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Moduli of spherical tori with one conical point

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We determine the topology of the moduli space $\mathcal{MS}_{1,1}(\vartheta)$ of surfaces of genus one with a Riemannian metric of constant curvature 1 and one conical point of angle $2\pi\vartheta$. In particular, for $\vartheta \in (2m-1, 2m+1)$ nonodd, $\mathcal{MS}_{1,1}(\vartheta)$ is connected, has orbifold Euler characteristic $-\frac{1}{12}m^2$, and its topology depends on the integer $m > 0$ only. For $\vartheta = 2m+1$ odd, $\mathcal{MS}_{1,1}(\vartheta)$ has $\left\lceil \frac{1}{6}m(m+1) \right\rceil$ connected components. For $\vartheta = 2m$ even, $\mathcal{MS}_{1,1}(\vartheta)$ has a natural complex structure and it is biholomorphic to $\mathbb{H}^2/G_m$ for a certain subgroup G_m of $\mathrm{SL}(2,\mathbb{Z})$ of index m^2, which is nonnormal for $m > 1$.

58D27; 32Q20, 52B70

1 Introduction and main results

A spherical metric on a surface S with *conical points* at the points $\boldsymbol{x} = \{x_1, \dots, x_n\} \in S$ is a Riemannian metric of curvature 1 on $\dot{S} := S \setminus \boldsymbol{x}$ such that a neighborhood of x_j is isometric to a cone with a conical angle $2\pi\vartheta_j > 0$.

Let us immediately specify what we mean by the moduli space $\mathcal{MS}_{g,n}(\vartheta)$ of spherical surfaces. As a set, $\mathcal{MS}_{g,n}(\vartheta)$ parametrizes compact connected oriented surfaces of genus g with a spherical metric that has conical angles $(2\pi\vartheta_1,\dots,2\pi\vartheta_n)$ at marked points $x_1,\dots,x_n$. Two surfaces correspond to the same point of the space if there is a marked isometry from one to the other. In order to define a topology on $\mathcal{MS}_{g,n}(\vartheta)$, we consider the bi-Lipschitz distance between marked surfaces; see Gromov [15]. Such a distance defines a metric, and the corresponding topology on $\mathcal{MS}_{g,n}(\vartheta)$ is called the *Lipschitz topology*; its properties are discussed in Section 6.

As a spherical metric defines a conformal structure on the surface, we have the *forgetful map* $F:\mathcal{MS}_{g,n}(\theta)\to\mathcal{M}_{g,n}$, where $\mathcal{M}_{g,n}$ is the moduli space of conformal structures on $(S,\boldsymbol{x})$.

Since a neighborhood of a smooth point on S is isometric to an open set on the sphere equipped with the standard spherical metric, by an analytic continuation we obtain an orientation-preserving locally isometric *developing map* $f:\dot{S}\to\mathbb{S}^2$. Strictly speaking, the developing map is defined on the universal cover of $\dot{S}$ but it is sometimes convenient to think of it as a multivalued function on $\dot{S}$.

The developing map defines a representation of the fundamental group of $\dot{S}$ to the group SO(3) of rotations of the unit sphere $\mathbb{S}^2$. The image of this representation is called the *monodromy group*.

Our goal is to provide an explicit description of the moduli space $\mathcal{MS}_{1,1}(\vartheta)$ of spherical tori with one conical point.

Spherical tori with one conical point were also studied by Chai, Lin and Wang [2], Chen and Lin [6], Chen, Kuo and Lin [5], Eremenko [10], Eremenko and Gabrielov [11] and Lin and Wang [19; 20].

1.1 Main results

Our main results consist of Theorems A–F, which are stated in the next three subsections.

1.1.1 ϑ not an odd integer

Theorem A (topology of $\mathcal{MS}_{1,1}(\vartheta)$ for ϑ not odd) *Take $\vartheta\in(1,\infty)$ that is not an odd integer and set $m=\left\lfloor\frac{1}{2}(\vartheta+1)\right\rfloor$. The moduli space $\mathcal{MS}_{1,1}(\vartheta)$ of spherical tori with a conical point of angle $2\pi\vartheta$ is a connected orientable 2–dimensional orbifold of finite type with the following properties*:

(i) *As a surface,* $\mathcal{MS}_{1,1}(\vartheta)$ *has genus* $\left\lfloor \frac{1}{12}(m^2-6m+12) \right\rfloor$ *and* m *punctures.*

(ii) $\mathcal{MS}_{1,1}(\vartheta)$ *has orbifold Euler characteristic* $\chi(\mathcal{MS}_{1,1}(\vartheta)) = -\frac{1}{12}m^2$*. Moreover, it has at most one orbifold point of order* 4 *and at most one orbifold point of order* 6*. All the other points are orbifold points of order* 2*.*

(iii) $\mathcal{MS}_{1,1}(\vartheta)$ *has one orbifold point of order* 6 *if and only if* $d_1(\vartheta, 6\mathbb{Z}) > 1$*.*

(iv) $\mathcal{MS}_{1,1}(\vartheta)$ *has one orbifold point of order* 4 *if and only if* $d_1(\vartheta, 4\mathbb{Z}) > 1$*.*

Note that for $\vartheta = 2m$ this theorem gives a positive answer to the question of Chai, Lin and Wang [2, Question 4.6.6(a)] as to whether $\mathcal{MS}_{1,1}(2m)$ is connected.

We refer to Cooper, Hodgson and Kerckhoff [7] for a general treatment of orbifolds. In fact we adopt a slightly more general definition of orbifolds that includes the case in which all points can have orbifold order greater than 1. The definition of orbifold Euler characteristic is given on page 29 of [7]. This is consistent with the definition used, for example, in Harer and Zagier [16]. A few properties of the orbifold Euler characteristic are listed in Remark 4.7.

Note that, in [13], with Gabrielov we used a different convention and we endowed our moduli spaces with an orbifold structure for which the order of each point is half the number of automorphisms of the corresponding object. Thus, the orbifold Euler characteristics computed in [13] are twice the ones that would be obtained following the convention here.

Remark 1.1 (orbifold structure and isometric involution) For ϑ not odd, spherical metrics in $\mathcal{MS}_{1,1}(\vartheta)$ are invariant under the unique conformal involution σ of tori (see Proposition 2.17). Thus every such spherical torus is a double cover of a spherical surface of genus 0 with conical points of angles $(\pi\vartheta, \pi, \pi, \pi)$, and so the moduli space $\mathcal{MS}_{1,1}(\vartheta)$ is homeomorphic to $\mathcal{MS}_{0,4}\big(\frac{1}{2}\vartheta, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\big)/\mathrm{S}_3$ as a topological space. On the other hand, the orbifold order of a point in $\mathcal{MS}_{1,1}(\vartheta)$ exactly corresponds to the number of (orientation-preserving) self-isometries of the corresponding spherical torus. This explains why every point of $\mathcal{MS}_{1,1}(\vartheta)$ has even orbifold order, as stated in Theorem A. Thus $\mathcal{MS}_{1,1}(\vartheta)$ is not isomorphic to the orbifold quotient $\mathcal{MS}_{0,4}\big(\frac{1}{2}\vartheta, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\big)/\mathrm{S}_3$.

An important geometric input on which Theorem A hinges is the notion of *balanced spherical triangles*; Theorem B describes the relation between spherical tori and balanced triangles.

Definition 1.2 (spherical polygons) A *spherical polygon* P with angles $\pi(\vartheta_1, \dots, \vartheta_n)$ is a closed disk equipped with a Riemannian metric of constant curvature 1, with n distinguished boundary points $x_1, \dots, x_n$ which are called *vertices*, and such that the arcs between the adjacent vertices are geodesics forming an interior angle $\pi\vartheta_i$ at the i^{th} vertex. Two polygons are *isometric* if there is an isometry between them that preserves the labeling.

Spherical polygons with two or three vertices are called *digons* or *triangles*.[1]

Definition 1.3 (balanced triangles) A spherical triangle Δ with angles $\pi(\vartheta_1, \vartheta_2, \vartheta_3)$ is called *balanced* if the numbers ϑ_1, ϑ_2 and ϑ_3 satisfy the three triangle inequalities. If the triangle inequalities are satisfied strictly, we call the triangle *strictly balanced*. If, for some permutation (i, j, k) of $(1, 2, 3)$, we have $\vartheta_i = \vartheta_j + \vartheta_k$, we call the triangle *semibalanced*. If $\vartheta_i > \vartheta_j + \vartheta_k$ for some i, we call the triangle *unbalanced*.

Semibalanced triangles are called *marginal* in Eremenko and Gabrielov [12] and [13].

Whenever a spherical triangle is realized as a subset of a surface, we will induce on it the orientation of the surface. We will say that two oriented spherical surfaces (or polygons) are *conformally isometric* (or *congruent*) if there is an orientation-preserving isometry from one surface (or polygon) to the other.

Terminology (integral angles) Throughout the paper angles will be measured in radians. Nevertheless, an angle $2\pi\vartheta$ at a conical point of a spherical surface is called *integral* if $\vartheta \in \mathbb{Z}_{>0}$; similarly, an angle $\pi\vartheta$ at a vertex of a spherical polygon is called *integral* if $\vartheta \in \mathbb{Z}_{>0}$.

Now we describe a construction that will be omnipresent:

Construction 1.4 To each spherical triangle Δ with vertices x_1, x_2 and x_3 one can associate a spherical torus $T(\Delta)$ with one conical point by taking a conformally isometric triangle Δ' with vertices x'_1, x'_2 and x'_3 and isometrically identifying each side $x_i x_j$ with the side $x'_j x'_i$ (in such a way that x_i is identified to x'_j and x_j is identified to x'_i) for $i, j \in \{1, 2, 3\}$. The angle at the conical point of $T(\Delta)$, which corresponds to the vertices of the triangles, is twice the sum of the angles of Δ. If Δ is endowed with an orientation, then $T(\Delta)$ canonically inherits an orientation.

[1] We note that spherical triangles in the sense of our definition are sometimes called *Schwarz–Klein triangles* to distinguish them from triangles understood as broken geodesic lines on the sphere; see for instance [12].

To state the next result we need two more notions. Let T be a spherical torus with one conical point. An isometric orientation-reversing involution on T will be called a *rectangular involution* if its set of fixed points consists of two connected components. By a *geodesic loop* γ based at a conical point x we mean a loop based at x which is geodesic in $\dot{T} = T \setminus \{x\}$ and which passes through x only at its endpoints.

Theorem B (canonical decomposition of a spherical torus for nonodd ϑ) *Let (T, x) be a spherical torus with one conical point of angle $2\pi\vartheta$ such that $\vartheta \in (1,\infty)\setminus(2\mathbb{Z}+1)$.*

(i) *If T does not have a rectangular involution, then there exists a unique (up to a reordering) triple of geodesic loops $(\gamma_1, \gamma_2, \gamma_3)$ based at x that cuts T into two congruent strictly balanced spherical triangles.*

(ii) *If T has a rectangular involution, there exist exactly two (unordered) triples of geodesic loops such that each of them cuts T into two congruent balanced triangles. Moreover, such triangles are semibalanced. These two triples are exchanged by the rectangular involution.*

We recall that, by Mondello and Panov [23, Section 4], the Voronoi graph associated to a spherical surface with n conical points decomposes such a surface into the union of n topological disks with one conical point each. Indeed, the role of this Voronoi graph is analogous to the role of the critical graph of a Jenkins–Strebel differential (a procedure that allows one to build a spherical surface out of a Jenkins–Strebel differential is described by Song, Cheng, Li and Xu [26]).

In order to prove Theorem B, we note that the complement of the Voronoi graph of the spherical torus (T, x) is one disk, and that this disk can be further split into two congruent triangles using the conformal involution of the torus. As a consequence of Theorem B, to each spherical torus T one can associate an essentially unique balanced spherical triangle $\Delta(T)$. Such uniqueness will permit us to reduce the description of the moduli space $\mathcal{MS}_{1,1}(\vartheta)$ to that of the moduli space of balanced triangles of area $\pi(\vartheta-1)$.

1.1.2 ϑ an odd integer The case when ϑ is an odd integer is quite different, as not all spherical metrics are invariant under the unique (nontrivial) conformal involution σ of the tori. We begin by stating our result for metrics that are σ–invariant:

Theorem C (topology of $\mathcal{MS}_{1,1}(2m+1)^\sigma$) *Fix an integer $m \geq 1$ and consider the moduli space $\mathcal{MS}_{1,1}(2m+1)^\sigma$ of tori with a σ–invariant spherical metric of area $4m\pi$.*

(a) *As a topological space, $\mathcal{MS}_{1,1}(2m+1)^\sigma$ is homeomorphic to the disjoint union of $\left\lceil \frac{1}{6}m(m+1)\right\rceil$ 2–dimensional open disks.*

(b) $\mathcal{MS}_{1,1}(2m+1)^\sigma$ *is naturally endowed with the structure of a* 2*–dimensional orbifold with* $\lceil \frac{1}{6}m(m+1) \rceil$ *connected components, which can be described as follows:*

(b-i) *If* $m \not\equiv 1 \pmod 3$, *then all components are isomorphic to the quotient* $\mathcal{D}$ *of* $\mathring{\Delta}^2 = \{y \in \mathbb{R}^3_+ \mid y_1 + y_2 + y_3 = 2\pi\}$ *by the trivial* $\mathbb{Z}_2$*–action. Hence, every point of* $\mathcal{MS}_{1,1}(2m+1)^\sigma$ *has orbifold order* 2.

(b-ii) *If* $m \equiv 1 \pmod 3$, *then one component is isomorphic to the quotient* $\mathcal{D}'$ *of* $\mathring{\Delta}^2$ *by* $\mathbb{Z}_2 \times \mathrm{A}_3$, *where* $\mathbb{Z}_2$ *acts trivially and* A_3 *acts by cyclically permuting the coordinates of* $\mathring{\Delta}^2$, *and all the other components are isomorphic to* $\mathcal{D}$. *Hence, one point of* $\mathcal{MS}_{1,1}(2m+1)^\sigma$ *has orbifold order* 6 *and all the other points have order* 2.

Remark 1.5 Similarly to Remark 1.1, as a topological space $\mathcal{MS}_{1,1}(2m+1)^\sigma$ is homeomorphic to $\mathcal{MS}_{0,4}\big(m+\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\big)/\mathrm{S}_3$ (though they are not isomorphic as orbifolds). Thus, Theorem C has a connection with the results of Chai, Lin and Wang [2], Chen and Lin [6], Eremenko [10], Eremenko and Gabrielov [11] and Lin and Wang [20].

The following description of the moduli space of tori with metrics that are not necessarily σ–invariant will be deduced from Theorem C:

Theorem D (topology of $\mathcal{MS}_{1,1}(2m+1)$) *For each positive integer* m, *the moduli space* $\mathcal{MS}_{1,1}(2m+1)$ *is a* 3*–dimensional orbifold with* $\lceil \frac{1}{6}m(m+1) \rceil$ *connected components.*

(i) *If* $m \not\equiv 1 \pmod 3$, *then all components of* $\mathcal{MS}_{1,1}(2m+1)$ *are isomorphic to the quotient* $\mathcal{M}$ *of* $\mathring{\Delta}^2 \times \mathbb{R}$ *by the involution* $(y,t) \mapsto (y,-t)$.

(ii) *If* $m \equiv 1 \pmod 3$, *then one component of* $\mathcal{MS}_{1,1}(2m+1)$ *is isomorphic to the quotient* $\mathcal{M}'$ *of* $\mathring{\Delta}^2 \times \mathbb{R}$ *by* $\mathbb{Z}_2 \times \mathrm{A}_3$, *where* $\mathbb{Z}_2$ *acts via the involution* $(y,t) \mapsto (y,-t)$ *and the alternating group* A_3 *acts by cyclically permuting the coordinates of* $\mathring{\Delta}^2$. *All the other components are isomorphic to* $\mathcal{M}$.

The locus $\mathcal{MS}_{1,1}(2m+1)^\sigma$ *of* σ*–invariant metrics correspond to* $t=0$.

In order to understand what happens for spherical metrics that are not necessarily σ–invariant, we recall:

Definition 1.6 (coaxiality) A monodromy is *coaxial* if and only if it is contained inside a one-parameter subgroup $\mathrm{SO}(3,\mathbb{R})$. A spherical surface is called *coaxial* if its monodromy is.

Note that every spherical metric with nontrivial coaxial monodromy on a surface belongs to a 1–parameter family of metrics that induce the same $\mathbb{CP}^1$–structure; we will say that metrics in the same 1–parameter family are *projectively equivalent*.

In the present case, a spherical metric on a torus T with one conical point of angle $2\pi\vartheta$ has nontrivial monodromy. Moreover, the monodromy is coaxial if and only if ϑ is odd. This fact is proven in [2, Theorem 5.2] and can be also deduced by combining the observations of Li, Song and Xu [18, page 8] with Chen, Wang, Wu and Xu [4, Proposition 1.4]. We reprove this statement using an argument based on monodromy considerations (see Corollary A.2).

The above discussion shows that every spherical surface in $\mathcal{MS}_{1,1}(2m+1)$ belongs to a 1–parameter family of projectively equivalent metrics, which thus traces a copy of $\mathbb{R}$ inside $\mathcal{MS}_{1,1}(2m+1)$. Moreover, in every family there exists exactly one metric which is σ–invariant (see Proposition 2.17). For this reason $\mathcal{MS}_{1,1}(2m+1)^\sigma$ is isomorphic to the moduli space $\mathcal{MS}_{1,1}(2m+1)/\mathrm{proj}$ of projective classes of spherical tori of area $4m\pi$, and so $\mathcal{MS}_{1,1}(2m+1)$ is 3–dimensional.

Another major difference from the nonodd case concerns the forgetful map: for ϑ nonodd, the forgetful map $\mathcal{MS}_{1,1}(\vartheta) \to \mathcal{M}_{1,1}$ is proper (see Mondello and Panov [23]) and surjective, whereas this is not so for odd ϑ; see Lin and Wang [19]. The boundary of $\mathcal{MS}_{1,1}(2m+1)/\mathrm{proj}$ inside the space of $\mathbb{CP}^1$–structures describes interesting real-analytic curves (see [2]) that are investigated in the sequel to this paper [13].

Theorems C and D will rely on the following result, which links moduli spaces of tori to moduli spaces of balanced triangles with integral angles:

Theorem E (canonical decomposition of a spherical torus with odd ϑ) *Fix a spherical torus with one conical point of angle $2\pi(2m+1)$. In the same projective class there exists a unique spherical torus (T, x) that admits an isometric orientation-preserving involution. Also, there exists a unique collection of three geodesic loops $(\gamma_1, \gamma_2, \gamma_3)$ based at x that cuts T into two congruent balanced spherical triangles Δ and Δ' with integral angles $\pi(m_1, m_2, m_3)$.*

1.1.3 ϑ an even integer Our final main result concerns moduli spaces $\mathcal{MS}_{1,1}(2m)$ where m is a positive integer. It is known (see Chai, Lin and Wang [2] and Eremenko and Tarasov [14]) that these moduli spaces have a natural holomorphic structure with respect to which they are compact Riemann surfaces with punctures. This is the unique conformal structure which makes the forgetful map to $\mathcal{M}_{1,1}$ holomorphic. With this structure $\mathcal{MS}_{1,1}(2m)$ is an algebraic curve.

Theorem F ($\mathcal{MS}_{1,1}(2m)$ is a Belyi curve) *For each integer $m > 0$ there exists a subgroup $G_m < \mathrm{SL}(2,\mathbb{Z})$ of index m^2 such that the orbifold $\mathcal{MS}_{1,1}(2m)$ is biholomorphic to the quotient $\mathbb{H}^2/G_m$. Such G_m is nonnormal for $m > 1$. Moreover, the points in $\mathbb{H}^2/G_m$ that project to the geodesic ray $[i,\infty)$ in the modular curve $\mathbb{H}^2/\mathrm{SL}(2,\mathbb{Z})$ correspond to tori T such that the triangle $\Delta(T)$ has one integral angle.*

1.2 Analytic representation of spherical metrics

Let (T,x) be a spherical torus with a conical singularity at x of angle $2\pi\vartheta$. The pullback of the spherical metric via the universal cover $\mathbb{C} = \widetilde{T} \to T$ has area element $e^u|dz|^2$. Then the function u satisfies the nonlinear PDE

$$\Delta u + 2e^u = 2\pi(\vartheta - 1)\delta_\Lambda, \tag{1}$$

where δ_Λ is the sum of delta functions over the lattice Λ and T is biholomorphic to $\mathbb{C}/\Lambda$. So our results describe the moduli spaces of pairs (Λ, u), where u is a Λ–periodic solution of (1).

Equation (1) is the simplest representative of the class of "mean field equations", which have important applications in physics; see Tarantello [27].

The general solution of (1) can be expressed in terms of a function $f : \mathbb{C} \to \mathbb{CP}^1$, the developing map, which is related to the conformal factor u by

$$u = \log \frac{4|f'|^2}{(1+|f|^2)^2}.$$

The developing map $f = w_1/w_2$ is the ratio of two linearly independent solutions w_1 and w_2 of the Lamé equation

$$w'' = \Big(\frac{\vartheta^2 - 1}{4}\wp - c\Big)w, \tag{2}$$

where $\wp$ is the Weierstrass function of the lattice Λ and $c \in \mathbb{C}$ is an accessory parameter. So our results can be also interpreted as a description of the moduli space of projective structures on tori whose monodromies are subgroups of $\mathrm{SO}(3,\mathbb{R})$.

Most of the known results on spherical tori are formulated in terms of (1) and (2). For example, it is proved in Chen and Lin [3] that when ϑ is not an odd integer, then the Leray–Schauder degree of the nonlinear operator in (1) equals $\big\lfloor \frac{1}{2}(\vartheta+1) \big\rfloor$. An especially well-studied case is the classical Lamé equation (2) where ϑ is an integer; see [2; 13]. Solutions of (2) with odd integer ϑ are special functions of mathematical physics; see Maier [21] and Whittaker and Watson [29].

1.3 The idea of the proof of Theorem A

Here we give a brief summary of the proof of Theorem A, since various parts of it stretch through the whole paper. Fix $\vartheta > 1$ not odd and consider spherical tori with a conical point of angle $2\pi\vartheta$, and area $2\pi(\vartheta-1)$.

- By Proposition 2.17(i), on every torus the unique nontrivial conformal involution is an isometry.

- Every spherical torus is obtained by gluing two isometric copies of a spherical balanced triangle with labeled vertices in an essentially unique way (Theorem B, proven in Section 2.4). This result has a clear refinement for tori with a 2–marking (namely, a labeling of its 2–torsion points); see Construction 4.5.

- The doubled space $\mathcal{MT}^{\pm}_{\mathrm{bal}}(\vartheta)$ of balanced triangles of area $\pi(\vartheta-1)$ is the double of the space $\mathcal{MT}_{\mathrm{bal}}(\vartheta)$ of balanced triangles of area $\pi(\vartheta-1)$ and it describes oriented balanced triangles up to some identifications that only involve semibalanced triangles (Definition 3.21).

- The space $\mathcal{MT}_{\mathrm{bal}}(\vartheta)$ is an orientable connected surface with boundary, and its topology is completely determined (see Proposition 3.20) and so is the topology of $\mathcal{MT}^{\pm}_{\mathrm{bal}}(\vartheta)$; see Proposition 3.22.

- As a topological space, the space $\mathcal{MS}^{(2)}_{1,1}(\vartheta)$ of isomorphism classes of 2–marked tori is homeomorphic to $\mathcal{MT}^{\pm}_{\mathrm{bal}}(\vartheta)$; see Theorem 6.5.

- As an orbifold, $\mathcal{MS}^{(2)}_{1,1}(\vartheta)$ is isomorphic to the quotient of $\mathcal{MT}^{\pm}_{\mathrm{bal}}(\vartheta)$ by the trivial $\mathbb{Z}_2$–action. This allows us to determine the topology and the orbifold Euler characteristic of $\mathcal{MS}^{(2)}_{1,1}(\vartheta)$; see Theorem 4.8.

- The map $\mathcal{MS}^{(2)}_{1,1}(\vartheta) \to \mathcal{MS}_{1,1}(\vartheta)$ that forgets the 2–marking is an unramified orbifold S_3–cover, where S_3 acts on $\mathcal{MS}^{(2)}_{1,1}(\vartheta)$ by permuting the 2–markings (see Remark 6.28). This allows us to describe the points in $\mathcal{MS}_{1,1}(\vartheta)$ of orbifold order greater than 2 (Proposition 4.4) and to determine the topology and the orbifold Euler characteristic of $\mathcal{MS}_{1,1}(\vartheta)$; see Theorem A, proved towards the end of Section 4.1.

1.4 Content of the paper

The relation between spherical tori with one conical point and balanced spherical triangles is established in Section 2, which culminates in the proof of Theorem B. The section contains a careful analysis of the Voronoi graph of a torus and of the action of the unique nontrivial conformal involution σ on its spherical metric.

In Section 3 we describe the topology of the space of balanced triangles of area $\pi(\vartheta-1)$ and of its double, separately considering the cases ϑ nonodd and ϑ odd. Here we visualize the space of spherical triangles with assigned area, which is a manifold, by looking at its image (which we call a *carpet*) through the angle map Θ. The balanced carpet will turn out to be a useful tool in computing the topological invariants of the space of balanced triangles.

In Section 4 we describe the topology of the moduli spaces of spherical tori with one conical point, endowed with the Lipschitz metric (which we study in Section 6). For ϑ nonodd, we first establish a homeomorphism between the doubled space of balanced triangles and the topological space of 2–marked tori using tools from Section 6. Then we prove Theorem A. For ϑ odd, we first prove Theorem E using results from Sections 2 and 3, which immediately allows us to prove part (a) of Theorem C. Then we endow our moduli space of σ–invariant spherical tori with a 2–dimensional orbifold structure and prove part (b) of Theorem C. Finally, using one-parameter projective deformations of σ–invariant spherical metrics, we put a 3–dimensional orbifold structure on the moduli space of (not necessarily σ–invariant) tori and prove Theorem D.

In Section 5 we analyze the moduli space of tori with ϑ even, and prove Theorem F by identifying it to a Hurwitz space of covers of $\mathbb{CP}^1$ branched at three points. This permits us to exhibit this moduli space as a Belyi curve and to characterize tori that sit on the 1–dimensional skeleton of its dessin.

Section 6 deals with properties of the Lipschitz metric on moduli spaces of spherical surfaces with conical points with area bounded from above. The main result of the section is Theorem 6.3 on properness of the inverse of the systole function. Then the treatment is specialized to tori with one conical point of angle $2\pi\vartheta$ with ϑ nonodd (or with ϑ odd and a σ–invariant metric). The section culminates with establishing the homeomorphism between the space of 2–marked tori and the doubled space of balanced triangles, needed in Section 4. A last remark explains how to use such results to endow our moduli spaces with an orbifold structure.

In the short appendix we prove a general SU(2)–lifting theorem for the monodromy of a spherical surface, and we apply it to the cases of ϑ odd and ϑ even to explain their special features.

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2 Voronoi diagrams and the proof of Theorem B

In this section we will study the Voronoi graphs of spherical tori (T, x, ϑ) with one conical point and prove Theorem B.

2.1 Properties of Voronoi graphs, functions and domains

In this subsection we recall the definition of a Voronoi graph [23, Section 4] and apply it to spherical tori with one conical point.

Definition 2.1 (Voronoi function and Voronoi graph) Let S be a surface with a spherical metric and conical points $\boldsymbol{x}$. The *Voronoi function* $\mathcal{V}_S\colon S \to \mathbb{R}$ is defined as $\mathcal{V}_S(p) := d(p, \boldsymbol{x})$. The *Voronoi graph* $\Gamma(S)$ is the locus of points $p \in \dot{S}$ at which the distance $d(p, \boldsymbol{x})$ is realized by two or more geodesic arcs joining p to $\boldsymbol{x}$. We will simply write $\Gamma = \Gamma(S)$ when no ambiguity is possible. The *Voronoi domains* of S are connected components of the complement $S \setminus \Gamma(S)$. Each Voronoi domain D_i contains a unique conical point x_i and this point is the closest conical point to all the points in the domain.

Various properties of Voronoi functions, graphs and domains of spherical surfaces were proven in [23, Section 4], and Proposition 2.3 lists some of the facts needed here. To formulate the last two properties we need one more definition:

Definition 2.2 (convex star-shaped polygons) Let D be a disk with a spherical metric, containing a unique conical point $x \in D$, such that its boundary is composed of a collection of geodesic segments. We say that D is a *convex and star-shaped polygon* if any two neighboring sides of D meet under an interior angle smaller than π and for any point $p \in D$ there is a unique geodesic segment that joins x with p.

Proposition 2.3 (basic properties of the Voronoi function and graph) *Let S be a spherical surface of genus g with conical points $x_1, \dots, x_n$.*

(i) *The Voronoi function is bounded from above by π, namely $\mathcal{V}_S < \pi$.*

(ii) *The Voronoi graph $\Gamma(S)$ is a graph with geodesic edges embedded in S and contains at most $-3\chi(\dot{S}) = 6g - 6 + 3n$ edges.*

(iii) *The valence of each vertex of $\Gamma(S)$ is at least three. For any point $p \in \Gamma(S)$ its valence coincides with the multiplicity μ_p, ie there exist exactly μ_p geodesic segments in S of length $\mathcal{V}_S(p)$ that join p with conical points of S.*

(iv) *The metric completion of each Voronoi domain[2] is a convex and star-shaped polygon with a unique conical point in its interior.*

(v) *Let γ be an open edge of $\Gamma(S)$. Let D_i and D_j be two Voronoi domains adjacent to γ. Let $\Delta \subset D_i$ and $\Delta' \subset D_j$ be the two triangles with one vertex x_i or x_j, respectively, and opposite side γ. Then Δ and Δ' are anticonformally isometric by an isometry fixing γ.*

Proof (i) This is proven in [23, Lemma 4.2].

(ii) This is proven in [23, Lemma 4.5 and Corollary 4.7].

(iii) The valence of vertices is at least three by [23, Corollary 4.7]. The valence of a point on $\Gamma(S)$ coincides with its multiplicity by [23, Lemma 4.5].

(iv) The convexity is proven in [23, Lemma 4.8]. The fact that each domain is star-shaped follows from the fact that each point p in it can be joined, by a unique geodesic segment of length $\mathcal{V}_S(p)$, with the conical point. Such a segment varies continuously with p, since $\mathcal{V}_S(p) < \pi$.

(v) To find the isometry between Δ and Δ' just notice that by definition each point $p \in \gamma$ can be joined by two geodesics of the same length with x_i and x_j. Also these two geodesics intersect γ under the same angle. The isometry between the triangles is obtained by the map exchanging each pair of such geodesics. □

Example 2.4 (Voronoi graph in a sphere with three conical points) Let S be a sphere with three conical points. It follows from Proposition 2.3(ii) that the Voronoi graph $\Gamma(S)$ is either a trefoil graph, an eight graph or an eyeglasses graph; see Figure 1. Indeed, $\Gamma(S)$ splits S into three disks, and it has at most three edges.

The next definition and remark explain how to define Voronoi functions and graphs for spherical polygons, mimicking Definition 2.1.

Definition 2.5 (Voronoi function and graph of a polygon) Let P be a spherical polygon with vertices $\boldsymbol{x}$. Then the Voronoi function $\mathcal{V}_P \colon P \to \mathbb{R}$ is defined as

[2] The metric completion can differ from the closure of the domain inside S; see the rightmost example in Figure 2.

