delta^B_{\omega_a}(\psi) \\
&\qquad + \frac{1}{2}\sum_{a=1}^{k} \frac{(n_a-1)}{(\langle\mu_\omega, p_a\rangle + c_a)^2}\Delta^X_{\omega,p}(\langle\mu_\omega, p_a\rangle)\Delta^B_{\omega_a}(\psi_x) \\
&\qquad + \sum_{a,b=1}^{k} \frac{n_b}{(\langle\mu_\omega, p_a\rangle + c_a)(\langle\mu_\omega, p_b\rangle + c_b)}\Delta^X_{\omega,p}(\langle\mu_\omega, p_b\rangle)\Delta^B_{\omega_a}(\psi_x).
\end{aligned} \tag{79}$$

Combining (75), (76) and (79) yields the desired expression.

The expression for $(\mathbb{H}^{\tilde{\rho}}_{\tilde{\omega},1})^Y(\psi)$ is obtained by similar arguments, using that

$$\begin{aligned}(\mathbb{H}^{\tilde{\rho}}_{\tilde{\omega},1})^Y(\psi) &= \langle\tilde{\rho}, d_Y d_Y^c\psi\rangle_{\tilde{\omega}} + \langle d_Y \operatorname{tr}_{\tilde{\omega}}(\tilde{\rho}), d_Y\psi\rangle_{\tilde{\omega}} \\ &= -\operatorname{tr}_{\tilde{\omega}}(\tilde{\rho})\Delta^Y_{\tilde{\omega}}(\psi) - \frac{\tilde{\rho}\wedge d_Y d_Y^c\psi\wedge\tilde{\omega}^{[n+m-2]}}{\tilde{\omega}^{[n+m]}} \\ &\qquad + \frac{d_Y \operatorname{tr}_{\tilde{\omega}}(\tilde{\rho})\wedge d_Y^c\psi\wedge\tilde{\omega}^{[n+m-1]}}{\tilde{\omega}^{[n+m]}}. \qquad \square\end{aligned}$$

Appendix B Weighted Futaki invariants

On a smooth Fano manifold $(X,\mathbb{T})$ as in the setting and notation of Section 2, we further relate the weighted Futaki obstruction $\operatorname{Fut}_{v,w}=0$ (see (3)) with weights $v(\mu)$ and $w(\mu)$ as in Proposition 1 with the Futaki-type obstructions studied by Tian and Zhu [71] in the case of Kähler–Ricci solitons (ie when $v=e^{\langle\xi,\mu\rangle}$):

Lemma B.1 *Let $(X,\mathbb{T})$ be a smooth Fano manifold $(X,\mathbb{T})$ with canonically normalized momentum polytope Δ, and $v>0$ and w smooth functions on Δ as in Proposition 1. Then, for any $\mathbb{T}$–invariant Kähler metric $\omega\in 2\pi c_1(X)$ with momentum map μ_ω and $\mathbb{T}$–invariant Ricci potential h (ie $\rho_\omega-\omega=\frac{1}{2}dd^ch$), the weighted Futaki invariant $\operatorname{Fut}_{v,w}$ introduced in (3) satisfies*

$$\operatorname{Fut}_{v,w}(\ell_\xi) = \int_X \big(\mathcal{L}_{J\xi}(\log v(\mu_\omega)-h)\big)v(\mu_\omega)\omega^{[m]} = -2\int_X\langle\zeta,\mu_\omega\rangle v(\mu_\omega)\omega^{[m]}$$

for $\ell_\xi=\langle\zeta,\mu\rangle+a$ and $\zeta\in\mathfrak{t}$.

