



# *Geometry & Topology*

Volume 28 (2024)

**Classification results for expanding and shrinking  
gradient Kähler–Ricci solitons**

RONAN J CONLON

ALIX DERUELLE

SONG SUN



## Classification results for expanding and shrinking gradient Kähler–Ricci solitons

RONAN J CONLON

ALIX DERUELLE

SONG SUN

We first show that a Kähler cone appears as the tangent cone of a complete expanding gradient Kähler–Ricci soliton with quadratic curvature decay with derivatives if and only if it has a smooth canonical model (on which the soliton lives). This allows us to classify two-dimensional complete expanding gradient Kähler–Ricci solitons with quadratic curvature decay with derivatives. We then show that any two-dimensional complete shrinking gradient Kähler–Ricci soliton whose scalar curvature tends to zero at infinity is, up to pullback by an element of  $GL(2, \mathbb{C})$ , either the flat Gaussian shrinking soliton on  $\mathbb{C}^2$  or the  $U(2)$ –invariant shrinking gradient Kähler–Ricci soliton of Feldman, Ilmanen and Knopf on the blowup of  $\mathbb{C}^2$  at one point. Finally, we show that up to pullback by an element of  $GL(n, \mathbb{C})$ , the only complete shrinking gradient Kähler–Ricci soliton with bounded Ricci curvature on  $\mathbb{C}^n$  is the flat Gaussian shrinking soliton, and on the total space of  $\mathcal{O}(-k) \rightarrow \mathbb{P}^{n-1}$  for  $0 < k < n$  is the  $U(n)$ –invariant example of Feldman, Ilmanen and Knopf. In the course of the proof, we establish the uniqueness of the soliton vector field of a complete shrinking gradient Kähler–Ricci soliton with bounded Ricci curvature in the Lie algebra of a torus. A key tool used to achieve this result is the Duistermaat–Heckman theorem from symplectic geometry. This provides the first step towards understanding the relationship between complete shrinking gradient Kähler–Ricci solitons and algebraic geometry.

53C25; 53C55, 53E30

1. Introduction	268
2. Preliminaries	276
3. Proof of general structure theorem	294
4. Classification results	305
5. Volume-minimising principle for complete shrinking gradient Kähler–Ricci solitons	309
6. The underlying manifold of a 2–dimensional shrinking gradient Kähler–Ricci soliton	324
7. Concluding remarks	326
Appendix. The Duistermaat–Heckman theorem	333
References	348

# 1 Introduction

## 1.1 Overview

A *Ricci soliton* is a triple  $(M, g, X)$ , where  $M$  is a Riemannian manifold with a complete Riemannian metric  $g$  and a complete vector field  $X$  satisfying the equation

$$(1-1) \quad \text{Ric}(g) + \frac{1}{2}\mathcal{L}_X g = \frac{1}{2}\lambda g$$

for some  $\lambda \in \{-1, 0, 1\}$ . If  $X = \nabla^g f$  for some real-valued smooth function  $f$  on  $M$ , then we say that  $(M, g, X)$  is *gradient*. In this case, the soliton equation (1-1) reduces to

$$(1-2) \quad \text{Ric}(g) + \text{Hess}_g(f) = \frac{1}{2}\lambda g.$$

If  $g$  is complete and Kähler with Kähler form  $\omega$ , then we say that  $(M, g, X)$  (or  $(M, \omega, X)$ ) is a *Kähler–Ricci soliton* if the vector field  $X$  is complete and real holomorphic and the pair  $(g, X)$  satisfies

$$(1-3) \quad \text{Ric}(g) + \frac{1}{2}\mathcal{L}_X g = \lambda g$$

for  $\lambda$  as above. If  $g$  is a Kähler–Ricci soliton and if  $X = \nabla^g f$  for some real-valued smooth function  $f$  on  $M$ , then we say that  $(M, g, X)$  is *gradient*. In this case, one can rewrite the soliton equation (1-3) as

$$(1-4) \quad \rho_\omega + i\partial\bar{\partial}f = \lambda\omega,$$

where  $\rho_\omega$  is the Ricci form of  $\omega$ .

For Ricci solitons and Kähler–Ricci solitons  $(M, g, X)$ , the vector field  $X$  is called the *soliton vector field*. Its completeness is guaranteed by the completeness of  $g$ ; see Zhang [69]. If the soliton is gradient, then the smooth real-valued function  $f$  satisfying  $X = \nabla^g f$  is called the *soliton potential*. It is unique up to a constant. Finally, a Ricci soliton and a Kähler–Ricci soliton are called *steady* if  $\lambda = 0$ , *expanding* if  $\lambda = -1$ , and *shrinking* if  $\lambda = 1$  in equations (1-1) and (1-3), respectively.

The study of Ricci solitons and their classification is important in the context of Riemannian geometry. For example, they provide a natural generalisation of Einstein manifolds. Also, to each soliton, one may associate a self-similar solution of the Ricci flow (see Chow and Knopf [14, Lemma 2.4]), which are candidates for singularity models of the flow. The difference in normalisations between (1-1) and (1-3) reflects the difference between the constants preceding the Ricci term in the Ricci flow and the Kähler–Ricci flow when one takes this dynamic point of view.

In this article, we are concerned with complete expanding and shrinking gradient Kähler–Ricci solitons. We consider primarily complete shrinking (resp. expanding) gradient Kähler–Ricci solitons with quadratic curvature decay (resp. with derivatives) as this assumption greatly simplifies the situation and already imposes some constraints on the solitons in question. Indeed, it is known that any complete shrinking (resp. expanding) gradient Ricci soliton whose curvature decays quadratically (resp. with derivatives) along an end must be asymptotic to a cone with a smooth link along that end; see Chen and Deruelle [12], Chow and Lu [15], Kotschwar and Wang [43] and Siepmann [62]. Furthermore, any complete shrinking

(resp. expanding) Ricci soliton whose Ricci curvature decays to zero (resp. quadratically with derivatives) at infinity must have quadratic curvature decay (resp. with derivatives) at infinity (see Deruelle [27] and Munteanu and Wang [53]) and consequently must be asymptotically conical along each of its ends. Kotschwar and Wang [44] have then shown that any two complete shrinking gradient Ricci solitons asymptotic along some end of each to the same cone with a smooth link, must in fact be isometric. Thus, at least for shrinking gradient Ricci solitons, classifying those that are complete with Ricci curvature decaying to zero at infinity reduces to classifying their possible asymptotic cone models. Here we are principally concerned with the classification of complete shrinking (resp. expanding) gradient Kähler–Ricci solitons whose curvature tensor has quadratic decay (resp. with derivatives). Such examples of expanding type have been constructed in Conlon and Deruelle [19] on certain equivariant resolutions of Kähler cones, whereas such examples of shrinking type include the flat Gaussian shrinking soliton on  $\mathbb{C}^n$  and those constructed by Feldman, Ilmanen and Knopf [30] on the total space of the holomorphic line bundles  $\mathcal{O}(-k)$  over  $\mathbb{P}^{n-1}$  for  $0 < k < n$ . These shrinking solitons are  $U(n)$ –invariant and in complex dimension two, they yield two known examples of complete shrinking gradient Kähler–Ricci solitons with scalar curvature tending to zero at infinity: the flat Gaussian shrinking soliton on  $\mathbb{C}^2$  and the aforementioned  $U(2)$ –invariant example of Feldman, Ilmanen and Knopf on the blowup of  $\mathbb{C}^2$  at the origin. One of our main results is that in fact, up to pullback by an element of  $GL(2, \mathbb{C})$ , these are the only two examples of complete shrinking gradient Kähler–Ricci solitons with scalar curvature tending to zero at infinity in two complex dimensions. Other examples of complete (and indeed incomplete) shrinking gradient Kähler–Ricci solitons with quadratic curvature decay have been constructed on the total space of certain holomorphic vector bundles; see for example Dancer and Wang [23], Futaki and Wang [34], Li [47] and Yang [68].

## 1.2 Main results

**1.2.1 General structure theorem** Our first result concerns the structure of complete shrinking (respectively expanding) gradient Kähler–Ricci solitons  $(M, g, X)$  with quadratic curvature decay (resp. with derivatives). By “quadratic curvature decay with derivatives”, we mean that the curvature  $Rm(g)$  of the Kähler–Ricci soliton  $g$  satisfies

$$A_k(g) := \sup_{x \in M} |(\nabla^g)^k Rm(g)|_g(x) d_g(p, x)^{2+k} < \infty \quad \text{for all } k \in \mathbb{N}_0,$$

where  $d_g(p, \cdot)$  denotes the distance to a fixed point  $p \in M$  with respect to  $g$ .

**Theorem A** (general structure theorem) *Let  $(M, g, X)$  be a complete expanding (resp. shrinking) gradient Kähler–Ricci soliton of complex dimension  $n \geq 2$  with complex structure  $J$  whose curvature  $Rm(g)$  satisfies*

$$A_k(g) := \sup_{x \in M} |(\nabla^g)^k Rm(g)|_g(x) d_g(p, x)^{2+k} < \infty \quad \text{for all } k \in \mathbb{N}_0 \text{ (resp. for } k = 0),$$

where  $d_g(p, \cdot)$  denotes the distance to a fixed point  $p \in M$  with respect to  $g$ . Then:

- (a)  $(M, g)$  has a unique tangent cone at infinity  $(C_0, g_0)$ .

(b) There exists a Kähler resolution  $\pi : M \rightarrow C_0$  of  $C_0$  with exceptional set  $E$  with  $g_0$  Kähler with respect to  $J_0 := \pi_* J$  such that

(i) the Kähler form  $\omega$  of  $g$  and the curvature form  $\Theta$  of the hermitian metric on  $K_M$  (resp.  $-K_M$ ) induced by  $\omega$  satisfy

$$(1-5) \quad \int_V (i\Theta)^k \wedge \omega^{\dim_{\mathbb{C}} V - k} > 0$$

for all positive-dimensional irreducible analytic subvarieties  $V \subset E$  and for all integers  $k$  such that  $1 \leq k \leq \dim_{\mathbb{C}} V$ ;

(ii) the real torus action on  $C_0$  generated by  $J_0 r \partial_r$  extends to a holomorphic isometric torus action of  $(M, g, J)$ , where  $r$  denotes the radial coordinate of  $g_0$ ;

(iii)  $d\pi(X) = r \partial_r$ .

(c) With respect to  $\pi$ , we have

$$\begin{aligned} |(\nabla^{g_0})^k (\pi_* g - g_0 - \text{Ric}(g_0))|_{g_0} &\leq C_k r^{-4-k} \quad \text{for all } k \in \mathbb{N}_0, \\ (\text{resp. } |(\nabla^{g_0})^k (\pi_* g - g_0 + \text{Ric}(g_0))|_{g_0} &\leq C_k r^{-4-k} \quad \text{for all } k \in \mathbb{N}_0). \end{aligned}$$

In the expanding case, this theorem provides a converse to [19, Theorem A]. Also notice that this theorem rules out the existence of shrinking (resp. expanding) gradient Kähler–Ricci solitons with quadratic curvature decay (resp. with derivatives) on smoothings of Kähler cones in contrast to the behaviour in the Calabi–Yau case [20]. This degree of flexibility for expanding and shrinking gradient Kähler–Ricci solitons is essentially ruled out due to the requirement of having a conical holomorphic soliton vector field. Finally, it has recently been shown by Kotschwar and Wang [44, Corollary 1.3] that the isometry group of the link of the asymptotic cone of a complete shrinking gradient Kähler–Ricci soliton with quadratic curvature decay embeds into the isometry group of the soliton itself; compare with statement (b)(ii) of the theorem above.

**1.2.2 Application to expanding gradient Kähler–Ricci solitons** As an application of Theorem A, we exploit the uniqueness [38, Proposition 8.2.5] of canonical models of normal varieties to obtain a classification theorem for complete expanding gradient Kähler–Ricci solitons whose curvature decays quadratically with derivatives. This provides a partial answer to question 7 of [30, Section 10].

**Corollary B** (strong uniqueness for expanders) *Let  $(C_0, g_0)$  be a Kähler cone of complex dimension  $n \geq 2$  with radial function  $r$ . Then there exists a unique (up to pullback by biholomorphisms) complete expanding gradient Kähler–Ricci soliton  $(M, g, X)$ , whose curvature  $\text{Rm}(g)$  satisfies*

$$(1-6) \quad A_k(g) := \sup_{x \in M} |(\nabla^g)^k \text{Rm}(g)|_g(x) d_g(p, x)^{2+k} < \infty \quad \text{for all } k \in \mathbb{N}_0,$$

where  $d_g(p, \cdot)$  denotes the distance to a fixed point  $p \in M$  with respect to  $g$ , with tangent cone  $(C_0, g_0)$  if and only if  $C_0$  has a smooth canonical model. When this is the case,

(a)  $M$  is the smooth canonical model of  $C_0$ , and

(b) there exists a resolution map  $\pi : M \rightarrow C_0$  such that  $d\pi(X) = r\partial_r$  and

$$|(\nabla^{g_0})^k(\pi_*g - g_0 - \text{Ric}(g_0))|_{g_0} \leq C_k r^{-4-k} \quad \text{for all } k \in \mathbb{N}_0.$$

This corollary yields an algebraic description of those Kähler cones appearing as the tangent cone of a complete expanding gradient Kähler–Ricci soliton with quadratic curvature decay with derivatives—such cones are precisely those admitting a smooth canonical model. In general, the canonical model will be singular and in particular, for a two-dimensional cone, it is obtained by contracting all exceptional curves with self-intersection  $(-2)$  in the minimal resolution. Applying Corollary B to this case yields a classification of two-dimensional complete expanding gradient Kähler–Ricci solitons with quadratic curvature decay with derivatives.

**Corollary C** (classification of two-dimensional expanders) *Let  $(C_0, g_0)$  be a two-dimensional Kähler cone with radial function  $r$ . Then there exists a unique (up to pullback by biholomorphisms) two-dimensional complete expanding gradient Kähler–Ricci soliton  $(M, g, X)$  whose curvature  $\text{Rm}(g)$  satisfies*

$$A_k(g) := \sup_{x \in M} |(\nabla^g)^k \text{Rm}(g)|_g(x) d_g(p, x)^{2+k} < \infty \quad \text{for all } k \in \mathbb{N}_0,$$

where  $d_g(p, \cdot)$  denotes the distance to a fixed point  $p \in M$  with respect to  $g$ , with tangent cone  $(C_0, g_0)$  if and only if  $C_0$  is biholomorphic to one of:

(I)  $\mathbb{C}^2/\Gamma$ , where  $\Gamma$  is a finite subgroup of  $U(2)$  acting freely on  $\mathbb{C}^2 \setminus \{0\}$  that is generated by the matrix

$$\begin{pmatrix} e^{2\pi i/p} & 0 \\ 0 & e^{2\pi i q/p} \end{pmatrix},$$

where  $p$  and  $q$  coprime integers with  $p > q > 0$ , and after writing

$$\frac{q}{p} = r_1 - \frac{1}{r_2 - \frac{1}{\dots - \frac{1}{r_k}}}$$

we have that  $r_j > 2$  for  $j = 1, \dots, k$ .

(II)  $L^\times$ , the blowdown of the zero section of a negative line bundle  $L \rightarrow C$  over a proper curve  $C$  of genus  $g > 0$ .

(III)  $L^\times/G$ , where  $G$  is a nontrivial finite group of automorphisms of a proper curve  $C$  of genus  $g > 0$  and  $L^\times$  is the blowdown of the zero section of a  $G$ -invariant negative line bundle  $L \rightarrow C$  over  $C$  with  $G$  acting freely on  $L^\times$  except at the apex, such that the (unique) minimal good resolution  $\pi : M \rightarrow L^\times/G$  contains no  $(-1)$ - or  $(-2)$ -curves.

When this is the case,

- (a) there exists a resolution map  $\pi: M \rightarrow C_0$  such that  $d\pi(X) = r\partial_r$  and

$$|(\nabla^{g_0})^k(\pi_*g - g_0 - \text{Ric}(g_0))|_{g_0} \leq C_k r^{-4-k} \quad \text{for all } k \in \mathbb{N}_0,$$

and

- (b)  $\pi: M \rightarrow C_0$  is
- (i) the minimal resolution  $\pi: M \rightarrow \mathbb{C}^2/\Gamma$  when  $C_0$  is as in (I);
  - (ii) the blowdown map  $\pi: L \rightarrow L^\times$  when  $C_0$  is as in (II);
  - (iii) the minimal good resolution  $\pi: M \rightarrow L^\times/G$  when  $C_0$  is as in (III).

Note that the Kähler cones of item (III) here are (up to analytic isomorphism) in one-to-one correspondence with the following data, which encodes the exceptional set of the minimal good resolution  $\pi: M \rightarrow L^\times/G$ :

- A weighted dual graph which is a star comprising a central vertex with  $n$  branches, with  $n \geq 1$ , each of finite length, with the  $j^{\text{th}}$  vertex of the  $i^{\text{th}}$  branch labelled with an integer  $b_{ij} \geq 3$  and the central curve labelled with an integer  $b \geq 1$ . The central vertex represents the curve  $C$  of genus  $g > 0$  with self-intersection  $-b$  and the  $j^{\text{th}}$  vertex of the  $i^{\text{th}}$  branch represents a  $\mathbb{P}^1$  with self-intersection  $-b_{ij}$ . The intersection matrix given by this graph must be negative-definite. There is a numerical criterion to determine when this is the case.
- The analytic type of  $C$ .
- A negative line bundle on  $C$  of degree  $-b$  (which is the normal bundle of  $C$  in  $M$ ).
- $n$  marked points on  $C$ .

The algebraic equations of  $C_0$  can be reconstructed from this data by the ansatz in [59, Section 5].

**1.2.3 Application to shrinking gradient Kähler–Ricci solitons** Tian and Zhu [64] showed that the soliton vector field of a compact shrinking gradient Kähler–Ricci soliton is unique up to holomorphic automorphisms of the underlying complex manifold. The method of proof there involved defining a weighted volume functional  $F$  which was strictly convex and was shown to be in fact independent of the metric structure of the soliton. The soliton vector field was then characterised as the unique critical point of this functional.

Tian and Zhu’s proof breaks down in the noncompact case due to the fact that, in general, one cannot a priori guarantee that the weighted volume functional (defined analytically in terms of a certain integral) is well-defined on a noncompact Kähler manifold, let alone investigate its convexity properties. Our key observation to circumvent this difficulty is to apply the *Duistermaat–Heckman theorem* from symplectic geometry, which provides a localisation formula to express the weighted volume functional in terms of an algebraic formula involving only the fixed point set of a torus action. This latter algebraic formula is



much more amenable. Combined with a version of Matsushima’s theorem [50] for complete noncompact shrinking gradient Kähler–Ricci solitons (cf Theorem 5.1) and Iwasawa’s theorem [39], we are able to implement Tian and Zhu’s strategy of proof for the compact case to obtain the following noncompact analogue of their uniqueness result.

**Theorem D** (uniqueness of the soliton vector field for shrinkers) *Let  $M$  be a noncompact complex manifold with complex structure  $J$  endowed with the effective holomorphic action of a real torus  $T$ . Denote by  $\mathfrak{t}$  the Lie algebra of  $T$ . Then there exists at most one element  $\xi \in \mathfrak{t}$  that admits a complete shrinking gradient Kähler–Ricci soliton  $(M, g, X)$  with bounded Ricci curvature with  $X = \nabla^g f = -J\xi$  for a smooth real-valued function  $f$  on  $M$ .*

We expect that the assumption of bounded Ricci curvature is superfluous in the statement of this theorem and that, given the uniqueness of the soliton vector field in the Lie algebra of the torus here, the corresponding shrinking gradient Kähler–Ricci soliton is also unique up to automorphisms of the complex structure commuting with the flow of the soliton vector field.

Not only does the Duistermaat–Heckman theorem imply the uniqueness of the soliton vector field in our case, it also provides a formula to compute the unique critical point of the weighted volume functional, the point at which the soliton vector field is achieved. Using this formula, we compute explicitly the soliton vector field of a shrinking gradient Kähler–Ricci soliton with bounded Ricci curvature on  $\mathbb{C}^n$  and on the total space of  $\mathcal{O}(-k)$  over  $\mathbb{P}^{n-1}$  for  $0 < k < n$ ; see Examples A.6 and A.8, respectively. This recovers the polynomials of Feldman, Ilmanen and Knopf [30, equation (36)]. Having identified the soliton vector field on these manifolds, we then use our noncompact version of Matsushima’s theorem (Theorem 5.1) together with an application of Iwasawa’s theorem [39] to deduce that, up to pullback by an element of  $\mathrm{GL}(n, \mathbb{C})$ , the corresponding soliton metrics have to be invariant under the action of  $U(n)$ . Consequently, thanks to a uniqueness theorem of Feldman, Ilmanen and Knopf [30, Proposition 9.3], up to pullback by an element of  $\mathrm{GL}(n, \mathbb{C})$ , the shrinking gradient Kähler–Ricci soliton must be the flat Gaussian shrinking soliton if on  $\mathbb{C}^n$ , or the unique  $U(n)$ –invariant shrinking gradient Kähler–Ricci soliton constructed by Feldman, Ilmanen and Knopf [30] if on the total space of  $\mathcal{O}(-k)$  over  $\mathbb{P}^{n-1}$  for  $0 < k < n$ .

In the complex two-dimensional case, we are actually able to identify the underlying complex manifold of a shrinking gradient Kähler–Ricci soliton whose scalar curvature tends to zero at infinity as either  $\mathbb{C}^2$  or  $\mathbb{C}^2$  blown up at the origin using the fact that the soliton, if nontrivial, has an asymptotic cone with strictly positive scalar curvature. Using a classification theorem for Sasaki manifolds in real dimension three (see Belgun [4]) and the fact that a shrinking Kähler–Ricci soliton can only contain  $(-1)$ –curves in complex dimension two, this is enough to identify the asymptotic cone at infinity as  $\mathbb{C}^2$ , from which the identification of the underlying complex manifold easily follows. Combined with the above discussion, this yields a complete classification of such solitons in two complex dimensions.

These conclusions are summarised in the following theorem.

**Theorem E** (classification of shrinkers) *Let  $(M, g, X)$  be a complete shrinking gradient Kähler–Ricci soliton.*

- (1) *If  $M = \mathbb{C}^n$  and  $g$  has bounded Ricci curvature, then up to pullback by an element of  $\mathrm{GL}(n, \mathbb{C})$ ,  $(M, g, X)$  is the flat Gaussian shrinking soliton.*
- (2) *If  $M$  is the total space of  $\mathcal{O}(-k)$  over  $\mathbb{P}^{n-1}$  for  $0 < k < n$  and  $g$  has bounded Ricci curvature, then up to pullback by an element of  $\mathrm{GL}(n, \mathbb{C})$ ,  $(M, g, X)$  is the unique  $U(n)$ –invariant shrinking gradient Kähler–Ricci soliton constructed by Feldman, Ilmanen and Knopf [30].*
- (3) *If  $\dim_{\mathbb{C}} M = 2$  and the scalar curvature of  $g$  tends to zero at infinity, then up to pullback by an element of  $\mathrm{GL}(2, \mathbb{C})$ ,  $(M, g, X)$  is the flat Gaussian shrinking soliton on  $\mathbb{C}^2$  or the unique  $U(2)$ –invariant shrinking gradient Kähler–Ricci soliton constructed by Feldman, Ilmanen and Knopf [30] on the total space of  $\mathcal{O}(-1)$  over  $\mathbb{P}^1$ .*

As exemplified in complex dimension two, in contrast to the expanding case, not many Kähler cones appear as tangent cones of complete shrinking gradient Kähler–Ricci solitons.

**Outline of paper** We begin in Section 2 by recalling the basics of Kähler cones, Ricci and Kähler–Ricci solitons, metric measure spaces, and the relevant algebraic geometry that we require. We also mention some important properties of the soliton vector field and of real vector fields that commute with the soliton vector field. In Section 3, we prove Theorem A for expanding gradient Kähler–Ricci solitons. The proof for shrinking Kähler–Ricci solitons is verbatim. By a theorem of Siepmann [62, Theorem 4.3.1], under our curvature assumption, a complete expanding gradient Ricci soliton flows out of a Riemannian cone. Our starting point is to prove some preliminary lemmas before providing a refinement of Siepmann’s theorem, namely Theorem 3.8, where, in the course of its proof, we construct a diffeomorphism between the cone and the end of the Ricci soliton using the flow of the soliton vector field that encapsulates the asymptotics of the soliton along the end. We then show in Proposition 3.10 that if the soliton is Kähler, then the cone is Kähler with respect to a complex structure that makes the aforementioned diffeomorphism a biholomorphism. In Theorem 3.11, this biholomorphism is then shown to extend to an equivariant resolution with the properties as stated in Theorem A.

In the first part of Section 4, we use Theorem A to prove Corollary B. This also requires an application of previous work from [19]. In the latter part of Section 4, we apply Corollary B to two-dimensional expanding gradient Kähler–Ricci solitons to conclude the statement of Corollary C, making use of the classification of two-dimensional Kähler cones, namely Theorem 2.5.

From Section 5 onwards, we turn our attention exclusively to complete shrinking gradient Kähler–Ricci solitons. We begin in Section 5.1 by proving a Matsushima-type theorem stating that the Lie algebra of real holomorphic vector fields commuting with the soliton vector field may be written as a direct sum. This is the statement of Theorem 5.1. Our proof of this theorem follows a manner similar to the proof of

Matsushima’s theorem on Kähler–Einstein Fano manifolds as presented in [40, Proof of Theorem 5.1, page 95]. After deriving some properties of the automorphism groups of a complete shrinking gradient Kähler–Ricci soliton  $(M, g, X)$ , we then apply Theorem 5.1 to prove the maximality of a certain compact Lie group acting on  $M$ . This is the content of Corollary 5.11.

We continue in Section 5.2 by showing in Proposition 5.12 that every real holomorphic Killing vector field on  $M$  admits a Hamiltonian potential satisfying a certain linear equation. This allows us to define a moment map in Definition 5.13, which is used in the definition of the weighted volume functional in Definition 5.14. The weighted volume functional is vital in proving the uniqueness statement of Theorem D, namely that the soliton vector field of a complete shrinking gradient Kähler–Ricci soliton with bounded Ricci curvature in the Lie algebra of a torus is unique. The weighted volume functional is the same as that defined by Tian and Zhu [64], although in our situation it is defined as an integral over the noncompact manifold  $M$ . This is compensated for by the fact that the domain of definition of the weighted volume functional is restricted to an open cone in the Lie algebra of the torus. Several important properties of the weighted volume functional are then derived in Lemma 5.16, including the crucial fact that it has a unique critical point in its open cone of definition given by the complex structure applied to the soliton vector field  $X$ . We conclude this subsection by taking note of the fact that the Duistermaat–Heckman formula may be used to compute the weighted volume functional. In particular, it follows that the weighted volume functional is independent of the complete shrinking gradient Kähler–Ricci soliton.

In Section 5.3, we prove the uniqueness statement of Theorem D, which has been recalled in the statement of Theorem 5.18. The proof of this theorem follows as in [64, page 322] using Iwasawa’s theorem [39] and the corollary of Matsushima’s theorem, namely Corollary 5.11 discussed above. Section 5.4 then comprises an application of Theorems A and D to classify complete shrinking gradient Kähler–Ricci solitons with bounded Ricci curvature on  $\mathbb{C}^n$  and on the total space of  $\mathcal{O}(-k) \rightarrow \mathbb{P}^{n-1}$  for  $0 < k < n$ . This completes the proof of items (1) and (2) of Theorem E.

In Section 6, we show that the underlying complex manifold  $M$  of a two-dimensional shrinking gradient Kähler–Ricci soliton with scalar curvature decaying to zero at infinity is either  $\mathbb{C}^2$  or  $\mathbb{C}^2$  blown up at the origin. This is the statement of Theorem 6.1. Combined with items (1) and (2) of Theorem E, Theorem 6.1 suffices to prove item (3) of Theorem E. The proof of Theorem 6.1 relies on first identifying the underlying complex space of the tangent cone. The fact that any nonflat shrinking gradient Ricci soliton has positive scalar curvature implies that the same property holds true on the tangent cone. From this we can identify the cone as a quotient of  $\mathbb{C}^2$ . Theorem A then tells us that  $M$  is a resolution of this cone which, by virtue of the shrinking Kähler–Ricci soliton equation, can only contain  $(-1)$ -curves. It turns out that the only possibility is that the cone is biholomorphic to  $\mathbb{C}^2$  and  $M$  is as stated in Theorem 6.1.

We conclude the paper in Section 7 with some closing remarks and open problems. In Section A.1 in the appendix, we recall the statement of the Duistermaat–Heckmann theorem on a noncompact symplectic manifold in Theorem A.3 as presented in [60]. We also provide an outline of its proof in Section A.3

after introducing some preliminaries in Section A.2. We then use it to compute the weighted volume functional and its unique critical point on  $\mathbb{C}^n$ , on the total space of  $\mathcal{O}(-k)$  over  $\mathbb{P}^{n-1}$  for  $0 < k < n$ , and on certain holomorphic line bundles over Fano manifolds in Section A.4. In Section A.5, we characterise algebraically, in the setting of asymptotically conical Kähler manifolds, those elements in the Lie algebra of a torus that admit a Hamiltonian potential that is proper and bounded below. A precise statement is given in Theorem A.10. For such elements of the Lie algebra of the torus, the weighted volume functional is defined. Finally, in Section A.6, we show directly that the weighted volume functional is defined on a complete shrinking gradient Kähler–Ricci soliton with bounded Ricci curvature without appealing to the Duistermaat–Heckman theorem. This conclusion follows from the estimates we derive in Proposition A.13 on the growth of those Hamiltonian potentials that are proper and bounded below.

**Acknowledgements** The authors wish to thank Richard Bamler, John Lott, Bing Wang and Yuanqi Wang for useful discussions and for their interest in this work, and Brett Kotschwar for his interest, for useful discussions involving [43; 44] and for comments on a preliminary version of this article. Part of this work was carried out while the first two authors were visiting the Institut Henri Poincaré as part of the Research in Paris program in July 2019. They wish to thank the institute for their hospitality and for the excellent working conditions provided. Finally, the authors wish to thank the anonymous referee for helpful comments regarding the proof of Theorem 5.1.

Conlon is supported by Simons Collaboration Grant #581984. Deruelle is supported by grant ANR-17-CE40-0034 of the French National Research Agency ANR (project CCEM) and Fondation Louis D, project *Jeunes géomètres*. Sun is supported by the Simons Collaboration Grant on *Special holonomy in geometry, analysis, and physics* #488633, an Alfred P Sloan fellowship, and NSF grant DMS-1708420.

## 2 Preliminaries

### 2.1 Riemannian cones

For us, the definition of a Riemannian cone will take the following form.

**Definition 2.1** Let  $(S, g_S)$  be a closed Riemannian manifold. The *Riemannian cone*  $C_0$  with *link*  $S$  is defined to be  $\mathbb{R}_{>0} \times S$  with metric  $g_0 = dr^2 \oplus r^2 g_S$  up to isometry. The radius function  $r$  is then characterised intrinsically as the distance from the apex in the metric completion.

The following is a simple computation.

**Lemma 2.2** Let  $(S, g_S)$  be a closed Riemannian manifold of real dimension  $m$  and let  $(C_0, g_0)$  be the Riemannian cone with link  $S$  and radial function  $r$ . Then the Ricci curvature  $\text{Ric}(g_S)$  of  $g_S$  and the Ricci curvature  $\text{Ric}(g_0)$  of the cone metric  $g_0$  over  $(S, g_S)$  are related by

$$\text{Ric}(g_0) = \text{Ric}(g_S) - (m - 1)g_S.$$

In particular, the scalar curvatures  $R_{g_0}$  and  $R_{g_S}$  of  $g_0$  and  $g_S$  respectively are related by

$$R_{g_0} = \frac{1}{r^2}(R_{g_S} - m(m - 1)).$$

## 2.2 Kähler cones

We may further impose that a Riemannian cone is Kähler, as the next definition demonstrates.

**Definition 2.3** A *Kähler cone* is a Riemannian cone  $(C_0, g_0)$  such that  $g_0$  is Kähler, together with a choice of  $g_0$ -parallel complex structure  $J_0$ . This will in fact often be unique up to sign. We then have a Kähler form  $\omega_0(X, Y) = g_0(J_0X, Y)$ , and  $\omega_0 = \frac{1}{2}i\partial\bar{\partial}r^2$  with respect to  $J_0$ .

The vector field  $r\partial_r$  is real holomorphic and  $\xi := J_0r\partial_r$  is real holomorphic and Killing [49, Appendix A]. This latter vector field is known as the *Reeb vector field*. The closure of its flow in the isometry group of the link of the cone generates the holomorphic isometric action of a real torus on  $C_0$  that fixes the apex of the cone. We call a Kähler cone “quasiregular” if this action is an  $S^1$ -action (and, in particular, “regular” if this  $S^1$ -action is free), and “irregular” if the action generated is that of a real torus of rank  $> 1$ .

Every Kähler cone is affine algebraic.

**Theorem 2.4** For every Kähler cone  $(C_0, g_0, J_0)$ , the complex manifold  $(C_0, J_0)$  is isomorphic to the smooth part of a normal algebraic variety  $V \subset \mathbb{C}^N$  with one singular point. In addition,  $V$  can be taken to be invariant under a  $\mathbb{C}^*$ -action  $(t, z_1, \dots, z_N) \mapsto (t^{w_1}z_1, \dots, t^{w_N}z_N)$  such that all of the  $w_i$  are positive integers.

This can be deduced from arguments written down by van Coevering in [17, Section 3.1].

Kähler cones of complex dimension two have been classified.

**Theorem 2.5** [4, Theorem 8; 59, Theorem 1.1] Let  $C_0$  be a Kähler cone of complex dimension two. Then  $C_0$  is biholomorphic to either

- (i)  $\mathbb{C}^2/\Gamma$ , where  $\Gamma$  is a finite subset of  $U(2)$  acting freely on  $\mathbb{C}^2 \setminus \{0\}$ ,
- (ii) the blowdown  $L^\times$  of the zero section of a negative line bundle  $L \rightarrow C$  over a smooth proper curve  $C$  of genus  $g$  with  $g > 0$ , or
- (iii)  $L^\times/G$ , where  $G$  is a nontrivial finite group of automorphisms of a proper curve  $C$  of genus  $g > 0$  and  $L^\times$  is the blowdown of the zero section of a  $G$ -invariant negative line bundle  $L \rightarrow C$  over  $C$  with  $G$  acting freely on  $L^\times$  except at the apex.

In cases (ii) and (iii), the corresponding Reeb vector field is quasiregular and is generated by a scaling of the standard  $S^1$ -action on the fibres of  $L$ .

Any automorphism of a resolution of a Kähler cone preserves the exceptional set of the resolution.

**Lemma 2.6** *Let  $\pi : M \rightarrow C_0$  be a resolution of a Kähler cone  $C_0$  with exceptional set  $E$ . Denote by  $J$  the complex structure on  $M$ . Then for any automorphism  $\sigma$  of  $(M, J)$ , it holds that  $\sigma(E) \subseteq E$ . In particular, real holomorphic vector fields on  $M$  are tangent to  $E$ .*

Such an automorphism of  $(M, J)$  therefore descends to an automorphism of the cone fixing the apex.

**Proof** If  $\sigma$  is an automorphism of  $(M, J)$ , then  $\pi \circ \sigma : M \rightarrow C_0$  is also a resolution of  $C_0$ . The exceptional set of this resolution is then a compact analytic subset of  $M$ . Since  $E$  is the maximal compact analytic subset of  $M$ , we must have that  $(\pi \circ \sigma)^{-1}(o) \subseteq E$ , where  $o$  denotes the apex of  $C_0$ , ie  $\sigma^{-1}(E) \subseteq E$ .  $\square$

The holomorphic torus action on a Kähler cone leads to the notion of an *equivariant resolution*.

**Definition 2.7** Let  $C_0$  be a Kähler cone with complex structure  $J_0$ , let  $\pi : M \rightarrow C_0$  be a resolution of  $C_0$ , and let  $G$  be a Lie subgroup of the automorphism group of  $(C_0, J_0)$  fixing the apex of  $C_0$ . We say that  $\pi : M \rightarrow C_0$  is an *equivariant resolution with respect to  $G$*  if the action of  $G$  on  $C_0$  extends to a holomorphic action on  $M$  in such a way that  $\pi(g \cdot x) = g \cdot \pi(x)$  for all  $x \in M$  and  $g \in G$ .

Such a resolution of a Kähler cone always exists; see [41, Proposition 3.9.1].

A closed Riemannian manifold  $(S, g_S)$  is *Sasaki* if and only if its Riemannian cone is a Kähler cone [8], in which case we identify  $(S, g_S)$  with the level set  $\{r = 1\}$  of its corresponding Kähler cone,  $r$  here denoting the radial function of the cone. The restriction of the Reeb vector field to this level set induces a nonzero vector field  $\xi \equiv J_0 r \partial_r|_{\{r=1\}}$  on  $S$ . Let  $\eta$  denote the  $g_S$ -dual one-form of  $\xi$ . Then we get a  $g_S$ -orthogonal decomposition  $TS = \mathcal{D} \oplus \langle \xi \rangle$ , where  $\mathcal{D}$  is the kernel of  $\eta$  and  $\langle \xi \rangle$  is the  $\mathbb{R}$ -span of  $\xi$  in  $TS$ , and correspondingly a decomposition of the metric  $g_S$  as  $g_S = \eta \otimes \eta + g^T$ , where  $g^T = g_S|_{\mathcal{D}}$ . The metric  $g^T$  is invariant under the flow of  $\xi$  and induces a Riemannian metric on the local leaf space of the foliation of  $S$  induced by the flow of  $\xi$ . We call  $g^T$  the *transverse metric*. We can then define the *transverse scalar curvature*  $R^T$  and the *transverse Ricci curvature*  $\text{Ric}^T$  as the corresponding curvatures of  $g^T$ . We also get an induced *transverse complex structure*  $J^T$  on the local leaf space of the foliation with respect to which  $g^T$  is Kähler given by the restriction of the complex structure of the cone to  $\mathcal{D}$ . In particular,  $\text{Ric}^T$  will be  $J^T$ -invariant. We have the following relationships between the various curvatures.

**Lemma 2.8** [8, Theorem 7.3.12] *Let  $(S, g_S)$  be a real  $(2n+1)$ -dimensional Sasaki manifold. Then the following identities hold:*

- (i)  $\text{Ric}(g_S)(X, \xi) = 2n\eta(X)$  for any vector field  $X$ .
- (ii)  $\text{Ric}(g_S)(X, Y) = \text{Ric}^T(X, Y) - 2g_S(X, Y)$  for any vector fields  $X, Y \in \mathcal{D}$ .

In particular, we deduce:

**Corollary 2.9** *Let  $(S, g_S)$  be a real  $(2n+1)$ –dimensional Sasaki manifold with scalar curvature  $R_{g_S}$ . Then*

$$R^T = R_{g_S} + 2n.$$

### 2.3 Canonical models

Resolutions of Kähler cones that are consistent with admitting an expanding Kähler–Ricci soliton are of the following type.

**Definition 2.10** [38, Definition 8.2.4] *A partial resolution  $\pi: M \rightarrow C_0$  of a normal isolated singularity  $x \in C_0$  is called a canonical model if*

- (i)  $M$  has at worst canonical singularities, and
- (ii)  $K_M$  is  $\pi$ –ample.

Note that the choice of partial resolution  $\pi$  is part of the data here. The existence of a canonical model  $\pi: M \rightarrow C_0$  is guaranteed by [7] and it is unique up to isomorphisms over  $C_0$  [38, Proposition 8.2.5]. We have the following criterion to determine when  $K_M$  is  $\pi$ –ample.

**Theorem 2.11** (Nakai’s criterion for a mapping [46, Corollary 1.7.9]) *Let  $\pi: M \rightarrow C_0$  be a proper morphism of schemes. A  $\mathbb{Q}$ –divisor  $D$  on  $M$  is  $\pi$ –ample if and only if  $(D^{\dim_{\mathbb{C}} V} \cdot V) > 0$  for every irreducible subvariety  $V \subset M$  of positive dimension that maps to a point in  $C_0$ .*

In particular, in complex dimension two, item (ii) of Definition 2.10 implies that the exceptional set of the canonical model cannot contain any  $(-1)$ – or  $(-2)$ –curves.

In our case,  $C_0$  will be a Kähler cone, hence is affine algebraic, and  $x$  will be the apex of the cone. As the next lemma shows, the canonical model of  $C_0$  is quasiprojective.

