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We study K–theoretic Gromov-Witten invariants of projective hypersurfaces using a virtual localization formula under finite group actions. In particular, it provides all K–theoretic Gromov-Witten invariants of the quintic threefold modulo 41, up to genus 19 and degree 40. As an illustration, we give an instance in genus one and degree one. Applying the same idea to a K–theoretic version of FJRW theory, we determine it modulo 205 for the quintic polynomial with minimal group and narrow insertions, in every genus.

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0 Introduction

One of the first achievements of Gromov–Witten (GW) theory is the celebrated formula of Candelas, de la Ossa, Green and Parkes [4] computing genus-0 invariants of the quintic threefold in terms of a hypergeometric series solution of a Picard–Fuchs equation. It was a first instance of mirror symmetry and was proved by Givental [14] and Lian, Liu and Yau [28].

The K–theoretic version of GW theory, which we refer to as KGW theory, was constructed in Lee [25], and it is only recently that mirror symmetry in this context was

developed by Givental in his series of preprints starting with [15]. It relates the KGW generating series to a q–hypergeometric function solution of a finite-difference equation.

Both GW and KGW theories rely on the notion of a perfect obstruction theory (see Behrend and Fantechi [2]), producing two fundamental objects on the moduli space $\overline{\mathcal{M}}_{g,n}(X, \beta)$ of stable maps to a given nonsingular variety X, namely the virtual cycle $[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\text{vir}}$ living in the Chow ring of the moduli space, and the virtual structure sheaf $\mathcal{O}^{\text{vir}}_{\overline{\mathcal{M}}_{g,n}(X,\beta)}$ living in its K–theory of coherent sheaves.

Given insertions $y_i \in \text{CH}_*(X)$, or $Y_i \in K^0(X)$, and psi-classes $\psi_i \in \text{CH}^1(\overline{\mathcal{M}}_{g,n})$, or psi-bundles $\Psi_i \in K^0(\overline{\mathcal{M}}_{g,n})$, we then form

$$a := \prod_{i=1}^{n} \text{ev}^*(y_i) \cdot \psi_i^{d_i} \quad \text{and} \quad A := \bigotimes_{i=1}^{n} \text{ev}^*(Y_i) \otimes \Psi_i^{\otimes d_i}, \quad \text{with } d_i \in \mathbb{Z},$$

using evaluation maps. It yields GW and KGW invariants

$$p_*(a \cdot [\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\text{vir}}) \in \mathbb{Q} \quad \text{and} \quad p_!(A \otimes \mathcal{O}^{\text{vir}}_{\overline{\mathcal{M}}_{g,n}(X,\beta)}) \in \mathbb{Z},$$

where p is the projection map to a point along which we take pushforwards in Chow or in K–theory.[1] Each theory has an important feature: the virtual cycle is pure-dimensional, leading to a degree condition on the insertions for the GW invariant to be nonzero, and KGW invariants are all integers. Moreover, the two theories are related via a Hirzebruch–Riemann–Roch theorem (see Tonita [38]), saying that all KGW invariants of a nonsingular variety X can be reconstructed from the knowledge of all GW invariants of the DM stacks $[X/(\mathbb{Z}/M\mathbb{Z})]$ for all $M \in \mathbb{N}^*$; see Givental [16, Main Theorem].

Let T be a torus. When the variety X carries a nontrivial T–action, so does the moduli space of stable maps, and the virtual cycle and virtual structure sheaf are T–equivariant. One then benefits from the virtual localization formula of Graber and Pandharipande [17] to reduce the computation of invariants to the T–fixed locus, which greatly simplifies the calculation. Unfortunately, the automorphism group of a smooth projective hypersurface such as the quintic threefold is finite (except in the special cases of quadrics, elliptic curves and K3 surfaces), so that there is no nontrivial T–action.

Let G be a finite cyclic group. In this paper, we take advantage of the fact that the (virtual) localization formula holds with no change under finite group actions. Since projective hypersurfaces X admit such actions, we can apply it to the study of KGW

[1]In K–theory, it is also known as the Euler characteristic.

theory of X. However, we still have two difficulties. First, the G–fixed moduli space is in general quite involved and we cannot guarantee that it is smooth, so even after applying the virtual localization formula, we may not be able to finish the computation. Second, the (virtual) localization formula gives an answer in a localized ring. For instance, the G–equivariant K–theory of a point is isomorphic to the representation ring $R(G)$, which in the case of a finite cyclic group of order M yields $\mathbb{Z}[X]/(1-X^M)$. Instead of providing an answer in $R(G)$, the (virtual) localization formula only gives us the image in the localized complexified ring $R(G)_{\mathbb{C},\mathrm{loc}}$, where we invert a maximal ideal corresponding to a nonzero element in G. The issue with the localized ring is that the map $R(G) \to R(G)_{\mathbb{C},\mathrm{loc}}$ is in general not injective. In our example, we have $R(G)_{\mathbb{C},\mathrm{loc}} \simeq \mathbb{C}$ and the map sends X to a given primitive M^{th} root of unity, so that the nonzero polynomial $1 + X + \cdots + X^{M-1} \in R(G)$ is sent to $0 \in \mathbb{C}$. Notice that when the group is a torus T, the localization map $R(T) \to R(T)_{\mathbb{C},\mathrm{loc}}$ is injective; that is why we have no such issue in the previous paragraph.

We overcome the first difficulty by means of an "equivariant quantum Lefschetz theorem" that we developed for GW theory in [19, Section 2] and that we adapt to KGW theory and to finite group actions in Section 1; see Theorem 1.6. It compares the G–equivariant virtual structure sheaf of a hypersurface $X \subset \mathbb{P}^N$ to that of the ambient space $\mathbb{P}^N$, and then we use the T–action on the ambient space to apply the virtual localization formula. However, Theorem 1.6 requires that for every G–fixed stable map from a curve C to X, all stable components of C are contracted to a point in X. This condition could fail if the automorphism group of the curve is too big, leading us to impose restrictions on the genus of the curve and on the degree of the stable map.

The second difficulty is more serious. Indeed, we know the G–equivariant KGW invariant is of the form $a_0 + a_1 X + \cdots + a_{M-1} X^{M-1}$ for some integers a_i, and our goal would be the "nonequivariant" limit $a_0 + \cdots + a_{M-1}$, but we only have access to the complex number $a_0 + a_1 \zeta + \cdots + a_{M-1} \zeta^{M-1}$, where ζ is a primitive M^{th} root of unity. Luckily, KGW are integers, so that when M is a prime number, we can sum all these complex numbers for primitive roots and obtain the KGW invariant modulo M.

As a conclusion, we seek automorphisms of X of prime order with isolated fixed points. For instance, the quintic threefold can be realized as the zero locus in $\mathbb{P}^4$ of the loop polynomial $x_0^4 x_1 + \cdots + x_4^4 x_0$ and the action

$$\zeta \cdot (x_0, \ldots, x_4) = (\zeta x_0, \zeta^{-4} x_1, \zeta^{16} x_2, \zeta^{-64} x_3, \zeta^{256} x_4), \quad \text{where } \zeta := e^{2i\pi/41},$$

yields an automorphism of X of prime order 41, whose fixed points are coordinate points. As a result of Corollary 2.9, we obtain[2] all KGW invariants of the quintic threefold up to genus 19 and degree 40, modulo 41.

Remark 0.1 It happens that 41 is the biggest prime number p for which there exists an automorphism of order p for a smooth quintic hypersurface; see Oguiso and Yu [30].

In Proposition 2.13, we provide an instance of this calculation in genus one. Precisely, we compute

$$\chi\left(\frac{\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_{1,0}(X,1)}}{1-q\mathbb{E}^{\vee}}\right) \equiv (120q^2 + 180q + 125)\frac{1-q^4-q^6}{(1-q^4)(1-q^6)} \in \mathbb{Z}/205\mathbb{Z}[\![q]\!],$$

where $\mathbb{E}$ denotes the Hodge bundle and q is a formal variable, so that the inverse of $1-q\mathbb{E}^{\vee}$ is defined as the geometric series in q of general term $\mathbb{E}^{-k}q^k$ (here $\mathbb{E}$ is a line bundle).

Interestingly, if we can find automorphisms of prime orders for infinitely many primes and if we can handle the respective localization formulas, then we are able to determine KGW invariants as integers instead of modulo a prime number. We apply this idea to elliptic curves. There are indeed p–torsion points for every prime number p, so that a translation by this point is an automorphism of order p. Furthermore, the localization formula is trivial since there are no fixed points. We deduce the vanishing of all KGW invariants of an elliptic curve, with homogeneous insertions.

Similarly to GW theory, Fan, Jarvis and Ruan [12; 11] developed a quantum singularity theory for Landau–Ginzburg orbifolds. It is known as the FJRW theory and an algebraic construction has been established by Polishchuk and Vaintrob in [33]. Precisely, they construct a matrix factorization over the moduli space of (W, G)–spin curves, where W is a nondegenerate quasihomogeneous polynomial and G is an admissible group of symmetries. We refer to Guéré [18] for details.

In Section 3, we explain how to construct a K–theoretic version of FJRW theory and we then pursue the same goal as for KGW theory: compute invariants by applying the localization theorem under finite group actions. We focus on the quintic polynomial

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[2]It means we write the answer as a sum over dual graphs of contributions involving only KGW invariants of the point. This computation can be encoded into a computer. Note however that the calculation of KGW invariants of the point is nontrivial in genus more than two. As explained in Proposition 2.13, we use the computer for the calculation of an explicit example in genus one.

with group μ_5 for clarity of the exposition and we find all its K–theoretic FJRW invariants with narrow insertions in every genus and modulo 205; see Corollary 3.11. Here, we do not have restriction bounds on the genus of the curve.

In [18], we compute genus-0 FJRW invariants of chain polynomials using a characteristic class[3] $\mathfrak{c}_t \colon K^0(S) \to H^*(S)[\![t]\!]$, which we can define for a line bundle L over a smooth DM stack S as

$$\mathfrak{c}_t(L) = \mathrm{Ch}(\lambda_{-t}L^{\vee})\,\mathrm{Td}(L) = c_1(L) \cdot \frac{e^{c_1(L)} - t}{e^{c_1(L)} - 1},$$

and then extend multiplicatively. Genus-0 FJRW invariants of a chain polynomial $x_1^{a_1}x_2 + \cdots + x_N^{a_N}$ are then equal to

(1) $$c_{\mathrm{vir}} = \lim_{t \to 1} \prod_{j=1}^{N} \mathfrak{c}_{t_j}(-R\pi_*\mathcal{L}_j), \quad \text{with } g = 0,$$

where $t_j := t^{(-a_1)\cdots(-a_{j-1})}$ and $R\pi_*\mathcal{L}_j$ are the derived pushforwards of the universal line bundles over the moduli space of (W, G)–spin curves. It is remarkable that such a limit exists and the author has wondered since then whether other limits could exist, for instance when t tends to some root of unity. Interestingly, we prove[4] for the quintic loop polynomial with group μ_5 and narrow insertions that such a limit exists for all genus when t tends to a 41st root of unity ζ_{41}. It then converges to a $\mathbb{Z}/41\mathbb{Z}$–equivariant version of the FJRW virtual cycle, defined as follows. The two-periodic complex obtained from the Polishchuk–Vaintrob matrix factorization naturally decouples as a direct sum of 41 two-periodic complexes.[5] Each one of them provides a (virtual) cycle a_k, for $0 \le k \le 40$, and we define

$$c_{\mathrm{vir}}^{\mathbb{Z}/41\mathbb{Z}} := \sum_{k=0}^{40} a_k \zeta_{41}^k = \lim_{t \to \zeta_{41}} \prod_{j=1}^{N} \mathfrak{c}_{t_j}(-R\pi_*\mathcal{L}_j), \quad \text{with } g \ge 0.$$

We easily find similar results for other loop polynomials.

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[3]Polishchuk and Vaintrob's definition of the virtual cycle in FJRW theory involves a Koszul complex of vector bundles; see [33]. The class $\mathfrak{c}_t$ then appears naturally from the definition of the Koszul complex, as it involves exterior powers of vector bundles. Note also that $\mathfrak{c}_1$ recovers the top Chern class of a vector bundle.

[4]The restriction to the group μ_5 is not necessary and we could consider other admissible groups. However, if one takes the maximal group μ_{1025}, then the action of $\mathbb{Z}/41\mathbb{Z}$ becomes trivial and gives no result.

[5]Precisely, the kth two-periodic complex is the one containing vector bundles $\mathrm{Sym}^{k+41\cdot l}A_1^{\vee}$, with $l \in \mathbb{N}$, in the notation of [18].