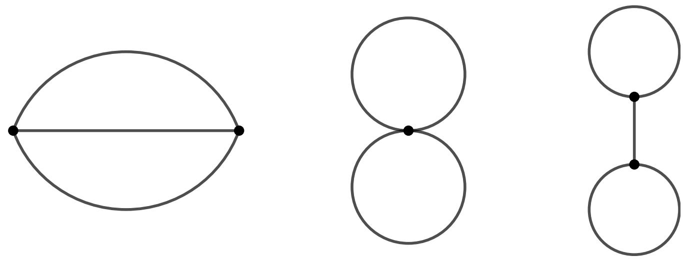

Figure 1: Voronoi graphs on a sphere with three conical points. From left to right: the trefoil, the eight graph and the eyeglass graph.

$\mathcal{V}_P(p) := d(p, \boldsymbol{x})$. The Voronoi graph $\Gamma(P)$ of P consists of points p of two types: first, the points for which there exist at least two geodesic segments of length $d(p, \boldsymbol{x})$ that join p with $\boldsymbol{x}$, and second, the points p on ∂P for which the closest vertex of P does not lie on the edge to which p belongs.

Remark 2.6 (doubling a polygon: Voronoi function and graph) To each spherical polygon P one can associate a sphere $S(P)$ with conical singularities by *doubling*[3] P across its boundary. Such a sphere has an anticonformal isometry that exchanges P and its isometric copy P', and fixes their boundary. It is easy to see that the function $\mathcal{V}_{S(P)}$ restricts to $\mathcal{V}_P$ on $P \subset S$ and to $\mathcal{V}_{P'}$ on $P' \subset S$. One can also check that the Voronoi graph $\Gamma(S(P))$ is the union $\Gamma(P) \cup \Gamma(P')$. As a result, the statements of Proposition 2.3 have their analogues for spherical polygons.

The following lemma gives an efficient criterion permitting one to verify whether a given geodesic graph on a spherical surface is in fact its Voronoi graph:

Lemma 2.7 (Voronoi graph criterion) *Let S be a spherical surface of genus g with conical points $x_1, \dots, x_n$ and let $\Gamma'(S) \subset S$ be a finite graph with geodesic edges embedded in S. Then $\Gamma'(S) = \Gamma(S)$ if and only if the following two conditions hold*:

(a) *$S \setminus \Gamma'(S)$ is a union of disks whose metric completions are convex and star-shaped polygons each with a unique conical point in its interior.*

(b) *For each point $p \in \Gamma'(S)$ all geodesic segments that join p with some conical point of S and intersect $\Gamma'(S)$ only at p have the same length.*

Proof Since by Proposition 2.3 the graph $\Gamma(S)$ satisfies the conditions (a) and (b), we only need to prove the "only if" direction.

[3]Given a topological space X and a closed subset A, the *doubling of X along A* is obtained from $X \times \{0, 1\}$ by making the identification $(a, 0) \sim (a, 1)$ for every $a \in A$.

For each conical point x_i let D_i be the Voronoi domain of x_i (namely the connected component of $S \setminus \Gamma(S)$ that contains x_i), and let D'_i be the component of $S \setminus \Gamma'(S)$ containing x_i. Let's assume, for contradiction, that there is a point $p \in D_i$ that is not contained in D'_i. By definition of D_i there is a unique geodesic segment $\gamma(p)$ of length $\mathcal{V}_S(p)$ that joins p with x_i. Denote by $\gamma'(p)$ the connected component of the intersection $\gamma(p) \cap D'_i$ that contains x_i and let $p' \notin D'_i$ be the point in its closure. Clearly p' belongs to $\Gamma'(S)$. By (a) each component of $S \setminus \Gamma'(S)$ is star-shaped, so using (b) we get a second (different from $\gamma'(p)$) geodesic segment of length $\mathcal{V}_S(p')$ that joins p' with a conical point. Hence $p' \in \Gamma(S)$, which contradicts the fact that p' is in D_i.

We proved that $D_i \subset D'_i$ for each i. It follows that $D_i = D'_i$, hence $\Gamma'(S) = \Gamma(S)$. □

Lemma 2.8 (Voronoi graphs of a sphere with three conical points) *Let S be a sphere with three conical points x_i of conical angles $2\pi\vartheta_i$.*

(i) *$\Gamma(S)$ is a trefoil if and only if ϑ_1, ϑ_2 and ϑ_3 satisfy the triangle inequality strictly.*

(ii) *$\Gamma(S)$ is an eight graph if and only if $\vartheta_i = \vartheta_j + \vartheta_k$ for some permutation (i, j, k) of $\{1, 2, 3\}$.*

(iii) *$\Gamma(S)$ is an eyeglasses graph if and only if $\vartheta_i > \vartheta_j + \vartheta_k$ for some permutation (i, j, k) of $\{1, 2, 3\}$.*

(iv) *In cases* (i) *and* (ii) *the vertices of $\Gamma(S)$ are equidistant from x_1, x_2 and x_3. In case* (iii) *the vertices of $\Gamma(S)$ are not equidistant from x_1, x_2 and x_3.*

Proof It is enough to prove the "only if" parts of claims (i), (ii) and (iii); the cases are mutually exclusive and so the "if" part will follow.

For the proof of the "only if" part, all three cases are treated in a similar way. Let us consider, for example, the case when $\Gamma(S)$ is a trefoil graph. Let's show that in this case the ϑ_i satisfy the triangle inequality strictly. Denote the two vertices of $\Gamma(S)$ by A and B. The three edges of $\Gamma(S)$ cut S into three Voronoi disks, each of which contains one conical point. Let us denote these three segments of $\Gamma(S)$ by γ_1, γ_2 and γ_3, as shown in the leftmost picture in Figure 2. Let us join each of the x_i with the vertices A and B by geodesics x_iA and x_iB of lengths $\mathcal{V}_S(A)$ and $\mathcal{V}_S(B)$, respectively. These geodesic segments are depicted in gray.

Consider now the spherical quadrilaterals Ax_3Bx_1, Ax_1Bx_2 and Ax_2Bx_3 into which the gray geodesics cut S. It follows from Proposition 2.3(v) for $i, j \in \{1, 2, 3\}$ that the

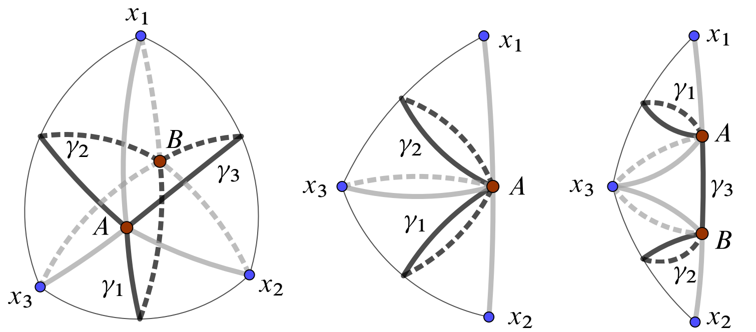

Figure 2: Three types of spheres.

angles of Ax_iBx_j at x_i and x_j are equal. This implies that ϑ_1, ϑ_2 and ϑ_3 satisfy the triangle inequality strictly.

(ii)–(iii) One treats the cases when $\Gamma(S)$ is an eight graph or an eyeglasses graph in a similar way; the corresponding pictures are shown in Figure 2.

(iv) This is clear from the way $\Gamma(S)$ is embedded in S; see Figure 2. In particular, if $\Gamma(S)$ is an eyeglasses graph, $d(A,x_1) = d(A,x_3) < d(A,x_2)$ and $d(B,x_2) = d(B,x_3) < d(B,x_1)$. □

2.2 The circumcenters of balanced triangles

It is well known that the circumcenter of a Euclidean triangle Δ is contained in Δ if and only if Δ is not obtuse. Moreover, in the case when Δ is right-angled, the circumcenter is the midpoint of the hypotenuse. It is also a classical fact that the circumcenter of a Euclidean triangle is the point of intersection of the axes[4] of its sides. The next theorem is a generalization of the above statements to spherical triangles. By an *involutive triangle* we mean a triangle that admits an anticonformal isometric involution that fixes one vertex and exchanges the other two.[5]

Theorem 2.9 (circumcenters of balanced triangles) *Let Δ be a spherical triangle with vertices x_1, x_2 and x_3.*

(i) *The triangle Δ contains a point O equidistant from x_1, x_2 and x_3 if and only if Δ is balanced.*

[4]The *axis* of a segment is the perpendicular through the midpoint of such segment.

[5]Note that every Euclidean or hyperbolic isosceles triangle admits an isometric involution exchanging the equal sides. This is not the case for spherical triangles; for example the triangle with angles $\frac{5}{2}\pi$, $\frac{13}{2}\pi$ and $\frac{9}{2}\pi$ is equilateral but clearly has no symmetries.

(ii) *The point O (equidistant from x_1, x_3 and x_3) is in the interior of Δ if and only if Δ is strictly balanced. The point O is the midpoint of a side of Δ if and only if Δ is semibalanced.*

(iii) *If Δ is strictly balanced, then the geodesic segments Ox_1, Ox_2 and Ox_3 cut Δ into three involutive triangles.*

(iv) *Suppose that Δ is semibalanced and the angle $\angle x_i = \pi\vartheta_i$ is the largest one. Then O is the midpoint of the side opposite to x_i, and $x_i O$ cuts Δ into two involutive triangles.*

To prove this theorem we need the following lemma.

Lemma 2.10 (some isosceles triangles are involutive triangles) *Let Δ be a spherical triangle with vertices q_1, q_2 and q_3 and denote by $|q_i q_j|$ the length of the side $q_i q_j$. Suppose that $|q_1q_2| = |q_1q_3| < \pi$ and $\angle q_1 < 2\pi$. Then there is an isometric reflection τ of Δ that fixes q_1 and exchanges q_2 with q_3. In particular, $\angle q_2 = \angle q_3$. Moreover, τ pointwise fixes a geodesic segment that joins q_1 with the midpoint of q_2q_3 and splits Δ into two isometric triangles. Furthermore, $|q_2q_3| < 2\pi$.*

Proof First, let $\angle q_1 = \pi$. In this case Δ can be isometrically identified with a digon so that q_1 is identified with the midpoint of one of its sides. Since each digon has an isometric reflection fixing the midpoints of both sides, the lemma holds.

From now on we assume that $\angle q_1 \neq \pi$. Consider the unique spherical triangle $\Delta' \subset \mathbb{S}^2$ with vertices q_1', q_2' and q_3' such that $|q_1'q_2'| = |q_1'q_3'| = |q_1q_2|$, $\angle q_1' = \angle q_1$, and $\mathrm{Area}(\Delta') < 2\pi$. We will show that Δ' admits an isometric embedding into Δ that sends q_i' to q_i. This will prove the lemma since this implies that Δ is isometric to a triangle obtained by gluing a digon to the side $q_2'q_3'$ of Δ'. And such a triangle clearly has an isometric reflection τ. This will also prove that $|q_2q_3| < 2\pi$, since $|q_2'q_3'| < 2\pi$ and either $|q_2q_3| = |q_2'q_3'|$ or $|q_2q_3| + |q_2'q_3'| = 2\pi$.

To prove the existence of the embedding, denote by $\iota\colon \Delta \to \mathbb{S}^2$ the developing map of triangle Δ. We may assume that $\iota(q_i) = q_i'$, $\iota(q_1q_2) = q_1'q_2'$ and $\iota(q_1q_3) = q_1'q_3'$. Note that ι sends q_2q_3 to the unique[6] geodesic circle that contains $\iota(q_2)$ and $\iota(q_3)$. Hence, it is not hard to see that the preimages of Δ' in Δ form a union of some number of isometric copies of Δ'. One of them, containing sides q_1q_2 and q_1q_3 of Δ, is the embedding we are looking for. □

[6]This circle is unique since $\angle q_1 \neq \pi$, and also it intersects the segments $q_1'q_2'$ and $q_1'q_3'$ only at the points q_2' and q_3'.

Remark 2.11 This lemma is sharp in the sense that neither of the two conditions $|q_1q_2| = |q_1q_3| < \pi$ and $\angle q_1 < 2\pi$ can be dropped.

Proof of Theorem 2.9 (i) Let $S(\Delta)$ be the sphere obtained by doubling Δ across its boundary, ie by gluing Δ with the triangle Δ' that is anticonformally isometric to Δ. Then, by Remark 2.6, the graph $\Gamma(S(\Delta))$ is the union of $\Gamma(\Delta)$ with $\Gamma(\Delta')$.

Suppose first that Δ contains a point O equidistant from all the x_i. Then, since the restriction of $\mathcal{V}_{S(\Delta)}$ to Δ equals $\mathcal{V}_\Delta$, we see that O is equidistant from x_i on S as well. So, by Proposition 2.3(iii), the point O corresponds to a vertex of $\Gamma(S(\Delta))$ of multiplicity at least 3. Furthermore, by Lemma 2.8(iv), we conclude that $\Gamma(S)$ is either a trefoil or an eight graph. Hence, again by Lemma 2.8, the triangle Δ is balanced.

Suppose now that Δ is balanced, ie ϑ_1, ϑ_2 and ϑ_3 satisfy the triangle inequality. Then, by Lemma 2.8(i)–(ii), the graph $\Gamma(S(\Delta))$ is a trefoil or a eight graph, and so by Lemma 2.8(iv) there is a point O in S equidistant from all x_i. It follows that Δ contains such a point as well.

(ii) We first prove the "only if" direction. Suppose that O is in the interior of Δ. Then $\Gamma(S(\Delta))$ has two vertices of valence 3. So according to (i), $\Gamma(S(\Delta))$ is a trefoil. Hence, Δ is strictly balanced by Lemma 2.8(i).

Suppose that O is on the boundary of Δ. Without loss of generality assume that O is on the side of Δ opposite to x_1. For $i = 1, 2, 3$ let γ_i be the geodesic segment of length $\mathcal{V}_\Delta(O)$ that joins O with x_i. Let γ_i' be the image of γ_i in $\Delta' \subset S(\Delta)$ under the anticonformal involution. Since the multiplicity of O in $\Gamma(S)$ is at most 4, we conclude that $\gamma_2 = \gamma_2'$ and $\gamma_3 = \gamma_3'$. Hence, O is the midpoint of the side x_2x_3.

To prove the "if" direction, one needs to apply Lemma 2.8(iv). Indeed, if Δ is strictly balanced, $\Gamma(S(\Delta))$ has two vertices of multiplicity 3 and one of them lies in Δ. If Δ is semibalanced, $\Gamma(S(\Delta))$ has one vertex and it has to lie on the boundary of Δ.

(iii) Since Δ is strictly balanced, by (ii) there is a point O in the interior of Δ equidistant from points x_1, x_2 and x_3. Since $\mathcal{V}_\Delta(O) < \pi$, we have $|Ox_1| = |Ox_1| = |Ox_3| < \pi$. Hence all three isosceles triangles x_iOx_j are involutive triangles by Lemma 2.10.

(iv) This proof is identical to the proof of (iii). □

Remark 2.12 Theorem 2.9 can be used to construct the Voronoi graph $\Gamma(\Delta)$ of a balanced triangle Δ with vertices x_1, x_2 and x_3. Indeed, according to this theorem, the geodesic segments Ox_i cut Δ into three or two involutive triangles, and, using a

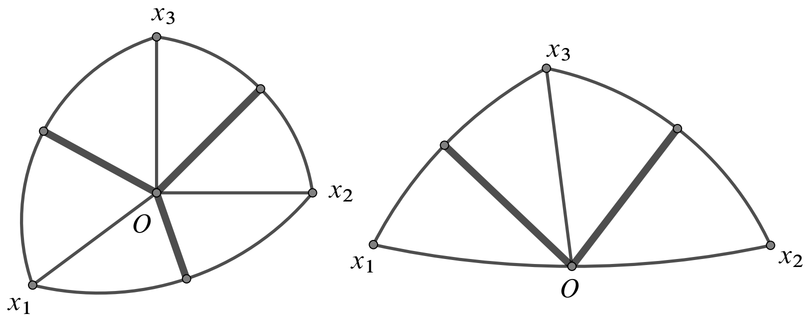

Figure 3: Voronoi graphs of balanced triangles.

variation of Lemma 2.7, one can show that $\Gamma(\Delta)$ is the union of symmetry axes of these triangles; see Figure 3.

We will see that some results we are interested in about balanced triangles indeed concern the following class of triangles.

Definition 2.13 (short-sided triangles) A spherical triangle *short-sided* if all its sides have length $l_i < 2\pi$. In this case, we set $\bar{l}_i := \min(l_i, 2\pi - l_i)$.

Theorem 2.9 has two simple corollaries:

Corollary 2.14 (balanced triangles are short-sided) *Let Δ be a balanced triangle with vertices x_1, x_2 and x_3. Then Δ is short-sided, ie $|x_i x_j| < 2\pi$.*

Proof Let us treat the case when Δ is strictly balanced. The semibalanced case is similar. By Theorem 2.9(iii), the triangle Δ can be cut into three involutive triangles $x_i O x_j$, where $\angle O < 2\pi$ and $|Ox_i| = |Ox_j| < \pi$. Applying Lemma 2.10 to the triangle $x_i O x_j$, we conclude that $|x_i x_j| < 2\pi$. □

Corollary 2.15 (short geodesic in a balanced triangle) *Let Δ be a balanced triangle with vertices x_1, x_2 and x_3. Suppose that $\{i, j, k\} = \{1, 2, 3\}$, ordered so that the value $\bar{l}_k = \min(|x_i x_j|, 2\pi - |x_i x_j|)$ is minimal. Then there is a geodesic segment γ_Δ in Δ that joins x_i with x_j and is such that $\ell(\gamma_\Delta) = \bar{l}_k \leq \frac{2}{3}\pi$, which in fact realizes the minimum distance between distinct vertices.*

Proof Let us again treat the case when Δ is strictly balanced. Let $x_i O x_j$ be three involutive triangles into which Δ is cut. Consider the developing map $\iota\colon \Delta \to \mathbb{S}^2$. Then, for each $\{i, j, k\} = \{1, 2, 3\}$, the value $\bar{l}_k$ is equal to the distance between $\iota(x_i)$ and $\iota(x_j)$ on $\mathbb{S}^2$, and so $d(x_i, x_j) \geq d(\iota(x_i), \iota(x_j)) = \bar{l}_k$. For this reason, it is not hard to see that the minimum of the value $\bar{l}_k$ is attained for the triangle $x_i O x_j$ for which the angle at O is the minimal one. In particular, in such a triangle the angle at O is at most $\frac{2}{3}\pi$. It follows that there is a geodesic segment γ_Δ in such a triangle $x_i O x_j$ of length less than $\frac{2}{3}\pi$ that joins x_i and x_j. Since it cuts out of $x_i O x_j$ a digon with one side $x_i x_j$, we conclude that $\ell(\gamma_\Delta) = \bar{l}_k = d(x_i, x_j)$. □

2.3 Isometric conformal involutions on tori

In this short section we prove the following useful proposition:

Lemma 2.16 (invariance of projective structures on one-pointed tori) *Let (T, x) be a flat one-pointed torus and let σ be its unique nontrivial conformal involution. Then every projective structure on T whose Schwarzian derivative has at worst a double pole at x is invariant under σ.*

Proof We represent our torus T as $\mathbb{C}/\Lambda$, where Λ is a lattice in $\mathbb{C}$, and suppose that x corresponds to the lattice points. We also endow T with the corresponding projective structure.

The involution σ pulls back to the map $z \mapsto -z$ on $\widetilde{T} = \mathbb{C}$. The Schwarzian derivative (see for example [25]) of a projective structure is a quadratic differential on the torus T. By hypothesis, it has at worst a double pole at x. The vector space of such quadratic differentials is 2–dimensional, generated by the constants and the Weierstrass elliptic function. Hence, all its elements are invariant under the involution σ, and so are all solutions of the associated Schwarz equations. As a consequence, all such projective structures are σ–invariant. □

Proposition 2.17 (spherical metrics and conformal involution) *Let σ be the unique conformal involution of a spherical torus T that fixes the unique conical point x.*

(i) *If $\vartheta \notin 2\mathbb{Z} + 1$, then σ is an isometry.*

(ii) *If $\vartheta \in 2\mathbb{Z} + 1$, then each projective equivalence class of spherical metrics is parametrized by a copy of $\mathbb{R}$, on which σ acts as an orientation-reversing diffeomorphism. Thus, σ is an isometry for a unique spherical metric in its projective equivalence class.*

Proof Consider the projective structure associated to a spherical metric on (T, x). By Lemma 2.16, such projective structure is σ–invariant.

(i) Every spherical metric is noncoaxial by Corollary A.2, and so in each projective equivalence class there is at most one spherical metric. Hence, this metric must be invariant under σ.

(ii) Fix a spherical metric h in $\mathcal{MS}_{1,1}(2m+1)$. Let $\tilde{\dot{T}}$ be the universal cover of $\dot{T}$ and let $\widehat{T}$ be its completion. Denote by $\hat{x}_i$ the points in $\partial\widehat{T} = \widehat{T} \setminus \tilde{\dot{T}}$ which project to $x \in T$. Pick a developing map ι for h, which in fact extends to $\hat{\iota}\colon \widehat{T} \to \mathbb{S}^2 \cong \mathbb{CP}^1$, and let ρ be the associated monodromy representation.

By Corollary A.2, the monodromy ρ is coaxial but nontrivial. Fix an element α of $\pi_1(T)$ such that $\rho(\alpha) = e^X \neq I$ with $X \in \mathfrak{su}_2$. Up to conjugation, we can assume that $\infty \in \mathbb{CP}^1$ is the attracting point and $0 \in \mathbb{CP}^1$ is the repelling point for $\rho(\alpha)' := e^{iX}$. The orbits of the group (e^{tX}) on $\mathbb{CP}^1 \setminus \{0,\infty\}$ will be called "parallels" and the unique geodetic orbit will be called the "equator".

First, we claim that $\hat{\iota}(\hat{x}_i) \neq 0, \infty$ for all $\hat{x}_i \in \partial\widehat{T}$, and they all sit on the same parallel. In fact, the holomorphic vector field $z(\partial/\partial z)$ on $\mathbb{CP}^1$ is invariant for the monodromy, and so its pullback descends to a nonzero holomorphic vector field V on $\dot{T}$, possibly with a pole in x. If $\hat{\iota}(\hat{x}_i) \in \{0,\infty\}$, then V would have a zero at x, contradicting $\chi(T) = 0$. The second assertion is clear, since $\hat{\iota}(\partial\widehat{T})$ is an orbit for the action of the monodromy.

Second, note that all spherical metrics $(h_t)_{t\in\mathbb{R}}$ projectively equivalent to h have developing maps $e^t\iota$ and monodromy representation ρ. Thus, up to replacing h by some h_{t_0}, we can assume that $\hat{\iota}(\partial\widehat{T})$ is contained inside the equator.

The function $d : \mathbb{CP}^1 \to [0,\pi]$ that measures the distance from the repelling point of $\rho(\alpha)'$ is invariant for the monodromy action, and so its pullback via ι_t to $\widehat{T}$ descends to a function $d_t\colon T \to [0,\pi]$. We observe that t can be recovered from $d_t(x)$ via $e^t = \frac{1}{2}\tan(d_t(x))$.

Now, $(\rho\circ\sigma)(\alpha) = \rho(\alpha)^{-1} = e^{-X}$. Thus, when considering the developing map $(e^t\iota)\circ\sigma$ with monodromy representation $\rho\circ\sigma$, the attracting point of $(\rho\circ\sigma)(\alpha)'$ is 0 and the repelling point is ∞. It follows that the distance of $(e^t\hat{\iota})\circ\sigma(x) = e^t\hat{\iota}(x)$ from the repelling point ∞ is $\pi - d_t(x)$. Hence, $(e^t\iota)\circ\sigma$ is a developing map for h_{-t}. It follows that σ acts on the family of metrics $(h_t)_{t\in\mathbb{R}}$ by sending h_t to h_{-t}, and so fixing the unique metric h_0 whose developing map sends $\partial\widehat{T}$ to the equator. It follows that σ acts on (T,x,h_t) as an isometry if and only if $t = 0$. □

Proposition 2.17(ii) was also proved in [2, Theorem 5.2]; see also [11, Theorem 1].

2.4 Proof of Theorem B

The goal of this section is to prove Theorem B and to make preparations for the proof of Theorem C. Throughout the section we will mainly consider the class of tori that have a conformal isometric involution. By Proposition 2.17, we know that such an involution exists automatically in the case when the conical angle is not $2\pi(2m+1)$. We start with the following simple lemma:

Lemma 2.18 (points of Γ fixed by a conformal isometric involution) *Let S be a spherical surface with conical points $\boldsymbol{x}$ that admits an isometric conformal involution σ. Let p be a point in $\dot{S} = S \setminus \boldsymbol{x}$ fixed by σ. Then p belongs to $\Gamma(S)$, its multiplicity μ_p is even, and there exist exactly $\frac{1}{2}\mu_p$ geodesic segments or loops*[7] *of lengths $2\mathcal{V}_S(p) < 2\pi$ based at $\boldsymbol{x}$ and passing through p. The point p cuts each such geodesic segment into two halves of equal length.*

Proof Consider any geodesic segment γ of length $\mathcal{V}_S(p)$ that joins p with one of the conical points. Since $\sigma(\gamma) \neq \gamma$ we see that p belongs to $\Gamma(S)$. If p is not a vertex of $\Gamma(S)$, then γ and $\sigma(\gamma)$ are the only two geodesic segments of length $\mathcal{V}_S(p)$ that join p with $\boldsymbol{x}$. Clearly, since σ is a conformal involution the union $\gamma \cup \sigma(\gamma)$ is a geodesic segment or loop based at $\boldsymbol{x}$. Its length is less than 2π by Proposition 2.3(i).

The case when p is a vertex of $\Gamma(S)$ is similar. Since σ is a conformal involution and it sends $\Gamma(S)$ to $\Gamma(S)$ we see that the valence of p in Γ_S is even. By Proposition 2.3(iii) the number μ_p of geodesic segments of length $\mathcal{V}_S(p)$ that join p with $\boldsymbol{x}$ is equal to this valence. Clearly, altogether these μ_p segments form $\frac{1}{2}\mu_p$ geodesic segments (or loops) of length $2\mathcal{V}_S(p)$, all of which have midpoint p. □

Now we concentrate on the case of spherical tori with one conical point. It will be convenient for us to recall first the construction of *hexagonal* and *square* flat tori.

Example 2.19 (hexagonal and square flat tori) Let T_6 and T_4 be the flat tori obtained by identifying opposite sides of a regular flat hexagon and a square, respectively. Denote by $\Gamma_6 \subset T_6$ and $\Gamma_4 \subset T_4$ the graphs formed by the images of the polygons' boundaries. Then it is easy to check that Γ_6 and Γ_4 are Voronoi graphs in T_6 and T_4 with respect to the images of the centers of the polygons.

Lemma 2.20 (Voronoi graph of a spherical torus) *Let T be a spherical torus with one conical point and let Γ be its Voronoi graph. Then Γ is either a trefoil or an eight*

[7] We always assume that a geodesic loop or segment can intersect $\boldsymbol{x}$ only at its endpoints.

graph. In the first case the pair (T,Γ) is homeomorphic to the pair (T_6,Γ_6). In the second case it is homeomorphic to the pair (T_4,Γ_4).

Proof By [23, Corollary 4.7] the Voronoi graph Γ has at most three edges and two vertices. Since the complement to the Voronoi graph is a disk, the graph has at least two edges.

Suppose first that Γ has three edges. By [23, Corollary 4.7] the vertices of Γ have multiplicity at least 3, so Γ is a trivalent graph with two vertices, ie a trefoil or an eyeglasses graph. Note that the punctured torus $\dot{T}$ is homeomorphic to a thickening $\mathrm{Th}(\Gamma)$ of Γ, and such $\mathrm{Th}(\Gamma)$ is uniquely determined by choosing a cyclic ordering of the half-edges incident at each vertex of Γ. Now, up to isomorphism, such a cyclic ordering is unique for the eyeglass graph, and its thickening is homeomorphic to a three-punctured sphere. Hence Γ must be a trefoil.

It is easy to see that $\mathrm{Th}(\Gamma)$ can be endowed with a metric such that, if we cut along Γ, we obtain a flat regular hexagon with its center removed. If $\widehat{\mathrm{Th}}(\Gamma)$ is the completion of $\mathrm{Th}(\Gamma)$ obtained by adding one point, then (T_6,Γ_6) is homeomorphic to $(\widehat{\mathrm{Th}}(\Gamma),\Gamma)$, which in turn is homeomorphic to (T,Γ).

The case when Γ has two edges is similar. □

The following is the main proposition on which the proof of Theorem B relies:

Proposition 2.21 (from tori to balanced triangles) *Let (T,x) be a spherical torus with one conical point x and suppose that T has a nontrivial isometric conformal involution σ. Let $\Gamma(T)$ be the Voronoi graph of T.*

(i) *Suppose $\Gamma(T)$ is a trefoil. Then σ permutes the two vertices of $\Gamma(T)$ and fixes the midpoints p_1, p_2 and p_3 of the three edges of $\Gamma(T)$. Moreover, there exist exactly three σ–invariant simple geodesic loops γ_1, γ_2 and γ_3 based at x such that γ_i intersects $\Gamma(T)$ orthogonally at p_i. These geodesic loops cut the torus into the union of two congruent strictly balanced triangles that are exchanged by σ.*

(ii) *Suppose $\Gamma(T)$ is an eight graph with the vertex A. Then σ fixes the vertex and the midpoints p_1 and p_2 of the two edges of $\Gamma(T)$. Moreover there exist four σ–invariant simple geodesic loops γ_1, γ_2, η_1 and η_2 based at x and uniquely characterized by the following properties: each geodesic γ_i intersects $\Gamma(T)$ orthogonally at p_i, and each geodesic η_i passes through A and has length $2d(A,x)$. Moreover, for $i=1,2$, the triple of loops $(\gamma_1,\gamma_2,\eta_i)$ cuts T into the union of two congruent semibalanced triangles that are exchanged by σ.*

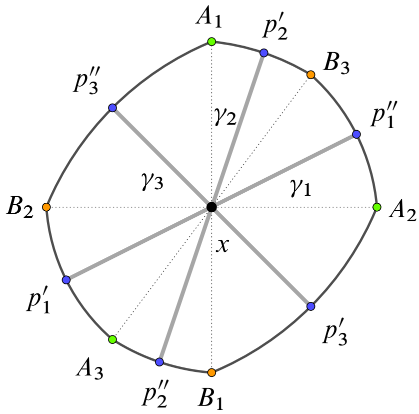

Figure 4: The trefoil case.

(iii) *T has a rectangular involution if and only if its Voronoi graph is an eight graph. For a torus T with a rectangular involution, the triangles into which γ_1, γ_2 and η_1 cut T are reflections of the triangles into which γ_1, γ_2 and η_2 cut T.*

Proof (i) Since σ is an isometry of T it sends $\Gamma(T)$ to itself. Let's denote the vertices of $\Gamma(T)$ by A and B. Since their valence is 3 and σ is a conformal isometric involution, σ can fix neither A nor B. Indeed, since σ is of order 2, if σ fixed A then it would fix at least one half-edge outgoing from A, and so it would be the identity. Hence σ permutes A and B, which implies in particular that A and B are at the same distance from x.

Next, since σ is an orientation-preserving involution and $\Gamma(T)$ is a trefoil, from simple topological considerations it follows that σ sends each edge γ_i of $\Gamma(T)$ into itself. It follows that the midpoints of the edges p_1, p_2 and p_3 are fixed by σ.

Let us now cut T along $\Gamma(T)$ and consider the completion $\overline{D}$ of the obtained open disk. Clearly $\overline{D}$ is a spherical hexagon with the conical point x in its interior. Moreover, σ induces an isometric involution on $\overline{D}$ without fixed points on $\partial\overline{D}$. It follows that σ sends each vertex of $\overline{D}$ to the opposite one.