**Lemma 2.12** *Let  $\pi: M \rightarrow C_0$  be the canonical model of a Kähler cone  $C_0$ . Then  $M$  is quasiprojective.*

**Proof** From Theorem 2.4, we see that  $C_0$  admits an affine embedding that is invariant under a  $\mathbb{C}^*$ –action with positive integer weights. Taking the weighted projective closure of  $C_0$  with respect to this action, we obtain a projective compactification  $\bar{C}_0$  of  $C_0$  by adding an ample divisor  $D$  at infinity. In particular,  $\bar{C}_0$  will have at worst orbifold singularities along  $D$ . Let  $\sigma: \bar{N} \rightarrow \bar{C}_0$  denote the canonical model of  $\bar{C}_0$ . By construction,  $\bar{N}$  is projective and the restricted map  $\sigma|_N: N \rightarrow C_0$ , where  $N := \bar{N} \setminus \sigma^{-1}(D)$ , is a canonical model of  $C_0$ . By uniqueness of canonical models,  $M$  must be biholomorphic to  $N$ , hence  $\bar{N}$  provides a projective compactification of  $M$  obtained by adjoining the set  $\sigma^{-1}(D)$  to  $M$  at infinity.  $M$  is therefore quasiprojective, as claimed. □

In addition, uniqueness of the canonical model implies that when smooth, the canonical model of a Kähler cone is equivariant with respect to the torus action on  $C_0$  generated by the flow of  $J_0 r \partial_r$ .

**Lemma 2.13** *Let  $C_0$  be a Kähler cone with complex structure  $J_0$  and radial function  $r$  and let  $\pi : M \rightarrow C_0$  denote the canonical model of  $C_0$ . If  $M$  is smooth, then the resolution  $\pi : M \rightarrow C_0$  is equivariant with respect to the holomorphic isometric torus action on  $C_0$  generated by  $J_0 r \partial_r$ . In particular, there exists a holomorphic vector field  $X$  on  $M$  such that  $d\pi(X) = r \partial_r$ .*

**Proof** Let  $T$  denote the torus generated by the flow of  $J_0 r \partial_r$  and let  $T_{\mathbb{C}}$  denote its complexification. We will show that the holomorphic action of  $T_{\mathbb{C}}$  on  $C_0$  lifts to a holomorphic action on  $M$ .

For  $h \in T_{\mathbb{C}}$ , let  $\psi_h : C_0 \rightarrow C_0$  denote the corresponding automorphism. Since  $\psi_h \circ \pi : M \rightarrow C_0$  is again a canonical model and since  $\pi : M \rightarrow C_0$  is unique up to isomorphisms over  $C_0$  [38, Proposition 8.2.5], there exists a unique biholomorphism  $\tilde{\psi}_h : M \rightarrow M$  such that  $\pi \circ \tilde{\psi}_h = \psi_h \circ \pi$ . This is the desired lift of  $\psi_h$ . Thus, we have a well-defined map  $\phi : T_{\mathbb{C}} \times M \rightarrow M$  defined by  $\phi(h, x) = \tilde{\psi}_h(x)$ . Since  $\tilde{\psi}_h$  coincides with  $\psi_h$  off of the exceptional set  $E$  of the resolution  $\pi : M \rightarrow C_0$ , the map  $\phi|_{T_{\mathbb{C}} \times (M \setminus E)}$  is holomorphic. We wish to show that  $\phi$  is holomorphic globally.

To this end, let  $h \in T_{\mathbb{C}}$ , let  $x \in E$ , let  $y = \tilde{\psi}_h(x) \in E$ , let  $B_x$  be an open ball in a chart containing  $x$  with  $x$  in its interior and let  $B_y$  be an open ball in a chart containing  $y$  with  $y$  in its interior. Since  $\tilde{\psi}_h$  is continuous, by shrinking  $B_x$  if necessary we may assume that  $\tilde{\psi}_h(B_x) \subset B_y$ . Let  $U$  be a neighbourhood of  $h$  in  $T_{\mathbb{C}}$  such that  $\tilde{\psi}_{h'}(B_x \setminus E) \subset B_y$  for all  $h' \in U$ . Again, this is possible because  $\phi|_{T_{\mathbb{C}} \times (M \setminus E)}$  is continuous. Then since  $\tilde{\psi}_{h'} : M \rightarrow M$  is itself continuous and preserves  $E$  for each fixed  $h' \in U$  and since  $B_x \cap E$  lies in the closure of  $B_x \setminus E$ , we have, after shrinking  $B_x$  further if necessary, that  $\tilde{\psi}_{h'}(B_x) \subset B_y$  for all  $h' \in U$ .

Next, let  $N_\varepsilon := \{x \in C_0 \mid r(x) < \varepsilon\}$  for  $\varepsilon > 0$ . Then  $N_\varepsilon \setminus \{o\}$  is foliated by disjoint punctured discs, obtained as the orbits in  $N_\varepsilon \setminus \{o\}$  of a  $\mathbb{C}^*$ -action from within  $T_{\mathbb{C}}$ . The open set  $\hat{N}_\varepsilon := \pi^{-1}(N_\varepsilon)$  will then be a neighbourhood of  $E$  in  $M$  with  $\hat{N}_\varepsilon \setminus E$  foliated by disjoint punctured discs. Let  $B'_x \subset B_x$  be an open ball containing  $x$  strictly contained in  $B_x$  such that  $\partial B_x \cap \partial B'_x = \emptyset$ , and let  $\varepsilon > 0$  be sufficiently small that each point of  $(\hat{N}_\varepsilon \cap B'_x) \setminus E$  is contained in a punctured holomorphic disk of radius  $\varepsilon$  which itself is contained in  $(\hat{N}_\varepsilon \cap B_x) \setminus E$ . Let  $V := \hat{N}_\varepsilon \cap B'_x$ . Then this is an open neighbourhood of  $x$  in  $M$  and each point  $z \in V \setminus E$  will lie on a unique punctured holomorphic disk, which we shall denote by  $D_z$ . We have that  $D_z \subseteq (\hat{N}_\varepsilon \cap B_x) \setminus E$  and that  $\partial D_z \subseteq B_x \cap \partial \hat{N}_\varepsilon \subset M \setminus E$ . Let  $\bar{D}_z$  denote the closure of  $D_z$  in  $M$ . Since  $\tilde{\psi}_{h'} : M \rightarrow M$  is holomorphic for each fixed  $h' \in U$ , after localising, we see from the maximum principle that for all  $z \in V \setminus E$  and all  $h' \in U$ ,

$$|\tilde{\psi}_{h'}(z)| \leq \sup_{w \in \bar{D}_z} |\tilde{\psi}_{h'}(w)| \leq \sup_{\{w \in \partial D_z\}} |\tilde{\psi}_{h'}(w)| \leq \sup_{\{w \in B_x \cap \partial \hat{N}_\varepsilon\}} |\phi(h', w)| \leq C$$

for some constant  $C > 0$ , where the last inequality follows from the fact that  $\phi|_{T_{\mathbb{C}} \times (M \setminus E)}$  is holomorphic, hence continuous. Thus,  $\phi|_{U \times (V \setminus E)}$  is a bounded holomorphic function. Since  $E \cap V$  is an analytic



subset of  $V$ , it follows from the Riemann extension theorem that  $\phi|_{U \times (V \setminus E)}$  has a unique extension to a holomorphic function  $\tilde{\phi}: U \times V \rightarrow B_y$ . Due to the fact that  $\phi(h', \cdot): M \rightarrow M$  is holomorphic for each fixed  $h' \in T_{\mathbb{C}}$ , we have from uniqueness of holomorphic extensions that  $\tilde{\phi}(h', \cdot) = \phi(h', \cdot): V \rightarrow B_y$  for all  $h' \in U$  so that in fact  $\tilde{\phi} = \phi|_{U \times V}$ . Thus, we see that  $\phi|_{U \times V}$  is holomorphic. Since being holomorphic is a local property, this suffices to show that  $\phi: T_{\mathbb{C}} \times M \rightarrow M$  is holomorphic, as desired.  $\square$

## 2.4 Minimal models

We consider two types of resolution of a normal isolated singularity of complex dimension two, the first being the minimal resolution.

**Definition 2.14** [38, Definition 7.1.14] A resolution  $\pi: M \rightarrow C_0$  of a normal isolated singularity  $x \in C_0$  is called a *minimal resolution* if for every resolution  $\pi': M' \rightarrow C_0$  of  $C_0$  there exists a unique morphism  $\varphi: M' \rightarrow M$  such that  $\pi'$  factors as  $\pi' = \pi \circ \varphi$ .

By definition, if there exists a minimal resolution, then it is unique up to isomorphisms over  $C_0$ . The following shows that there exists a minimal resolution for two-dimensional isolated normal singularities.

**Theorem 2.15** [38, Theorem 7.1.15] Assume that  $\dim_{\mathbb{C}} C_0 = 2$ . A resolution  $\pi: M \rightarrow C_0$  of an isolated normal singularity  $x \in C_0$  is the minimal resolution if and only if  $\pi^{-1}(\{x\})$  does not contain a  $(-1)$ -curve. In particular, there exists a minimal resolution.

We also consider “good” resolutions. We henceforth follow [56; 59].

**Definition 2.16** A resolution  $\pi: M \rightarrow C_0$  of a normal isolated surface singularity  $x \in C_0$  is called *good* if

- (i) all of the components of the exceptional divisor of  $\pi: M \rightarrow C_0$  are smooth and intersect transversely;
- (ii) not more than two components pass through any given point;
- (iii) two different components intersect at most once.

It is known that there is a unique resolution which is minimal among all good resolutions for two-dimensional isolated normal singularities [45, Theorem 5.12], which we henceforth refer to as the “minimal good resolution” (not to be confused with the “minimal resolution”). In general, the minimal resolution of a two-dimensional isolated normal singularity will not be the minimal good resolution of the singularity; by Theorem 2.15, the two coincide precisely when the minimal good resolution does not contain any  $(-1)$ -curves.

For the Kähler cones of Theorem 2.5(ii), the minimal good resolution is given by  $\pi: L \rightarrow L^\times$ , that is, by contracting the zero section of  $L$ . By adjunction, this resolution will be the canonical model of the singularity.

As for the Kähler cones of Theorem 2.5(iii), the situation is slightly more complicated. The minimal good resolution of  $L^\times/G$  is obtained as follows. A partial resolution is given by the induced map  $\pi: L/G \rightarrow L^\times/G$  between the quotient spaces. The variety  $L/G$  will only have isolated cyclic quotient singularities along the exceptional set of  $\pi$  [59, Lemma 3.5], which comprises a single curve  $C$  of genus  $g > 0$ . Each of these cyclic quotient singularities has a minimal resolution with exceptional set a string of  $\mathbb{P}^1$ 's. Resolving them with this resolution then yields the minimal good resolution of  $L^\times/G$ , the exceptional set of which will then comprise the curve  $C$  (of genus  $g > 0$ ) with branches of  $\mathbb{P}^1$ 's stemming from finitely many points of  $C$ . The original singularity is determined up to analytic isomorphism by this data which can be succinctly stored in a “weighted dual graph”. This we now explain.

The *weighted dual graph* of a good resolution is a graph each vertex of which represents a component of the exceptional divisor, weighted by self-intersection. Two vertices are connected if the corresponding components intersect.

In our case, the weighted dual graph of the minimal good resolution of  $L^\times/G$  is represented by a *star*, that is, a connected tree where at most one vertex is connected to no more than two other vertices.  $C$  itself is contained in the exceptional set of the minimal good resolution, hence one of the vertices of the star will represent  $C$ . We call  $C$  the *central curve*. The connected components of the graph minus the central curve are called the *branches* of the graph and are indexed by  $i$ , where  $1 \leq i \leq n$ . The curves of the  $i^{\text{th}}$  branch are denoted by  $C_{ij}$  for  $1 \leq j \leq r_i$ , where  $C_{i1}$  intersects  $C$  and  $C_{ij}$  intersects  $C_{i,j+1}$ . Let  $b = -C.C$  and  $b_{ij} = -C_{ij}.C_{ij}$ . Then  $b_{ij} \geq 2$  and  $b \geq 1$ . Finally, set

$$\frac{d_i}{e_i} = b_{i1} - \frac{1}{b_{i2} - \frac{1}{\ddots - \frac{1}{b_{ir_i}}}}$$

with  $e_i < d_i$  and  $e_i$  and  $d_i$  relatively prime. Then one has:

**Theorem 2.17** [59, Theorem 2.1] *The singularity  $L^\times/G$  with  $G$  nontrivial is determined up to analytic isomorphism by the following data:*

- (i) *The weighted dual graph of the minimal good resolution.*
- (ii) *The analytic type of the central curve  $C$  (of genus  $g > 0$ ).*
- (iii) *The conormal bundle of  $C$  in the resolution.*
- (iv) *The  $n$  points  $P_i = C \cap C_{i1}$  on  $C$  with  $n \geq 1$ .*

Conversely, given any data as above, there exists a unique singularity of the form  $L^\times/G$  having this data, provided that the intersection matrix given by the graph in (i) is negative-definite; this condition can be written as

$$b - \sum_{i=1}^n \frac{e_i}{d_i} > 0.$$

Indeed, the algorithm that recovers the algebraic equations cutting out  $L^\times/G$  is laid out in [59, Section 5]. However, it does not identify the group  $G$ .

For the Kähler cones of Theorem 2.5(iii), the minimal good resolution does not contain any  $(-1)$ –curves, hence it coincides with the minimal model of the singularity. Moreover, since the central curve has trivial or negative anticanonical bundle, adjunction tells us that the canonical model is obtained from the minimal good resolution by further contracting all of its  $(-2)$ –curves. However, the result of this will be singular unless the minimal good resolution does not contain any  $(-2)$ –curves. Thus, the canonical model will be smooth and coincide with the minimal good resolution if the minimal good resolution does not contain any  $(-2)$ –curves. Conversely, if the canonical model of the singularity is smooth, then, since it cannot contain any  $(-1)$ –curves, it coincides with the minimal model, which itself coincides with the minimal good resolution for the cones in question, so that the minimal good resolution does not contain any  $(-2)$ –curves since the canonical model cannot contain any  $(-2)$ –curves. Combining this observation with Theorem 2.17, we are able to characterise those cones of Theorem 2.5(iii) that admit a smooth canonical model.

**Proposition 2.18** *A Kähler cone of Theorem 2.5(iii) admits a smooth canonical model if and only if the minimal good resolution does not contain any  $(-2)$ –curves. These cones are in one-to-one correspondence with the data (i)–(iv) listed in Theorem 2.17 with the intersection matrix of the graph in (i) being negative-definite and with the labels  $b_{ij}$  of this graph being  $\geq 3$ . Moreover, the canonical model and the minimal resolution of such a cone are given by the minimal good resolution.*

## 2.5 Ricci solitons

Recall the definitions given at the beginning of Section 1.1. For a gradient Kähler–Ricci soliton  $(M, g, X)$  with complex structure  $J$ , the vector field  $JX$  is Killing by [32, Lemma 2.3.8]. We also have the following asymptotics on the soliton potential of a complete expanding gradient Ricci soliton with quadratic Ricci curvature decay.

**Proposition 2.19** *Let  $(M^n, g, \nabla^g f)$  be a complete expanding gradient Ricci soliton of real dimension  $n$ , ie  $2 \operatorname{Ric}(g) - \mathcal{L}_{\nabla^g f}(g) = -g$ . If  $|\operatorname{Ric}(g)|_g = O(d_g(p, \cdot)^{-2})$ , where  $d_g(p, \cdot)$  denotes the distance to a fixed point  $p \in M$ , then the function  $(-f)$  is equivalent to  $\frac{1}{4}d_g(p, \cdot)^2$  as  $d_g(p, \cdot)$  tends to  $+\infty$ .*

**Proof** See [12] or [62, Lemma 4.2.1]. □

Because of Proposition 2.19, we prefer to deal with an asymptotically positive soliton potential. Henceforth, an *expanding* gradient Ricci soliton will be a triple  $(M, g, X)$ , where  $X = \nabla^g f$  for some real-valued smooth function  $f$  on  $M$ , such that the equation

$$(2-1) \quad 2 \operatorname{Ric}(g) - \mathcal{L}_X g = -g$$

is satisfied. When the Ricci curvature of  $g$  decays quadratically, the bound of Proposition 2.19 on  $f$  may then be given as

$$(2-2) \quad \frac{1}{4}d_g^2(p, x) - c_1 d_g(p, x) - c_2 \leq f(x) \leq \frac{1}{4}d_g^2(p, x) + c_1 d_g(p, x) + c_2,$$

where  $p \in M$  is fixed and  $c_1$  and  $c_2$  are positive constants depending on  $p$ . In particular,  $f$  is proper under the assumption of quadratic Ricci curvature decay on the expanding soliton metric.

In the case of a shrinking gradient Ricci soliton, the quadratic growth of the soliton potential is always satisfied without further conditions on the decay of the Ricci tensor at infinity. More precisely, one has the following.

**Theorem 2.20** *Let  $(M, g, X)$  be a complete noncompact shrinking gradient Ricci soliton satisfying (1-1) with  $\lambda = 1$ , with soliton vector field  $X = \nabla^g f$  for a smooth real-valued function  $f: M \rightarrow \mathbb{R}$ . Then the following properties hold true.*

- (i) **Growth of the soliton potential** [11, Theorem 1.1] *For  $x \in M$ ,  $f$  satisfies the estimates*

$$\frac{1}{4}(d_g(p, x) - c_1)^2 - C \leq f(x) \leq \frac{1}{4}(d_g(p, x) + c_2)^2$$

*for some  $C > 0$ , where  $d_g(p, \cdot)$  denotes the distance to a fixed point  $p \in M$  with respect to  $g$ . Here,  $c_1$  and  $c_2$  are positive constants depending only on the real dimension of  $M$  and the geometry of  $g$  on the unit ball  $B_p(1)$  based at  $p$ .*

- (ii) **Polynomial volume growth** [11, Theorem 1.2] *For each  $x \in M$ , there exists a positive constant  $C > 0$  such that*

$$\text{vol}_g(B_r(x)) \leq Cr^n \quad \text{for } r > 0 \text{ sufficiently large,}$$

*where  $n = \dim_{\mathbb{R}} M$ .*

- (iii) **Regularity at infinity** *If the curvature tensor decays quadratically, ie if  $A_0(g) < +\infty$ , then the soliton metric has quadratic curvature decay with derivatives, ie  $A_k(g) < +\infty$  for all  $k \in \mathbb{N}$ .*

**Proof** References for items (i) and (ii) have been provided above. Item (iii) concerning the covariant derivatives of the curvature tensor follows from Shi's estimates for ancient solutions of the Ricci flow; see [43, Section 2.2.3] for a proof.  $\square$

**Remark 2.21** The regularity at infinity stated in Theorem 2.20(iii) does not hold for expanding gradient Ricci solitons; see [26] for examples of expanding gradient Ricci solitons coming out of metric cones with a finite amount of regularity at infinity.

The next lemma collects together some well-known Ricci soliton identities concerning expanding gradient Ricci solitons and shrinking gradient Kähler–Ricci solitons that we require. Their proofs are standard.

**Lemma 2.22** (Ricci soliton identities) *Let  $(M, g, X)$  be a shrinking gradient Kähler–Ricci soliton of complex dimension  $n$  satisfying (1-4) with  $\lambda = 1$  (resp. an expanding gradient Ricci soliton of real dimension  $n$  satisfying (2-1)) with soliton vector field  $X = \nabla^g f$  for a smooth real-valued function  $f : M \rightarrow \mathbb{R}$ . Then the trace and first-order soliton identities are:*

$$\begin{aligned} \Delta_\omega f + \frac{1}{2} R_g &= n && (\text{resp. } -\Delta_g f + R_g = -\frac{1}{2}n), \\ \nabla^g R_g - 2 \operatorname{Ric}(g)(X) &= 0 && (\text{resp. } \nabla^g R_g + 2 \operatorname{Ric}(g)(X) = 0), \\ |\nabla^g f|^2 + R_g - 2f &= \text{const.} && (\text{resp. } |\nabla^g f|^2 + R_g - f = \text{const.}), \end{aligned}$$

where  $R_g$  denotes the scalar curvature of  $g$  and  $|\nabla^g f|^2 := g^{ij} \partial_i f \partial_j f$ .

**Remark 2.23** We henceforth normalise the soliton potential  $f$  of a shrinking gradient Kähler–Ricci soliton of complex dimension  $n$  satisfying (1-4) with  $\lambda = 1$  so that  $|\nabla^g f|^2 + R_g - 2f = 2n$ . The choice of constant  $2n$  is dictated by the following equation satisfied by  $f$ :

$$\Delta_\omega f - \frac{1}{2} X \cdot f = -f.$$

This choice of constant also implies that  $f + n$  is nonnegative on  $M$ , since the scalar curvature  $R_g$  of  $g$  is necessarily nonnegative.

Kähler cones are quasiprojective. This property is inherited by complete expanding and shrinking gradient Kähler–Ricci solitons on resolutions of Kähler cones.

**Proposition 2.24** *Let  $(M, g, X)$  be a complete expanding or shrinking gradient Kähler–Ricci soliton on a resolution  $\pi : M \rightarrow C_0$  of a Kähler cone  $C_0$ . Then  $M$  is quasiprojective.*

**Proof** We prove this proposition in the case that  $(M, g, X)$  is an expanding gradient Kähler–Ricci soliton. The proof for the shrinking case is similar.

As explained in the proof of Lemma 2.12, by adding an appropriate ample divisor  $D$  to  $C_0$  at infinity, we obtain a projective compactification  $\bar{C}_0$  of  $C_0$  so that  $\bar{C}_0$  will have at worst orbifold singularities along  $D$ . Using  $D$ , we then compactify  $M$  at infinity to obtain a compact complex orbifold  $\bar{M}$  such that  $M = \bar{M} \setminus D$ . We claim that  $\bar{M}$  admits an ample line bundle, hence is projective.

Indeed, since the normal orbibundle of  $D$  in  $\bar{C}_0$  is positive, the normal orbibundle of  $D$  in  $\bar{M}$  will also be positive, hence by the proof of [21, Lemma 2.3], we may endow the line orbibundle  $[D]$  on  $\bar{M}$  with a nonnegatively curved hermitian metric with strictly positive curvature on some tubular neighbourhood  $U$  of  $D$  in  $\bar{M}$ . Next note that the curvature of the hermitian metric  $h$  induced on  $K_M$  by the expanding gradient Kähler–Ricci soliton metric  $g$  is  $-\rho_\omega$ , where  $\rho_\omega$  is the Ricci form of the Kähler form  $\omega$  associated to  $g$ . Let  $f$  denote the soliton potential so that  $X = \nabla^g f$ . Then by virtue of the expanding soliton equation, the curvature of the hermitian metric  $e^f h$  on  $K_M$  is precisely the Kähler form  $\omega$  of  $g$ . In particular, the curvature of  $e^f h$  on  $K_M$  is a positive form. Extend the hermitian metric  $e^f h$  on  $K_M$  to a hermitian

metric on  $K_{\overline{M}}$  by amalgamating  $e^f h$  with an arbitrary hermitian metric on  $K_{\overline{M}}|_U$  using an appropriate bump function supported on  $U$ . Then the line orbibundle  $K_{\overline{M}} + p[D]$  will be ample for  $p$  sufficiently large. A high tensor power of the resulting line orbibundle will then be an ample line bundle on  $\overline{M}$  so that  $\overline{M}$  is projective and  $M$  is quasiprojective, as claimed.  $\square$

Finally, note that to each complete gradient Ricci soliton, one can associate a Ricci flow that evolves via diffeomorphisms and scaling. We describe this picture for an expanding gradient Ricci soliton next.

For a complete expanding gradient Ricci soliton  $(M, g, X)$  with soliton potential  $f$ , set

$$g(t) := t\varphi_t^* g \quad \text{for } t > 0,$$

where  $\varphi_t$  is a family of diffeomorphisms generated by the gradient vector field  $-\frac{1}{t}X$  with  $\varphi_1 = \text{id}$ , ie

$$(2-3) \quad \frac{\partial \varphi_t}{\partial t}(x) = -\frac{\nabla^g f(\varphi_t(x))}{t} \quad \text{with } \varphi_1 = \text{id}.$$

Then  $\partial_t g(t) = -2 \text{Ric}(g(t))$  for  $t > 0$ ,  $g(1) = g$ , and if we define  $f(t) = \varphi_t^* f$  so that  $f(1) = f$ , then  $g(t)$  satisfies

$$(2-4) \quad \text{Ric}(g(t)) - \text{Hess}_{g(t)} f(t) + \frac{g(t)}{2t} = 0 \quad \text{for all } t > 0.$$

Taking the divergence of this equation and using the Bianchi identity yields

$$(2-5) \quad R_{g(t)} + |\nabla^{g(t)} f(t)|_{g(t)}^2 - \frac{f(t)}{t} = \frac{C_1}{t}$$

for some constant  $C_1$ , where  $R_{g(t)}$  denotes the scalar curvature of  $g(t)$ .

Similarly, for a complete expanding gradient Kähler–Ricci soliton with Kähler form  $\omega$ , one obtains a solution of the Kähler–Ricci flow  $\partial_t \omega(t) = -\rho_{\omega(t)}$ , where  $\rho_{\omega(t)}$  denotes the Ricci form of  $\omega(t)$ . The difference in normalisations between (1-2) and (1-4) is accounted for by the fact that the constant preceding the Ricci term in the Ricci flow is  $-2$  and that preceding the Ricci term in the Kähler–Ricci flow is  $-1$ . In the same way that a Kähler–Ricci flow yields a solution of the Ricci flow and, vice versa, a solution of the Ricci flow which is Kähler yields a solution of the Kähler–Ricci flow, the same holds true for gradient Ricci solitons and gradient Kähler–Ricci solitons. Indeed, a solution  $(M, g, X)$  of (1-4) yields a solution of (1-2) by replacing  $g$  with  $2g$  and composing (1-4) with the complex structure in the first arguments. Conversely, a solution  $(M, g, X)$  of (1-2) for which  $g$  is Kähler and  $X$  is real holomorphic defines a solution of (1-4) after replacing  $g$  with  $\frac{1}{2}g$  and composing (1-2) with the complex structure in the first arguments.

## 2.6 Properties of the soliton vector field

In this subsection, we provide sufficient conditions for which the zero set of the soliton vector field of a complete shrinking gradient Kähler–Ricci soliton is compact. We begin with the following simple observation.

**Lemma 2.25** *Let  $(M, g, X)$  be a complete shrinking gradient Kähler–Ricci soliton with bounded scalar curvature. Then the zero set of  $X$  is compact.*

**Proof** With  $f$  denoting the soliton potential, the boundedness of the scalar curvature  $R_g$  of  $g$  together with the properness of  $f$  as a consequence of Theorem 2.20(i) imply that  $2f - R_g$  is proper. From the soliton identity  $|\nabla^g f|^2 + R_g = 2f$  (see Lemma 2.22), we then see that the function  $|\nabla^g f|^2$  is proper. The compactness of the zero set of  $X$  is now immediate.  $\square$

In the case that  $M$  is in addition “1–convex”, meaning that  $M$  carries a plurisubharmonic exhaustion function which is strictly plurisubharmonic outside of a compact set, we can be more precise. Since a 1–convex space is in particular holomorphically convex,  $M$  in this case will admit a “Remmert reduction”  $p: M \rightarrow M'$  [36], ie a proper holomorphic map  $p: M \rightarrow M'$  onto a normal Stein space  $M'$  with finitely many isolated singularities obtained by contracting the maximal compact analytic subset  $E$  of  $M$ . As a Stein space with only finitely many isolated singularities, [3, Theorem 3.1] asserts that  $M'$  admits an embedding  $h: M' \rightarrow \mathbb{C}^P$  into  $\mathbb{C}^P$  for some  $P \in \mathbb{N}$ . We have:

**Proposition 2.26** *Let  $(M, g, X)$  be a complete shrinking gradient Kähler–Ricci soliton of complex dimension  $n$  with bounded scalar curvature. Assume that  $M$  is 1–convex with maximal compact analytic subset  $E$ . Then the zero set of  $X$  is compact and*

- (i) *if  $E = \emptyset$ , then the zero set of  $X$  comprises a single point and  $M$  is biholomorphic to  $\mathbb{C}^n$ ;*
- (ii) *if  $E \neq \emptyset$ , then the zero set of  $X$  is contained in  $E$ .*

Before we prove this proposition, an auxiliary result is required, which will be used several times throughout.

**Proposition 2.27** *Let  $(N, g)$  be a complete Riemannian manifold and let  $u: N \rightarrow \mathbb{R}$  be a  $C^2$  function that is proper and bounded below. Assume that the flow  $\phi_x(t)$  of  $\nabla^g u$  with  $\phi_x(0) = x \in N$  exists for all  $t \in (-\infty, 0]$ . Then, for any  $x \in N$ , the orbit  $(\phi_x(t))_{t \leq 0}$  accumulates in the critical set of  $u$ , ie for all sequences  $(t_i)_i$  diverging to  $-\infty$ , there exists a subsequence  $(t'_i)_i$  such that  $(\phi_{t'_i}(x))_i$  converges to a point  $x_\infty \in N$  satisfying  $(\nabla^g u)(x_\infty) = 0$ .*

**Proof** Let  $x \in N$  and let  $(\phi_x(t))_{t \leq 0}$  denote the flow of  $\nabla^g u$  which passes through  $x$  at  $t = 0$  and is defined for all nonpositive times. Since  $\partial_t \phi_x(t) = (\nabla^g u)(\phi_x(t))$ , the function  $t \in (-\infty, 0] \mapsto u(\phi_x(t)) \in \mathbb{R}$  is a nondecreasing function and for all nonpositive times  $t$ ,

$$(2-6) \quad u(x) - u(\phi_x(t)) = \int_t^0 |\nabla^g u|_g^2(\phi_x(\tau)) d\tau \geq 0.$$

In particular, the orbit  $(\phi_x(t))_{t \leq 0}$  lies in the sublevel set  $\{y \in M \mid u(y) \leq u(x)\}$  of  $u$ , which is compact since  $u$  is proper and bounded below. Moreover, since  $u(\phi_x(t))$  is bounded from below, the estimate

(2-6) implies that the function  $\tau \in (-\infty, 0] \mapsto |\nabla^g u|^2(\phi_x(\tau)) \in \mathbb{R}$  is integrable on  $(-\infty, 0]$ ; that is,

$$(2-7) \quad \int_{-\infty}^0 |\nabla^g u|^2(\phi_x(\tau)) d\tau < +\infty.$$

Now, since  $u$  is  $C^2$  and the orbit  $(\phi_x(t))_{t \leq 0}$  lies in a compact subset of  $M$ , the function  $\tau \in (-\infty, 0] \mapsto |\nabla^g u|^2(\phi_x(\tau)) \in \mathbb{R}$  is Lipschitz, ie there is a positive constant  $C$  such that

$$||\nabla^g u|^2(\phi_x(t)) - |\nabla^g u|^2(\phi_x(s))| \leq C|t - s| \quad \text{for all } s, t \in (-\infty, 0].$$

This fact, together with (2-7), implies that  $\lim_{\tau \rightarrow -\infty} |\nabla^g u|^2(\phi_x(\tau)) = 0$ . This allows us to conclude that any accumulation point of  $(\phi_x(t))_{t \leq 0}$  lies in the critical set of  $u$ .  $\square$

**Proof of Proposition 2.26** Let  $f$  denote the soliton potential and let  $M_0(X)$  denote the zero set of  $X$ , a set which is compact by Lemma 2.25. Our first claim encapsulates the structure of  $M_0(X)$ .

**Claim 2.28** *Each connected component of  $M_0(X)$  is a smooth compact complex submanifold of  $M$  contained in a level set of  $f$ .*

**Proof** Let  $J$  denote the complex structure of  $M$  and let  $F$  be a connected component of  $M_0(X)$ . Then since  $F$  is locally the zero set of the holomorphic vector field  $X^{1,0} = \frac{1}{2}(X - iJX)$ , it is a complex-analytic subvariety of  $M$ . Furthermore, as a connected component of the zero set of the Killing vector field  $JX$ , it is a totally geodesic submanifold by [40, Theorem 5.3, page 60]. Hence  $F$  is a smooth complex submanifold of  $M$ .

Next observe that along any geodesic  $\gamma(t)$  in  $F$ , we have for the soliton potential  $f$ ,

$$\frac{d}{dt} f(\gamma(t)) = df(\gamma'(t)) = g(X, \gamma'(t)) = 0,$$

so that  $f$  is constant on  $F$ . Consequently,  $F$  is contained in a level set of  $f$ . From Theorem 2.20(i), we know that  $f$  is proper so that the level sets of  $f$  are compact. Thus, as a closed subset of a compact set,  $F$  is compact.  $\triangleleft$

Now note that, by [31, Proof of Lemma 1],  $f$  is a Morse–Bott function on  $M$ . The critical submanifolds of  $f$  are precisely the connected components of  $M_0(X)$ . Since  $M$  is Kähler, the Morse indices — ie the number of negative eigenvalues of  $\text{Hess}(f)$  — of the critical submanifolds are all even [31]. Write

$$M_0(X) = M^{(0)} \cup \bigcup_{k=1}^n M^{(2k)},$$

where  $M^{(j)}$  denotes the disjoint union of the critical submanifolds of  $M_0(X)$  of index  $j$ . As a consequence of Claim 2.28, we see that each connected component of  $M_0(X)$ , being a compact complex submanifold of  $M$ , is either contained in the maximal compact analytic subset  $E$  of  $M$  or is an isolated point contained in  $M \setminus E$ . Suppose that there exists an isolated point  $x \in M^{(j)} \cap (M \setminus E)$  for some  $j \geq 2$ .



Using ideas from [13, page 3332] in this paragraph, we see from [10] that the holomorphic vector field  $X^{1,0}$  is linearisable at each of its critical points, meaning in particular that there exist local holomorphic coordinates  $(z_1, \dots, z_n)$  centred at  $x$  such that  $X^{1,0} = \sum_{j=1}^n a_j z_j \partial_{z_j}$  with  $a_j \in \mathbb{R}$  for all  $j = 1, \dots, n$ . Since  $\text{Hess}(f)$  has at least one negative eigenvalue at  $x$ , we have that  $a_i < 0$  for some  $i$ . Without loss of generality, we may assume that  $i = n$  so that  $a_n < 0$ . Now, clearly the orbits of  $JX$  on the  $z_n$ -axis are all periodic. Fix one such orbit  $\theta: S^1 \rightarrow M$ . Then we can construct a map  $R: S^1 (\simeq \mathbb{R}/T\mathbb{Z}) \times \mathbb{R} \rightarrow M$  by defining  $R(s, t)$  to be  $\phi_t(\theta(s))$ , where  $\phi_t$  is the integral curve of the negative gradient flow of  $f$  and  $T$  is the period of the orbit of  $\theta$ . Since  $[X, JX] = 0$ ,  $R$  is holomorphic and after reparametrising, extends to a nontrivial holomorphic map  $\bar{R}: \mathbb{C} \rightarrow M$  with  $\bar{R}(0) = x$  by the Riemann removable singularity theorem.

Since  $f$  is decreasing along its negative gradient flow and is bounded from below, we see that  $f(\bar{R}(z))$  is bounded for all  $z \in \mathbb{C}$ . Hence, by properness of  $f$ , the set  $\{\bar{R}(z) \mid z \in \mathbb{C}\}$  is contained in a compact subset of  $M$ . Letting  $p: M \rightarrow M'$  denote the Remmert reduction of  $M$  and recalling that  $M'$  admits an embedding  $h: M' \rightarrow \mathbb{C}^P$  into  $\mathbb{C}^P$  for some  $P \in \mathbb{N}$ , we therefore obtain a bounded nontrivial holomorphic map  $h \circ p \circ \bar{R}: \mathbb{C} \rightarrow \mathbb{C}^P$ . By Liouville’s theorem, such a map is constant. This is a contradiction. Thus  $M_0(X) \cap (M \setminus E)$ , if nonempty, is contained in  $M^{(0)}$ .

The next claim concerns the structure of  $M^{(0)}$ .

**Claim 2.29**  $M^{(0)}$  is a nonempty, connected, compact complex submanifold of  $M$  that comprises the global minima of  $f$ .

**Proof**  $M^{(0)}$  is clearly nonempty since  $f$  attains a global minimum and, as a closed subset of the compact set  $M_0(X)$ , comprises finitely many connected, compact, complex submanifolds of  $M$  by Claim 2.28. To see that  $M^{(0)}$  comprises one connected component only, recall that the soliton vector field  $X$  is complete. Then by Proposition 2.27, for any point  $x \in M$ , the forward orbit of the negative gradient flow of  $f$  beginning at  $x$  converges to a point of  $M_0(X)$ . This gives rise to a stratification of  $M$ , namely  $M = \bigsqcup_{k=0}^n W^s(M^{(2k)})$ , where

$$W^s(M^{(2k)}) = \{x \in M \mid \lim_{t \rightarrow -\infty} \phi_x(t) \in M^{(2k)}\},$$

$\phi_x: \mathbb{R} \rightarrow M$  here denoting the gradient flow of  $f$  beginning at  $x$ . Note that

$$W^s(M^{(0)}) = M \setminus \bigsqcup_{k=1}^n W^s(M^{(2k)}).$$

Now, since  $M_0(X)$  is compact, for each  $k$ ,  $W^s(M^{(2k)})$  comprises finitely many connected components, each of which is an open submanifold of  $M$  of real dimension  $2n - 2k$ ; see [1, Proposition 3.2]. The complement of finitely many submanifolds of real codimension at least two in a connected manifold is still connected. Hence  $W^s(M^{(0)})$ , and consequently  $M^{(0)}$ , is connected. Finally, since  $M^{(0)}$  contains all of the local minima of  $f$  and, comprising only one connected component, is contained in a level set of  $f$  by Claim 2.28, it must be the set of global minima of  $f$ . ◁

Now, we have already established the fact that  $M_0(X) \cap (M \setminus E)$ , if nonempty, is contained in  $M^{(0)}$ . Thus, if  $E = \emptyset$ , then, since  $M_0(X)$  is nonempty as  $f$  attains a global minimum, we must have that  $M_0(X) = M^{(0)}$ , so that  $M_0(X)$  is a nonempty, connected, compact complex submanifold of  $M$  by Claim 2.29. Since  $M$  is affine if  $E = \emptyset$ , we deduce that  $M_0(X)$  must comprise a single point if  $E = \emptyset$ . It then follows from [10] that  $M$  is biholomorphic to  $\mathbb{C}^n$  if  $E = \emptyset$ . This is case (i) of the proposition.

Next consider the case when  $E \neq \emptyset$ . If  $M_0(X) \cap (M \setminus E) = \emptyset$ , then  $M_0(X) \subseteq E$  and we are in case (ii) of the proposition. So, to derive a contradiction, suppose that  $E \neq \emptyset$  and  $M_0(X) \cap (M \setminus E) \neq \emptyset$ . In light of the above, we must have that  $M^{(0)} \cap E = \emptyset$  and that  $M_0(X) \cap (M \setminus E) = M^{(0)}$ , which comprises a single point  $x$ , say. Moreover,  $(\bigcup_{j=1}^n M^{(2j)}) \cap E \neq \emptyset$  since otherwise  $M$  would be biholomorphic to  $\mathbb{C}^n$  by [10], thereby yielding a contradiction. Thus, noting that  $f(M^{(0)})$  is the global minimum value of  $f$  by Claim 2.29, let  $A$  be the smallest critical value of  $f$  with  $A > f(M^{(0)})$  and let  $y \in f^{-1}(\{A\})$ . Then we must have that  $y \in M^{(k)} \subseteq E$  for some  $k \geq 2$  by what we have just said. As before, we can construct a holomorphic map  $\bar{R}: \mathbb{C} \rightarrow M$  with  $\bar{R}(0) = y$ . Since  $f$  is decreasing along its negative gradient flow and since there are no critical values of  $f$  in the open interval  $(f(M^{(0)}), A)$ , we see from Proposition 2.27 that necessarily  $\lim_{z \rightarrow +\infty} \bar{R}(z) = x$ . The Riemann removable singularity theorem then applies and allows us to extend  $\bar{R}$  to a holomorphic map  $\bar{R}': \mathbb{P}^1 \rightarrow M$ . Since  $x \neq y$  and  $x \notin E$ , what we have constructed is a nontrivial holomorphic curve in  $M$  that is not contained in  $E$ . This contradicts the maximality of  $E$ . Thus, cases (i) and (ii) of the proposition are the only two possibilities that can occur. This completes the proof.  $\square$

## 2.7 Properties of real vector fields commuting with the soliton vector field

In this subsection, we mention some properties of real vector fields that commute with the soliton vector field on a complete shrinking gradient Kähler–Ricci soliton. As the next proposition demonstrates, a bound on the Ricci curvature yields control on the growth of the norm of these vector fields.