Proof We have

$$\begin{aligned}\int_X\big(\mathcal{L}_{J\xi}(\log v(\mu_\omega)-h)\big)v(\mu_\omega)\omega^{[m]} &= \int_X g_\omega\big(d\ell_\xi, d\log(v(\mu_\omega)-h)\big)v(\mu_\omega)\omega^{[m]} \\ &= \int_X \ell_\xi\big(\Delta_{\omega,v}(\log v(\mu_\omega)-h)\big)v(\mu_\omega)\omega^{[m]} \\ &= \int_X \ell_\xi\big(\operatorname{Scal}_v(\omega)-w(\mu_\omega)\big)\omega^{[m]} = \operatorname{Fut}_{v,w}(\ell_\xi),\end{aligned}$$

where for the last equality we have used (13). The second equality in the lemma follows from the first, the second relation in (11) and integration by parts. $\square$

References

[1] **V Apostolov**, *The Kähler geometry of toric manifolds* (2022) arXiv 2208.12493

[2] **V Apostolov**, **D M J Calderbank**, *The CR geometry of weighted extremal Kähler and Sasaki metrics*, Math. Ann. 379 (2021) 1047–1088 MR Zbl

[3] **V Apostolov**, **D M J Calderbank**, **P Gauduchon**, *Hamiltonian 2–forms in Kähler geometry, I: General theory*, J. Differential Geom. 73 (2006) 359–412 MR Zbl

[4] **V Apostolov**, **D M J Calderbank**, **P Gauduchon**, **C W Tønnesen-Friedman**, *Hamiltonian 2–forms in Kähler geometry, II: Global classification*, J. Differential Geom. 68 (2004) 277–345 MR Zbl

[5] **V Apostolov**, **D M J Calderbank**, **P Gauduchon**, **C W Tønnesen-Friedman**, *Hamiltonian 2–forms in Kähler geometry, IV: Weakly Bochner-flat Kähler manifolds*, Comm. Anal. Geom. 16 (2008) 91–126 MR Zbl

[6] **V Apostolov**, **D M J Calderbank**, **P Gauduchon**, **C W Tønnesen-Friedman**, *Extremal Kähler metrics on projective bundles over a curve*, Adv. Math. 227 (2011) 2385–2424 MR Zbl

[7] **V Apostolov**, **D M J Calderbank**, **E Legendre**, *Weighted K–stability of polarized varieties and extremality of Sasaki manifolds*, Adv. Math. 391 (2021) art. id. 107969 MR Zbl

[8] **C Arezzo**, **F Pacard**, **M Singer**, *Extremal metrics on blowups*, Duke Math. J. 157 (2011) 1–51 MR Zbl

[9] **M F Atiyah**, *Convexity and commuting Hamiltonians*, Bull. London Math. Soc. 14 (1982) 1–15 MR Zbl

[10] **R J Berman**, *K–polystability of $\mathbb{Q}$–Fano varieties admitting Kähler–Einstein metrics*, Invent. Math. 203 (2016) 973–1025 MR Zbl

[11] **R J Berman**, **B Berndtsson**, *Convexity of the K–energy on the space of Kähler metrics and uniqueness of extremal metrics*, J. Amer. Math. Soc. 30 (2017) 1165–1196 MR Zbl

[12] **R J Berman**, **S Boucksom**, **P Eyssidieux**, **V Guedj**, **A Zeriahi**, *Kähler–Einstein metrics and the Kähler–Ricci flow on log Fano varieties*, J. Reine Angew. Math. 751 (2019) 27–89 MR Zbl

[13] **R J Berman**, **T Darvas**, **C H Lu**, *Convexity of the extended K–energy and the large time behavior of the weak Calabi flow*, Geom. Topol. 21 (2017) 2945–2988 MR Zbl

[14] **R J Berman**, **T Darvas**, **C H Lu**, *Regularity of weak minimizers of the K–energy and applications to properness and K–stability*, Ann. Sci. Éc. Norm. Supér. 53 (2020) 267–289 MR Zbl

[15] **R J Berman**, **D W Nyström**, *Complex optimal transport and the pluripotential theory of Kähler–Ricci solitons*, preprint (2014) arXiv 1401.8264

[16] **Z Błocki**, *On geodesics in the space of Kähler metrics*, from "Advances in geometric analysis" (S Janeczko, J Li, D H Phong, editors), Adv. Lect. Math. 21, International, Somerville, MA (2012) 3–19 MR Zbl

[17] **Z Błocki**, **S Kołodziej**, *On regularization of plurisubharmonic functions on manifolds*, Proc. Amer. Math. Soc. 135 (2007) 2089–2093 MR Zbl