As a conclusion, we mention a future line of research. Chiodo and Ruan [9] and Chiodo, Iritani and Ruan [8] studied the so-called genus-0 Landau–Ginzburg/Calabi–Yau (LG/CY) correspondence, which provides a striking relation between GW theory of a projective hypersurface and FJRW theory of the defining polynomial. Following Chiodo and Ruan [10], there is a similar correspondence in higher genus as well. Since we expect the LG/CY correspondence to hold in K–theory as well, it would be interesting to probe a K–theoretic version for the quintic threefold, up to genus 19, degree 40, and modulo 41.

Another question we may ask is: what information do we get on GW invariants of the quintic threefold up to genus 19? The quintic threefold X is special, its virtual cycle (with no markings) is 0–dimensional, so that a lot of its GW invariants vanish. In fact, they are all deduced from some rational numbers $n_{g,d} \in \mathbb{Q}$ for nonnegative integers g and d, corresponding to its GW invariants without markings. As a consequence, we expect some simplifications in the Hirzebruch–Riemann–Roch theorem of Tonita [38] and Givental [16], and to find formulas expressing KGW invariants of X in terms of the $n_{g,d}$. Moreover, it is proven in Fan and Lee [13], Guo, Janda and Ruan [20] and Chang, Guo and Li [5] that all values of $n_{g,d}$ are expressed in terms of low degrees, where $5d \leq 2g - 2$. Up to genus 19, there are exactly 61 unknowns:

$$n_{4,1}, n_{5,1}, \ldots, n_{18,6}, n_{19,7} \in \mathbb{Q}.$$

As we are able to compute all KGW invariants modulo 41 up to genus 19 and degree 40, we expect a lot of relations among these 61 unknowns. Moreover, KGW is not restricted by a degree condition on insertions, so we can also insert K–classes from $\mathbb{P}^4$, yielding indeed infinitely many relations among these 61 unknowns. Of course, we do not know yet how many of these relations are nontrivial. It would also be enlightening to express KGW invariants in terms of BPS numbers, which are integers as well; see [22] for a formula in genus zero.

Notation In this paper, we work over the complex numbers. We denote by $G_0(X)$ the Grothendieck group of coherent sheaves on a DM stack X and by $K^0(X)$ the Grothendieck ring of vector bundles on X. If a linear algebraic group G acts on X, then we denote by $G_0(G, X)$ and $K^0(G, X)$ the Grothendieck groups of G–equivariant coherent sheaves and vector bundles. They are identified when X is smooth, by Thomason [36]. When X is a point, then it equals the representation ring $R(G)$ of the group G. The G–fixed locus inside X is denoted by X^G. For an element $h \in G$, we

denote by X^h the h–fixed locus. If V and W are G–equivariant vector bundles over X, then we denote by

$$\lambda^G_{-t}(V - W) = \sum_{k,l \geq 0} (-1)^k \Lambda^k V \otimes \mathrm{Sym}^l W \, t^{k+l} \in K^0(G, X)[\![t]\!]$$

the lambda-structure in K–theory. We extend multiplicatively the notation to any element $V \in K^0(G, X)$. When we forget the group action, we simply denote it by $\lambda_{-t}(V) \in K^0(X)[\![t]\!]$.

Let G be a diagonalizable group. The complexified representation ring $R(G)_{\mathbb{C}} := R(G) \otimes \mathbb{C}$ is identified with the coordinate ring $\mathcal{O}(G)$ of G. Hence, for every $h \in G$, there is a corresponding maximal ideal $\mathfrak{m}_h \subset R(G)_{\mathbb{C}}$. Let

$$G_0(G, X)_{\mathrm{loc}} := G_0(G, X) \otimes_{\mathbb{Z}} R(G)_{\mathbb{C},\mathfrak{m}_h},$$
$$K^0(G, X)_{\mathrm{loc}} := K^0(G, X) \otimes_{\mathbb{Z}} R(G)_{\mathbb{C},\mathfrak{m}_h}$$

denote the localizations. Assume X is smooth and let $\iota \colon X^h \subset X$ be the inclusion of the h–fixed locus. The localization theorem says

$$(2) \qquad A = \iota_! \left(\frac{\iota^* A}{\lambda^G_{-1}(N_\iota^\vee)} \right) \in K^0(G, X)_{\mathrm{loc}} \quad \text{for all } A \in K^0(G, X)_{\mathrm{loc}};$$

see Thomason [37]. Note that λ^G_{-1} is the evaluation of the formula above at $t = 1$. In general, it is not defined in $K^0(G, X)$ and it is only partially defined in $K^0(G, X)_{\mathrm{loc}}$. Precisely, for a vector bundle V, the term $\lambda^G_{-1}(V)$ is invertible if V has no G–fixed part. This is the case in equation (2).

Equation (2) is in particular true for finite groups G, even though the localization map $R(G)_{\mathbb{C}} \to R(G)_{\mathrm{loc}}$ is not injective in that case. Moreover, we can relax the smoothness condition on X. Indeed, if X is singular but carries a G–equivariant perfect obstruction theory $[E_{-1} \to E_0]$, then there is a G–equivariant virtual structure sheaf $\mathcal{O}_X^{\mathrm{vir},G} \in G_0(G, X)$; see Lee [25]. The obstruction theory pulls back to the G–fixed locus X^G and we denote by $N_\iota^{\mathrm{vir}} \in K^0(G, X^G)$ the K–theoretic class of the dual of its G–moving part. The G–fixed part gives a perfect obstruction theory on X^G and yields a virtual structure sheaf $\mathcal{O}_{X^G}^{\mathrm{vir}}$. Furthermore, we have the virtual localization formula

$$(3) \qquad \mathcal{O}_X^{\mathrm{vir},G} = \iota_! \left(\frac{\mathcal{O}_{X^G}^{\mathrm{vir}}}{\lambda^G_{-1}(N_\iota^{\mathrm{vir}\vee})} \right) \in G_0(G, X)_{\mathrm{loc}}.$$

See Qu [34, Theorem 3.3] for the proof in the case where the group is a torus T, but the same proof holds word for word when we replace T by any diagonalizable group G. In particular, it applies to the moduli space $\overline{\mathcal{M}}(\mathcal{X})$ of stable maps to a smooth DM stack $\mathcal{X}$.

Here, we specify the genus, the degree and the number of markings as $\overline{\mathcal{M}}_{g,n}(\mathcal{X}, \beta)$ when needed.

The letters GW stand for Gromov–Witten and KGW for K–theoretic Gromov–Witten.

1 Equivariant quantum Lefschetz theorem

This section is a generalization of [19, Section 2] to K–theory and to more general group actions. The main result is an "equivariant quantum Lefschetz" theorem which is of first importance in the next section.

1.1 Virtual localization formula

Let G be a linear algebraic group and $\mathcal{X}$ be a smooth DM stack equipped with a G–action. The moduli space $\overline{\mathcal{M}}(\mathcal{X})$ of stable maps to $\mathcal{X}$ carries a G–action, a G–equivariant perfect obstruction theory, and thus a G–equivariant virtual structure sheaf $\mathcal{O}^{\mathrm{vir},G}_{\overline{\mathcal{M}}(\mathcal{X})} \in G_0(G, \overline{\mathcal{M}}(\mathcal{X}))$.

Denote by $\iota\colon \overline{\mathcal{M}}(\mathcal{X})^G \hookrightarrow \overline{\mathcal{M}}(\mathcal{X})$ the embedding of the G–fixed locus. By definition, the virtual normal bundle $N_\iota^{\mathrm{vir}} \in K^0(G, \overline{\mathcal{M}}(\mathcal{X})^G)$ is the moving part of the pullback of the perfect obstruction theory to the fixed locus.[6] The virtual localization formula (3) states

$$\mathcal{O}^{\mathrm{vir},G}_{\overline{\mathcal{M}}(\mathcal{X})} = \iota_!\left(\frac{\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}(\mathcal{X})^G}}{\lambda^G_{-1}(N_\iota^{\mathrm{vir}\vee})} \right) \in G_0(G, \overline{\mathcal{M}}(\mathcal{X}))_{\mathrm{loc}}.$$

1.2 Enhancement of the group

Let $G \subset T$ be an embedding of linear algebraic groups and $\mathcal{X} \hookrightarrow \mathcal{P}$ be an embedding of smooth DM stacks equipped with a G–action. We assume that

- the G–fixed loci of $\mathcal{X}$ and of $\mathcal{P}$ are equal;
- for every G–fixed stable map to $\mathcal{P}$, all stable components of the source curve are sent to $\mathcal{P}^G$;
- $\mathcal{P}$ is equipped with a T–action extending the G–action;
- the normal bundle of $\mathcal{X} \hookrightarrow \mathcal{P}$ is the pullback of a T–equivariant vector bundle $\mathcal{N}$ over $\mathcal{P}$;
- $\mathcal{X}$ is the zero locus of a G–invariant section of the vector bundle $\mathcal{N}$; and

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[6]The perfect obstruction theory admits a global presentation by a two-term complex of vector bundles, hence the virtual normal bundle is an element in K^0 rather than an element in G_0.

- the vector bundle $\mathcal{N}$ is convex up to two markings, ie for every stable map $f \colon \mathcal{C} \to \mathcal{P}$, where $\mathcal{C}$ is a smooth genus-0 orbifold curve with at most two markings, we have $H^1(\mathcal{C}, f^*\mathcal{N}) = 0$.

Let us first consider the G–fixed loci of the moduli spaces of stable maps and observe the fibered diagram

$$
\begin{array}{ccc}
\overline{\mathcal{M}}(\mathcal{X})^G & \xrightarrow{\ j\ } & \overline{\mathcal{M}}(\mathcal{P})^G \\
{\scriptstyle \iota}\big\downarrow & \square & \big\downarrow{\scriptstyle \tilde{\tau}} \\
\overline{\mathcal{M}}(\mathcal{X}) & \xrightarrow[\ \tilde{j}\ ]{} & \overline{\mathcal{M}}(\mathcal{P})
\end{array}
$$

Writing $i \colon \mathcal{X} \hookrightarrow \mathcal{P}$, we have a G–equivariant short exact sequence

$$
0 \to T_{\mathcal{X}} \to i^* T_{\mathcal{P}} \to i^* \mathcal{N} \to 0
$$

inducing a distinguished triangle

(4) $$R\pi_* f^* T_{\mathcal{X}} \to R\pi_* f^* T_{\mathcal{P}} \to R\pi_* f^* \mathcal{N} \to (R\pi_* f^* T_{\mathcal{X}})[1].$$

Note that their duals are parts of the perfect obstruction theories of $\overline{\mathcal{M}}(\mathcal{X})$ and of $\overline{\mathcal{M}}(\mathcal{P})$, the remaining parts being the perfect obstruction theory of the moduli space of stable curves itself.

The term $\mathcal{E} := R\pi_* f^* \mathcal{N}$, pulled back to $\overline{\mathcal{M}}(\mathcal{P})^G$, has a fixed and a moving part, that we denote respectively by $\mathcal{E}_{\mathrm{fix}}$ and $\mathcal{E}_{\mathrm{mov}}$.

Proposition 1.1 *The fixed part $\mathcal{E}_{\mathrm{fix}}$ is a vector bundle over the fixed moduli space $\overline{\mathcal{M}}(\mathcal{P})^G$.*

Proof Let $f \colon \mathcal{C} \to \mathcal{P}$ be a stable map belonging to $\overline{\mathcal{M}}(\mathcal{P})^G$. We denote by $\rho \colon \mathcal{C} \to C$ the coarse map. It is enough to prove that

$$
H^1(C, \rho_* f^* \mathcal{N})^{\mathrm{fix}} = 0.
$$

Take the normalization $\nu \colon C^\nu \to C$ of the curves at all their nodes. We have

$$
C^\nu = \bigsqcup_{i \in I} C_i^{\mathrm{fix}} \sqcup \bigsqcup_{j \in J} C_j^{\mathrm{nf}},
$$

where the superscripts refer respectively to fixed/nonfixed components of C^ν under the map f. In particular, nonfixed components are unstable curves, ie the projective

line with one or two special points. By the normalization exact sequence, we obtain an exact sequence

$$\bigoplus_{\text{nodes}} H^0(\text{node}, f^*\mathcal{N}_{|\text{node}}) \to H^1(C, \rho_* f^*\mathcal{N}) \to H^1(C^\nu, \nu^* \rho_* f^*\mathcal{N}) \to 0,$$

with

$$H^1(C^\nu, \nu^* \rho_* f^*\mathcal{N}) = \bigoplus_{i \in I} H^1(C_i^{\text{fix}}, \nu^* \rho_* f^*\mathcal{N}) \oplus \bigoplus_{j \in J} H^1(C_j^{\text{nf}}, \nu^* \rho_* f^*\mathcal{N}).$$

Since the normal bundle has a nontrivial G–action once restricted to the fixed locus of $\mathcal{X}$ (or equivalently of $\mathcal{P}$), we have

$$H^0(\text{node}, f^*\mathcal{N}_{|\text{node}})^{\text{fix}} = 0 \quad \text{and} \quad H^1(C_i^{\text{fix}}, \nu^* \rho_* f^*\mathcal{N})^{\text{fix}} = 0.$$

Therefore, it remains to see the vanishing of H^1 for nonfixed unstable curves C_j^{nf}, for $j \in J$. The curve C_j^{nf} is isomorphic to $\mathbb{P}^1$ with either one or two markings, hence $H^1(C_j^{\text{nf}}, \nu^* \rho_* f^*\mathcal{N}) = 0$ by our assumption of convexity up to two markings. $\qquad\square$

Denote by $\mathcal{O}_{\overline{\mathcal{M}}(\mathcal{P})^G}^{\text{vir}}$ the virtual structure sheaf obtained by the G–fixed part of the perfect obstruction theory $R\pi_* f^* T_\mathcal{P}$.