Next, let's denote the vertices of $\overline{D}$ by A_1, B_2, A_3, B_1, A_2 and B_3, as is shown in Figure 4. Here all the points A_i correspond to A and B_i to B when we reassemble T from the disk. In a similar way we mark midpoints of the sides of $\overline{D}$ by p_i' and p_i''.

According to Lemma 2.18, for each i there is a geodesic loop γ_i of length $2d(p_i, x)$ based at x for which p_i is the midpoint. Let us show that γ_1, γ_2 and γ_3 cut T into two equal strictly balanced triangles whose vertices are identified to the point x.

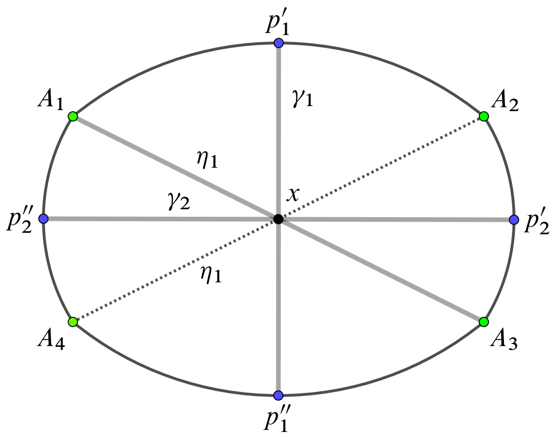

Figure 5: The eight graph case.

Indeed, the first triangle, which we will call Δ_A, is assembled from three quadrilaterals $A_1 p_3'' x p_2'$, $A_2 p_1'' x p_3'$ and $A_3 p_2'' x p_1'$. The second triangle Δ_B is assembled from the remaining three quadrilaterals. Clearly $\sigma(\Delta_A) = \Delta_B$, so these two triangles are congruent.

Finally, Δ_A is strictly balanced according to Theorem 2.9(i). Indeed the point A lies in the interior of Δ_A and is at distance $d(A, x)$ from all the vertices of Δ_A.

(ii) Let us now consider the case when $\Gamma(T)$ is an eight graph with a vertex labeled by A. Clearly, A is fixed by σ since this is the unique point of $\Gamma(T)$ of valence 4.

As before, we see that the midpoints p_1 and p_2 of the two edges of $\Gamma(T)$ are fixed by σ, and this gives us two σ–invariant geodesic loops γ_1 and γ_2. To construct η_1 and η_2 we apply Lemma 2.18 to the point A.

Now let us cut T along the Voronoi graph $\Gamma(T)$ and consider the completion $\overline{D}$ of the obtained open disk. Clearly this disk is a quadrilateral with one conical point in the interior. Let us mark the vertices of this quadrilateral and the midpoints of its edges as shown in Figure 5.

As before, the loops γ_1, γ_2 and η_1 cut T into two congruent triangles, exchanged by σ. To show that these triangles are semibalanced consider one of these triangles obtained as a union of two triangles $A_1 x p_2''$ and $A_3 x p_1''$ and the quadrilateral $x p_1' A_2 p_2'$. To assemble this triangle one has to identify the pairs of sides $(A_1 p_2'', A_2 p_2')$ and $(A_2 p_1', A_3 p_1'')$. The resulting triangle is semibalanced by Theorem 2.9(ii).

(iii) Suppose first that $\Gamma(T)$ is an eight graph. Then we are in the setting of case (ii) of this proposition. Let us construct an involution τ_1 of $\overline{D}$ that pointwise fixes γ_1. We define τ_1 so that $\tau_1(A_1) = A_2$ and $\tau_1(A_3) = A_4$. Then in order show that τ_1 extends to $\overline{D}$ it is enough to show that the triangle A_1xA_4 is isometric to A_2xA_3 and that the geodesic γ_1 is the axis of symmetry of both triangles A_1xA_2 and A_3xA_4. The former statement follows from Proposition 2.3(v). To prove the latter statement, note again that A_1xA_2 is isometric to A_4xA_3 by Proposition 2.3(v) and then compose this isometry with σ. This induces the desired reflections on both triangles A_1xA_2 and A_4xA_3. The involution τ_2 fixing γ_2 is constructed in the same way.

Suppose now that T has a rectangular involution τ. Let us show that $\Gamma(T)$ is an eight graph. Since τ is a rectangular involution, its fixed locus is a union of two disjoint geodesic loops. One of these loops passes through x while the other one, say ξ, is a simple smooth closed geodesic. For any point $p \in \xi$ there exist at least two length-minimizing geodesic segments that join it with x (they are exchanged by τ). It follows that ξ lies in $\Gamma(T)$. And since a trefoil graph can't contain a smooth simple closed geodesic, we conclude that $\Gamma(T)$ is an eight graph. □

Later we will need the following, which is a part of the proof of Proposition 2.21:

Remark 2.22 Suppose we are in case (ii) of Proposition 2.21. Consider the four sectors into which geodesic loops η_1 and η_2 cut a neighborhood of x. Then, for each $i = 1, 2$, the geodesic loop γ_i bisects two of these sectors.

The final preparatory proposition of this subsection is the converse to Proposition 2.21:

Proposition 2.23 (from balanced triangles to tori) *Let Δ be a balanced triangle and let Δ' be a triangle congruent to it. Let $T(\Delta)$ be the torus obtained by identifying the sides of Δ and Δ' through orientation-reversing isometries.*

(i) *The Voronoi graph $\Gamma(T(\Delta))$ coincides with the union in $T(\Delta)$ of $\Gamma(\Delta)$ and $\Gamma(\Delta')$.*

(ii) *If Δ is strictly balanced then the Voronoi graph $\Gamma(T(\Delta))$ has two vertices. Moreover, the images of the three sides of Δ in $T(\Delta)$ coincide with three canonical geodesic loops γ_1, γ_2 and γ_3 on $T(\Delta)$ constructed in Proposition 2.21*(i).

(iii) *If Δ is semibalanced then $\Gamma(T(\Delta))$ has one vertex. Moreover, the images of the three sides of Δ in $T(\Delta)$ coincide with three canonical geodesic loops γ_1, γ_2 and η_i on $T(\Delta)$ constructed in Proposition 2.21*(ii). *Here the side of Δ opposite to the largest angle of Δ corresponds to η_i.*

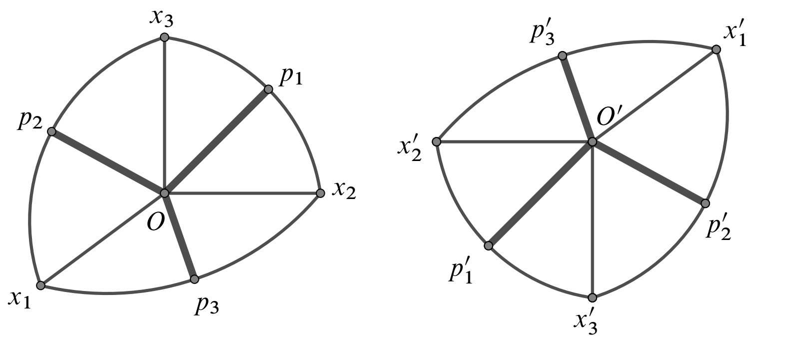

Figure 6: Two isomorphic triangles Δ and Δ'.

Proof (i) Assume first that Δ is strictly balanced. Let $\check{\Gamma}$ be the graph obtained as the union $\Gamma(\Delta) \cup \Gamma(\Delta')$. In order to prove that $\check{\Gamma} = \Gamma(T(\Delta))$, it is enough to show that $\check{\Gamma}$ satisfies properties (a) and (b) of Lemma 2.7.

Recall that by Theorem 2.9(ii) there is a point O in the interior of Δ that is equidistant from the points x_i. Denote by p_i and p_i' the midpoints of sides opposite to x_i and x_i', as in Figure 6. Then, by Remark 2.12, $\Gamma(\Delta)$ is the union of the segments Op_i and $\Gamma(\Delta')$ is the union of the segments Op_i'. It follows that $T(\Delta) \setminus \check{\Gamma}$ is convex and star-shaped with respect to x, which means that property (a) of Lemma 2.7 holds. As for property (b), it holds since $\Gamma(\Delta)$ and $\Gamma(\Delta')$ are Vornoi graphs of Δ and Δ'.

The case when Δ is semibalanced is treated in the same way, so we omit it.

(ii) Since Δ is strictly balanced, it follows from (i) that $\Gamma(T(\Delta))$ has two vertices. Now, it follows from (i) that for any permutation $\{i, j, k\}$ the side $x_i x_j \subset T(\Delta)$ intersects an edge of $\Gamma(T(\Delta))$ at its midpoint and it is orthogonal to it at this point. Hence, by Proposition 2.21(ii), each geodesic $x_i x_j$ coincides with the geodesic loop γ_k.

(iii) The proof of this result is similar to case (ii) and we omit it. □

Remark 2.24 Proposition 2.23 does not hold for any unbalanced triangle. Indeed, if Δ is unbalanced one can still construct a torus $T(\Delta)$ from Δ and its copy of Δ'. However, the union of the Voronoi graphs of Δ and Δ' will be an eyeglasses graph in $T(\Delta)$. Such a graph can never be the Voronoi graph of a torus with one conical point.

Proof of Theorem B Let T be a spherical torus with one conical point of angle $2\pi\vartheta$ with $\vartheta \notin 2\mathbb{Z}+1$. By Proposition 2.17, there exists a conformal isometric involution σ on T. Hence we can apply Proposition 2.21. In particular, by Proposition 2.21(iii), the torus T has a rectangular involution if and only if $\Gamma(T)$ is an eight graph.

(i) The Voronoi graph $\Gamma(T)$ of T is a trefoil, and we get a collection of three geodesics γ_1, γ_2 and γ_3 that cut T into two congruent strictly balanced triangles. Such a collection of geodesics is unique on T by Proposition 2.23.

(ii) The Voronoi graph $\Gamma(T)$ is an eight graph, and by Proposition 2.21 we get two triples of geodesics, $(\gamma_1, \gamma_2, \eta_1)$ and $(\gamma_1, \gamma_2, \eta_2)$, both cutting T into two congruent semibalanced triangles. Again, it follows from Proposition 2.23 that these two triples are the only ones that cut T into two isometric balanced triangle, and they are exchanged by the rectangular involution. □

3 Balanced spherical triangles

The main goal of this section is to describe the space of balanced spherical triangles with assigned area. To do this, we recall in Section 3.1 several theorems describing the inequalities satisfied by the angles of spherical triangles. We also give an explicit constructions of such triangles. Section 3.2 is mainly expository. It recalls the results from [12] that the space $\mathcal{MT}$ of all (unoriented) spherical triangles has the structure of a 3–dimensional real-analytic manifold. From this we deduce that the space of balanced triangles of a fixed noneven area is a smooth-bordered surface. In Section 3.3 we describe a natural cell decomposition of the space $\mathcal{MT}_{\mathrm{bal}}(\vartheta)$ of all balanced triangles of fixed area $\pi(\vartheta - 1)$ with $\vartheta \notin 2\mathbb{Z} + 1$.

3.1 The shape of spherical triangles

We start this section by recalling the classifications [9] of spherical triangles. In fact, such triangles are in one-to-one correspondence with spheres with a spherical metric with three conical points, provided we exclude spheres and triangles with all integral angles. Indeed, for each S^2 with a spherical metric and three conical points that are not all integral, there is a unique isometric anticonformal involution τ such that S^2/τ is a spherical triangle. Conversely, for each spherical triangle Δ we can take the sphere $S(\Delta)$ formed by gluing together two copies of Δ.

It will be useful to introduce the following notation:

Notation Let $\mathbb{Z}^3_e$ be the subset of $\mathbb{Z}^3$ consisting of triples (n_1, n_2, n_3) with $n_1+n_2+n_3$ even. By d_1 we denote the ℓ_1–distance in $\mathbb{R}^3$ defined by $d_1(\boldsymbol{v}, \boldsymbol{w}) = \sum_i |v_i - w_i|$. If a spherical triangle has angles $\pi(\vartheta_1, \vartheta_2, \vartheta_3)$, then we call $(\vartheta_1, \vartheta_2, \vartheta_3) \in \mathbb{R}^3$ its associated *angle vector*.

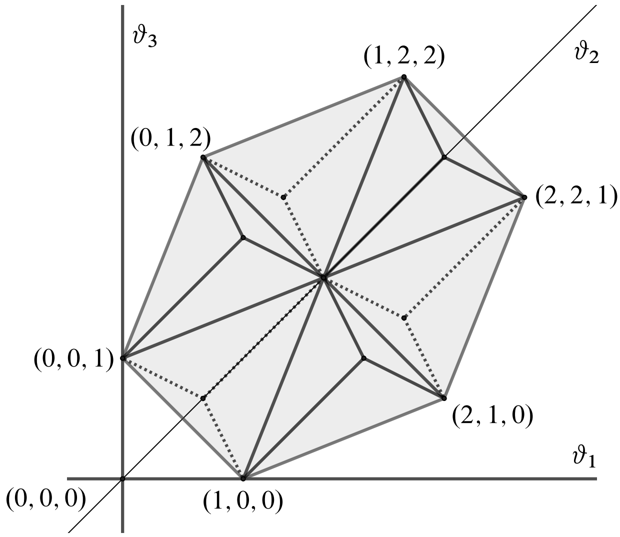

Figure 7: Angle vectors of spherical triangles.

We collect the results into three subsections, depending on the number of integral angles, recalling that there cannot be a triangle with exactly two integral angles.

3.1.1 Triangle with no integral angle The first result we want to recall from [9] is the following:

Theorem 3.1 (triangles with nonintegral angles [9]) *Suppose ϑ_1, ϑ_2 and ϑ_3 are positive and not integers. A spherical triangle with angles $\pi(\vartheta_1,\vartheta_2,\vartheta_3)$ exists if and only if*

$$d_1((\vartheta_1,\vartheta_2,\vartheta_3),\mathbb{Z}^3_e) > 1. \tag{3}$$

Moreover, such a triangle is unique, when it exists.

The unique triangle with three nonintegral angles $\pi(\vartheta_1,\vartheta_2,\vartheta_3)$ will be denoted by $\Delta(\vartheta_1,\vartheta_2,\vartheta_3)$.

Remark 3.2 Let us decipher (3). Note that the subset $d_1((\vartheta_1,\vartheta_2,\vartheta_3),\mathbb{Z}^3_e) \le 1 \subset \mathbb{R}^3$ is a union of octahedra of diameter 2 centered at points of $\mathbb{Z}^3_e$. The complement to this set is a disjoint union of open tetrahedra, each contained in a unit cube with integer vertices. This collection of tetrahedra is invariant under translations of $\mathbb{R}^3$ by elements of $\mathbb{Z}^3_e$. Theorem 3.1 states that if a point $(\vartheta_1,\vartheta_2,\vartheta_3) \in \mathbb{R}^3_{>0}$ lies in one of these tetrahedra, the corresponding spherical triangle exists and is unique. Figure 7 depicts the union of six such tetrahedra in the octant $\mathbb{R}^3_{>0}$.

Section 3.1.2 of [22] contains an explicit construction of balanced spherical triangles. In fact, this was used previously by Klein [17].

3.1.2 Triangles with one integral angle The second result we wish to recall from [9] is the following:

Theorem 3.3 (triangles with one integral angle [9]) *If ϑ_1 is an integer and ϑ_2 and ϑ_3 are not integers, then a spherical triangle with angles $\pi(\vartheta_1, \vartheta_2, \vartheta_3)$ exists if and only if at least one of the following conditions is satisfied:*

(a) *$|\vartheta_2 - \vartheta_3|$ is an integer n of opposite parity from ϑ_1 and $n = |\vartheta_2 - \vartheta_3| \le \vartheta_1 - 1$.*

(b) *$\vartheta_2 + \vartheta_3$ is an integer n of opposite parity from ϑ_1 and $n = \vartheta_2 + \vartheta_3 \le \vartheta_1 - 1$.*

Moreover, when the ϑ_i satisfy (a) or (b) there is a one-parameter family of triangles with angles $\pi(\vartheta_1, \vartheta_2, \vartheta_3)$ that is parametrized by the length $|x_1x_2|$ (or $|x_1x_3|$).

It is obvious that triangles satisfying the hypotheses of Theorem 3.3(b) are never balanced.

Remark 3.4 It is easy to see that, in the case when a triple $(\vartheta_1, \vartheta_2, \vartheta_3)$ of positive numbers satisfies the triangle inequality and the integrality constraints of Theorem 3.3(a), there are integers $n_1, n_2, n_3 \ge 0$ and a number $\theta \in (0, 1)$ such that $\vartheta_1 = n_2 + n_3 + 1$, $\vartheta_2 = n_1 + n_3 + \theta$ and $\vartheta_3 = n_1 + n_2 + \theta$.

Finally, we give a full description of balanced triangles with exactly one integral angle:

Proposition 3.5 (balanced triangles with one integral angle) *Let Δ be a balanced spherical triangle with vertices x_1, x_2 and x_3 and angles $\pi(\vartheta_1, \vartheta_2, \vartheta_3)$, where ϑ_1 is an integer while ϑ_2 and ϑ_3 are not integers. Let n_1, n_2, n_3 and θ be as in Remark 3.4. Then the following hold:*

(i) $|x_2x_3| = \pi$.

(ii) *There is a unique pair of geodesic segments $\gamma_{12}, \gamma_{13} \subset \Delta$, with $|\gamma_{12}| + |\gamma_{13}| = \pi$, that cut Δ into three domains. The first is a digon with angles πn_3 bounded by the sides x_1x_2 and γ_{13}. The second is a digon with angles πn_2 bounded by the sides x_1x_3 and γ_{13}. The third is a triangle with sides γ_{12}, γ_{13} and x_2x_3, and angles $\pi(\theta + n_1, \theta + n_1, 1)$ opposite to the sides.*

(iii) *All balanced triangles with angles $\pi(\vartheta_1, \vartheta_2, \vartheta_3)$ are parametrized by the interval $(0, \pi)$, where one can choose either $|x_1x_2|$ or $2\pi - |x_1x_2|$ as a parameter, depending on whether n_3 is even or odd.*

Proof (i) Since Δ is balanced, by Corollary 2.14 we have $|x_1x_2|, |x_2x_3|, |x_3x_1| < 2\pi$. Consider the developing map $\iota\colon \Delta \to \mathbb{S}^2$. Since ϑ_1 is integer, the images $\iota(x_1x_2)$ and $\iota(x_1x_3)$ belong to one great circle C in $\mathbb{S}^2$. At the same time, since the angle ϑ_2 is not an integer, the image $\iota(x_2x_3)$ does not belong to C. This means that $\iota(x_2)$ and $\iota(x_3)$ are opposite points on $\mathbb{S}^2$, and so $|x_2x_3| = \pi$.

(ii) Since $|x_2x_3| = \pi$ by part (i), there exists a maximal digon embedded in Δ, with one edge equal to x_2x_3. The other edge of such a digon must pass through x_1 by maximality, and so it is the concatenation of two geodesics, γ_{12} from x_1 to x_2 and γ_{13} from x_1 to x_3, that form an angle π at x_1. It is easy to see that these are the geodesics we are looking for. The uniqueness of γ_{12} and γ_{13} follows, because n_1 and θ are uniquely determined.

(iii) This follows from part (ii). □

The next lemma is a partial converse to Proposition 3.5(i).

Lemma 3.6 (balanced triangles with one edge of length π) *Let Δ be a balanced spherical triangle with vertices x_1, x_2 and x_3 and angles $\pi(\vartheta_1, \vartheta_2, \vartheta_3)$. Suppose that $|x_2x_3| = \pi$. Then ϑ_1 is an integer.*

Proof Consider the developing map $\iota\colon \Delta \to \mathbb{S}^2$. Since $|x_ix_j| < 2\pi$ by Corollary 2.14, we see that $\iota(x_i) \neq \iota(x_j)$ for $i \neq j$. In order to show that ϑ_1 is an integer, it is enough to prove that both images $\iota(x_1x_2)$ and $\iota(x_1x_3)$ lie on the same great circle. But this is clear, since the points $\iota(x_2)$ and $\iota(x_3)$ are opposite on $\mathbb{S}^2$, while $\iota(x_1)$ is different from both points. □

The last lemma concerns semibalanced triangles.

Lemma 3.7 (semibalanced triangles with one integral angle) *Suppose Δ is a semibalanced triangle with angles $\pi(\vartheta_1, \vartheta_2, \vartheta_3)$.*

(i) *If ϑ_i is an integer, then $\vartheta_1 + \vartheta_2 + \vartheta_3$ is an even integer $2m$ and ϑ_j and ϑ_k are half-integers.*

(ii) *If $\vartheta_1 + \vartheta_2 + \vartheta_3 = 2m$, then one, ϑ_i, is an integer and the other two, ϑ_j and ϑ_k, are half-integers.*

Proof Without loss of generality, we can assume that $\vartheta_1 = \vartheta_2 + \vartheta_3$. So certainly $\vartheta_1 + \vartheta_2 + \vartheta_3$ cannot be an odd integer. It follows from [9, Theorem 2] that ϑ_1, ϑ_2 and ϑ_3 cannot be three integers.

(i) Note that ϑ_2 cannot be an integer, because the relation $\vartheta_1 - \vartheta_3 = \vartheta_2$ would violate Theorem 3.3(a). Similarly, ϑ_3 cannot be an integer. Hence ϑ_1 is an integer, and so Theorem 3.3(a) implies that ϑ_2 and ϑ_3 are half-integers.

(ii) Our hypotheses imply that $\vartheta_1 = m$ is an integer. By (i), we obtain that ϑ_2 and ϑ_3 are half-integers. □

3.1.3 Triangles with three integral angles We begin by giving a description of all triangles with integral angles.

Proposition 3.8 (triangles with three integral angles) *For any spherical triangle Δ with integral angles $\pi(m_1, m_2, m_3)$:*

(i) *There exists a unique triple (n_1, n_2, n_3) of nonnegative integers such that $m_1 = n_2 + n_3 + 1$, $m_2 = n_3 + n_1 + 1$ and $m_3 = n_1 + n_2 + 1$. Moreover, there exist unique geodesic segments $\gamma_{12}, \gamma_{23}, \gamma_{13} \subset \Delta$, with $|\gamma_{12}| + |\gamma_{23}| + |\gamma_{13}| = 2\pi$, that join points x_i and cut Δ into the following four domains:*
   - *the central disk Δ_0 isometric to a half-sphere and bounded by segments γ_{12}, γ_{23} and γ_{13};*
   - *digons B_1, B_2 and B_3, where each B_i is bounded by segments γ_{jk} and $x_j x_k$ and has angle πn_i.*

(ii) *The space of triangles with angles $\pi(n_1, n_2, n_3)$ can be identified with the set of triples of positive numbers (l_{12}, l_{13}, l_{23}) satisfying $l_{12} + l_{23} + l_{13} = 2\pi$ (where the l_{ij} are interpreted as the lengths of the sides of Δ_0).*

(iii) *All sides of Δ are shorter than 2π. Moreover, there is at most one side of length π.*

Proof (i) Consider the developing map: $\iota\colon \Delta \to \mathbb{S}^2$. Since all the angles of Δ are integral, all its sides are sent to one great circle on $\mathbb{S}^2$. The full preimage of this circle cuts Δ into a collection of hemispheres. It is easy to see that only one of these hemispheres contains all three conical points; this is the disk Δ_0 in Δ. The conical points cut the boundary of the disk into three geodesic segments, γ_{12}, γ_{23} and γ_{13}. The complement of Δ_0 in Δ is the union of the three digons B_1, B_2 and B_3.

(ii) It is clear from (i) that Δ is uniquely defined by the three lengths $l_{ij} = |\gamma_{ij}|$ as well as n_1, n_2 and n_3. Conversely, for each positive triple l_{ij} with $l_{12} + l_{23} + l_{13} = 2\pi$ and each integer triple (n_1, n_2, n_3), one constructs a unique spherical triangle.

(iii) Since $|\gamma_{12}| + |\gamma_{23}| + |\gamma_{31}| = 2\pi$, all the γ_{ij} are shorter than 2π. If $n_k = 0$, then $x_i x_j = \gamma_{ij}$. If $n_k > 0$, then $x_i x_j$ bounds a digon B_k with angles πn_k. In both cases, $x_i x_j$ has length $|\gamma_{ij}|$ (if n_k is even) or $2\pi - |\gamma_{ij}|$ (if n_k is odd). Thus, $|x_i x_j| < 2\pi$.

Moreover, suppose that one of the sides $x_i x_j$, say $x_2 x_3$, has length π. It follows that $|\gamma_{23}| = \pi$ and so $|\gamma_{12}|, |\gamma_{13}| < \pi$. As a consequence, $x_1 x_2$ and $x_1 x_3$ have length different from π. $\square$

Remark 3.9 (existence of balanced triangles with integral angles) If (m_1, m_2, m_3) is a triple of positive integers that satisfies the triangle inequality, then there exist integers $n_1, n_2, n_3 \geq 0$ such that $m_i = 1 + n_j + n_k$ for $\{i, j, k\} = \{1, 2, 3\}$. Then the construction described in Proposition 3.8(i) shows that there exists a balanced spherical triangle with angles $\pi(m_1, m_2, m_3)$.

We thus obtain a characterization of such triangles (see also [9; 12]):

Corollary 3.10 (balanced triangles of area $2m\pi$) *Let Δ be a triangle.*

(i) *If Δ has integral angles $\pi(m_1, m_2, m_3)$, then Δ is strictly balanced and it has area $2m\pi$ with $m = \frac{1}{2}(m_1 + m_2 + m_3 - 1) \in \mathbb{Z}$.*

(ii) *If Δ has area $2m\pi$ for some integer $m > 0$ and it is balanced, then Δ has integral angles $\pi(m_1, m_2, m_3)$, with $m_1 + m_2 + m_3 = 2m + 1$.*

Proof (i) By Proposition 3.8, the central disk Δ_0 has angles $\pi(1, 1, 1)$ and so it is strictly balanced. Since Δ is obtained from Δ_0 by gluing digons along its edges, Δ is strictly balanced. The second claim is a consequence of [9, Theorem 2].

(ii) Suppose that Δ has angles $\pi(\vartheta_1, \vartheta_2, \vartheta_3)$. Since $\mathrm{Area}(\Delta) = \pi(\vartheta_1 + \vartheta_2 + \vartheta_3 - 1)$, we see that $\vartheta_1 + \vartheta_2 + \vartheta_3 = 2m + 1$. It follows easily that $d_1((\vartheta_1, \vartheta_2, \vartheta_3), \mathbb{Z}^3_e) = 1$. Hence, from Theorem 3.1, we conclude that at least one of the ϑ_i, say ϑ_1, is an integer.

Assume, for contradiction, that ϑ_2 and ϑ_3 are not integers, and so we are in the setting of Theorem 3.3. The possibility (b) can't hold because Δ is balanced. Assume that possibility (a) holds, in which case $\vartheta_2 - \vartheta_3$ is an integer, and $\vartheta_1 + \vartheta_2 - \vartheta_3$ is odd. But then, since $\vartheta_1 + \vartheta_2 + \vartheta_3$ is also odd, we see that ϑ_3 is an integer. This is a contradiction.

We conclude that all the ϑ_i are integers. $\square$

3.1.4 Final considerations The last statement of the section can be derived in many ways. Here we obtain it as a consequence of Theorems 3.1 and 3.3 and Proposition 3.8:

Corollary 3.11 (triangles are determined the side lengths and angles) *Let Δ be a spherical triangle with angles $\pi(\vartheta_1, \vartheta_2, \vartheta_3)$, and let l_i be the length of the side opposite to the vertex x_i. Then Δ is uniquely determined by the ϑ_i and l_i.*

Proof If none of the ϑ_i is an integer, then Δ is uniquely determined by $(\vartheta_1, \vartheta_2, \vartheta_3)$ by Theorem 3.1.

If ϑ_1 is an integer while ϑ_2 and ϑ_3 are not integers, then the triangle Δ is uniquely determined by the angles ϑ_i and the length l_3 by Theorem 3.3.

If ϑ_1, ϑ_2 and ϑ_3 are integers, then it follows from Proposition 3.8 that all triangles with angles ϑ_i are uniquely determined by the lengths of their sides. □

3.2 The space of spherical triangles and its coordinates

Let us denote by $\mathcal{MT}$ be space of all (unoriented) spherical triangles with vertices labeled by x_1, x_2 and x_3, up to isometries that preserve the labeling. This space has a natural topology induced by the Lipschitz distance (see Section 6). We will denote by ϑ_1, ϑ_2, ϑ_3, l_1, l_2 and l_3 the functions on $\mathcal{MT}$ defined by requiring that $\pi\vartheta_i(\Delta)$ is the angle of the spherical triangle Δ at x_i and $l_i(\Delta)$ is the length of the side of Δ opposite to x_i.

By Corollary 3.11, the map $\Psi\colon \mathcal{MT} \to \mathbb{R}^6$ that associates to each triangle its angles and side lengths is one-to-one onto its image. Moreover:

Theorem 3.12 (space of spherical triangles [12, Theorem 1.2]) *Let $\mathcal{MT}$ be the space of spherical triangles. The image $\Psi(\mathcal{MT}) \subset \mathbb{R}^6$ is a smooth connected orientable real-analytic 3–dimensional submanifold of $\mathbb{R}^6$.*

This theorem says that the space $\mathcal{MT}$ has the structure of a smooth connected analytic manifold, and moreover at each point $\Delta \in \mathcal{MT}$ one can choose three functions among the ϑ_i and l_i as local analytic coordinates. It also follows from Theorem 3.12 that formulas of spherical trigonometry, that are usually stated for convex spherical triangles, hold for all spherical triangles. In particular, for any permutation (i, j, k) of $(1, 2, 3)$ and any $\Delta \in \mathcal{MT}$, the following cosine formula for lengths holds:[8]

$$\cos l_i \sin(\pi\vartheta_j)\sin(\pi\vartheta_k) = \cos(\pi\vartheta_i) + \cos(\pi\vartheta_j)\cos(\pi\vartheta_k). \tag{4}$$

Lemma 3.13 (some coordinates on the space $\mathcal{MT}$) *Consider the functions ϑ_1, ϑ_2 and ϑ_3 on $\mathcal{MT}$.*

(i) *The functions ϑ_1, ϑ_2 and ϑ_3 form global analytic coordinates on the (open dense) subset of $\mathcal{MT}$ consisting of triangles with nonintegral angles.*

(ii) *Suppose $\Delta \in \mathcal{MT}$ is short-sided and the angle sum $\vartheta_1(\Delta) + \vartheta_2(\Delta) + \vartheta_3(\Delta)$ is not an odd integer. Then the function $\vartheta_1 + \vartheta_2 + \vartheta_3$ has nonzero differential at Δ.*

[8] Indeed, an analytic function vanishing on an open subset of an irreducible analytic variety vanishes identically.

Proof (i) Consider the projection map from $\Psi(\mathcal{MT})$ to the angle space $\mathbb{R}^3$. According to Theorem 3.1, this map is one-to-one over the subset of $(\vartheta_1, \vartheta_2, \vartheta_3)$ in $\mathbb{R}^3_{>0}$ that satisfy (3). We need to show that this projection is in fact a diffeomorphism over this set. However, using the cosine formula (4) and the fact that none of the ϑ_i are integers, we see that the lengths l_i depend analytically on the ϑ_i.