**Proposition 2.30** *Let  $(M, g, X)$  be a complete shrinking gradient Kähler–Ricci soliton with bounded Ricci curvature, and let  $d_g(p, \cdot)$  denote the distance to a fixed point  $p \in M$  with respect to  $g$ . Then there exists  $a > 0$  such that  $|Y|_g^2(x) = O(d_g(p, x)^a)$  for every real vector field  $Y$  on  $M$  with  $[X, Y] = 0$ .*

**Remark 2.31** The growth rate obtained in Proposition 2.30 will be sharpened in Claim A.14 for real holomorphic vector fields commuting with  $X$ , as exemplified by shrinking cylinders of the form  $\mathbb{C}^{n-k} \times N^{2k}$ , where  $\mathbb{C}^{n-k}$  is endowed with its Gaussian soliton metric and where  $N^{2k}$  supports a closed Kähler–Einstein metric of positive scalar curvature.

**Proof** Let  $|\cdot|$  denote the norm with respect to  $g$  and let  $\text{Ric}$  denote the Ricci curvature of  $g$ . Since  $|\text{Ric}|$  is bounded so that the scalar curvature of  $g$  is bounded, it follows from Lemma 2.25 that the zero set of  $X$  is contained in a compact subset of  $M$ . For  $A > 0$ , let  $K := f^{-1}([2A, 4A])$  and  $N = f^{-1}((-\infty, 3A])$ .

Since  $f$  is proper and bounded below as a consequence of Theorem 2.20(i),  $K$  and  $N$  are compact subsets of  $M$ . Choose  $A$  sufficiently large so that all of the critical points of  $f$  are contained in  $f^{-1}((-\infty, A])$  and so that  $A > \sup_M |R_g|$ . Let  $\gamma_x(t)$  denote the integral curve of  $X$  with  $\gamma_x(0) = x \in M$ . We begin with the following claim.

**Claim 2.32** *Let  $y \in M \setminus N$ . Then there exists  $x \in K$  and  $t_0 > 0$  such that  $y = \gamma_x(t_0)$ .*

That is to say, every point of  $M \setminus N$  lies on an integral curve of  $X$  passing through  $K$ .

**Proof** For  $y \in M \setminus K$ , we see from the soliton identity  $|\nabla^g f|^2 + R_g = 2f$  that

$$\frac{d}{dt} f(\gamma_y(t)) = |\nabla^g f|_g^2(\gamma_y(t)) = 2f(\gamma_y(t)) - R_g(\gamma_y(t)).$$

Using the upper bound on  $|R_g|$ , we deduce that

$$\left| \frac{d}{dt} f(\gamma_y(t)) - 2f(\gamma_y(t)) \right| \leq 2A.$$

Integrating this differential inequality for  $t < 0$  then yields the inequalities

$$(2-8) \quad (f(y) + A)e^{2t} - A \leq f(\gamma_y(t)) \leq (f(y) - A)e^{2t} + A \quad \text{for } t < 0.$$

Set

$$t_0 = -\frac{1}{2} \ln\left(\frac{3A}{f(y) + A}\right) > 0.$$

Then from (2-8) we see that

$$2A \leq f(\gamma_y(-t_0)) \leq 3A\left(\frac{f(y) - A}{f(y) + A}\right) + A = 3A\left(1 - \frac{2A}{f(y) + A}\right) + A \leq 4A.$$

Thus,  $y = \gamma_x(t_0)$  where  $x = \gamma_y(-t_0) \in K$ . This proves the claim. ◁

Next observe that

$$\mathcal{L}_X(|Y|^2) = (\mathcal{L}_X g)(Y, Y) = g(Y, Y) - \text{Ric}(Y, Y) = |Y|^2 - \text{Ric}(Y, Y).$$

For  $x \in M$  a point where  $X \neq 0$ , let  $h(t) := |Y|^2(\gamma_x(t))$ . Then we can rewrite the previous equation as

$$h'(t) = h(t) - \text{Ric}(Y, Y)(\gamma_x(t)),$$

so that

$$(2-9) \quad \frac{h'(t)}{h(t)} = 1 - \frac{\text{Ric}(Y, Y)(\gamma_x(t))}{h(t)}.$$

Analysing the error term here, we have that

$$\frac{\text{Ric}(Y, Y)(\gamma_x(t))}{h(t)} = \frac{\text{Ric}(Y, Y)}{|Y|^2}.$$

Since  $|\text{Ric}|$  is bounded by assumption, we then have that

$$\left| \frac{\text{Ric}(Y, Y)}{|Y|^2} \right| \leq C$$

for a constant  $C > 0$ , so that (2-9) gives us the bound

$$\left| \frac{h'(t)}{h(t)} \right| \leq 2a$$

for some  $a > 0$ . Solving this for  $t > 0$  yields

$$-2at \leq \ln(h(t)) - \ln(h(0)) \leq 2at,$$

so that, in particular,

$$h(t) \leq |Y|^2(x)e^{2at} \quad \text{for all } t > 0.$$

Hence,

$$(2-10) \quad |Y|^2(\gamma_x(t)) \leq |Y|^2(x)e^{2at} \quad \text{for all } t > 0.$$

Let  $y \in M \setminus N$ . Then by Claim 2.32, there is an  $x \in K$  and  $t_0 > 0$  such that  $y = \gamma_x(t_0)$ . Applying the above inequality to this choice of  $x$  and  $t_0$ , we deduce that

$$|Y|^2(y) \leq |Y|^2(x)e^{2at_0}.$$

Now, as in the proof of Claim 2.32, we have that

$$\left| \frac{d}{dt} f(\gamma_x(t)) - 2f(\gamma_x(t)) \right| \leq 2A.$$

Integrating this for  $t > 0$  yields the fact that

$$(f(x) - A)e^{2t} + A \leq f(\gamma_x(t)) \leq (f(x) + A)e^{2t} - A \quad \text{for all } t > 0.$$

Since  $x \in K$  so that  $f(x) \geq 2A$ , we see from the left-hand side of this inequality that  $f(\gamma_x(t)) \geq A(1 + e^{2t})$ , so that

$$(2-11) \quad e^{2t} \leq \frac{f(\gamma_x(t))}{A} - 1 \quad \text{for all } t > 0.$$

Plugging this into (2-10) and setting  $t = t_0$  results in the bound

$$|Y|^2(y) \leq |Y|^2(x)e^{2at_0} \leq |Y|^2(x) \left( \frac{f(y)}{A} - 1 \right)^a \leq (\sup_K |Y|^2) \left( \frac{f(y)}{A} - 1 \right)^a.$$

Since  $K$  is compact and  $f$  grows quadratically with respect to the distance to a fixed point  $p \in M$  by Theorem 2.20(i), we arrive at the estimate

$$|Y|(z) \leq c_1 d_g(p, z)^a + c_2 \quad \text{for all } z \in M$$

for some positive constants  $c_1, c_2 > 0$ . This leads to the desired conclusion. □

**Remark 2.33** In the case that the Ricci curvature decays quadratically at infinity, the constant  $a$  may be taken to be equal to 2 in Proposition 2.30.

We can also show that such vector fields are complete when the zero set of the soliton vector field is compact.

**Lemma 2.34** *Let  $(M, g, X)$  be a complete shrinking gradient Kähler–Ricci soliton. Assume that the zero set of  $X$  is compact (which, by Lemma 2.25, is the case when the scalar curvature of  $g$  is bounded). Then every real vector field  $Y$  on  $M$  with  $[X, Y] = 0$  is complete.*

**Proof** Let  $Y$  be as in the statement of the lemma and let  $K$  be any compact subset of  $M$  containing the zero set of  $X$  in its interior. Then note the following:

- Since  $K$  is compact, there exists  $\varepsilon_0 > 0$  such that the flow of  $Y$  beginning at any point of  $K$  exists on the open interval  $(-\varepsilon_0, \varepsilon_0)$ .
- By Proposition 2.27, for any  $p \in M$ , there exists  $T(p) > 0$  such that the image of  $p$  under the forward flow of  $-X$  for time  $T$  will be contained in  $K$ .

Consequently, for any point  $p \in M$ , by flowing first along  $-X$  into  $K$  for time  $T(p)$ , then flowing along  $Y$ , then flowing along  $X$  for time  $T$ , one sees from the fact that  $[X, Y] = 0$  that the flow of  $Y$  beginning at any point of  $M$  exists on the interval  $(-\varepsilon_0, \varepsilon_0)$ . This observation suffices to prove the completeness of  $Y$ .  $\square$

## 2.8 Basics of metric measure spaces

We take the following from [33]; the notions introduced in this section will be used in Section 5.1.

A smooth metric measure space is a Riemannian manifold endowed with a weighted volume.

**Definition 2.35** *A smooth metric measure space is a triple  $(M, g, e^{-f} dV_g)$ , where  $(M, g)$  is a complete Riemannian manifold with Riemannian metric  $g$ ,  $dV_g$  is the volume form associated to  $g$ , and  $f : M \rightarrow \mathbb{R}$  is a smooth real-valued function.*

A shrinking gradient Ricci soliton  $(M, g, X)$  with  $X = \nabla^g f$  naturally defines a smooth metric measure space  $(M, g, e^{-f} dV_g)$ . On such a space, we define the weighted Laplacian  $\Delta_f$  by

$$\Delta_f u := \Delta u - g(\nabla^g f, \nabla u)$$

on smooth real-valued functions  $u \in C^\infty(M, \mathbb{R})$ . There is a natural  $L^2$ -inner product  $\langle \cdot, \cdot \rangle_{L_f^2}$  on the space  $L_f^2$  of square-integrable smooth real-valued functions on  $M$  with respect to the measure  $e^{-f} dV_g$ , defined by

$$\langle u, v \rangle_{L_f^2} := \int_M uv e^{-f} dV_g \quad \text{for } u, v \in L_f^2.$$

As one can easily verify, the operator  $\Delta_f$  is self-adjoint with respect to  $\langle \cdot, \cdot \rangle_{L_f^2}$ .

In the Kähler case, we have:

**Definition 2.36** *If  $(M, g, e^{-f} dV_g)$  is a smooth metric measure space and  $(M, g)$  is Kähler, we say that  $(M, g, e^{-f} dV_g)$  is a Kähler metric measure space.*

A shrinking gradient Kähler–Ricci soliton naturally defines such a space.

Unlike the real case, on a Kähler metric measure space we have the weighted  $\bar{\partial}$ -Laplacian  $\Delta_f$  defined on smooth complex-valued functions  $u \in C^\infty(M, \mathbb{C})$  by

$$\Delta_f u := \Delta_{\bar{g}} u - (\nabla^{1,0} u) f = g^{i\bar{j}} \bar{\partial}_{i\bar{j}}^2 u - g^{i\bar{j}} (\partial_i f) (\partial_{\bar{j}} u).$$

This may be a complex-valued function even if  $u$  is real-valued. We define a hermitian inner product on the space  $C_0^\infty(M, \mathbb{C})$  by

$$\langle u, v \rangle_{L_f^2} := \int_M u \bar{v} e^{-f} dV_g \quad \text{for } u, v \in C_0^\infty(M, \mathbb{C}).$$

Then  $\Delta_f$  is symmetric with respect to this inner product. In fact, we have that

$$\int_M (\Delta_f u) \bar{v} e^{-f} dV_g = \int_M u \overline{\Delta_f v} e^{-f} dV_g = - \int_M g(\bar{\partial} u, \bar{\partial} v) e^{-f} dV_g = - \langle \bar{\partial} u, \bar{\partial} v \rangle_{L_f^2},$$

where

$$g(\bar{\partial} u, \bar{\partial} v) = g^{i\bar{j}} (\partial_{\bar{j}} u) (\partial_i \bar{v}).$$

See [33] for further details.

### 3 Proof of Theorem A

We first consider Theorem A in the expanding case.

#### 3.1 Construction of a map to the tangent cone

By a result of Siepmann [62, Theorem 4.3.1], a complete expanding gradient Ricci soliton  $(M, g, X)$  with quadratic curvature decay with derivatives has a unique tangent cone along each end. We first prove a series of lemmas before providing a refinement of Siepmann’s result in Theorem 3.8 by using the flow of the soliton vector field  $X$  to construct a diffeomorphism between each end of the expanding Ricci soliton and its tangent cone  $(C_0, g_0)$  along that end, with respect to which  $r \partial_r$  pushes forward to  $2X$  (here  $r$  denotes the radial coordinate of  $g_0$ ), and with respect to which  $g - g_0 - \text{Ric}(g_0) = O(r^{-4})$  with derivatives.

Our set-up in this section is as follows:

$(M, g, X)$  is a complete expanding gradient Ricci soliton with soliton vector field  $X = \nabla^g f$  for a smooth real-valued function  $f: M \rightarrow \mathbb{R}$  such that for some point  $p \in M$  and all  $k \in \mathbb{N}_0$ ,

$$A_k(g) := \sup_{x \in M} |(\nabla^g)^k \text{Rm}(g)|_g(x) d_g(p, x)^{2+k} < \infty,$$

where  $\text{Rm}(g)$  denotes the curvature of  $g$  and  $d_g(p, x)$  denotes the distance between  $p$  and  $x$  with respect to  $g$ .

The diffeomorphisms  $(\varphi_t)_{t \in (0,1]}$  will be as in (2-3). We begin with:

**Lemma 3.1** *The one-parameter family of functions  $(tf \circ \varphi_t)_{t>0}$  converges to a nonnegative continuous real-valued function  $q(x) := \lim_{t \rightarrow 0^+} tf(\varphi_t(x))$  on  $M$  as  $t \rightarrow 0^+$ .*

**Proof** Since

$$\frac{\partial}{\partial t} f(\varphi_t(x)) = -\frac{|\nabla^g f|_g^2(\varphi_t(x))}{t},$$

so that

$$\frac{\partial}{\partial t} (tf(\varphi_t(x))) = (f - |\nabla^g f|_g^2)(\varphi_t(x)) = R_g(\varphi_t(x)),$$

by the soliton identities for expanding gradient Ricci solitons (see Lemma 2.22), where  $R_g$  denotes the scalar curvature of  $g$ , we see after integrating that

$$(3-1) \quad tf(\varphi_t(x)) - sf(\varphi_s(x)) = \int_s^t R_g(\varphi_\tau(x)) d\tau \quad \text{for } 0 < s \leq t.$$

Since  $R_g$  is bounded on  $M$ , it follows that for all  $x \in M$ ,

$$(3-2) \quad |tf(\varphi_t(x)) - sf(\varphi_s(x))| \leq C(t - s) \quad \text{for } 0 < s \leq t$$

for some positive constant  $C$ . Thus,  $\{t(f \circ \varphi_t)\}_{t \in (0,1]}$  is a Cauchy sequence in  $C^0(M)$  and hence converges uniformly as  $t \rightarrow 0^+$  to a continuous real-valued function  $q$  on  $M$  as in the statement of the lemma.

To see that  $q(x) \geq 0$  for all  $x \in M$ , note from the soliton identities that

$$tf(\varphi_t(x)) = t|X|^2(\varphi_t(x)) + tR_g(\varphi_t(x)) \geq tR_g(\varphi_t(x)) \geq t \inf_M R_g$$

since  $t \in (0, 1]$  and  $R_g$  is bounded from below. Letting  $t \rightarrow 0^+$  in this inequality yields the desired conclusion. □

For  $a > 0$ , set

$$M_a := \{x \in M \mid \lim_{t \rightarrow 0^+} tf(\varphi_t(x)) > a\}, \quad M_0 := \{x \in M \mid \lim_{t \rightarrow 0^+} tf(\varphi_t(x)) > 0\}.$$

In view of (3-2) and the consequences discussed thereafter, the sets  $M_0$  and  $M_a$  are well-defined for  $a > 0$ . Furthermore, note that

- $M_a$  and  $M_0$  are preserved by  $\varphi_t$  for all  $t \in (0, 1]$  and  $a > 0$ , since  $\varphi_{ts} = \varphi_t \circ \varphi_s$  for all positive times  $s$  and  $t$ .

Hence for any  $a \geq 0$ ,  $t(f \circ \varphi_t)$  defines a family of smooth functions  $t(f \circ \varphi_t): M_a \rightarrow \mathbb{R}$  for  $t \in (0, 1]$ .

**Lemma 3.2** *As  $t \rightarrow 0^+$ ,  $t(f \circ \varphi_t)$  converges to  $q$  in  $C_{\text{loc}}^\infty(M_0)$ . In particular,  $q$  is smooth on  $M_0$ .*

**Proof** For each  $a > 0$ , [62, Lemma 4.3.3] ensures that along the Ricci flow  $g(t)$  defined by  $g$ , the norm of the curvature tensor  $\text{Rm}_{g(t)}$  of  $g(t)$  is bounded with respect to  $g(t)$  when restricted to  $M_a$ . In particular, there exists a positive constant  $C$  (depending on  $a$ ) such that for all  $t \in (0, 1]$ ,

$$\sup_{x \in M_a} |\text{Ric}(g(t))|_{g(t)}(x) \leq C.$$

By definition of a Ricci flow, this implies that the metrics  $(g(t))_{t \in (0,1]}$  are uniformly equivalent, ie there exists a positive constant  $C$  (that may vary from line to line) such that

$$(3-3) \quad C^{-1}g(x) \leq g(t)(x) \leq Cg(x) \quad \text{for all } t \in (0, 1], x \in M_a.$$

An induction argument (see [62, Lemma 4.3.6]) then shows that for all  $x \in M_a, t \in (0, 1]$  and  $k \geq 0$ ,

$$|(\nabla^g)^k(g(t))|_g(x) \leq C(k, a).$$

Similarly, one obtains that

$$(3-4) \quad |(\nabla^g)^k(\text{Rm}_{g(t)})|_g(x) \leq C(k, a)$$

for all  $x \in M_a, t \in (0, 1]$  and  $k \geq 0$ . As a consequence,

$$(3-5) \quad |(\nabla^g)^k(tf(\varphi_t(x)))|_g \leq C(k, a)$$

for all  $x \in M_a, t \in (0, 1]$  and  $k \geq 0$ . Indeed, by (3-1) with  $t = 1$  and  $s = t$ ,

$$f(x) - tf(\varphi_t(x)) = \int_t^1 R_g(\varphi_s(x)) ds = \int_t^1 sR_{g(s)}(x) ds \quad \text{for all } t \in (0, 1], x \in M.$$

In particular, by deriving  $k$  times at a point  $x \in M_a$ , we see that

$$(\nabla^g)^k(t\varphi_t^* f) = (\nabla^g)^k f - \int_t^1 s(\nabla^g)^k R_{g(s)} ds \quad \text{for all } t \in (0, 1],$$

which implies the desired inequality (3-5) after invoking (3-4). As a result,  $t(f \circ \varphi_t)$  converges in  $C_{\text{loc}}^\infty(M_0)$  as  $t \rightarrow 0^+$ , so that  $q$  is smooth on  $M_0$ , as claimed.  $\square$

Since  $\varphi_t$  preserves  $M_a$  for every  $a > 0$ , we also have that for any  $a \geq 0$ , the Ricci flow  $g(t)$  determined by  $g$ , namely  $g(t) := t\varphi_t^* g$ , defines a family of smooth metrics on  $M_a$  for all  $t \in (0, 1]$ . This family converges in  $C_{\text{loc}}^\infty(M_0)$  as  $t \rightarrow 0^+$  as well.

**Lemma 3.3** *The family of metrics  $g(t)$  converges to a Riemannian metric  $\tilde{g}_0$  in  $C_{\text{loc}}^\infty(M_0)$  as  $t \rightarrow 0^+$ . Moreover,  $\tilde{g}_0 = 2 \text{Hess}_{\tilde{g}_0} q$ .*

**Proof** From the definition of the Ricci flow, one deduces from the curvature bounds (3-4) that  $g(t)$  is a Cauchy sequence in  $C^k(M_a)$  for every  $k \geq 0$  and  $a > 0$ , hence converges uniformly locally as  $t \rightarrow 0^+$  in  $C^k(M_0)$  for every  $k \geq 0$  to a Riemannian metric  $\tilde{g}_0$  on  $M_0$ . To see that  $\tilde{g}_0 = 2 \text{Hess}_{\tilde{g}_0} q$ , multiply (2-4) across by  $t$  and take the limit as  $t \rightarrow 0^+$ , recalling that  $\lim_{t \rightarrow 0^+} tf(t) = q$  in  $C_{\text{loc}}^\infty(M_0)$  by Lemma 3.2.  $\square$

We have the following properties of  $q$ .

**Lemma 3.4** *The function  $q$  is proper and bounded below.*

**Proof** Lemma 3.1 already implies that  $q$  is nonnegative. In particular, it is bounded from below. Now, one sees from the quadratic growth of the soliton potential  $f$  given by (2-2) that for  $p$  in the critical set of the soliton potential  $f$ ,

$$\frac{1}{4}d_{g(t)}^2(p, x) - c_1\sqrt{t}d_{g(t)}(p, x) - c_2t \leq tf(\varphi_t(x)) \leq \frac{1}{4}d_{g(t)}^2(p, x) + c_1\sqrt{t}d_{g(t)}(p, x) + c_2t$$



for all  $t > 0$  and  $x \in M$ , for some constants  $c_1, c_2 > 0$ , where  $d_{g(t)}$  denotes the distance with respect to  $g(t)$ . Using this inequality and taking the limit as  $s \rightarrow 0^+$  in (3-2), one finds that

$$(3-6) \quad \frac{1}{4}d_{g(t)}^2(p, x) - c_1\sqrt{t}d_{g(t)}(p, x) - c_2t \leq q(x) \leq \frac{1}{4}d_{g(t)}^2(p, x) + c_1\sqrt{t}d_{g(t)}(p, x) + c_2t$$

for all  $t > 0$  and  $x \in M$ , for some constants  $c_1, c_2 > 0$  that may now vary from line to line. Thus,  $q(x) \rightarrow +\infty$  as  $x \rightarrow \infty$  and the result follows.  $\square$

Moreover, we have:

**Lemma 3.5** *On  $M_0$ ,*

- (i)  $|\nabla^{\tilde{g}_0} q|_{\tilde{g}_0}^2 = q$ , so that the integral curves of  $\nabla^{\tilde{g}_0}(2\sqrt{q})$  on  $M_0$  are geodesics, and
- (ii)  $\nabla^{\tilde{g}_0} q = X$ .

*In particular,  $X$  is nowhere vanishing on  $M_0$ .*

**Proof** To prove the first part of (i), namely that  $|\nabla^{\tilde{g}_0} q|_{\tilde{g}_0}^2 = q$ , we multiply equation (2-5) across by  $t^2$  and take the limit as  $t \rightarrow 0^+$ . Next, let  $\gamma_x(t)$  denote the integral curve of  $\nabla^{\tilde{g}_0}(2\sqrt{q})$  with  $\gamma_x(0) = x \in M_0$ . Then  $\gamma_x(t)$  remains in  $M_0$  for  $t \geq 0$ . Indeed, so long as  $t \geq 0$  is such that  $\gamma_x(t)$  lies in  $M_0$ , we have that

$$\partial_t(2\sqrt{q})(\gamma_x(t)) = \tilde{g}_0(\nabla^{\tilde{g}_0}(2\sqrt{q}), \nabla^{\tilde{g}_0}(2\sqrt{q}))(\gamma_x(t)) = |\nabla^{\tilde{g}_0}(2\sqrt{q})|_{\tilde{g}_0}^2(\gamma_x(t)) = 1.$$

After integrating, we deduce that  $(2\sqrt{q})(\gamma_x(t)) > 0$  for  $t \geq 0$ , as desired. To see that these integral curves are in fact geodesics with respect to the metric  $\tilde{g}_0$ , we compute: for any  $t \geq 0$  and for any tangent vector  $v$  to  $M_0$  at  $\gamma_x(t)$ , we have that

$$(3-7) \quad \tilde{g}_0(\nabla_{\dot{\gamma}_x(t)}^{\tilde{g}_0} \dot{\gamma}_x(t), v) = \tilde{g}_0(\nabla_{\nabla^{\tilde{g}_0}(2\sqrt{q})}^{\tilde{g}_0} \nabla^{\tilde{g}_0}(2\sqrt{q}), v)|_{\gamma_x(t)} = \text{Hess}_{\tilde{g}_0}(2\sqrt{q})(\nabla^{\tilde{g}_0}(2\sqrt{q}), v)|_{\gamma_x(t)}.$$

Since  $\tilde{g}_0 = 2 \text{Hess}_{\tilde{g}_0} q$  on  $M_0$  by Lemma 3.3, we also have the identity

$$\frac{1}{2}\tilde{g}_0 = \text{Hess}_{\tilde{g}_0} q = \text{Hess}_{\tilde{g}_0}(\sqrt{q})^2 = (2\sqrt{q}) \text{Hess}_{\tilde{g}_0}(\sqrt{q}) + 2d(\sqrt{q}) \otimes d(\sqrt{q}).$$

Plugging this into (3-7) then leads to the following sequence of equalities on  $M_0$ :

$$\begin{aligned} (2\sqrt{q})(\gamma_x(t)) \cdot \tilde{g}_0(\nabla_{\dot{\gamma}_x(t)}^{\tilde{g}_0} \dot{\gamma}_x(t), v) &= \tilde{g}_0(\nabla^{\tilde{g}_0}(2\sqrt{q}), v)|_{\gamma_x(t)} - 4(d(\sqrt{q}) \otimes d(\sqrt{q}))(\nabla^{\tilde{g}_0}(2\sqrt{q}), v)|_{\gamma_x(t)} \\ &= \tilde{g}_0(\nabla^{\tilde{g}_0}(2\sqrt{q}), v)|_{\gamma_x(t)} - |\nabla^{\tilde{g}_0}(2\sqrt{q})|_{\tilde{g}_0}^2(\gamma_x(t)) \cdot \tilde{g}_0(\nabla^{\tilde{g}_0}(2\sqrt{q}), v)|_{\gamma_x(t)} = 0, \end{aligned}$$

where we have used in the last line the already established fact that  $|\nabla^{\tilde{g}_0}(2\sqrt{q})|_{\tilde{g}_0}^2 = 1$  on  $M_0$ .

As for part (ii), let  $\varphi_t$  be as in (2-3). Then on compact subsets of  $M_0$ , we have that

$$\begin{aligned} \nabla^{\tilde{g}_0} q &= \lim_{t \rightarrow 0^+} \nabla^{g(t)}(tf(t)) = \lim_{t \rightarrow 0^+} \nabla^{t\varphi_t^* g}(t\varphi_t^* f) \\ &= \lim_{t \rightarrow 0^+} \frac{1}{t} \nabla^{\varphi_t^* g}(t\varphi_t^* f) = \lim_{t \rightarrow 0^+} \nabla^{\varphi_t^* g}(\varphi_t^* f) \\ &= \lim_{t \rightarrow 0^+} \varphi_t^*(\nabla^g f) = \lim_{t \rightarrow 0^+} \varphi_t^* X = \lim_{t \rightarrow 0^+} X = X, \end{aligned}$$

where the penultimate equality follows from the fact that  $\varphi_t$  is generated by the flow of  $-(1/t)X$  and  $\mathcal{L}_X X = 0$ . □

The above observations then imply:

**Lemma 3.6**  *$M$  has only finitely many ends.*

**Proof** For any  $a > 0$ ,  $q$  is a smooth function on  $M_a$  by Lemma 3.2. Furthermore, by Lemma 3.5,  $q$  has no critical points in  $M_a$ . Consequently, using the Morse flow  $(\psi_t^q)_{t \geq 0}$  associated to  $q$ , one sees that all the level sets of  $q$  of the form  $q^{-1}(\{b\})$  with  $b \geq b_0$  for some  $b_0 \in \mathbb{R}$  large enough are diffeomorphic. Since  $q$  is proper by Lemma 3.1, such level sets are compact and the map

$$\psi^q : (0, +\infty) \times q^{-1}(\{b_0\}) \rightarrow q^{-1}((b_0, +\infty)), \quad (t, x) \mapsto \psi_t^q(x),$$

is a diffeomorphism of a neighbourhood of  $M$  at infinity. Again, since  $q$  is proper, the level set  $q^{-1}(\{b_0\})$  is compact hence has a finite number of connected components. Thus,  $M$  has a finite number of ends. □

Our final lemma is then:

**Lemma 3.7** *There exists  $A > 0$  such that for all  $c > A$ , the intersection of each end of  $M$  with  $q^{-1}(\{c\})$  is compact, connected, and nonempty.*

**Proof** By Lemma 3.6,  $M$  has only a finite number of ends. Thus, since  $q$  is proper and bounded below by Lemma 3.4, there exists  $A > 0$  such that all of the ends of  $M$  are contained in  $M \setminus q^{-1}((-\infty, A])$ . For any  $c > A$ , the intersection of  $q^{-1}(\{c\})$  with each end of  $M$  is then compact and nonempty and comprises one connected component only since, as a consequence of Lemma 3.5,  $q$  is strictly increasing along the flow lines of the (nowhere vanishing) vector field  $X$  on  $M_0$ . □

Using the above lemmas, we can now construct our map to the tangent cone at infinity.

**Theorem 3.8** (map to the tangent cone for expanding Ricci solitons) *Let  $A > 0$  be as in Lemma 3.7, let  $\rho : M_0 \rightarrow \mathbb{R}_+$  be defined by  $\rho := 2\sqrt{q}$ , and let  $S := \rho^{-1}(\{c\})$  for any  $c > 0$  with  $\frac{1}{4}c^2 > A$  (so that the intersection of each end of  $M$  with  $S$  is compact, connected, and nonempty). Then there exists a diffeomorphism  $\iota : (c, \infty) \times S \rightarrow M_{c^2/4}$  such that  $g_0 := \iota^* \tilde{g}_0 = dr^2 + r^2 g_S / c^2$  and  $d\iota(r \partial_r) = 2X$ , where  $r$  is the coordinate on the  $(c, \infty)$ -factor and  $g_S$  is the restriction of  $\tilde{g}_0$  to  $S$ . Moreover, along  $M_{c^2/4}$ , we have that*

$$(3-8) \quad |(\nabla^{g_0})^k (\iota^* g - g_0 - 2 \operatorname{Ric}(g_0))|_{g_0} = O(r^{-4-k}) \quad \text{for all } k \in \mathbb{N}_0.$$

*In particular,  $(M, g)$  has a unique tangent cone along each end.*

**Proof** To prove the first part of this statement, we follow the proof of [22, Theorem 1.7.2].

We have that  $\text{Hess}_{\tilde{g}_0}(\rho^2) = 2\tilde{g}_0$  from Lemma 3.3 and we know from Lemma 3.5(i) that  $|\nabla^{\tilde{g}_0}\rho^2|_{\tilde{g}_0}^2 = 4\rho^2$  is constant along the level sets of  $\rho$  and that the integral curves of  $\nabla^{\tilde{g}_0}\rho$  are geodesics. Then we have that

$$\nabla^{\tilde{g}_0}\rho^2 = 2\rho\nabla^{\tilde{g}_0}\rho \quad \text{and} \quad \text{Hess}_{\tilde{g}_0}(\rho^2) = 2d\rho^2 + 2\rho\text{Hess}_{\tilde{g}_0}(\rho),$$

so that

$$2\tilde{g}_0 = \text{Hess}_{\tilde{g}_0}(\rho^2) = 2d\rho^2 + 2\rho\text{Hess}_{\tilde{g}_0}(\rho).$$

Hence,

$$\text{Hess}_{\tilde{g}_0}(\rho) = \frac{\tilde{g}_0}{\rho} \quad \text{on the } \tilde{g}_0\text{-orthogonal complement of } \nabla^{\tilde{g}_0}\rho.$$

On the other hand,  $\tilde{g}_0 = d\rho^2 + \tilde{g}_\rho$  with  $\tilde{g}_\rho$  the restriction of  $\tilde{g}_0$  to the level set of  $\rho$ , and

$$\mathcal{L}_{\nabla^{\tilde{g}_0}\rho}\tilde{g}_\rho = \mathcal{L}_{\nabla^{\tilde{g}_0}\rho}(\tilde{g}_0 - d\rho^2) = 2\text{Hess}_{\tilde{g}_0}(\rho) - 16\rho d\rho^2 = \frac{2\tilde{g}_\rho}{\rho} + \left(\frac{2}{\rho} - 16\rho\right)d\rho^2,$$

so that

$$\mathcal{L}_{\nabla^{\tilde{g}_0}\rho}\tilde{g}_\rho = \frac{2\tilde{g}_\rho}{\rho} \quad \text{on the } \tilde{g}_0\text{-orthogonal complement of } \nabla^{\tilde{g}_0}\rho.$$

Thus,

$$(3-9) \quad \mathcal{L}_{\rho\nabla^{\tilde{g}_0}\rho}\tilde{g}_\rho = 2\tilde{g}_\rho \quad \text{on the } \tilde{g}_0\text{-orthogonal complement of } \nabla^{\tilde{g}_0}\rho.$$

Next define a map  $\iota: (c, \infty) \times S \rightarrow M_{c^2/4}$  by

$$(r, x) \mapsto \Phi_x(r - c),$$

where  $\Phi_x(\cdot)$  denotes the flow of  $\nabla^{\tilde{g}_0}\rho$  with  $\Phi_x(0) = x$ . By choice of  $c$ , this map is well-defined. Moreover,  $d\iota(\partial_r) = \nabla^{\tilde{g}_0}\rho$  by construction, and since  $\rho(\Phi_x(t)) = t + c$ , we have that  $\iota^*\rho = r$  so that  $d\iota(r\partial_r) = \rho\nabla^{\tilde{g}_0}\rho = 2X$ . In this new frame, we thus have that

$$\iota^*\tilde{g}_0 = dr^2 + \iota^*\tilde{g}_\rho,$$

where we find from (3-9) that

$$\mathcal{L}_{r\partial_r}\iota^*\tilde{g}_\rho = 2\iota^*\tilde{g}_\rho \quad \text{on the } \iota^*\tilde{g}_0\text{-orthogonal complement of } \partial_r.$$

Hence,  $\iota^*\tilde{g}_\rho = r^2g_S/c^2$  so that  $\iota^*\tilde{g}_0 = dr^2 + r^2g_S/c^2$ , as claimed.

As for the fact that (3-8) holds true along  $M_{c^2/4}$ , we have from Young’s inequality applied to (3-6) that

$$(3-10) \quad C^{-1}d_{g(t)}(p, x) - C\sqrt{t} \leq \rho(x) \leq Cd_{g(t)}(p, x) + C\sqrt{t} \quad \text{for all } t \in (0, 1], x \in M_0.$$

Using this, we can now prove an estimate less sharp than (3-8).

**Claim 3.9** For all  $x \in M_{c^2/4}$  and  $k \in \mathbb{N}_0$ ,

$$(3-11) \quad |(\nabla^g)^k(g - \tilde{g}_0)|_g(x) \leq C_k\rho(x)^{-2-k}.$$

**Proof** Let us prove the claim first for  $k = 0$ . Since the curvature tensor of  $g$  (and hence that of  $g(s)$ ) decays quadratically with derivatives, we have that for any  $p$  in the critical set of  $f$  and for any  $x \in M_{c^2/4}$ ,

$$|g - \tilde{g}_0|_g(x) \leq \int_0^1 |\partial_s g(s)|_g(x) ds \leq C \int_0^1 |\text{Ric}(g(s))|_g(x) ds \leq C \int_0^1 d_{g(s)}(p, x)^{-2} ds \leq C\rho(x)^{-2},$$

where we have used (3-3) and (3-10) after increasing  $C$  if necessary.

As for the case  $k = 1$ , we must work slightly harder. Recall that if  $T$  is a tensor on  $M$ , then  $\nabla^{g(t)} T = \nabla^g T + g(t)^{-1} * \nabla^g(g(t) - g) * T$ , since at the level of Christoffel symbols, one has that

$$\Gamma(g(t))_{ij}^k = \Gamma(g)_{ij}^k + \frac{1}{2} g(t)^{km} (\nabla_i^g(g(t) - g)_{jm} + \nabla_j^g(g(t) - g)_{im} - \nabla_m^g(g(t) - g)_{ij}).$$

Thus, for all  $x \in M_{c^2/4}$  and  $t \in (0, 1]$ , we have that

$$\begin{aligned} \partial_t |\nabla^g(g(t) - g)|_g^2(x) &\geq -4 |\nabla^g \text{Ric}(g(t))|_g(x) |\nabla^g(g(t) - g)|_g(x) \\ &\geq -4 (|\nabla^{g(t)} \text{Ric}(g(t))|_g(x) + (|\nabla^g - \nabla^{g(t)}| \text{Ric}(g(t))|_g(x))) |\nabla^g(g(t) - g)|_g(x) \\ &\geq -C (d_{g(t)}(p, x)^{-3} + |\nabla^g - \nabla^{g(t)}| \text{Ric}(g(t))|_g(x)) |\nabla^g(g(t) - g)|_g(x) \\ &\geq -C (\rho(x)^{-3} + |\nabla^g(g(t) - g)|_g(x) |\text{Ric}(g(t))|_g(x)) |\nabla^g(g(t) - g)|_g(x) \\ &\geq -C (\rho(x)^{-3} + |\nabla^g(g(t) - g)|_g(x)) |\nabla^g(g(t) - g)|_g(x) \\ &\geq -C |\nabla^g(g(t) - g)|_g^2(x) - C\rho(x)^{-6}, \end{aligned}$$

where  $C$  denotes a positive constant that may vary from line to line, and where we used Young's inequality in the last line. Recalling that  $|\nabla^g(g(t) - g)|_g^2 = 0$  when  $t = 1$ , one can integrate the previous differential inequality between a time  $t \in (0, 1)$  and  $t = 1$  to obtain

$$|\nabla^g(g(t) - g)|_g^2(x) \leq C\rho(x)^{-6} \quad \text{for } x \in M_{c^2/4}, t \in (0, 1],$$

for some positive constant  $C$  uniform in time. This fact implies the desired estimate (3-11) for  $k = 1$  by letting  $t \rightarrow 0^+$ .

The cases  $k \geq 2$  are proved by induction on  $k$ . ◁

It follows from Claim 3.9 that

$$|(\nabla^{\tilde{g}_0})^k (g - \tilde{g}_0)|_{\tilde{g}_0} \leq C_k \rho^{-2-k} \quad \text{for all } k \in \mathbb{N}_0,$$

so that, after pulling back by  $\iota$ , we have that

$$|(\nabla^{g_0})^k (\iota^* g - g_0)|_{g_0} \leq C_k r^{-2-k} \quad \text{for all } k \in \mathbb{N}_0.$$

We now prove (3-8). To this end, recall that  $\varphi_t(x)$  satisfies

$$\frac{\partial \varphi_t}{\partial t}(x) = -\frac{\nabla^g f(\varphi_t(x))}{t} \quad \text{and} \quad \varphi_1 = \text{id}.$$

Since  $4\nabla^g f = \nabla^{g_0} \rho^2 = 2\rho\nabla^{g_0} \rho$  by Lemma 3.5(ii), we have that

$$\begin{aligned} \frac{d}{dt} \rho(\varphi_t(x)) &= d\rho|_{\varphi_t(x)} \left( \frac{\partial \varphi_t}{\partial t}(x) \right) = d\rho|_{\varphi_t(x)} \left( -\frac{\nabla^g f(\varphi_t(x))}{t} \right) \\ &= -\frac{\rho(\varphi_t(x))}{2t} d\rho|_{\varphi_t(x)} (\nabla^{g_0} \rho|_{\varphi_t(x)}) = -\frac{\rho(\varphi_t(x))}{2t}, \end{aligned}$$

so that

$$(3-12) \quad \rho(\varphi_t(x)) = \frac{\rho(x)}{t^{1/2}}.$$

Let  $\widehat{\varphi}_t(x)$  satisfy

$$\frac{\partial \widehat{\varphi}_t}{\partial t}(x) = -\frac{r}{2t} \frac{\partial}{\partial r}(\widehat{\varphi}_t(x)) \quad \text{and} \quad \widehat{\varphi}_1 = \text{id}.$$

Then since  $dt(r\partial_r) = 2X$ , we have that  $\iota \circ \widehat{\varphi}_t = \varphi_t \circ \iota$ , and in light of (3-12), we see that  $\widehat{\varphi}_t^* r = r/t^{1/2}$ , so that  $t\widehat{\varphi}_t^* g_0 = g_0$ . Recall that the Ricci flow  $g(t)$  defined by  $g$  is given by  $g(t) = t\varphi_t^* g$  for  $t \in (0, 1]$ . Together with the scaling properties of the norm induced on tensors by  $g_0$  and the invariance of the Levi-Civita connection under rescalings, these observations imply that

$$\begin{aligned} |(\nabla^{\widehat{\varphi}_t^* g_0})^k(\iota^* g(t) - g_0)|_{\widehat{\varphi}_t^* g_0}(x) &= |(\nabla^{t^{-1} g_0})^k(\iota^* g(t) - g_0)|_{t^{-1} g_0}(x) \\ &= t^{1+(k/2)} |(\nabla^{g_0})^k(\iota^* g(t) - g_0)|_{g_0}(x), \end{aligned}$$

so that

$$\begin{aligned} |(\nabla^{g_0})^k(\iota^* g(t) - g_0)|_{g_0}(x) &= t^{-1-k/2} |(\nabla^{\widehat{\varphi}_t^* g_0})^k(\iota^* g(t) - g_0)|_{\widehat{\varphi}_t^* g_0}(x) \\ &= t^{-1-k/2} \cdot t |(\nabla^{\widehat{\varphi}_t^* g_0})^k(\widehat{\varphi}_t^* \iota^* g - \widehat{\varphi}_t^* g_0)|_{\widehat{\varphi}_t^* g_0}(x) \\ &= t^{-1-k/2} \cdot t |(\nabla^{g_0})^k(\iota^* g - g_0)|_{g_0}(\widehat{\varphi}_t(x)) \\ &\leq C_k t \cdot t^{-1-k/2} \cdot (r(\widehat{\varphi}_t(x)))^{-2-k} \\ &= C_k t \cdot t^{-1-k/2} \cdot t^{1+k/2} (r(x))^{-2-k} \quad \text{for all } k \in \mathbb{N}_0; \end{aligned}$$

that is,

$$|(\nabla^{g_0})^k(\iota^* g(t) - g_0)|_{g_0}(x) \leq C_k t r^{-2-k} \quad \text{for all } k \in \mathbb{N}_0.$$

In particular,

$$|(\nabla^{g_0})^k(\text{Ric}(\iota^* g(t)) - \text{Ric}(g_0))|_{g_0} \leq C_k t r^{-4-k} \quad \text{for all } k \in \mathbb{N}_0,$$

which is clear from the expression of the components of the Ricci curvature in local coordinates. Consequently, we have the improved estimate

$$|(\nabla^{g_0})^k(\iota^* g - g_0 - 2\text{Ric}(g_0))|_{g_0}(x) \leq C_k \int_0^1 |(\nabla^{g_0})^k(\text{Ric}(\iota^* g(s)) - \text{Ric}(g_0))|_{g_0}(x) ds \leq C_k r(x)^{-4-k}.$$

This is precisely (3-8). □

Thus, an expanding gradient Ricci soliton  $M$  with quadratic curvature decay with derivatives has a unique tangent cone  $C_0$  along each of its ends  $V$ . Moreover, there is a diffeomorphism

$$\iota: C_0 \setminus K \rightarrow V,$$

where  $K \subset C_0$  is a compact subset containing the apex of  $C_0$ , induced by the flow of the vector field  $2X/\rho$ . The statement of Theorem 3.8 is verbatim the same for expanding gradient Kähler–Ricci solitons, except that  $d\iota(r\partial_r) = X$  rather than  $2X$  and  $2\text{Ric}(g_0)$  is replaced by  $\text{Ric}(g_0)$  in (3-8), accounting for the difference in normalisation between Ricci solitons and Kähler–Ricci solitons.