Proposition 1.2 *We have*

$$j_! \mathcal{O}_{\overline{\mathcal{M}}(\mathcal{X})^G}^{\text{vir}} = \lambda_{-1}^G(\mathcal{E}_{\text{fix}}^\vee) \otimes \mathcal{O}_{\overline{\mathcal{M}}(\mathcal{P})^G}^{\text{vir}} \in G_0(\overline{\mathcal{M}}(\mathcal{P})^G).$$

Furthermore, in the localized equivariant K–theoretic ring, we have

$$\frac{1}{\lambda_{-1}^G(N_\iota^{\text{vir}\vee})} = j^* \left(\frac{\lambda_{-1}^G(\mathcal{E}_{\text{mov}}^\vee)}{\lambda_{-1}^G(N_{\tilde{\iota}}^{\text{vir}\vee})} \right) \in K^0(G, \overline{\mathcal{M}}(\mathcal{P})^G)_{\text{loc}}.$$

Proof It follows from the standard proof using convexity; we recall here the main arguments.

The DM stack $\mathcal{X}$ is the zero locus of a G–invariant section of the vector bundle $\mathcal{N}$ over the ambient space $\mathcal{P}$. This section induces a map s from the moduli space of stable maps to $\mathcal{P}$ to the direct image cone $\pi_* f^*\mathcal{N}$; see [6, Definition 2.1]. Since the moduli space $\overline{\mathcal{M}}(\mathcal{P})^G$ is fixed by the action of G, it maps to the fixed part of the direct image cone, that is, the vector bundle $\mathcal{E}_{\text{fix}}$. Hence we have the fibered diagram

$$
\begin{array}{ccc}
\overline{\mathcal{M}}(\mathcal{X})^G & \xrightarrow{\quad j \quad} & \overline{\mathcal{M}}(\mathcal{P})^G \\
\downarrow & \square & \downarrow{\scriptstyle s} \\
\overline{\mathcal{M}}(\mathcal{P})^G & \xrightarrow[\quad 0 \quad]{} & \mathcal{E}_{\text{fix}}
\end{array}
$$

where the bottom map is the embedding as the zero section. The fixed part of the distinguished triangle (4) gives a compatibility datum of perfect obstruction theories for the fixed moduli spaces. Functoriality of the virtual structure sheaf gives

$$0^! \mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}(\mathcal{P})^G} = \mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}(\mathcal{X})^G};$$

see [34]. Applying the projection formula via the map j on both sides and using the Koszul resolution gives the first result. The second part of the statement follows from the moving part of the distinguished triangle (4). □

By the virtual localization formula, the G–equivariant virtual structure sheaf satisfies

$$\widetilde{j}_! \mathcal{O}^{\mathrm{vir},G}_{\overline{\mathcal{M}}(\mathcal{X})} = \widetilde{j}_! \iota_! \left(\frac{\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}(\mathcal{X})^G}}{\lambda^G_{-1}(N_\iota^{\mathrm{vir}\vee})} \right) = \widetilde{\iota}_! j_! \left(\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}(\mathcal{X})^G} \otimes j^* \left(\frac{\lambda^G_{-1}(\mathcal{E}_{\mathrm{mov}}{}^\vee)}{\lambda^G_{-1}(N_{\widetilde{\iota}}^{\mathrm{vir}\vee})} \right) \right)$$

$$= \widetilde{\iota}_! \left(\lambda^G_{-1}(\mathcal{E}_{\mathrm{fix}}{}^\vee) \otimes \mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}(\mathcal{P})^G} \otimes \frac{\lambda^G_{-1}(\mathcal{E}_{\mathrm{mov}}{}^\vee)}{\lambda^G_{-1}(N_{\widetilde{\iota}}^{\mathrm{vir}\vee})} \right)$$

$$= \widetilde{\iota}_! \left(\lambda^G_{-1}(\mathcal{E}^\vee) \otimes \frac{\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}(\mathcal{P})^G}}{\lambda^G_{-1}(N_{\widetilde{\iota}}^{\mathrm{vir}\vee})} \right),$$

where equalities happen in $G_0(G, \overline{\mathcal{M}}(\mathcal{P}))_{\mathrm{loc}}$.

Remark 1.3 If it were defined, the right-hand side would equal

$$\lambda^G_{-1}(R\pi_* f^* \mathcal{N})^\vee \otimes \mathcal{O}^{\mathrm{vir},G}_{\overline{\mathcal{M}}(\mathcal{P})},$$

using the virtual localization formula, but it is not clear that the G–lambda class of $R\pi_* f^* \mathcal{N}$ is defined in $G_0(G, \overline{\mathcal{M}}(\mathcal{P}))_{\mathrm{loc}}$. However, we say that $\lambda^G_{-1}(R\pi_* f^* \mathcal{N})^\vee$ is defined after localization[7] to mean that its pullback to the fixed locus is defined.

Now, we aim to extend the right-hand side of the equality to the T action. The inclusion of groups $G \hookrightarrow T$ yields a morphism

$$\xi_* : G_0(T, \overline{\mathcal{M}}(\mathcal{P})) \to G_0(G, \overline{\mathcal{M}}(\mathcal{P})),$$

under which we get

$$\xi_*(\mathcal{O}^{\mathrm{vir},T}_{\overline{\mathcal{M}}(\mathcal{P})}) = \mathcal{O}^{\mathrm{vir},G}_{\overline{\mathcal{M}}(\mathcal{P})}.$$

Unfortunately, the map ξ_* is only partially defined when we localize equivariant parameters: the denominators could be nonzero in the T–localization but vanish in the G–localization. It is easier to work out this issue on the fixed locus of the moduli space.

---

[7] We find this definition for the formal quintic; see [27].

Let $\overline{\mathcal{M}}(\mathcal{P})^T \hookrightarrow \overline{\mathcal{M}}(\mathcal{P})$ denote the T–fixed locus of the moduli space. In particular, we have the inclusion $\hat{\imath} \colon \overline{\mathcal{M}}(\mathcal{P})^T \hookrightarrow \overline{\mathcal{M}}(\mathcal{P})^G$. We notice that the moduli space $\overline{\mathcal{M}}(\mathcal{P})^G$ is stable under the T–action from $\overline{\mathcal{M}}(\mathcal{P})$ and that the map $\hat{\imath}$ is T–equivariant. Moreover, we have a T–equivariant virtual structure sheaf

$$\mathcal{O}^{\mathrm{vir},T}_{\overline{\mathcal{M}}(\mathcal{P})^G} \in G_0(T, \overline{\mathcal{M}}(\mathcal{P})^G)$$

and the equality

$$\xi_*(\mathcal{O}^{\mathrm{vir},T}_{\overline{\mathcal{M}}(\mathcal{P})^G}) = \mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}(\mathcal{P})^G} \in G_0(G, \overline{\mathcal{M}}(\mathcal{P})^G).$$

By the virtual localization formula, we have

$$\mathcal{O}^{\mathrm{vir},T}_{\overline{\mathcal{M}}(\mathcal{P})^G} = \hat{\imath}_!\left(\frac{\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}(\mathcal{P})^T}}{\lambda^T_{-1}(N^{\mathrm{vir}\vee}_{\hat{\imath}})} \right) \in G_0(T, \overline{\mathcal{M}}(\mathcal{P})^G)_{\mathrm{loc}}.$$

Furthermore, in K–theory on the space $\overline{\mathcal{M}}(\mathcal{P})^T$ we have the equality

$$(5) \qquad\qquad N^{\mathrm{vir}}_{\hat{\imath}\circ\hat{\imath}} = \hat{\imath}^* N^{\mathrm{vir}}_{\hat{\imath}} + N^{\mathrm{vir}}_{\hat{\imath}}.$$

Indeed, let $\mathcal{F}$ be the pullback of the perfect obstruction theory from $\overline{\mathcal{M}}(\mathcal{P})$ to $\overline{\mathcal{M}}(\mathcal{P})^T$. By definition, the virtual normal bundle $N^{\mathrm{vir}}_{\hat{\imath}\circ\hat{\imath}}$ is the T–moving part $\mathcal{F}^{\mathrm{mov}}$, which decomposes as $\mathcal{F}^{\mathrm{mov}} = \mathcal{F}^{\mathrm{mov}}_{\mathrm{fix}} + \mathcal{F}^{\mathrm{mov}}_{\mathrm{mov}}$, where the subscript denotes the G–fixed/moving part. By definition, the virtual normal bundle $\hat{\imath}^* N^{\mathrm{vir}}_{\hat{\imath}}$ is the G–moving part of $\mathcal{F}$, ie $\mathcal{F}^{\mathrm{mov}}_{\mathrm{mov}}$, since there is no G–moving T–fixed part in $\mathcal{F}$. The virtual normal bundle $N^{\mathrm{vir}}_{\hat{\imath}}$ identifies with $\mathcal{F}^{\mathrm{mov}}_{\mathrm{fix}}$.

Remark 1.4 The virtual normal bundle $N^{\mathrm{vir}}_{\hat{\imath}}$ is defined on $\overline{\mathcal{M}}(\mathcal{P})^G$ and we have a well-defined equality

$$\xi_*(\lambda^T_{-1}(N^{\mathrm{vir}\vee}_{\hat{\imath}})^{-1}) = \lambda^G_{-1}(N^{\mathrm{vir}\vee}_{\hat{\imath}})^{-1} \in K^0(G, \overline{\mathcal{M}}(\mathcal{P})^G)_{\mathrm{loc}}.$$

We also have seen the G–decomposition $\mathcal{E} = \mathcal{E}_{\mathrm{fix}} + \mathcal{E}_{\mathrm{mov}}$ over $\overline{\mathcal{M}}(\mathcal{P})^G$ with $\mathcal{E}_{\mathrm{fix}}$ being a T–equivariant vector bundle. Indeed, the vector bundle $\mathcal{N}$ over $\mathcal{P}$ is T–equivariant, thus so are $\mathcal{E}$ and $\mathcal{E}_{\mathrm{fix}}$. As a consequence, the equality

$$\xi_*(\lambda^T_{-1}(\mathcal{E})) = \lambda^G_{-1}(\mathcal{E}) \in K^0(G, \overline{\mathcal{M}}(\mathcal{P})^G)_{\mathrm{loc}}$$

is well defined.