[18] **S Boucksom**, **T Hisamoto**, **M Jonsson**, *Uniform K–stability, Duistermaat–Heckman measures and singularities of pairs*, Ann. Inst. Fourier (Grenoble) 67 (2017) 743–841 MR Zbl

[19] **C P Boyer**, **K Galicki**, **S R Simanca**, *Canonical Sasakian metrics*, Comm. Math. Phys. 279 (2008) 705–733 MR Zbl

[20] **C P Boyer**, **C W Tønnesen-Friedman**, *The Sasaki join, Hamiltonian 2–forms, and constant scalar curvature*, J. Geom. Anal. 26 (2016) 1023–1060 MR Zbl

[21] **E Calabi**, *Extremal Kähler metrics*, from "Seminar on differential geometry" (S T Yau, editor), Ann. of Math. Stud. 102, Princeton Univ. Press (1982) 259–290 MR Zbl

[22] **X Chen**, *The space of Kähler metrics*, J. Differential Geom. 56 (2000) 189–234 MR Zbl

[23] **X Chen**, **J Cheng**, *On the constant scalar curvature Kähler metrics, II: Existence results*, J. Amer. Math. Soc. 34 (2021) 937–1009 MR Zbl

[24] **X Chen**, **M Paun**, **Y Zeng**, *On deformation of extremal metrics*, preprint (2015) arXiv 1506.01290

[25] **J Chu**, **V Tosatti**, **B Weinkove**, *On the $C^{1,1}$ regularity of geodesics in the space of Kähler metrics*, Ann. PDE 3 (2017) art. id. 15 MR Zbl

[26] **A S Dancer**, **M Y Wang**, *On Ricci solitons of cohomogeneity one*, Ann. Global Anal. Geom. 39 (2011) 259–292 MR Zbl

[27] **T Darvas**, *The Mabuchi geometry of finite energy classes*, Adv. Math. 285 (2015) 182–219 MR Zbl

[28] **T Darvas**, *The Mabuchi completion of the space of Kähler potentials*, Amer. J. Math. 139 (2017) 1275–1313 MR Zbl

[29] **T Darvas**, **Y A Rubinstein**, *Tian's properness conjectures and Finsler geometry of the space of Kähler metrics*, J. Amer. Math. Soc. 30 (2017) 347–387 MR Zbl

[30] **V Datar**, **G Székelyhidi**, *Kähler–Einstein metrics along the smooth continuity method*, Geom. Funct. Anal. 26 (2016) 975–1010 MR Zbl

[31] **T Delcroix**, *K–stability of Fano spherical varieties*, Ann. Sci. Éc. Norm. Supér. 53 (2020) 615–662 MR Zbl

[32] **F Delgove**, *Kähler–Ricci solitons on horospherical variety*, preprint (2017) arXiv 1710.05623

[33] **R Dervan**, *Relative K–stability for Kähler manifolds*, Math. Ann. 372 (2018) 859–889 MR Zbl

[34] **R Dervan**, **J Ross**, *K–stability for Kähler manifolds*, Math. Res. Lett. 24 (2017) 689–739 MR Zbl

[35] **R Dervan**, **L M Sektnan**, *Optimal symplectic connections on holomorphic submersions*, Comm. Pure Appl. Math. 74 (2021) 2132–2184 MR Zbl

[36] **S K Donaldson**, *Lower bounds on the Calabi functional*, J. Differential Geom. 70 (2005) 453–472 MR Zbl

[37] **S K Donaldson**, *Kähler geometry on toric manifolds*, *and some other manifolds with large symmetry*, from "Handbook of geometric analysis" (L Ji, P Li, R Schoen, L Simon, editors), Adv. Lect. Math. 7, International, Somerville, MA (2008) 29–75 MR Zbl

[38] **A Fujiki**, *On automorphism groups of compact Kähler manifolds*, Invent. Math. 44 (1978) 225–258 MR Zbl

[39] **A Futaki**, **H Ono**, *Conformally Einstein–Maxwell Kähler metrics and structure of the automorphism group*, Math. Z. 292 (2019) 571–589 MR Zbl