(ii) As mentioned just before Section 3.1.1, there cannot be a spherical triangle with exactly two integral angles. Moreover, Proposition 3.8(i) implies that Δ cannot have three integral angles if $\vartheta_1(\Delta) + \vartheta_2(\Delta) + \vartheta_3(\Delta)$ is not an odd integer. Thus Δ can have at most one integral angle.

If all the ϑ_i are not integers, the statement follows immediately from (i). Suppose finally that exactly one of the ϑ_i, say ϑ_1, is an integer. Then, since Δ is short-sided, using exactly the same reasoning as in the proof of Proposition 3.5(i), we deduce that $l_i = \pi$. Now, for any $\theta > 0$, we can glue the digon with two sides of length π and angles $\pi\theta$ to the side x_2x_3 of Δ. The family of triangles thus constructed, which depends on θ, determines a straight segment in $\Psi(\mathcal{MT})$ starting from $\Psi(\Delta)$, and the linear function $\vartheta_1 + \vartheta_2 + \vartheta_3$ restricted to this segment has nonzero derivative. □

Definition 3.14 (spaces of triangles with assigned area) For any $\vartheta > 1$ we denote by $\mathcal{MT}(\vartheta) \subset \mathcal{MT}$ the surface consisting of triangles with $\vartheta_1 + \vartheta_2 + \vartheta_3 = \vartheta$. We denote by $\mathcal{MT}_{\mathrm{bal}}(\vartheta)$ and $\mathcal{MT}_{\mathrm{sh}}(\vartheta)$ the subsets of balanced and short-sided triangles, respectively.

The following statement is a corollary of Theorem 3.12 and Lemma 3.13:

Corollary 3.15 (space of balanced triangles with assigned area) *For any $\vartheta > 1$, the set $\mathcal{MT}_{\mathrm{bal}}(\vartheta)$ is a nonsingular real-analytic orientable bordered submanifold of the manifold $\mathcal{MT}$ of all spherical triangles. The boundary of $\mathcal{MT}_{\mathrm{bal}}(\vartheta)$ consists of semibalanced triangles.*

Proof Suppose first that $\vartheta_1 + \vartheta_2 + \vartheta_3 = 2m + 1$. Balanced spherical triangles of area $2m\pi$ are classified in Corollary 3.10 and Proposition 3.8. They have integral angles, and each connected component forms an open Euclidean triangle in $\mathbb{R}^6$. Clearly such a subset of $\mathcal{MT} \subset \mathbb{R}^6$ is a smooth submanifold.

Assume now that $\vartheta = \vartheta_1 + \vartheta_2 + \vartheta_3$ is not an odd integer. Clearly $\mathcal{MT}_{\mathrm{sh}}$ is an open subset of $\mathcal{MT}$, and so we deduce from Lemma 3.13(ii) that $\mathcal{MT}_{\mathrm{sh}}(\vartheta)$ is an open smooth

2–dimensional submanifold of $\mathcal{MT}$. The set $\mathcal{MT}_{\text{bal}}(\vartheta)$ is contained in $\mathcal{MT}_{\text{sh}}(\vartheta)$ and its boundary is composed of semibalanced triangles. We need to show that such triangles form a smooth curve in $\mathcal{MT}_{\text{sh}}(\vartheta)$.

Let $\Delta \in \mathcal{MT}_{\text{sh}}(\vartheta)$ be a semibalanced triangle, say $\vartheta_1 = \vartheta_2 + \vartheta_3$. If ϑ_1, ϑ_2 and ϑ_3 are not integers, from Lemma 3.13(i) it follows immediately that the curve $\vartheta_1 - \vartheta_2 - \vartheta_3 = 0$ is smooth in a neighborhood of Δ. Suppose that one of the ϑ_i is an integer. Then we are in the setting of Lemma 3.7. In particular, by Lemma 3.7(i), $\vartheta_1 + \vartheta_2 + \vartheta_3 = 2m$. But then, applying Lemma 3.7(ii), all semibalanced triangles in $\mathcal{MT}_{\text{bal}}(2m)$ have one integral and two half-integral angles. Such triangles are governed by Proposition 3.5, and their image under the map Ψ forms a collection of straight segments in $\mathbb{R}^6$. It follows that semibalanced triangles form a smooth curve in $\mathcal{MT}_{\text{sh}}(2m)$.

Finally, let's show that $\mathcal{MT}_{\text{bal}}(\vartheta)$ is orientable. This is clear if ϑ is an odd integer, because a disjoint union of open triangles is orientable. If ϑ is not an odd integer, it suffices to show that $\mathcal{MT}_{\text{bal}}(\vartheta)$ can be co-oriented, since $\mathcal{MT}$ is orientable. A co-orientation can indeed be chosen since the function $\vartheta_1 + \vartheta_2 + \vartheta_3 = \vartheta$ has nonzero differential along the surface $\mathcal{MT}_{\text{bal}}(\vartheta)$ by Lemma 3.13(ii). □

3.3 Balanced spherical triangles of fixed area

The goal of this section is to describe the topology of the moduli space $\mathcal{MT}_{\text{bal}}(\vartheta)$ of balanced triangles with marked vertices of fixed area $\pi(\vartheta - 1)$, where $\vartheta > 1$. To better visualize the structure of this space, we introduce the following object:

Definition 3.16 (angle carpet) Take $\vartheta > 1$ such that $\vartheta \notin 2\mathbb{Z} + 1$. The *angle carpet* is the subset of the plane $\Pi(\vartheta) := \{(\vartheta_1, \vartheta_2, \vartheta_3) \in \mathbb{R}^3_{>0} \mid \vartheta_1 + \vartheta_2 + \vartheta_3 = \vartheta\}$ consisting of points such that there exists a spherical triangle with angles $\pi(\vartheta_1, \vartheta_2, \vartheta_3)$, and is denoted by $\mathrm{Crp}(\vartheta)$. Points in $\mathrm{Crp}(\vartheta)$ with one integral coordinate are called *nodes*. The *balanced angle carpet* is the subset $\mathrm{Crp}_{\text{bal}}(\vartheta) := \mathrm{Crp}(\vartheta) \cap \mathrm{Bal}(\vartheta)$, where $\mathrm{Bal}(\vartheta) = \{(\vartheta_1, \vartheta_2, \vartheta_3) \mid \vartheta_i \le \vartheta_j + \vartheta_k\}$. A node in $\mathrm{Crp}_{\text{bal}}(\vartheta)$ is *internal* if it does not lie on $\partial\,\mathrm{Bal}(\vartheta)$.

Now we separately treat the cases ϑ not odd and ϑ odd.

3.3.1 Case $\boldsymbol{\vartheta}$ not odd Throughout the section, assume $\vartheta \notin 2\mathbb{Z} + 1$. We will denote by $\mathcal{MT}^{\mathbb{Z}}_{\text{bal}}(\vartheta)$ the subset of $\mathcal{MT}_{\text{bal}}(\vartheta)$ consisting of triangles with at least one integral angle. By Proposition 3.5, this subset is a disjoint union of smooth open intervals in

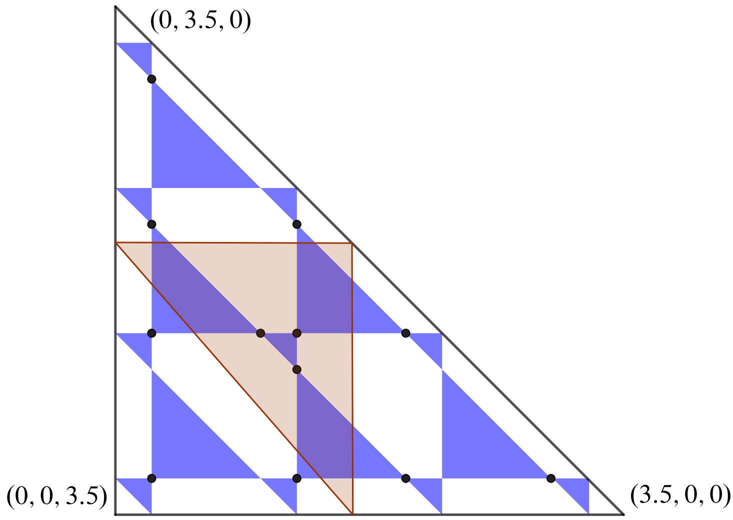

Figure 8: The angle carpet $\mathrm{Crp}(\frac{7}{2})$, composed of 16 open triangles and 12 nodes. The shaded area represents points in $\mathrm{Bal}(\vartheta)$.

$\mathcal{MT}_{\mathrm{bal}}(\vartheta)$. We will see that it cuts $\mathcal{MT}_{\mathrm{bal}}(\vartheta)$ into a union of topological disks. This decomposition is very well reflected in the structure of the associated balanced carpet, as we will see below.

The carpet $\mathrm{Crp}(\vartheta)$ is composed of a disjoint union of open triangles with a subset of their vertices (the nodes). In order to better visualize such carpets, we will often identify $\mathrm{Crp}(\vartheta)$ with its projection to the horizontal $(\vartheta_1, \vartheta_2)$–plane. Figure 8 shows the projection of $\mathrm{Crp}(3.5)$. It is a union of 16 disjoint open triangles (singled out by inequality (3) of Theorem 3.1) and a subset of 12 nodes (governed by condition (a) of Theorem 3.3) marked as black dots. Figure 9 depicts the projection of balanced angle carpets for five different values of ϑ.

The following lemma is a consequence of Theorems 3.1 and 3.3:

Lemma 3.17 (description of the angle carpets) *Take $\vartheta \in (1, \infty) \setminus \{2\mathbb{Z}+1\}$ and set $m = \lfloor \frac{1}{2}(\vartheta+1) \rfloor$.*

(i) $\mathrm{Crp}(\vartheta)$ *is the union of $4m^2$ open triangles with $3m^2$ nodes $(\vartheta_1, \vartheta_2, \vartheta_3)$ such that the unique integer coordinate ϑ_i of a node satisfies $\vartheta_i \geq |\vartheta_j - \vartheta_k| + 2l + 1$ for some integer $l \leq 0$.*

(ii) *All points $(\vartheta_1, \vartheta_2, \vartheta_3) \in \mathrm{Bal}(\vartheta)$ with one positive integer coordinate are nodes in $\mathrm{Crp}_{\mathrm{bal}}(\vartheta)$. Hence, the balanced carpet $\mathrm{Crp}_{\mathrm{bal}}(\vartheta)$ is a connected set.*

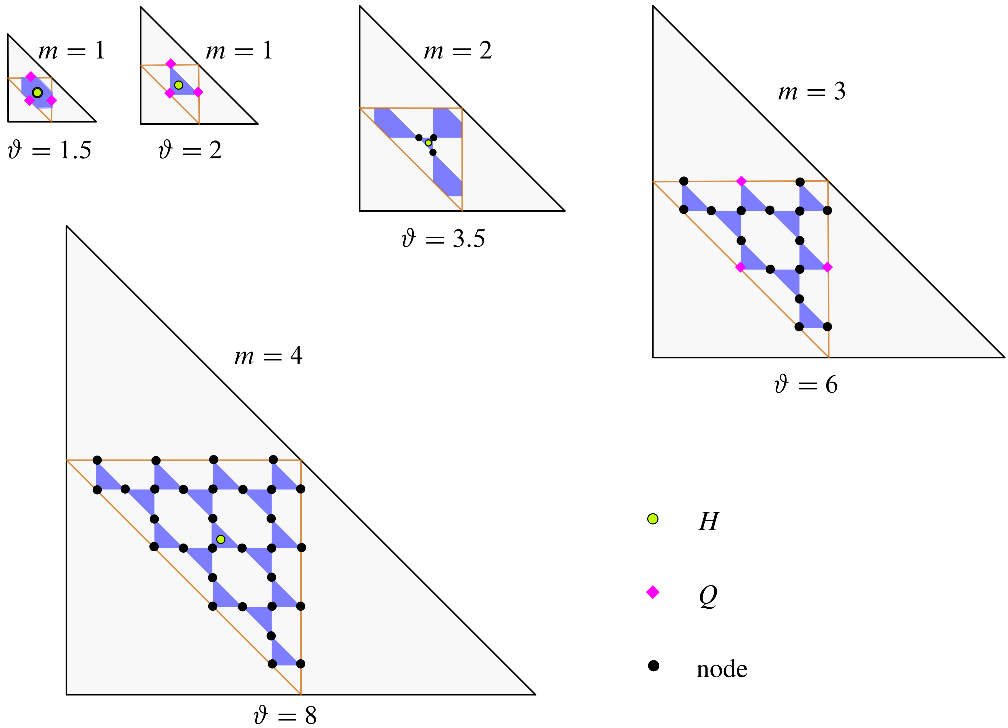

Figure 9: Balanced carpets for $\vartheta = 1.5, 2, 3.5, 6, 8$.

(iii) *The balanced carpet* $\mathrm{Crp}_{\mathrm{bal}}(\vartheta)$ *intersects* E *open triangles and it contains* N *internal nodes, where*

$$E = \begin{cases} m^2 & \text{if } \vartheta \le 2m, \\ m^2 + 3m & \text{if } \vartheta > 2m, \end{cases} \qquad N = \begin{cases} \frac{3}{2}m(m-1) & \text{if } \vartheta \le 2m, \\ \frac{3}{2}m(m+1) & \text{if } \vartheta > 2m. \end{cases}$$

Hence $E - N = -\frac{1}{2}m(m-3)$.

(iv) *There exists a point in* $\mathrm{Crp}_{\mathrm{bal}}(\vartheta)$ *with noninteger coordinates at which* $\vartheta_2 = \vartheta_3$.

Proof (i) Let us split the carpet into two subsets. The first subset consists of points such that none of the coordinates ϑ_i are integers, and the second subset is where one of the coordinates ϑ_i is an integer.

It is clear that the first subset is the union of open triangles given by intersecting the plane $\vartheta_1 + \vartheta_2 + \vartheta_3 = \vartheta$ with the open tetrahedra that are given by inequality (3) of Theorem 3.1. Since this plane does not pass through any vertex of the tetrahedra for ϑ nonodd, it follows that the number of triangles only depends on m, and so we can compute it for $\vartheta = 2m$. Look at the projection of $\mathrm{Crp}(2m)$ inside the $(\vartheta_1, \vartheta_2)$–plane and enumerate the open triangles as follows: to points of type $\left(0, l + \frac{1}{2}\right)$ with $l \in \{0, 1, \dots, 2m-1\}$ we can associate a unique triangle, and to points of type $\left(n, l + \frac{1}{2}\right)$ with $n \in \{1, \dots, 2m-1\}$ and $l \in \{0, \dots, 2m-n-1\}$ we can associate two triangles. The number of such triangles is thus $4m^2$.

The second subset is governed by Theorem 3.3. Since $\vartheta_1 + \vartheta_2 + \vartheta_3 = \vartheta$ is not an odd integer, only the nodes that satisfy condition (a) of Theorem 3.3 lie in $\mathrm{Crp}(\vartheta)$. Again it's enough to count the nodes for $\vartheta = 2m$. Suppose first that ϑ_1 is an integer. We must have $|2\vartheta_2 + \vartheta_1 - 2m| = |\vartheta_2 - \vartheta_3| = \vartheta_1 - 1 - 2l$ for some integer l. If $\vartheta_1 \in \{1, 2, \dots, m\}$, then $\vartheta_2 \in \frac{1}{2} + \{m - \vartheta_1, \dots, m-1\}$ and so we have $\frac{1}{2}m(m+1)$ nodes. If $\vartheta_1 \in \{m+1, \dots, 2m-1\}$, then $\vartheta_2 \in \frac{1}{2} + \{0, \dots, 2m-1-\vartheta_1\}$ and so we have $\frac{1}{2}m(m-1)$ nodes. Thus, we have m^2 nodes with integral ϑ_1, and we conclude that we have $3m^2$ nodes in total.

(ii) Again, it is enough to consider the case where $\vartheta = 2m$. In the balanced carpet, $\vartheta_i \le m$ for all i and so the first claim follows from the above enumeration of the nodes. Hence, $\mathrm{Crp}_{\mathrm{bal}}(\vartheta)$ is connected.

(iii) Let us first consider N. For $\vartheta = 2m$ the enumeration in part (i) shows that $N = \frac{3}{2}m(m-1)$. If $\vartheta < 2m$, then N does not change. If $\vartheta > 2m$, then $N = \frac{3}{2}m(m-1) + 3m$, and the extra $3m$ is exactly the number of nodes sitting in $\partial\,\mathrm{Bal}(2m)$.

As for E, the enumeration in (i) for $\vartheta = 2m$ shows that $4E = 4m^2$, and so $E = m^2$. For $\vartheta < 2m$, the value of E does not change. For $\vartheta > 2m$, there $3m$ extra triangles intersected by $\mathrm{Bal}(\vartheta)$, which is exactly the number of nodes sitting in $\partial\,\mathrm{Bal}(2m)$.

(iv) The point with $\vartheta_1 = \frac{1}{4}(c+3)$ and $\vartheta_2 = \vartheta_3 = m - \frac{3}{8}(1-c)$ belongs to the interior of $\mathrm{Crp}_{\mathrm{bal}}(\vartheta)$ and it is not a node. □

In order to understand the topology of $\mathcal{MT}_{\mathrm{bal}}(\vartheta)$, we consider the natural projection map $\Theta \colon \mathcal{MT}_{\mathrm{bal}}(\vartheta) \to \mathrm{Crp}_{\mathrm{bal}}(\vartheta)$ that sends Δ to $(\vartheta_1(\Delta), \vartheta_2(\Delta), \vartheta_3(\Delta))$.

Analysis of the map Θ By Lemma 3.17, the balanced carpet $\mathrm{Crp}_{\mathrm{bal}}(\vartheta)$ consists of E polygons $\{P_l\}$, bounded by some *semibalanced edges* that sit in $\partial\,\mathrm{Bal}(\vartheta)$ and some nodes. Note that we are considering P_l as closed subsets of $\mathrm{Crp}_{\mathrm{bal}}(\vartheta)$; in fact, P_l is not a closed subset of the plane $\Pi(\vartheta)$ as it misses the edges sitting on the lines $\vartheta_i = a + \frac{1}{2}(c+1)$ with $i \in \{1, 2, 3\}$ and $a \in \{0, 1, \dots, m-1\}$. Such edges will be called *ideal edges*. In Figure 10 the polygon P_l on the right has two nodes, one semibalanced edge and three ideal edges. (Note that a node can be semibalanced too.)

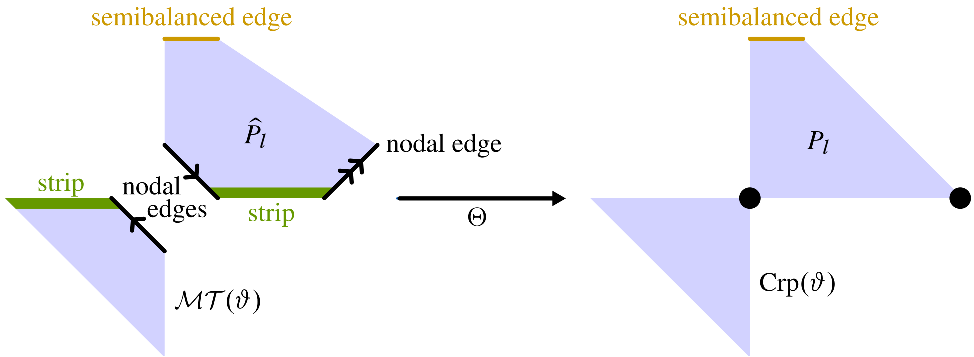

Figure 10: The map Θ. Unmarked edges are ideal edges.

For each polygon P_l, the real blow-up $\widehat{P}_l$ of P_l at its nodes is obtained from P_l by replacing each node by an open interval (*nodal edge*). The natural projection $\widehat{P}_l \to P_l$ contracts each nodal edge to the corresponding node. (Note that a nodal edge can also be semibalanced.) For every l we can fix a realization of $\widehat{P}_l$ inside $\mathbb{R}^2$ as the union of an open convex polygon with some of its open edges (nodal edges and semibalanced edges). Again, such a $\widehat{P}_l$ is not a closed subset of $\mathbb{R}^2$, as it misses the edges corresponding to the ideal edges of P_l. Such missing edges will be referred to as the ideal edges of $\widehat{P}_l$. In Figure 10 the polygon $\widehat{P}_l$ has two nodal edges, one semibalanced edge and three ideal edges.

We recall that $\mathcal{MT}_{\mathrm{bal}}(\vartheta)$ is a surface by Corollary 3.15 and its boundary consists of semibalanced triangles, and that the map Θ contracts each open interval in $\mathcal{MT}_{\mathrm{bal}}^{\mathbb{Z}}(\vartheta)$ to a node by Proposition 3.5 and it is a homeomorphism elsewhere by Lemma 3.13(i).

It is easy then to see that $\Theta^{-1}(P_l \setminus \{\text{nodes}\})$ is homeomorphic to $\widehat{P}_l \setminus \{\text{nodes}\}$. Suppose now that two distinct polygons P_l and P_h intersect in a node $\bar{\boldsymbol{\vartheta}}$. The preimage $\Theta^{-1}(\bar{\boldsymbol{\vartheta}})$ is an open segment and $\Theta^{-1}(P_l \cup P_h)$ is homeomorphic to the space obtained from $\widehat{P}_l \sqcup \widehat{P}_h$ by identifying the nodal edges that correspond to $\bar{\boldsymbol{\vartheta}}$.

To understand this identification, choose an orientation of $\mathcal{MT}_{\mathrm{bal}}(\vartheta)$ in a neighborhood of $\Theta^{-1}(\bar{\boldsymbol{\vartheta}})$ and an orientation of the plane $\Pi(\vartheta)$, so that P_l and $\widehat{P}_l$ inherit an orientation from $\Pi(\vartheta)$, and each nodal edge of $\widehat{P}_l$ inherits an orientation from $\widehat{P}_l$. Together with Corollary 3.15, the last paragraph of the proof of [12, Proposition 4.7] shows that Θ is orientation-preserving on one of the two polygons P_l or P_h and orientation-reversing on the other. Hence, the two nodal edges corresponding to $\bar{\boldsymbol{\vartheta}}$ are identified through a map that preserves their orientation; we can also prescribe that such an identification is a homothety in the chosen realizations of $\widehat{P}_l$ and $\widehat{P}_h$.

Part of the above analysis can be rephrased:

Lemma 3.18 *The space $\mathcal{MT}_{\mathrm{bal}}(\vartheta)$ is homeomorphic to the real blow-up of* $\mathrm{Crp}_{\mathrm{bal}}(\vartheta)$ *at its nodes.*

A further step in describing the topology of $\mathcal{MT}_{\mathrm{bal}}(\vartheta)$ is to study its ends:

Construction 3.19 (the strips $\mathcal{S}_{i,a}(\vartheta)$) As remarked above, every ideal edge of P_l has equation $\vartheta_i = a + \frac{1}{2}(c+1)$ for some $a \in \{0, \dots, m-1\}$ and $i \in \{1,2,3\}$. Viewing $\widehat{P}_l$ inside $\mathbb{R}^2$, an open thickening of the corresponding ideal edge intersects $\widehat{P}_l$ in a region $\mathcal{S}^l_{i,a}(\vartheta)$ homeomorphic to $[0,1]\times\mathbb{R}$, where $\{0,1\}\times\mathbb{R}$ corresponds to portions of nodal or semibalanced segments. In every $\widehat{P}_l$, such thickenings can be chosen so that the corresponding regions are disjoint and their ends $\{0\}\times\mathbb{R}$ and $\{1\}\times\mathbb{R}$ cover $\frac{1}{4}$ of the corresponding nodal or semibalanced segment. The complement inside $\widehat{P}_l$ of such strips is clearly compact. (One example of the region $\mathcal{S}^l_{i,a}(\vartheta)$ is illustrated in Figure 10 on the left: it is the darker thickening of the horizontal ideal edge of $\widehat{P}_l$.)

It follows that, for fixed $i \in \{1,2,3\}$ and $a \in \{0,1,\dots,m-1\}$, the regions $\{\mathcal{S}^l_{i,a}(\vartheta)\}$ glue to give a strip $\mathcal{S}_{i,a}(\vartheta)$ homeomorphic to $[0,1]\times\mathbb{R}$, with $\{0,1\}\times\mathbb{R}$ corresponding to semibalanced triangles. Thus there are $3m$ disjoint such strips, each one associated to a pair (i,a).

We are now ready to completely determine the topology of the space $\mathcal{MT}_{\mathrm{bal}}(\vartheta)$:

Proposition 3.20 (topology of the space of balanced triangles with assigned area) *Suppose that $\vartheta = 2m + c$ where $c \in (-1,1)$.*

(i) $\mathcal{MT}_{\mathrm{bal}}(\vartheta)$ *is a connected orientable smooth-bordered surface of finite type whose boundary is the set of semibalanced triangles.*

(ii) *The boundary of $\mathcal{MT}_{\mathrm{bal}}(\vartheta)$ is a union of $3m$ disjoint open intervals.*

(iii) *The surface $\mathcal{MT}_{\mathrm{bal}}(\vartheta)$ has $3m$ ends, namely the strips $\mathcal{S}_{i,a}(\vartheta)$. Each strip corresponds in* $\mathrm{Crp}_{\mathrm{bal}}(\vartheta)$ *to a line $\vartheta_i = a + \frac{1}{2}(c+1)$ for some $a \in \{0,1,\dots,m-1\}$ and $i \in \{1,2,3\}$. Moreover, each $\mathcal{S}_{i,a}(\vartheta)$ is homeomorphic to* $[0,1]\times\mathbb{R}$ *and* $\{0,1\}\times\mathbb{R}$ *corresponds to semibalanced triangles.*

(iv) *The Euler characteristic of $\mathcal{MT}_{\mathrm{bal}}(\vartheta)$ is $\chi(\mathcal{MT}_{\mathrm{bal}}(\vartheta)) = -\frac{1}{2}m(m-3)$.*

Proof (i) Thanks to Corollary 3.15 we only need to prove that $\mathcal{MT}_{\mathrm{bal}}(\vartheta)$ is connected and of finite type. Since the balanced carpet $\mathrm{Crp}_{\mathrm{bal}}(\vartheta)$ is connected by Lemma 3.17(ii) and consists of finitely many nodes and polygons, both claims follow from Lemma 3.18.

(ii) It will be enough to show that the set of semibalanced triangles with angles ϑ_1, ϑ_2 and ϑ_3, satisfying $\vartheta_1 = \vartheta_2 + \vartheta_3$ and $\vartheta_1 + \vartheta_2 + \vartheta_3 = \vartheta$, is a union of m open intervals. In case $c = 0$ these m intervals correspond to m types of triangles with angles $\pi\big(m, \frac{1}{2} + l, \frac{1}{2} + m - l - 1\big)$ where $l \in [0, m-1]$ is an integer number. In case $c \neq 0$ these intervals correspond to the intersection of the line $\vartheta_1 = \vartheta_2 + \vartheta_3$ with m open triangles of the carpet $\mathrm{Crp}(\vartheta)$.

(iii) This follows from Construction 3.19.

(iv) The internal part of $\mathcal{MT}_{\mathrm{bal}}(\vartheta)$ is an orientable surface without boundary, and so the Euler characteristic of its cohomology with compact support coincides with its Euler characteristic by Poincaré duality. Decompose the interior of $\mathcal{MT}_{\mathrm{bal}}(\vartheta)$ into a finite union of open 1–cells $\mathcal{MT}^{\mathbb{Z}}_{\mathrm{bal}}(\vartheta)$ (corresponding to internal nodes in the balanced carpet) and open 2–cells (corresponding to the intersection of $\mathrm{Bal}(\vartheta)$ with open triangles in the carpet). By Lemma 3.17(iii), the space $\mathcal{MT}_{\mathrm{bal}}(\vartheta)$ is a union of E open 2–cells and N open 1–cells. Thus, its Euler characteristic is $E - N = -\frac{1}{2}m(m-3)$. □

Let us now consider balanced triangles (with labeled vertices, as usual) endowed with an orientation. We stress that the orientation and the labeling of the vertices are unrelated. Let $\mathcal{MT}^{+}_{\mathrm{bal}}(\vartheta)$ be the set of oriented balanced triangles of area $\pi(\vartheta - 1)$ in which the vertices are labeled anticlockwise, and let $\mathcal{MT}^{-}_{\mathrm{bal}}(\vartheta)$ be the analogous space in which the vertices are labeled clockwise. Both sets can be given the topology induced by the identification with $\mathcal{MT}_{\mathrm{bal}}(\vartheta)$. The space of oriented balanced triangles is then $\mathcal{MT}^{+}_{\mathrm{bal}}(\vartheta) \sqcup \mathcal{MT}^{-}_{\mathrm{bal}}(\vartheta)$.

Definition 3.21 (doubled space of balanced triangles) The *doubled space of balanced triangles* of area $\pi(\vartheta - 1)$ is the space $\mathcal{MT}^{\pm}_{\mathrm{bal}}(\vartheta)$ obtained from $\mathcal{MT}^{+}_{\mathrm{bal}}(\vartheta) \sqcup \mathcal{MT}^{-}_{\mathrm{bal}}(\vartheta)$ by identifying an oriented semibalanced triangle Δ to the triangle obtained from Δ by reversing its orientation.

It follows that $\mathcal{MT}^{\pm}_{\mathrm{bal}}(\vartheta)$ is homeomorphic to the double of $\mathcal{MT}_{\mathrm{bal}}(\vartheta)$.

Proposition 3.22 (the doubled space of balanced triangles of assigned area) *Let $\vartheta > 1$ be a nonodd real number and let $m = \big\lfloor \frac{1}{2}(\vartheta + 1) \big\rfloor$.*

(i) *$\mathcal{MT}^{\pm}_{\mathrm{bal}}(\vartheta)$ is a connected orientable surface of finite type, without boundary.*

(ii) *$\mathcal{MT}^{\pm}_{\mathrm{bal}}(\vartheta)$ has Euler characteristic $-m^2$, genus $\frac{1}{2}(m-1)(m-2)$, and $3m$ punctures.*

(iii) *The action of* S_3 *by relabeling the vertices of the triangles consists of orientation-preserving homeomorphisms of* $\mathcal{MT}^{\pm}_{\mathrm{bal}}(\vartheta)$.

(iv) *The action of* S_3 *on the set of punctures of* $\mathcal{MT}^{\pm}_{\mathrm{bal}}(\vartheta)$ *has* m *orbits of length* 3.

Proof (i) This is a consequence Proposition 3.20(i), since $\mathcal{MT}^{\pm}_{\mathrm{bal}}(\vartheta)$ is the double of $\mathcal{MT}_{\mathrm{bal}}(\vartheta)$.

(ii) Since $\mathcal{MT}^{\pm}_{\mathrm{bal}}(\vartheta)$ is an orientable surface without boundary, the Euler characteristic agrees with the Euler characteristic with compact support. By Proposition 3.20(ii), $\mathcal{MT}_{\mathrm{bal}}(\vartheta)$ has boundary consisting of $3m$ open segments. Hence, $\chi(\mathcal{MT}^{\pm}_{\mathrm{bal}}(\vartheta)) = 2\chi(\mathcal{MT}_{\mathrm{bal}}(\vartheta)) - 3m = -m(m-3) - 3m = -m^2$.

By Proposition 3.20(iii), each end of $\mathcal{MT}_{\mathrm{bal}}(\vartheta)$ is associated to a strip $\mathcal{S}^i_a(\vartheta)$ with $a \in \{0, 1, \dots, m\}$ and $i \in \{1, 2, 3\}$, and it is homeomorphic to $[0,1] \times \mathbb{R}$, so it doubles to punctured disk $S^1 \times \mathbb{R}$ inside $\mathcal{MT}^{\pm}_{\mathrm{bal}}(\vartheta)$, which will be denoted by $\mathcal{E}^i_a(\vartheta)$. Hence, we obtain $3m$ punctures. The genus of $g(\mathcal{MT}^{\pm}_{\mathrm{bal}}(\vartheta)) = 1 - \frac{3}{2}m - \frac{1}{2}\chi(\mathcal{MT}^{\pm}_{\mathrm{bal}}(\vartheta))$ is then easily computed.