Proposition 1.5 *Consider the well-defined class*

$$C_T := \hat{\imath}^*(\lambda^T_{-1}(\mathcal{E}^\vee)) \otimes \frac{\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}(\mathcal{P})^T}}{\lambda^T_{-1}(N^{\mathrm{vir}\vee}_{\hat{\imath}\circ\hat{\imath}})} \in G_0(T, \overline{\mathcal{M}}(\mathcal{P})^T)_{\mathrm{loc}}.$$

Then its pushforward under the inclusion $\hat{\iota}$ equals

$$\hat{\iota}_!(C_T) = \lambda_{-1}^T(\mathcal{E}^\vee) \otimes \frac{\mathcal{O}_{\overline{\mathcal{M}}(\mathcal{P})^G}^{\mathrm{vir},T}}{\lambda_{-1}^T(N_{\tilde{\iota}}^{\mathrm{vir}\vee})} \in G_0(T, \overline{\mathcal{M}}(\mathcal{P})^G)_{\mathrm{loc}}.$$

In particular, we have

$$\xi_*(\hat{\iota}_!(C_T)) = \lambda_{-1}^G(\mathcal{E}^\vee) \otimes \frac{\mathcal{O}_{\overline{\mathcal{M}}(\mathcal{P})^G}^{\mathrm{vir}}}{\lambda_{-1}^G(N_{\tilde{\iota}}^{\mathrm{vir}\vee})} \in G_0(G, \overline{\mathcal{M}}(\mathcal{P})^G)_{\mathrm{loc}}.$$

Proof By the virtual localization above and equation (5), we have

$$\hat{\iota}_!(C_T) = \lambda_{-1}^T(\mathcal{E}^\vee) \otimes \hat{\iota}_! \left(\frac{\mathcal{O}_{\overline{\mathcal{M}}(\mathcal{P})^T}^{\mathrm{vir}}}{\lambda_{-1}^T(N_{\tilde{\iota}\circ\hat{\iota}}^{\mathrm{vir}\vee})} \right)$$

$$= \lambda_{-1}^T(\mathcal{E}^\vee) \otimes \hat{\iota}_! \left(\frac{\mathcal{O}_{\overline{\mathcal{M}}(\mathcal{P})^T}^{\mathrm{vir}}}{\hat{\iota}^*(\lambda_{-1}^T(N_{\tilde{\iota}}^{\mathrm{vir}\vee})) \otimes \lambda_{-1}^T(N_{\tilde{\iota}}^{\mathrm{vir}\vee})} \right)$$

$$= \frac{\lambda_{-1}^T(\mathcal{E}^\vee)}{\lambda_{-1}^T(N_{\tilde{\iota}}^{\mathrm{vir}\vee})} \otimes \hat{\iota}_! \left(\frac{\mathcal{O}_{\overline{\mathcal{M}}(\mathcal{P})^T}^{\mathrm{vir}}}{\lambda_{-1}^T(N_{\tilde{\iota}}^{\mathrm{vir}\vee})} \right) = \frac{\lambda_{-1}^T(\mathcal{E}^\vee) \otimes \mathcal{O}_{\overline{\mathcal{M}}(\mathcal{P})^G}^{\mathrm{vir},T}}{\lambda_{-1}^T(N_{\tilde{\iota}}^{\mathrm{vir}\vee})}.$$

The last sentence follows from the following property of ξ_*. Let Z be a DM stack with a T–action and take $A \in K^0(T, Z)_{\mathrm{loc}}$, $B \in G_0(T, Z)_{\mathrm{loc}}$, $a \in K^0(G, Z)_{\mathrm{loc}}$ and $b \in G_0(G, Z)_{\mathrm{loc}}$. If $\xi_*(A) = a$ and $\xi_*(B) = b$ are well-defined equalities, then $\xi_*(A \otimes B)$ is well defined and equals the localized class $a \otimes b$. □

The pushforward maps $\tilde{\iota}_!$ and ξ_* commute when the latter is well defined. Precisely, the map $\tilde{\iota}$ is T–equivariant and for any localized class $C \in G_0(T, \overline{\mathcal{M}}(\mathcal{P})^G)_{\mathrm{loc}}$ such that $\xi_*(C)$ is well defined in $G_0(G, \overline{\mathcal{M}}(\mathcal{P})^G)_{\mathrm{loc}}$, the localized class $\tilde{\iota}_!(C)$ is well defined under ξ_* and we have

$$\tilde{\iota}_!\xi_*(C) = \xi_*\tilde{\iota}_!(C) \in G_0(G, \overline{\mathcal{M}}(\mathcal{P}))_{\mathrm{loc}}.$$

1.3 Equivariant quantum Lefschetz formula

Summarizing our discussion, we obtain the following.

Theorem 1.6 (equivariant quantum Lefschetz) *Let* $\mathcal{X} \hookrightarrow \mathcal{P}$ *be a* G–*equivariant embedding of smooth DM stacks satisfying assumptions listed at the beginning of this section. Then we have*

$$\tilde{j}_!\mathcal{O}_{\overline{\mathcal{M}}(\mathcal{X})}^{\mathrm{vir},G} = \xi_*\big(\lambda_{-1}^T(R\pi_* f^*\mathcal{N})^\vee \otimes \mathcal{O}_{\overline{\mathcal{M}}(\mathcal{P})}^{\mathrm{vir},T}\big) \in G_0(G, \overline{\mathcal{M}}(\mathcal{P}))_{\mathrm{loc}},$$

where $\widetilde{j}$ is the embedding of moduli spaces and ξ_* is the specialization of T–equivariant parameters into G–equivariant parameters. Here, the T–equivariant lambda class $\lambda_{-1}^T(R\pi_* f^*\mathcal{N})^\vee$ is defined after localization; see Remark 1.3.

Proof Using previous equalities, we get

$$\widetilde{j}_! \mathcal{O}_{\overline{\mathcal{M}}(\mathcal{X})}^{\mathrm{vir},G} = \widetilde{\iota}_! \left(\lambda_{-1}^G(\mathcal{E}^\vee) \otimes \frac{\mathcal{O}_{\overline{\mathcal{M}}(\mathcal{P})^G}^{\mathrm{vir}}}{\lambda_{-1}^G(N_{\widetilde{\iota}}^{\mathrm{vir}\vee})} \right) = \xi_* \widetilde{\iota}_! \widehat{\iota}_! \left(\widehat{\iota}^*(\lambda_{-1}^T(\mathcal{E}^\vee)) \otimes \frac{\mathcal{O}_{\overline{\mathcal{M}}(\mathcal{P})^T}^{\mathrm{vir}}}{\lambda_{-1}^T(N_{\widetilde{\iota}\circ\widehat{\iota}}^{\mathrm{vir}\vee})} \right).$$

Following Remark 1.3, the meaning of "defined after localization" is precisely

$$\xi_* \left(\lambda_{-1}^T(R\pi_* f^*\mathcal{N})^\vee \otimes \mathcal{O}_{\overline{\mathcal{M}}(\mathcal{P})}^{\mathrm{vir},T} \right) = \xi_* \left(\lambda_{-1}^T(R\pi_* f^*\mathcal{N})^\vee \otimes \widetilde{\iota}_! \widehat{\iota}_! \left(\frac{\mathcal{O}_{\overline{\mathcal{M}}(\mathcal{P})^T}^{\mathrm{vir}}}{\lambda_{-1}^T(N_{\widetilde{\iota}\circ\widehat{\iota}}^{\mathrm{vir}\vee})} \right) \right)$$

$$= \xi_* \widetilde{\iota}_! \widehat{\iota}_! \left(\widehat{\iota}^*(\lambda_{-1}^T(\mathcal{E}^\vee)) \otimes \frac{\mathcal{O}_{\overline{\mathcal{M}}(\mathcal{P})^T}^{\mathrm{vir}}}{\lambda_{-1}^T(N_{\widetilde{\iota}\circ\widehat{\iota}}^{\mathrm{vir}\vee})} \right). \qquad \square$$

2 K–theoretic Gromov–Witten theory

2.1 Automorphisms of loop hypersurfaces

Let X be a smooth degree-d hypersurface in $\mathbb{P}^N$. K–theoretic Gromov–Witten (KGW) theory is invariant under smooth deformations, so that we can choose any degree-d homogeneous polynomial P to define X, as long as it satisfies the Jacobian criterion for smoothness. Here, we will focus on loop polynomials, ie we take

$$X := \{x_0^{d-1}x_1 + \cdots + x_N^{d-1}x_0 = 0\} \subset \mathbb{P}^N.$$

Let $\overline{M} := |1 - (1-d)^{N+1}|/d$ and consider on $\mathbb{P}^N$ the $\mathbb{Z}/\overline{M}\mathbb{Z}$–action

$$\zeta \cdot (x_0, \ldots, x_N) = (x_0, \zeta x_1, \zeta^{u_2} x_2, \ldots, \zeta^{u_N} x_N),$$

where $u_0 := 0$ and $u_{j+1} := 1 - (d-1)u_j$. We have $(d-1)u_N \equiv 1$ modulo $\overline{M}$, so that the hypersurface X is $\mathbb{Z}/\overline{M}\mathbb{Z}$–invariant. Explicitly, we have

$$u_j = \sum_{l=0}^{j-1} (1-d)^l \quad \text{and} \quad \overline{M} = \sum_{l=0}^{N} (1-d)^l.$$

Notation 2.1 Let $\overline{M}_1 := \overline{M}$ and

$$\overline{M}_{j+1} := \frac{\overline{M}_j}{\gcd(u_{j+1}, \overline{M}_j)} \quad \text{for } 1 \le j < N.$$

We write $M := \overline{M}_N$ and $G := \mathbb{Z}/M\mathbb{Z}$.

Proposition 2.2 *The group G acts on $\mathbb{P}^N$, leaves the hypersurface X invariant, and for all nonzero $g \in G$, the g–fixed locus in $\mathbb{P}^N$ consists of all coordinate points. Furthermore, assuming the Calabi–Yau condition $N + 1 = d$ and assuming d is a prime number, then we have $M = \overline{M}$. It holds in particular for the quintic hypersurface in $\mathbb{P}^4$.*

Proof By the construction of M, we see that every u_j with $1 \leq j \leq N$ is coprime with M. It implies that every pairwise difference $u_i - u_j$ is coprime with M. Indeed, let $0 \leq i < j \leq N$, then we have

$$u_j - u_i = (1 - d)^i u_{j-i},$$

so that it is enough to prove that $d - 1$ is coprime with M. If a nonzero integer p divides M and $d - 1$, then it divides $\overline{M}$, and from its expression in terms of powers of $d - 1$, we get $1 \equiv 0$ modulo p, so that $p = 1$.

Therefore, we have, for all $0 \leq i < j \leq N$ and all $0 < k < M$,

$$\zeta^{ku_i} \neq \zeta^{ku_j}, \quad \text{where } \zeta = e^{2i\pi/M},$$

and hence the statement about the fixed locus.

For the second statement, let $d = N + 1$ be a prime number. Then we have $\overline{M} \equiv 0$ modulo d. Thus, if we have $ku_j \equiv 0$ modulo $\overline{M}$, then we have $ku_j \equiv 0$ modulo d. But $u_j \equiv j$ modulo d, so that $k = 0$. As a consequence, every u_j is coprime with $\overline{M}$, and $\overline{M} = M$. □

Example 2.3 We realize the quintic hypersurface in $\mathbb{P}^4$ as

$$P = x_0^4 x_2 + \cdots + x_4^4 x_0.$$

Then the group is $G = (\mathbb{Z}/205\mathbb{Z})$, acting as

$$\zeta \cdot \underline{x} = (x_0, \zeta x_1, \zeta^{-3} x_2, \zeta^{13} x_3, \zeta^{-51} x_4).$$

2.2 Virtual localization formula

Gromov–Witten (GW) theory of $\mathbb{P}^N$ and its K–theoretic version is computed by the virtual localization formula under the natural action of the torus $T = (\mathbb{C}^*)^N$. Unfortunately, there are in general no nontrivial torus-actions preserving a smooth degree-d hypersurface X, leading to many difficulties in the computation of its GW and KGW invariants. Nevertheless, we have an action of the finite group G on X.

In cohomology or in Chow theory, the action of the finite group G is useless with respect to the localization formula. Indeed, we have, for example, $A_*^G(\text{pt}) = \mathbb{C}$. On the other hand, we have $K^0(G, \text{pt}) = R(G)$, the representation ring of the group G. Moreover, there exists in K–theory a (virtual) localization formula under finite group actions.

Unfortunately, the (virtual) localization formula does not give a result in $K^0(G, \text{pt})$, but in a localized ring where we invert equivariant parameters. For instance, in the case of an abelian group $G = \mathbb{Z}/M\mathbb{Z}$, the representation ring (taken with complex coefficients) is

$$R(G)_\mathbb{C} \simeq \mathbb{C}[X]/(1 - X^M),$$

and the multiplicative set we use for localization is generated by

$$\{1 - X, \dots, 1 - X^{M-1}\}.$$

As a consequence, the localized ring is isomorphic to $\mathbb{C}$ and the map $R(G)_\mathbb{C} \to R(G)_{\mathbb{C},\text{loc}}$ is not injective. Precisely, the map sends X to a primitive M^{th} root of unity ζ, so that for every prime divisor p of M, the polynomial

$$\sum_{k=0}^{p} X^{kM/p} \mapsto \sum_{k=0}^{p} \zeta^{kM/p} = 0.$$

In conclusion, the (virtual) localization formula successfully computes a G–equivariant K–class expressed using roots of unity, but we cannot extract the "nonequivariant" limit corresponding to the map

$$K^0(G, \text{pt}) \simeq \mathbb{C}[X]/(1 - X^M) \to \mathbb{C} \simeq K^0(\text{pt}), \quad X \mapsto 1.$$

Nevertheless, we find a way to extract some information. Indeed, K–theoretic invariants have another important feature: they are integers. Therefore, when the order of the group is a prime number p, the defect of injectivity of the map $R(G) \to R(G)_{\mathbb{C},\text{loc}}$ amounts to the uncertainty

$$1 + X + \cdots + X^{p-1},$$

which equals p in the nonequivariant limit $X \mapsto 1$. To conclude, we are left with the desired integer modulo p. Furthermore, if we have several finite actions of different prime orders, we can increase our knowledge about the result.