### 3.2 Existence of a resolution map to the tangent cone

In the case that  $(M, g, X)$  is a complete expanding gradient Kähler–Ricci soliton with quadratic curvature decay with derivatives, the soliton potential is proper [12], hence  $M$  has only one end  $V$  [52] with tangent cone  $C_0$  along the end. (Also note from [52] that any complete shrinking gradient Kähler–Ricci soliton has only one end without any curvature assumption on the metric.) Along  $V$ , we have from Theorem 3.8 the diffeomorphism  $\iota: C_0 \setminus K \rightarrow V$  for  $K \subset C_0$  a compact subset containing the apex of  $C_0$ . As we will now see, the inverse of this map actually extends to define a resolution  $\pi: M \rightarrow C_0$  with respect to which  $d\pi(X) = r\partial_r$ . We first show that  $\iota$  is a biholomorphism with respect to a complex structure on  $C_0$  that makes the cone metric  $g_0$  Kähler. As the next proposition demonstrates, the Kählerity of the soliton implies this fact.

**Proposition 3.10** *Let  $(M, g, X)$  be a complete expanding gradient Kähler–Ricci soliton with complex structure  $J$  such that for some point  $p \in M$  and all  $k \in \mathbb{N}_0$ ,*

$$A_k(g) := \sup_{x \in M} |(\nabla^g)^k \text{Rm}(g)|_g(x) d_g(p, x)^{2+k} < \infty,$$

where  $\text{Rm}(g)$  denotes the curvature of  $g$  and  $d_g(p, x)$  denotes the distance between  $p$  and  $x$  with respect to  $g$ . For the unique end  $V$  of  $M$ , let  $\tilde{g}_0 = \lim_{t \rightarrow 0^+} g(t)$  be the limit of the Kähler–Ricci flow  $g(t)$  defined by  $(M, g, X)$ , let  $(C_0, g_0)$  be the unique tangent cone along  $V$  with radial function  $r$  and let  $\iota: (C_0 \setminus K, g_0) \rightarrow (V, \tilde{g}_0)$ , for  $K \subset C_0$  compact containing the apex of  $C_0$ , be the isometry of Theorem 3.8. Then  $(C_0, g_0)$  is a Kähler cone with respect to  $\iota^*J$ . In particular,  $\iota: (C_0 \setminus K, \iota^*J) \rightarrow (V, J)$  is a biholomorphism.

**Proof** Since  $\lim_{t \rightarrow 0^+} g(t) = \tilde{g}_0$  smoothly on compact subsets of  $V$  and  $g(t)$  is Kähler with respect to  $J$ , we have that on  $V$ ,

$$\nabla^{\tilde{g}_0} J = \lim_{t \rightarrow 0^+} \nabla^{g(t)} J = 0,$$

so that  $\tilde{g}_0$  is Kähler with respect to  $J$ . The metric  $g_0$  is therefore Kähler with respect to  $\iota^*J$  away from a compact subset of  $C_0$ . Recall that the radial vector field on a Kähler cone is holomorphic with respect to its complex structure. Thus,  $r\partial_r$  is holomorphic on the subset of  $C_0$  for which  $\iota^*J$  is defined. Flowing along  $-r\partial_r$  then extends  $\iota^*J$  to a global complex structure on  $C_0$ , with respect to which  $g_0$  is Kähler.  $\square$

In fact the converse of Proposition 3.10 holds true for shrinking gradient Kähler–Ricci solitons; see [42] for details.

The previous proposition implies that  $M$  is 1–convex. This property is what allows us to extend the biholomorphism  $\iota^{-1}$  to a resolution  $\pi : M \rightarrow C_0$  that is equivariant with respect to the torus action on  $C_0$  generated by the flow of  $J_0 r \partial_r$ . The details are contained in the next theorem.

**Theorem 3.11** *Let  $(M, g, X)$  be a complete expanding gradient Kähler–Ricci soliton with complex structure  $J$  such that for some point  $p \in M$  and all  $k \in \mathbb{N}_0$ ,*

$$A_k(g) := \sup_{x \in M} |(\nabla^g)^k \text{Rm}(g)|_g(x) d_g(p, x)^{2+k} < \infty,$$

*and let  $(C_0, g_0)$  be its unique tangent cone with radial function  $r$  and complex structure  $J_0$ . Then there exists a holomorphic map  $\pi : M \rightarrow C_0$  that is a resolution of  $C_0$  with the property that  $d\pi(X) = r \partial_r$ . Furthermore, the holomorphic isometric real torus action on  $(C_0, g_0, J_0)$  generated by  $J_0 r \partial_r$  extends to a holomorphic isometric torus action of  $(M, g, J)$ .*

**Proof** The proof of Theorem 3.11 comprises several steps. From Proposition 3.10, we know that along the unique end  $V$  of  $M$ , there is a biholomorphism  $\iota : C_0 \setminus K \rightarrow V$  for  $K \subset C_0$  a compact subset containing the apex of  $C_0$ . Thus, by [20, Lemma 2.15], this in particular implies that  $M$  is 1–convex, hence holomorphically convex, so that there is a Remmert reduction  $p : M \rightarrow M'$  of  $M$ . Recall that this is a proper holomorphic map  $p : M \rightarrow M'$  from  $M$  onto a normal Stein space  $M'$  with finitely many isolated singularities obtained by contracting the maximal compact analytic subset of  $M$ . By construction,  $M'$  is biholomorphic to  $M$  outside compact sets, therefore we have a biholomorphism given by  $F := p \circ \iota : \{x \in C_0 \mid r(x) > R\} \rightarrow M' \setminus K'$  for some compact subset  $K' \subset M'$  and for some  $R > 0$ . We claim that this biholomorphism extends globally.

**Claim 3.12** *The biholomorphism  $F : \{x \in C_0 \mid r(x) > R\} \rightarrow M' \setminus K'$  extends to a biholomorphism  $\bar{F} : C_0 \rightarrow M'$ .*

**Proof** Since  $M'$ , as a Stein space with finitely many isolated singularities, admits an embedding  $h : M' \rightarrow \mathbb{C}^P$  for some  $P$  by [3, Theorem 3.1], we have a holomorphic function

$$h \circ F : \{x \in C_0 \mid r(x) > R\} \rightarrow \mathbb{C}^P.$$

Since  $C_0$  is in particular an example of a Stein space, this holomorphic function extends to a unique holomorphic function  $\bar{F} : C_0 \rightarrow \mathbb{C}^P$  by Hartogs’ theorem for Stein spaces [61, Theorem 6.6]. The fact that  $\bar{F}(C_0) \subseteq M'$  follows from Hartogs’ theorem. To show that  $\bar{F}$  is in fact a biholomorphism, we construct an inverse map  $\bar{F}^{-1} : M' \rightarrow C_0$  as an extension of the map

$$F^{-1} : M' \setminus K' \rightarrow \{x \in C_0 \mid r(x) > R\}$$

by applying the previous argument beginning with the fact that  $C_0$  is affine algebraic. ◁

Thus, the Remmert reduction of  $M$  is actually  $C_0$ , ie the composition  $\pi := \bar{F}^{-1} \circ p : M \rightarrow C_0$  is a proper holomorphic map contracting the maximal compact analytic subset  $E$  of  $M$  to obtain the cone  $C_0$ . Denote the connected components of  $E$  by  $E_1, \dots, E_k$ . Then  $\pi$  contracts each  $E_i$  to a point  $p_i \in C_0$  and

restricts to a biholomorphism  $\pi: M \setminus E \rightarrow C_0 \setminus \{p_1, \dots, p_k\}$ . We next show that  $\pi$  defines a resolution of  $C_0$  for which  $d\pi(X) = r\partial_r$ . Note that at infinity,  $\pi = (p \circ \iota)^{-1} \circ p = \iota^{-1}$ .

**Claim 3.13** *The map  $\pi := \bar{F}^{-1} \circ p: M \rightarrow C_0$  is a resolution of  $C_0$  with respect to which  $d\pi(X) = r\partial_r$ .*

**Proof** Consider the biholomorphism  $\pi: M \setminus E \rightarrow C_0 \setminus \{p_1, \dots, p_k\}$ . This map allows us to lift the holomorphic vector field  $r\partial_r$  to a holomorphic vector field  $Y := (d\pi)^{-1}(r\partial_r)$  on  $M \setminus E$ . Since at infinity  $\pi = \iota^{-1}$ , and so identifies the vector field  $r\partial_r$  on  $C_0$  with the vector field  $X$  on  $M$  outside compact subsets of each,  $Y$  will agree with  $X$  outside of a compact subset of  $M$ . Thus, analyticity implies that  $X = Y$  on  $M \setminus E$ . The next observation is that since the flow lines of  $Y$  (and hence  $X$ ) foliate  $M \setminus E$ , the flow of  $X$  must preserve  $E$ . Via  $\pi$  therefore, the flow of  $X$  induces a flow on  $C_0$  that fixes the points  $p_1, \dots, p_k$ , where as before each  $p_i$  denotes the image of a connected component  $E_i$  of  $E \subset M$  under  $\pi$ . The result of this induced flow on  $C_0$  is a holomorphic vector field  $\hat{X}$  that coincides with  $r\partial_r$  on  $C_0 \setminus \{p_1, \dots, p_k\}$  and which is equal to zero at each  $p_i$ . By analyticity again,  $\hat{X} = r\partial_r$ , so that  $E$  comprises one connected component only, which is mapped to the apex of the cone by  $\pi$ . Thus,  $\pi: M \rightarrow C_0$  is a resolution of the singularity of the cone, and the vector field  $X$  on  $M$  is an extension of  $(d\pi)^{-1}(r\partial_r)$  from  $M \setminus E$  to  $M$  so that  $d\pi(X) = r\partial_r$ , as claimed.  $\triangleleft$

The resolution  $\pi: M \rightarrow C_0$  is clearly equivariant with respect to the flow of  $J_0 r\partial_r$  on  $C_0$  and the flow of  $JX$  on  $M$ . We wish to show next that  $\pi: M \rightarrow C_0$  is in fact equivariant with respect to the holomorphic isometric torus action on  $C_0$  induced by the flow of  $J_0 r\partial_r$  and that the lift of this torus action to  $M$  acts isometrically on  $g$ . This will conclude the proof of Theorem 3.11.

**Claim 3.14** *The holomorphic isometric torus action on  $(C_0, g_0)$  generated by  $J_0 r\partial_r$  extends to a holomorphic isometric action of  $(M, g, J)$  so that, in particular,  $\pi: M \rightarrow C_0$  is equivariant with respect to this torus action.*

**Proof** Consider the isometry group of  $(M, g)$  that fixes  $E$  endowed with the topology induced by uniform convergence on compact subsets of  $M$ . By the Arzelà–Ascoli theorem, this is a compact Lie group. Taking the closure of the flow of  $JX$  in this group therefore yields the holomorphic isometric action of a torus  $T$  on  $(M, J, g)$ . Since the action of  $T$  preserves  $E$ , this action pushes down via  $\pi$  to a holomorphic action of  $T$  on  $C_0$  fixing the apex  $o$  of  $C_0$ . Now, by Theorem 3.8, after noting again that  $\pi = \iota^{-1}$  at infinity, we see that the soliton metric  $g$  and the cone metric  $g_0$  are asymptotic at infinity. Therefore these metrics are quasi-isometric on  $C_0 \setminus K$ , where  $K \subset C_0$  is any compact subset of  $C_0$  containing the apex  $o$  of  $C_0$ , so that uniform convergence on compact subsets of  $C_0 \setminus \{o\}$  measured with respect to  $g$  and  $g_0$  are equivalent. Recall that  $d\pi(X) = r\partial_r$ , so that the flow of  $J_0 r\partial_r$  is dense in  $T$  and the flow of  $J_0 r\partial_r$  is isometric with respect to  $g_0$ . Consequently, every automorphism of  $(C_0, J_0)$  induced by  $T$  is obtained as a limit of automorphisms of  $(C_0 \setminus \{o\}, g_0, J_0)$  with respect to uniform convergence on compact subsets measured using  $g_0$ . Since a uniform limit of isometries is itself an isometry, it follows



that  $T$  acts isometrically with respect to  $g_0$  on  $C_0 \setminus \{o\}$ , so that the action of  $T$  on  $C_0$  preserves the slices of  $C_0$  and defines a torus in the isometry group of the link of  $C_0$  in which the flow of  $J_0 r \partial_r$  is dense. This final observation concludes the proof of the claim and Theorem 3.11.  $\square$

### 3.3 Conclusion of the proof of Theorem A

We now conclude the proof of Theorem A for complete expanding gradient Kähler–Ricci solitons. Conclusion (a) follows from [62, Theorem 4.3.1], whereas the Kählerity of  $(C_0, g_0)$  as stated in conclusion (b) follows from Proposition 3.10. The remainder of conclusion (b), apart from (b)(i), then follows from Theorem 3.11. Conclusion (c) follows from Theorem 3.8 after noting that  $\pi = \iota^{-1}$  at infinity as above.

As for conclusion (b)(i), the Kähler form  $\omega$  of the expanding gradient Kähler–Ricci soliton satisfies the expanding soliton equation  $\rho_\omega + i \partial \bar{\partial} f = -\omega$  on  $M$ , where  $\rho_\omega$  is the Ricci form of  $\omega$  and  $f$  is the soliton potential. In  $H^2(M)$ , this equation yields  $[-\rho_\omega] = [\omega]$ . Since  $i\rho_\omega$  is the curvature form  $\Theta$  resulting from the hermitian metric on  $K_M$  induced by  $\omega$ , we have that  $[i\Theta] = [-\rho_\omega] = [\omega]$ , so that (1-5) is seen to hold true for the expanding soliton Kähler form  $\omega$  and the curvature form  $i\Theta$  it induces on  $K_M$ .

For a complete shrinking gradient Ricci soliton  $(M, g, X)$  with soliton potential  $f$ , we define a Kähler–Ricci flow via

$$g(t) = -t\varphi_t^* g \quad \text{for } t < 0,$$

where  $\varphi_t$  is a family of diffeomorphisms generated by the gradient vector field  $-(1/t)X$  with  $\varphi_{-1} = \text{id}$ , ie

$$\frac{\partial \varphi_t}{\partial t}(x) = -\frac{\nabla^g f(\varphi_t(x))}{t} \quad \text{with } \varphi_{-1} = \text{id}.$$

Then  $(\partial g / \partial t)(t) = -2 \text{Ric}(g(t))$  for  $t < 0$  and  $g(-1) = g$ . Such a soliton with quadratic curvature decay has quadratic curvature decay with derivatives by Theorem 2.20(iii), and hence, as proved in [43, Sections 2.2–2.3], has a unique tangent cone at infinity. These observations provide the starting point for the proof of Theorem A for complete shrinking gradient Kähler–Ricci solitons with quadratic curvature decay. The proof then follows the proof for the expanding case, verbatim.

## 4 Classification results for expanding gradient Kähler–Ricci solitons with quadratic curvature decay with derivatives

### 4.1 Proof of Corollary B

Let  $(M, g, X)$  be a complete expanding gradient Kähler–Ricci soliton satisfying (1-6) with tangent cone  $(C_0, g_0)$ , as in Corollary B. Let  $\omega$  denote the Kähler form of  $g$ .

To see that  $M$  is the canonical model of  $C_0$ , note first that Theorem A asserts that there is a Kähler resolution  $\pi : M \rightarrow C_0$  with exceptional set  $E$  such that

$$(4-1) \quad \int_V (i\Theta)^k \wedge \omega^{\dim_{\mathbb{C}} V - k} > 0$$

for all positive-dimensional irreducible analytic subvarieties  $V \subset E$  of  $\pi : M \rightarrow C_0$  and for all integers  $k$  such that  $1 \leq k \leq \dim_{\mathbb{C}} V$ , where  $\Theta$  denotes the curvature form of the hermitian metric on  $K_M$  induced by  $\omega$ . In particular, (4-1) implies that

$$\int_V (i\Theta)^k \wedge \omega^{\dim_{\mathbb{C}} V - k} > 0$$

for all positive-dimensional irreducible algebraic subvarieties  $V \subset E$  and for all integers  $k$  such that  $1 \leq k \leq \dim_{\mathbb{C}} V$ . Setting  $k = \dim_{\mathbb{C}} V$ , we then see that

$$\int_V (i\Theta)^{\dim_{\mathbb{C}} V} > 0 \quad \text{for every irreducible algebraic subvariety } V \subset E \text{ of positive dimension.}$$

But since  $M$  is quasiprojective by Proposition 2.24, this is the same as saying that

$$(D^{\dim_{\mathbb{C}} V} \cdot V) > 0 \quad \text{for every irreducible algebraic subvariety } V \subset E \text{ of positive dimension,}$$

where  $D$  is now a canonical divisor of  $M$ . Nakai’s criterion for a mapping (Theorem 2.11) now tells us that  $K_M$  is  $\pi$ -ample, so that by definition,  $\pi : M \rightarrow C_0$  is the canonical model of  $C_0$ . Hence  $C_0$  has a smooth canonical model, namely  $M$ .

Conversely, suppose that  $(C_0, g_0)$  is a Kähler cone with radial function  $r$  and with a smooth canonical model  $\pi : M \rightarrow C_0$ . We begin by explaining that [19, Theorem A] holds true without hypothesis (b) of that theorem. This hypothesis was required in the proof of [19, Proposition 3.2] to show that  $\mathcal{L}_X \omega = i\partial\bar{\partial}\theta_X$ , where  $\omega$  is the Kähler form of [19, Proposition 3.1],  $X$  is the lift of the radial vector field on the cone, and  $\theta_X$  is a smooth real-valued function. The following claim asserts that this in fact always holds true.

**Claim 4.1** *Let  $(C_0, g_0)$  be a Kähler cone with complex structure  $J_0$  and radial function  $r$  and let  $\pi : M \rightarrow C_0$  be an equivariant resolution with respect to the real torus action on  $C_0$  generated by  $J_0 r \partial_r$ . Let  $X$  be the unique holomorphic vector field on  $M$  with  $d\pi(X) = r \partial_r$  and let  $\omega$  be the Kähler form of [19, Proposition 3.1]. Then  $\mathcal{L}_X \omega = i\partial\bar{\partial}\theta_X$  for a smooth real-valued function  $\theta_X : M \rightarrow \mathbb{R}$ .*

**Proof** Denote the complex structure of  $M$  by  $J$  and let  $X^{1,0} = \frac{1}{2}(X - iJX)$ . Then since  $\mathcal{L}_{JX} \omega = 0$  by construction, we have that

$$(4-2) \quad \frac{1}{2}\mathcal{L}_X \omega = \frac{1}{2}d(\omega \lrcorner X) = d(\omega \lrcorner X^{1,0}).$$

Now by construction,  $\omega$  takes the form  $\omega = i\tilde{\Theta}_h + i\partial\bar{\partial}u$ , where  $u : M \rightarrow \mathbb{R}$  is a smooth real-valued function and  $\tilde{\Theta}_h$  is the average over the action of the torus on  $M$  of the curvature form  $\Theta_h$  of a hermitian metric  $h$  on  $K_M$ . Thus,

$$(4-3) \quad \omega \lrcorner X^{1,0} = i\tilde{\Theta}_h \lrcorner X^{1,0} + i\bar{\partial}(X^{1,0} \cdot u).$$

Studying the term  $i\tilde{\Theta}_h \lrcorner X^{1,0}$ , let  $n = \dim_{\mathbb{C}} M$ , let  $\Omega$  be a local holomorphic volume form on  $M$ , ie a nowhere vanishing locally defined holomorphic  $(n, 0)$ -form (defined in some local holomorphic coordinate chart, for example), and set

$$v := X^{1,0} \cdot \log(\|\Omega\|_h^2) - \frac{\mathcal{L}_{X^{1,0}} \Omega}{\Omega},$$

where  $\|\cdot\|_h$  denotes the norm with respect to  $h$ . We claim that  $v$  is independent of the choice of  $\Omega$  and hence is globally defined. Indeed, any other local holomorphic volume form takes the form  $q\Omega$  for some holomorphic function  $q$ . Then

$$\begin{aligned} X^{1,0} \cdot \log(\|q\Omega\|_h^2) - \frac{\mathcal{L}_{X^{1,0}}(q\Omega)}{q\Omega} &= X^{1,0} \cdot \log|q|^2 + X^{1,0} \cdot \log(\|\Omega\|_h^2) - \frac{(X^{1,0} \cdot q)\Omega + q\mathcal{L}_{X^{1,0}}\Omega}{q\Omega} \\ &= X^{1,0} \cdot \log(\|\Omega\|_h^2) - \frac{\mathcal{L}_{X^{1,0}}\Omega}{\Omega} + \underbrace{X^{1,0} \cdot \log|q|^2 - \frac{X^{1,0} \cdot q}{q}}_{=0} \\ &= X^{1,0} \cdot \log(\|\Omega\|_h^2) - \frac{\mathcal{L}_{X^{1,0}}\Omega}{\Omega}, \end{aligned}$$

as required. Next observe that

$$i\Theta_{h \lrcorner} X^{1,0} = -i\bar{\partial}(X^{1,0} \cdot \log(\|\Omega\|_h^2)) = -i\bar{\partial}\left(X^{1,0} \cdot \log(\|\Omega\|_h^2) - \frac{\mathcal{L}_{X^{1,0}}\Omega}{\Omega}\right) = -i\bar{\partial}v,$$

since  $(\mathcal{L}_{X^{1,0}}\Omega)/\Omega$  is a holomorphic function. Averaging this equation over the action of  $T$  then yields the fact that  $i\tilde{\Theta}_{h \lrcorner} X^{1,0} = i\bar{\partial}\tilde{v}$  for a smooth function  $\tilde{v}$  on  $M$ . Plugging this into (4-2), we thus see from (4-3) that

$$\mathcal{L}_X\omega = 2d(i\bar{\partial}(\tilde{v} + X^{1,0} \cdot u)) = i\partial\bar{\partial}(2(\tilde{v} + X^{1,0} \cdot u)).$$

Hence  $\mathcal{L}_X\omega = i\partial\bar{\partial}\theta_X$ , where  $\theta_X := 2\operatorname{Re}(\tilde{v} + X^{1,0} \cdot u)$ , because  $\mathcal{L}_X\omega$  is a real  $(1, 1)$ -form and  $i\partial\bar{\partial}$  is a real operator. ◁

**Remark 4.2** The existence of the function  $v$  satisfying  $i\Theta_{h \lrcorner} X^{1,0} = -i\bar{\partial}v$  is essentially due to the fact that  $X$  has a canonical lift to the total space of  $K_M$  and  $\Theta_h$  is the curvature form of a hermitian metric on  $K_M$ .

Returning now to our smooth canonical model  $\pi: M \rightarrow C_0$  of  $C_0$ , we will verify the hypotheses of [19, Theorem A] (apart from the redundant hypothesis (b) of this theorem) for this resolution to show that  $M$  admits a complete expanding gradient Kähler–Ricci soliton  $g$  with the desired asymptotics. By Lemma 2.13, the radial vector field  $r\partial_r$  on  $C_0$  lifts to a holomorphic vector field  $X$  on  $M$  with  $d\pi(X) = r\partial_r$ , and by Lemma 2.12,  $M$  is quasiprojective, hence Kähler. Moreover, there exists a Kähler form  $\sigma$  on  $M$  and a hermitian metric on  $K_M$  with curvature form  $\Theta$  such that

$$(4-4) \quad \int_V (i\Theta)^k \wedge \sigma^{\dim_{\mathbb{C}} V - k} > 0$$

for all positive-dimensional irreducible analytic subvarieties  $V$  contained in the exceptional set  $E$  of  $\pi: M \rightarrow C_0$  and for all integers  $k$  such that  $1 \leq k \leq \dim_{\mathbb{C}} V$ . Indeed, proceeding as in [25], let  $\sigma$  be the curvature form of a very ample line bundle  $L$  on the projective variety which contains  $M$  as an open subset and let  $\Theta$  be the curvature form of the hermitian metric induced on  $K_M$  by  $\sigma$ . Then observe that for any analytic subvariety  $V \subset E$  of dimension  $k$ ,

$$(4-5) \quad \int_V (i\Theta)^k \wedge \sigma^{\dim_{\mathbb{C}} V - k} = \int_{V \cap H_1 \cap \dots \cap H_{\dim_{\mathbb{C}} V - k}} (i\Theta)^k$$

for generic members  $H_1, \dots, H_{\dim_{\mathbb{C}} V - k}$  of the linear system  $|L|$ , so that  $V \cap H_1 \cap \dots \cap H_{\dim V - k} \subset E$  is an irreducible subvariety of dimension  $k$ . Since  $E$  is projective (as  $M$  is quasiprojective), this intersection is a projective algebraic variety by Chow's theorem. The right-hand side of (4-5) may therefore be written as  $D^k \cdot (V \cap H_1 \cap \dots \cap H_{\dim V - k})$ , where  $D$  is a canonical divisor of  $M$ . By definition of the canonical model,  $K_M$  is  $\pi$ -ample, which by Nakai's criterion for a mapping (see Theorem 2.11) implies that this intersection is strictly positive. Thus, we have that (4-4) holds true for the Kähler form  $\sigma$  and the curvature form  $\Theta$  that it induces on  $K_M$ . The hypotheses required for the application of [19, Theorem A] are therefore satisfied and so  $M$  admits a complete expanding gradient Kähler–Ricci soliton  $(M, g, X)$  with

$$|(\nabla^{g_0})^k (\pi_* g - g_0 - \text{Ric}(g_0))|_{g_0} \leq C_k r^{-4-k} \quad \text{for all } k \in \mathbb{N}_0,$$

as required.

As for the uniqueness of  $(M, g, X)$ , let  $(M_i, g_i, X_i)$  for  $i = 1, 2$  be two complete expanding gradient Kähler–Ricci solitons satisfying (1-6) with tangent cone  $(C_0, g_0)$ . As initially proved, both  $M_1$  and  $M_2$  are equal to the unique (smooth) canonical model  $M$  of  $C_0$ . Moreover, Theorem A asserts that for  $i = 1, 2$  there exists a resolution map  $\pi_i: M \rightarrow C_0$  with  $d\pi_i(X_i) = r\partial_r$  such that

$$(4-6) \quad |(\nabla^{g_0})^k ((\pi_i)_* g_i - g_0 - \text{Ric}(g_0))|_{g_0} \leq C_k r^{-4-k} \quad \text{for all } k \in \mathbb{N}_0.$$

The composition  $H := \pi_2 \circ \pi_1^{-1}: C_0 \rightarrow C_0$  induces an automorphism of  $C_0$  fixing the vertex. As in the proof of Lemma 2.13, uniqueness of the canonical model implies that there exists a unique biholomorphism  $F: M \rightarrow M$  such that  $\pi_1 \circ F = H \circ \pi_1$ . Unravelling the definition of  $H$ , this yields the fact that  $\pi_1 \circ F = \pi_2$ . Consequently,  $d\pi_2((dF)^{-1}(X_1)) = d\pi_1(X_1) = r\partial_r$  so that  $(dF)^{-1}(X_1) = X_2$ . Furthermore, in light of (4-6), we have that

$$(4-7) \quad |(\nabla^{g_0})^k ((\pi_2)_*(F^* g_1) - g_0 - \text{Ric}(g_0))|_{g_0} \leq C_k r^{-4-k} \quad \text{for all } k \in \mathbb{N}_0.$$

Thus,  $(M, F^* g_1, X_2)$  and  $(M, g_2, X_2)$  are two expanding gradient Kähler–Ricci solitons with the same soliton vector field which from (4-6) for  $i = 2$  and (4-7) in addition satisfy  $|F^* g_1 - g_2| = O(r^{-4})$ . The uniqueness theorem [19, Theorem C(ii)] therefore applies (where, in studying the proof of [19, Theorem C(ii)], one sees that finite fundamental group is not actually required) and asserts that  $F^* g_1 = g_2$ . Thus,  $(M, g, X)$  is unique up to pullback by biholomorphisms of  $M$ , as claimed.

As for the remainder of Corollary B, item (a) is now clear and item (b) follows from Theorem A.

## 4.2 Proof of Corollary C

Corollary C follows from Corollary B once we identify the two-dimensional Kähler cones that admit smooth canonical models as those stated in Corollary C(I)–(III) and realise their respective smooth canonical models as those stated in Corollary C(b)(i)–(iii).

To this end, let  $C_0$  be a two-dimensional Kähler cone with a smooth canonical model  $M$ . By adjunction,  $M$  cannot contain any  $(-1)$ - or  $(-2)$ -curves. In particular, by Theorem 2.15,  $M$  coincides with the minimal model of  $C_0$ . Using this information, we can identify  $C_0$  and  $M$  as follows.

Since  $C_0$  is a two-dimensional Kähler cone, it must be prescribed as in Theorem 2.5. We henceforth work on a case-by-case basis. If  $C_0$  is as in Theorem 2.5(i), then  $\Gamma$  must be as prescribed in Corollary C(I) since  $M$  cannot contain any  $(-2)$ -curves; indeed, see [48, Figure 2.1 and Theorem 4.1] for details. In this case,  $M$  will be the minimal model of  $C_0$  as stated in Corollary C(b)(i). Otherwise,  $C_0$  may be as in Theorem 2.5(ii) which is precisely the statement of Corollary C(II). In this case, the minimal model  $M$  is given as in the statement of Corollary C(b)(ii). Finally,  $C_0$  may be as in Theorem 2.5(iii). Those cones of Theorem 2.5(iii) that admit a smooth canonical model have been identified in Proposition 2.18, which yields the statement of Corollary C(III). For these cones, the minimal resolution is the minimal good resolution which identifies  $M$  as in the statement of Corollary C(b)(iii).

## 5 A volume-minimising principle for complete shrinking gradient Kähler–Ricci solitons

We now focus our attention solely on shrinking gradient Kähler–Ricci solitons for the remainder of the article. The set-up of this section is as follows. Let  $(M, g, X)$  be a complete shrinking gradient Kähler–Ricci soliton of complex dimension  $n$  with complex structure  $J$ , Kähler form  $\omega$ , and with soliton vector field  $X = \nabla^g f$  for a smooth real-valued function  $f: M \rightarrow \mathbb{R}$ . We assume that a real torus  $T$  with Lie algebra  $\mathfrak{t}$  acts holomorphically, effectively and isometrically on  $(M, g, J)$ . Then  $\mathfrak{t}$  can be identified with real holomorphic Killing vector fields on  $M$ . We furthermore assume that  $JX \in \mathfrak{t}$ .

The goal of this section is to prove the uniqueness of the soliton vector field  $JX$  in  $\mathfrak{t}$  by characterising  $JX$  as the unique critical point of a soon-to-be-defined weighted volume functional.

### 5.1 A Matsushima-type theorem

Let  $\text{aut}^X(M)$  denote the Lie algebra of real holomorphic vector fields on  $M$  that commute with  $X$  and hence  $JX$ , and let  $\mathfrak{g}^X$  denote the Lie algebra of real holomorphic  $g$ -Killing vector fields on  $M$  that commute with  $X$  and hence  $JX$ . Clearly  $\mathfrak{g}^X$  is a Lie subalgebra of  $\text{aut}^X(M)$ . In order to prove the uniqueness of the soliton vector field  $X$ , we need to show that the connected component of the identity of the Lie group of holomorphic isometries of  $(M, g, J)$  commuting with the flow of  $X$  is maximal compact in the connected component of the identity of the Lie group of automorphisms of  $(M, J)$  commuting with the flow of  $X$ . This fact will follow from the next theorem, an analogue of Matsushima’s theorem [50] for shrinking gradient Kähler–Ricci solitons stating that the Lie algebra  $\text{aut}^X(M)$  is reducible, after we prove that the aforementioned groups are indeed Lie groups.

**Theorem 5.1** (a Matsushima theorem for shrinking Kähler–Ricci solitons) *Let  $(M, g, X)$  be a complete shrinking gradient Kähler–Ricci soliton with complex structure  $J$  endowed with the holomorphic, effective, isometric action of a real torus  $T$  with Lie algebra  $\mathfrak{t}$  with  $JX \in \mathfrak{t}$ . If  $|\text{Ric}(g)|_g$  is bounded, then we have that*

$$\text{aut}^X(M) = \mathfrak{g}^X \oplus J\mathfrak{g}^X.$$

We expect this theorem to hold true without the assumption of bounded Ricci curvature.

The proof of Theorem 5.1 consists of several steps. Beginning with any real holomorphic vector field  $Y \in \text{aut}^X(M)$ , Hörmander's  $L^2$ -estimates allow for a complex-valued potential, that is, a smooth complex-valued function  $u_Y$  such that  $Y^{1,0} = \nabla^{1,0} u_Y$ , where  $Y^{1,0}$  is the  $(1, 0)$ -part of  $Y$ . Thanks to the defining equation of a shrinking gradient Kähler–Ricci soliton, we can then modify  $u_Y$  by a holomorphic function if necessary so that  $\Delta_\omega u_Y + u_Y - Y^{1,0} \cdot f = 0$ , where  $f$  is the soliton potential with  $X = \nabla^g f$ . We further average  $u_Y$  over the action of  $T$  so that  $\mathcal{L}_{JX} u_Y = 0$ , which results in the commutator relation  $(u_Y)_{\bar{k}} f_k = (u_Y)_k f_{\bar{k}}$ . Using this, we then apply a Bochner formula followed by an integration by parts argument to deduce that  $\nabla^{0,2} \bar{u}_Y = 0$  so that  $\nabla^{1,0} \bar{u}_Y$  is a holomorphic vector field. The bound on the norm of the Ricci curvature is required to control the boundary term in the integration by parts argument. The gradient of the real and imaginary parts of  $u_Y$  will therefore be real holomorphic vector fields so that, once one applies the complex structure to these vector fields, they become real holomorphic and Killing. From this, the stated decomposition follows. To conclude that the sum is direct, we make use of a splitting theorem for shrinking gradient Ricci solitons.

**Proof** Write  $\langle \cdot, \cdot \rangle$ ,  $|\cdot|$  and  $\nabla$ , respectively, for the inner product, norm and Levi-Civita connection determined by  $g$ , and let  $Y \in \text{aut}^X(M)$ . Then  $Y$  defines a real holomorphic vector field on  $M$  with  $[X, Y] = 0$ . Take the  $(1, 0)$ -part  $Y^{1,0}$  of  $Y$ , ie let  $Y^{1,0} = \frac{1}{2}(Y - iJY)$ . Then  $\bar{\partial} Y^{1,0} = 0$ , so that  $\omega \lrcorner Y^{1,0}$  is a  $\bar{\partial}$ -closed  $(0, 1)$ -form, where  $\omega$  denotes the Kähler form of  $g$ . We first claim that  $\omega \lrcorner Y^{1,0}$  admits a smooth complex potential.

**Claim 5.2** *There exists a smooth complex-valued function  $u_Y$  on  $M$  such that  $-i\omega \lrcorner Y^{1,0} = \bar{\partial} u_Y$ , or equivalently, such that  $Y^{1,0} = \nabla^{1,0} u_Y$ .*

Note that  $u_Y$  is unique up to the addition of a holomorphic function.

**Proof** Let  $h$  denote the metric on  $-K_M$  induced by  $\omega$ . Then the curvature of the metric  $e^{-f} h$  on  $-K_M$  is precisely  $\omega$  by virtue of the defining equation of a shrinking gradient Kähler–Ricci soliton. Treat  $\omega \lrcorner Y^{1,0}$  as a  $-K_M$ -valued  $(n, 1)$ -form. Then since the norm of  $\text{Ric}(g)$  is bounded so that  $|Y^{1,0}|$  grows at most polynomially by Proposition 2.30, we see from the growth on  $f$  dictated by Theorem 2.20(i) that the  $L^2$ -norm of  $\omega \lrcorner Y^{1,0}$  measured with respect to  $e^{-f} h$  is finite. An application of Hörmander's  $L^2$ -estimates [24, Theorem 6.1, page 376] now yields the desired conclusion.  $\triangleleft$

Next, contracting (1-4) with  $\lambda = 1$  with  $Y^{1,0}$  and using the Bochner formula, we see that

$$-i\bar{\partial}\Delta_\omega u_Y + i\bar{\partial}(Y^{1,0} \cdot f) = i\bar{\partial}u_Y,$$

so that

$$\bar{\partial}(\Delta_\omega u_Y + u_Y - Y^{1,0} \cdot f) = 0.$$

By adding a holomorphic function to  $u_Y$  if necessary, we may therefore assume that

$$(5-1) \quad \Delta_\omega u_Y + u_Y - Y^{1,0} \cdot f = 0.$$

Furthermore, by averaging  $u_Y$  over the action of  $T$ , we may assume that  $\mathcal{L}_{JX}u_Y = 0$ . These two operations normalise  $u_Y$ . Notice that

$$(u_Y)_{\bar{k}} f_k = Y^{1,0} \cdot f = \nabla^{1,0} u_Y \cdot f = \frac{1}{2} \langle \nabla u_Y - iJ \nabla u_Y, X \rangle = \frac{1}{2} \langle \nabla u_Y, X \rangle,$$

by virtue of the fact that  $\mathcal{L}_{JX}u_Y = 0$ . For the same reason, we also have that

$$(u_Y)_k f_{\bar{k}} = Y^{0,1} \cdot f = (\nabla^{0,1} u_Y) \cdot f = \frac{1}{2} \langle \nabla u_Y, X \rangle.$$

Hence,

$$(u_Y)_{\bar{k}} f_k = (u_Y)_k f_{\bar{k}}.$$

In particular, from (5-1) we deduce that

$$(5-2) \quad \Delta_\omega u_Y + u_Y - (u_Y)_k f_{\bar{k}} = 0.$$

Before continuing, we need to establish some estimates on  $u_Y$  together with its covariant derivatives. We will divide these estimates up into three claims.