(iii) Choose an arbitrary orientation of $\mathcal{MT}^{\pm}_{\mathrm{bal}}(\vartheta)$. We want to show that every transposition $(i\ j) \in S_3$ acts on $\mathcal{MT}^{\pm}_{\mathrm{bal}}(\vartheta)$ through an orientation-preserving homeomorphism. Consider, for instance, the transposition $(2\,3)$, that sends a triangle in $\mathcal{MT}^{+}_{\mathrm{bal}}(\vartheta)$ with nonintegral angles $(\vartheta_1, \vartheta_2, \vartheta_3)$ to the triangle in $\mathcal{MT}^{-}_{\mathrm{bal}}(\vartheta)$ with nonintegral angles $(\vartheta_1, \vartheta_3, \vartheta_2)$. Since $\mathcal{MT}^{+}_{\mathrm{bal}}(\vartheta)$ and $\mathcal{MT}^{-}_{\mathrm{bal}}(\vartheta)$ have opposite orientations when viewed as subsets of $\mathcal{MT}^{\pm}_{\mathrm{bal}}(\vartheta)$, it is enough to show that $(2\,3)$ acts on $\mathcal{MT}_{\mathrm{bal}}(\vartheta)$ by reversing its orientation.

By Lemma 3.17(iv), there exists a point in $\mathrm{Crp}_{\mathrm{bal}}(\vartheta)$ with noninteger coordinates $(\vartheta_1, \vartheta_2, \vartheta_2)$, and so a corresponding balanced triangle Δ in $\mathcal{MT}^{\pm}_{\mathrm{bal}}(\vartheta)$. It is clear that the transformation $(\vartheta_1, \vartheta_2, \vartheta_3) \mapsto (\vartheta_1, \vartheta_3, \vartheta_2)$ of $\mathrm{Crp}_{\mathrm{bal}}(\vartheta)$ reverses the orientation at $(\vartheta_1, \vartheta_2, \vartheta_2)$. Hence, (23) acts on $\mathcal{MT}_{\mathrm{bal}}(\vartheta)$ by reversing its orientation.

(iv) Each orbit of the S_3–action on the ends $\mathcal{E}^i_a(\vartheta)$ is of type $\{\mathcal{E}^1_a(\vartheta), \mathcal{E}^2_a(\vartheta), \mathcal{E}^3_a(\vartheta)\}$. Since $a \in \{0, 1, \dots, m-1\}$, there are m orbits of length 3. □

3.3.2 Case ϑ odd The case where $\vartheta = 2m+1$ for some integer $m \geq 0$ is much easier to handle.

Lemma 3.23 (description of the balanced carpet) *The balanced carpet* $\mathrm{Crp}_{\mathrm{bal}}(2m+1)$ *consists of* $\frac{1}{2}m(m+1)$ *internal nodes.*

Proof Triangles in $\mathcal{MT}_{\mathrm{bal}}(2m+1)$ have area $2m\pi$ by the Gauss–Bonnet theorem. By Corollary 3.10 and Remark 3.9, the balanced carpet $\mathrm{Crp}_{\mathrm{bal}}(2m+1)$ consists just of triples $(\vartheta_1,\vartheta_2,\vartheta_3)\in\mathbb{Z}^3$ such that $\vartheta_1+\vartheta_2+\vartheta_3=2m+1$ and $1\le\vartheta_i\le m$ for all i. It is easy to see that such points are $\frac{1}{2}m(m+1)$ internal nodes. □

This easily leads to the description of the moduli space of balanced triangles:

Proposition 3.24 (topology of the space of balanced triangles) *$\mathcal{MT}_{\mathrm{bal}}(2m+1)$ is diffeomorphic to the disjoint union of $\frac{1}{2}m(m+1)$ copies of the open 2–simplex $\mathring{\Delta}^2$.*

Proof Fix $(\vartheta_1,\vartheta_2,\vartheta_3)\in\mathrm{Crp}_{\mathrm{bal}}(2m+1)$. By Proposition 3.8, the locus of triangles Δ in $\mathcal{MT}_{\mathrm{bal}}(2m+1)$ with $\vartheta_i(\Delta)=\vartheta_i$ for $i=1,2,3$ is real-analytically diffeomorphic to the set of triples $(l_1,l_2,l_3)\in(0,2\pi)^3$ such that $l_1+l_2+l_3=2\pi$, which is clearly homothetic to $\mathring{\Delta}^2$. The conclusion then follows from Lemma 3.23. □

Let $\mathrm{Crp}^{\pm}_{\mathrm{bal}}(2m+1)$ be the disjoint union of two copies of $\mathrm{Crp}_{\mathrm{bal}}(2m+1)$. Namely its elements are of type $(\boldsymbol{\vartheta},\epsilon)$, where $\boldsymbol{\vartheta}\in\mathrm{Crp}^{\pm}_{\mathrm{bal}}(2m+1)$ and $\epsilon=\pm1$. We denote by $\mathcal{MT}^{\pm}_{\mathrm{bal}}(2m+1)$ the doubled space of spherical triangles of area $2m\pi$ and by $\Theta^{\pm}\colon\mathcal{MT}^{\pm}_{\mathrm{bal}}(2m+1)\to\mathrm{Crp}^{\pm}_{\mathrm{bal}}(2m+1)$ the map that sends an oriented triangle Δ to $(\boldsymbol{\vartheta}(\Delta),\epsilon(\Delta))$, where $\epsilon(\Delta)=1$ if the vertices of Δ are numbered anticlockwise, and $\epsilon(\Delta)=-1$ otherwise.

Proposition 3.25 (topology of the doubled space of balanced triangles) *The space $\mathcal{MT}^{\pm}_{\mathrm{bal}}(2m+1)$ is diffeomorphic to $\mathrm{Crp}^{\pm}_{\mathrm{bal}}(2m+1)\times\mathring{\Delta}^2$, namely to the disjoint union of $m(m+1)$ open 2–simplices. The permutation group S_3 that relabels the vertices of a triangle in $\mathcal{MT}^{\pm}_{\mathrm{bal}}(2m+1)$ acts on an element $(\boldsymbol{\vartheta},\epsilon,\boldsymbol{y})$ of $\mathrm{Crp}^{\pm}_{\mathrm{bal}}(2m+1)\times\mathring{\Delta}^2$ by permuting the coordinates of $\boldsymbol{\vartheta}$ and $\boldsymbol{y}$, and through its sign on ϵ.*

Proof The first claim relies on Proposition 3.24. The others are straightforward. □

4 Moduli spaces of spherical tori

The goal of this section is to describe the topology of the moduli space $\mathcal{MS}_{1,1}(\vartheta)$ and so to prove Theorem A (case ϑ nonodd) and Theorems C and D (case ϑ odd).

We recall that, by *isomorphism* between two spherical tori, we mean an orientation-preserving isometry. We refer to Section 6 for the definition of Lipschitz distance and topology on $\mathcal{MS}_{1,1}(\vartheta)$ and $\mathcal{MS}^{(2)}_{1,1}(\vartheta)$ needed below.

The object of our interest is the following:

Definition 4.1 ($\mathcal{MS}_{1,1}(\vartheta)$ as a topological space) The space $\mathcal{MS}_{1,1}(\vartheta)$ is the set of isomorphism classes of spherical tori with one conical point of angle $2\pi\vartheta$, endowed with the Lipschitz topology.

In order to prove Theorem A it will be convenient to introduce the notion of 2–marking:

Definition 4.2 (2–marking) A 2–*marking* of a spherical torus T with one conical point x is a labeling of its nontrivial 2–torsion points or, equivalently, an isomorphism $H_1(T;\mathbb{Z}_2) \cong (\mathbb{Z}_2)^2$.

There is a bijective correspondence between isomorphisms $\mu\colon (\mathbb{Z}_2)^2 \to H_1(T;\mathbb{Z}_2)$ and orderings of the three nontrivial elements of $H_1(T;\mathbb{Z}_2)$, sending μ to the triple $(\mu(e_1), \mu(e_2), \mu(e_1+e_2))$. In fact, the action of $\mathrm{SL}(2,\mathbb{Z}_2)$ on 2–markings corresponds to the S_3–action that permutes the orderings. If the torus T has a spherical metric with conical point x, the nontrivial conformal involution σ fixes x and its three nontrivial 2–torsion points. The above ordering is then equivalent to the labeling of these three points. In this case, an isomorphism between two 2–marked spherical tori is an orientation-preserving isometry compatible with the 2–markings.

Definition 4.3 ($\mathcal{MS}_{1,1}^{(2)}$ as a topological space) The space $\mathcal{MS}_{1,1}^{(2)}(\vartheta)$ is the set isomorphisms classes of 2–marked spherical tori with one conical point of angle $2\pi\vartheta$, endowed with the Lipschitz topology.

In Remark 6.28 we show that $\mathcal{MS}_{1,1}(\vartheta)$ and $\mathcal{MS}_{1,1}^{(2)}(\vartheta)$ can be given the structure of orbifolds in such a way that the map $\mathcal{MS}_{1,1}^{(2)}(\vartheta) \to \mathcal{MS}_{1,1}(\vartheta)$ that forgets the 2–marking is a Galois cover with group S_3 (which is unramified in the orbifold sense).

4.1 The case when ϑ is not an odd integer

Because of the relevance for the orbifold structure of the moduli spaces we are interested in, we first classify all possible automorphisms of spherical tori with one conical point:

Proposition 4.4 (automorphisms group of a spherical torus (ϑ nonodd)) *Suppose that $\vartheta \notin 2\mathbb{Z}+1$. For any spherical torus (T,x) of area $2\pi(\vartheta-1)$, the group of automorphisms G_T is isomorphic to $\mathbb{Z}_2$, $\mathbb{Z}_4$ or $\mathbb{Z}_6$.*

(i) *A torus with automorphism group $\mathbb{Z}_6$ exists if and only if $d_1(\vartheta, 6\mathbb{Z}) > 1$.*

(ii) *A torus with automorphism group $\mathbb{Z}_4$ exists if and only if $d_1(\vartheta, 4\mathbb{Z}) > 1$.*

(iii) *For each ϑ, there can be at most one torus with automorphism group $\mathbb{Z}_4$ and one torus with automorphism group $\mathbb{Z}_6$.*

(iv) *The subgroup of G_T of automorphisms that fix the 2–torsion points of T is isomorphic to $\mathbb{Z}_2$ and generated by the conformal involution.*

In Figure 9 we have highlighted with Q or H the triples $\Theta(\Delta)$ such that $T(\Delta)$ has automorphism group isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_6$, respectively.

Proof Recall that, by Proposition 2.17, each torus has an automorphism of order 2, namely the conformal involution. Clearly this involution fixes the 2–torsion points of the torus. This implies (iv) and it proves that $|G_T|$ is even.

To bound the automorphism group we note that the action of G_T fixes x and preserves the conformal structure on T. Hence, when $|G_T| > 2$, the torus T is biholomorphic to either $T_4 = \mathbb{C}/(\mathbb{Z} \oplus \zeta_4\mathbb{Z})$ or $T_6 = \mathbb{C}/(\mathbb{Z} \oplus \zeta_6\mathbb{Z})$, where $\zeta_k = \exp(2\pi i/k)$, and its automorphism group is isomorphic to $\mathbb{Z}_4$ (generated by the multiplication by ζ_4) or to $\mathbb{Z}_6$ (generated by the multiplication by ζ_6), respectively.

Let us now prove the existence part of (i) and (ii).

(i) Suppose that $d_1(\vartheta, 6\mathbb{Z}) > 1$. According to Theorem 3.1, this condition is equivalent to the existence of a spherical triangle Δ with angles $\frac{1}{3}\pi\vartheta$. Such a triangle has a rotational $\mathbb{Z}_3$–symmetry. It follows that the torus $T(\Delta)$ has an automorphism of order 6.

(ii) Suppose that $d_1(\vartheta, 4\mathbb{Z}) > 1$. According to Theorem 3.1, this condition is equivalent to the existence of a spherical triangle Δ with angles $\pi\left(\frac{1}{2}\vartheta, \frac{1}{4}\vartheta, \frac{1}{4}\vartheta\right)$. This triangle has a reflection, ie an anticonformal isometry that exchanges two vertices of angles $\frac{1}{4}\pi\vartheta$. Gluing two copies of Δ along the edge that faces the angle $\frac{1}{2}\pi\vartheta$, we obtain a quadrilateral with four edges of the same length and four angles $\frac{1}{2}\pi\vartheta$. It is easy to see that such a quadrilateral has a rotational $\mathbb{Z}_4$–symmetry, and so $T(\Delta)$ has an order-4 automorphism.

Now let (T, x, ϑ) be any spherical torus with $|G_T| > 2$, and let us show that it has to be one of the two tori constructed above. Consider two cases.

First, suppose that the Voronoi graph $\Gamma(T)$ is a trefoil. In this case, by Proposition 2.21 and Theorem B, there is a unique collection of three geodesic loops $(\gamma_1, \gamma_2, \gamma_3)$ based at x that cuts T into two isometric strictly balanced triangles Δ and Δ'. This collection is sent by G_T to itself, and so $|G_T|$ is divisible by three; hence $|G_T| = 6$. It is easy to see then that the subgroup $\mathbb{Z}_3 \subset G_T$ sends Δ to itself and permutes its vertices. So Δ has angles $\frac{1}{3}\pi\vartheta$ and so we are in case (i). Since $\frac{1}{3}\vartheta$ cannot be integer, this also proves the uniqueness of a torus with automorphism group $\mathbb{Z}_6$.

Suppose now that the Voronoi graph $\Gamma(T)$ is an eight graph. Again by Proposition 2.21 and Theorem B, there is a canonical collection of four geodesic loops γ_1, γ_2, η_1 and η_2. Since G_T sends the pair (η_1, η_2) to itself, we see that geodesics η_1 and η_2 cut a neighborhood of x into four sectors of angles $\frac{1}{2}\pi\vartheta$. The same holds for the pair of loops γ_1 and γ_2. Since, by Remark 2.22, each γ_i bisects two sectors formed by η_1 and η_2, we see that, taken together, the geodesics γ_1, γ_2, η_1 and η_2 cut a neighborhood of x into eight sectors of angles $\frac{1}{4}\pi\vartheta$. Hence, γ_1, γ_2 and η_1 cut Δ into two semibalanced triangles with angles $\pi\left(\frac{1}{2}\vartheta, \frac{1}{4}\vartheta, \frac{1}{4}\vartheta\right)$, and so we are in case (ii). The uniqueness of a torus with automorphism group $\mathbb{Z}_4$ follows from the uniqueness of an isosceles triangle with angles $\pi\left(\frac{1}{2}\vartheta, \frac{1}{4}\vartheta, \frac{1}{4}\vartheta\right)$. □

We recall in more detail the construction mentioned in the introduction:

Construction 4.5 Consider the maps of sets

$$\mathcal{MT}^{\pm}_{\mathrm{bal}}(\vartheta) \underset{\Delta^{(2)}}{\overset{T^{(2)}}{\rightleftarrows}} \mathcal{MS}^{(2)}_{1,1}(\vartheta).$$

The map $T^{(2)}$ is defined by sending an oriented triangle Δ to the torus $T(\Delta)$, where we mark by p_i the midpoint of the side opposite to the vertex x_i of Δ.

As for $\Delta^{(2)}$, we proceed as follows. Let $(T, x, \boldsymbol{p})$ be a torus with its order-2 points marked by p_1, p_2 and p_3.

Suppose first that T does not have a rectangular involution. By Theorem B, there is a unique collection of three geodesics loops γ_i that cuts T into two congruent strictly balanced triangles Δ and Δ'. We enumerate the geodesics so that each p_i is the midpoint of γ_i. Next, we label the vertices of Δ by x_1, x_2 and x_3 so that x_i is opposite to γ_i. Hence, we associate to T a unique strictly balanced triangle with enumerated vertices. If the vertices of Δ go in anticlockwise order, we associate to Δ the corresponding point in the interior of $\mathcal{MT}^{+}_{\mathrm{bal}}(\vartheta)$, otherwise we associate to Δ a point in the interior of $\mathcal{MT}^{-}_{\mathrm{bal}}(\vartheta)$.

Suppose now that T has a rectangular involution. Then, by Theorem B, the torus T can be cut into two isomorphic semibalanced triangles in two different ways. At the same time, the rectangular involution sends one pair to the other by reversing the orientation and fixing the labeling of the vertices. This means that the two points associated to T in the boundaries of $\mathcal{MT}^{+}_{\mathrm{bal}}(\vartheta)$ and $\mathcal{MT}^{-}_{\mathrm{bal}}(\vartheta)$ are identified in $\mathcal{MT}^{\pm}_{\mathrm{bal}}(\vartheta)$.

At this point we have the tools to prove the following preliminary fact:

Lemma 4.6 ($T^{(2)}$ is bijective) *The map* $T^{(2)}\colon \mathcal{MT}^{\pm}_{\mathrm{bal}}(\vartheta) \to \mathcal{MS}^{(2)}_{1,1}(\vartheta)$ *is a bijection and* $\Delta^{(2)}$ *is its inverse.*

Proof It is very easy to see that $T^{(2)} \circ \Delta^{(2)}$ is the identity of $\mathcal{MS}^{(2)}_{1,1}(\vartheta)$. Conversely, $\Delta^{(2)} \circ T^{(2)}$ is the identify of $\mathcal{MT}^{\pm}_{\mathrm{bal}}(\vartheta)$ by Theorem B. □

Remark 4.7 (orbifold Euler characteristic) We recall from the introduction that we are using the definition of orbifold Euler characteristic given by [7, page 29]. We are particularly interested in two properties enjoyed by the orbifold Euler characteristic:

(a) If $\mathcal{Y} \to \mathcal{Z}$ is an orbifold cover of degree d, then $\chi(\mathcal{Y}) = d \cdot \chi(\mathcal{Z})$.

(b) If $\mathcal{Y}$ is a connected orientable 2–dimensional orbifold with underlying topological space Y, then

$$\chi(\mathcal{Y}) = \frac{1}{\mathrm{ord}(\mathcal{Y})}\chi(Y) - \sum_{y}\Big(\frac{1}{\mathrm{ord}(\mathcal{Y})} - \frac{1}{\mathrm{ord}(y)}\Big),$$

where $\mathrm{ord}(\mathcal{Y})$ is the orbifold order of a general point of Y, $\mathrm{ord}(y)$ is the orbifold order of $y \in Y$, and the sum ranges over points $y \in Y$ that have orbifold order strictly greater than $\mathrm{ord}(\mathcal{Y})$.

Since we only compute χ for 2–dimensional connected orientable orbifolds, property (b) could even be taken as a definition.

The main ingredient for the proof of Theorem A is to show that the map $T^{(2)}$ is a homeomorphism, so, as a topological space, $\mathcal{MS}^{(2)}_{1,1}(\vartheta)$ is a surface. As a consequence, we can endow $\mathcal{MS}^{(2)}_{1,1}(\vartheta)$ with an orbifold structure (as done in Remark 6.28) in such a way that every point has orbifold order 2, which is consistent with Proposition 4.4(iv).

Theorem 4.8 (moduli space of spherical tori with 2–marking) *Let* $\vartheta > 1$ *be a real number such that* $\vartheta \notin 2\mathbb{Z}+1$ *and let* $m = \lfloor \frac{1}{2}(\vartheta+1) \rfloor$. *As a topological space,* $\mathcal{MS}^{(2)}_{1,1}(\vartheta)$ *has the following properties*:

(i) *The map* $T\colon \mathcal{MT}^{\pm}_{\mathrm{bal}}(\vartheta) \to \mathcal{MS}^{(2)}_{1,1}(\vartheta)$ *is a homeomorphism, and so* $\mathcal{MS}^{(2)}_{1,1}(\vartheta)$ *is a connected orientable surface of finite type without boundary.*

(ii) *It has genus* $\frac{1}{2}(m-1)(m-2)$ *and* $3m$ *punctures.*

(iii) *The group* S_3 *that permutes the* 2*–torsion points of a torus acts on* $\mathcal{MS}^{(2)}_{1,1}(\vartheta)$ *by orientation-preserving homeomorphisms.*

(iv) *The action of* S_3 *on the set of punctures of* $\mathcal{MS}^{(2)}_{1,1}(\vartheta)$ *has* m *orbits of length* 3.

As an orbifold, $\mathcal{MS}_{1,1}^{(2)}(\vartheta)$ *is isomorphic to the quotient of its underlying topological space by the trivial* $\mathbb{Z}_2$*–action, and its orbifold Euler characteristic is* $-\frac{1}{2}m^2$.

Proof The map $T^{(2)}$ is bijective by Lemma 4.6, and in fact a homeomorphism by Theorem 6.5. Hence, (i)–(iv) follow from Proposition 3.22(i)–(iv). The orbifold structure was described just above the statement of the theorem: the involution σ is the only nontrivial automorphism of a point in $\mathcal{MS}_{1,1}^{(2)}(\vartheta)$ by Proposition 4.4(iv), and it acts trivially on $\mathcal{MT}_{\mathrm{bal}}^{\pm}(\vartheta)$. Hence, $\mathcal{MS}_{1,1}^{(2)}(\vartheta)$ is isomorphic to the quotient of $\mathcal{MT}_{\mathrm{bal}}^{\pm}(\vartheta)$ by the trivial $\mathbb{Z}_2$–action. As a consequence, the orbifold Euler characteristic satisfies $\chi(\mathcal{MS}_{1,1}^{(2)}(\vartheta)) = \frac{1}{2}\chi(\mathcal{MT}_{\mathrm{bal}}^{\pm}(\vartheta))$. □

As above, we can endow $\mathcal{MS}_{1,1}(\vartheta)$ with an orbifold structure as in Remark 6.28, in such a way that the orbifold order of a point in $\mathcal{MS}_{1,1}(\vartheta)$ agrees with the number of automorphisms of the corresponding spherical torus.

Proof of Theorem A By Remark 6.28, the map $\mathcal{MS}_{1,1}^{(2)}(\vartheta) \to \mathcal{MS}_{1,1}(\vartheta)$ that forgets the 2–marking is an unramified S_3–cover of orbifolds. Hence, $\mathcal{MS}_{1,1}(\vartheta)$ is a smooth connected 2–dimensional orbifold of finite type by Theorem 4.8(i), and orientability follows from Proposition 4.4.

(ii)–(iv) Clearly $\chi(\mathcal{MS}_{1,1}(\vartheta)) = \chi(\mathcal{MS}_{1,1}^{(2)}(\vartheta))/|\mathrm{S}_3| = -\frac{1}{12}m^2$ by Theorem 4.8. Also, (iii)–(iv) and the remaining claim of (ii) are established in Proposition 4.4.

(i) The space $\mathcal{MS}_{1,1}(\vartheta)$ has m punctures by Theorem 4.8(ii) and (iv). Moreover, its (nonorbifold) Euler characteristic is $2\bigl(\frac{1}{12} - m^2 + \epsilon\bigr)$, where $\epsilon \in \bigl\{0, \frac{1}{4}, \frac{1}{3}, \frac{7}{12} = \frac{1}{4} + \frac{1}{3}\bigr\}$. Indeed, a point of order 4 in $\mathcal{MS}_{1,1}(\vartheta)$ contributes to ϵ with $\frac{1}{4} = \frac{1}{2} - \frac{1}{4}$ and a point of order 6 contributes with $\frac{1}{3} = \frac{1}{2} - \frac{1}{6}$. Hence, the genus of $\mathcal{MS}_{1,1}(\vartheta)$ is $1 - \frac{1}{2}\bigl(m + 2\bigl(-\frac{1}{12}m^2 + \epsilon\bigr)\bigr) = \bigl\lfloor \frac{1}{6}(m^2 - 6m + 12)\bigr\rfloor$. □

Let us finish this subsection with a simple corollary of Theorem 4.8. As a topological space, we denote by $\overline{\mathcal{MS}}_{1,1}^{(2)}(\vartheta)$ the unique smooth compactification of the surface $\mathcal{MS}_{1,1}^{(2)}(\vartheta)$ obtained by filling in the $3m$ punctures. As above, we endow $\overline{\mathcal{MS}}_{1,1}^{(2)}(\vartheta)$ with the orbifold structure given by taking the quotient of its underlying topological space by the trivial $\mathbb{Z}_2$–action.

Corollary 4.9 (a cell decomposition of $\overline{\mathcal{MS}}_{1,1}^{(2)}(\vartheta)$) *Suppose that* $\vartheta = 2m + c$, *where* $c \in (-1, 1)$. *As a topological space,* $\overline{\mathcal{MS}}_{1,1}^{(2)}(\vartheta)$ *has the following properties:*

(i) *It is a compact connected orientable surface of genus* $\frac{1}{2}(m-1)(m-2)$.

(ii) *It has a natural structure of a CW complex, where*
- *its* 0*–cells are the* $3m$ *added points*;
- *its* 1*–cells are formed by tori* T *such that* $\Delta(T)$ *is ether a semibalanced triangle, or a triangle with one integral angle*;
- *its* 2*–cells are the complement of the union of the* 0*–cells and* 1*–cells.*

Moreover, for $c \le 0$, *the cell decomposition is a triangulation into* $2m^2$ *triangles.*

Proof Let us comment on the last claim, since the other claims are rather immediate after Theorem 4.8. Recall that in the proof of Proposition 3.20(iv), for $c \le 0$, we constructed a decomposition of $\mathcal{MT}_{\mathrm{bal}}(\vartheta)$ into the union of $\frac{3}{2}m(m+1)$ 1-cells and m^2 2-cells. One can check that each of theses m^2 cells has exactly three 1–cells in its boundary. Hence, we get a triangulation of the topological space $\overline{\mathcal{MS}}_{1,1}^{(2)}(\vartheta)$.

Note, however, that for $c > 0$ the total number of 2–cells is $2m^2+6m$, and the additional $6m$ cells are digons rather than triangles. □

4.2 The case when ϑ is an odd integer

In this subsection we prove Theorems C and D. Our first step will be to prove Theorem E, from which part (a) of Theorem C is easily obtained.

Proof of Theorem E According to Proposition 2.17, there is a unique curvature-1 metric on T with angle $2\pi(2m+1)$ in a given projective equivalence class, which is invariant under the conformal involution σ of T. Hence we can apply Proposition 2.21 to T endowed with such a σ–invariant metric. According to this proposition, there exist three geodesic loops based at the conical point x that cut T into two isometric balanced triangles Δ and Δ'. By the Gauss–Bonnet formula $\mathrm{Area}(\Delta) = 2\pi m$, and so we can apply Corollary 3.10 to obtain that Δ is a balanced triangle with angles $2\pi(m_1, m_2, m_3)$ where $m_1+m_2+m_3 = 2m+1$. □

This result directly allows us to describe $\mathcal{MS}_{1,1}(2m+1)^\sigma$ as a topological space:

Proof of Theorem C(a) As in the proof of Theorem E, we can associate to each torus with a σ–invariant metric a unique oriented balanced spherical triangle with integral angles and unmarked vertices. Clearly an orientation on a triangle is equivalent to a numbering of its vertices up to cyclic permutations, and this correspondence determines a bijective map

$$T\colon \mathcal{MT}_{\mathrm{bal}}(2m+1)/\mathrm{A}_3 \to \mathcal{MS}_{1,1}(2m+1)^\sigma,$$

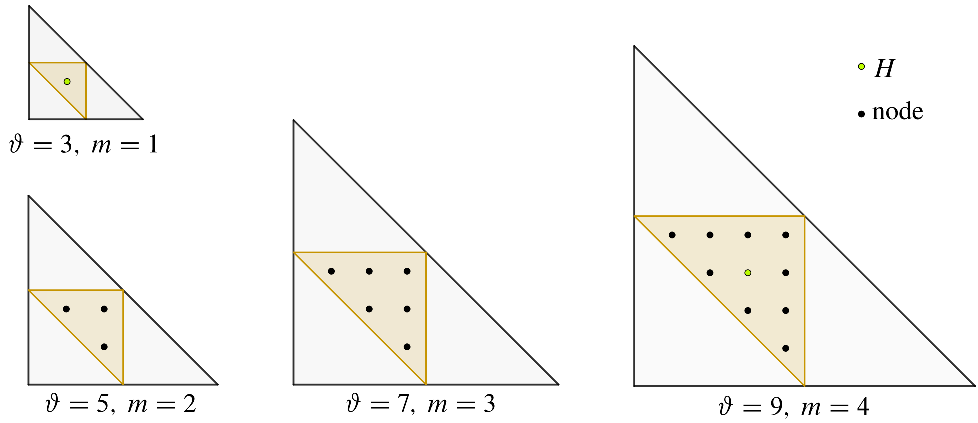

Figure 11: Angle carpets for $\vartheta = 2m + 1$ an odd integer.

where the alternating group A_3 acts by relabeling the vertices of the triangle. Arguments entirely analogous to the ones used in Theorem 6.5(ii) show that T is continuous and proper, and hence a homeomorphism of topological spaces.

By Proposition 3.24, the space $\mathcal{MT}_{\mathrm{bal}}(2m+1)$ is homeomorphic to the disjoint union of $\frac{1}{2}m(m+1)$ copies of the open standard simplex $\mathring{\Delta}^2$. Each component represents triangles of angles $\pi(m_1, m_2, m_3)$ with $m_1+m_2+m_3 = 2m+1$, where (m_1, m_2, m_3) is a triple of positive integers that satisfy the three triangle inequalities (see Figure 11).

Consider two cases:

(i) Suppose that $m \not\equiv 1 \pmod 3$. In this case, the integer $2m+1$ is not divisible by 3 and so neither of the spherical triangles in $\mathcal{MT}_{\mathrm{bal}}(2m+1)$ has all equal angles. It follows that the action of A_3 does not send any component to itself. So the number of components of $\mathcal{MS}_{1,1}(2m+1)^{\sigma}$ is $\frac{1}{6}m(m+1)$ and each one is homeomorphic to the open 2–disk $\mathring{\Delta}^2$.

(ii) Suppose that $m \equiv 1 \pmod 3$. Then the component corresponding to triangles with angles $m_1 = m_2 = m_3 = \frac{1}{3}(2m+1)$ is the only one that is sent to itself. It contains a unique point fixed by A_3, namely the equilateral spherical triangle, and the quotient of this component by A_3 is homeomorphic to an open 2–disk. All the other $\frac{1}{2}(m(m+1)-2)$ components of $\mathcal{MT}_{\mathrm{bal}}(2m+1)$ are nontrivially permuted by A_3; hence, they give $\frac{1}{6}(m(m+1)-2)$ components of $\mathcal{MS}_{1,1}(2m+1)^{\sigma}$ homeomorphic to $\mathring{\Delta}^2$. Therefore the total number of connected components of $\mathcal{MS}_{1,1}(2m+1)^{\sigma}$ is $\frac{1}{6}(m(m+1)+4)$. □

The rest of the subsection is devoted to a careful analysis of the orbifold structures on our moduli spaces, the proof of part (b) of Theorem C and the proof of Theorem D.

4.2.1 Voronoi graphs and decorations The orbifold structure on our moduli spaces is defined in Remark 6.28, but a more explicit interpretation of the structure for moduli spaces of tori of area $4m\pi$ relies on the notion of decoration.