Let us go back to the degree-d hypersurface $X \subset \mathbb{P}^N$. The action of $G = \mathbb{Z}/M\mathbb{Z}$ on $\mathbb{P}^N$ leaving X invariant induces a G–action on the moduli spaces of stable maps to $\mathbb{P}^N$ and to X, so that their virtual structure sheaves are G–equivariant, namely

$$\mathcal{O}_{\overline{\mathcal{M}}_g(\mathbb{P}^N, \beta)}^{\text{vir}} \in G_0(G, \overline{\mathcal{M}}_g(\mathbb{P}^N, \beta)) \quad \text{and} \quad \mathcal{O}_{\overline{\mathcal{M}}_g(X, \beta)}^{\text{vir}} \in G_0(G, \overline{\mathcal{M}}_g(X, \beta)).$$

By the virtual localization formula, we then obtain

$$
\mathcal{O}^{\text{vir}}_{\overline{\mathcal{M}}_g(X,\beta)} = \iota_! \left(\frac{\mathcal{O}^{\text{vir}}_{\overline{\mathcal{M}}_g(X,\beta)^{\text{fix}}}}{\lambda_{-1}(N_\iota^{\text{vir}\vee})} \right) \in G_0(G, \overline{\mathcal{M}}_g(X,\beta))_{\text{loc}},
$$

where $\iota \colon \overline{\mathcal{M}}_g(X,\beta)^{\text{fix}} \hookrightarrow \overline{\mathcal{M}}_g(X,\beta)$ denotes the G–fixed locus and N_ι^{vir} denotes the moving part of the perfect obstruction theory on the fixed locus. At last, we get

$$
\chi(\mathcal{O}^{\text{vir}}_{\overline{\mathcal{M}}_g(X,\beta)}) = \chi \left(\frac{\mathcal{O}^{\text{vir}}_{\overline{\mathcal{M}}_g(X,\beta)^{\text{fix}}}}{\lambda_{-1}(N_\iota^{\text{vir}\vee})} \right) \in \mathbb{C}.
$$

The next step is to use Theorem 1.6 to relate this formula to a formula for $\mathbb{P}^N$, where an explicit localization formula is available via the torus action.

2.3 Fixed locus

We easily check all the conditions listed in Section 1.2 but the second:

- every stable component of a fixed stable map is contracted.

We are able to prove it under the following restrictions on genus of the source curve and degree of the map.

Proposition 2.4 *Let $G = \mathbb{Z}/M\mathbb{Z}$ act on a smooth projective variety X so that, for every nonzero element $h \in G$, the h–fixed locus X^h consists of isolated points in X. Let $f \colon C \to X$ be a stable map corresponding to a G–fixed point in the moduli space $\overline{\mathcal{M}}_{g,n}(X,\beta)$. We assume*

$$
(6) \qquad\qquad g < \tfrac{1}{2}(p-1) \quad \text{and} \quad \beta < M,
$$

where p is the greatest prime divisor of M. Then every stable component of the curve C is mapped to one of the G–fixed points in X.

Proof First, we claim that if $f \colon C \to X$ is a G–fixed stable map of positive degree, then the group G is a subgroup of the group $\text{Aut}(C)$ of automorphisms of C. Indeed, let $\zeta \in G$ be a primitive element. Since the stable map is fixed, we can choose an automorphism $\phi_1 \in \text{Aut}(C)$ of the curve C such that

$$
\zeta \cdot f(x) = f(\phi_1(x)) \quad \text{for all } x \in C.
$$

We then define $\phi_k := \phi_1^k$ for any $k \in \mathbb{N}$. Since the degree of the map f is positive and all but a finite number of points in X are not fixed by any element of G, we can choose a point $x \in C$ such that the points

$$
\zeta^k \cdot f(x) = f(\phi_k(x))
$$

are all distinct for $0 \leq k < M$. Since the automorphism ϕ_M is an automorphism of the stable map f, it is of finite order. Thus we can consider the smallest integer $K \in \mathbb{N}^*$ such that $\phi_K = \mathrm{id}$, so that we have

$$\zeta^K \cdot f(x) = f(\phi_K(x)) = f(x),$$

and the integer M divides the integer K. As a consequence, the map sending $0 \leq k < M$ to $\phi_{K/M}^k$ embeds the group G in $\mathrm{Aut}(C)$.

Secondly, let us assume C is a stable curve and is not contracted. Since G is a subgroup of $\mathrm{Aut}(C)$, the prime number p divides the order of $\mathrm{Aut}(C)$. By [29, Proposition 3.6], we get[8] $p \leq 2g + 1$, which is a contradiction.

Lastly, we consider the case where C is not a stable curve (and therefore the degree of the map is positive). Let Γ_f be a dual graph representing the stable map f, where we represent every stable component of C by a vertex and every unstable component by an edge. Furthermore, we add on the graph labels to keep track of the genus, number of markings and degree. It is clear that every automorphism ϕ_k of the curve C induces an automorphism of the dual graph Γ_f. Moreover, for each stable component D of C whose corresponding vertex is fixed in Γ_C, the restriction $f_{|D}\colon D \to X$ is fixed by the group G.

We aim to show that the set $V_{>0}$ of vertices with positive degree is empty. Assume it is not. Then, if the group G acts on $V_{>0}$ without fixed points, the total degree of the map is at least M, which is a contradiction. Therefore, there is at least one fixed point, ie there exists a stable component D of C such that the restriction $f_{|D}$ is G–fixed. As we have seen above, the stable component D is then contracted to a point, which contradicts the fact that its corresponding vertex is in $V_{>0}$. $\qquad\square$

Remark 2.5 Proposition 2.4 also holds if the condition $g < \frac{1}{2}(p-1)$ is replaced by "for every stable curve of genus less than g, there is no automorphism of order equal to M".

2.4 Equivariant and congruent formulas

Let us apply Theorem 1.6 to our situation.

---

[8]The reference is only for $g \geq 2$. Nevertheless, it also holds for genus-0 stable curves, as their automorphism groups are all trivial. For genus-1 stable curves, the automorphism group is the largest when there is only one marking. In this case, we have three possibilities: $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/6\mathbb{Z}$. As a consequence, the prime number p can only be 2 or 3 and is indeed less than $2g + 1 = 3$.

Theorem 2.6 Let g, n and β be nonnegative integers. Let X be a degree-d loop hypersurface in $\mathbb{P}^N$ and take a subgroup $H \subset \mathbb{Z}/M\mathbb{Z}$ of order q acting on X via the action

$$(7) \quad k \cdot (x_0, \ldots, x_N) = (x_0, \zeta^k x_1, \zeta^{k \cdot u_2} x_2, \ldots, \zeta^{k \cdot u_N} x_N) \quad \text{for } k \in H \subset \mathbb{Z}/M\mathbb{Z};$$

see Section 2.1. This action depends on the choice of a primitive q^{th} root of unity ζ. Moreover, we have the usual $T := (\mathbb{C}^*)^N$–action on $\mathbb{P}^N$ and we see that it extends the H–action via the embedding $\varphi \colon H \hookrightarrow T := (\mathbb{C}^*)^N$ sending k to $(\zeta^k, \zeta^{k \cdot u_2}, \ldots, \zeta^{k \cdot u_N})$.

Assume the bounds

$$g < \tfrac{1}{2}(p-1) \quad \text{and} \quad \beta < q,$$

where p is the greatest prime divisor of q. Let $A := \bigotimes_{i=1}^n \Psi_i^{a_i} \otimes \mathrm{ev}^*(Y_i)$ denote some insertions of Psi-classes and K–classes $Y_i \in K^0(T, \mathbb{P}^N)$ coming from the ambient space. Then the corresponding H–equivariant K–theoretic GW invariant equals

$$\chi^T\big(\lambda_{-1}^T(R\pi_* f^* \mathcal{O}(d))^\vee \otimes A \otimes \mathcal{O}_{\overline{\mathcal{M}}_{g,n}(\mathbb{P}^N, \beta)}^{\mathrm{vir}, T}\big) = \chi^H\big(A \otimes \mathcal{O}_{\overline{\mathcal{M}}_{g,n}(X, \beta)}^{\mathrm{vir}, H}\big) \in \mathbb{C}.$$

Precisely, the class $\lambda_{-1}^T(R\pi_* f^* \mathcal{O}(d))^\vee$ is only defined after localization, so we first apply the virtual localization formula to the left-hand side, then we compute it in $K^0(T, \overline{\mathcal{M}}_{g,n})_{\mathrm{loc}} = K^0(\overline{\mathcal{M}}_{g,n}) \otimes \mathbb{C}(t_1, \ldots, t_N)$ as rational fractions in the T–equivariant parameters, then we specialize them to $(t_1, \ldots, t_N) = (\zeta, \zeta^{u_2}, \ldots, \zeta^{u_N})$ using $\varphi \colon H \hookrightarrow T$ and obtain a well-defined K–class in $K^0(\overline{\mathcal{M}}_{g,n}) \otimes \mathbb{C}$. Eventually we take its Euler characteristic and land in $R(H)_{\mathbb{C}, \mathrm{loc}} \simeq \mathbb{C}$, where the last isomorphism depends on the primitive q^{th} root of unity ζ.

Remark 2.7 The localization map $R(H) \to R(H)_{\mathbb{C}, \mathrm{loc}}$ corresponds to the map $\mathbb{Z}[X]/(1 - X^q) \to \mathbb{C}$ sending the variable X to ζ.

Remark 2.8 In Theorem 2.6, it is important that for every nonzero element $h \in H$, the h–fixed locus consists of coordinate points in $\mathbb{P}^N$. It is guaranteed by Proposition 2.2 and the fact that $H \subset \mathbb{Z}/M\mathbb{Z}$.

Corollary 2.9 We take the same notation and assumptions as in Theorem 2.6. We further assume that the order q of the group H is a prime number. For each $1 \le k < q$, denote by $B_k \in \mathbb{C}$ the result of Theorem 2.6 when $\zeta = e^{2ik\pi/q}$. Then the K–theoretic GW invariant of X equals

$$\chi(A \otimes \mathcal{O}_{\overline{\mathcal{M}}_{g,n}(X, \beta)}^{\mathrm{vir}}) \equiv -(B_1 + \cdots + B_{q-1}) \in \mathbb{Z}/q\mathbb{Z}.$$

Proof The H–equivariant Euler characteristic $\chi^H(A \otimes \mathcal{O}^{\mathrm{vir},H}_{\overline{\mathcal{M}}_{g,n}(X,\beta)})$ lies in the representation ring $R(H) \simeq \mathbb{Z}[X]/(1 - X^q)$, so there exist integers $a_0, \ldots, a_{q-1}$ such that

$$\chi^H(A \otimes \mathcal{O}^{\mathrm{vir},H}_{\overline{\mathcal{M}}_{g,n}(X,\beta)}) = \sum_{l=0}^{q-1} a_l X^l \in \mathbb{Z}[X]/(1 - X^q).$$

Our goal would be to compute

$$\chi(A \otimes \mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_{g,n}(X,\beta)}) = \sum_{l=0}^{q-1} a_l \in \mathbb{Z},$$

but Theorem 2.6 only gives us $\sum_{l=0}^{q-1} a_l \zeta^l \in \mathbb{C}$. However, since q is a prime number, we can apply Theorem 2.6 to every primitive q^{th} root of unity ζ^k for $1 \le k < q$. Summing the various results, we obtain

$$\sum_{k=1}^{q-1} \sum_{l=0}^{q-1} a_l \zeta^{kl} = qa_0 - \sum_{l=0}^{q-1} a_l \in \mathbb{Z},$$

leading to the congruence. $\qquad\square$

Remark 2.10 Assume the order q of the group H is not a prime number and choose a nonzero element $h \in H$. Even when h is not a primitive element, we can apply Theorem 2.6 to the subgroup $\langle h \rangle$, but we then have the bounds

$$g < \tfrac{1}{2}(p-1) \quad \text{and} \quad \beta < \mathrm{ord}(h),$$

where $\mathrm{ord}(h)$ denotes the order of the element h, and p is its greatest prime divisor. In order to obtain the KGW invariant in H, we then need to sum all the results of Theorem 2.6 for all nonzero elements $h \in H$. Therefore, we have to restrict to the bounds

$$g < \tfrac{1}{2}(p-1) \quad \text{and} \quad \beta < p,$$

where p is the smallest prime divisor of q.

Example 2.11 For the quintic threefold of Example 2.3, the specialization of equivariant parameters corresponding to $G \hookrightarrow T$ is

$$(t_0, \ldots, t_4) = (1, \zeta, \zeta^{-3}, \zeta^{13}, \zeta^{-51}), \quad \text{where } \zeta^{205} = 1.$$

Moreover, we have a subgroup $H := \mathbb{Z}/41\mathbb{Z} \subset \mathbb{Z}/205\mathbb{Z}$, so that by Corollary 2.9, we are able to compute all KGW invariants modulo 41 up to genus 19 and degree 40. Moreover, by Remark 2.10, we are able to compute all KGW invariants modulo 205 in genera 0 and 1 up to degree 4.