[40] **A Futaki**, **H Ono**, **G Wang**, *Transverse Kähler geometry of Sasaki manifolds and toric Sasaki–Einstein manifolds*, J. Differential Geom. 83 (2009) 585–635 MR Zbl

[41] **P Gauduchon**, *Calabi's extremal metrics*: *an elementary introduction*, lecture notes, in preparation

[42] **J P Gauntlett**, **D Martelli**, **J Sparks**, **D Waldram**, *A new infinite class of Sasaki–Einstein manifolds*, Adv. Theor. Math. Phys. 8 (2004) 987–1000 MR Zbl

[43] **V Guedj**, **A Zeriahi**, *The weighted Monge–Ampère energy of quasiplurisubharmonic functions*, J. Funct. Anal. 250 (2007) 442–482 MR Zbl

[44] **V Guillemin**, **S Sternberg**, *Convexity properties of the moment mapping*, Invent. Math. 67 (1982) 491–513 MR Zbl

[45] **J Han**, **C Li**, *On the Yau–Tian–Donaldson conjecture for generalized Kähler–Ricci soliton equations*, Comm. Pure Appl. Math. 76 (2023) 1793–1867 MR Zbl

[46] **Y Hashimoto**, *Existence of twisted constant scalar curvature Kähler metrics with a large twist*, Math. Z. 292 (2019) 791–803 MR Zbl

[47] **W He**, *On Calabi's extremal metric and properness*, Trans. Amer. Math. Soc. 372 (2019) 5595–5619 MR Zbl

[48] **W He**, **J Li**, *Geometric pluripotential theory on Sasaki manifolds*, J. Geom. Anal. 31 (2021) 1093–1179 MR Zbl

[49] **T Hisamoto**, *Stability and coercivity for toric polarizations*, preprint (2016) arXiv 1610.07998

[50] **E Inoue**, *Constant μ–scalar curvature Kähler metric–formulation and foundational results*, J. Geom. Anal. 32 (2022) art. id. 145 MR Zbl

[51] **D Jerison**, **J M Lee**, *Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem*, J. Amer. Math. Soc. 1 (1988) 1–13 MR Zbl

[52] **S Jubert**, *A Yau–Tian–Donaldson correspondence on a class of toric fibrations*, preprint (2021) arXiv 2108.12297

[53] **N Koiso**, *On rotationally symmetric Hamilton's equation for Kähler–Einstein metrics*, from "Recent topics in differential and analytic geometry" (T Ochiai, editor), Adv. Stud. Pure Math. 18, Academic, Boston, MA (1990) 327–337 MR Zbl

[54] **A Lahdili**, *Automorphisms and deformations of conformally Kähler, Einstein–Maxwell metrics*, J. Geom. Anal. 29 (2019) 542–568 MR Zbl

[55] **A Lahdili**, *Kähler metrics with constant weighted scalar curvature and weighted K–stability*, Proc. Lond. Math. Soc. 119 (2019) 1065–1114 MR Zbl

[56] **A Lahdili**, *Convexity of the weighted Mabuchi functional and the uniqueness of weighted extremal metrics*, preprint (2020) arXiv 2007.01345

[57] **C Li**, *G–uniform stability and Kähler–Einstein metrics on Fano varieties*, Invent. Math. 227 (2022) 661–744 MR Zbl

[58] **T Mabuchi**, *Kähler–Einstein metrics for manifolds with nonvanishing Futaki character*, Tohoku Math. J. 53 (2001) 171–182 MR Zbl

[59] **T Mabuchi**, *Multiplier Hermitian structures on Kähler manifolds*, Nagoya Math. J. 170 (2003) 73–115 MR Zbl

[60] **T Mabuchi**, **Y Nakagawa**, *New examples of Sasaki–Einstein manifolds*, Tohoku Math. J. 65 (2013) 243–252 MR Zbl

[61] **D Martelli**, **J Sparks**, **S-T Yau**, *Sasaki–Einstein manifolds and volume minimisation*, Comm. Math. Phys. 280 (2008) 611–673 MR Zbl

[62] **Y Odaka**, *A generalization of the Ross–Thomas slope theory*, Osaka J. Math. 50 (2013) 171–185 MR Zbl