We begin with a simple lemma:

Lemma 4.10 (Voronoi graphs of tori of area $4m\pi$) *The Voronoi graph $\Gamma(T)$ of a spherical torus T of area $4m\pi$ has two vertices and three edges of lengths $(2m_i+1)\pi$ for integers $m_i \geq 0$. The two vertices are exchanged by the conformal involution σ. Also, projectively equivalent spherical metrics on a torus have the same Voronoi graph.*

Proof Consider first the case $m = 1$. A spherical triangle Δ_0 with vertices x_1, x_2 and x_3 of angles (π, π, π) is isometric to a hemisphere, and its circumcenter O is at distance $\frac{1}{2}\pi$ from the boundary of the hemisphere. So the rotations of the hemisphere that take x_i to x_j fix O. A torus T_0 with a σ–invariant metric h of area 4π is isometric to $T(\Delta_0)$ and so it has three edges and two vertices. Since σ fixes the Voronoi graph $\Gamma(T_0)$ and pointwise fixes the conical point and the midpoints of the three edges of $\Gamma(T_0)$, it does not fix any other point. In particular, σ exchanges the two vertices of $\Gamma(T_0)$. Moreover, the vertices of $\Gamma(T_0)$ are at distance $\frac{1}{2}\pi$ from $\partial\Delta_0$, and so the edges of $\Gamma(T_0)$ have length π. It follows that a (multivalued) developing map for T_0 sends the vertices of $\Gamma(T_0)$ to the two fixed points O and O' for the monodromy, and the edges of $\Gamma(T_0)$ to meridians running between O and O'. Note that another spherical metric on T_0 projectively equivalent to h is obtained by postcomposing the developing map of h by a Möbius transformation that fixes O and O'. Since such transformations preserve the meridians between O and O', the two metrics have the same Voronoi graph.

Suppose now $m > 1$. By Theorem E and Proposition 3.8, a torus T with σ–invariant metric of area $4m\pi$ is obtained from a torus $T_0 = T(\Delta_0)$ of area 4π as above by gluing a sphere S_i with two conical points of angles $2m\pi$ at distance $|x_j x_k|$ along the geodesic segment $x_j x_k$ of T_0. The conclusion then follows from the analysis of the case $m = 1$. □

In order to make the role of the conformal involution σ in Constructions 4.14 and 4.16 below more transparent, we will need:

Definition 4.11 (decorations on strictly balanced tori) A *decoration* v of a spherical torus (T, x) is a vertex v of its Voronoi graph $\Gamma(T)$.

The main reason for introducing decorations relies on the following fact:

Lemma 4.12 (rigidity of 2–marked decorated spherical tori) *Decorated 2–marked spherical tori of area $4m\pi$ have nontrivial automorphisms.*

Proof Being an isometry, an automorphism is in particular biholomorphic. It is a classical fact that the only nontrivial biholomorphism of a 2–marked conformal torus (T, x) is the involution σ. By Lemma 4.10, the Voronoi graph $\Gamma(T)$ has two vertices and they are exchanged by σ. □

As a consequence, we obtain the following modular interpretation of $\mathcal{MS}_{1,1}^{(2)}(2m+1)$ as a topological space:

Remark 4.13 The topological space $\mathcal{MS}_{1,1}^{(2)}(2m+1)$ is the moduli space of decorated 2–marked spherical tori of area $4m\pi$.

By Lemma 4.10, σ induces an action σ^* on $\mathcal{MS}_{1,1}^{(2)}(2m+1)$ by sending $(T, \boldsymbol{p}, v, h)$ to $(T, \boldsymbol{p}, v, \sigma^* h)$. Since $\sigma\colon (T, \boldsymbol{p}, v, \sigma^* h) \to (T, \boldsymbol{p}, \sigma(v), h)$ is an isomorphism, we also have $\sigma^*(T, \boldsymbol{p}, v, h) = (T, \boldsymbol{p}, \sigma(v), h)$.

4.2.2 Moduli spaces of σ–invariant spherical metrics of area $4m\pi$ Similarly to Section 4.1, we first discuss the space of decorated 2–marked tori:

Construction 4.14 (tori with σ–invariant metrics) If σ is the unique (nontrivial) conformal involution of a conformal torus, denote by $\mathcal{MS}_{1,1}^{(2)}(2m+1)^\sigma$ the set of 2–marked decorated tori $(T, x, \boldsymbol{p})$ with a σ–invariant spherical metric of angle $(2m+1)2\pi$ at x. We recall that triangles in $\mathcal{MT}_{\mathrm{bal}}^{\pm}(2m+1)$ have area $2m\pi$ and integral angles and they are strictly balanced. We then define the maps

$$\mathcal{MT}_{\mathrm{bal}}^{\pm}(2m+1) \underset{\Delta^{(2)}}{\overset{T^{(2)}}{\rightleftarrows}} \mathcal{MS}_{1,1}^{(2)}(2m+1)^\sigma$$

as in Construction 4.5. In particular, $T^{(2)}$ sends an oriented triangle Δ to the 2–marked torus $T(\Delta)$ obtained as the union of Δ and Δ', with the decoration given by the vertex of $\Gamma(T(\Delta))$ that sits inside Δ.

We easily have the following preliminary result:

Theorem 4.15 (moduli space of 2–marked σ–invariant tori of area $4m\pi$) *For $m > 0$ an integer, the space $\mathcal{MS}_{1,1}^{(2)}(2m+1)^\sigma$ of decorated 2–marked tori with a σ–invariant spherical metric has the following properties:*

(i) *The map* $T^{(2)}\colon \mathcal{MT}^{\pm}_{\mathrm{bal}}(2m+1) \to \mathcal{MS}^{(2)}_{1,1}(2m+1)^{\sigma}$ *is a homeomorphism, with inverse* $\Delta^{(2)}$.

(ii) $\mathcal{MS}^{(2)}_{1,1}(2m+1)^{\sigma}$ *is a disjoint union of* $m(m+1)$ *open* 2*–disks* $\mathring{\Delta}^2$.

(iii) S_3 *acts on* $\mathcal{MS}^{(2)}_{1,1}(2m+1)^{\sigma}$ *by permuting its components. If* $m \not\equiv 1 \pmod 3$, *then all orbits have length* 6. *If* $m \equiv 1 \pmod 3$, *then one orbit has length* 2 *and all the others have length* 6.

(iv) *The action of* σ^* *on the topological space* $\mathcal{MS}^{(2)}_{1,1}(2m+1)^{\sigma}$ *is trivial.*

As an orbifold, the moduli space of 2*–marked tori with a* σ*–invariant spherical metric is isomorphic to the quotient of* $\mathcal{MT}^{\pm}_{\mathrm{bal}}(2m+1)$ *by the trivial* $\mathbb{Z}_2$*–action.*

Proof (i) It is very easy to see that $T^{(2)} \circ \Delta^{(2)}$ is the identity on $\mathcal{MS}^{(2)}_{1,1}(2m+1)^{\sigma}$. Conversely $\Delta^{(2)} \circ T^{(2)}$ is the identify on $\mathcal{MT}^{\pm}_{\mathrm{bal}}(2m+1)$ by Theorem E. Hence $T^{(2)}$ is bijective. Moreover, $T^{(2)}$ is a homeomorphism by Theorem 6.5.

(ii)–(iii) These follow from Propositions 3.25 and 3.24.

(iv) This is clear, since σ is an isomorphism between the 2–marked decorated spherical tori $(T, \boldsymbol{p}, v, h)$ and $(T, \boldsymbol{p}, v, \sigma^* h)$.

In view of Remark 6.28, the final claim follows from (iv). □

Now we discuss the moduli space $\mathcal{MS}_{1,1}(2m+1)^{\sigma}$ of σ–invariant spherical tori:

Proof of Theorem C(b) Recall $\mathcal{MS}_{1,1}(2m+1)^{\sigma}$ is endowed with a 2–dimensional orbifold structure by Remark 6.28. By Theorem 4.15(i), the space $\mathcal{MT}^{\pm}_{\mathrm{bal}}(2m+1)$ is isomorphic to the moduli space of decorated 2–marked tori with a σ–invariant metric. Fix such a torus. Then 2–markings are permuted by S_3 and the decorations are exchanged by σ. Hence, the moduli space $\mathcal{MS}_{1,1}(2m+1)^{\sigma}$ is isomorphic (as an orbifold) to the quotient of $\mathcal{MT}^{\pm}_{\mathrm{bal}}(2m+1)$ by $\mathrm{S}_3 \times \langle 1, \sigma^* \rangle$. By Proposition 3.25, this quotient can be identified to $\mathcal{MT}_{\mathrm{bal}}(2m+1)/\mathrm{A}_3 \times \mathbb{Z}_2$, where the alternating group A_3 acts by cyclically relabeling the vertices of the triangles and $\mathbb{Z}_2$ acts trivially by Theorem 4.15(iv). By Proposition 3.24, the space $\mathcal{MT}_{\mathrm{bal}}(2m+1)$ consists of $\frac{1}{2}m(m+1)$ connected components and is diffeomorphic to $\mathrm{Crp}_{\mathrm{bal}}(2m+1) \times \mathring{\Delta}^2$.

Consider two cases.

(b-i) Suppose $2m+1$ is not divisible by 3. In this case, neither of the spherical triangles in $\mathcal{MT}_{\mathrm{bal}}(2m+1)$ have all equal angles, so the action of A_3 does not send any component to itself. So the number of components of $\mathcal{MS}_{1,1}(2m+1)^{\sigma}$ is $\frac{1}{6}m(m+1)$

and each one is homeomorphic to the quotient $\mathcal{D}$ of $\mathring{\Delta}^2$ by the trivial $\mathbb{Z}_2$–action, and so all points have orbifold order 2.

(b-ii) Suppose $2m+1$ is divisible by 3. Then the component corresponding to triangles with angles $m_1 = m_2 = m_3 = \frac{1}{3}(2m+1)$ is the only one that is sent to itself. It contains a unique point fixed by A_3, namely the equilateral spherical triangle. This point gives rise to an orbifold point of order 6 on $\mathcal{MS}_{1,1}(2m+1)^\sigma$, which belongs to a component homeomorphic to the quotient $\mathcal{D}'$ of $\mathring{\Delta}^2$ by $\mathbb{Z}_2 \times \mathrm{A}_3$, where $\mathbb{Z}_2$ acts trivially. All the other $\frac{1}{2}m(m+1)-1$ components are nontrivially permuted by A_3, and they are all homeomorphic to $\mathcal{D}$. Hence, there are $\left\lceil \frac{1}{6}m(m+1) \right\rceil$ connected components, and all points except the equilateral spherical triangle have orbifold order 2. □

4.2.3 Moduli spaces of spherical metrics of area $4m\pi$ In order to treat spherical metrics that are not σ–invariant, we need a further construction.

Construction 4.16 Given a point $O \in \mathbb{S}^2$, let $R \in \mathfrak{su}(2)$ be the unique element with $\mathrm{tr}(R^2) = -\frac{1}{2}$ that generates anticlockwise rotations of $\mathbb{S}^2$ at O.

We view the topological space $\mathcal{MS}_{1,1}^{(2)}(2m+1)$ as a moduli space of decorated, 2–marked tori and we define the pair of maps

$$\mathcal{MT}^{\pm}_{\mathrm{bal}}(2m+1) \times \mathbb{R} \underset{\nu}{\overset{\Xi}{\rightleftarrows}} \mathcal{MS}_{1,1}^{(2)}(2m+1)$$

as follows.

In order to define Ξ, let Δ be an oriented triangle in $\mathcal{MT}^{\pm}_{\mathrm{bal}}(2m+1)$ and fix a developing map ι for Δ that sends its circumcenter v to $O \in \mathbb{S}^2$. Extend ι to the universal cover of the torus $T(\Delta)$, which has a σ–invariant metric h, and is given a 2–marking as in Construction 4.5. For every $t \in \mathbb{R}$, the map $e^{itR} \circ \iota \colon \widetilde{T} \to \mathbb{S}^2$ has the same equivariance of ι, and so the pullback of the metric of $\mathbb{S}^2$ via such a map descends to a spherical metric h_t on T. We then define $\Xi(T, x, \boldsymbol{p}, v, h, t) := (T, x, \boldsymbol{p}, v, h_t)$.

In order to define ν, consider a 2–marked decorated spherical torus $(T, x, \boldsymbol{p}, v, \hat{h})$, whose metric $\hat{h}$ is not necessarily invariant under the conformal involution σ. Its developing map $\iota \colon \widetilde{T} \to \mathbb{S}^2$ has monodromy contained in a 1–parameter subgroup that fixes $O = \iota(\tilde{v})$, where $\tilde{v}$ is a lift of v, and a maximal circle E. Note that points in $e^{itR}E$ sit at constant distance $\arctan(2e^{-t})$ from O and that the distance from O corresponds to the distance function $d_v \colon T \to [0, \pi]$ from the vertex v. Thus we also have the function $t = -\log\tan\left(\frac{1}{2}d_v\right) \colon T \to [-\infty, \infty]$. We remark that a developing map of $\sigma^*(T, x, \boldsymbol{p}, v, \hat{h})$ can be obtained by postcomposing ι with an isometry of $\mathbb{S}^2$ that exchanges O with $-O$. Hence, $t \circ \sigma^* = -t$. It follows that $\hat{h}$ is σ–invariant if and

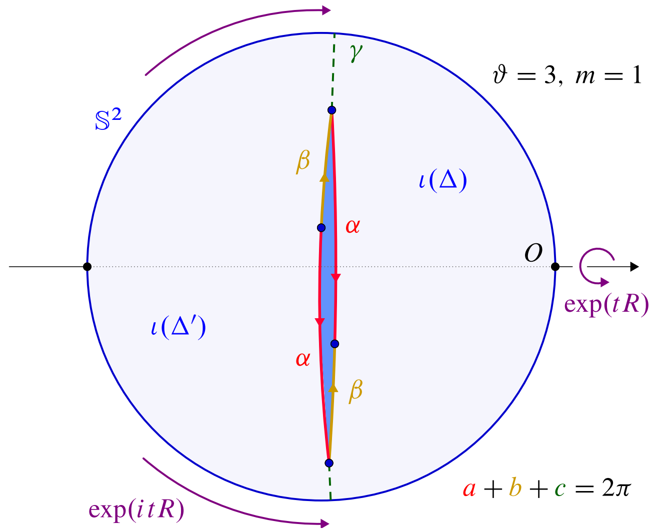

Figure 12: The developing map ι for $T(\Delta) \setminus (\alpha \cup \beta)$, where Δ is a triangle with $\vartheta = 3$ and edges α, β and γ of lengths a, b and c. The two congruent triangles Δ and Δ' are mapped to antipodal hemispheres, and their edges are mapped to the separating equator.

only if $t(x) = 0$, namely $\iota(\tilde{x}) \in E$ for any lift $\tilde{x}$ of x. It is easy to see that the modified developing map $e^{-it(x)R} \circ \iota$ has the same invariance as ι and sends $\tilde{x}$ to E. Hence, the round metric on $\mathbb{S}^2$ pulls back and descends to a σ–invariant metric h on T. We define $\nu(T, x, \boldsymbol{p}, v, \hat{h}) := \Delta^{(2)}(T, x, \boldsymbol{p}, v, h, t(x))$.

Before proceeding, we need a very simple lemma:

Lemma 4.17 (Lipschitz constant of projective transformations) *For every $t \in \mathbb{R}$, the transformation e^{itR} of $\mathbb{S}^2$ has (bi)Lipschitz constant $\cosh(t)$. Moreover, along the maximal circle E it has Lipschitz constant $1/\cosh(t)$.*

Proof If O is the origin of $\mathbb{C}$ and $2|dz|/(1+|z|^2)$ is the spherical line element, then the transformation e^{itR} can be written as $z \mapsto e^{-t}z$. Through the map e^{itR} the metric decreases the most at $E = \{|z| = 1\}$, where the Lipschitz constant is exactly $1/\cosh(t)$. □

The first fact about Construction 4.16 is the following:

Proposition 4.18 (the homeomorphism Ξ) *The map Ξ is a homeomorphism and ν is its inverse.*

Proof It is routine to check that the maps Ξ and ν are set-theoretic inverses of each other. Note that the restriction of Ξ to $\mathcal{MT}^{\pm}_{\mathrm{bal}}(2m+1)\times\{0\}$ is a homeomorphism by Theorem 4.15(i). Hence, the continuity of Ξ follows from Lemma 4.17.

To show that Ξ is proper, consider a diverging sequence in $\mathcal{MT}^{\pm}_{\mathrm{bal}}(2m+1)\times\mathbb{R}$, which we can assume to be contained in a fixed connected component. By Proposition 3.25, an element of this component can be identified by a quadruple (l_1,l_2,l_3,t) with $0<l_i<2\pi$ and $l_1+l_2+l_3=2\pi$. A sequence of quadruples diverges if and only if some $\bar{l}_i\to 0$ or if $|t|\to\infty$ (up to subsequences). Since the systole of the triangle corresponding to (l_1,l_2,l_3) is $\min\{\bar{l}_i\}$ by Lemma 6.24, the systole of the torus $\Xi(l_1,l_2,l_3,t)$ is at most $\min\{\bar{l}_i\}/\log\cosh(t)\to 0$ by Lemma 4.17. It follows that Ξ sends diverging sequences to diverging sequences by Theorem 6.3. $\square$

Since $\mathcal{MS}^{(2)}_{1,1}(2m+1)$ is a manifold by Proposition 4.18, it can be endowed with an orbifold structure as in Remark 6.28. We then have the following preliminary result:

Theorem 4.19 (moduli space of 2–marked tori of area $4m\pi$) *For $m>0$ an integer, the moduli space $\mathcal{MS}^{(2)}_{1,1}(2m+1)$ of 2–marked tori with spherical metric of area $4m\pi$ has the following properties:*

- (i) *As an orbifold, it is isomorphic to the quotient of $\mathcal{MT}^{\pm}_{\mathrm{bal}}(2m+1)\times\mathbb{R}$ by the action of the involution σ^* that flips the sign of the $\mathbb{R}$ factor. Hence it consists of $m(m+1)$ components isomorphic to $\mathring{\Delta}^2\times(\mathbb{R}/\{\pm1\})$.*
- (ii) *The locus in $\mathcal{MS}^{(2)}_{1,1}(2m+1)$ of metrics that are invariant under the conformal involution σ corresponds to $\mathcal{MT}^{\pm}_{\mathrm{bal}}(2m+1)\times\{0\}$.*
- (iii) *The group S_3 that permutes the 2–torsion points of the torus acts trivially on $\mathbb{R}$ and as in Proposition 3.25 on $\mathcal{MT}^{\pm}_{\mathrm{bal}}(2m+1)$.*

Proof (i) The action of σ is described in Construction 4.16. The claim follows from Theorem 4.15(i) and Proposition 4.18.

(ii) This is also clear by Construction 4.16.

(iii) This follows by noting that relabeling the 2–torsion points does not affect the decoration. $\square$

Proof of Theorem D The forgetful map $\mathcal{MS}^{(2)}_{1,1}(2m+1)\to\mathcal{MS}_{1,1}(2m+1)$ is an unramified S_3–cover of orbifolds. By Theorem 4.19 such a quotient can be identified to

$(\mathcal{MT}_{\mathrm{bal}}(2m+1)\times\mathbb{R})/(\mathrm{A}_3\times\mathbb{Z}_2)$, where $\mathbb{Z}_2$ acts by flipping the sign of the $\mathbb{R}$ factor, and the alternating group A_3 acts by cyclically relabeling the vertices of the triangles.

The rest of argument is entirely analogous to the one used in the proof of Theorem C. $\square$

5 $\mathcal{MS}_{1,1}(2m)$ and $\mathcal{MS}^{(2)}_{1,1}(2m)$ as Belyi curves

The goal of this section is to identify the moduli spaces $\mathcal{MS}_{1,1}(2m)$ and $\mathcal{MS}^{(2)}_{1,1}(2m)$ with Belyi curves, and to relate their cell decompositions constructed in Corollary 4.9 with the corresponding dessins. We recall [2, Section 2; 14] that these two spaces have a *canonical complex structure*. This structure is the unique one with respect to which the forgetful maps to $\mathcal{M}_{1,1}$ and $\mathcal{M}^{(2)}_{1,1}$ are holomorphic. We also recall that the compactification $\overline{\mathcal{MS}}^{(2)}_{1,1}(2m)$, obtained from $\mathcal{MS}^{(2)}_{1,1}(2m)$ by filling in the $3m$ punctures, has orbifold structure that makes it isomorphic to the quotient of its underlying topological space (which is in fact a Riemann surface) by the trivial $\mathbb{Z}_2$–action. The respective forgetful maps extend to the smooth compactifications of all the four orbifolds.

The following definition slightly differs from the usual definition of a dessin d'enfant, though it is very similar in spirit.

Definition 5.1 (Belyi functions and dessins) A *Belyi function* is a holomorphic map $\psi\colon S\to\mathbb{CP}^1$ from a compact Riemann surface S to the complex projective line $\mathbb{CP}^1$, ramified only over points 0, 1 and ∞. The *dessin* associated to ψ is the 3–partite graph embedded in S obtained as the preimage of the real line $\mathbb{RP}^1\subset\mathbb{CP}^1$ under ψ.

The dessin of ψ can also be seen as the 1–skeleton of the triangulation of S whose open cells are the preimages through ψ of the two open disks into which $\mathbb{RP}^1$ cuts $\mathbb{CP}^1$.

The main result of this section concerns the underlying Riemann surface $\overline{\mathcal{MS}}^{(2)}_{1,1}(2m)$:

Theorem 5.2 (the topological space $\overline{\mathcal{MS}}^{(2)}_{1,1}(2m)$ as a Belyi curve) *Let m be a positive integer. Then there is a holomorphic Belyi map $\psi_{\mathrm{Bel}}\colon\overline{\mathcal{MS}}^{(2)}_{1,1}(2m)\to\mathbb{CP}^1$ of degree m^2 from the Riemann surface underlying $\overline{\mathcal{MS}}^{(2)}_{1,1}(2m)$ with the following properties*:

(i) *The preimage of $\mathbb{CP}^1\setminus\{0,1,\infty\}$ under ψ_{Bel} coincides with the Riemann surface $\mathcal{MS}^{(2)}_{1,1}(2m)$.*

(ii) *The cycle type of ramification of ψ_{Bel} over points $\{0,1,\infty\}$ is $(1,3,\dots,2m-1)$.*

(iii) *The dessin of ψ_{Bel} is composed of tori T such that the triangle $\Delta(T)$ has one integral angle. In particular, the triangulation given by this dessin is the one described in Corollary 4.9.*

Definition 5.3 (Klein group and Klein sphere) The *Klein group* K_4 is the subgroup of diagonal matrices in $\mathrm{SO}(3,\mathbb{R})$. The *Klein sphere* S_{Kl} is the sphere with three conical points (y_1, y_2, y_3) of angles (π,π,π), obtained by taking the quotient of the unit sphere $\mathbb{S}^2$ by the action of $K_4 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. We denote by $S_{\mathrm{Kl}}(\mathbb{R})$ the circle in S_{Kl} which is invariant under the unique antiholomorphic isometric involution of S_{Kl}.

Using the conformal structure on S_{Kl} given by the spherical metric, we can view S_{Kl} as $\mathbb{CP}^1$, where $y_1 = 0$, $y_2 = 1$ and $y_3 = \infty$, and $S_{\mathrm{Kl}}(\mathbb{R})$ as $\mathbb{RP}^1$.

Remark 5.4 (Klein sphere as a doubled triangle) The Klein sphere S_{Kl} can also be obtained by doubling the spherical triangle Δ with angles (π,π,π) across its boundary. This way $\partial\Delta$ corresponds to the circle $S_{\mathrm{Kl}}(\mathbb{R})$ in S_{Kl}. Recall that, in the triangle with three angles π, each vertex is at distance exactly $\frac{1}{2}\pi$ from each point of the opposite side. For this reason, points of $S_{\mathrm{Kl}}(\mathbb{R})$ are exactly the points on S_{Kl} that are at distance $\frac{1}{2}\pi$ from one conical point.

The key result to parametrize spherical tori using a Hurwitz space is the following:

Proposition 5.5 (tori of area $(2m-1)\pi$ cover the Klein sphere) *Let (T,x) be a spherical torus with a conical point of angle $4\pi m$ and with points of order 2 marked by p_1, p_2 and p_3. There exists a unique branched cover map $\varphi_{\mathrm{Kl}}\colon T \to S_{\mathrm{Kl}}$ of degree $4m-2$ which is a local isometry outside of branching points, and such that $\varphi_{\mathrm{Kl}}(p_i) = y_i$. Moreover, $\varphi_{\mathrm{Kl}}(x) \neq y_i$ for $i = 1,2,3$.*

Proof We first construct the map and then count its degree. Recall [2, Proposition 1.5.1] that the image of the monodromy map $\rho\colon \pi_1(T,x) \to \mathrm{SO}(3,\mathbb{R})$ is the Klein group (see also Corollary A.3). Consider the developing map $\iota\colon \widetilde{T} \to \mathbb{S}^2$ from the universal cover $\widetilde{T}$ of T. This map is equivariant with respect to the action of $\pi_1(T,x)$ on $\widetilde{T}$ by deck transformation and on $\mathbb{S}^2$ by the monodromy representation. Hence, by taking the quotient, we get a map $\varphi_{\mathrm{Kl}}\colon T \to S_{\mathrm{Kl}} \cong \mathbb{S}^2/K_4$.

We now prove that the constructed map φ_{Kl} sends points p_i to the three distinct orbifold points of S_{Kl}. This will permit us to label these three points so that $\varphi_{\mathrm{Kl}}(p_i) = y_i$. In order to do this, consider the order-two automorphisms σ of T and denote by S the quotient T/σ. The surface S is a sphere with three conical points of angle π that are the images of the points p_i, and one conical point of angle $2\pi m$. Let us take a lift $\tilde{x} \in \widetilde{T}$ of x and let $\tilde{\sigma}$ be the lift of σ to $\widetilde{T}$ that fixes $\tilde{x}$. Since the conical angle at x is an even multiple of 2π, the maps ι and $\iota \circ \tilde{\sigma}$ coincide in a neighborhood of $\tilde{x}$. It follows that ι is $\tilde{\sigma}$–invariant and the map φ_{Kl} descends to a map $\varphi'_{\mathrm{Kl}}\colon S \to S_{\mathrm{Kl}}$. Now, by construction, the map φ'_{Kl} is a local isometry outside of ramification points. This implies that all three conical points of angle π on S are sent by φ'_{Kl} to conical points of angle π on S_{Kl}. Finally, to see that the images of the three conical points are distinct, we use the fact that the monodromy of S is generated by three loops winding simply around these points, and that it is isomorphic to K_4. Hence, we proved that points $\varphi_{\mathrm{Kl}}(p_i)$ in S_{Kl} are the three distinct conical points of S_{Kl}, and so we can label each $\varphi_{\mathrm{Kl}}(p_i)$ by y_i. This finishes the construction of the map. Its uniqueness is clear.

To prove that $\deg(\varphi_{\mathrm{Kl}}) = 4m-2$, we use the fact that φ_{Kl} is a local isometry outside of branching points, so $\deg(\varphi_{\mathrm{Kl}}) = \mathrm{Area}(T)/\mathrm{Area}(S_{\mathrm{Kl}}) = 2\pi(2m-1)/\pi = 4m-2$. Finally, if φ_{Kl} mapped x to some y_i, its local degree at x would be $(4m\pi)/\pi = 4m > \deg(\varphi_{\mathrm{Kl}})$. This contradiction proves the last claim. □

Corollary 5.6 (moduli space of 2–marked tori as a Hurwitz space) *As a differentiable orbifold, the moduli space $\mathcal{MS}_{1,1}^{(2)}(2m)$ is isomorphic to the Hurwitz space $\mathcal{H}_m$ of connected degree $4m-2$ covers, ramified over points 0, 1 and ∞ with cyclic type $(2,\dots,2)$, and over $\lambda \neq 0,1,\infty$ with cyclic type $(1,\dots,1,2m)$.*

Proof To construct $\mathcal{MS}_{1,1}^{(2)}(2m) \to \mathcal{H}_m$ we use Proposition 5.5, which associates to each spherical torus (T,x) a 2–marking of the branched cover $\varphi_{\mathrm{Kl}}\colon T \to S_{\mathrm{Kl}}$. Using the conformal structure on S_{Kl} given by the spherical metric, we view it as $\mathbb{CP}^1$, where $y_1 = 0$, $y_2 = 1$ and $y_3 = \infty$. By Proposition 5.5, we know that $\lambda = \varphi_{\mathrm{Kl}}(x) \neq 0,1,\infty$. To find the cyclic type of ramification over points $(0,1,\infty,\lambda)$, we recall that the map φ_{Kl} is a local isometry outside of the branching locus, and so for each preimage of the points 0, 1 and ∞ the map has branching of order 2. Finally, there is only one conical point in the preimage of λ, hence the cyclic type over λ is $(1,\dots,1,2m)$.

To define the inverse map $\mathcal{H}_m \to \mathcal{MS}_{1,1}^{(2)}(2m)$, for each ramified cover $T \to \mathbb{CP}^1 \cong S_{\mathrm{Kl}}$ with the prescribed cyclic type we pull back the spherical metric of S_{Kl} to T. By Proposition 5.5, the 2–torsion points of T are mapped to y_1, y_2 and y_3, and we call p_i the unique 2–torsion point of T that is sent to y_i. □

In view of Corollary 5.6, we can give $\mathcal{MS}_{1,1}^{(2)}(2m)$ the unique structure of a complex-analytic orbifold that makes the isomorphism $\mathcal{MS}_{1,1}^{(2)}(2m) \cong \mathcal{H}_m$ complex-analytic.

Now, there are two interesting holomorphic maps. The first map $F\colon \mathcal{H}_w \to \mathcal{M}_{1,1}$ sends a cover $(T,x) \to (\mathbb{CP}^1, \lambda)$ to the isomorphism class of (T,x), and so it has finite fibers. Since F can be interpreted as the forgetful map $\mathcal{MS}^{(2)}_{1,1}(2m) \to \mathcal{M}_{1,1}$, which is proper and surjective (see [23]), the map F is a finite (possibly branched) holomorphic cover. The second map $\psi_{\mathrm{Bel}}\colon \mathcal{H}_w \to \mathbb{CP}^1 \setminus \{0,1,\infty\}$ sends a $(4m-2)$–cover branched over 0, 1, ∞ and λ with cyclic types (2^{2m-1}), (2^{2m-1}), (2^{2m-1}) and $(2m, 1^{2m-2})$ to λ. Since the cyclic types are fixed, ψ_{Bel} is a finite unramified cover. In view of the complex isomorphism between $\mathcal{H}_w$ and $\mathcal{MS}^{(2)}_{1,1}(2m)$, we have proven:

Corollary 5.7 ($\mathcal{MS}^{(2)}_{1,1}(2m)$ covers the 3–punctured sphere) *The map*

$$\psi_{\mathrm{Bel}}\colon \mathcal{MS}^{(2)}_{1,1}(2m) \to \mathbb{CP}^1 \setminus \{0,1,\infty\}$$

is a finite unramified holomorphic cover.

We need one last lemma:

Lemma 5.8 (dessin of ψ_{Bel}) *A torus T in the topological space $\mathcal{MS}^{(2)}_{1,1}(2m)$ belongs to the dessin of ψ_{Bel} if and only if the balanced triangle $\Delta(T)$ has one integral angle.*

Proof Let us prove the "if" direction. Suppose that Δ has an integral angle. Then it has one side of length π. This means that, for some i, the distance on T from x to p_i is $\frac{1}{2}\pi$. This means that the distance on S_{Kl} between y_i and $\varphi_{\mathrm{Kl}}(x)$ is $\frac{1}{2}\pi$. Using Remark 5.4, we deduce that $\varphi_{\mathrm{Kl}}(x)$ belongs to $S_{\mathrm{Kl}}(\mathbb{R})$. By the definition of the dessin of ψ_{Bel}, we see that T belongs to the dessin.