Remark 2.12 Another way to realize the quintic hypersurface in $\mathbb{P}^4$ is

$$X = \{x_0^5 + \cdots + x_4^5 = 0\} \subset \mathbb{P}^4.$$

Then the group is $(\mathbb{Z}/5\mathbb{Z})^4$, but to ensure that the g–fixed locus consists of isolated points for every element g of the group, we need to consider the subgroup $G = \mathbb{Z}/5\mathbb{Z}$, acting as

$$\zeta \cdot \underline{x} = (x_0, \zeta x_1, \zeta^2 x_2, \zeta^3 x_3, \zeta^4 x_4).$$

Furthermore, we observe that the G–fixed locus is empty. We then deduce that all KGW invariants in genera 0 and 1 and up to degree 4 vanish modulo 5.

2.5 Example of the quintic threefold

We illustrate Theorem 2.6 and Corollary 2.9 by a computation of the genus-one degree-one unmarked KGW invariant in the case of the quintic hypersurface in $\mathbb{P}^4$, modulo 205.

Proposition 2.13 *Let $X \subset \mathbb{P}^4$ be a smooth quintic hypersurface. We find that*

$$\chi\left(\frac{\mathcal{O}_{\overline{\mathcal{M}}_{1,0}(X,1)}^{\mathrm{vir}}}{1 - q\mathbb{E}^\vee}\right) \equiv (120q^2 + 180q + 125)\frac{1 - q^4 - q^6}{(1 - q^4)(1 - q^6)} \in \mathbb{Z}/205\mathbb{Z}[\![q]\!].$$

In order to prove Proposition 2.13, we first write the general graph sum formula coming from torus localization and we then specialize to $(g, n, \beta) = (1, 0, 1)$.

Following the general scheme of Theorem 2.6, we compute the K–theoretic class

$$\chi^T\left(\lambda_{-1}^T(R\pi_* f^*\mathcal{O}(5))^\vee \otimes A \otimes \mathcal{O}_{\overline{\mathcal{M}}_{g,n}(\mathbb{P}^4,\beta)}^{\mathrm{vir},T}\right) \in K^0(\overline{\mathcal{M}}_{g,n}) \otimes \mathbb{C}(t_0, \dots, t_4).$$

It is done via the standard virtual localization formula of [17], lifted to K–theory, as a sum over dual graphs. Indeed, the class λ_{-1} is multiplicative in K–theory, just as the Euler class in cohomology, so that the whole proof of [17, Section 4] holds. Therefore, we take the same notation as in [17], to which we refer, for instance, for the description of graphs, except that we take the convention $t_j = e^{-\lambda_j}$ with respect to their T–weights.

Let Γ be a graph in the localization formula of $\mathbb{P}^4$. We denote by $\overline{\mathcal{M}}_\Gamma$ the associated moduli space of stable curves and by A_Γ the group of automorphisms coming from the graph Γ and from degrees of the edges, so that the corresponding fixed locus in $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^4, \beta)$ is the quotient stack $[\overline{\mathcal{M}}_\Gamma/A_\Gamma]$; see [17]. The contribution of the graph Γ to the localization formula is of the form

$$\chi\left([\overline{\mathcal{M}}_\Gamma/A_\Gamma]; \mathrm{Contr(flags)} \cdot \mathrm{Contr(vertices)} \cdot \mathrm{Contr(edges)}\right),$$

where we have

$$\text{Contr(flags)} = \frac{1}{1 - \left(\dfrac{t_{i(F)}}{t_{j(F)}}\right)^{1/d_e} \Psi_F} \cdot \frac{\displaystyle\prod_{m \neq i(F)} 1 - \frac{t_{i(F)}}{t_m}}{1 - \dfrac{t_{i(F)}^5}{t_0^4 t_1}},$$

$$\text{Contr(vertices)} = \frac{1 - \dfrac{t_{i(v)}^5}{t_0^4 t_1}}{\displaystyle\sum_{k=0}^{g(v)} (-1)^k \left(\dfrac{t_{i(v)}^5}{t_0^4 t_1}\right)^k \Lambda^k \mathbb{E}} \cdot \prod_{m \neq i(v)} \frac{\displaystyle\sum_{k=0}^{g(v)} (-1)^k \left(\dfrac{t_{i(v)}}{t_m}\right)^k \Lambda^k \mathbb{E}}{1 - \dfrac{t_{i(v)}}{t_m}},$$

$$\text{Contr(edges)} = \prod_{a=1}^{d_e} \left(2 - \left(\frac{t_{j'}}{t_j}\right)^{a/d_e} - \left(\frac{t_j}{t_{j'}}\right)^{a/d_e}\right)^{-1} \cdot \prod_{\substack{a+b=d_e \\ k \neq j, j'}} \left(1 - \frac{(t_j^a t_{j'}^b)^{1/d_e}}{t_k}\right)^{-1}$$
$$\cdot \prod_{a+b=5d_e} \left(1 - \frac{(t_j^a t_{j'}^b)^{1/d_e}}{t_0^4 t_1}\right),$$

and where we write here the contribution of an edge linking the coordinate points p_j and $p_{j'}$. These formulae follow exactly from [17, Section 4], replacing the Euler class with the lambda class in K–theory.

Remark 2.14 In the contribution of vertices, we can rewrite the sum in terms of the lambda-structure as $\lambda_{-u}(\mathbb{E})$, with $u := t_{i(v)}^5 / t_0^4 t_1$.

Remark 2.15 All individual contributions are in $K^0(\overline{\mathcal{M}}_\Gamma) \otimes \mathbb{C}(t_0^{1/d}, \ldots, t_4^{1/d})_{d \in \mathbb{N}^*}$, but it is a consequence of the virtual localization formula that the final answer is in $K^0(\overline{\mathcal{M}}_\Gamma) \otimes \mathbb{C}(t_0, \ldots, t_4)$.

Let us now specialize the formula to $(g, n, \beta) = (1, 0, 1)$. The graph Γ has only two vertices v_1 and v_2, of respective genera 1 and 0, and one degree-one edge in between. Moreover, as the vertex v_2 has valence one, it corresponds to a free point (not marked, not a node) rather than to a stable component of the curve. We denote by $0 \leq i_1 \neq i_2 \leq 4$ the indices of the coordinate points p_{i_1} and p_{i_2} to which the vertices v_1 and v_2 are sent by the stable map. Note also that such a graph has no automorphisms and the moduli space $\overline{\mathcal{M}}_\Gamma$ is isomorphic to $\overline{\mathcal{M}}_{1,1}$. Furthermore, we recall that the Hodge bundle $\mathbb{E}$ over $\overline{\mathcal{M}}_{1,1}$ is identified with the cotangent line Ψ_1 at the marking. As a consequence,

the virtual localization formula equals

$$\sum_{0 \leq i_1 \neq i_2 \leq 4} \frac{\prod_{a+b=5} \left(1 - \frac{t_{i_1}^a t_{i_2}^b}{t_0^4 t_1}\right)}{\left(2 - \frac{t_{i_1}}{t_{i_2}} - \frac{t_{i_2}}{t_{i_1}}\right) \cdot \prod_{k \neq i_1, i_2} \left(1 - \frac{t_{i_1}}{t_k} - \frac{t_{i_2}}{t_k} + \frac{t_{i_1} t_{i_2}}{t_k^2}\right)}$$

$$\cdot \chi\left(\overline{\mathcal{M}}_{1,1}; \frac{1}{1 - q\mathbb{E}^\vee} \frac{\prod_{m \neq i_1, i_2}\left(1 - \frac{t_{i_1}}{t_m}\Psi_1\right)}{1 - \frac{t_{i_1}^5}{t_0^4 t_1}\Psi_1}\right).$$

Once we specialize to $(t_0, \dots, t_4) = (1, \zeta, \zeta^{-3}, \zeta^{13}, \zeta^{-51})$, where ζ is any primitive root of unity of order 41, we notice that denominators never vanish, but the numerator could vanish; precisely,

$$1 - \frac{t_{i_1}^a t_{i_2}^b}{t_0^4 t_1} = 0 \iff \begin{cases} i_2 = i_1 + 1 \text{ and } (a, b) = (4, 1), \text{ or} \\ i_1 = i_2 + 1 \text{ and } (a, b) = (1, 4), \end{cases}$$

with the cyclic convention on indices, ie $t_5 := t_0$. Moreover, we have

$$1 - \frac{t_{i_1}}{t_{i_1+1}}\Psi_1 = 1 - \frac{t_{i_1}^5}{t_0^4 t_1}\Psi_1,$$

so that the specialization of the localization formula gives

$$\sum_{\substack{0 \leq i_1 \neq i_2 \leq 4 \\ i_2 \neq i_1 + 1 \\ i_1 \neq i_2 + 1}} \left[\frac{\prod_{a+b=5} \left(1 - \frac{t_{i_1}^a t_{i_2}^b}{t_0^4 t_1}\right)}{\left(2 - \frac{t_{i_1}}{t_{i_2}} - \frac{t_{i_2}}{t_{i_1}}\right) \cdot \prod_{k \neq i_1, i_2} \left(1 - \frac{t_{i_1}}{t_k} - \frac{t_{i_2}}{t_k} + \frac{t_{i_1} t_{i_2}}{t_k^2}\right)} \right.$$

$$\left. \cdot \chi\left(\overline{\mathcal{M}}_{1,1}; \frac{1}{1 - q\mathbb{E}^\vee} \prod_{m \neq i_1, i_1+1, i_2} \left(1 - \frac{t_{i_1}}{t_m}\Psi_1\right)\right) \right]_{(t_0, \dots, t_4) = (1, \zeta, \zeta^{-3}, \zeta^{13}, \zeta^{-51})}.$$

By [26, Proposition 2.9], we have

$$\chi\left(\overline{\mathcal{M}}_{1,1}; \frac{1}{1 - q\mathbb{E}^\vee} \frac{1}{1 - q_1\Psi_1}\right)$$

$$= \frac{(1 - qq_1)(1 - q^4 - q^6 - q_1^2 q^6 - q_1^2 q^8 - q_1^4 q^8 + q^2 q_1^2 + q^4 q_1^4 + q^6 q_1^6 + q^8 q_1^8)}{(1 - q^4)(1 - q^6)(1 - q_1^4)(1 - q_1^6)}.$$

Hence, we get

$$\chi\left(\overline{\mathcal{M}}_{1,1}; \frac{\prod\limits_{m\neq i_1,i_1+1,i_2}\left(1-\dfrac{t_{i_1}}{t_m}\Psi_1\right)}{1-q\mathbb{E}^{\vee}}\right) = \frac{(1-q^4-q^6)}{(1-q^4)(1-q^6)}\prod\limits_{m\neq i_1,i_1+1,i_2}\left(1+\frac{t_{i_1}}{t_m}q\right).$$

As a consequence, our formula simplifies as

$$\frac{1-q^4-q^6}{(1-q^4)(1-q^6)}$$

$$\cdot\sum_{\substack{0\leq i_1\neq i_2\leq 4\\ i_2\neq i_1+1\\ i_1\neq i_2+1}}\left[\frac{\prod\limits_{a+b=5}\left(1-\dfrac{t_{i_1}^a t_{i_2}^b}{t_0^4 t_1}\right)\prod\limits_{m\neq i_1,i_1+1,i_2}\left(1+\dfrac{t_{i_1}}{t_m}q\right)}{\left(2-\dfrac{t_{i_1}}{t_{i_2}}-\dfrac{t_{i_2}}{t_{i_1}}\right)\cdot\prod\limits_{k\neq i_1,i_2}\left(1-\dfrac{t_{i_1}}{t_k}-\dfrac{t_{i_2}}{t_k}+\dfrac{t_{i_1}t_{i_2}}{t_k^2}\right)}\right]_{(t_0,\ldots,t_4)=(1,\zeta,\zeta^{-3},\zeta^{13},\zeta^{-51})}.$$

Finally, we must take the opposite of the sum of these expressions over all primitive roots ζ of order 41. First, we notice that the term inside the sum is a polynomial in q of degree at most two, so that it is enough to evaluate it at $q \in \{0, 1, 2\}$. Using Sagemath, we find

$$\chi\left(\frac{\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_{1,0}(X,1)}}{1-q\mathbb{E}^{\vee}}\right) \equiv (-85q^2+590q-80)\frac{1-q^4-q^6}{(1-q^4)(1-q^6)}$$

$$\equiv (38q^2+16q+2)\frac{1-q^4-q^6}{(1-q^4)(1-q^6)} \in \mathbb{Z}/41\mathbb{Z}[\![q]\!].$$

Furthermore, using Remark 2.12, we obtain the result of Proposition 2.13.

2.6 Special case of elliptic curves

In this section, we use the ideas behind Corollary 2.9 to prove that KGW theory with homogeneous insertions of an elliptic curve is trivial.