**Claim 5.3** *There exists a positive constant  $A$  such that  $u_Y(x) = O(d_g(p, x)^A)$  as  $d_g(p, x)$  tends to  $+\infty$ .*

**Proof** By Proposition 2.30,  $Y^{1,0}$  grows polynomially, ie  $|Y^{1,0}|(x) = O(d_g(p, x)^a)$  for some  $a > 0$ , where  $d_g(p, \cdot)$  denotes the distance with respect to  $g$  to a fixed point  $p \in M$ , so that  $|\bar{\partial}u_Y|(x) = O(d_g(p, x)^a)$ . Then

$$\bar{\partial}u_Y(X) = \frac{1}{2} (du_Y(X) + \underbrace{i du_Y(JX)}_{=0}) = \frac{1}{2} X \cdot u_Y.$$

Thus,

$$(5-3) \quad |X \cdot u_Y| = 2|\bar{\partial}u_Y(X)| = O(d_g(p, x)^{a+1}).$$

Let  $\gamma_x(t)$  be an integral curve of  $X$  with  $\gamma_x(0) = x \in M$ . Then

$$u_Y(\gamma_x(t)) = u_Y(\gamma_x(0)) + \int_0^t \frac{d}{ds} u_Y(\gamma_x(s)) ds = u_Y(\gamma_x(0)) + \int_0^t (X \cdot u_Y)(\gamma_x(s)) ds = C + O(e^{(a+1)t}),$$

so that, by (2-11) and Theorem 2.20(i),

$$|u_Y(x)| = O(d_g(p, x)^{a+1}). \quad \triangleleft$$

The next claim concerns the weighted  $L^2$ -integrability of the total gradient and second covariant derivatives of  $u$ .

**Claim 5.4** *The gradient  $\nabla u_Y$  and the second covariant derivatives  $\nabla^2 u_Y$  of  $u_Y$  belong to  $L^2(e^{-f} \omega^n)$ .*

**Proof** Since  $\Delta_\omega u_Y = Y^{1,0} \cdot f - u_Y$ , the estimate established in Claim 5.3 together with the polynomial growth of  $X$  and  $Y$  at infinity show that  $\Delta_\omega u_Y$  is growing at most polynomially at infinity as well. By (5-3), the same holds true for the drift term  $X \cdot u_Y$ . Therefore the drift Laplacian  $\Delta_\omega u_Y - \frac{1}{2} X \cdot u_Y$  is growing at most polynomially at infinity ensuring its weighted  $L^2$ -integrability, ie  $\Delta_{\omega, X} u_Y := \Delta_\omega u_Y - \frac{1}{2} X \cdot u_Y \in L^2(e^{-f} \omega^n)$ . This implies in turn that  $\nabla \operatorname{Re}(u_Y)$  and  $\nabla \operatorname{Im}(u_Y)$  belong to  $L^2(e^{-f} \omega^n)$ . Indeed, by the previous arguments, it suffices to show that if a smooth real-valued function  $v: M \rightarrow \mathbb{R}$  satisfies  $v \in L^2(e^{-f} \omega^n)$  and  $\Delta_{\omega, X} v \in L^2(e^{-f} \omega^n)$ , then  $\nabla v \in L^2(e^{-f} \omega^n)$ .

To this end, let  $R$  be a positive real number and let  $\phi_R: M \rightarrow [0, 1]$  be a cut-off function with compact support in the geodesic ball  $B_g(p, 2R)$  such that  $\phi_R = 1$  on  $B_g(p, R)$  and  $|\nabla\phi_R|_g \leq c/R$ . Then since  $(\Delta_{\omega, X}v)^2 = 2|\nabla v|^2 + 2\langle\Delta_{\omega, X}v, v\rangle$ , integration by parts leads to the inequality

$$\begin{aligned} 2 \int_M |\nabla v|^2 \phi_R^2 e^{-f} \omega^n &= \int_M \Delta_{\omega, X} v^2 \phi_R^2 e^{-f} \omega^n - 2 \int_M \langle\Delta_{\omega, X}v, v\rangle \phi_R^2 e^{-f} \omega^n \\ &= - \int_M \langle\nabla v^2, \nabla\phi_R^2\rangle e^{-f} \omega^n - 2 \int_M \langle\Delta_{\omega, X}v, v\rangle \phi_R^2 e^{-f} \omega^n \\ &\leq \int_M |\nabla v|^2 \phi_R^2 e^{-f} \omega^n + \frac{c}{R^2} \int_M v^2 e^{-f} \omega^n + \int_M (|\Delta_{\omega, X}v|^2 + |v|^2) e^{-f} \omega^n, \end{aligned}$$

which yields

$$\int_M |\nabla v|^2 \phi_R^2 e^{-f} \omega^n \leq \frac{c}{R^2} \int_M v^2 e^{-f} \omega^n + \int_M (|\Delta_{\omega, X}v|^2 + |v|^2) e^{-f} \omega^n.$$

One then obtains the expected result for the gradient by letting  $R$  tend to  $+\infty$ .

Similarly, for the second covariant derivatives, it suffices to show that if a smooth real-valued function  $v: M \rightarrow \mathbb{R}$  satisfies  $\nabla v \in L^2(e^{-f} \omega^n)$  and  $\Delta_{\omega, X}v \in L^2(e^{-f} \omega^n)$ , then  $\nabla^2 v \in L^2(e^{-f} \omega^n)$ . To this end, we apply the Bochner formula and use the soliton equation as follows:

$$\begin{aligned} (5-4) \quad \Delta_{\omega, X}|\nabla v|^2 &= |\nabla^2 v|^2 + (\text{Ric}(g) + \frac{1}{2}\mathcal{L}_X g)(\nabla v, \nabla v) + \langle\nabla(\Delta_{\omega, X}v), \nabla v\rangle \\ &= |\nabla^2 v|^2 + |\nabla v|^2 + \langle\nabla(\Delta_{\omega, X}v), \nabla v\rangle \\ &\geq |\nabla^2 v|^2 + \langle\nabla(\Delta_{\omega, X}v), \nabla v\rangle. \end{aligned}$$

Next, let  $\phi_R: M \rightarrow [0, 1]$  be the cut-off function defined as above. Then using integration by parts, the identity (5-4) leads to the inequalities

$$\begin{aligned} 2 \int_M |\nabla^2 v|^2 \phi_R^2 e^{-f} \omega^n &\leq 2 \int_M \Delta_{\omega, X}|\nabla v|^2 \phi_R^2 e^{-f} \omega^n - 2 \int_M \langle\nabla(\Delta_{\omega, X}v), \nabla v\rangle \phi_R^2 e^{-f} \omega^n \\ &= - \int_M \langle\nabla|\nabla v|^2, \nabla\phi_R^2\rangle e^{-f} \omega^n + 2 \int_M (2|\Delta_{\omega, X}v|^2 \phi_R^2 + \Delta_{\omega, X}v \langle\nabla v, \nabla\phi_R^2\rangle) e^{-f} \omega^n \\ &\leq \int_M |\nabla^2 v|^2 \phi_R^2 e^{-f} \omega^n + \frac{c}{R^2} \int_M |\nabla v|^2 e^{-f} \omega^n + c \int_M (|\Delta_{\omega, X}v|^2 + |\nabla v|^2) e^{-f} \omega^n \end{aligned}$$

for some positive constant  $c$  independent of  $R$ . Thus,

$$\int_M |\nabla^2 v|^2 \phi_R^2 e^{-f} \omega^n \leq \frac{c}{R^2} \int_M |\nabla v|^2 e^{-f} \omega^n + \int_M (|\Delta_{\omega, X}v|^2 + |\nabla v|^2) e^{-f} \omega^n.$$

The desired result for  $\nabla^2 v$  now follows by letting  $R$  tend to  $+\infty$ . ◁

Finally, we show that some components of the Hessian of  $\bar{u}_Y$  vanish identically.

**Claim 5.5** *The  $(0, 2)$ -part  $\nabla^{0,2}\bar{u}_Y$  of the Hessian of  $\bar{u}_Y$  vanishes identically on  $M$ .*

**Proof** For clarity, we suppress the dependence of the potential  $u_Y$  on the vector field  $Y$  in what follows.



Let  $R > 0$  and let  $\phi_R$  be a cut-off function as in the proof of Claim 5.4. Reminiscent of [63, equation (2.7)], from (5-2) we then find, in normal holomorphic coordinates at a point where the Ricci form  $\rho_\omega$  of  $\omega$  has components  $\rho_{i\bar{j}}$ , that

$$\begin{aligned} 0 &= \int_M (\Delta_\omega u + u - f_{\bar{k}} u_k)_i \bar{u}_{\bar{i}} \phi_R^2 e^{-f} \omega^n \\ &= \int_M (u_{k\bar{k}i} + u_i - f_{i\bar{k}} u_k - f_{\bar{k}} u_{ik}) \bar{u}_{\bar{i}} \phi_R^2 e^{-f} \omega^n \\ &= \int_M (u_{ki\bar{k}} - \rho_{i\bar{s}} u_s + u_i - f_{i\bar{k}} u_k - f_{\bar{k}} u_{ik}) \bar{u}_{\bar{i}} \phi_R^2 e^{-f} \omega^n \quad \text{since } u_{j\bar{i}\bar{j}} = u_{j\bar{j}i} + \rho_{i\bar{s}} u_s, \\ &= \int_M (u_{ik\bar{k}} + \underbrace{(-\rho_{i\bar{s}} u_s + u_i - f_{i\bar{k}} u_k)}_{=0} - f_{\bar{k}} u_{ik}) \bar{u}_{\bar{i}} \phi_R^2 e^{-f} \omega^n \\ &= \int_M (u_{ik\bar{k}} - f_{\bar{k}} u_{ik}) \bar{u}_{\bar{i}} \phi_R^2 e^{-f} \omega^n = \int_M (\overline{\Delta_\omega X \bar{u}_{\bar{i}}}) \bar{u}_{\bar{i}} \phi_R^2 e^{-f} \omega^n \\ &= - \int_M \bar{u}_{\bar{i}\bar{j}} u_{ij} \phi_R^2 e^{-f} \omega^n - \frac{1}{2} \int_M \langle \bar{\partial} \phi_R^2, \bar{\partial} |\bar{\partial} \bar{u}|^2 \rangle e^{-f} \omega^n. \end{aligned}$$

Therefore, by the Cauchy–Schwarz inequality,

$$\begin{aligned} \int_M |\nabla^{0,2} \bar{u}|_\omega^2 \phi_R^2 e^{-f} \omega^n &\leq c \int_M (|\bar{\partial} \phi_R| |\bar{\partial} \bar{u}|) \cdot (\phi_R |\nabla^2 \bar{u}|_\omega) e^{-f} \omega^n \\ &\leq c \left( \int_M |\nabla \phi_R|^2 |\nabla \bar{u}|^2 e^{-f} \omega^n \right)^{\frac{1}{2}} \left( \int_M |\nabla^2 \bar{u}|_\omega^2 \phi_R^2 e^{-f} \omega^n \right)^{\frac{1}{2}} \end{aligned}$$

for some positive constant  $c$  independent of  $R$  that may vary from line to line. By Claim 5.4, the previous inequality leads to the bound

$$\int_M |\nabla^{0,2} \bar{u}|_\omega^2 \phi_R^2 e^{-f} \omega^n \leq \frac{c}{R}$$

for some positive constant  $c$  independent of  $R$ . Letting  $R$  tend to  $+\infty$ , this shows that

$$\int_M |\nabla^{0,2} \bar{u}|_\omega^2 e^{-f} \omega^n = 0,$$

as desired. ◁

Consequently,  $\nabla^{0,2} \bar{u}_Y = 0$ , from which it follows that  $\nabla^{1,0} \bar{u}_Y$  is a holomorphic vector field.

Thus,  $\nabla^{1,0} u_Y$  and  $\nabla^{1,0} \bar{u}_Y$  are holomorphic vector fields. Write  $u_Y = v_Y + i w_Y$ , where  $v_Y$  and  $w_Y$  are smooth real-valued functions on  $M$ . Then we deduce that  $\nabla^{1,0} v_Y = \frac{1}{2}(\nabla v_Y - i J \nabla v_Y)$  and  $\nabla^{1,0} w_Y = \frac{1}{2}(\nabla w_Y - i J \nabla w_Y)$  are holomorphic. In particular,  $\nabla v_Y$  and  $\nabla w_Y$  are real holomorphic vector fields on  $M$  so that by [32, Lemma 2.3.8],  $J \nabla v_Y$  and  $J \nabla w_Y$  are real holomorphic  $g$ -Killing vector fields. Therefore we have the decomposition

$$\frac{1}{2}(Y - i J Y) = Y^{1,0} = \nabla^{1,0} u_Y = \nabla^{1,0} (v_Y + i w_Y) = \frac{1}{2}(\nabla v_Y + J \nabla w_Y) - \frac{1}{2}i(J \nabla v_Y - \nabla w_Y),$$

so that

$$(5-5) \quad Y = \nabla v_Y + J\nabla w_Y = J\nabla w_Y + J(-J\nabla v_Y).$$

Moreover, since  $\mathcal{L}_{JX}u_Y = 0$ , we have that  $\mathcal{L}_{JX}v_Y = \mathcal{L}_{JX}w_Y = 0$  so that  $[JX, \nabla v_Y] = [JX, \nabla w_Y] = 0$ , and consequently  $[X, J\nabla v_Y] = [X, J\nabla w_Y] = 0$ . Hence  $J\nabla v_Y$  and  $J\nabla w_Y$  lie in  $\mathfrak{g}^X$ , leaving (5-5) as the desired decomposition.

To show that this decomposition is direct, suppose that  $Z \in \mathfrak{g}^X \cap J\mathfrak{g}^X$ . Then  $Z = JW$ , where  $W$  and  $JW$  are real holomorphic and Killing. Since  $W$  is holomorphic and  $JW$  is Killing,  $\nabla W$  is symmetric. Since  $W$  is Killing,  $\nabla W$  is skew-symmetric. Thus,  $W$  is parallel. If  $W$  is nontrivial, then by [29, Corollary 3.2],  $(M, g)$  splits off a line, with  $W$  the generator of this line. In particular, we may write  $M = N \times \mathbb{R}$  for a manifold  $N$  with  $g = g_N \oplus dt^2$  and  $W = \partial_t$ , where  $t$  is the coordinate on the  $\mathbb{R}$ -direction and  $g_N$  a shrinking Ricci soliton on  $N$ . Now the soliton vector field  $X$  must split as a direct sum with the summand in the  $\mathbb{R}$ -direction necessarily  $t\partial_t$ . Since  $[W, X] = 0$  as  $Z \in \mathfrak{g}^X$ , this yields a contradiction, so that  $W = 0$ . Hence the stated decomposition of  $\text{aut}^X(M)$  is direct.  $\square$

Since  $M$  is noncompact, we need to verify that the various automorphism groups in question are indeed Lie groups. This is necessary for the applications of Theorem 5.1 that we have in mind. We begin with:

**Proposition 5.6** *Let  $(M, g, X)$  be a complete shrinking gradient Kähler–Ricci soliton with bounded Ricci curvature. Then there exists a unique connected Lie group  $\text{Aut}_0^X(M)$  (endowed with the compact–open topology) of diffeomorphisms acting effectively on  $M$  with Lie algebra  $\text{aut}^X(M)$ .*

$\text{Aut}_0^X(M)$  is of course the connected component of the identity of the holomorphic automorphisms of  $M$  that commute with the flow of  $X$ .

The fact that there is a unique Lie group  $\text{Aut}_0^X(M)$  with the stated properties follows from Palais’ integrability theorem [57] (see also [40, Theorem 3.1, page 13]), once we establish the completeness and finite-dimensionality of  $\text{aut}^X(M)$ . However, this theorem only asserts that  $\text{Aut}_0^X(M)$  is a Lie group with respect to the “modified” compact–open topology. In order to see that it is a Lie group with respect to the compact–open topology, we must appeal to [35, Theorem 5.14], using the fact that  $\text{Aut}_0^X(M)$  is closed with respect to the compact–open topology. Now, the completeness of the vector fields in  $\text{aut}^X(M)$  is clear from Lemma 2.34. As for their finite-dimensionality, we have the following.

**Proposition 5.7** *Let  $(M, g, X)$  be a complete shrinking gradient Kähler–Ricci soliton with bounded Ricci curvature and let  $\text{aut}^X(M)$  denote the space of all real holomorphic vector fields  $Y$  on  $M$  with  $[X, Y] = 0$ . Then  $\text{aut}^X(M)$  is finite-dimensional.*

**Proof** We provide an analytic proof of this fact. Letting  $|\cdot|$  denote the norm with respect to  $g$  and writing  $f$  for the soliton potential, we have a natural norm  $\|\cdot\|_{L_f^2}$  on  $\text{aut}^X(M)$  defined by

$$\|Y\|_{L_f^2}^2 := \int_M |Y|^2 e^{-f} \omega^n.$$

It suffices to show that the unit ball is compact with respect to this norm. To this end, suppose that we have a sequence  $(Y_i)_{i \geq 0}$  with  $\|Y_i\|_{L_f^2} = 1$ . Then by elliptic estimates, we get uniform  $C^k$ –bounds on  $|Y_i|$  over a fixed ball  $B_g(p, R) \subset M$  once a  $C^0$ –estimate is established. Now, by a Nash–Moser iteration applied to the norm of  $Y_i$ , one obtains the estimate

$$\sup_{B_g(p, R/2)} |Y_i| \leq C(n, R) \|Y_i\|_{L^2(B_g(p, R))}.$$

Since  $\|Y_i\|_{L_f^2} \leq 1$ , then, a fortiori,

$$\sup_{B_g(p, R/2)} |Y_i| \leq C(n, R) e^{cR^2} \|Y_i\|_{L_f^2(B_g(p, R))} \leq C'(n, R).$$

Finally, according to (the proof) of Proposition 2.30, there is some large radius  $R_0 > 0$  such that

$$(5-6) \quad |Y_i|(x) \leq C \left( n, \sup_M |\text{Ric}(g)|, \sup_{B_g(p, R_0)} |Y_i| \right) \cdot (d_g(p, x) + 1)^a \quad \text{for all } x \in M,$$

for some uniform positive constant  $a$ , where  $d_g(p, x)$  denotes the distance between  $p$  and  $x$  with respect to  $g$ . Since  $\sup_{B_g(p, R_0)} |Y_i| \leq C(n, R_0)$ , passing to a subsequence if necessary we may assume that  $(Y_i)_{i \geq 0}$  converges to some  $Y_\infty$  on the whole of  $M$  in the  $C_{\text{loc}}^\infty(M)$ –topology. The question is whether this convergence is strong in the above norm. Thanks to (5-6), given  $\varepsilon > 0$ , there exists some positive radius  $R$  such that for all indices  $i \geq 0$ ,

$$\|Y_i\|_{L_f^2(M \setminus B_g(p, R))} \leq \varepsilon,$$

since the soliton potential grows quadratically by Theorem 2.20(i), and the volume growth of geodesic balls is at most polynomial by Theorem 2.20(ii). This shows that if  $R$  is chosen sufficiently large, then the remainder of the norm outside  $B_g(p, R)$  is uniformly small; hence we do indeed have strong convergence. □

**Remark 5.8** Munteanu and Wang [51, Theorem 1.4] proved that the space of polynomial growth holomorphic functions of a fixed degree on a shrinking gradient Kähler–Ricci soliton is finite-dimensional without assuming a Ricci curvature bound. We therefore expect that the above proposition holds true in more generality. We also expect that the ring of holomorphic functions of polynomial growth on  $M$  is finitely generated and that  $M$  is algebraic, at least under a Ricci bound assumption.

Recall that  $\mathfrak{g}^X$  denotes the Lie algebra comprising real holomorphic  $g$ –Killing vector fields that commute with  $X$  and hence with  $JX$ . We next consider the existence of a Lie group with Lie algebra  $\mathfrak{g}^X$ .

**Proposition 5.9** *Let  $(M, g, X)$  be a complete shrinking gradient Kähler–Ricci soliton with bounded Ricci curvature. Then there exists a unique connected Lie group  $G_0^X$  (endowed with the compact–open topology) of diffeomorphisms acting effectively on  $M$  with Lie algebra  $\mathfrak{g}^X$ .*

**Proof** Since  $\mathfrak{g}^X$  is a Lie subalgebra of the Lie algebra of  $g$ –Killing vector fields on  $M$ ,  $\mathfrak{g}^X$  is a finite-dimensional Lie algebra. Furthermore, vector fields induced by  $\mathfrak{g}^X$  on  $M$  are complete by Lemma 2.34.

By Palais’ integrability theorem [57] therefore, there exists a unique connected Lie group  $G_0^X$  of diffeomorphisms acting effectively on  $M$  with Lie algebra  $\mathfrak{g}^X$ .  $G_0^X$  is precisely the connected component of the identity of the Lie group of holomorphic isometries on  $M$  that commute with the flow of  $X$ . Since  $G_0^X$  is closed with respect to the compact–open topology, [35, Theorem 5.14] guarantees that  $G_0^X$  is a Lie group with respect to this topology, as stated.  $\square$

We next prove that  $G_0^X$  is a compact Lie subgroup of  $\text{Aut}_0^X(M)$ .

**Lemma 5.10** *Let  $(M, g, X)$  be a complete shrinking gradient Kähler–Ricci soliton with complex structure  $J$  and with soliton vector field  $X = \nabla^g f$  for a smooth real-valued function  $f: M \rightarrow \mathbb{R}$ . Then elements of  $\mathfrak{g}^X$  are tangent to the level sets of  $f$ . Moreover, if  $g$  has bounded Ricci curvature, then  $G_0^X$  is a compact Lie subgroup of  $\text{Aut}_0^X(M)$  (with respect to the compact–open topology).*

**Proof** Let  $Y \in \mathfrak{g}^X$ . For the first part of the lemma, we will show that  $\mathcal{L}_Y f = 0$ , so that the flow of  $Y$  preserves the level sets of  $f$ , thereby forcing  $Y$  to be tangent to the level sets of  $f$ .

Applying  $\mathcal{L}_Y$  to the shrinking Kähler–Ricci soliton equation, we find that  $i\partial\bar{\partial}(\mathcal{L}_Y f) = 0$ . Notice that since  $[JX, Y] = 0$ , we have that  $\mathcal{L}_{JX}(\mathcal{L}_Y f) = \mathcal{L}_Y(\mathcal{L}_{JX} f) = 0$ . The function  $X \cdot (\mathcal{L}_Y f)$  is therefore holomorphic. It is also real-valued, hence must be equal to a constant, say  $X \cdot (\mathcal{L}_Y f) = c_0$ . Since  $X = \nabla^g f$  and  $f$  has a minimum, we deduce that in fact  $c_0 = 0$ , so that  $X \cdot (\mathcal{L}_Y f) = 0$ .

Next, deriving with respect to the Killing vector field  $Y$  the soliton identity from Lemma 2.22, namely

$$\mathcal{L}_X f + R_g = X \cdot f + R_g = |\nabla^g f|^2 + R_g = 2f,$$

making use of the fact that  $X$  and  $Y$  commute, we obtain

$$2\mathcal{L}_Y f = \mathcal{L}_Y(\mathcal{L}_X f) + \underbrace{\mathcal{L}_Y R_g}_{=0} = \mathcal{L}_X(\mathcal{L}_Y f) = X \cdot (\mathcal{L}_Y f),$$

where we have just seen that this last term vanishes. Hence  $\mathcal{L}_Y f = 0$ , as desired.

As for the second part of the lemma, note that under the assumption of bounded Ricci curvature of  $g$ , both  $G_0^X$  and  $\text{Aut}_0^X(M)$  are Lie groups endowed with the compact–open topology, by Propositions 5.6 and 5.9, respectively. In addition,  $G_0^X$  is a subgroup of  $\text{Aut}_0^X(M)$  since  $\mathfrak{g}^X$  is a Lie subalgebra of  $\mathfrak{aut}^X(M)$ . Compactness of  $G_0^X$  with respect to the compact–open topology follows from the Arzelà–Ascoli theorem because the level sets of  $f$  are compact by properness of  $f$  and, as we have just seen, are preserved by  $G_0^X$ . Being compact,  $G_0^X$  is then a closed subgroup of  $\text{Aut}_0^X(M)$ , hence is a compact Lie subgroup of  $\text{Aut}_0^X(M)$ , with everything being relative to the compact–open topology.  $\square$

Finally, we can now deduce from Theorem 5.1 that  $G_0^X$  is a maximal compact Lie subgroup of  $\text{Aut}_0^X(M)$ .

**Corollary 5.11** *Let  $(M, g, X)$  be a complete shrinking gradient Kähler–Ricci soliton with bounded Ricci curvature. Then  $G_0^X$  is a maximal compact Lie subgroup of  $\text{Aut}_0^X(M)$ .*

**Proof** First note that  $G_0^X$  is a compact Lie subgroup of  $\text{Aut}_0^X(M)$  by Lemma 5.10. Now suppose that  $G_0^X$  is not maximal in  $\text{Aut}_0^X(M)$ . Then there exists a compact Lie subgroup  $K$  of  $\text{Aut}_0^X(M)$  strictly containing  $G_0$ . In particular, the real dimension of  $K$  must be strictly greater than  $G_0^X$ . On the Lie algebra level, since the decomposition of Theorem 5.1 is direct, there exists a nonzero real holomorphic vector  $Z$  in the Lie algebra of  $K$  which is not contained in  $\mathfrak{g}^X$ , yet is contained in  $J\mathfrak{g}^X$ . Since  $K$  is compact, the closure of the flow of  $Z$  in  $K$  will define a real torus  $T^k$  of real dimension  $k$  in  $K$  in which the flow of  $Z$  is dense.

Consider the vector field  $Z$ . This is a real holomorphic vector field with  $JZ$  Killing. Since a shrinking soliton has finite fundamental group [67], we have that  $H^1(M) = 0$ . Hence  $JZ$  admits a Hamiltonian potential  $u: M \rightarrow \mathbb{R}$  so that  $Z = \nabla^g u$ . If the real dimension  $k$  of  $T$  is equal to one, then the orbits of  $Z$  are all closed, but a gradient flow has no nontrivial closed integral curves since

$$\frac{d}{dt}u(\gamma_x(t)) = |\nabla^g u|^2 \geq 0,$$

where  $\gamma_x(t)$  denotes the integral curve of  $Z$  with  $\gamma_x(0) = x \in M$ . Hence  $k$  is strictly greater than one. But this is impossible as well. Indeed, let  $x$  be any point of  $M$  where  $Z(x) \neq 0$ . Then  $u(\gamma_x(t))$  is an increasing function of  $t$ , so that  $u(\gamma_x(t)) - u(x) > c$  for some constant  $c > 0$  say, for all  $t > 1$ . On the other hand, since the flow of  $Z$  is dense in  $T$ ,  $\gamma_x(t)$  intersects any neighbourhood of  $x$  in  $M$  for some  $t > 1$ . This yields another contradiction. Thus,  $G_0^X$  is maximal in  $\text{Aut}_0^X(M)$ , as claimed.  $\square$

## 5.2 The weighted volume functional

Recall that  $(M, g, X)$  is a complete shrinking gradient Kähler–Ricci soliton of complex dimension  $n$  with complex structure  $J$ , Kähler form  $\omega$  and soliton vector field  $X = \nabla^g f$  for  $f: M \rightarrow \mathbb{R}$  smooth, and that by assumption, we have a real torus  $T$  with Lie algebra  $\mathfrak{t}$  acting holomorphically, effectively and isometrically on  $(M, g, J)$  with  $JX \in \mathfrak{t}$ .

In order to make sense of the weighted volume functional of a shrinking gradient Kähler–Ricci soliton, we need to define a moment map for the action of  $T$  on  $M$ . This comes down to showing that every element of  $\mathfrak{t}$  admits a real Hamiltonian potential, as demonstrated in the next proposition. Such a potential exists essentially because  $T$  acts by isometries and  $H^1(M) = 0$ . However, a Hamiltonian potential is only defined up to a constant. Therefore a normalisation is required to determine the potential uniquely. We normalise so that the potential lies in the kernel of a certain linear operator, precisely the condition required to show that  $JX$  is the unique critical point of the weighted volume functional.

**Proposition 5.12** *In the above situation, let  $Y \in \mathfrak{t}$  so that  $Y$  defines a real holomorphic  $g$ –Killing vector field on  $M$  with  $[X, Y] = 0$ . Then there exists a unique smooth real-valued function  $u_Y: M \rightarrow \mathbb{R}$  with  $\mathcal{L}_{JX}u_Y = 0$  such that  $\Delta_\omega u_Y + u_Y + \frac{1}{2}(JY) \cdot f = 0$  and  $du_Y = -\omega \lrcorner Y$ .*

**Proof** Let  $Z := -JY$ . Then  $Z$  is real holomorphic and  $JZ$  is  $g$ –Killing. Since a shrinking soliton has finite fundamental group [67], we have that  $H^1(M) = 0$ . This implies that there exists a smooth

real-valued function  $u_Y : M \rightarrow \mathbb{R}$  such that  $Z = \nabla^g u_Y$ . Then  $du_Y \circ J = -\omega \lrcorner Z$ . Let  $Z^{1,0} = \frac{1}{2}(Z - iJZ)$ . Then we have that

$$\omega \lrcorner Z^{1,0} = \frac{1}{2}\omega \lrcorner Z - \frac{1}{2}i\omega \lrcorner JZ = -\frac{1}{2}du_Y \circ J + \frac{1}{2}i du_Y = \frac{1}{2}i(du_Y + i du_Y \circ J) = i\bar{\partial}u_Y,$$

ie  $\bar{\partial}(-iu_Y) = -\omega \lrcorner Z^{1,0}$ . After noting from the proof of Lemma 5.10 that  $Y \in \mathfrak{t}$  implies that  $Y \cdot f = 0$ , we automatically have that  $\mathcal{L}_{JX}u_Y = 0$ . Using the Bochner formula, contracting (1-4) with  $\lambda = 1$  with  $Z^{1,0}$  then results in

$$-i\bar{\partial}\Delta_\omega u_Y + i\bar{\partial}(Z^{1,0} \cdot f) = i\bar{\partial}u_Y.$$

In other words,

$$\bar{\partial}(\Delta_\omega u_Y + u_Y - Z^{1,0} \cdot f) = 0.$$

Now, the fact that  $\mathcal{L}_{JX}u_Y = 0$  implies that

$$0 = du_Y(JX) = g(\nabla^g u_Y, JX) = g(Z, JX) = -g(JZ, X) = -df(JZ) = -(JZ) \cdot f.$$

In particular,  $\Delta_\omega u_Y + u_Y - Z^{1,0} \cdot f$  is a real-valued holomorphic function, hence is equal to a constant. By subtracting this constant from  $u_Y$  and plugging in the definition of  $Z$ , we arrive at our desired normalisation of  $u_Y$ , namely

$$\Delta_\omega u_Y + u_Y + \frac{1}{2}(JY) \cdot f = 0.$$

Since  $u_Y$  is defined up to a constant, this condition determines  $u_Y$  uniquely. □

With this proposition, we can now define our moment map for the action of  $T$  on  $M$ .

**Definition 5.13** Let  $\langle \cdot, \cdot \rangle$  denote the natural pairing between  $\mathfrak{t}$  and  $\mathfrak{t}^*$ . Then we define the *moment map*  $\mu : M \rightarrow \mathfrak{t}^*$  for the action of  $T$  on  $M$  as follows: for  $x \in M$ ,  $\mu(x)$  is defined by the equation

$$u_Y(x) = \langle \mu(x), Y \rangle \quad \text{for all } Y \in \mathfrak{t},$$

where  $u_Y$  is such that  $\nabla^g u_Y = -JY$ ,  $\mathcal{L}_{JX}u_Y = 0$  and  $\Delta_\omega u_Y + u_Y + \frac{1}{2}(JY) \cdot f = 0$ .

We next define the weighted volume functional for complete shrinking gradient Kähler–Ricci solitons.

**Definition 5.14** (weighted volume functional, see [64, equation (2.3)]) Let  $(M, g, X)$  be a complete shrinking gradient Kähler–Ricci soliton of complex dimension  $n$  with complex structure  $J$ , Kähler form  $\omega$ , and with soliton vector field  $X = \nabla^g f$  for a smooth real-valued function  $f : M \rightarrow \mathbb{R}$ , endowed with the holomorphic, effective, isometric action of a real torus  $T$  with Lie algebra  $\mathfrak{t}$  and with a compact fixed point set. Let  $\mu$  denote the moment map of the action as prescribed in Definition 5.13 and assume that  $JX \in \mathfrak{t}$ . Let  $Y \in \mathfrak{t}$  and let  $u_Y := \langle \mu, Y \rangle$  be the Hamiltonian potential of  $Y$ , so that  $\mathcal{L}_{JX}u_Y = 0$  and

$$(5-7) \quad \Delta_\omega u_Y + u_Y + \frac{1}{2}(JY) \cdot f = 0.$$

Finally, let

$$\Lambda := \{Y \in \mathfrak{t} \mid u_Y \text{ is proper and bounded below}\} \subseteq \mathfrak{t}.$$

Then the *weighted volume functional*  $F: \Lambda \rightarrow \mathbb{R}_{>0}$  is defined by

$$F(Y) = \int_M e^{-\langle \mu, Y \rangle} \omega^n = \int_M e^{-u_Y} \omega^n.$$

The set  $\Lambda$  is an open cone in  $\mathfrak{t}$  which is determined by the image of  $M$  under  $\mu$ ; see Proposition A.4 for details. Since  $f$  grows quadratically at infinity by Theorem 2.20(i), we know that it is also proper. Hence  $JX$ , which by assumption lies in  $\mathfrak{t}$ , lies in  $\Lambda$ , so that  $\Lambda$  is nonempty. Thus, by the Duistermaat–Heckman theorem (Theorem A.3),  $F$  is seen to be well-defined. (Proposition A.13 provides an alternative argument, without using the Duistermaat–Heckman theorem, for why  $F$  is well-defined.) As the next lemma shows, the value of  $F$  is also independent of the choice of shrinking gradient Kähler–Ricci soliton. This relies on the normalisation (5-7) of the Hamiltonian potentials.

**Lemma 5.15** *Let  $(M, g_i, X_i)$  for  $i = 1, 2$  be two shrinking gradient Kähler–Ricci solitons, both satisfying the hypotheses of Definition 5.14 with respect to a fixed real torus  $T$ . Let  $F_i$  denote the weighted volume functional of  $(M, g_i, X_i)$  and let  $\Lambda_i$  denote the domain of  $F_i$ . Then  $F_1 = F_2$  on  $\Lambda_1 \cap \Lambda_2$ .*

**Proof** Let  $\omega_i$  denote the Kähler form of  $g_i$  and let  $Y \in \Lambda_1 \cap \Lambda_2$ . Write  $u_Y^{(i)}$  for the Hamiltonian potential of  $Y$  with respect to  $\omega_i$ . Then analysing the expression given in Theorem A.3 for each  $F_i$ , namely (A-1), one sees that the right-hand side depends only on the value of  $u_Y^{(i)}$  on the (compact) zero set  $M_0(Y)$  of  $Y$  and integrals over this set with respect to  $\omega_i$ . Now, the normalisation condition (5-7) infers that on  $M_0(Y)$ ,  $u_Y^{(i)} = -\Delta_{\omega_i} u_Y^{(i)} = -\operatorname{div}(Y)$ , a quantity that, on  $M_0(Y)$ , is independent of the choice of metric. Moreover,  $du_Y^{(i)} = -\omega_i \lrcorner Y$  so  $u_Y^{(i)}$  is evidently constant on each connected component of  $M_0(Y)$ . Thus, we deduce that  $u_Y^{(1)} = u_Y^{(2)}$  on  $M_0(Y)$ , both being equal to a fixed constant on each connected component of  $M_0(Y)$ . As for the integrals on the right-hand side of (A-1), these involve integrating a closed form  $\omega_i$  over the compact boundary-less set  $M_0(Y)$ . Both being shrinking Kähler–Ricci solitons,  $\omega_1$  and  $\omega_2$  lie in the same cohomology class, hence integrating over  $M_0(Y)$  with respect to either  $\omega_1$  or  $\omega_2$  does not change the value of the integral. This brings us to the desired conclusion.  $\square$

We next list some more elementary properties of  $F$ , in particular the desired property that characterises  $JX$  as the unique critical point of  $F$ . Here our normalisation of the Hamiltonian potentials also comes into play.

**Lemma 5.16** (volume-minimising principle) *Let  $(M, g, X)$  be a complete shrinking gradient Kähler–Ricci soliton of complex dimension  $n$  with Kähler form  $\omega$  and with soliton vector field  $X = \nabla^g f$  for a smooth real-valued function  $f: M \rightarrow \mathbb{R}$ , endowed with the holomorphic, effective, isometric action of a real torus  $T$  with Lie algebra  $\mathfrak{t}$ . Assume that  $JX \in \mathfrak{t}$  and that the Ricci curvature of  $g$  is bounded. Then*

- (i)  $F$  is strictly convex on  $\Lambda$ , and
- (ii)  $JX$  is the unique critical point of  $F$  in  $\Lambda$ .

**Remark 5.17** The boundedness of the scalar curvature of  $g$  guarantees here that  $F$  is well-defined on  $\Lambda$ . Indeed, this is clear from the Duistermaat–Heckman theorem (Theorem A.3) after noting that the zero set of  $X$ , which contains the fixed point set of  $T$  as a closed subset, is compact by Lemma 2.25.

**Proof of Lemma 5.16** (i) Let  $Y_1, Y_2 \in \Lambda$ . Then the line segment  $tY_1 + (1-t)Y_2, t \in [0, 1]$ , is contained in  $\Lambda$  because  $\Lambda$  is convex, as one sees from its definition. Moreover, by the linearity of the moment map, we have that

$$u_{tY_1+(1-t)Y_2} = tu_{Y_1} + (1-t)u_{Y_2} \quad \text{for all } t \in [0, 1].$$

Thus, since the function  $x \in \mathbb{R} \mapsto e^{-x} \in \mathbb{R}$  is strictly convex, we find that

$$F(t \cdot Y_1 + (1-t) \cdot Y_2) < t \cdot F(Y_1) + (1-t) \cdot F(Y_2) \quad \text{for all } t \in (0, 1), \text{ unless } Y_1 = Y_2.$$

(ii) As a strictly convex function on the convex set  $\Lambda$ ,  $F$  has at most one critical point. The claim is that this critical point is obtained at  $JX$ . Indeed, let  $Y \in \mathfrak{t}$  and let  $u_Y$  denote the Hamiltonian potential of  $Y$ , normalised so that  $\Delta_\omega u_Y + u_Y + \frac{1}{2}(JY) \cdot f = 0$ . Recall that  $-J(JX) = \nabla^g f$ , so that

$$d_{JX} F(Y) = - \int_M u_Y e^{-f} \omega^n.$$

Let  $R$  be a positive real number and let  $\phi_R: M \rightarrow [0, 1]$  be a cut-off function with compact support in the geodesic ball  $B_g(p, 2R)$  such that  $\phi_R = 1$  on  $B_g(p, R)$  and  $|\nabla \phi_R|_g \leq c/R$  for some  $c > 0$ . Then, using integration by parts, we have that

$$\begin{aligned} \left| \int_M u_Y \phi_R^2 e^{-f} \omega^n \right| &= \left| \int_M (\Delta_\omega u_Y + \frac{1}{2}(JY) \cdot f) \phi_R^2 e^{-f} \omega^n \right| = \left| \frac{1}{2} \int_M (\Delta_g - \nabla^g f \cdot) u_Y \phi_R^2 e^{-f} \omega^n \right| \\ &= \frac{1}{2} \left| \int_M g(\nabla u_Y, \nabla(\phi_R^2)) e^{-f} \omega^n \right| \\ &\leq \frac{1}{2} \left( \int_M |\nabla u_Y|^2 e^{-f} \omega^n \right)^{\frac{1}{2}} \left( \int_M |\nabla(\phi_R^2)|^2 e^{-f} \omega^n \right)^{\frac{1}{2}} \\ &= \left( \int_M |\nabla u_Y|^2 e^{-f} \omega^n \right)^{\frac{1}{2}} (|\nabla \phi_R|^2 |\phi_R|^2 e^{-f} \omega^n)^{\frac{1}{2}} \\ &\leq \frac{c^2}{R^2} \left( \int_M |\nabla u_Y|^2 e^{-f} \omega^n \right)^{\frac{1}{2}} \left( \int_M e^{-f} \omega^n \right)^{\frac{1}{2}}, \end{aligned}$$

where the fact that  $\nabla u_Y \in L^2(e^{-f} \omega^n)$  follows as in the proof of Claim 5.4. Letting  $R \rightarrow +\infty$ , we see that  $d_{JX} F(Y) = 0$ , as required.  $\square$

The main tool we use to compute the weighted volume functional is the Duistermaat–Heckman theorem. The statement of this theorem and a discussion have been relegated to the appendix. It expresses the weighted volume functional in terms of data determined by the induced action on  $M$  of the element  $Y \in \Lambda$ . In particular, this data is independent of the metric  $\omega$ . Consequently,  $F$  is independent of the particular shrinking gradient Kähler–Ricci soliton. It is this observation that will allow us to ascertain the uniqueness of the soliton vector field  $X$  under certain assumptions. This is the content of the next subsection.