Let us now prove the "only if" direction. Suppose that $\varphi_{\mathrm{Kl}}(x)$ belongs to $S_{\mathrm{Kl}}(\mathbb{R})$. For example, assume $\varphi_{\mathrm{Kl}}(x) \in y_1y_2$. Let γ_3 be the geodesic loop on T based at x whose midpoint is p_3. Since half of this geodesic is projected by φ_{Kl} to the segment that joins y_3 with the segment y_1y_2, we see that $|\gamma_3| = \pi$. From Lemma 3.6, it follows that the angle of Δ opposite to γ_3 is integral. □

Proof of Theorem 5.2 (i) The ramified cover is the extension of the cover constructed in Corollary 5.7 to the compactified spaces.

(ii) Recall $\mathcal{MS}^{(2)}_{1,1}(2m)$ is glued from two copies of $\mathcal{MT}_{\mathrm{bal}}(2m)$ and that $\mathcal{MT}_{\mathrm{bal}}(2m)$ is obtained by gluing m^2 polygons $\widehat{P}_l$ as in Figure 10 (see also the case $\vartheta = 6$ in Figure 9). Let us call each connected component of $\mathbb{CP}^1 \setminus \mathbb{RP}^1$ a "hemisphere" and the intersection of a neighborhood of p with a closed hemisphere a "half-neighborhood" of a point $p \in \mathbb{RP}^1$. Recall now, from Construction 3.19, that the ends of $\mathcal{MS}^{(2)}_{1,1}(2m)$ are described by the strips $\mathcal{S}_{i,a}(2m)$ with $i = 1,2,3$ and $0 \le a \le m-1$. It is easy to see that the "length" of the strip $\mathcal{S}_{i,a}(2m)$, namely the number of regions $\mathcal{S}^l_{i,a}(2m)$ such a strip is made of, is exactly $2(2a+1)$.

By Lemma 5.8, the finite unramified cover ψ_{Bel} maps the interior of each $\widehat{P}_l$ onto a hemisphere, and the three nodal edges of $\widehat{P}_l$ are mapped to $\mathbb{RP}^1 \setminus \{0,1,\infty\}$. It follows that, up to labeling the coordinates, ψ_{Bel} maps each region $\mathcal{S}^l_{1,a}(2m)$ to a half-neighborhood of 0. Hence, ψ_{Bel} maps a strip $\mathcal{S}_{1,a}(2m)$ of length $2(2a+1)$ onto a (punctured) neighborhood of 0 with degree $2a+1$. It follows that the cycle type ramification of ψ_{Bel} over 0 is $(1,3,5,\dots,2m-1)$. Analogous considerations hold for the cycle type ramification over 1 and over ∞.

(iii) This is proven in Lemma 5.8. □

Proof of Theorem F To prove this result we will realize $\mathcal{MS}_{1,1}(2m)$ as an unramified orbifold cover of the modular curve $\mathbb{H}^2/\mathrm{SL}(2,\mathbb{Z})$. Recall that in Theorem 5.2 we constructed the unramified covering map ψ_{Bel} of degree m^2 from the topological space $\mathcal{MS}^{(2)}_{1,1}(2m)$ to $\mathbb{CP}^1 \setminus \{0,1,\infty\}$. Note that the quotient of $\mathbb{CP}^1 \setminus \{0,1,\infty\}$ by the trivial $\mathbb{Z}_2$–action is an orbifold isomorphic to $\mathbb{H}^2/\Gamma(2)$, where

$$\Gamma(2) = \{A \in \mathrm{SL}(2,\mathbb{Z}) \mid A \equiv I \pmod 2\}.$$

So the above cover can be promoted to an unramified cover of orbifolds $\mathcal{MS}^{(2)}_{1,1}(2m) \to \mathbb{H}^2/\Gamma(2)$ of the same degree.

The symmetric group S_3 acts on $\mathcal{MS}^{(2)}_{1,1}(2m)$ by relabeling the 2–torsion points of the tori, and it acts on $\mathbb{H}^2/\Gamma(2)$ through the isomorphism $\mathrm{S}_3 \cong \mathrm{SL}(2,\mathbb{Z})/\Gamma(2)$.

Since $\mathcal{MS}^{(2)}_{1,1}(2m)/\mathrm{S}_3 = \mathcal{MS}_{1,1}(2m)$ as orbifolds, the covering map then descends to an unramified orbifold covering $\mathcal{MS}_{1,1}(2m) \to \mathbb{H}^2/\mathrm{SL}(2,\mathbb{Z})$ of degree m^2. Note that the cycle type ramification of this cover at infinity is $(1,3,\dots,2m-1)$ by Theorem 5.2(ii). It follows that, for $m>1$, such a cover is not Galois and so G_m is not a normal subgroup.

The last claim follows from Theorem 5.2(iii), noting that the real locus $\mathbb{RP}^1 \setminus \{0,1,\infty\}$ inside $\mathbb{CP}^1 \setminus \{0,1,\infty\} \cong \mathbb{H}^2/\Gamma(2)$ descends to $\{[it] \mid t \geq 1\}$ inside $\mathbb{H}^2/\mathrm{SL}(2,\mathbb{Z})$. □

6 Lipschitz topology on $\mathcal{MS}_{g,n}$

In this section we define a natural topology on the set of spherical surfaces with conical singularities and establish some of its basic properties. We choose the approach using Lipschitz distance, described, for example in [15, Example on page 71].

We first recall the definition of the Lipschitz distance between two marked metric spaces:

Definition 6.1 Let $(X, x_1, \dots, x_n; d_X)$ and $(Y, y_1, \dots, y_n; d_Y)$ be two metric spaces with distinct marked points x_i and y_i. The *Lipschitz distance* between them is defined by

$$d_{\mathcal{L}}((X, \boldsymbol{x}), (Y, \boldsymbol{y})) = \inf_f \log \max\{\mathrm{dil}(f), \mathrm{dil}(f^{-1})\},$$

where

$$\mathrm{dil}(f) = \sup_{p_1 \neq p_2 \in X} \frac{d_Y(f(p_1), f(p_2))}{d_X(p_1, p_2)}$$

and the infimum runs over bi-Lipschitz homeomorphisms between X and Y that send each x_i to y_i. The value $\max\{\mathrm{dil}(f), \mathrm{dil}(f^{-1})\}$ is called the *bi-Lipschitz constant* of the map f.

Furthermore, we say that a map $f : X \to Y$ is a *bi-Lipschitz embedding* with constant $c \geq 1$ if, for any two points x_1, x_2, we have

$$c^{-1} \cdot d_Y(f(x_1), f(x_2)) \leq d_X(x_1, x_2) \leq c \cdot d_Y(f(x_1), f(x_2)).$$

We will denote by $\mathcal{MS}_{g,n}$ the space of genus-g surfaces with n marked conical points up to a marked isometry. By $\mathcal{MS}_{g,n}(\leq A)$ we denote the subspace of surfaces with area bounded by $A > 0$. To state the main two results of this section we recall the notion of the systole of a spherical surface:

Definition 6.2 (systole) The *systole* $\mathrm{sys}(S)$ of a spherical surface S is the half length of the shortest geodesic segment or geodesic loop on S whose endpoints are conical points of S.

The *systole* $\mathrm{sys}(P)$ of a spherical polygon P is the minimum of half-distances between all vertices of P and the distances between a vertex of P with the unions of edges not adjacent to the vertex. Such a systole is clearly equal to the systole of the sphere obtained by doubling P along its boundary.

Let $\mathcal{MS}^{\geq s}_{g,n}(\leq A)$ be the subspace of $\mathcal{MS}_{g,n}(\leq A)$ of surfaces with systole at least s.

Theorem 6.3 *$\mathcal{MS}_{g,n}$ is a complete metric space with respect to Lipschitz distance. The function $\mathrm{sys}(S)^{-1}$ is proper on $\mathcal{MS}_{g,n}(\leq A)$ in the Lipschitz topology.*

Let us denote by $\mathcal{MP}_n$ the space of all spherical polygons with n cyclically labeled vertices up to isometries that preserve the labeling. We have the following similar result:

Corollary 6.4 *The space $\mathcal{MP}_n$ of spherical polygons with n vertices is complete with respect to Lipschitz distance. For any positive $A > 0$, the function $\mathrm{sys}^{-1}(P)$ is proper on the subset $\mathcal{MP}_n$ of polygons with area at most A.*

To prove Theorem 6.3, we show that surfaces from $\mathcal{MS}^{\geq s}_{g,n}(\leq A)$ admit triangulations into a finite number of *relatively large* triangles. This is done in Theorem 6.23, which itself relies on Delaunay triangulations, constructed in Proposition 6.15. The proof of Corollary 6.4 is similar.

As an application of Theorem 6.3 and Corollary 6.4, we get a result on the topology of the space $\mathcal{MS}^{(2)}_{1,1}(\vartheta)$ of 2–marked tori induced by Lipschitz metric $\mathcal{L}$:

Theorem 6.5 (i) *Suppose ϑ is not odd. Then $T^{(2)}\colon \mathcal{MT}^{\pm}_{\mathrm{bal}}(\vartheta) \to (\mathcal{MS}^{(2)}_{1,1}(\vartheta), \mathcal{L})$ is a homeomorphism of surfaces.*

(ii) *Let m be a positive integer. Then $T^{(2)}\colon \mathcal{MT}^{\pm}_{\mathrm{bal}}(2m+1) \to (\mathcal{MS}^{(2)}_{1,1}(2m+1)^{\sigma}, \mathcal{L})$ is a homeomorphism of surfaces.*

Recall that the bijective map $T^{(2)}$ was defined in Construction 4.5, whereas the Lipschitz distance between two 2–marked tori is measured among maps that preserve 2–marking.

6.1 Lipschitz metric and its basic properties

Here we collect basic results concerning the Lipschitz metric, with an emphasis on spherical surfaces.

Lemma 6.6 *Lipschitz distance defines a metric on the space $\mathcal{MS}_{g,n}$ of spherical surfaces of genus g with n conical points.*

Proof Let $(S, \boldsymbol{x}, h)$ and $(S', \boldsymbol{x}', h')$ be genus-g spherical surfaces with n conical points. Let's show that $d_{\mathcal{L}}(S, S') < \infty$, ie that there is a bi-Lipschitz map $\varphi\colon (S, \boldsymbol{x}) \to (S', \boldsymbol{x}')$. By definition, every point x_i has a contractible neighborhood U_i with polar coordinates (r_i, ϕ_i) on which $h = dr_i^2 + \vartheta_i^2 r_i^2 d\phi_i^2$, and similarly for the points x_i'. Pick a small $\varepsilon > 0$ such that the subsets $U_i(\varepsilon) = \{r_i \leq \varepsilon\} \subset U_i$ and $U_i'(\varepsilon) = \{r_i' \leq \varepsilon\} \subset U_i'$ are compact. Define a map $\varphi_i : U_i(\varepsilon) \to U_i'(\varepsilon)$ such that it is the identity in polar coordinates. Manifestly, φ_i has bi-Lipschitz constant $\max\{\vartheta_i'/\vartheta_i,\ \vartheta_i/\vartheta_i'\}$, and it is a diffeomorphism away from x_i. Moreover, it can be extended to a homeomorphism $\varphi\colon (S, \boldsymbol{x}) \to (S', \boldsymbol{x}')$ that is a diffeomorphism from $\dot{S}$ to $\dot{S}'$. Such a map is clearly bi-Lipschitz.

Note that $d_{\mathcal{L}}(S, S') = 0$ if and only if S and S' are isometric by [1, Theorem 7.2.4].

All the other properties of the metric are obvious. □

Definition 6.7 The *Lipschitz topology* on the moduli space $\mathcal{MS}_{g,n}$ of spherical surfaces is the topology induced by the Lipschitz metric.

The next lemma explains how differences in the values of conical angles of two surfaces affects the Lipschitz distance between them.

Lemma 6.8 (continuity of angle functions) *Let U and U' be neighborhoods of conical points x and x' with conical angles ϑ and ϑ'. Suppose $f\colon U \to U'$ is a bi-Lipschitz homeomorphism. Then*

$$\max\{\mathrm{dil}(f), \mathrm{dil}(f^{-1})\} \geq \max\left(\frac{\vartheta}{\vartheta'}, \frac{\vartheta'}{\vartheta}\right)^{1/2}. \tag{5}$$

In particular, functions $\vartheta_i\colon \mathcal{MS}_{g,n} \to \mathbb{R}_+$ are continuous for the Lipschitz topology.

Proof After scaling by a large constant and passing to the limit, we can assume that the metrics on U and U' are flat; moreover both U and U' are flat cones with conical angles $2\pi\vartheta$ and $2\pi\vartheta'$, respectively. Note that as a result, the limit quantity $\max\{\mathrm{dil}(f), \mathrm{dil}(f^{-1})\}$ can only decrease. Replacing f by f^{-1} if necessary, we can assume that $\vartheta \leq \vartheta'$.

Let us now reason by contradiction. Assume that (5) is not satisfied. Consider the radius-1 circle S^1 centered at x on U. Since $\mathrm{dil}(f^{-1}) < (\vartheta'/\vartheta)^{1/2}$, the image $f(S^1)$ lies at distance c from x', where $c > (\vartheta/\vartheta')^{1/2}$. Hence, $l(f(S^1)) \geq 2\pi c\vartheta'$. At the same time,

$$\mathrm{dil}(f) \geq \frac{l(f(S^1))}{l(S^1)} = \frac{l(f(S^1))}{2\pi\vartheta} \geq \frac{2\pi c\vartheta'}{2\pi\vartheta} > \left(\frac{\vartheta'}{\vartheta}\right)^{1/2}.$$

This contradicts our assumption. □

Lemma 6.9 (continuity of systole function) *Let (S,h) and (S',h') be spherical surfaces from $\mathcal{MS}_{g,n}$ such that $d_{\mathcal{L}}(S,S') \leq d$. Then*

$$e^{-d}\,\mathrm{sys}(S,h) \leq \mathrm{sys}(S',h') \leq e^{d}\,\mathrm{sys}(S,h).$$

In particular, the function $\mathrm{sys}\colon \mathcal{MS}_{g,n} \to \mathbb{R}_+$ is continuous for the Lipschitz topology.

Proof Let S be a spherical surface with conical points $x_1, \dots, x_n$. According to [23], $\mathrm{sys}(S)$ is equal to the minimum of half-distances between conical points and half-lengths of all (rectifiable) simple loops based at some conical point x_i contained in $\dot{S} \cup x_i$ and noncontractible in $\dot{S} \cup x_i$. Any bi-Lipschitz homeomorphism f from S to S' that sends conical points x_i of S to the corresponding points x_i' of S' also sends rectifiable loops based at x_i to rectifiable loops based at x_i'. By definition, for any

$\varepsilon > 0$, there exists a homeomorphism $f_\varepsilon \colon S \to S'$ with bi-Lipschitz constant $e^{d+\varepsilon}$. This clearly explains the above inequalities. □

6.2 Injectivity radius

Here we prove Proposition 6.11, which gives an estimate on the injectivity radius of points on spherical surfaces in terms of the value of the Voronoi function and the systole of the surface.

Definition 6.10 Let S be a spherical surface and $y \in \dot{S}$ be a nonconical point. The *injectivity radius* $\mathrm{inj}(y)$ is the supremum of r such that S contains an isometric copy of a spherical disk of radius r embedded in S and centered at y.

For a conical point $x_i \in S$, the injectivity radius is defined to be the minimum of all distances from x_i to other conical points and half lengths of geodesic loops based at x_i.

Proposition 6.11 *Let S be a spherical surface with conical angles $2\pi(\vartheta_1, \dots, \vartheta_n)$. Then, for any $y \in \dot{S}$,*

$$\mathrm{inj}(y) \geq \min(\mathrm{sys}(S), \mathcal{V}_S(y), \min_i \vartheta_i \mathcal{V}_S(y)). \tag{6}$$

Moreover:

- (i) *If $\mathrm{inj}(y) < \mathcal{V}_S(y)$, then there exists a closed geodesic loop $\gamma \subset \dot{S}$ of length $2\,\mathrm{inj}(y)$ based at y. Also, $l(\gamma) = 2\,\mathrm{inj}(y) < \pi$.*
- (ii) *If $\mathcal{V}_S(y) > \frac{1}{2}\pi$ then $\mathrm{inj}(y) = \mathcal{V}_S(y)$.*
- (iii) *If $\mathrm{inj}(y) < \mathcal{V}_S(y)$ and so $\mathcal{V}_S(y) \leq \frac{1}{2}\pi$, then at least one of the following holds:*
  - (a) $\mathrm{inj}(y) > \mathrm{sys}(S)$.
  - (b) *There exists i such that $\vartheta_i < \frac{1}{2}$ and $\mathrm{inj}(y) > \min_i \vartheta_i \mathcal{V}_S(y)$.*

We will need one lemma to prove this result:

Lemma 6.12 *Let D be a spherical disk with one conical point x in its interior. Suppose that the boundary γ of D satisfies $\ell(\gamma) < 2\pi$ and γ is a geodesic loop with a unique nonsmooth point y. Then there is an orientation-reversing isometric involution τ on D.*

Proof Note first that the angle at x is not an integer, otherwise the univalent developing map from D to $\mathbb{S}^2$ would send γ onto a great circle. Consider the sphere S obtained from D by doubling along γ, and denote by τ_γ the corresponding isometric involution. Since not all conical angles of S are integers, there exists a unique anticonformal isometry τ of S fixing its conical points. Clearly τ commutes with τ_γ, and so τ leaves $\gamma \subset S$ invariant. Hence τ induces the desired involution on $D \subset S$. □

Proof of Proposition 6.11 Since clearly $\mathrm{inj}(y) \le \mathcal{V}_S(y)$, (6) immediately follows from (iii), so we only need to prove (i)–(iii).

(i) Since $\mathrm{inj}(y) < \mathcal{V}_S(y)$, the existence of a geodesic loop of length $2\,\mathrm{inj}(y)$, based at y is straightforward. Indeed, the midpoint of such a loop is a point at distance $\mathrm{inj}(y)$ from y, where the disk centered at y of radius $\mathrm{inj}(y)$ touches itself. One can check that $l(\gamma) \le \pi$, since otherwise there would be points close to the midpoint of γ that could be joined with y by two distinct geodesic segments of length less than $\mathrm{inj}(y)$. To see that $\mathrm{inj}(y) < \frac{1}{2}\pi$ we note that, in case $\mathrm{inj}(y) = \frac{1}{2}\pi$, the boundary of the open disk centered at y of radius $\frac{1}{2}\pi$ is a closed geodesic to which the disk is adjacent twice. This means that S is a standard $\mathbb{RP}^2$, which is impossible since S is orientable.

(ii) Assume $\mathcal{V}_S(y) > \frac{1}{2}\pi$ and suppose, for contradiction, that $\mathrm{inj}(y) < \mathcal{V}_S(y)$. Let γ be a geodesic constructed in (i). Let $2\pi\theta$ and $2\pi(1-\theta)$ be the angles into which γ cuts the neighborhood of y, and assume, without loss of generality, that $\theta \le \frac{1}{2}$.

Take a point $O \in \mathbb{S}^2$ and consider a spherical kite OP_1QP_2 in $\mathbb{S}^2$ with $\angle O = 2\pi\theta$, $\angle P_1 = \angle P_2 = \frac{1}{2}\pi$ and $l([OP_1]) = l([OP_2]) = \frac{1}{2}l(\gamma)$. Since $\theta \le \frac{1}{2}$ and $l([OP_1]) \le \frac{1}{2}\pi$, one can check that $l([OQ]) \le \frac{1}{2}\pi$. In particular, the kite lies in the interior of a disk $\mathbb{D}_r$ centered at O for any $r \in \left(\frac{1}{2}\pi, \mathcal{V}_S(y)\right)$. Since $\mathcal{V}_S(y) > r$, there exists a locally isometric immersion $\iota\colon \mathbb{D}_r \to \dot{S}$ such that $\iota(O) = y$. By precomposing ι with a rotation, we can arrange so that ι sends the sides OP_1 and OP_2 to γ, and $\iota(P_1) = \iota(P_2)$ is the midpoint of γ. It is clear then that the segments P_1Q and P_2Q are sent by ι to the same geodesic segment in $\dot{S}$. It follows that ι is not a locally isometric immersion in any neighborhood of Q. This is a contraction.

(iii) Since $\mathrm{inj}(y) < \mathcal{V}_S(y)$, by (i) there is a simple geodesic loop γ on $\dot{S}$ based at y of length $2\,\mathrm{inj}(y) < \pi$. We will consider separately two possibilities, depending on whether γ is *essential* (it doesn't bound on $\dot{S}$ a disk with at most one puncture) on $\dot{S}$.

If γ is essential on $\dot{S}$, it follows from [23] that $\mathrm{inj}(y) = \frac{1}{2}l(\gamma) > \mathrm{sys}(S)$, and so we are in case (a).

Let's assume now that γ is nonessential on $\dot{S}$. Then γ encircles on S a disk D with at most one conical point in its interior. Since $l(\gamma) < \pi$ by (i), the disk D should contain exactly one conical point, which we denote by x_i. Denote by $2\pi\theta$ the angle that γ forms at y in D.

Suppose first that $\theta \ge \frac{1}{2}$. In this case γ forms a convex boundary of the surface $S \setminus D$. Thanks to this, using exactly the same method as in [23, Corollary 3.11], one proves that $l(\gamma) > 2\,\mathrm{sys}(S)$, and we are in case (a).

Suppose now $\theta < \frac{1}{2}$. Since $\ell(\gamma) < \pi$, we can apply Lemma 6.12 to D to get its isometric involution τ. This involution fixes the midpoint p of γ, and fixes two geodesic segments yx_i and px_i that cut D into two isometric right-angled spherical triangles. Let yp be one of two halves of γ. The segments yx_i, px_i and yp border a triangle $x_i yp$ in D with $\angle x_i = \pi\vartheta_i$, $\angle y = \frac{1}{2}\pi\theta$ and $\angle p = \frac{1}{2}\pi$. Since the side yp of the triangle is shorter than π and two adjacent angles are less than π, the triangle is convex. Since $|yx_i| > |yp|$, we have $\theta_i < \frac{1}{2}$. Applying the sine rule to the triangle $x_i yp$ we get $\sin(|yp|) = \sin(\pi\vartheta_i)\sin(|x_i y|)$. Hence

$$\operatorname{inj}(y) = |yp| > \sin(\pi\vartheta_i)\sin(|x_i y|) > 2\vartheta_i \sin(\mathcal{V}_S(y)) > \frac{4}{\pi}\vartheta_i \mathcal{V}_S(y),$$

which proves that we are in case (b). □

6.3 Equivalence of Lipschitz and analytic topologies on $\mathcal{MT}$

In this section we prove that Lipschitz distance between triangles induces the same topology on $\mathcal{MT}$ as the topology induced by the embedding in $\mathbb{R}^6$ described in Theorem 3.12.

Definition 6.13 The *relative Lipschitz distance* $d_{\overline{\mathcal{L}}}$ (or $\overline{\mathcal{L}}$–distance) between two spherical triangles is the infimum of $\log\max(\operatorname{dil}(f), \operatorname{dil}(f^{-1}))$ over all the marked bi-Lipschitz homeomorphisms $f\colon \Delta_1 \to \Delta_2$ that restrict to a homothety on each edge of Δ_1.

The $\overline{\mathcal{L}}$–distance defines a metric on the space $\mathcal{MT}$ of spherical triangles, which we call the $\overline{\mathcal{L}}$–metric. We have the following natural statement.

Proposition 6.14 *The topologies defined on $\mathcal{MT}$ by the $\mathcal{L}$– and $\overline{\mathcal{L}}$–metrics coincide with the analytic topology given by the angle–side length embedding $\Psi\colon \mathcal{MT} \to \mathbb{R}^6$.*

Proof Note that the side lengths of Δ are clearly continuous functions in both the $\overline{\mathcal{L}}$ and $\mathcal{L}$ topologies. The angles of Δ are continuous in these topologies thanks to Lemma 6.8, applied to the double of Δ. Furthermore the $\overline{\mathcal{L}}$–distance is greater than or equal to the $\mathcal{L}$–distance. Hence, the $\overline{\mathcal{L}}$–topology is finer than the $\mathcal{L}$–topology, which is finer than the analytic topology. For this reason, we only need to show that, for any spherical triangle Δ and a sequence of triangles Δ_i converging to Δ in $\mathbb{R}^6$ (ie in the analytic topology), we have $\lim d_{\overline{\mathcal{L}}}(\Delta_i, \Delta) = 0$. This claim can be proven by exhibiting explicit bi-Lipschitz maps between spherical triangles. We will only treat the case when Δ is short-sided, since this is the only case needed for our purposes.

Following [12, Lemma 4.1], denote by U the open subset of $\mathcal{MT}$ consisting of triangles with angles $\pi\vartheta_i$, where $\vartheta_i < 2$. This subset consists of spherical triangles that admit an isometric embedding into $\mathbb{S}^2$. In particular, U lies in $\mathcal{MT}_{\mathrm{sh}}$, the space of all short-sided triangles. We first prove that the $\overline{\mathcal{L}}$–topology coincides with the analytic topology on U.

For two spherical triangles $\Delta = x_1x_2x_3$ and $\Delta' = x_1'x_2'x_3'$ embedded into $\mathbb{S}^2$ with incenters I_Δ and $I_{\Delta'}$, respectively, define the *incentric* map $\Phi\colon \Delta \to \Delta'$ as the unique map satisfying:

- $\Phi(x_i) = x_i'$, $\Phi(I_\Delta) = \Phi(I_{\Delta'})$.
- Φ is a homothety on each edge x_ix_j.
- For any point $p \in \partial\Delta$, Φ sends the geodesic segment $pI_{\Delta'}$ to a geodesic segment and restricts to a homothety on it.

Suppose now we have a sequence of embedded triangles $\Delta_i \in U$ whose angles and side lengths converge to those of $\Delta \in U$. Then it is not hard to see that the bi-Lipschitz constant of the incentric maps $\Phi\colon \Delta_i \to \Delta$ tends to 1. Hence Δ_i converges to Δ in the $\overline{\mathcal{L}}$–topology as well. This proves the statement for U.

Let us denote by $U_{klm} \subset \mathcal{MT}_{\mathrm{sh}}$ the subspace of triangles which can be obtained from an embedded triangle Δ by repeated gluing of $k-1$, $l-1$ and $m-1$ hemispheres to the sides x_1x_2, x_2x_3 and x_3x_1, respectively, of Δ. From [12, Theorem 4.7 and Lemma 5.2] it follows that the sets U_{klm} give an open cover of $\mathcal{MT}_{\mathrm{sh}}$. At the same time, the incentric map Φ between any two triangles Δ and Δ' from U can be naturally extended to a map $\tilde{\Phi}\colon \tilde{\Delta} \to \tilde{\Delta}'$ between triangles with attached hemispheres. Namely, a radius of each hemisphere is sent isometrically to a radius and the restriction of $\tilde{\Phi}$ to both sides of each hemisphere are homotheties. Since the Lipschitz constants of Φ and $\tilde{\Phi}$ clearly coincide, the statement about the topologies is proven for each U_{klm}, and so for the whole space $\mathcal{MT}_{\mathrm{sh}}$. □

6.4 Delaunay triangulations

We now turn to triangulations of spherical surfaces into convex spherical triangles. We will not require the triangulation to induce the structure of a simplicial complex on the surface. In particular, a triangle can be adjacent to a vertex up to three times, and to an edge up to two times.

The first result is a variation of the famous Delaunay triangulations of the plane [8] (see also [24, Section 14] for a modern exposition).

Proposition 6.15 (Delaunay triangulations) *Let S be a spherical surface with conical points $x_1, \dots, x_n$, some of which might have angle 2π. Suppose that the Voronoi function $\mathcal{V}_S$ is bounded by $\frac{1}{2}\pi$. Then there exists a triangulation of S into convex spherical triangles with the following "empty circle" property: for each triangle $x_i x_j x_k$ of the triangulation, there exists a vertex $v \in \Gamma(S)$ at equal distance r from x_i, x_j and x_k such that $d(x_l, v) \ge r$ for all $l \in \{1, \dots, n\}$.*

The proof will follow the proof by Thurston of a similar result [28, Proposition 3.1] concerning triangulations of surfaces with flat metric and conical singularities. We will need the following elementary lemma:

Lemma 6.16 *Let $D, D' \subset \mathbb{S}^2$ be two disks of radius less than $\frac{1}{2}\pi$. Let $x_1, x_2 \in \partial D$ and $x_1', x_2' \in \partial D'$ be four distinct points. Suppose x_1 and x_2 don't lie in the interior of D', and x_1' and x_2' don't lie in the interior of D. Then the geodesic segments $x_1x_2 \subset D$ and $x_1'x_2' \subset D'$ are disjoint in $\mathbb{S}^2$.*

Proof If D and D' are disjoint, there is nothing to prove. Suppose D and D' intersect, and let y_1 and y_2 be the two points of intersection of the boundary circles ∂D and $\partial D'$. Let γ be the unique great circle on $\mathbb{S}^2$ passing through y_1 and y_2. It is now easy to see that the complements $D \setminus D'$ and $D' \setminus D$ lie in different hemispheres of $\mathbb{S}^2$ with respect to γ. It follows that the segments x_1x_2 and $x_1'x_2'$ also lie in different hemispheres, and so they can intersect only in their endpoints. However, the points x_i and x_i' are distinct, so x_1x_2 and $x_1'x_2'$ are disjoint. □

Proof of Proposition 6.15 The proof closely follows the proof of [28, Proposition 3.1]. Let $\Gamma(S)$ be the Voronoi graph of S. Let us first explain how to associate to each edge $e \subset \Gamma(S)$ a *dual* geodesic segment $\check{e}$ with conical endpoints.

Let $p \in \Gamma(S)$ be a point in the interior of an edge $e \subset \Gamma(S)$, and set $r = \mathcal{V}_S(p)$. Then there exists a locally isometric immersion $\iota_p \colon \mathbb{D}_r \to S$, from a radius $r < \frac{1}{2}\pi$ spherical disk, that sends the center of $\mathbb{D}_r$ to p. Exactly two of the boundary points of $\mathbb{D}_r$, say y and z, are sent to two conical points x_i and x_j of S. Denote by $\check{e}$ the image $\iota_p(yz)$. It is easy to see that the segment $\check{e}$ is independent of the choice of $p \in e$.

Let us now deduce from Lemma 6.16 that, for any two edges $e, e' \subset \Gamma_S$, their dual edges $\check{e}$ and $\check{e}'$ do not intersect in their interior points. This is similar the proof of [28, Proposition 3.1]. Let D and D' be the disks immersed in S that correspond to e and e'. Assume, for contradiction, that $\check{e}$ and $\check{e}'$ intersect in their interior point p. Consider the (multivalued) developing map $\iota \colon S \to \mathbb{S}^2$. The images of D and D' under this map are embedded disks, and the images of $\check{e}$ and $\check{e}'$ are chords of these disks, intersecting in $\iota(p)$. This contradicts Lemma 6.16. Indeed, the endpoints of $\check{e}$ are conical points that belong to $\partial D \setminus D'$, and the endpoints of $\check{e}'$ are conical points that belong to $\partial D' \setminus D$. Hence, Lemma 6.16 is applicable to the 4–tuple $\iota(D, \check{e}, D', \check{e}')$.