Proposition 2.16 *Let E be an elliptic curve. Then for every genus g, degree β, number of markings n and insertions $A := \bigotimes_{i=1}^n \Psi_i^{a_i} \otimes Y_i$, with $2g-2+n > 0$ and $Y_i \in K^0(E)$ homogeneous K–classes, the corresponding KGW invariant vanishes:*

$$\chi(A \otimes \mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_{g,n}(E,\beta)}) = 0.$$

Proof Let M be the largest possible order of an automorphism of a stable curve of genus g. Let p be any prime number larger than $M+1$ and $\beta+1$. Define $G := \mathbb{Z}/p\mathbb{Z}$

and take a G–torsion point $x \in E$. Then the group G acts on the elliptic curve E by translation $y \mapsto y + x$, and for every nonzero element $h \in G$, the h–fixed locus is empty. By Remark 2.5 and Proposition 2.4, the G–fixed locus in the moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(E, \beta)$ is empty. Therefore, by the localization formula, the G–equivariant KGW invariant vanishes, so that we get

$$\chi(A \otimes \mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_{g,n}(E,\beta)}) \equiv 0 \in \mathbb{Z}/p\mathbb{Z}$$

for the nonequivarant limit. Since it is true for infinitely many prime numbers p, we obtain the vanishing in $\mathbb{Z}$. $\qquad\square$

Remark 2.17 Interestingly, KGW invariants are deduced from GW invariants via a Kawazaki–Riemann–Roch theorem; see [38; 16]. It would be instructive to compare Proposition 2.16 with GW theory of elliptic curves, which is nontrivial and described in [31; 32].

Remark 2.18 The same proof holds for abelian varieties. However, when the dimension of the abelian variety is greater than 2 and the degree-class β is nonzero, there is a trivial quotient of the obstruction theory, so that both GW and KGW theories are trivial. However, for degree-0 invariants, GW theory is nontrivial, but KGW theory is.

Remark 2.19 The main idea in the proof of Proposition 2.16 is to use congruence relations for infinitely many prime numbers. Indeed, if we were able to find, for a smooth DM stack X, automorphisms of prime orders for infinitely many primes and to compute the localization formulae, then we would be able to know all KGW invariants of X. Therefore, a necessary condition is that the automorphism group of X must be infinite. However, it is not sufficient. For instance, some K3 surfaces have infinitely many symmetries, but it was shown by [23] that the maximal order of a finite group acting faithfully on a K3 surface is 3840.

Remark 2.20 Here are a few remarks on finiteness of automorphism groups. For projective hypersurfaces (except quadrics, elliptic curves, and K3 surfaces), every automorphism is projective and the automorphism group is finite. All Batyrev Calabi–Yau (CY) 3–folds have finite automorphism groups [35]. Every projective variety of general type has finite automorphism group. CY varieties with Picard numbers 1 or 2 have finite automorphism groups. It is expected that most CY varieties with Picard number more than 4 have infinitely many automorphisms. In particular, it would be interesting to know whether the Schoen CY 3–fold has automorphisms of prime order for infinitely many primes and to study its KGW theory; see [21].

3　K–theoretic FJRW theory

Similarly to KGW theory, we aim in this section to compute the K–theoretic FJRW invariants of a Landau–Ginzburg (LG) orbifold modulo prime numbers. For simplicity of the exposition, we focus in this paper on the quintic polynomial with minimal group of symmetries. However, it is straightforward to apply the same ideas to an LG orbifold (W, H), where W is an invertible polynomial and H is an admissible group, as long as we only insert $\mathrm{Aut}(W)$–invariant states in the correlator. We refer to [18] for details.

3.1　Sketch of Polishchuk–Vaintrob construction

Let $W(x_1, \ldots, x_5)$ be a quintic polynomial in five variables and let μ_5 act on $\mathbb{C}^5$ by multiplication by a fifth root of unity. The moduli space used in FJRW theory of (W, μ_5) is the moduli space $\mathcal{S}_{g,n}^{1/5}$, which parametrizes $(\mathcal{C}, \sigma_1, \ldots, \sigma_n, \mathcal{L}, \phi)$. Precisely, the curve $\mathcal{C}$ is an orbifold genus-g stable curve with isotropy group μ_5 at the markings $\sigma_1, \ldots, \sigma_n$ and at the nodes (and trivial everywhere else), $\mathcal{L}$ is a line bundle on $\mathcal{C}$, and $\phi \colon \mathcal{L} \to \omega_{\log} := \omega_{\mathcal{C}}(\sigma_1 + \cdots + \sigma_n)$ is an isomorphism.

Let π be the projection from the universal curve to $\mathcal{S}_{g,n}^{1/5}$ and $\mathcal{L}$ be the universal line bundle. In [33], Polishchuk and Vaintrob constructed resolutions $R\pi_*(\mathcal{L}^{\oplus 5}) = [A \to B]$ by vector bundles over $\mathcal{S}_{g,n}^{1/5}$ such that there exists some morphism

$$\alpha \colon \mathrm{Sym}^4 A \to B^\vee$$

corresponding to the differentiation of the polynomial W; see [18] for details. Taking $p \colon X \to \mathcal{S}_{g,n}^{1/5}$ to be the total space of the vector bundle A, then the morphism α is interpreted as a global section of $p^* B^\vee$ over X, and the map $\beta \colon A \to B$ coming from the resolution is interpreted as a global section of $p^* B$. As a consequence, we obtain a Koszul matrix factorization

$$\mathbf{PV} := \{\alpha, \beta\} := (\Lambda^\bullet B^\vee, \alpha \wedge \cdot + \iota_\beta) \in D(X, \alpha(\beta))$$

of potential $\alpha(\beta)$ over the space X, and the support of this matrix factorization is exactly the moduli space $\mathcal{S}_{g,n}^{1/5}$.

The moduli space $\mathcal{S}_{g,n}^{1/5}$ has several components depending on the monodromies

$$\underline{\gamma} := (\gamma_1, \ldots, \gamma_n) \in \mu_5^n$$

at the markings; we denote by $\mathcal{S}_{g,n}^{1/5}(\underline{\gamma})$ the corresponding component. Assume all monodromies are nonzero; this is known as the *narrow condition*. Then the pairing $\alpha(\beta)$ is the zero function over X, and the matrix factorization $\mathbf{PV}$ becomes a two-periodic

complex, exact off the moduli space $\mathcal{S}_{g,n}^{1/5}(\underline{\gamma})$. Therefore, we can define the pushforward along the projection map p in the category of matrix factorizations, yielding

$$p_*(\mathbf{PV}) \in D(\mathcal{S}_{g,n}^{1/5}(\underline{\gamma}), 0) \simeq D^b(\mathcal{S}_{g,n}^{1/5}(\underline{\gamma})),$$

where on the right we have the derived category of coherent sheaves.

Remark 3.1 If one allows trivial monodromies (ie one considers broad insertions), then the pairing $\alpha(\beta)$ does not vanish and we rather end with a functor

$$\Phi \colon D_\Gamma(\mathbb{A}^{\underline{\gamma}}, W_{\underline{\gamma}}) \to D_{\overline{H}}(\mathcal{S}_{g,n}^{\mathrm{rig}}(\underline{\gamma}), 0) \simeq D([\mathcal{S}_{g,n}^{\mathrm{rig}}(\underline{\gamma})/\overline{H}]),$$
$$U \mapsto p_*(\mathrm{ev}^*(U) \otimes \mathbf{PV}),$$

where we need to consider rigidified moduli spaces; see [33] for details and notation.

In general, to any triangulated category $\mathcal{C}$ we associate a Grothendieck group $K_0(\mathcal{C})$ by taking the free abelian group generated by the objects of the category and then modding out the relation

$$[A] - [B] + [C] = 0$$

for every distinguished triangle $A \to B \to C$. Furthermore, any functor $f \colon \mathcal{C}_1 \to \mathcal{C}_2$ of triangulated dg categories induces a morphism of groups

$$f_! \colon K_0(\mathcal{C}_1) \to K_0(\mathcal{C}_2).$$

When the category is the derived category of coherent sheaves on a smooth DM stack, we recover the usual K–theory of the stack.

Remark 3.2 If we apply it to the functor of Remark 3.1, we get a morphism of groups

$$\Phi_! := K_0(\Phi) \colon K_0(D_\Gamma(\mathbb{A}^{\underline{\gamma}}, W_{\underline{\gamma}})) \to K^0([\mathcal{S}_{g,n}^{\mathrm{rig}}(\underline{\gamma})/\overline{H}]).$$

Definition 3.3 We define the K–theoretic class

$$\mathcal{O}_{\mathcal{S}_{g,n}^{1/5}(\underline{\gamma})}^{\mathrm{vir}} := p_*(\mathbf{PV}) \in K^0(\mathcal{S}_{g,n}^{1/5}(\underline{\gamma}))$$

and we call it the virtual structure sheaf of the moduli space $\mathcal{S}_{g,n}^{1/5}(\underline{\gamma})$.

Definition 3.4 Fix $d_1, \ldots, d_n \in \mathbb{Z}$ and nontrivial monodromies $\gamma_1, \ldots, \gamma_n \in \mu_5$. The K–theoretic FJRW invariant of the LG orbifold (W, μ_5) is

$$\chi(\Psi_1^{\otimes d_1} \otimes \cdots \otimes \Psi_n^{\otimes d_n} \otimes \mathcal{O}_{\mathcal{S}_{g,n}^{1/5}(\underline{\gamma})}^{\mathrm{vir}}) \in \mathbb{Z},$$

where $\chi \colon K^0(\mathcal{S}_{g,n}^{1/5}(\underline{\gamma})) \to K^0(\mathrm{pt}) = \mathbb{Z}$ is the Euler characteristic and the line bundle Ψ_i is the relative cotangent line at the ith marked point.

Remark 3.5 A special feature of the LG orbifold (W, μ_5), and more generally when the polynomial has degree d and the group is μ_d, is that FJRW invariants do not depend on the polynomial W, as long as it is nondegenerate with the same weights and degree; see [12, Proposition 4.1.7]. The same result holds for the K–theoretic version, as it holds at the matrix factorization level. Hence, we can consider several choices for our quintic polynomial.

3.2 Invertible polynomials

In the context of mirror symmetry, a well-behaved class of polynomials has been introduced by Berglund and Hübsch [3]. We say that a polynomial is invertible when it is nondegenerate and has as many variables as monomials. According to Kreuzer and Skarke [24], every invertible polynomial is a (Thom–Sebastiani) sum of invertible polynomials, with disjoint sets of variables, of the following three types:

$$\textbf{Fermat} \qquad x^{a+1},$$

$$(8) \qquad \textbf{chain of length } c \quad x_1^{a_1} x_2 + \cdots + x_{c-1}^{a_{c-1}} x_c + x_c^{a_c+1}, \quad \text{where } c \geq 2,$$

$$\textbf{loop of length } l \quad x_1^{a_1} x_2 + \cdots + x_{l-1}^{a_{l-1}} x_l + x_l^{a_l} x_1, \quad \text{where } l \geq 2.$$

In the case of the quintic polynomial, we have for example the following choices:

$$(9) \quad \begin{array}{lll} \textbf{Fermat} & x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5, & (\mu_5)^5, \\ \textbf{loop} & x_1^4 x_2 + x_2^4 x_3 + x_3^4 x_4 + x_4^4 x_5 + x_5^4 x_1, & \mu_{1025}, \\ \textbf{chain} & x_1^4 x_2 + x_2^4 x_3 + x_3^4 x_4 + x_4^4 x_5 + x_5^5, & \mu_{1280}, \\ \textbf{2--loops} & x_1^4 x_2 + x_2^4 x_3 + x_3^4 x_1 + x_4^4 x_5 + x_5^4 x_4, & \mu_{15} \times \mu_{65}, \\ \textbf{loop-Fermat} & x_1^4 x_2 + x_2^4 x_3 + x_3^4 x_4 + x_4^4 x_1 + x_5^5, & \mu_{255} \times \mu_5, \end{array}$$

where on the right we have written the group $\mathrm{Aut}(W)$ of diagonal matrices leaving the polynomial invariant.