### 5.3 A general uniqueness theorem

As an application of Corollary 5.11, we prove the uniqueness statement of Theorem D for the soliton vector field  $X$  of a shrinking gradient Kähler–Ricci soliton, the precise statement of which we now recall below.

**Theorem 5.18** (Theorem D) *Let  $M$  be a noncompact complex manifold with complex structure  $J$ , endowed with the effective holomorphic action of a real torus  $T$ . Denote by  $\mathfrak{t}$  the Lie algebra of  $T$ . Then there exists at most one element  $\xi \in \mathfrak{t}$  that admits a complete shrinking gradient Kähler–Ricci soliton  $(M, g, X)$  with bounded Ricci curvature, with  $X = \nabla^g f = -J\xi$  for a smooth real-valued function  $f$  on  $M$ .*

The outline of the proof of this theorem is as follows. Suppose that  $M$  admitted two soliton vector fields  $X_1$  and  $X_2$ . Then the maximal tori in the Lie groups  $\text{Aut}_0^{X_i}(M)$  for  $i = 1, 2$  will be conjugate to  $T$  by Iwasawa’s theorem [39]. After choosing an appropriate gauge,  $JX_1$  and  $JX_2$  will then be contained in the Lie algebra  $\mathfrak{t}$  of  $T$  and both vector fields will be critical points of their respective weighted volume functional. But since the weighted volume functional is independent of the shrinking Kähler–Ricci soliton by the Duistermaat–Heckman theorem, both weighted volume functionals must coincide, so that  $JX_1 = JX_2$  by uniqueness of the critical point.

**Proof** Suppose that  $M$  admitted two complete shrinking gradient Kähler–Ricci solitons,  $(M, g_i, X_i)$  for  $i = 1, 2$ , with bounded Ricci curvature and with  $X_i = \nabla^{g_i} f_i$  for  $f_i: M \rightarrow \mathbb{R}$  smooth such that  $X_i = -J\xi_i$  for  $\xi_i \in \mathfrak{t}$ . Let  $G_0^{X_i}$  denote the connected component of the identity of the group of holomorphic isometries of  $(M, g_i, J)$  that commute with the flow of  $X_i$ . Corollary 5.11 then asserts that  $G_0^{X_i}$  is a maximal compact Lie subgroup of the Lie group  $\text{Aut}_0^{X_i}(M)$ , the connected component of the identity of the group of automorphisms of  $(M, J)$  that commute with the flow of  $X_i$ . Denote by  $T_i$  the maximal real torus in  $G_0^{X_i}$ . Then  $T_i$  is maximal in  $\text{Aut}_0^{X_i}(M)$ . For each  $v \in \mathfrak{t}$ , we have that  $[v, \xi_i] = 0$ , so that  $[v, X_i] = 0$ . Hence each element of  $T$  commutes with the flow of  $X_i$  and so  $T$  itself is a Lie subgroup of  $\text{Aut}_0^{X_i}(M)$ . Without loss of generality, we may assume that  $T$  is maximal in  $\text{Aut}_0^{X_i}(M)$ . Then, by Iwasawa’s theorem [39], there exists an element  $\alpha_i \in \text{Aut}_0^{X_i}(M)$  such that  $\alpha_i T_i \alpha_i^{-1} = T$ . Since  $\alpha_i$  commutes with the flow of  $X_i$ , necessarily  $d\alpha_i^{-1}(X_i) = X_i$ . Moreover,  $\alpha_i^* g_i$  is invariant under  $T$ . Thus,  $(M, \tilde{g}_i, \tilde{X}_i)$  with  $\tilde{g}_i = \alpha_i^* g_i$  and  $\tilde{X}_i = d\alpha_i^{-1}(X_i)$  is a  $T$ –invariant shrinking gradient Kähler–Ricci soliton with soliton vector field  $\tilde{X}_i = X_i = -J\xi_i$  as before. Hence, by considering this pullback, we may assume that each  $(M, g_i, X_i)$  is invariant under  $T$ .

Now, by assumption we have that  $\xi_1, \xi_2 \in \mathfrak{t}$ . Since the corresponding Hamiltonian potentials are the soliton potentials, which themselves are proper and bounded below, we have that  $\xi_i \in \Lambda_i \subset \mathfrak{t}$ , where  $\Lambda_i$  denotes the open cone of elements of  $\mathfrak{t}$  admitting Hamiltonian potentials with respect to the Kähler form  $\omega_i$  of  $g_i$  that are proper and bounded below. We wish to show that  $\xi_1, \xi_2 \in \Lambda_1 \cap \Lambda_2 \neq \emptyset$ . The result will then follow from an application of the Duistermaat–Heckman theorem. So let  $u_1$  denote the

Hamiltonian potential of  $\xi_1 = JX_1$  with respect to  $g_2$ , that is,  $\nabla^{g_2}u_1 = X_1$ , and for  $x \in M$ , let  $\gamma_1(t)$  denote the integral curve of  $X_1$  through  $x$  at  $t = 0$ . Then we have that

$$\begin{aligned} u_1(\gamma_1(t)) &= u_1(\gamma_1(0)) + \int_0^t du_1(\dot{\gamma}_1(s)) ds \\ &= u_1(x) + \int_0^t g_2(X_1, X_1)(\gamma_1(s)) ds \\ &= u_1(x) + \int_0^t |X_1|_{g_2}^2(\gamma_1(s)) ds. \end{aligned}$$

Since  $g_1$  has bounded Ricci curvature, so that the zero set of  $X_1$  is compact by Lemma 2.25, and since each forward orbit of the negative gradient flow of  $f_1$  converges to a point in the zero set of  $X_1$  by Proposition 2.27, it is clear that every point of  $M$  lies on an integral curve of  $X_1$  passing through a fixed compact set. Thus, we see that  $u_1$  is bounded from below. For  $t \in [0, 1]$ , consider the vector field  $Y_t := t\xi_1 + (1-t)\xi_2$ . The Hamiltonian potential of  $Y_0$  with respect to  $g_2$  is  $f_2$ , whereas that of  $Y_1$  is  $u_1$ . By linearity of the moment map, the Hamiltonian potential of  $Y_t$  with respect to  $g_2$  is  $h_t := tu_1 + (1-t)f_2$ . Since  $u_1$  is bounded from below and  $f_2$  is proper,  $h_t$  is proper and bounded below for  $t \in [0, 1]$ , so that  $Y_t \in \Lambda_2$  for  $t \in [0, 1]$ . In a similar manner, one can show that  $Y_t \in \Lambda_1$  for  $t \in (0, 1]$ . The upshot is that  $Y_t \in \Lambda_1 \cap \Lambda_2 \neq \emptyset$  for  $t \in (0, 1)$  with  $\xi_1, \xi_2 \in \overline{\Lambda_1 \cap \Lambda_2}$ .

Define a real-valued function  $F$  on  $[0, 1]$  as follows:  $F(t) := F_2(Y_t)$  if  $t \in [0, 1)$  and  $F(t) := F_1(Y_t)$  if  $t \in (0, 1]$ , where  $F_i$  is the weighted volume functional with respect to  $\omega_i$ . Then  $F$  is well-defined as both  $F_1$  and  $F_2$  are well-defined because of the Ricci curvature bound (see Remark 5.17) and by Lemma 5.15 they are equal on  $\Lambda_1 \cap \Lambda_2$ . Moreover,  $F$  is convex and continuous on  $[0, 1]$  and strictly convex on  $(0, 1)$ . Finally, observe that

$$F(0) = F_2(\xi_2) \leq \min_{[0,1]} F_2 \leq F_2(Y_t) = F_1(Y_t)$$

for every  $t \in (0, 1)$ . By letting  $t$  tend to 1, one sees that  $F(0) \leq F(1)$ . By symmetry, one also sees that  $F(1) \leq F(0)$ , which implies that  $F(1) = F(0) = \min_{[0,1]} F$ . Since  $F$  is convex,  $F$  must be constant on  $[0, 1]$ , which contradicts the fact that  $F$  is strictly convex on  $(0, 1)$  unless  $(Y_t)_{t \in (0,1)}$  is reduced to a single point, ie unless  $\xi_1 = \xi_2$ . This concludes the proof.  $\square$

#### 5.4 Shrinking gradient Kähler–Ricci solitons on $\mathbb{C}^n$ and $\mathcal{O}(-k) \rightarrow \mathbb{P}^{n-1}$ for $0 < k < n$

Using Theorem D, we are now able to classify shrinking gradient Kähler–Ricci solitons with bounded Ricci curvature on  $\mathbb{C}^n$  and on the total space of the line bundle  $\mathcal{O}(-k) \rightarrow \mathbb{P}^{n-1}$  for  $0 < k < n$ , and in doing so, prove items (1) and (2) of Theorem E.

**Theorem 5.19** (items (1) and (2) of Theorem E) *Let  $(M, g, X)$  be a complete shrinking gradient Kähler–Ricci soliton with bounded Ricci curvature.*

- (1) *If  $M = \mathbb{C}^n$ , then up to pullback by an element of  $\mathrm{GL}(n, \mathbb{C})$ ,  $(M, g, X)$  is the flat Gaussian shrinking soliton.*

- (2) If  $M$  is the total space of the line bundle  $\mathcal{O}(-k) \rightarrow \mathbb{P}^{n-1}$  for  $0 < k < n$ , then up to pullback by an element of  $GL(n, \mathbb{C})$ ,  $(M, g, X)$  is the unique  $U(n)$ –invariant shrinking gradient Kähler–Ricci soliton constructed by Feldman, Ilmanen and Knopf [30] on this complex manifold.

**Proof** Let  $f$  denote the soliton potential of  $X$ , so that  $f: M \rightarrow \mathbb{R}$  is a smooth real-valued function with  $X = \nabla^g f$ , and let  $M$  be as in item (1) or (2) of the theorem. We make no distinction as of yet. Denote the complex structure of  $M$  by  $J$  and let  $G_0^X$  denote the connected component of the identity of the holomorphic isometries of  $(M, J, g)$  that commute with the flow of  $X$ . Since  $g$  has bounded Ricci curvature,  $G_0^X$  is a compact Lie group by Lemma 5.10, hence the closure of the flow of  $JX$  in  $G_0^X$  yields the holomorphic isometric action of a real torus  $T$  on  $(M, J, g)$  with Lie algebra  $\mathfrak{t}$  containing  $JX$ . Since  $M$  is 1–convex by [20, Lemma 2.15] and the Ricci curvature of  $g$  is bounded, Proposition 2.26 tells us that the zero set of  $X$ , and correspondingly the fixed-point set of  $T$ , comprises a single point in item (1) and is contained in the zero section of the line bundle in item (2). Furthermore, Proposition 2.27 implies that each forward orbit of the negative gradient flow of  $f$  converges to a point in this fixed-point set. By contracting the zero section of the line bundle in item (2), we see that the action of  $T$  on  $M$  induces an action of  $T$  on  $\mathbb{C}^n/\mathbb{Z}_k$  for  $k = 1, \dots, n - 1$ , as appropriate with fixed-point set the apex, and that this action further lifts to an action of  $T$  on  $\mathbb{C}^n$  with an isolated fixed point. The lift of  $X$  to  $\mathbb{C}^n$  then defines a holomorphic vector field on  $\mathbb{C}^n$  with  $J_0X \in \mathfrak{t}$ , where  $J_0$  denotes the standard complex structure on  $\mathbb{C}^n$ , and with each forward orbit of  $-X$  converging to this isolated fixed point.

By [17, Section 3.1], we may choose global holomorphic coordinates  $(z_1, \dots, z_n)$  on  $\mathbb{C}^n$  with respect to which the action of  $T$  on  $\mathbb{C}^n$  is linear, that is, lies in  $GL(n, \mathbb{C})$ . These coordinates descend to coordinates on  $\mathbb{C}^n/\mathbb{Z}_k$ , then lift to coordinates on  $M$  with respect to which the action of  $T$  on  $M$  lies in  $GL(n, \mathbb{C})$ . Without loss of generality, we may assume that  $T$  is maximal in  $GL(n, \mathbb{C})$ . Then we still have that  $JX \in \mathfrak{t}$ . Since any two maximal tori in  $GL(n, \mathbb{C})$  are conjugate by Iwasawa’s theorem [39], there exists  $\alpha \in GL(n, \mathbb{C})$  such that  $\alpha T \alpha^{-1}$  is equal to  $\{\text{diag}(e^{i\eta_1}, \dots, e^{i\eta_n}) \mid \eta_i \in \mathbb{R}\}$ . By considering  $\alpha^* \omega$ , we can therefore assume that  $JX$  lies in the Lie algebra  $\mathfrak{t}$  of a torus of the form  $T = \{\text{diag}(e^{i\eta_1}, \dots, e^{i\eta_n}) \mid \eta_i \in \mathbb{R}\}$  acting on  $M$ . We will then have induced coordinates  $(\eta_1, \dots, \eta_n)$  on  $\mathfrak{t}$ , where  $(1, 0, \dots, 0) \in \mathfrak{t}$  will generate the vector field  $\text{Im}(z_1 \partial_{z_1})$  on  $M$ , and so on.

Since the fixed-point set of  $T$  is compact, we can now apply Theorem D, which tells us that there is at most one element of  $\mathfrak{t}$  that admits a complete shrinking gradient Kähler–Ricci soliton with bounded Ricci curvature. On  $\mathbb{C}^n$ , we have the flat Gaussian shrinking soliton and Feldman, Ilmanen and Knopf [30] have constructed a complete shrinking gradient Kähler–Ricci soliton with bounded Ricci curvature on  $M$  for  $M$  as in item (2) of the theorem. In all cases, the soliton vector field  $X$  of these solitons satisfies  $JX \in \mathfrak{t}$ , and each is proportional to  $(1, \dots, 1)$  in our coordinates on  $\mathfrak{t}$ . Therefore we deduce that  $JX = \lambda(1, \dots, 1)$  for some  $\lambda > 0$ , so that on  $M$ , we have  $\frac{1}{2}(X - iJX) = \lambda z_i \partial_{z_i}$ . The automorphism group of  $(M, J)$  commuting with the flow of this vector field is precisely the Lie group  $GL(n, \mathbb{C})$ . Thus, Corollary 5.11 asserts that  $G_0^X$  is maximal compact in  $GL(n, \mathbb{C})$  and so, by Iwasawa’s theorem [39] again, there exists

$\beta \in \mathrm{GL}(n, \mathbb{C})$  such that  $\beta G_0^X \beta^{-1} = U(n)$ . The  $(1, 1)$ -form  $\beta^* \omega$  will then be  $U(n)$ -invariant and by [30, Proposition 9.3], the only such complete shrinking gradient Kähler–Ricci soliton on  $M$  is the flat Gaussian shrinking soliton if  $M = \mathbb{C}^n$ , and that constructed by Feldman, Ilmanen and Knopf if  $M$  is as in item (2) of the theorem.  $\square$

## 6 The underlying manifold of a two-dimensional shrinking gradient Kähler–Ricci soliton

Item (3) of Theorem E will result from items (1) and (2) of Theorem E once we establish the following theorem.

**Theorem 6.1** *Let  $(M, g, X)$  be a two-dimensional complete shrinking gradient Kähler–Ricci soliton whose scalar curvature decays to zero at infinity. Then  $C_0$  is biholomorphic to  $\mathbb{C}^2$ , and  $M$  is biholomorphic to either  $\mathbb{C}^2$  or  $\mathbb{C}^2$  blown up at one point.*

The key observation in proving this theorem is that the scalar curvature of the asymptotic cone is strictly positive if the shrinking soliton is not flat. Since we are working in complex dimension two, this allows us to identify the tangent cone at infinity as a quotient singularity using a classification theorem of Belgun [4, Theorem 8] for 3-dimensional Sasaki manifolds. The fact that  $M$  is a resolution of  $C_0$  by Theorem A, combined with the fact that the exceptional set of this resolution must contain only  $(-1)$ -curves as imposed by the shrinking Kähler–Ricci soliton equation, then allows us to identify  $M$  and  $C_0$ .

### 6.1 Properties of shrinking Ricci solitons

We begin by noting some important features of shrinking Ricci solitons that we require in this section. We have the following condition on the scalar curvature of a shrinking gradient Ricci soliton.

**Theorem 6.2** [16] *Let  $(M, g, X)$  be a complete noncompact nonflat shrinking gradient Ricci soliton with scalar curvature  $R_g$ . Then for any given point  $o \in M$ , there exists a constant  $C > 0$  such that  $R_g(x) d_g(x, o)^2 > C^{-1}$  wherever  $d_g(x, o) > C$ , where  $d_g$  denotes the distance function with respect to  $g$ .*

This yields the following condition on the scalar curvature of an asymptotic cone of a shrinking Ricci soliton.

**Corollary 6.3** *Let  $(M, g, X)$  be a complete noncompact nonflat shrinking gradient Ricci soliton with tangent cone  $(C_0, g_0)$  along an end. Then the scalar curvature  $R_{g_0}$  of the cone metric  $g_0$  is strictly positive.*

**Proof** The tangent cone at infinity is obtained as a Gromov–Hausdorff limit of a pointed sequence  $(M, g_k, o) := (M, \lambda_k^{-2} g, o)$  for  $o \in M$  fixed, where  $\lambda_k \rightarrow \infty$  as  $k \rightarrow \infty$ . By our asymptotic assumption, the tangent cone is unique and this process recovers the asymptotic cone  $(C_0, g_0)$ . Indeed, an arbitrary

point  $p \in C_0$  with  $r(p) = r_0 > 0$  is associated with a sequence  $p_k \rightarrow p$ , where  $d_g(o, p_k) = \lambda_k r_0 \rightarrow \infty$  as  $k \rightarrow \infty$ , where  $r$  denotes the radial coordinate of  $g_0$  and  $d_g$  denotes the distance measured with respect to  $g$ . In particular, we see that

$$R_{g_0}(p) = \lim_{k \rightarrow \infty} R_{\lambda_k^{-2}g}(p_k) = \lim_{k \rightarrow \infty} \lambda_k^2 R_g(p_k),$$

where  $R_{\lambda_k^{-2}g}$  denotes the scalar curvature of the rescaled metric  $\lambda_k^{-2}g$ . Using the lower bound of Theorem 6.2, we then have that

$$R_{g_0}(p) = \lim_{k \rightarrow \infty} \lambda_k^2 R_g(p_k) > \lim_{k \rightarrow \infty} \frac{\lambda_k^2 C^{-1}}{d_g(o, p_k)^2} = \lim_{k \rightarrow \infty} \frac{\lambda_k^2 C^{-1}}{(\lambda_k r_0)^2} = \frac{1}{Cr_0^2} > 0$$

for some positive constant  $C$ . Since  $p$  is arbitrary, it follows that  $R_{g_0} > 0$  away from the apex of  $C_0$ , as claimed. □

### 6.2 Proof of Theorem 6.1

Let  $(M, g, X)$  be a complete noncompact shrinking gradient Kähler–Ricci soliton of complex dimension  $n + 1$  with quadratic curvature decay and with tangent cone along its end the Kähler cone  $(C_0, g_0)$  given by Theorem A. Let  $r$  denote the radial function of the cone. Then the link of the cone  $\{r = 1\}$ , which we denote by  $(S, g_S)$ , is a Sasaki manifold of real dimension  $2n + 1$  foliated by the orbits of the flow of  $\xi$ , the restriction of the Reeb vector field of the cone to its link.

We know from [58, Theorem 3] that if the scalar curvature  $R_g$  of  $g$  is zero at a point, then  $(M, g)$  is isometric to Euclidean space. So we henceforth assume that  $R_g \neq 0$  everywhere, so that  $(M, g)$  is nonflat. Then Corollary 6.3 tells us that the scalar curvature of the cone  $R_{g_0}$  is strictly positive. Next we see from Lemma 2.2 that  $R_{g_S} > 2n(2n + 1)$  and so it follows from Corollary 2.9 that

$$R^T > 2n(2n + 1) + 2n = 4n(n + 1).$$

**Identification of  $C_0$**  In our case,  $(M, g, X)$  is of complex dimension two and the scalar curvature of  $g$  decays to zero at infinity. By [54], the scalar curvature decay implies that the norm of the curvature tensor of  $g$  decays quadratically. Thus, the above applies with  $n = 1$  and we have the lower bound  $R^T > 8$ . From the classification of 3–dimensional Sasaki manifolds by Belgun [4, Theorem 8], it then follows that  $C_0$  is biholomorphic to  $\mathbb{C}^2/\Gamma$ , with  $\Gamma$  a finite subgroup of  $U(2)$  acting freely on  $\mathbb{C}^2 \setminus \{0\}$ . We next wish to show that  $\Gamma = \{\text{id}\}$ .

Recall from Theorem A that there is a resolution  $\pi: M \rightarrow C_0$  of the singularity of  $C_0$  with  $d\pi(X) = r\partial_r$ . Since  $C_0$  is biholomorphic to  $\mathbb{C}^2/\Gamma$  for  $\Gamma \subset U(2)$  a finite subgroup acting freely on  $\mathbb{C}^2 \setminus \{0\}$ , it is in particular a rational singularity. It is well-known that the exceptional set of a resolution of such a singularity contains a string of  $\mathbb{P}^1$ 's [9, Lemma 1.3]. Since  $g$  is a shrinking Kähler–Ricci soliton, each of these  $\mathbb{P}^1$ 's must have self-intersection  $(-1)$  by adjunction. Moreover, since  $C_0$  is obtained from  $M$  by blowing down all of these  $(-1)$ –curves,  $C_0$  must in fact be smooth at the apex, so that  $\Gamma = \{\text{id}\}$  and  $C_0$  is biholomorphic to  $\mathbb{C}^2$ .

**Identification of  $M$**  It follows that  $M$  is then an iterated blowup of  $\mathbb{C}^2$  at the origin containing only  $(-1)$ -curves. The only iterated blowups of  $\mathbb{C}^2$  at the origin containing complex curves of this type are  $\mathbb{C}^2$  and  $\mathbb{C}^2$  blown up at one point, since any further iterated blowup would introduce at least one  $\mathbb{P}^1$  with self-intersection  $(-k)$  for some  $k \geq 2$ . The conclusion is then that  $M$  must be biholomorphic to either  $\mathbb{C}^2$  or  $\mathbb{C}^2$  blown up at one point.

## 7 Concluding remarks

We conclude with a discussion of future directions of research emanating from the results within this paper.

### 7.1 The conjectural picture

The results on shrinking gradient Kähler–Ricci solitons presented here allow us to speculate on possible deeper connections between such metrics and algebraic geometry. In the compact case, Berman, Witt and Nyström [6] gave an algebraic formula for the weighted volume functional and its derivative. We generalise this result to the noncompact case under suitable assumptions, making use of the results of Wu [66]. We begin with the definition of an anticanonically polarised Kähler manifold, the underlying complex manifold of a shrinking Kähler–Ricci soliton.

**Definition 7.1** An *anticanonically polarised Kähler manifold* is a Kähler manifold  $M$  admitting a Kähler form  $\omega$  together with a hermitian metric on  $-K_M$  with curvature form  $\Theta$  such that

$$\int_V (i\Theta)^k \wedge \omega^{\dim_{\mathbb{C}} V - k} > 0$$

for all positive-dimensional irreducible compact analytic subvarieties  $V$  of  $M$  and for all integers  $k$  such that  $1 \leq k \leq \dim_{\mathbb{C}} V$ .

By [25, Theorem 4.2], a compact anticanonically polarised Kähler manifold is a Fano manifold. Moreover, any shrinking Kähler–Ricci soliton naturally lives on an anticanonically polarised Kähler manifold.

Under certain criteria, we can write an algebraic formula for the weighted volume functional.

**Proposition 7.2** Let  $(M, \omega)$  be a (possibly noncompact) Kähler manifold of complex dimension  $n$  with Kähler form  $\omega$  on which there is a Hamiltonian action of a real torus  $T$  with moment map  $\mu: M \rightarrow \mathfrak{t}^*$ , where  $\mathfrak{t}$  is the Lie algebra of  $T$  and  $\mathfrak{t}^*$  its dual. Assume that the fixed-point set of  $T$  is compact and that

- (i)  $H^p(M, \mathcal{O}(-kK_M)) = 0$  for all  $p > 0$  and for all  $k$  sufficiently large, and
- (ii)  $\omega$  is the curvature form of a hermitian metric on  $-K_M$ .

If there exists an element  $\zeta_0 \in \mathfrak{t}$  such that the component of the moment map  $u_{\zeta_0} = \langle \mu, \zeta_0 \rangle$  is proper and bounded below, then

$$(7-1) \quad \int_M e^{-\langle \mu, \zeta \rangle} \frac{\omega^n}{n!} = \lim_{k \rightarrow \infty} \frac{1}{k^n} \text{char } H^0(M, \mathcal{O}(-kK_M)) \left( \frac{\zeta}{k} \right)$$

for all  $\zeta$  in an open cone  $\Lambda \subset \mathfrak{t}$ .

In this situation, the character  $\text{char } H^0(M, \mathcal{O}(-kK_M))$  is well-defined by [66]. Moreover, it follows from [55, Theorem 4.5] that the vanishing condition (i) holds true for any 1-convex anticanonically polarised Kähler manifold and condition (ii) holds true for any shrinking gradient Kähler–Ricci soliton. In particular, if  $(M, \omega, X)$  is a complete shrinking gradient Kähler–Ricci soliton with Ricci curvature decaying to zero at infinity, endowed with the holomorphic, effective, isometric action of a real torus  $T$  with Lie algebra  $\mathfrak{t}$  containing  $JX$ , then the above theorem applies. The volume minimising principle (Lemma 5.16) then tells us that for such a soliton, the unique minimum of the weighted volume functional is obtained at  $JX$ .

Before we present the proof of equation (7-1), it is necessary to introduce some notation. Our notation will mostly follow [66]. We denote by  $M^T$  the fixed-point set of  $T$  in  $M$ . By assumption, this is compact. If nonempty, it is a complex submanifold of  $M$ . Let  $F$  be the set of connected components of  $M^T$ . Then  $M^T = \bigcup_{\alpha \in F} M_\alpha^T$ , where  $M_\alpha^T$  is the component labelled by  $\alpha \in F$ . Let  $n_\alpha = \dim_{\mathbb{C}} M_\alpha^T$  and let  $N_\alpha \rightarrow M_\alpha^T$  be the holomorphic normal bundle of  $M_\alpha^T$  in  $M$ .  $T$  acts on  $N_\alpha$  preserving the base  $M_\alpha^T$  pointwise. The weights of the isotropy representation on the normal fibre remain constant within any connected component. Let  $\ell$  be the integral lattice in the Lie algebra  $\mathfrak{t}$  of  $T$ , let  $\ell^* \subset \mathfrak{t}^*$  denote the dual lattice, and let  $\lambda_{\alpha,i} \in \ell^* \setminus \{0\}$  for  $1 \leq i \leq n - n_\alpha$  be the isotropy weights on  $N_\alpha$ . The hyperplanes  $(\lambda_{\alpha,i})^\perp \subset \mathfrak{t}$  cut  $\mathfrak{t}$  into open polyhedral cones called *action chambers* [60]. Choose an action chamber  $C$ . We define  $\nu_\alpha^C$  as the number of weights  $\lambda_{\alpha,i} \in C^*$ , where  $C^*$  is the dual cone in  $\mathfrak{t}^*$  defined by  $C^* = \{\xi \in \mathfrak{t}^* \mid \langle \xi, C \rangle > 0\}$ . Let  $N_\alpha^C$  be the direct sum of the subbundles corresponding to the weights  $\lambda_{\alpha,i} \in C^*$ . Then  $N_\alpha = N_\alpha^C \oplus N_\alpha^{-C}$ . The rank of the holomorphic vector bundle  $N_\alpha^C$  is  $\nu_\alpha^C$ ; that of  $N_\alpha^{-C}$  is  $\nu_\alpha^{-C} = n - n_\alpha - \nu_\alpha^C$ .

**Proof of Proposition 7.2** For  $k \in \mathbb{N}$  sufficiently large, the vanishing assumption (i) together with [66, equation (3.41)] implies that

$$\text{char } H^0(M, \mathcal{O}(-kK_M)) = \sum_{\alpha \in F} (-1)^{n-n_\alpha-\nu_\alpha^C} \int_{M_\alpha^T} \text{ch}^T \left( \frac{-kK_M|_{M_\alpha^T} \otimes \det(N_\alpha^{-C})}{\det(1 - (N_\alpha^C)^*) \otimes \det(1 - N_\alpha^{-C})} \right) \text{td}(M_\alpha^T),$$

where, if  $R$  is a finite-dimensional representation of  $T$ ,

$$\frac{1}{\det(1 - R)} := \bigoplus_{m=0}^{\infty} \text{Sym}^m(R),$$

and where  $\text{ch}^T$  denotes the equivariant Chern character. For a fixed  $\alpha \in F$ , we therefore have that

$$\begin{aligned} \text{ch}^T \left( \frac{-kK_M|_{M_\alpha^T} \otimes \det(N_\alpha^{-C})}{\det(1 - (N_\alpha^C)^*) \otimes \det(1 - N_\alpha^{-C})} \right) \text{td}(M_\alpha^T) \\ = \text{ch}^T(-kK_M|_{M_\alpha^T}) \text{ch}^T \left( \frac{1}{\det(1 - (N_\alpha^C)^*)} \right) \text{ch}^T \left( \frac{1}{\det(1 - N_\alpha^{-C})} \right) \text{ch}^T(\det(N_\alpha^{-C})) \text{td}(M_\alpha^T). \end{aligned}$$

Now,

$$\text{td}(M_\alpha^T) = 1 + \frac{1}{2}c_1(-K_{M_\alpha^T}) + \dots$$

Analysing the term  $\text{ch}^T(-kK_M|_{M_\alpha^T})$ , we have by adjunction that

$$K_M|_{M_\alpha^T} = K_{M_\alpha^T} - \det(N_\alpha),$$

so that

$$-kK_M|_{M_\alpha^T} = (-kK_{M_\alpha^T}) + k \det(N_\alpha).$$

Now,  $M_\alpha^T$  is fixed under the action of  $T$  and so the action of  $T$  on  $-kK_{M_\alpha^T}$  is trivial. The torus  $T$  therefore acts on  $-kK_M|_{M_\alpha^T}$  as multiplication by  $e^{k \sum_{i=1}^{n-n_\alpha} \lambda_{\alpha,i}}$ , where  $n_\alpha$  is the dimension of  $M_\alpha^T$ . Thus, we have that

$$c_1^T(-kK_M|_{M_\alpha^T}) = kc_1(-K_{M_\alpha^T}) + k \sum_{i=1}^{n-n_\alpha} \lambda_{\alpha,i},$$

where  $c_1^T$  is the equivariant first Chern class, so that

$$\text{ch}^T(-kK_M|_{M_\alpha^T}) \left( \frac{\xi}{k} \right) = e^{c_1^T(-kK_M|_{M_\alpha^T})(\xi/k)} = e^{kc_1(-K_{M_\alpha^T})} e^{\sum_{i=1}^{n-n_\alpha} \lambda_{\alpha,i}(\xi)}.$$

Next analysing the second term, we may write  $N_\alpha^C = \bigoplus_{\{i|\lambda_{\alpha,i} \in \mathbb{C}^*\}} L_{\alpha,i}$ , where  $L_{\alpha,i}$  is the line subbundle of  $N_\alpha$  with isotropy weight  $\lambda_{\alpha,i}$ . Then we have that

$$\frac{1}{\det(1 - (N_\alpha^C)^*)} = \bigoplus_{m=0}^{\infty} \text{Sym}^m((N_\alpha^C)^*) = \bigotimes_{\{i|\lambda_{\alpha,i} \in \mathbb{C}^*\}} \frac{1}{\det(1 - L_{\alpha,i}^*)},$$

so that

$$\text{ch}^T \left( \frac{1}{\det(1 - (N_\alpha^C)^*)} \right) = \text{ch}^T \left( \bigotimes_{\{i|\lambda_{\alpha,i} \in \mathbb{C}^*\}} \frac{1}{\det(1 - L_{\alpha,i}^*)} \right) = \prod_{\{i|\lambda_{\alpha,i} \in \mathbb{C}^*\}} \text{ch}^T \left( \frac{1}{\det(1 - L_{\alpha,i}^*)} \right).$$

Now observe that, for each  $i$ ,

$$\begin{aligned} \text{ch}^T \left( \frac{1}{\det(1 - L_{\alpha,i}^*)} \right) &= \text{ch}^T \left( \bigoplus_{m=0}^{\infty} \text{Sym}^m(L_{\alpha,i}^*) \right) = \text{ch}^T \left( \bigoplus_{m=0}^{\infty} (L_{\alpha,i}^*)^m \right) \\ &= \sum_{m=0}^{\infty} (\text{ch}^T(L_{\alpha,i}^*))^m = \sum_{m=0}^{\infty} (e^{-\lambda_{\alpha,i} + c_1(L_{\alpha,i}^*)})^m \\ &= \frac{1}{1 - e^{-\lambda_{\alpha,i} + c_1(L_{\alpha,i}^*)}}. \end{aligned}$$



Hence,

$$\text{ch}^T\left(\frac{1}{\det(1 - (N_\alpha^C)^*)}\right) = \prod_{\{i|\lambda_{\alpha,i} \in C^*\}} \frac{1}{1 - e^{-\lambda_{\alpha,i} + c_1(L_{\alpha,i}^*)}}.$$

Consequently,

$$\begin{aligned} \text{ch}^T\left(\frac{1}{\det(1 - (N_\alpha^C)^*)}\right)\left(\frac{\xi}{k}\right) &= \prod_{\{i|\lambda_{\alpha,i} \in C^*\}} \frac{1}{1 - e^{-(1/k)\lambda_{\alpha,i}(\xi) - c_1(L_{\alpha,i})}} \\ &= \prod_{\{i|\lambda_{\alpha,i} \in C^*\}} \frac{1}{1 - \left(1 + \left(-\frac{1}{k}\lambda_{\alpha,i}(\xi) - c_1(L_{\alpha,i})\right) + \sum_{l=2}^{\infty} \frac{1}{l!} \left(-\frac{1}{k}\lambda_{\alpha,i}(\xi) - c_1(L_{\alpha,i})\right)^l\right)} \\ &= \prod_{\{i|\lambda_{\alpha,i} \in C^*\}} \frac{1}{\frac{1}{k}\lambda_{\alpha,i}(\xi) + c_1(L_{\alpha,i}) - \sum_{l=2}^{\infty} \frac{1}{l!} \left(-\frac{1}{k}\lambda_{\alpha,i}(\xi) - c_1(L_{\alpha,i})\right)^l} \\ &= \prod_{\{i|\lambda_{\alpha,i} \in C^*\}} \frac{k}{\lambda_{\alpha,i}(\xi) + kc_1(L_{\alpha,i}) - k \sum_{l=2}^{\infty} \frac{1}{l!} \left(-\frac{1}{k}\lambda_{\alpha,i}(\xi) - c_1(L_{\alpha,i})\right)^l} \\ &= \prod_{\{i|\lambda_{\alpha,i} \in C^*\}} \frac{k}{\lambda_{\alpha,i}(\xi) \left(1 + \frac{kc_1(L_{\alpha,i})}{\lambda_{\alpha,i}(\xi)} + \frac{k}{\lambda_{\alpha,i}(\xi)} \sum_{l=2}^{\infty} \frac{(-1)^{l+1}}{l!} \left(\frac{1}{k}\lambda_{\alpha,i}(\xi) + c_1(L_{\alpha,i})\right)^l\right)}. \end{aligned}$$

Now,

$$\left(\frac{1}{k}\lambda_{\alpha,i}(\xi) + c_1(L_{\alpha,i})\right)^l = c_1(L_{\alpha,i})^l + \frac{l}{k}c_1(L_{\alpha,i})^{l-1}\lambda_{\alpha,i}(\xi) + O(k^{-2}),$$

so that

$$k\left(\frac{1}{k}\lambda_{\alpha,i}(\xi) + c_1(L_{\alpha,i})\right)^l = kc_1(L_{\alpha,i})^l + lc_1(L_{\alpha,i})^{l-1}\lambda_{\alpha,i}(\xi) + O(k^{-1}).$$

Since  $l \geq 2$ , we have that

$$\frac{k}{\lambda_{\alpha,i}(\xi)} \sum_{l=2}^{\infty} \frac{(-1)^{l+1}}{l!} \left(\frac{1}{k}\lambda_{\alpha,i}(\xi) + c_1(L_{\alpha,i})\right)^l = O(k)c_1(L_{\alpha,i})^2 P_1 + c_1(L_{\alpha,i})P_2 + O(k^{-1}),$$

where  $P_1$  and  $P_2$  are polynomials in  $c_1(L_{\alpha,i})$ . Therefore, we see that

$$\begin{aligned} \text{ch}^T\left(\frac{1}{\det(1 - (N_\alpha^C)^*)}\right)\left(\frac{\xi}{k}\right) &= \prod_{\{i|\lambda_{\alpha,i} \in C^*\}} \frac{k}{\lambda_{\alpha,i}(\xi) \left(1 + \frac{kc_1(L)}{\lambda_{\alpha,i}(\xi)}\right) + O(k)c_1(L_{\alpha,i})^2 P_1 + c_1(L_{\alpha,i})P_2 + O(k^{-1})} \\ &= k^{v_\alpha^C} \prod_{\{i|\lambda_{\alpha,i} \in C^*\}} \frac{1}{\lambda_{\alpha,i}(\xi) \left(1 + \frac{kc_1(L)}{\lambda_{\alpha,i}(\xi)}\right) + O(k)c_1(L_{\alpha,i})^2 P_1 + c_1(L_{\alpha,i})P_2 + O(k^{-1})}. \end{aligned}$$

A similar argument also shows that

$$\begin{aligned} & \text{ch}^T \left( \frac{1}{\det(1 - N_\alpha^{-C})} \right) \left( \frac{\xi}{k} \right) \\ &= (-k)^{n-n_\alpha - \nu_\alpha^C} \prod_{\{i|\lambda_{\alpha,i} \in -C^*\}} \frac{1}{\lambda_{\alpha,i}(\xi) \left( 1 + \frac{k c_1(L_{\alpha,i})}{\lambda_{\alpha,i}(\xi)} \right) + O(k) c_1(L_{\alpha,i})^2 Q_1 + c_1(L_{\alpha,i}) Q_2 + O(k^{-1})} \end{aligned}$$

for polynomials  $Q_1$  and  $Q_2$  in  $c_1(L_{\alpha,i})$ .