Next, we associate to each vertex v of $\Gamma(S)$ a convex polygon embedded in S whose edges $\check{e}_1, \dots, \check{e}_k$ are dual to the half-edges of $\Gamma(S)$ adjacent to v. To do so, consider the immersion $\iota_v \colon \mathbb{D}_r \to S$ of a disk of radius $r = \mathcal{V}_S(v)$ that sends the center of $\mathbb{D}_r$ to v. There will be exactly k points, say $y_1, \dots, y_k$, on $\partial\mathbb{D}_r$ whose images in S are conical points. Let P_v be the convex hull of the points y_i in $\mathbb{D}_r$. Then the map ι_v is an embedding on the interior $\mathring{P}_v$ of the polygon P_v; it may identify some vertices and it may identify an edge to at most one other edge of P_v.

Our last observation is that the union of the $\iota_v(\mathring{P}_v)$ over all vertices v of $\Gamma(S)$ coincides with the complement in S of the union of edges $\check{e}$. Indeed, since the edges $\check{e}$ can only intersect at endpoints, each $\iota_v(\mathring{P}_v)$ is a connected component of the complement of edges $\check{e}$. At the same time, each edge $\check{e}$ is adjacent to one or two open polygons $\iota_v(\mathring{P}_v)$ corresponding to the vertices of the edge e dual to $\check{e}$. It follows that polygons $\iota_v(P_v)$ cover the whole S.

Finally, if some of convex polygons $\iota_v(P_v)$ are not triangles, we subdivide them by diagonals into a collection of triangles. This gives the desired triangulation of S, where for each triangle $x_i x_j x_k$, the point v is the corresponding vertex of the Voronoi graph. □

Remark 6.17 Let Δ be a triangle from a Delaunay triangulation with vertices x_i, x_j and x_k, and let v be the corresponding vertex of $\Gamma(S)$. Then the circumscribed radius of Δ is equal to $\mathcal{V}_S(v) = d(v, x_i)$.

6.4.1 Compact subsets of $\mathcal{MS}_{g,n}(\le A)$ In this subsection we prove Proposition 6.22, which singles out a class of compact subsets of $\mathcal{MS}_{g,n}(\le A)$ consisting of surfaces that admit triangulations into triangles of *bounded shapes*.

Definition 6.18 ((l,r)–bounded triangles and surfaces) Fix constants $l \in (0, \pi)$ and $r \in \left(0, \frac{1}{2}\pi\right)$. We say that a convex spherical triangle is (l,r)*–bounded* if all its sides have length at least l and its circumscribed circle has radius at most r. A spherical surface is (l,r)*-bounded* if it admits a triangulation into (l,r)–bounded spherical triangles.

We will denote by $\mathcal{MT}_{l,r}$ the subset of $\mathcal{MT}$ consisting of (l,r)–bounded triangles.

Remark 6.19 (compactness of $\mathcal{MT}_{l,r}$) The set $\mathcal{MT}_{l,r}$ is compact in the analytic topology of $\mathcal{MT}$. Indeed, let $\Delta_i \subset \mathbb{S}^2$ be a sequence of convex triangles from $\mathcal{MT}_{l,r}$

with vertices (x_1^i, x_2^i, x_3^i). Passing to a subsequence, we can assume that the sequences of vertices converge to x_1, x_2 and x_3. We have $|x_i x_j| \geq l$, and the circle on $\mathbb{S}^2$ containing x_1, x_2 and x_3 has radius at most r. Hence $x_1x_2x_3$ is a triangle from $\mathcal{MT}_{l,r}$.

Definition 6.20 (space of (l,r)–triangulated surfaces) Let μ be a combinatorial type of triangulations of a genus-g surface with n marked points such that the marked points are vertices of the triangulation. Denote by $Y_{l,r}^{\mu}(\leq A)$ the set of all spherical surfaces of area at most A with a chosen triangulation of type μ consisting of (l,r)–bounded triangles. The $\overline{\mathcal{L}}$–distance between two triangulated surfaces from $Y_{l,r}^{\mu}(\leq A)$ is the Lipschitz distance with respect to all the maps that send the triangulation to the triangulation and restrict to homotheties on the edges.

We recall that, given a compact surface S of genus g with n marked points $\boldsymbol{x}$, there always exists a triangulation of S whose set of vertices contains $\boldsymbol{x}$ as in Definition 6.20. Indeed, it is possible to pick a point $b \in S$ and $2g$ loops $\{\gamma_j\}$ based at b such that no γ_j passes through $\boldsymbol{x}$ and $S \setminus \bigcup_j \gamma_j$ is a topological disk. This shows that $(S, \boldsymbol{x})$ can be obtained from a $2g$–gon P with n marked points $\boldsymbol{x}'$ in its interior via pairwise identification of its edges. Thus, every triangulation of P whose vertices include $\boldsymbol{x}'$ descends to a triangulation of S whose vertices include $\boldsymbol{x}$. The existence of such a triangulation of P is obvious.

Lemma 6.21 *The set $Y_{l,r}^{\mu}(\leq A)$ is compact in the $\overline{\mathcal{L}}$–metric.*

Proof From Remark 6.19 and Proposition 6.14 it follows that the subset $\mathcal{MT}_{l,r} \subset \mathcal{MT}$ of (l,r)–bounded triangles is compact in the $\overline{\mathcal{L}}$ metric. At the same time, $Y_{l,r}^{\mu}(\leq A)$ can be identified with a closed subset of the set of $(\mathcal{MT}_{l,r})^{|\mu|}$, where $|\mu|$ is the number of triangles in μ. □

Proposition 6.22 ($\mathcal{L}$–compactness of (l,r)–bounded surfaces) *Fix $A > 0$, $l \in (0,\pi)$ and $r \in \left(0, \frac{1}{2}\pi\right)$. Then the subset $X_{l,r}(\leq A)$ of $\mathcal{MS}_{g,n}(\leq A)$ consisting of (l,r)–bounded surfaces is compact in the Lipschitz topology. The analogous statement holds for $\mathcal{MP}_n(\leq A)$.*

Proof Since the area of an (l,r)–bounded triangle is bounded from below, there exists only a finite number of combinatorial triangulations μ of surfaces from $\mathcal{MS}_{g,n}(\leq A)$. Note that, for each μ, the natural map $Y_{l,r}^{\mu}(\leq A) \to \mathcal{MS}_{g,n}(\leq A)$, that forgets the triangulation is continuous since it contracts the metric. Hence $X_{l,r}(\leq A)$ is a finite union of images of compact sets under continuous maps. □

6.5 Properness of the function $\mathrm{sys}(S)^{-1}$ on $\mathcal{MS}_{g,n}(\le A)$

In this section we deduce Theorem 6.3 and Corollary 6.4 from the following result.

Theorem 6.23 (bounded Delaunay triangulations) *For any* $s > 0$:

(i) *Any spherical surface from* $\mathcal{MS}^{\ge s}_{g,n}$ *can be triangulated into* $\left(\frac{1}{2}s, \frac{1}{4}\pi\right)$*–bounded spherical triangles.*

(ii) *Any spherical polygon* P *with* $\mathrm{sys}(P) \ge s$ *can be triangulated into* $\left(f(s), \frac{1}{4}\pi\right)$*–bounded spherical triangles, where* f *is a positive and continuous function.*

Proof of Theorem 6.3 We start with the properness of sys^{-1}. Since $\mathrm{sys}: \mathcal{MS}_{g,n} \to \mathbb{R}_+$ is continuous by Lemma 6.9, the subset $\mathcal{MS}^{\ge s}_{g,n}(\le A)$ is closed inside $\mathcal{MS}_{g,n}(\le A)$. Furthermore, $\mathcal{MS}^{\ge s}_{g,n}(\le A)$ is contained in the subset $X_{s/2,\pi/4}(\le A)$ of $\mathcal{MS}_{g,n}(\le A)$ consisting of $\left(\frac{1}{2}s, \frac{1}{4}\pi\right)$–bounded surfaces by Theorem 6.23(i). Since $X_{s/2,\pi/4}(\le A)$ is compact by Proposition 6.22, it follows that $\mathcal{MS}^{\ge s}_{g,n}(\le A)$ is compact too, and so the restriction of sys^{-1} to $\mathcal{MS}_{g,n}(\le A)$ is proper.

For the completeness of $\mathcal{MS}_{g,n}$, it is enough to show that, for every $r > 0$ and spherical surface S in $\mathcal{MS}_{g,n}$, the closed ball $\bar{B}(S, r) = \{S' \in \mathcal{MS}_{g,n} \mid d_{\mathcal{L}}(S, S') \le r\}$ is compact. By Lemma 6.9, $\bar{B}(S, r)$ is contained in $\mathcal{MS}^{\ge s}_{g,n}(\le A)$ with $s = e^{-r}\,\mathrm{sys}(S)$ and $A = e^{2r}\,\mathrm{Area}(S)$. Since $\mathcal{MS}^{\ge s}_{g,n}(\le A)$ was shown above to be compact and $\bar{B}(S, r)$ is closed, it follows that $\bar{B}(S, r)$ is compact. □

Proof of Corollary 6.4 The proof is identical to the proof of Theorem 6.3, where instead of using Theorem 6.23(i) one applies Theorem 6.23(ii). □

Proof of Theorem 6.23 (i) We will prove that, for any $S \in \mathcal{MS}^{\ge s}_{g,n}$, there exists a collection of regular points $x_{n+1}, \dots, x_{n+m} \in S$, such that the surface $(S, x_1, \dots, x_{n+m})$ has the following three properties:

(a) For any $i \ne j$, $d(x_i, x_j) \ge \frac{1}{2}s$ for all $i \ne j \in \{1, \dots, n+m\}$.

(b) For each i the injectivity radius of x_i on S is at least $\frac{1}{4}s$.

(c) For any $x \in S$ there is a point x_i such that $d(x, x_i) \le \frac{1}{4}\pi$.

Before proving this claim, let us explain why the statement of the theorem follows from it. Indeed, suppose that we have such a collection of points. Then let us consider the Delaunay triangulation of S with respect to points $x_1, \dots, x_{n+m}$ that exists thanks to Proposition 6.15. We claim that all the triangles of the triangulation are $\left(\frac{1}{2}s, \frac{1}{4}\pi\right)$–bounded. Indeed, by condition (c) and Remark 6.17, each such triangle is isometric to a triangle that can be inscribed in a circle of radius at most $\frac{1}{4}\pi$. At the same time, by conditions (a) and (b), all sides of the triangle have length at least $\frac{1}{2}s$.

Let us now show how to find such a collection of points $x_{n+1},\dots,x_{n+m}\in S$. We will add points $x_{n+1},\dots,x_{n+m}$ by induction. Note first that $x_1,\dots,x_n$ satisfy conditions (a) and (b). Suppose that there is a point $x\in S$ at distance more than $\frac{1}{4}\pi$ from $x_1,\dots,x_n$. Let us denote this x by x_{n+1}, and let us show that $(S,x_1,\dots,x_{n+1})$ satisfies conditions (a) and (b) for $m=1$. Note that by [23, Lemma 3.10] we have $\mathrm{sys}(S)\le\frac{1}{2}\pi$, which means $\frac{1}{4}\pi\ge\frac{1}{2}s$, and so when we add x_{n+1} we don't violate (a). It remains to show that the injectivity radius of x_{n+1} of S is at least $\frac{1}{4}s$. We apply (6) from Proposition 6.11 to get

$$\mathrm{inj}(x_{n+1})\ge\min\bigl(s,\tfrac{1}{4}\pi,\min_i\vartheta_i\tfrac{1}{4}\pi\bigr).$$

But, by [23, Lemma 3.13], we know that $\mathrm{sys}(S)\le\min_i\vartheta_i\pi$. So we get $\mathrm{inj}(x_{n+1})\ge\frac{1}{4}s$. Hence, condition (b) is satisfied for $x_1,\dots,x_{n+1}$. In this way we can go on adding points x_{n+i} until condition (c) is satisfied. Indeed, the process must terminate since the $\frac{1}{8}s$–neighborhoods of points x_{n+i} are disjoint disks on S and the area of S is finite.

(ii) To prove the second part of the theorem we work with the double $S(P)$ of P. We construct a collection of regular points $x_{n+1},\dots,x_{n+m}\in S(P)$ such that the surface $(S(P),x_1,\dots,x_{n+m})$ has the following four properties:

- (o) The set of points x_i is invariant under the isometric involution τ of $S(P)$.
- (a) For any $i\ne j$, $d(x_i,x_j)\ge\frac{1}{4}s$ for all $i\ne j\in\{1,\dots,n+m\}$.
- (b) For each i the injectivity radius of x_i on S is at least $\frac{1}{8}s$.
- (c) For any $x\in S$ there is a point x_i such that $d(x,x_i)\le\frac{1}{4}\pi$.

Let us explain how to make the first step. Consider P and ∂P as subsets of $S(P)$. Suppose there is a point $y\in S(P)$ at distance greater than $\frac{1}{4}\pi$ from $x_1,\dots,x_n$. If its distance from ∂P is more than $\frac{1}{8}\pi$, we set $x_{n+1}=y$ and $x_{n+2}=\tau(y)$. In this case conditions (o)–(b) are still satisfied for points $x_1,\dots,x_{n+2}$, since $d(x_{n+1},x_{n+2})\ge\frac{1}{4}\pi$. Suppose now that $d(y,\partial P)<\frac{1}{8}\pi$. Let y' be a point on ∂P closest to y and set $x_{n+1}=y'$. Clearly the distance from x_{n+1} to $x_1,\dots,x_n$ is at least $\frac{1}{8}\pi$. For this reason, as in (i), conditions (b) and (c) are still satisfied. This finishes the first step.

Now, we repeat the above step until we get a collection of points $x_1,\dots,x_{n+m}$ in $S(P)$ that satisfy conditions (o)–(c). As in the proof of Proposition 6.15, we get a canonical decomposition of $S(P)$ into convex spherical polygons, invariant under the action of τ, and such that each polygon has side lengths at least $\frac{1}{4}s$ and can be inscribed in a circle of radius at most $\frac{1}{4}\pi$. Those polygons whose interior doesn't intersect ∂P should be further cut into triangles by diagonals. Suppose that the interior of a polygon Q intersects ∂P. Then $\tau(Q) = Q$, and using a τ–invariant subset of diagonals of Q, one can cut it into a union of triangles exchanged by τ and either a triangle or a trapezoid Q' satisfying $\tau(Q') = Q'$. If Q' is a triangle, we take $Q' \cap P$ as one of the triangles of the triangulation of P. If Q' is a trapezoid, we subdivide further $Q' \cap P$ into two triangles along a diagonal. It is not hard to see that the resulting triangles are $\bigl(f(s), \frac{1}{4}\pi\bigr)$–bounded for some positive function $f(s)$. That concludes the decomposition of P into triangles. □

6.6 Systole of balanced triangles

In this section we calculate the systole of a balanced triangle and show that, for a balanced triangle Δ, we have $\operatorname{sys}(\Delta) = \operatorname{sys}(T(\Delta))$.

Lemma 6.24 *Let Δ be a balanced spherical triangle with vertices x_1, x_2 and x_3. Then*

$$2\operatorname{sys}(\Delta) = \min_{i,j}\bigl(\min(|x_i x_j|, 2\pi - |x_i x_j|)\bigr). \tag{7}$$

Moreover:

(i) *For any vertex x_i of Δ, the distance to the opposite side $x_i x_j$ is larger than* $\operatorname{sys}(\Delta)$.

(ii) *Let $p \in \partial\Delta$ be a point that is not a vertex of Δ. Suppose that η is a geodesic segment in Δ that joins p with x_i and doesn't belong to $\partial\Delta$. Then $l(\eta) > \operatorname{sys}(\Delta)$.*

(iii) *There exists a geodesic segment $\gamma_\Delta \subset \Delta$ of length $2\operatorname{sys}(\Delta)$ that joins two vertices of Δ.*

Proof We will first prove statements (i)–(iii) and then will deduce (7).

(i) Let us show that, for any p in x_2x_3, we have $d(p, x_1) > \operatorname{sys}(\Delta)$. From Remark 2.12 it follows that p lies either in the Voronoi domain of x_2 or of x_3. Assume the former. Then, by definition of Voronoi domains, $d(p, x_1) \geq d(p, x_2)$.

Suppose first that the strict inequality $d(p, x_1) > d(p, x_2)$ holds. Applying the triangle inequality to the points x_1, x_2 and p and using $d(x_1, x_2) \geq 2\operatorname{sys}(\Delta)$, we get

$$d(p, x_1) \geq d(x_1, x_2) - d(p, x_2) > d(x_1, x_2) - d(p, x_1) \geq 2\operatorname{sys}(\Delta) - d(p, x_1).$$

It follows that $d(p, x_1) > \operatorname{sys}(\Delta)$.

Suppose now that $d(p, x_1) = d(p, x_2)$. Then, by Remark 2.12, Δ is semibalanced, p is the midpoint of the segment x_1x_2, and there is a geodesic segment $x_1 p$ that joins x_1 with p. It is clear then that $2|x_1 p| = |x_1 p| + |x_2 p| > 2d(x_1, x_2) \geq 2\operatorname{sys}(\Delta)$.

(ii) Consider two cases. If p lies on the side of Δ opposite to x_i then by (i) we have $\ell(\eta) \ge d(x_i, p) > \operatorname{sys}(\Delta)$. Suppose now p lies on a side adjacent to x_j. In this case η cuts out of Δ a digon with angles less than π (since p is an interior point of an edge). So $e(\eta) = \pi$ and the statement follows from Corollary 2.15.

(iii) Using (i) and Definition 6.2, we see that $2\operatorname{sys}(S) = \min_{i,j} d(x_i, x_j)$. Hence there is a geodesic segment γ_Δ of length $2\operatorname{sys}(S)$ that joins two vertices of Δ.

To prove (7), take the geodesic γ_Δ given by (iii). It cuts out of Δ a digon, one of whose sides is a side $x_i x_j$ of the triangle Δ. If follows that either $2\operatorname{sys}(\Delta) = |x_i x_j|$ or $2\pi - |x_i x_j|$. This shows that $2\operatorname{sys}(\Delta)$ is no smaller than the right-hand expression in (7). The opposite inequality follows immediately from Corollary 2.15. □

Lemma 6.25 *For any balanced triangle Δ and the corresponding spherical torus $(T(\Delta), x)$, we have $\operatorname{sys}(\Delta) = \operatorname{sys}(T(\Delta))$. Conversely, for any spherical torus T and the corresponding balanced spherical triangle $\Delta(T)$, we have $\operatorname{sys}(T) = \operatorname{sys}(\Delta(T))$.*

Proof The first and the second statements are equivalent, so we prove just the first. By Lemma 6.24(iii), there is a geodesic segment γ_Δ in Δ of length $2\operatorname{sys}(\Delta)$ that joins two vertices of Δ. Such a γ_Δ is embedded as a geodesic loop in $T(\Delta)$, which clearly implies $\operatorname{sys}(\Delta) \ge \operatorname{sys}(T(\Delta))$. To get $\operatorname{sys}(\Delta) \le \operatorname{sys}(T(\Delta))$, let $\gamma_{T(\Delta)}$ be the systole geodesic loop in $T(\Delta)$, and let Δ_1 and Δ_2 be two balanced triangles isometric to Δ from which $T(\Delta)$ is glued. It will be enough to prove that $\gamma_{T(\Delta)}$ lies entirely in Δ_1 or Δ_2. Assume, for contradiction, that this is not so. Then $\gamma_{T(\Delta)}$ contains two subsegments η and η' whose interiors lie in the interior of Δ_1 or Δ_2 and which satisfy the conditions of Lemma 6.24(ii). Applying this lemma, we get $l(\gamma_{T(\Delta)}) \ge l(\eta) + l(\eta') > 2\operatorname{sys}(\Delta)$, which contradicts the established inequality $\operatorname{sys}(\Delta) \ge \operatorname{sys}(T(\Delta))$. □

Corollary 6.26 *The function $\operatorname{sys}(\Delta)^{-1} = 2\min_{i,j}\big(\min(|x_i, x_j|, 2\pi - |x_i, x_j|)\big)^{-1}$ is proper on $\mathcal{MT}_{\mathrm{bal}}(\vartheta)$ in the analytic topology.*

Proof The function $\operatorname{sys}(\Delta)^{-1}$ is proper on $\mathcal{MT}(\vartheta)$ in the $\mathcal{L}$–topology by Corollary 6.4. At the same time, by Proposition 6.14, the $\mathcal{L}$–topology and the analytic topology coincide on $\mathcal{MT}_{\mathrm{bal}}(\vartheta)$. □

6.7 Proof of Theorem 6.5

Here we finally prove Theorem 6.5, concerning 2–marked tori. We note first that Theorem 6.3 holds for 2–marked tori as well; namely, the function sys^{-1} is proper in the Lipschitz topology on the space $\mathcal{MS}_{1,1}^{(2)}(\le A)$ of such tori of area at most A.

We will use the following standard lemma, whose proof we omit:

Lemma 6.27 *Let X and Y be locally compact Hausdorff topological spaces and let $\varphi\colon X \to Y$ be a continuous bijective map.*

(i) *If φ is proper then it is a homeomorphism.*

(ii) *Suppose there exist proper functions $s_X\colon X \to \mathbb{R}$ and $s_Y\colon Y \to \mathbb{R}$ such that $s_X = s_Y \circ \varphi$. Then φ is a homeomorphism.*

Proof of Theorem 6.5 (i) By Proposition 3.22(i) $\mathcal{MT}^{\pm}_{\mathrm{bal}}(\vartheta)$ is a surface, so we need to show that $T^{(2)}$ is a homeomorphism. To show we can apply Lemma 6.27, we note:

(a) Every bi-Lipschitz map $\Delta \to \Delta'$ that restricts to a homothety on the edges gives rise to a σ–equivariant bi-Lipschitz map $T^{(2)}(\Delta) \to T^{(2)}(\Delta')$ with the same Lipschitz constant. Hence, the map $T^{(2)}$ is contracting with respect to the $\overline{\mathcal{L}}$–metric on $\mathcal{MT}^{\pm}_{\mathrm{bal}}(\vartheta)$, namely $d_{\mathcal{L}}(T^{(2)}(\Delta), T^{(2)}(\Delta')) \le d_{\overline{\mathcal{L}}}(\Delta, \Delta')$. It follows that $T^{(2)}$ is continuous. Moreover, $T^{(2)}$ is bijective by Lemma 4.6.

(b) Since $\mathcal{MT}^{\pm}_{\mathrm{bal}}(\vartheta)$ is a surface it is locally compact, and the function sys^{-1} is proper on it by Corollary 6.26.

(c) The space $(\mathcal{MS}^{(2)}_{1,1}(\vartheta), \mathcal{L})$ is locally compact, and the function sys^{-1} is proper on it by Theorem 6.3.

(d) The map $T^{(2)}$ preserves the function sys^{-1} by Lemma 6.25.

To sum up, the map $T^{(2)}$ satisfies all the properties of Lemma 6.27, which proves the claim.

(ii) The proof of this claim is the same and so we omit it. □

Remark 6.28 (orbifold structures on $\mathcal{MS}_{1,1}(\vartheta)$ and $\mathcal{MS}^{(2)}_{1,1}(\vartheta)$) Let $\mathcal{MS}^{(4)}_{1,1}(\vartheta)$ be the set of spherical tori T endowed with a 4–marking, namely an isomorphism $H_1(T;\mathbb{Z}_4) \cong (\mathbb{Z}_4)^2$. We endow $\mathcal{MS}^{(4)}_{1,1}(\vartheta)$ with the Lipschitz distance measured among maps between tori that respect the 4–marking. Since 4–marked tori have no nontrivial conformal automorphisms, $\mathcal{MS}^{(4)}_{1,1}(\vartheta)$ is a moduli space for such 4–marked tori. It is easy to see that the forgetful map $\mathcal{MS}^{(4)}_{1,1}(\vartheta) \to \mathcal{MS}^{(2)}_{1,1}(\vartheta)$ is a local isometry, and in fact an unramified Galois cover with group $K/\{\pm 1\}$, where $K = \ker(\mathrm{SL}(2,\mathbb{Z}_4) \to \mathrm{SL}(2,\mathbb{Z}_2))$.

Assume first that ϑ is not odd. The space $\mathcal{MS}^{(2)}_{1,1}(\vartheta)$ is an orientable surfaces of finite type by Theorem 6.5 and Proposition 3.22(i), and so the same holds for $\mathcal{MS}^{(4)}_{1,1}(\vartheta)$. We endow $\mathcal{MS}^{(2)}_{1,1}(\vartheta)$ with the orbifold structure given by $\mathcal{MS}^{(4)}_{1,1}(\vartheta)/K$, and $\mathcal{MS}_{1,1}(\vartheta)$

with the orbifold structure $\mathcal{MS}_{1,1}^{(4)}(\vartheta)/\mathrm{SL}(2,\mathbb{Z}_4)$. As a consequence, $\mathcal{MS}_{1,1}^{(2)}(\vartheta) \to \mathcal{MS}_{1,1}(\vartheta)$ is an unramified Galois cover with group $\mathrm{SL}(2,\mathbb{Z}_2) \cong \mathrm{S}_3$.

Assume that $\vartheta = 2m+1$ is odd. Again, $\mathcal{MS}_{1,1}^{(2)}(2m+1)^\sigma$ is a disjoint union of finitely many 2–dimensional disks by Theorem 6.5 and Proposition 3.25, and so the same holds for the moduli space $\mathcal{MS}_{1,1}^{(4)}(2m+1)^\sigma$. The same argument as in Construction 4.16 shows that $\mathcal{MS}_{1,1}^{(4)}(2m+1)$ fibers over $\mathcal{MS}_{1,1}^{(4)}(2m+1)^\sigma$ with fiber $\mathbb{R}$, and so is a 3–dimensional manifold. We then put on $\mathcal{MS}_{1,1}^{(2)}(2m+1)$ and $\mathcal{MS}_{1,1}(2m+1)$ the orbifold structures induced by $\mathcal{MS}_{1,1}^{(2)}(2m+1) = \mathcal{MS}_{1,1}^{(4)}(2m+1)/K$ and $\mathcal{MS}_{1,1}(2m+1) = \mathcal{MS}_{1,1}^{(4)}(2m+1)/\mathrm{SL}(2,\mathbb{Z}_4)$. We put a similar structure on the moduli spaces of σ–invariant metrics.

In all cases, the orbifold order of a point in such moduli spaces is the number of automorphisms of the corresponding (possibly marked) spherical torus.

Appendix Monodromy and coaxiality

In this section we prove that a spherical torus with one conical point of angle $2\pi\vartheta$ is coaxial if and only if ϑ is an odd integer. This was already shown in [2].

In order to prove this, we recall that monodromy representation of spherical surfaces can be lifted to $\mathrm{SU}(2)$:

Proposition A.1 (lift of the monodromy to $\mathrm{SU}(2)$) *Let $(S, \boldsymbol{x})$ be a spherical surface with conical points of angles $2\pi\boldsymbol{\vartheta}$ and let $p \in \dot{S}$ be a basepoint. Let $(\tilde{\dot{S}}, \tilde{p})$ be a universal cover of $(\dot{S}, p)$, endowed with the pullback spherical metric, and let $\iota\colon \tilde{\dot{S}} \to \mathbb{S}^2 \cong \mathbb{CP}^1$ be an associated developing map with monodromy representation $\rho\colon \pi_1(\dot{S}, p) \to \mathrm{SO}_3(\mathbb{R})$. Then there exists a lift $\hat{\rho}\colon \pi_1(\dot{S}, p) \to \mathrm{SU}(2)$ of ρ such that*

- (a) *the developing map ι extends to the completion $\widehat{S}$ of $\tilde{\dot{S}}$ and each point of $\widehat{S} \setminus \tilde{\dot{S}}$ corresponds to a loop based at p that simply winds about some x_j,*
- (b) *if $\gamma_j \in \pi_1(\dot{S}, p)$ is a loop that simply winds about x_j corresponding to a point $\hat{x}_j$ in $\widehat{S} \setminus \tilde{\dot{S}}$, then $\hat{\rho}(\gamma_j) \in \mathrm{SU}(2)$ acts on the complex line $\iota(\hat{x}_j) \subset \mathbb{C}^2$ as multiplication by $e^{i\pi(\vartheta_j - 1)}$.*

Moreover, two such lifts multiplicatively differ by a homomorphism $\pi_1(S, p) \to \{\pm I\}$.

Proof In [22, Proposition 2.12] the statement was proven for a surface S of genus 0. For a surface of arbitrary genus, the proof of existence for a lift is analogous, with minor modifications. In particular, D will be the complement $S \setminus \{q\}$ of an unmarked point q in S, and the vector field V is chosen to be nowhere vanishing on D and have vanishing order $2-2g$ at q, so that the unit normalized vector field $\widehat{V}$ on D has even winding number about q.

Finally, two lifts certainly differ by multiplication by a homomorphism $\pi(\dot{S}, p) \to \{\pm I\}$. Since the eigenvalues of the monodromy about the punctures are fixed by (b), such a homomorphism factors through $\pi(S, p) \to \{\pm I\}$. □

We use the above SU(2)–lifting property to characterize 1–punctured tori (S, x) with integral angles. In order to do that, choose standard generators $\{\alpha, \beta, \gamma\}$ of $\pi_1(\dot{S})$ such that $\gamma = [\alpha, \beta]$. Given a spherical metric on (S, x), its monodromy representation ρ can be lifted to an SU_2–valued representation $\hat{\rho}$ by Proposition A.1. Write $A = \hat{\rho}(\alpha)$, $B = \hat{\rho}(\beta)$ and $C = \hat{\rho}(\gamma)$, and note that C has eigenvalues $e^{\pm i\pi(\vartheta-1)}$.

Corollary A.2 (monodromy of tori with odd ϑ) *Let (S, x) be a spherical torus with one conical point of angle $2\pi\vartheta$. Then its monodromy is nontrivial. Moreover (S, x) has coaxial monodromy if and only if ϑ is an odd integer.*

Proof As for the first claim, if the monodromy of (S, x) were trivial, then the developing map of (S, x) would descend to a cover $S \to \mathbb{S}^2$ ramified at x only. This is clearly absurd.

As for the second claim, the monodromy ρ is coaxial if and only if $\hat{\rho}$ is. On the other hand, since elements in SU(2) are diagonalizable, $\hat{\rho}$ is coaxial if and only if it is abelian. Finally, $\hat{\rho}$ is abelian if and only if $\hat{\rho}(\gamma) = I$, which implies that ϑ is an odd integer. □

Corollary A.3 (monodromy of tori with even ϑ) *Let (S, x) be a spherical torus with one conical point of angle $2\pi\vartheta$. Then the monodromy of (S, x) is isomorphic to the Klein group $K_4 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ if and only if ϑ is an even integer. In this case, the three nontrivial elements in the monodromy group are rotations of angle π along mutually orthogonal axes of $\mathbb{S}^2$.*

Proof The monodromy is isomorphic to K_4 if and only if

$$\rho(\alpha)^2 = \rho(\beta)^2 = [\rho(\alpha), \rho(\beta)] = I.$$

If ϑ is even an even integer, then $C = -I$. Up to conjugacy, we can assume that A is diagonal. The relation $AB = -BA$ gives that A has eigenvalues $\pm i$ and B has zero entries on the diagonal. It follows that $A^2 = B^2 = -I$, and so $\rho(\alpha)^2 = \rho(\beta)^2 = [\rho(\alpha), \rho(\beta)] = I$. It can be observed that A, B and AB act on $\mathbb{S}^2$ as rotations of angle π along mutually orthogonal axes.

Conversely, suppose the monodromy is isomorphic to the Klein group. Then $C = \pm I$ and so ϑ must be integral, but ϑ cannot be odd by Corollary A.2. Hence, ϑ is even. □

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