Let W be an invertible quintic polynomial and $\mathrm{Aut}(W)$ be its maximal group of diagonal symmetries. Recall that the space X is the total space of the vector bundle A over the moduli space $\mathcal{S}_{g,n}^{1/5}(\gamma)$, and that we have

$$R\pi_*(\mathcal{L}^{\oplus 5}) = [A \to B].$$

Therefore, the vector bundles A and B are direct sums of five copies, which we write as

$$A = A_1 \oplus \cdots \oplus A_5 \quad \text{and} \quad B = B_1 \oplus \cdots \oplus B_5.$$

We then have a natural action of $\mathrm{Aut}(W)$ on the vector bundles A and B by rescaling the fibers. Precisely, the actions on X in the examples (9) are

$$
\begin{array}{lll}
\textbf{Fermat} & (\zeta_1 x_1, \zeta_2 x_2, \zeta_3 x_3, \zeta_4 x_4, \zeta_5 x_5), & \zeta_j = e^{2i\pi/5}, \\[4pt]
\textbf{loop} & (\zeta x_1, \zeta^{-4} x_2, \zeta^{16} x_3, \zeta^{-64} x_4, \zeta^{256} x_5), & \zeta = e^{2i\pi/1025}, \\[4pt]
(10) \quad \textbf{chain} & (\zeta x_1, \zeta^{-4} x_2, \zeta^{16} x_3, \zeta^{-64} x_4, \zeta^{256} x_5), & \zeta = e^{2i\pi/1280}, \\[4pt]
\textbf{2–loops} & (\zeta_1 x_1, \zeta_1^{-4} x_2, \zeta_1^{16} x_3, \zeta_2 x_4, \zeta_2^{-4} x_5), & \zeta_1 = e^{2i\pi/65}, \zeta_2 = e^{2i\pi/15}, \\[4pt]
\textbf{loop-Fermat} & (\zeta_1 x_1, \zeta_1^{-4} x_2, \zeta_1^{16} x_3, \zeta_1^{-64} x_4, \zeta_2 x_5), & \zeta_1 = e^{2i\pi/255}, \zeta_2 = e^{2i\pi/5}.
\end{array}
$$

By construction, since the polynomial W is $\mathrm{Aut}(W)$–invariant, the matrix factorization **PV** is $\mathrm{Aut}(W)$–equivariant and so is the virtual structure sheaf.

However, we need to be careful when we compute the $\mathrm{Aut}(W)$–fixed locus. Indeed, the group of automorphisms of a (W, μ_5)–spin curve $(\mathcal{C}, \sigma_1, \ldots, \sigma_n, \mathcal{L})$ fixing the coarse curve of $\mathcal{C}$ is $\mu_5 \times (\mu_5)^{\#\mathrm{nodes}}$, where the first factor rescales the line bundle $\mathcal{L}$, and the second factor acts only on the orbifold curve $\mathcal{C}$ — it is the so-called ghost automorphism; see [1, Proposition 7.1.1] and [7, Section 2.1.4]. As a consequence, we would rather consider the action of the group

$$
G := \mathrm{Aut}(W)/\mu_5
$$

on the space X over the moduli space $\mathcal{S}_{g,n}^{1/5}(\gamma)$. Here, the subgroup μ_5 is generated by $(e^{2i\pi/5}, \ldots, e^{2i\pi/5})$ in the Fermat case, by $e^{2i\pi/5}$ in the loop and chain cases, and by $(e^{2i\pi/5}, e^{2i\pi/5})$ in the 2–loops and loop-Fermat cases.

Unfortunately, we still have too many fixed points. For instance, in the Fermat example, the point

$$
(\mathcal{C}, \sigma_1, \ldots, \sigma_n, \mathcal{L}; x_1, 0, \ldots, 0) \in X
$$

is fixed. Another example is the chain polynomial, for which the point

$$
(\mathcal{C}, \sigma_1, \ldots, \sigma_n, \mathcal{L}; 0, \ldots, 0, x_5) \in X
$$

is fixed. In both cases, the G–fixed locus is noncompact. We easily check that, among all invertible quintic polynomials, the only cases where the G–fixed locus is compact are

- the loop polynomial with group $G = \mathrm{Aut}(W)/\mu_5 = \mu_{205}$,
- the 2–loops polynomial with group $G = (\mu_{65} \times \mu_{15})/\mu_5$.

Moreover, the G–fixed locus in the space X equals the base $\mathcal{S}_{g,n}^{1/5}(\gamma)$. We are therefore able to apply the (nonvirtual) localization formula on the smooth space X to get the next theorem.

Remark 3.6 It is more convenient to work with cyclic groups. Therefore, in the 2–loops polynomial case, we prefer to use $G = \mu_{195}$, where the G–action on X is

$$(\zeta^{15}x_1, \zeta^{-60}x_2, \zeta^{240}x_3, \zeta^{65}x_4, \zeta^{-260}x_5), \quad \text{where } \zeta = e^{2i\pi/195}.$$

Definition 3.7 Let $l \in \mathbb{Z}$. The Adams operation Ψ^l in K–theory is defined on a line bundle L over a space S as

$$\Psi^l(L) := L^{\otimes l},$$

and then extended as a ring homomorphism

$$\Psi^l \colon K^0(S) \to K^0(S).$$

Theorem 3.8 *Consider the two following situations*:

- *W is the loop polynomial, $G := \mu_{205}$, ζ a primitive 205th root of unity, and $(a_1, \dots, a_5) = (1, -4, 16, -64, 256)$.*

- *W is the 2–loops polynomial, $G := \mu_{195}$, ζ is a primitive 195th root of unity, and $(a_1, \dots, a_5) = (15, -60, 240, 65, -260)$.*

Let g and n be nonnegative integers in the stable range $2g - 2 + n > 0$, and let $\underline{\gamma} \in \mu_5^n$ be nontrivial monodromies. Then the G–equivariant virtual structure sheaf equals

$$(11) \qquad \mathcal{O}^{\mathrm{vir},G}_{\mathcal{S}^{1/5}_{g,n}(\underline{\gamma})} = \exp\left(\sum_{l \le -1} \sum_{j=1}^5 \frac{\zeta^{a_j \cdot l}}{l} \Psi^l(-R\pi_*\mathcal{L}) \right) \in K^0(\mathcal{S}^{1/5}_{g,n}(\underline{\gamma})) \otimes \mathbb{C}.$$

Proof In the G–equivariant K–theory of the space X, the matrix factorization equals

$$\mathbf{PV} = \lambda^G_{-1}p^*B^\vee \in K^0(G, X),$$

and by the localization formula we get

$$\mathbf{PV} = \iota_!\left(\frac{\lambda^G_{-1}B^\vee}{\lambda^G_{-1}A^\vee} \right) = \iota_!(\lambda^G_{-1}(B^\vee - A^\vee)) \in K^0(G, X)_{\mathrm{loc}}$$

in the localized ring, where $\iota \colon \mathcal{S}^{1/5}_{g,n}(\underline{\gamma}) \hookrightarrow X$ is the zero section. Taking the pushforward along the projection map p, we obtain the G–equivariant virtual structure sheaf

$$\mathcal{O}^{\mathrm{vir},G}_{\mathcal{S}^{1/5}_{g,n}(\underline{\gamma})} = \lambda^G_{-1}(B^\vee - A^\vee) \in K^0(G, \mathcal{S}^{1/5}_{g,n}(\underline{\gamma}))_{\mathrm{loc}} \simeq K^0(\mathcal{S}^{1/5}_{g,n}(\underline{\gamma})) \otimes \mathbb{C}.$$

If V is a vector bundle, we can express the λ–structure in terms of Adams operators via the formula

$$\lambda_{-p}(V^\vee) = \exp\left(\sum_{l \le -1} \frac{p^{-l}}{l} \Psi^l(V) \right).$$

Moreover, if the action of a group G on the vector bundle V is by rescaling fibers with $\zeta \in G$, then

$$\lambda^G_{-1}(V^\vee) = \lambda_{-\zeta^{-1}}(V^\vee) = \exp\left(\sum_{l \leq -1} \frac{\zeta^l}{l} \Psi^l(V) \right).$$

In our situation, we find formula (11). $\qquad\square$

Remark 3.9 In [18], we define the characteristic class $\mathfrak{c}_t \colon K^0(S) \to H^*(S)[\![t]\!]$ by

$$\mathfrak{c}_t(B - A) = \mathrm{Ch}(\lambda_{-t}(B^\vee - A^\vee)) \mathrm{Td}(B - A),$$

and we then obtain the formula

$$\lim_{t \to 1} \prod_{i=1}^5 \mathfrak{c}_{t_j}(-R\pi_*\mathcal{L}) = c_{\mathrm{vir}} \in H^*(\mathcal{S}^{1/5}_{0,n}(\underline{\gamma}))$$

for the FJRW virtual cycle of (W, μ_5), where $t_j := t^{a_j}$. This formula is only valid in genus 0 and we do not expect the left-hand side to converge in positive genus when $t \to 1$. However, by Theorem 3.8, we see that the formula converges for every genus when $t \to \zeta$.

In order to get congruences for the nonequivariant limit, we need to consider a subgroup of $\mathrm{Aut}(W)$ with prime order and whose fixed locus in X is compact. The only invertible polynomial for which it is possible is the loop polynomial, together with the subgroup μ_{41} acting on X as

$$(\zeta_{41}x_1, \zeta_{41}^{37}x_2, \zeta_{41}^{16}x_3, \zeta_{41}^{18}x_4, \zeta_{41}^6 x_5), \quad \text{where } \zeta_{41} := e^{2i\pi/41}.$$

Remark 3.10 The prime decomposition of 205 is $5 \cdot 41$, so we could also hope for a congruence modulo 5. However, the subgroup μ_5 acts trivially on X. Indeed, it acts as

$$(\zeta_5 x_1, \zeta_5 x_2, \zeta_5 x_3, \zeta_5 x_4, \zeta_5 x_5), \quad \text{where } \zeta_5 := e^{2i\pi/5},$$

which is rescaled by the automorphism group of the (W, μ_5)–spin curve, so that the fixed locus is X. Nevertheless, from its definition using the quintic Fermat polynomial, we observe that the virtual structure sheaf decomposes into five identical summands, each one corresponding to the so-called 5–spin theory. It is then divisible by five in the K–theoretic ring with $\mathbb{Z}$ coefficients. As a consequence, all FJRW correlators of the quintic vanish modulo 5.

Corollary 3.11 *Let W be the loop polynomial and let*

$$(a_1, \ldots, a_5) = (1, -4, 16, -64, 256).$$

For any nonnegative integers g and n in the stable range $2g - 2 + n > 0$, nontrivial monodromies $\underline{\gamma} \in \mu_5^n$, and integers $d_1, \ldots, d_n \in \mathbb{Z}$, the K–theoretic FJRW invariant

of (W, μ_5),

$$\chi(\Psi_1^{\otimes d_1} \otimes \cdots \otimes \Psi_n^{\otimes d_n} \otimes \mathcal{O}^{\mathrm{vir}}_{\mathcal{S}^{1/5}_{g,n}(\underline{\gamma})}) \in \mathbb{Z},$$

equals

$$-\sum_{k=1}^{40} \chi\left(\Psi_1^{\otimes d_1} \otimes \cdots \otimes \Psi_n^{\otimes d_n} \otimes \exp\left(\sum_{l \leq -1} \sum_{j=1}^{5} \frac{e^{2\mathrm{i}\pi kl \cdot a_j/41}}{l} \Psi^l(-R\pi_*\mathcal{L})\right)\right) \in \mathbb{Z}/41\mathbb{Z}.$$

Using [Remark 3.10](#), the correlator vanishes modulo 5, and we can then compute it modulo 205.

Remark 3.12 More generally, the K–class

$$B_{41} := -\sum_{k=1}^{40} \exp\left(\sum_{l \leq -1} \sum_{j=1}^{5} \frac{e^{2\mathrm{i}\pi kl \cdot a_j/41}}{l} \Psi^l(-R\pi_*\mathcal{L})\right) \in K^0(G, \mathcal{S}^{1/5}_{g,n}(\underline{\gamma}))$$

lies in the K–theoretic ring with $\mathbb{Z}$–coefficients and we know there exists another K–class R in $K^0(G, \mathcal{S}^{1/5}_{g,n}(\underline{\gamma}))$ with $\mathbb{Z}$–coefficients such that

$$\mathcal{O}^{\mathrm{vir}}_{\mathcal{S}^{1/5}_{g,n}(\underline{\gamma})} = B_{41} + 41 \cdot R \in K^0(G, \mathcal{S}^{1/5}_{g,n}(\underline{\gamma})).$$

This yields the following formula for the FJRW virtual cycle of (W, μ_5):

$$c_{\mathrm{vir}} = \sum_{k=1}^{40}\left(\lim_{t \to \zeta_{41}^k} \prod_{i=1}^{5} \mathfrak{c}_{t_j}(-R\pi_*\mathcal{L}) + 41 \cdot \mathrm{Ch}(R)\right) \mathrm{Td}(-R\pi_*\mathcal{L})^5 \in H^*(\mathcal{S}^{1/5}_{g,n}(\underline{\gamma}))_{\mathbb{Q}}.$$

However, since the cohomology is taken with $\mathbb{Q}$–coefficients, we do not obtain congruence results on the virtual cycle. An idea would be to guess a formula for the K–class R as an integral linear combination of (natural) vector bundles over $\mathcal{S}^{1/5}_{g,n}(\underline{\gamma})$. Since the virtual cycle is pure-dimensional and the right-hand side of the formula above is most likely not pure-dimensional when we take a generic R, only special integral coefficients in this linear combination would work.

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Institut Fourier, CNRS, Université de Grenoble Alpes
Grenoble, France

jeremy.guere@univ-grenoble-alpes.fr

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