Finally,

$$\text{ch}^T(\det(N_\alpha^{-C})) = e^{c_1^T(\det(N_\alpha^{-C}))} = e^{c_1(\det(N_\alpha^{-C})) + \sum_{\{i|\lambda_{\alpha,i} \in -C^*\}} \lambda_{\alpha,i}},$$

so that

$$\text{ch}^T(\det(N_\alpha^{-C})) \left( \frac{\xi}{k} \right) = e^{c_1(N_\alpha^{-C}) + (1/k) \sum_{\{i|\lambda_{\alpha,i} \in -C^*\}} \lambda_{\alpha,i}(\xi)}.$$

Putting all of the above observations together, we find that

$$\begin{aligned} & \frac{1}{k^n} \text{char } H^0(M, \mathcal{O}(-kK_M)) \left( \frac{\xi}{k} \right) \\ &= \sum_{\alpha \in F} \frac{(-1)^{n-n_\alpha - \nu_\alpha^C}}{k^n} \int_{M_\alpha^T} \text{ch}^T \left( \frac{-kK_M|_{M_\alpha^T} \otimes \det(N_\alpha^{-C})}{\det(1 - (N_\alpha^C)^*) \otimes \det(1 - N_\alpha^{-C})} \right) \text{td}(M_\alpha^T) \left( \frac{\xi}{k} \right) \\ &= \sum_{\alpha \in F} \frac{1}{k^{n_\alpha}} \int_{M_\alpha^T} e^{k c_1(-K_M|_{M_\alpha^T})} e^{\sum_{i=1}^{n-n_\alpha} \lambda_{\alpha,i}(\xi)} e^{c_1(N_\alpha^{-C}) + (1/k) \sum_{\{i|\lambda_{\alpha,i} \in -C^*\}} \lambda_{\alpha,i}(\xi)} \\ & \quad \prod_{\{i|\lambda_{\alpha,i} \in C^*\}} \frac{1}{\lambda_{\alpha,i}(\xi) \left( 1 + \frac{k c_1(L_{\alpha,i})}{\lambda_{\alpha,i}(\xi)} \right) + O(k) c_1(L_{\alpha,i})^2 P_1 + c_1(L_{\alpha,i}) P_2 + O(k^{-1})} \\ & \quad \prod_{\{i|\lambda_{\alpha,i} \in -C^*\}} \frac{1}{\lambda_{\alpha,i}(\xi) \left( 1 + \frac{k c_1(L_{\alpha,i})}{\lambda_{\alpha,i}(\xi)} \right) + O(k) c_1(L_{\alpha,i})^2 Q_1 + c_1(L_{\alpha,i}) Q_2 + O(k^{-1})} \text{td}(M_\alpha^T) \\ &= \sum_{\alpha \in F} \frac{1}{k^{n_\alpha}} \int_{M_\alpha^T} e^{k c_1(-K_M|_{M_\alpha^T})} e^{\sum_{i=1}^{n-n_\alpha} \lambda_{\alpha,i}(\xi)} e^{c_1(N_\alpha^{-C}) + (1/k) \sum_{\{i|\lambda_{\alpha,i} \in -C^*\}} \lambda_{\alpha,i}(\xi)} \prod_{i=1}^{n-n_\alpha} \frac{1}{\lambda_{\alpha,i}(\xi)} \\ & \quad \prod_{\{i|\lambda_{\alpha,i} \in C^*\}} \frac{1}{1 + \frac{k c_1(L_{\alpha,i})}{\lambda_{\alpha,i}(\xi)} + O(k) c_1(L_{\alpha,i})^2 P_1 + c_1(L_{\alpha,i}) P_2 + O(k^{-1})} \\ & \quad \prod_{\{i|\lambda_{\alpha,i} \in -C^*\}} \frac{1}{1 + \frac{k c_1(L_{\alpha,i})}{\lambda_{\alpha,i}(\xi)} + O(k) c_1(L_{\alpha,i})^2 Q_1 + c_1(L_{\alpha,i}) Q_2 + O(k^{-1})} (1 + \frac{1}{2} c_1(-K_{M_\alpha^T}) + \dots) \\ & \xrightarrow{k \rightarrow \infty} \sum_{\alpha \in F} \int_{M_\alpha^T} e^{c_1(-K_M|_{M_\alpha^T})} e^{\sum_{i=1}^{n-n_\alpha} \lambda_{\alpha,i}(\xi)} \prod_{i=1}^{n-n_\alpha} \frac{1}{\lambda_{\alpha,i}(\xi) \left( 1 + \frac{c_1(L_{\alpha,i})}{\lambda_{\alpha,i}(\xi)} \right)}, \end{aligned}$$

where, in taking the limit, we use the fact that any integrand involving terms not of the form  $k^j \sigma_j$  for  $\sigma_j$  a real  $(j, j)$ -form vanishes. The result now follows from an application of Theorem A.3, making use of assumption (ii) of the proposition.  $\square$

Given this proposition, it is tempting to define a notion of K–stability that characterises algebraically the existence of a shrinking gradient Kähler–Ricci soliton on a complete anticanonically polarised Kähler manifold  $M$  endowed with a complete holomorphic vector field following the strategy as implemented in the Fano case. For this purpose, we make the following definition.

**Definition 7.3** Let  $M$  be a quasiprojective manifold endowed with the effective holomorphic action of a real torus  $T$  whose fixed-point set is compact. Denote by  $\mathfrak{t}$  the Lie algebra of  $T$ , let  $\mathcal{O}_M(M)$  denote the global algebraic sections of the structure sheaf of  $M$ , and write

$$\mathcal{O}_M(M) = \bigoplus_{\alpha \in \mathfrak{t}^*} \mathcal{H}_\alpha$$

for the weight decomposition under the action of  $T$ . Then we say that a vector field  $Y \in \mathfrak{t}$  on  $M$  is *positive* if  $\alpha(Y) > 0$  for all  $\alpha \in \mathfrak{t}^*$  such that  $\mathcal{H}_\alpha \neq \emptyset$  and  $\alpha \neq 0$ .

**Remark 7.4** If  $\pi : M \rightarrow C_0$  is a quasiprojective equivariant resolution of a Kähler cone  $(C_0, g_0)$  with respect to the holomorphic isometric torus action on  $(C_0, g_0)$  generated by the flow of the Reeb vector field of  $g_0$ , and  $g$  is a Kähler metric on  $M$  that is asymptotic to  $g_0$  and with respect to which the induced torus action on  $(M, g)$  is isometric and Hamiltonian, then in the terminology just introduced, the weighted volume functional  $F$  for  $(M, g)$  is defined on the open cone of positive vector fields in the Lie algebra of the torus if this open cone is nonempty. This fact follows from Theorem A.3 after noting Theorem A.10.

Roughly speaking, one considers equivariant degenerations (or test configurations) of the pair  $(M, X)$ , where  $M$  is a quasiprojective anticanonically polarised Kähler manifold with complex structure  $J$  endowed with the holomorphic effective action of a real torus  $T$  whose fixed-point set is compact, and where  $X$  is a vector field on  $M$  with  $JX$  a positive vector field lying in the Lie algebra of  $T$ . Then one defines a Futaki invariant in the usual manner as the derivative of the algebraic realisation of the weighted volume functional which is given by the right-hand side of (7-1). Of course, one must verify that this formula is well-defined in general. One subsequently defines  $(M, X)$  as above to be K–stable if and only if the Futaki invariant is nonnegative on all test configurations, and positive if and only if the test configuration is nontrivial. This then allows one to make the following conjecture generalising the Yau–Tian–Donaldson conjecture for Fano manifolds.

**Conjecture 7.5** Let  $M$  be a quasiprojective anticanonically polarised Kähler manifold endowed with the holomorphic effective action of a real torus  $T$  whose fixed-point set is compact. Denote by  $\mathfrak{t}$  the Lie algebra of  $T$  and let  $X$  be a vector field on  $M$  such that  $JX \in \mathfrak{t}$  is a positive vector field. Then  $M$  admits a complete shrinking gradient Kähler–Ricci soliton with soliton vector field  $X$  if and only if  $(M, X)$  is K–stable.

Thus, in light of this conjecture, one may view an anticanonically polarised Kähler manifold as a “noncompact Fano manifold”. (In a similar manner, one may also define a noncompact manifold of general type, etc.) We expect that the well-developed machinery in the study of Kähler–Einstein metrics may be suitably adapted to study this conjecture. We leave this for future work.

## 7.2 Open problems

There are also various other interesting open problems that we raise here.

(1) Is a complete expanding or shrinking gradient Kähler–Ricci soliton necessarily algebraic (or quasi-projective)? In particular, is the canonical ring of an expanding gradient Kähler–Ricci soliton finitely generated? Is the anticanonical ring of a shrinking gradient Kähler–Ricci soliton finitely generated? What we can say here is that if the curvature tensor of a shrinking gradient Kähler–Ricci soliton decays quadratically, or if that of an expanding gradient Kähler–Ricci soliton decays quadratically with derivatives, then the soliton lives on a resolution of a Kähler cone by Theorem A, hence is quasiprojective by Proposition 2.24.

(2) Is there at most one complete shrinking Kähler–Ricci soliton for a given holomorphic vector field on an anticanonically polarised Kähler manifold up to automorphisms of the complex structure commuting with the flow of the vector field? More speculatively, is a complete shrinking Kähler–Ricci soliton on such a manifold unique up to automorphisms of the complex structure? A noncompact Kähler manifold may admit many nonisometric complete expanding gradient Kähler–Ricci solitons even for a fixed holomorphic soliton vector field, as demonstrated by [19, Theorem A].

(3) What are the constraints on a Kähler cone to appear as the tangent cone of a complete shrinking gradient Kähler–Ricci soliton with quadratic curvature decay? Is the underlying complex manifold of the shrinking soliton then determined uniquely by its tangent cone? By Theorem A, we know that the shrinking soliton must live on a resolution of its tangent cone that is, moreover, an anticanonically polarised Kähler manifold. For complete expanding gradient Kähler–Ricci solitons with quadratic curvature decay with derivatives, we know from Corollary B that a Kähler cone appears as the tangent cone if and only if the Kähler cone has a smooth canonical model (on which the soliton lives).

(4) Related to the previous question, modulo automorphisms of the complex structure, how many shrinking gradient Kähler–Ricci solitons with quadratic curvature decay have a given affine cone appearing as the underlying complex space of the tangent cone? For  $\mathbb{C}^2$ , we have shown in Theorem E that the answer is two;  $\mathbb{C}^2$  only appears as the underlying complex space of the tangent cone of the flat Gaussian shrinking soliton on  $\mathbb{C}^2$  and of the  $U(2)$ -invariant shrinking gradient Kähler–Ricci soliton of Feldman, Ilmanen and Knopf on  $\mathbb{C}^2$  blown up at a point [30]. In general, we expect the answer to be finitely many for any given affine cone. By Corollary B, the answer to this question for complete expanding gradient Kähler–Ricci solitons with quadratic curvature decay with derivatives is infinitely many for any given affine cone admitting a smooth canonical model.

- (5) In Corollary B, we have seen that when the canonical model of a Kähler cone is smooth, it admits a complete expanding gradient Kähler–Ricci soliton. Is this also true when the canonical model is singular? (We thank John Lott for raising this question.)
- (6) Let  $M$  be a complete quasiprojective Kähler manifold endowed with the holomorphic Hamiltonian action of a real torus  $T$  with Lie algebra  $\mathfrak{t}$  whose fixed-point set is compact. Are the elements of  $\mathfrak{t}$  admitting Hamiltonian potentials that are proper and bounded below precisely those elements in  $\mathfrak{t}$  that are positive in the sense of Definition 7.3? Equivalently, does the open cone of positive vector fields in  $\mathfrak{t}$  coincide with the open cone  $\text{int}(\mathcal{C}(\mu(M))') \subset \mathfrak{t}$  of Proposition A.4? These questions have an affirmative answer in the setting of asymptotically conical Kähler manifolds; see Theorem A.10 for a precise statement.
- (7) Does Theorem D still hold true without the assumption of bounded Ricci curvature?
- (8) Given a complete shrinking gradient Kähler–Ricci soliton  $(M, g, X)$ , is the zero set of  $X$  always compact? As demonstrated in Lemma 2.25, this is the case if  $g$  has bounded scalar curvature.
- (9) Let  $M$  be a complete Kähler manifold endowed with the holomorphic Hamiltonian action of a real torus  $T$  with Lie algebra  $\mathfrak{t}$  whose fixed-point set is compact. By the Duistermaat–Heckman theorem (Theorem A.3), the weighted volume functional  $F$  is defined on the open cone  $\Lambda$  of elements of  $\mathfrak{t}$  admitting Hamiltonian potentials that are proper and bounded below. Is  $F$  necessarily proper on  $\Lambda$ ? If so, then it would have a unique minimiser on  $\Lambda$ . Properness of the volume functional on the set of normalised Reeb vector fields of a Sasaki manifold was shown in [37, Proposition 3.3].

## Appendix The Duistermaat–Heckman theorem

### A.1 Statement of the theorem

The material in this section has been taken verbatim from various sources in the literature, including [5; 28; 49; 60]. We begin with the definition of a moment map, which is required for the statement of the Duistermaat–Heckman theorem.

**Definition A.1** Let  $(M, \omega)$  be a symplectic manifold and let  $T$  be a real torus acting on  $(M, \omega)$  by symplectomorphisms. Denote by  $\mathfrak{t}$  the Lie algebra of  $T$  and by  $\mathfrak{t}^*$  its dual. Then we say that the action of  $T$  is *Hamiltonian* if there exists a smooth map  $\mu: M \rightarrow \mathfrak{t}^*$  such that for all  $\zeta \in \mathfrak{t}$ ,  $-\omega_{\lrcorner}\zeta = du_{\zeta}$ , where  $u_{\zeta}(x) = \langle \mu(x), \zeta \rangle$  for all  $\zeta \in \mathfrak{t}$  and  $x \in M$ . We call  $\mu$  the *moment map* of the  $T$ -action and we call  $u_{\zeta}$  the *Hamiltonian (potential)* of  $\zeta$ .

Notice that  $u_{\zeta}$  is invariant under the flow of  $\zeta$ . Indeed, we have that

$$\mathcal{L}_{\zeta}u_{\zeta} = du_{\zeta}\lrcorner\zeta = -\omega(\zeta, \zeta) = 0.$$

Consequently, each integral curve of  $\zeta$  must be contained in a level set of  $u_{\zeta}$ .

Now consider the Hamiltonian action of a real torus  $T$  of rank  $s$  on a symplectic manifold  $(M, \omega)$  of real dimension  $2n$ . Identify  $T$  with  $(S^1)^s \subset (\mathbb{C}^*)^s$  and introduce complex coordinates  $(\phi_1, \dots, \phi_s)$  on  $T$  via this identification. This induces coordinates  $(\eta_1, \dots, \eta_s) \in \mathbb{R}^s$  on the Lie algebra  $\mathfrak{t}$  of  $T$ , where  $(\eta_1, \dots, \eta_s)$  corresponds to the vector  $\sum_{i=1}^s \eta_i (\partial/\partial \phi_i)$ , each  $\partial/\partial \phi_i$  the vector field on  $M$  induced by the coordinate  $\phi_i$  on  $T$ . For  $\zeta \in \mathfrak{t}$  with coordinates  $(b_1, \dots, b_s)$ , say, the flow on  $M$  generated by  $\zeta$  will have a fixed-point set  $M_0(\zeta)$  corresponding to the zero set of the vector field  $\zeta$ . This set has the following properties.

**Proposition A.2** [5, Proposition 7.12] *The connected components  $\{F_i\}$  of  $M_0(\zeta)$  are smooth submanifolds of  $M$ . The dimensions of different connected components do not have to be the same. The normal bundles  $\mathcal{E}_i$  of the  $F_i$  in  $M$  are orientable vector bundles with even-dimensional fibres.*

For a disconnected component  $F$  of  $M_0(\zeta)$  of real codimension  $2k$  in  $M$ , let  $\iota: F \rightarrow M$  denote the inclusion. Then  $\iota^*\omega$  is a symplectic form on  $F$  so that  $F$  is a symplectic submanifold of  $M$ . The normal bundle  $\mathcal{E}$  of  $F$  in  $M$  has the structure of a symplectic vector bundle and will have real dimension  $2k$ . We denote this induced symplectic form on  $\mathcal{E}$  by  $\tau$ . The flow of  $\zeta$  will generate a fibre-preserving linear action  $L\zeta: \mathcal{E} \rightarrow \mathcal{E}$  on  $\mathcal{E}$ , which is an automorphism of  $\mathcal{E}$  leaving  $\tau$  invariant in the infinitesimal sense. We introduce an almost complex structure  $I: \mathcal{E} \rightarrow \mathcal{E}$ , ie an automorphism of  $\mathcal{E}$  such that  $I^2 = -\text{id}$ , commuting with  $L\zeta$  and compatible with  $\tau$  in the sense that  $\tau(I \cdot, \cdot)$  defines an inner product on  $\mathcal{E}$ . This gives  $\mathcal{E}$  the structure of a complex vector bundle over  $F$  with  $L\zeta$  an automorphism of  $\mathcal{E}$  preserving the complex structure.

Next, denote by  $u_1, \dots, u_R \in \mathbb{Z}^s \subset \mathfrak{t}^*$  the weights of the induced representation of  $\mathfrak{t}$  on  $\mathcal{E}$ . Then we have a direct sum decomposition of vector bundles

$$\mathcal{E} = \bigoplus_{\lambda=1}^R \mathcal{E}_\lambda,$$

where

$$\mathcal{E}_\lambda := \{v \in \mathcal{E} \mid (e^{i\eta_1}, \dots, e^{i\eta_s}) \cdot v = u_\lambda(\eta_1, \dots, \eta_s) I v \text{ for all } (\eta_1, \dots, \eta_s) \in \mathfrak{t}\}.$$

Each  $\mathcal{E}_\lambda$  is a vector bundle of rank  $2n_\lambda$ , say. Clearly we must have  $k = \sum_{\lambda=1}^R n_\lambda$ . Consider now the complex vector bundle  $\mathcal{E}^{1,0}$  of complex dimension  $k$ , endowed with the action of  $L\zeta$  extended by  $\mathbb{C}$ -linearity. Then we have an induced decomposition

$$\mathcal{E}^{1,0} = \bigoplus_{\lambda=1}^R \mathcal{E}_\lambda^{1,0},$$

where  $L\zeta$  acts on the  $\lambda^{\text{th}}$  factor by  $i u_\lambda(b_1, \dots, b_n)$ , and so the action of  $L\zeta$  on  $\mathcal{E}^{1,0}$  will take the form

$$L\zeta = i \text{diag}(1_{n_1} u_1(b), \dots, 1_{n_R} u_R(b)),$$

where  $1_{n_\lambda}$  denotes the  $n_\lambda \times n_\lambda$  identity matrix and  $b = (b_1, \dots, b_s)$  are the coordinates of  $\zeta$ . Thus,

$$\det\left(\frac{L\zeta}{i}\right) = \prod_{\lambda=1}^R u_\lambda(b)^{n_\lambda}.$$

Note that this is homogeneous of degree  $k$  in  $b$ .

We next choose any  $L\xi$ -invariant connection on  $\mathcal{E}^{1,0}$  with curvature matrix  $\Omega$ . Finally, for a polyhedral set  $U$  of a vector space  $V$ , we define the *asymptotic cone*

$$\mathcal{C}(U) := \{v \in V \mid \text{there is a } v_0 \in V \text{ such that } v_0 + tv \in U \text{ for } t > 0 \text{ sufficiently large}\},$$

and for a subset  $W$  of  $V$ , we define the *dual cone*  $W' := \{\alpha \in V^* \mid \alpha(W) \subseteq \mathbb{R}_{\geq 0}\}$ . Now we can state the Duistermaat–Heckman theorem.

**Theorem A.3** (the Duistermaat–Heckman theorem [60, Theorem 2.2]) *Let  $(M, \omega)$  be a (possibly noncompact) symplectic manifold of real dimension  $2n$  with symplectic form  $\omega$  on which there is a Hamiltonian action of a real torus  $T$  with moment map  $\mu: M \rightarrow \mathfrak{t}^*$ , where  $\mathfrak{t}$  is the Lie algebra of  $T$  and  $\mathfrak{t}^*$  its dual. Assume that the fixed-point set of  $T$  is compact. If there exists an element  $\zeta_0 \in \mathfrak{t}$  such that the component of the moment map  $u_{\zeta_0} = \langle \mu, \zeta_0 \rangle$  is proper and bounded below, then*

$$(A-1) \quad \int_M e^{-\langle \mu, \zeta \rangle} \frac{\omega^n}{n!} = \sum_{F \in M_0(\zeta)} \int_F \frac{e^{-i^* \langle \mu, \zeta \rangle} e^{i^* \omega}}{\det\left(\frac{L\xi - \Omega}{2\pi i}\right)}$$

for all  $\zeta$  in the open cone  $\text{int}(\mathcal{C}(\mu(M))') \subset \mathfrak{t}$ , where the sum on the right-hand side is taken over the connected components  $F$  of the zero set  $M_0(\zeta)$  of  $\zeta$ .

Under the assumptions on the moment map  $\mu$  as in the theorem,  $\mu(M)$  is a proper polyhedral set in  $\mathfrak{t}^*$  and the elements of  $\text{int}(\mathcal{C}(\mu(M))') \subset \mathfrak{t}$  are characterised as follows.

**Proposition A.4** [60, Proposition 1.4] *Under the assumptions on  $T$  and  $\mu$  as in Theorem A.3,  $u_\zeta = \langle \mu, \zeta \rangle$  is proper if and only if  $\zeta \in \pm \text{int}(\mathcal{C}(\mu(M))') \subset \mathfrak{t}$ . Moreover, if  $\zeta \in \text{int}(\mathcal{C}(\mu(M))') \subset \mathfrak{t}$ , then  $u_\zeta(M) = [m_\zeta, +\infty)$  for a suitable  $m_\zeta \in \mathbb{R}$ .*

That is, elements of  $\text{int}(\mathcal{C}(\mu(M))')$  are precisely those elements of  $\mathfrak{t}$  whose Hamiltonian is proper and bounded below. Notice that this cone is nonempty because it contains  $\zeta_0$  by assumption. Then for each  $\zeta \in \text{int}(\mathcal{C}(\mu(M))')$ , each connected component of the zero set  $M_0(\zeta)$  of  $\zeta$  must be compact because  $u_\zeta = \langle \mu, \zeta \rangle$  is proper, and moreover, it must contain a fixed point of the torus action by [60, Proposition 1.2]. Hence, since the fixed-point set of  $T$  is assumed to be compact in Theorem A.3, the sum on the right-hand side of (A-1) is over a finite set and so is itself finite for all such  $\zeta$ .

Now, the sum on the right-hand side of (A-1) is over each connected component  $F$  of the zero set  $M_0(\zeta)$  of  $\zeta$ . The determinant is a  $k \times k$  determinant and should be expanded formally into a differential form of mixed degree. Moreover, the inverse is understood to mean one should expand this formally in a Taylor series, as is standard in index theory. We next study the right-hand side of (A-1) in more detail.

Under the decomposition

$$\mathcal{E}^{1,0} = \bigoplus_{\lambda=1}^R \mathcal{E}_\lambda^{1,0},$$

let  $\Omega_\lambda$  be the component of the curvature matrix of  $\Omega_\mathcal{E}$  corresponding to  $\mathcal{E}_\lambda$ . Then we have that

$$\det\left(\frac{L\xi - \Omega}{2\pi i}\right) = \det\left(\frac{L\xi}{2\pi i}\right) \det(1 - (L\xi)^{-1}\Omega) = \det\left(\frac{L\xi}{2\pi i}\right) \prod_{\lambda=1}^R \det(1 - (L\xi)^{-1}\Omega_\lambda).$$

Fix one of the bundles  $\mathcal{E}_\lambda$ . Then

$$\det(1 - (L\xi)^{-1}\Omega_\lambda) = \det(1 + w i \Omega_\lambda) = \sum_{a \geq 0} c_a(\mathcal{E}_\lambda) w^a \in H^*(F, \mathbb{R}),$$

where  $w = 1/u_\lambda(b)$  and  $c_a(\mathcal{E}_\lambda)$  are the Chern classes of  $\mathcal{E}_\lambda$  for  $0 \leq a \leq n_\lambda$  with  $c_0 = 1$ . Thus,

$$\det\left(\frac{L\xi - \Omega}{2\pi i}\right) = \prod_{\lambda=1}^R u_\lambda(b)^{n_\lambda} \left(\sum_{a \geq 0} c_a(\mathcal{E}_\lambda) w^a\right).$$

In particular, if  $F$  is an isolated fixed point, in which case  $k = n$  and  $\mathcal{E}$  is the trivial bundle, then we may write the  $n$  (possibly indistinct) weights as  $u_1, \dots, u_n$ . The Chern classes and the measure  $e^{t^*\omega}$  contribute nontrivially, and we arrive at the contribution

$$e^{-t^*\langle \mu, \zeta \rangle} \prod_{\lambda=1}^R \frac{1}{u_\lambda(b)^{n_\lambda}}$$

of an isolated fixed point to the Duistermaat–Heckman formula.

We next wish to sketch the proof of Theorem A.3. Before we do so, however, we must first discuss invariant forms on a symplectic manifold.

### A.2 Invariant forms

Consider a symplectic manifold  $(M, \omega)$  of real dimension  $2n$  endowed with the Hamiltonian action of a real torus  $T$ . For  $\zeta$  in the Lie algebra  $\mathfrak{t}$  of  $T$ , denote by  $\Omega_\zeta^k(M)$  the space of smooth  $k$ -forms on  $M$  which are invariant under the flow of  $\zeta$ , ie  $\alpha \in \Omega_\zeta^k(M)$  if and only if  $\mathcal{L}_\zeta \alpha = 0$ . The wedge product of two invariant forms is also invariant, therefore we have an algebra  $\Omega_\zeta^*(M)$  of invariant forms on  $M$ . We define the *equivariant derivative*  $d_\zeta$  on  $\Omega_\zeta^*(M)$  by

$$d_\zeta \alpha = d\alpha - \alpha \lrcorner \zeta.$$

This derivative has the properties that  $d_\zeta^2 = 0$  and

$$d_\zeta(\alpha \wedge \beta) = d_\zeta \alpha \wedge \beta + (-1)^p \alpha \wedge d_\zeta \beta$$

for  $\alpha$  a  $p$ -form and  $\beta$  another differential form.

For  $\alpha \in \Omega_T^*(M)$ , we can write

$$\alpha = \alpha_{[0]} + \alpha_{[1]} + \dots + \alpha_{[2n]},$$



with  $\alpha_{[i]}$  a differential form of degree  $i$  in  $\Omega_T^*(M)$ . Then integration of invariant forms is defined by integrating over the highest-degree part of the form, ie

$$\int : \Omega_T^*(M) \rightarrow \mathbb{R}, \quad \int \alpha := \int_M \alpha_{[2n]}.$$

This leads to a version of Stokes’ theorem for invariant forms: if an invariant form  $\alpha$  is  $d_\zeta$ –exact, ie if  $\alpha = d_\zeta \beta$  for another form  $\beta$ , then  $\alpha_{[2n]} = d\beta_{[2n-1]}$ , since contracting with  $\zeta$  decreases the degree of a form. We then have that

$$\int_M \alpha := \int_M \alpha_{[2n]} = \int_M d\beta_{[2n-1]} = \int_{\partial M} \beta_{[2n-1]}.$$

Recall that  $M_0(\zeta)$  denotes the zero locus on  $M$  of  $\zeta \in \mathfrak{t}$ .

**Lemma A.5** *Let  $\alpha \in \Omega_\zeta^*(M)$  be  $d_\zeta$ –closed. Then  $\alpha_{[2n]}$  is exact on  $M \setminus M_0(\zeta)$ .*

**Proof** Let  $\theta$  be a one-form on  $M \setminus M_0(\zeta)$  such that

$$(A-2) \quad \mathcal{L}_\zeta \theta = 0 \quad \text{and} \quad \theta \lrcorner \zeta \neq 0.$$

Such a one-form can be constructed explicitly. Indeed, let  $g$  be a  $T$ –invariant Riemannian metric on  $M$ , let  $\tilde{\zeta}$  be any nonzero positive smooth function times  $\zeta$ , and define

$$\theta(v) = g(\tilde{\zeta}, v) \quad \text{for any vector field } v \text{ on } M.$$

This is well-defined on  $M \setminus M_0(\zeta)$  as  $\zeta$ , and hence  $\tilde{\zeta}$ , are nonzero on this set, and is easily seen to satisfy (A-2). We can then invert  $d_\zeta \theta$  on  $M \setminus M_0(\zeta)$  using a geometric series:

$$(d_\zeta \theta)^{-1} = \frac{1}{(d\theta - \theta \lrcorner \zeta)} = \frac{1}{(\theta \lrcorner \zeta)((\theta \lrcorner \zeta)^{-1} d\theta - 1)} = -(\theta \lrcorner \zeta)^{-1} - (\theta \lrcorner \zeta)^{-2} d\theta - (\theta \lrcorner \zeta)^{-3} (d\theta)^2 - \dots .$$

Note that this geometric series is finite because the  $(d\theta)^k$  vanish if  $2k > 2n = \dim M$ , and we have that

$$d_\zeta \theta \wedge (d_\zeta \theta)^{-1} = 1.$$

Applying  $d_\zeta$  to this yields

$$d_\zeta \theta \wedge d_\zeta((d_\zeta \theta)^{-1}) = 0.$$

Further taking the wedge product with  $(d_\zeta \theta)^{-1}$  on the left then leaves us with

$$d_\zeta[(d_\zeta \theta)^{-1}] = 0.$$

Define  $v$  by

$$v := \theta \wedge (d_\zeta \theta)^{-1} \wedge \alpha.$$

Then, since  $d_\zeta \alpha = 0$  by assumption, we have that

$$d_\zeta v = d_\zeta \theta \wedge (d_\zeta \theta)^{-1} \wedge \alpha = \alpha.$$

Taking the highest-degree part of each side of this equality, we obtain the result. □

### A.3 Sketch of the proof of Theorem A.3

Since the left-hand side of (A-1) is analytic on  $\text{int}(\mathcal{C}(\mu(M))') \subset \mathfrak{t}$ , it suffices to prove (A-1) for rational elements in this open cone. So let  $\zeta \in \text{int}(\mathcal{C}(\mu(M))')$  be rational and recall that the zero set  $M_0(\zeta)$  of  $\zeta$  is compact because the fixed-point set of  $T$  is compact by assumption; see the discussion after Proposition A.4. Write  $\zeta = t\eta$  for some integral point  $\eta \in \text{int}(\mathcal{C}(\mu(M))')$  and some  $t > 0$ , and let  $H := \langle \mu, \eta \rangle$  denote the Hamiltonian of  $\eta$ , which serves as a moment map of the induced  $S^1$ -action of  $\{e^{i\eta}\}$  on  $M$ . Recall that  $H$  is a proper function bounded from below and so must tend to infinity as  $x \rightarrow \infty$  in  $M$ .

Next, observe that

$$e^{\omega-tH} = e^{-tH} \left( 1 + \omega + \dots + \frac{\omega^n}{n!} \right) \in \Omega_\zeta^*(M),$$

$$d_\zeta e^{\omega-tH} = e^{\omega-tH} (d_\zeta(\omega - tH)) = e^{\omega-tH} (-d(tH) - \omega \lrcorner \zeta) = t e^{\omega-tH} (-dH - \omega \lrcorner \eta) = 0,$$

so that  $e^{\omega-tH}$  is  $d_\zeta$ -closed. An immediate consequence of Lemma A.5 is therefore that  $e^{-tH} \omega^n/n!$  is exact off of the zero set  $M_0(\zeta)$  of  $\zeta$ . Indeed, fix a  $T$ -invariant metric  $g$  on  $M$ . Then tracing through the proof of Lemma A.5, we see that

$$e^{-tH} \frac{\omega^n}{n!} = d\nu_{[2n-1]}, \quad \text{where } \nu = \theta \wedge (d_\zeta \theta)^{-1} \wedge e^{\omega-tH} \text{ and } \theta = g(\tilde{\zeta}, \cdot),$$

with  $\tilde{\zeta}$  denoting any nonzero positive function times  $\zeta$ . We take  $\tilde{\zeta} = \eta/g(\eta, \eta)$  in what follows.

Let  $F$  denote each of the connected components of  $M_0(\zeta)$  and recall that each is a smooth submanifold of  $M$ . Using the exponential map of the  $T$ -invariant metric  $g$  on  $M$ , we obtain a diffeomorphism  $\psi$  from a neighbourhood  $U$  of the zero section of the normal bundle  $\mathcal{E}$  of  $F$  in  $E$  onto a neighbourhood  $\psi(U)$  of  $F$  in  $M$ . For  $\varepsilon > 0$ , denote by  $B_\varepsilon$  the  $\varepsilon$ -ball bundle in  $\mathcal{E}$  and by  $S_\varepsilon$  its boundary. Since  $M_0(\zeta)$  is compact and  $H(x) \rightarrow +\infty$  as  $x \rightarrow \infty$  in  $M$ , we have by Stokes' theorem that

$$\begin{aligned} \text{(A-3)} \quad \int_M e^{-tH} \frac{\omega^n}{n!} &= \lim_{a \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \int_{H^{-1}((-\infty, a]) \cup_{F \in M_0(\zeta)} \psi(B_\varepsilon)} e^{-tH} \frac{\omega^n}{n!} \\ &= \lim_{a \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \int_{H^{-1}((-\infty, a]) \cup_{F \in M_0(\zeta)} \psi(B_\varepsilon)} d\nu_{[2n-1]} \\ &= \lim_{\varepsilon \rightarrow 0} \sum_{F \in M_0(\zeta)} \int_{\psi(S_\varepsilon)} \nu_{[2n-1]} + \lim_{a \rightarrow +\infty} \int_{H^{-1}(a)} \nu_{[2n-1]}, \end{aligned}$$

where we recall the fact that  $H$  is proper and that  $a$  is a regular value of  $H$  for all  $a$  sufficiently large by [60, Proposition 1.2] because the fixed-point set of the torus action is compact, so that  $H^{-1}(a)$  is a smooth compact submanifold of  $M$  for all such values of  $a$ .

Now, we have that

$$\nu_{[2n-1]} = -e^{-tH} \theta \wedge \sum_{j=0}^{n-1} \frac{(d\theta)^j}{(\theta \lrcorner \zeta)^{j+1}} \wedge \frac{\omega^{n-1-j}}{(n-1-j)!} = -e^{-tH} \theta \wedge \sum_{j=0}^{n-1} \frac{(d\theta)^j}{j^{j+1}} \wedge \frac{\omega^{n-1-j}}{(n-1-j)!},$$

where again  $\theta = g(\eta, \cdot)/g(\eta, \eta)$ . For one connected component  $F \in M_0(\zeta)$  of codimension  $k$ , say, as in the proof of the Duistermaat–Heckman formula in the compact case [28], the only summand contributing to  $\int_{\psi(S_\varepsilon)} \nu_{[2n-1]}$  in the limit as  $\varepsilon \rightarrow 0$  is the one with  $j = k - 1$ . Therefore, computing as in [28], one sees that

$$\begin{aligned}
 \text{(A-4)} \quad \lim_{\varepsilon \rightarrow 0} \int_{\psi(S_\varepsilon)} \nu_{[2n-1]} &= - \lim_{\varepsilon \rightarrow 0} \int_F e^{-tH} \theta \wedge \frac{(d\theta)^{k-1}}{t^k} \wedge \frac{\omega^{n-k}}{(n-k)!} \\
 &= - \lim_{\varepsilon \rightarrow 0} \int_F e^{-tH} \tilde{\theta} \wedge (d\tilde{\theta})^{k-1} \wedge \frac{\omega^{n-k}}{(n-k)!} \\
 &= \frac{e^{-t^*(tH)} e^{t^*\omega}}{\det\left(\frac{1}{2\pi i}(L\zeta - \Omega)\right)},
 \end{aligned}$$

where  $\tilde{\theta} = g(\zeta, \cdot)/g(\zeta, \zeta)$  on the second line.

We finally deal with the term  $\lim_{a \rightarrow +\infty} \int_{H^{-1}(a)} \nu_{[2n-1]}$ . Since every  $a$  sufficiently large is a regular value of  $H$ , the moment map of the  $S^1$ -action  $\{e^{i\eta}\}$ , the set  $H^{-1}(a)$  is a connected compact submanifold of  $M$  on which the  $S^1$ -action is locally free. Let  $M_a = H^{-1}(a)/S^1$  be the symplectic quotient with canonical symplectic form  $\omega_a$ . The preimage  $H^{-1}(a) \rightarrow M_a$  then has the structure of a orbibundle over  $M_a$ . Moreover, since  $a$  is a regular value of  $H$ , there exists a number  $\delta > 0$  such that  $H^{-1}((a - \delta, a + \delta))$  is diffeomorphic to  $H^{-1}(a) \times (-\delta, \delta)$ . With respect to this diffeomorphism, the symplectic form  $\omega$  on  $H^{-1}(a) \times (-\delta, \delta)$  is, up to an exact form, equal to

$$\alpha \wedge dH - (H - a)F_a + \omega_a$$

for one (and hence any) connection one-form  $\alpha$  on the orbibundle  $H^{-1}(a) \rightarrow M_a$  with curvature  $F_a$ . Now, when restricted to  $H^{-1}(a)$ , one can verify that  $\omega|_{H^{-1}(a)} = \omega_a$ , that  $\theta|_{H^{-1}(a)} =: \alpha$  defines a connection 1-form, and that  $d\theta|_{H^{-1}(a)} =: F_a$  is the curvature form of  $\alpha$ , so that  $d_\zeta\theta|_{H^{-1}(a)} = F_a - t$ . So we have that

$$\begin{aligned}
 \int_{H^{-1}(a)} \nu_{[2n-1]} &= \int_{H^{-1}(a)} \theta \wedge (d_\zeta\theta)^{-1} \wedge e^{\omega - tH} = \int_{H^{-1}(a)} \alpha \wedge (F_a - t)^{-1} \wedge e^{\omega_a - ta} \\
 &= -\frac{e^{-ta}}{t} \int_{M_a} \left(1 - \frac{F_a}{t}\right)^{-1} \wedge e^{\omega_a} \\
 &= -\sum_{j=1}^{n-1} \frac{e^{-ta}}{t^{j+1}} \int_{M_a} \frac{\omega_a^{n-1-j}}{(n-1-j)!} \wedge F_a^j.
 \end{aligned}$$

As  $a \rightarrow +\infty$ , the cohomology class of  $\omega_a$  depends linearly on  $a$  [28; 65], whereas that of  $F_a$  remains fixed since the topology of the bundle  $H^{-1}(a) \rightarrow M_a$  does not change as  $a$  runs through a set of regular values. So the integral over  $M_a$  here is a polynomial in  $a$ . Consequently,  $\int_{H^{-1}(a)} \nu_{[2n-1]} \rightarrow 0$  exponentially as  $a \rightarrow +\infty$ . Thus, combining this fact with (A-3) and (A-4), and noting that  $tH = \langle \mu, \zeta \rangle$ , we arrive at the desired conclusion.

### A.4 Examples

We next consider some simple examples and see what formula (A-1) yields for the weighted volume functional  $F$ .

**Example A.6** Let  $M = \mathbb{C}^n$  and consider the action of the maximal torus

$$T = \{\text{diag}(e^{i\eta_1}, \dots, e^{i\eta_n}) \mid \eta_i \in \mathbb{R}\}$$

in  $GL(n, \mathbb{C})$  acting on  $M$  with induced coordinates  $(\eta_1, \dots, \eta_n)$  on the Lie algebra  $\mathfrak{t}$  of  $T$ , where  $(1, 0, \dots, 0) \in \mathfrak{t}$  generates the vector field  $\text{Im}(z_1 \partial_{z_1})$  on  $M$ , etc. The fixed-point set of  $T$  is clearly compact.

For any  $Y \in \{(\eta_1, \dots, \eta_n) \in \mathfrak{t} \mid \eta_i > 0\}$  and for any  $T$ -invariant complete shrinking gradient Kähler–Ricci soliton  $(M, \omega, X)$  with  $X = \nabla^g f$  for  $f: M \rightarrow \mathbb{R}$  smooth, let  $u_Y$  be the Hamiltonian potential of  $Y$  normalised as in Definition 5.13, so that, in particular,  $\Delta_\omega u_Y + u_Y + \frac{1}{2}(JY) \cdot f = 0$ . Then

$$-u_Y(0) = (\Delta_\omega u_Y)(0) + \underbrace{\frac{1}{2}((JY) \cdot f)(0)}_{=0} = \text{div}(Y) = \sum_j \eta_j,$$

and so the Duistermaat–Heckman theorem yields

$$F(\eta_1, \dots, \eta_n) = \int_M e^{-u_Y} \omega^n = \prod_j \eta_j^{-1} \cdot e^{\sum_j \eta_j}.$$

Since this function is symmetric in its components, its unique critical point must be of the form  $\lambda(1, \dots, 1)$  for some  $\lambda > 0$ . It is then easy to show that  $\lambda = 1$ . The corresponding shrinking gradient Kähler–Ricci soliton is the flat Gaussian shrinking soliton on  $\mathbb{C}^n$ .

**Example A.7** Let  $M$  be  $\mathbb{C}^2$  blown up at the origin and again consider the action of the maximal torus  $T = \{\text{diag}(e^{i\eta_1}, e^{i\eta_2}) \mid \eta_1, \eta_2 \in \mathbb{R}\}$  in  $GL(2, \mathbb{C})$  acting on  $M$ , with induced coordinates  $(\eta_1, \eta_2)$  on the Lie algebra  $\mathfrak{t}$  of  $T$ , where  $(1, 0) \in \mathfrak{t}$  generates the vector field  $\text{Im}(z_1 \partial_{z_1})$  on  $M$ , etc. In this case, the weighted volume functional is given by

$$F: \{(\eta_1, \eta_2) \in \mathfrak{t} \mid \eta_1, \eta_2 > 0\} \rightarrow \mathbb{R}_{>0}, \quad F(\eta_1, \eta_2) = \begin{cases} \frac{e^{\eta_1}}{(\eta_1 - \eta_2)\eta_2} + \frac{e^{\eta_2}}{(\eta_2 - \eta_1)\eta_1} & \text{if } \eta_1 \neq \eta_2, \\ e^{\eta_1}(\eta_1^{-1} + \eta_1^{-2}) & \text{if } \eta_1 = \eta_2. \end{cases}$$

Again by symmetry, the unique critical point of  $F$  here must have  $\eta_1 = \eta_2$ , and a computation shows that  $\eta_1 = \eta_2 = \sqrt{2}$  in this case. The corresponding shrinking gradient Kähler–Ricci soliton is that of Feldman, Ilmanen and Knopf [30] on this space.