[63] **F Podestà**, **A Spiro**, *Kähler–Ricci solitons on homogeneous toric bundles*, J. Reine Angew. Math. 642 (2010) 109–127 MR Zbl

[64] **S Semmes**, *Complex Monge–Ampère and symplectic manifolds*, Amer. J. Math. 114 (1992) 495–550 MR Zbl

[65] **Z Sjöström Dyrefelt**, *K–semistability of cscK manifolds with transcendental cohomology class*, J. Geom. Anal. 28 (2018) 2927–2960 MR Zbl

[66] **Z Sjöström Dyrefelt**, *On K–polystability of cscK manifolds with transcendental cohomology class*, Int. Math. Res. Not. 2020 (2020) 2769–2817 MR Zbl

[67] **J Stoppa**, *K–stability of constant scalar curvature Kähler manifolds*, Adv. Math. 221 (2009) 1397–1408 MR Zbl

[68] **J Stoppa**, **G Székelyhidi**, *Relative K–stability of extremal metrics*, J. Eur. Math. Soc. 13 (2011) 899–909 MR Zbl

[69] **A Swann**, *Twisting Hermitian and hypercomplex geometries*, Duke Math. J. 155 (2010) 403–431 MR Zbl

[70] **G Tian**, *Kähler–Einstein metrics with positive scalar curvature*, Invent. Math. 130 (1997) 1–37 MR Zbl

[71] **G Tian**, **X Zhu**, *A new holomorphic invariant and uniqueness of Kähler–Ricci solitons*, Comment. Math. Helv. 77 (2002) 297–325 MR Zbl

[72] **X Wang**, *Height and GIT weight*, Math. Res. Lett. 19 (2012) 909–926 MR Zbl

[73] **X-J Wang**, **X Zhu**, *Kähler–Ricci solitons on toric manifolds with positive first Chern class*, Adv. Math. 188 (2004) 87–103 MR Zbl

Département de Mathématiques, Université du Québec à Montréal
Montréal, QC, Canada

Current address: *Institute of Mathematics and Informatics, Bulgarian Academy of Sciences*
Sofia, Bulgaria

Département de Mathématiques, Université du Québec à Montréal
Montréal, QC, Canada

Current address: *Institut de Mathématiques de Toulouse*
Toulouse, France

Department of Mathematics, Aarhus University
Aarhus, Denmark

apostolov.vestislav@uqam.ca, simonjubert@gmail.com,
lahdili.abdellah@gmail.com

Proposed: Gang Tian
Seconded: Frances Kirwan, Simon Donaldson

Received: 17 April 2021
Revised: 4 December 2021

Geometry & Topology

msp.org/gt

See inside back cover or msp.org/gt for submission instructions.

The subscription price for 2023 is US $740/year for the electronic version, and $1030/year (+$70, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Geometry & Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Geometry & Topology (ISSN 1465-3060 printed, 1364-0380 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

GT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Geometry & Topology

Volume 27 Issue 8 (pages 2937–3385) 2023

Formal groups and quantum cohomology 2937

PAUL SEIDEL

AGT relations for sheaves on surfaces 3061

ANDREI NEGUȚ

Partially hyperbolic diffeomorphisms homotopic to the identity in dimension 3, II: Branching foliations 3095

THOMAS BARTHELMÉ, SÉRGIO R FENLEY, STEVEN FRANKEL and RAFAEL POTRIE

The Weil–Petersson gradient flow of renormalized volume and 3–dimensional convex cores 3183

MARTIN BRIDGEMAN, JEFFREY BROCK and KENNETH BROMBERG

Weighted K–stability and coercivity with applications to extremal Kähler and Sasaki metrics 3229

VESTISLAV APOSTOLOV, SIMON JUBERT and ABDELLAH LAHDILI

Anosov representations with Lipschitz limit set 3303

MARIA BEATRICE POZZETTI, ANDRÉS SAMBARINO and ANNA WIENHARD

The deformation space of geodesic triangulations and generalized Tutte's embedding theorem 3361

YANWEN LUO, TIANQI WU and XIAOPING ZHU