**Example A.8** More generally, let  $M$  be the total space of the line bundle  $\mathcal{O}(-k)$  over  $\mathbb{P}^{n-1}$  for  $0 < k < n$  and consider the induced action of the maximal torus  $T = \{\text{diag}(e^{i\eta_1}, \dots, e^{i\eta_n}) \mid \eta_i \in \mathbb{R}\}$  in  $GL(n, \mathbb{C})$  acting on  $M$ , with induced coordinates  $(\eta_1, \dots, \eta_n)$  on the Lie algebra  $\mathfrak{t}$  of  $T$ , where  $(1, 0, \dots, 0) \in \mathfrak{t}$  generates the vector field  $\text{Im}(z_1 \partial_{z_1})$  on  $M$ , etc. In this case, the weighted volume

functional  $F: \{(\eta_1, \dots, \eta_n) \in \mathfrak{t} \mid \eta_i > 0\} \rightarrow \mathbb{R}_{>0}$  is given by

$$\begin{aligned}
 F(\eta_1, \dots, \eta_n) &= \sum_{i=1}^n \frac{e^{(k+1-n)\eta_i + \sum_{j \neq i} \eta_j}}{k \eta_i \prod_{j \neq i} (\eta_j - \eta_i)} \\
 &= \frac{\sum_{i=1}^n (-1)^{i+1} \prod_{j \neq i} \eta_j \prod_{k, l \neq i, k > l} (\eta_k - \eta_l) e^{(k+1-n)\eta_i + \sum_{j \neq i} \eta_j}}{k \prod_{i=1}^n \eta_i \prod_{i < j} (\eta_i - \eta_j)} \quad \text{if } \eta_k \neq \eta_l \text{ for } k \neq l.
 \end{aligned}$$

Again, by symmetry, the unique critical point of  $F$  here must satisfy  $\eta_1 = \dots = \eta_n$ . By taking limits, one can write down an expression for  $F$  when this is the case. Differentiating the resulting expression and setting it equal to zero, one obtains the polynomials of [30, equation (36)]. For example, in low dimensions, when  $\eta_1 = \dots = \eta_n =: \eta$ , we obtain the formulae for  $F$  in Table 1. The corresponding shrinking gradient Kähler–Ricci solitons are those of Feldman, Imanen and Knopf [30] on these spaces.

**Example A.9** Let  $L$  be the total space of a negative holomorphic line bundle over a Fano manifold  $D$  of complex dimension  $n$ . By adjunction, in order for  $L$  to admit a shrinking gradient Kähler–Ricci soliton, we must have  $c_1(-K_D \otimes L) > 0$ . Assuming that this is the case, consider the action of the torus  $T$  given by rotating the fibres of  $L$ . We have an induced coordinate  $w$  on the Lie algebra  $\mathfrak{t}$  of  $T$ , where  $1 \in \mathfrak{t}$  will generate the vector field  $\text{Im}(z_i \partial_{z_i})$  in a local trivialising chart of  $L$ . The zero set of every element of  $\mathfrak{t}$  will be  $D$ , the zero section of  $L$ , and in this case the weighted volume functional  $F$  on the domain  $\{\eta \in \mathfrak{t} \mid \eta > 0\}$  is given by

$$\begin{aligned}
 \text{(A-5)} \quad F(\eta) &= \int_{D^n} \frac{e^\eta e^{t^* \omega}}{\eta \left(1 + \frac{c_1(L)}{\eta}\right)} = \frac{e^\eta}{\eta} \int_{D^n} e^{t^* \omega} \left(1 + \frac{c_1(L)}{\eta}\right)^{-1} = \frac{e^\eta}{\eta} \int_{D^n} e^{t^* \omega} \left(1 - \frac{c_1(L^*)}{\eta}\right)^{-1} \\
 &= \frac{e^\eta}{\eta} \int_{D^n} \left(1 + \omega + \frac{\omega^2}{2!} + \dots\right) \left(1 + \frac{c_1(L^*)}{\eta} + \frac{c_1(L^*)^2}{\eta^2} + \dots\right) \\
 &= \frac{e^\eta}{\eta} \int_{D^n} \sum_{i=0}^n \frac{\omega^i}{i!} \wedge \left(\frac{c_1(L^*)^{n-i}}{\eta^{n-i}}\right) = \frac{e^\eta}{\eta} \sum_{i=0}^n \frac{1}{\eta^{n-i} i!} \int_{D^n} \omega^i \wedge c_1(L^*)^{n-i} \\
 &= \frac{e^\eta}{\eta^{n+1}} \sum_{i=0}^n \frac{\eta^i}{i!} \int_{D^n} c_1(K_D^{-1} \otimes L)^i \wedge c_1(L^*)^{n-i}.
 \end{aligned}$$

This formula in particular applies to the total space of the line bundle  $\mathcal{O}(-k)$  over  $\mathbb{P}^{n-1}$  for  $0 < k < n$ . Its relationship to the formulae of Example A.8 is as follows. On the total space of  $\mathcal{O}(-k)$ , we have two torus actions, one given by the standard action of a torus  $T_1$  rotating the fibres of  $\mathcal{O}(-k)$ , and another given by the action of a torus  $T_2$  induced from the standard torus action on  $\mathcal{O}(-1)$  that rotates the fibres. The formulae of Example A.8 with  $\eta_1 = \dots = \eta_n$  are given with respect to the action of  $T_2$ , whereas formula (A-5) is with respect to  $T_1$ . Consequently, the formulae of Example A.8 with  $\eta_1 = \dots = \eta_n$  are given by  $F(k\eta)$ , where  $F$  is as in (A-5).

line bundle	$F(\eta)$
$\mathcal{O}(-1) \rightarrow \mathbb{P}^1$	$\frac{(\eta + 1)e^\eta}{\eta^2}$
$\mathcal{O}(-1) \rightarrow \mathbb{P}^2$	$\frac{(2\eta^2 + 2\eta + 1)e^\eta}{\eta^3}$
$\mathcal{O}(-2) \rightarrow \mathbb{P}^2$	$\frac{(\eta^2 + 2\eta + 2)e^{2\eta}}{\eta^3}$
$\mathcal{O}(-1) \rightarrow \mathbb{P}^3$	$\frac{(9\eta^3 + 9\eta^2 + 6\eta + 2)e^\eta}{\eta^4}$
$\mathcal{O}(-2) \rightarrow \mathbb{P}^3$	$\frac{(4\eta^3 + 6\eta^2 + 6\eta + 3)e^{2\eta}}{\eta^4}$
$\mathcal{O}(-3) \rightarrow \mathbb{P}^3$	$\frac{(\eta^3 + 3\eta^2 + 6\eta + 6)e^{3\eta}}{\eta^4}$

Table 1

### A.5 The domain of definition of the weighted volume functional

By Theorem A.3 and Proposition A.4, we see that the weighted volume functional  $F$  is defined on the open cone  $\Lambda$  of elements of the Lie algebra of the torus admitting Hamiltonian potentials that are proper and bounded below if  $\Lambda$  is nonempty. In this subsection, we characterise  $\Lambda$  algebraically in the setting of asymptotically conical Kähler manifolds.

Our precise set-up is as follows. Let  $(C_0, g_0)$  be a Kähler cone with apex  $o$ , complex structure  $J_0$ , and radial function  $r$  such that  $g_0 = dr^2 + r^2g_S$  for a Riemannian metric  $g_S$  on the link  $S = \{r = 1\}$  of  $C_0$ . Let  $\pi : M \rightarrow C_0$  be a quasiprojective resolution of  $C_0$  that is equivariant with respect to the holomorphic isometric action on  $C_0$  of the torus  $T$  with Lie algebra  $\mathfrak{t}$  generated by  $\xi := J_0r\partial_r$ , and let  $g$  be a Kähler metric on  $M$  with

$$(A-6) \quad |\pi_*g - g_0|_{g_0} = O(r^{-2}),$$

with respect to which  $T$  acts isometrically in a Hamiltonian fashion with moment map  $\mu : M \rightarrow \mathfrak{t}^*$ . Write  $u_Y(x) := \langle \mu(x), Y \rangle$  for  $x \in M$  for the Hamiltonian potential of  $Y \in \mathfrak{t}$ , so that  $du_Y = -\omega \lrcorner Y$ , where  $\omega$  is the Kähler form of  $g$ , and set

$$\Lambda := \{Y \in \mathfrak{t} \mid u_Y \text{ is proper and bounded below}\}.$$

Next, let  $\mathcal{O}_M(M)$  (resp.  $\mathcal{O}_{C_0}(C_0)$ ) denote the global algebraic sections of the structure sheaf of  $M$  (resp. of  $C_0$ ), and write

$$\mathcal{O}_M(M) = \bigoplus_{\alpha \in \mathfrak{t}^*} \mathcal{H}_\alpha$$

for the weight decomposition under the action of  $T$ . Then we have:

**Theorem A.10** *In the above situation,*

$$\Lambda = \{Y \in \mathfrak{t} \mid \alpha(Y) > 0 \text{ for all } \alpha \in \mathfrak{t}^* \text{ such that } \mathcal{H}_\alpha \neq \emptyset \text{ and } \alpha \neq 0\}.$$

**Proof** Let  $E$  denote the exceptional set of the resolution  $\pi : M \rightarrow C_0$ . In what follows, we will identify  $M \setminus E$  with  $C_0 \setminus \{0\}$  via  $\pi$ . Let us begin by making some useful observations. Let  $X$  be the unique vector field on  $M$  such that  $d\pi(X) = r\partial_r$ . Then  $d\pi(JX) = \xi$ , where  $J$  denotes the complex structure on  $M$ , so that  $JX \in \mathfrak{t}$  and  $[X, Y] = 0$  for every  $Y \in \mathfrak{t}$ . Then we have:

**Lemma A.11** *Let  $Y \in \mathfrak{t}$ , so that  $Y$  defines a real holomorphic  $g$ -Killing vector field on  $M$  with  $[X, Y] = 0$ . Then  $Y$  is tangent to the level sets of  $r$  on  $C_0 \setminus \{0\}$ .*

**Proof** Since  $T$  acts isometrically on  $g$  and  $g_0$ ,  $Y$  will define a holomorphic  $g_0$ -Killing vector field on  $C_0$ . We claim that such a vector field is tangent to the level sets of  $r$ . Indeed, simply note that

$$0 = \mathcal{L}_Y g_0 = d(Y \cdot r) \otimes dr + dr \otimes d(Y \cdot r) + 2rdr(Y)g_S + r^2\mathcal{L}_Y g_S.$$

Then, plugging  $\xi$  into both arguments on the right-hand side and observing that

$$[Y, \xi] = [Y, Jr\partial_r] = [Y, JX] = J[Y, X] = 0$$

along the end of  $C_0$  since  $Y$  is holomorphic, we arrive at the fact that

$$-dr(Y) = \frac{1}{2}r(\mathcal{L}_Y g_S)(\xi, \xi) = \frac{1}{2}r(\mathcal{L}_Y(g_S(\xi, \xi)) - 2g_S([Y, \xi], \xi)) = 0,$$

as required. □

We now demonstrate that

$$\Lambda \subseteq \{Y \in \mathfrak{t} \mid \alpha(Y) > 0 \text{ for all } \alpha \in \mathfrak{t}^* \text{ such that } \mathcal{H}_\alpha \neq \emptyset \text{ and } \alpha \neq 0\}.$$

To this end, let  $Y \in \Lambda$ , so that the Hamiltonian potential  $u_Y$  of  $Y$  is proper and bounded below, let  $f$  be a nonconstant holomorphic function on  $M$  on which  $Y$  acts with weight  $\lambda$  so that  $JY(f) = -\lambda f$ , let  $x \in M \setminus E$  be a point where  $f(x) \neq 0$ , and denote by  $\gamma_x(t)$  the flow line of  $-JY$  with  $\gamma_x(0) = x$ . Then

$$\frac{d}{dt} f(\gamma_x(t)) = \lambda f(\gamma_x(t)),$$

so that

$$(A-7) \quad f(\gamma_x(t)) = f(x)e^{-\lambda t} \quad \text{for all } t < 0.$$

Now, by definition, we have that  $-JY = \nabla^g u_Y$  and so from Proposition 2.27 we deduce that there is a sequence  $t_i \rightarrow -\infty$  as  $i \rightarrow +\infty$  such that  $(\gamma_x(t_i))_i$  converges to a point  $x_\infty \in M$  satisfying  $\nabla^g u(x_\infty) = 0$ . Since the fixed-point set of  $T$  is contained in  $E$ , we must have that  $x_\infty \in E$ . Let  $x_i := \gamma_x(t_i)$ . Then plugging  $t_i$  into (A-7) yields the fact that

$$|f(x_i)| = |f(x)|e^{-\lambda t_i} \xrightarrow{i \rightarrow \infty} \begin{cases} +\infty & \text{if } \lambda < 0, \\ 0 & \text{if } \lambda > 0. \end{cases}$$

Since  $x_i \rightarrow x_\infty \in E$  as  $i \rightarrow \infty$ , we conclude from the maximum principle that  $\lambda > 0$ , as required.

Next we show that

$$(A-8) \quad \{Y \in \mathfrak{t} \mid \alpha(Y) > 0 \text{ for all } \alpha \in \mathfrak{t}^* \text{ such that } \mathcal{H}_\alpha \neq \emptyset \text{ and } \alpha \neq 0\} \subseteq \Lambda.$$

By [20, Lemma 2.15],  $M$  is 1-convex. By construction, then,  $\pi: M \rightarrow C_0$  will be the Remmert reduction of  $M$ . In particular, we have that  $\pi^*\mathcal{O}_{C_0}(C_0) = \mathcal{O}_M(M)$  by the properties of the Remmert reduction. Since  $\pi: M \rightarrow C_0$  is equivariant with respect to the action of  $T$ , we thus see that  $Y \in \mathfrak{t}$  acts with weight  $\lambda$  on  $f \in \mathcal{O}_{C_0}(C_0)$  if and only if it acts with weight  $\lambda$  on the unique lift  $\pi^*f$  of  $f$  to  $\mathcal{O}_M(M)$ . Applying [18, Proposition 2.7], we therefore deduce that

$$\{Y \in \mathfrak{t} \mid g_S(Y, \xi) > 0\} = \{Y \in \mathfrak{t} \mid \alpha(Y) > 0 \text{ for all } \alpha \in \mathfrak{t}^* \text{ such that } \mathcal{H}_\alpha \neq \emptyset \text{ and } \alpha \neq 0\}.$$

Consequently, in order to prove the inclusion (A-8), it suffices to show that

$$\{Y \in \mathfrak{t} \mid g_S(Y, \xi) > 0\} \subseteq \Lambda.$$

This inclusion is established by the following proposition.

**Proposition A.12** *Let  $Y \in \{Z \in \mathfrak{t} \mid g_S(Z, \xi)(x) > 0 \text{ for all } x \in S\}$  with Hamiltonian potential  $u_Y$ . Then  $u_Y \geq cr^2$  along the end of  $C_0$  for some  $c > 0$ . In particular,  $u_Y$  is proper and bounded below.*

**Proof** Let  $x \in \{r = 1\}$  and let  $\gamma_x(t)$  denote the integral curve of  $X$ , the vector field on  $M$  satisfying  $d\pi(X) = r\partial_r$ , with  $\gamma_x(0) = x$ . Then we have that

$$(A-9) \quad \begin{aligned} u_Y(\gamma_x(t)) &= u_Y(\gamma_x(0)) + \int_0^t du_Y(\dot{\gamma}_x(s)) ds \\ &= u_Y(x) + \int_0^t g(-JY, X)(\gamma_x(s)) ds \\ &= u_Y(x) + \int_0^t g(Y, JX)(\gamma_x(s)) ds. \end{aligned}$$

Next observe that since  $Y \in \mathfrak{t}$ ,  $Y$  is tangent to the level sets of  $r$  by Lemma A.11. Hence, the asymptotics (A-6) give us that

$$\begin{aligned} g(Y, JX)(\gamma_x(s)) &= g_0(Y, JX) + \overbrace{O(r^{-2})|Y|_{g_0}|JX|_{g_0}}^{=O(1)} \\ &= O(1) + r(\gamma_x(s))^2 g_S(Y, \xi) \\ &= O(1) + r(x)^2 e^{2s} g_S(Y, \xi), \end{aligned}$$

where the final equality follows from the fact that  $r(\gamma_x(s)) = r(x)e^s$  because

$$\frac{\partial}{\partial s}(r(\gamma_x(s))) = r(\gamma_x(s)) \quad \text{and} \quad \gamma_x(0) = x.$$

Plugging this into (A-9) yields

$$\begin{aligned} u_Y(\gamma_x(t)) &= u_Y(x) + \frac{1}{2}r(x)^2 g_S(Y, \xi)(e^{2t} - 1) + O(t) \\ &= u_Y(x) - \frac{1}{2}g_S(Y, \xi)r(x)^2 + \frac{1}{2}g_S(Y, \xi)r(\gamma_x(t))^2 + O(\ln r(\gamma_x(t))) \\ &\geq cr(\gamma_x(t))^2 \end{aligned}$$

along the end of  $C_0$  for some  $c > 0$ , since  $g_S(Y, \xi) > 0$ . From this, the assertion follows. □



### A.6 Coercive estimates on Hamiltonian potentials

The goal of this subsection is to prove sharp positive bounds on the growth of the Hamiltonian potential of a real holomorphic Killing vector field on a complete shrinking gradient Kähler–Ricci soliton  $(M, g, X)$  that commutes with the soliton vector field  $X$  under certain conditions. Since  $H^1(M) = 0$  by [67], such a vector field always admits a Hamiltonian potential. Let  $\omega$  denote the Kähler form of  $g$  and recall that for each real holomorphic Killing vector field  $Y$  on  $M$  commuting with  $X$ , the Hamiltonian  $u_Y$  of  $Y$  is normalised so that  $\Delta_\omega u_Y + u_Y + \frac{1}{2}JY \cdot f = 0$ . Since  $du_Y = -\omega_\perp Y$  by definition, one sees that

$$(A-10) \quad \Delta_\omega u_Y + u_Y = -\frac{1}{2}g(JY, X) = \frac{1}{2}g(\nabla u_Y, X) = \frac{1}{2}X \cdot u_Y,$$

an identity that shall prove useful in what follows. We will prove:

**Proposition A.13** *Let  $(M, g, X)$  be a complete shrinking gradient Kähler–Ricci soliton with bounded Ricci curvature with soliton vector field  $X = \nabla^g f$  for a smooth real-valued function  $f : M \rightarrow \mathbb{R}$ . Let  $Y$  be a real holomorphic Killing vector field on  $M$  commuting with  $X$  and assume that the Hamiltonian potential  $u_Y$  of  $Y$  is proper and bounded below. Then there exist positive constants  $c_1$  and  $c_2$  such that  $c_1 f \leq u_Y \leq c_2 f$  outside of a compact set.*

As a consequence of this proposition, we see, without appealing to the Duistermaat–Heckman theorem, that the weighted volume functional is defined on a complete shrinking gradient Kähler–Ricci soliton  $(M, g, X)$  with bounded Ricci curvature on those elements admitting a Hamiltonian potential that is proper and bounded below in the Lie algebra of any torus that acts in a holomorphic Hamiltonian fashion on  $M$  and contains the flow of  $JX$ , where  $J$  denotes the complex structure of  $M$ .

**Proof** Let  $|\cdot|_g$  denote the norm with respect to  $g$ . The following claim provides a sharp growth rate on  $|Y|_g$  which improves Proposition 2.30 for real holomorphic vector fields commuting with  $X$ .

**Claim A.14** *There exists a positive constant  $c$  such that  $|Y|_g^2 \leq cf$  outside a compact set.*

**Proof** We start by taking note of the following crucial equation that holds pointwise on  $M$  and is independent of the soliton structure. For a real holomorphic vector field  $Y$ , it holds that

$$\Delta_g Y + \text{Ric}(Y) = 0.$$

This observation is due to Lichnerowicz and its proof can be found for instance in [2, Proposition 4.79]. Since we allow  $Y$  to commute with  $X$ ,  $Y$  satisfies the elliptic equation

$$\Delta_g Y - \nabla_X Y + Y = \Delta_g Y - \nabla_X Y + \text{Ric}(Y) + \nabla_Y X = 0,$$

where we have used the soliton equation (1-4) with  $\lambda = 1$ . Recall also that  $\Delta_g f - X \cdot f = -2f$ .

Using these facts, we compute

$$\begin{aligned}
 \text{(A-11)} \quad (\Delta_g - X \cdot) \left( \frac{|Y|_g^2}{f} \right) &= 2f^{-1} |\nabla Y|_g^2 - 2 \frac{|Y|_g^2}{f} + |Y|_g^2 (\Delta_g - X \cdot) f^{-1} + 2g(\nabla^g |Y|_g^2, \nabla^g f^{-1}) \\
 &\geq -2 \frac{|Y|_g^2}{f} + \frac{|Y|_g^2}{f} \left( -\frac{(\Delta_g - \nabla^g f \cdot) f}{f} + 2 \frac{|\nabla^g f|^2}{f^2} \right) \\
 &\quad - 2g \left( \nabla^g \left( \frac{|Y|_g^2}{f} \right), \nabla^g \ln f \right) - 2 |\nabla^g \ln f|^2 \frac{|Y|_g^2}{f} \\
 &\geq -2g \left( \nabla^g \left( \frac{|Y|_g^2}{f} \right), \nabla^g \ln f \right),
 \end{aligned}$$

a computation valid on the set where  $f$  is strictly positive. For  $\varepsilon > 0$  a given constant and  $a > 0$  a constant such that  $|Y|_g^2 = O(f^a)$  (see Proposition 2.30), we consider the function  $|Y|_g^2 \cdot f^{-1} - \varepsilon f^a$  defined on the complement of a compact set of the form  $\{f \leq C(n)\}$  such that  $f$  is strictly positive on  $\{f \geq C(n)\}$ . By Proposition 2.30, we know that

$$\lim_{f \rightarrow +\infty} (|Y|_g^2 \cdot f^{-1} - \varepsilon f^a) = -\infty,$$

so that this function attains a maximum on the set  $\{f \geq C(n)\}$ . Now, if  $f \geq 2(a + 1)$ , then using (A-11), we compute that

$$\begin{aligned}
 (\Delta_g - X \cdot) \left( \frac{|Y|_g^2}{f} - \varepsilon f^a \right) &\geq -2g \left( \nabla^g \left( \frac{|Y|_g^2}{f} - \varepsilon f^a \right), \nabla^g \ln f \right) + \varepsilon a f^a \left( 2 - (a + 1) \frac{|\nabla^g f|^2}{f^2} \right) \\
 &\geq -2g \left( \nabla^g \left( \frac{|Y|_g^2}{f} - \varepsilon f^a \right), \nabla^g \ln f \right) + \varepsilon a f^a \left( 2 - (a + 1) \frac{2}{f} \right) \\
 &> -2g \left( \nabla^g \left( \frac{|Y|_g^2}{f} - \varepsilon f^a \right), \nabla^g \ln f \right),
 \end{aligned}$$

where we used the first-order soliton identity from Lemma 2.22 in the penultimate inequality. As already noted,  $|Y|_g^2 \cdot f^{-1} - \varepsilon f^a$  attains a maximum on  $\{f \geq C(n)\}$  and so the maximum principle applied to this function implies that it must attain its maximum at a point on the boundary of the set  $\{f \geq \max\{2(a + 1), C(n)\}\}$ . In particular, we find that

$$\text{(A-12)} \quad |Y|_g^2 \cdot f^{-1} - \varepsilon f^a \leq \max_{f=C(a,n)} (|Y|_g^2 \cdot f^{-1} - \varepsilon f^a) \leq \max_{f=C(a,n)} (|Y|_g^2 \cdot f^{-1}) = C(a, n) \max_{f=C(a,n)} |Y|_g^2,$$

where  $C(a, n)$  is a positive constant that may vary from line to line. As the right-hand side of (A-12) is independent of  $\varepsilon$ , by letting  $\varepsilon$  tend to 0 on the left-hand side, we reach the desired conclusion.  $\triangleleft$

Since  $|\nabla u_Y|_g = |JY|_g = |Y|_g$ , one obtains the expected growth on  $u_Y$  by integrating the estimate on  $|Y|_g$  established in the previous claim.

We next prove the lower bound on  $u_Y$ . First notice that since  $u_Y$  is proper and bounded below,  $u_Y$  is strictly positive outside a sufficiently large compact set of the form  $\{f \leq c_0\}$ . Recall from Remark 2.23 that

the normalisation of  $f$  is determined by the soliton identities, which in this case yield  $\Delta_g f - X \cdot f = -2f$ . Using  $f$  as a barrier function together with (A-10), we compute the weighted Laplacian of the difference of the inverses of  $u_Y$  and  $f$  on the region where this difference makes sense. We have

$$(A-13) \quad (\Delta_\omega - \frac{1}{2}X \cdot) \left( \frac{1}{u_Y} - \frac{C}{f} \right) = \frac{1}{u_Y} - \frac{C}{f} + 2 \frac{|\nabla u_Y|^2}{u_Y^3} - 2C \frac{|\nabla f|^2}{f^3},$$

where  $C$  is a positive constant to be specified later. Assume that the function  $u_Y^{-1} - Cf^{-1}$  attains its maximum at an interior point  $x_0$  of a domain of the form  $\{r_1 \leq f \leq r_2\}$  with  $r_1$  sufficiently large so that both  $u_Y$  and  $f$  are strictly positive. At such a point  $x_0$ , one sees that  $\nabla(u_Y^{-1} - Cf^{-1})(x_0) = 0$ , and from the maximum principle that

$$0 \geq (\Delta_\omega - \frac{1}{2}X \cdot) \left( \frac{1}{u_Y} - \frac{C}{f} \right) (x_0).$$

This information, together with (A-13), implies that at  $x_0$ ,

$$\begin{aligned} 0 &\geq \frac{1}{u_Y} - \frac{C}{f} + 2 \frac{|\nabla u_Y|^2}{u_Y^3} - 2C \frac{|\nabla f|^2}{f^3} = \frac{1}{u_Y} - \frac{C}{f} + 2u_Y |\nabla u_Y^{-1}|^2 - 2Cf |\nabla f^{-1}|^2 \\ &= \frac{1}{u_Y} - \frac{C}{f} + 2u_Y C^2 |\nabla f^{-1}|^2 - 2Cf |\nabla f^{-1}|^2 = \left( \frac{1}{u_Y} - \frac{C}{f} \right) (1 - 2Cu_Y f |\nabla f^{-1}|^2) \\ &= \left( \frac{1}{u_Y} - \frac{C}{f} \right) \left( 1 - 2C \frac{u_Y}{f^3} |\nabla f|^2 \right). \end{aligned}$$

Next, using the fact that  $|\nabla f|^2$  grows quadratically by the soliton identities, we see from the upper bound on  $u_Y$  that on  $M$ ,

$$2C \frac{u_Y}{f^3} |\nabla f|^2 \leq \frac{2Cd}{f}$$

for some positive constant  $d$  uniform in  $r_1$  and  $r_2$ . In particular, the term  $(1 - 2Cu_Y f^{-3} |\nabla f|^2)$  is positive on  $\{r_1 \leq f \leq r_2\}$  as long as  $2Cf^{-1}d$  is strictly less than 1, or equivalently, as long as  $C < (2d)^{-1}r_1$ .

In summary, for any heights  $r_1 < r_2$  and any constant  $C$  such that  $C < (2d)^{-1}r_1$ , we have that

$$\max_{r_1 \leq f \leq r_2} \left( \frac{1}{u_Y} - \frac{C}{f} \right) \leq \max \left\{ 0, \max_{f=r_1} \left( \frac{1}{u_Y} - \frac{C}{f} \right), \max_{f=r_2} \left( \frac{1}{u_Y} - \frac{C}{f} \right) \right\}.$$

Since  $u_Y$  (and  $f$ ) tend to  $+\infty$  as  $f$  approaches  $+\infty$ , one sees, by letting  $r_2$  tend to  $+\infty$ , that

$$(A-14) \quad \max_{r_1 \leq f} \left( \frac{1}{u_Y} - \frac{C}{f} \right) \leq \max \left\{ 0, \max_{f=r_1} \left( \frac{1}{u_Y} - \frac{C}{f} \right) \right\}.$$

We choose  $C$  and  $r_1$  so that the right-hand side of (A-14) is nonpositive and so that  $C < (2d)^{-1}r_1$ . This is possible because by properness of  $u_Y$ , there exists some positive height  $r_1$  such that  $\min_{f=r_1} u_Y \geq 4d$ . Then for  $C := (4d)^{-1}r_1$ , we have that  $C < (2d)^{-1}r_1$  and

$$\max_{f=r_1} \left( \frac{1}{u_Y} - \frac{C}{f} \right) \leq 0,$$

as desired. This completes the proof of the proposition. □

## References

- [1] **DM Austin, P J Braam**, *Morse–Bott theory and equivariant cohomology*, from “The Floer memorial volume” (H Hofer, C H Taubes, A Weinstein, E Zehnder, editors), Progr. Math. 133, Birkhäuser, Basel (1995) 123–183 MR Zbl
- [2] **W Ballmann**, *Lectures on Kähler manifolds*, Eur. Math. Soc., Zürich (2006) MR Zbl
- [3] **W Barth, R Narasimhan** (editors), *Several complex variables, VI: Complex manifolds*, Encyclopaedia of Mathematical Sciences 69, Springer (1990) MR Zbl
- [4] **F A Belgun**, *On the metric structure of non-Kähler complex surfaces*, Math. Ann. 317 (2000) 1–40 MR Zbl
- [5] **N Berline, E Getzler, M Vergne**, *Heat kernels and Dirac operators*, corrected 1st edition, Grundle. Math. Wissen. 298, Springer (2004) MR Zbl
- [6] **R J Berman, D W Nystrom**, *Complex optimal transport and the pluripotential theory of Kähler–Ricci solitons*, preprint (2014) arXiv 1401.8264
- [7] **C Birkar, P Cascini, C D Hacon, J McKernan**, *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc. 23 (2010) 405–468 MR Zbl
- [8] **C P Boyer, K Galicki**, *Sasakian geometry*, Oxford Univ. Press (2008) MR Zbl
- [9] **E Brieskorn**, *Rationale Singularitäten komplexer Flächen*, Invent. Math. 4 (1967/68) 336–358 MR Zbl
- [10] **R L Bryant**, *Gradient Kähler Ricci solitons*, from “Géométrie différentielle, physique mathématique, mathématiques et société, I” (O Hijazi, editor), Astérisque 321, Soc. Math. France, Paris (2008) 51–97 MR Zbl
- [11] **H-D Cao, D Zhou**, *On complete gradient shrinking Ricci solitons*, J. Differential Geom. 85 (2010) 175–185 MR Zbl
- [12] **C-W Chen, A Deruelle**, *Structure at infinity of expanding gradient Ricci soliton*, Asian J. Math. 19 (2015) 933–950 MR Zbl
- [13] **X Chen, S Sun, G Tian**, *A note on Kähler–Ricci soliton*, Int. Math. Res. Not. 2009 (2009) 3328–3336 MR Zbl
- [14] **B Chow, D Knopf**, *The Ricci flow: an introduction*, Mathematical Surveys and Monographs 110, Amer. Math. Soc., Providence, RI (2004) MR Zbl
- [15] **B Chow, P Lu**, *Uniqueness of asymptotic cones of complete noncompact shrinking gradient Ricci solitons with Ricci curvature decay*, C. R. Math. Acad. Sci. Paris 353 (2015) 1007–1009 MR Zbl
- [16] **B Chow, P Lu, B Yang**, *Lower bounds for the scalar curvatures of noncompact gradient Ricci solitons*, C. R. Math. Acad. Sci. Paris 349 (2011) 1265–1267 MR Zbl
- [17] **C van Coevering**, *Examples of asymptotically conical Ricci-flat Kähler manifolds*, Math. Z. 267 (2011) 465–496 MR Zbl
- [18] **T C Collins, G Székelyhidi**, *K-semistability for irregular Sasakian manifolds*, J. Differential Geom. 109 (2018) 81–109 MR Zbl
- [19] **R J Conlon, A Deruelle**, *Expanding Kähler–Ricci solitons coming out of Kähler cones*, J. Differential Geom. 115 (2020) 303–365 MR Zbl
- [20] **R J Conlon, H-J Hein**, *Asymptotically conical Calabi–Yau manifolds, I*, Duke Math. J. 162 (2013) 2855–2902 MR Zbl

- [21] **R J Conlon, H-J Hein**, *Asymptotically conical Calabi–Yau metrics on quasi-projective varieties*, *Geom. Funct. Anal.* 25 (2015) 517–552 MR Zbl
- [22] **X Dai, G Wei**, *Comparison geometry for Ricci curvature*, online book (2008) Available at <http://web.math.ucsb.edu/~dai/Ricci-book.pdf>
- [23] **A S Dancer, M Y Wang**, *On Ricci solitons of cohomogeneity one*, *Ann. Global Anal. Geom.* 39 (2011) 259–292 MR Zbl
- [24] **J-P Demailly**, *Complex analytic and differential geometry*, online book (2012) Available at <http://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf>
- [25] **J-P Demailly, M Paun**, *Numerical characterization of the Kähler cone of a compact Kähler manifold*, *Ann. of Math.* 159 (2004) 1247–1274 MR Zbl
- [26] **A Deruelle**, *Smoothing out positively curved metric cones by Ricci expanders*, *Geom. Funct. Anal.* 26 (2016) 188–249 MR Zbl
- [27] **A Deruelle**, *Asymptotic estimates and compactness of expanding gradient Ricci solitons*, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* 17 (2017) 485–530 MR Zbl
- [28] **J J Duistermaat, G J Heckman**, *On the variation in the cohomology of the symplectic form of the reduced phase space*, *Invent. Math.* 69 (1982) 259–268 MR Zbl
- [29] **F Fang, X-D Li, Z Zhang**, *Two generalizations of Cheeger–Gromoll splitting theorem via Bakry–Emery Ricci curvature*, *Ann. Inst. Fourier (Grenoble)* 59 (2009) 563–573 MR Zbl
- [30] **M Feldman, T Ilmanen, D Knopf**, *Rotationally symmetric shrinking and expanding gradient Kähler–Ricci solitons*, *J. Differential Geom.* 65 (2003) 169–209 MR Zbl
- [31] **T Frankel**, *Fixed points and torsion on Kähler manifolds*, *Ann. of Math.* 70 (1959) 1–8 MR Zbl
- [32] **A Futaki**, *Kähler–Einstein metrics and integral invariants*, *Lecture Notes in Math.* 1314, Springer (1988) MR Zbl
- [33] **A Futaki**, *The weighted Laplacians on real and complex metric measure spaces*, from “Geometry and analysis on manifolds” (T Ochiai, T Mabuchi, Y Maeda, J Noguchi, A Weinstein, editors), *Progr. Math.* 308, Springer (2015) 343–351 MR Zbl
- [34] **A Futaki, M-T Wang**, *Constructing Kähler–Ricci solitons from Sasaki–Einstein manifolds*, *Asian J. Math.* 15 (2011) 33–52 MR Zbl
- [35] **A M Gleason, R S Palais**, *On a class of transformation groups*, *Amer. J. Math.* 79 (1957) 631–648 MR Zbl
- [36] **H Grauert**, *Über Modifikationen und exzeptionelle analytische Mengen*, *Math. Ann.* 146 (1962) 331–368 MR Zbl
- [37] **W He, S Sun**, *Frankel conjecture and Sasaki geometry*, *Adv. Math.* 291 (2016) 912–960 MR Zbl
- [38] **S Ishii**, *Introduction to singularities*, Springer (2014) MR Zbl
- [39] **K Iwasawa**, *On some types of topological groups*, *Ann. of Math.* 50 (1949) 507–558 MR Zbl
- [40] **S Kobayashi**, *Transformation groups in differential geometry*, *Ergebnisse der Math.* 70, Springer (1972) MR Zbl
- [41] **J Kollár**, *Lectures on resolution of singularities*, *Annals of Mathematics Studies* 166, Princeton Univ. Press (2007) MR Zbl

- [42] **B Kotschwar**, *Kählerity of shrinking gradient Ricci solitons asymptotic to Kähler cones*, J. Geom. Anal. 28 (2018) 2609–2623 MR Zbl
- [43] **B Kotschwar, L Wang**, *Rigidity of asymptotically conical shrinking gradient Ricci solitons*, J. Differential Geom. 100 (2015) 55–108 MR Zbl
- [44] **B Kotschwar, L Wang**, *Isometries of asymptotically conical shrinking Ricci solitons*, preprint (2019) arXiv 1901.00044
- [45] **H B Laufer**, *Normal two-dimensional singularities*, Annals of Mathematics Studies 71, Princeton Univ. Press (1971) MR Zbl
- [46] **R Lazarsfeld**, *Positivity in algebraic geometry, I: Classical setting: line bundles and linear series*, Ergebnisse der Math. (3) 48, Springer (2004) MR Zbl
- [47] **C Li**, *On rotationally symmetric Kähler–Ricci solitons*, preprint (2010) arXiv 1004.4049
- [48] **M T Lock, J A Viaclovsky**, *A smörgåsbord of scalar-flat Kähler ALE surfaces*, J. Reine Angew. Math. 746 (2019) 171–208 MR Zbl
- [49] **D Martelli, J Sparks, S-T Yau**, *Sasaki–Einstein manifolds and volume minimisation*, Comm. Math. Phys. 280 (2008) 611–673 MR Zbl
- [50] **Y Matsushima**, *Sur la structure du groupe d’homéomorphismes analytiques d’une certaine variété kählérienne*, Nagoya Math. J. 11 (1957) 145–150 MR Zbl
- [51] **O Munteanu, J Wang**, *Holomorphic functions on Kähler–Ricci solitons*, J. Lond. Math. Soc. 89 (2014) 817–831 MR Zbl
- [52] **O Munteanu, J Wang**, *Topology of Kähler Ricci solitons*, J. Differential Geom. 100 (2015) 109–128 MR Zbl
- [53] **O Munteanu, J Wang**, *Conical structure for shrinking Ricci solitons*, J. Eur. Math. Soc. 19 (2017) 3377–3390 MR Zbl
- [54] **O Munteanu, J Wang**, *Structure at infinity for shrinking Ricci solitons*, Ann. Sci. Éc. Norm. Supér. 52 (2019) 891–925 MR Zbl
- [55] **T Ohsawa**, *Vanishing theorems on complete Kähler manifolds*, Publ. Res. Inst. Math. Sci. 20 (1984) 21–38 MR Zbl
- [56] **P Orlik, P Wagreich**, *Isolated singularities of algebraic surfaces with  $C^*$  action*, Ann. of Math. 93 (1971) 205–228 MR Zbl
- [57] **R S Palais**, *A global formulation of the Lie theory of transformation groups*, Mem. Amer. Math. Soc. 22, Amer. Math. Soc., Providence, RI (1957) MR Zbl
- [58] **S Pigola, M Rimoldi, A G Setti**, *Remarks on noncompact gradient Ricci solitons*, Math. Z. 268 (2011) 777–790 MR Zbl
- [59] **H Pinkham**, *Normal surface singularities with  $C^*$  action*, Math. Ann. 227 (1977) 183–193 MR Zbl
- [60] **E Prato, S Wu**, *Duistermaat–Heckman measures in a noncompact setting*, Compositio Math. 94 (1994) 113–128 MR Zbl
- [61] **H Rossi**, *Vector fields on analytic spaces*, Ann. of Math. 78 (1963) 455–467 MR Zbl
- [62] **M Siepmann**, *Ricci flows of Ricci flat cones*, PhD thesis, ETH Zürich (2013) Available at <http://e-collection.library.ethz.ch/eserv/eth:7556/eth-7556-02.pdf>

- [63] **G Tian, X Zhu**, *Uniqueness of Kähler–Ricci solitons*, Acta Math. 184 (2000) 271–305 MR Zbl
- [64] **G Tian, X Zhu**, *A new holomorphic invariant and uniqueness of Kähler–Ricci solitons*, Comment. Math. Helv. 77 (2002) 297–325 MR Zbl
- [65] **S Wu**, *An integration formula for the square of moment maps of circle actions*, Lett. Math. Phys. 29 (1993) 311–328 MR Zbl
- [66] **S Wu**, *On the instanton complex of holomorphic Morse theory*, Comm. Anal. Geom. 11 (2003) 775–807 MR Zbl
- [67] **W Wylie**, *Complete shrinking Ricci solitons have finite fundamental group*, Proc. Amer. Math. Soc. 136 (2008) 1803–1806 MR Zbl
- [68] **B Yang**, *A characterization of noncompact Koiso-type solitons*, Internat. J. Math. 23 (2012) 1250054, 13 MR Zbl
- [69] **Z-H Zhang**, *On the completeness of gradient Ricci solitons*, Proc. Amer. Math. Soc. 137 (2009) 2755–2759 MR Zbl

RJC: *Department of Mathematical Sciences, The University of Texas at Dallas  
Richardson, TX, United States*

AD: *Université Paris-Saclay, Laboratoire de mathématiques d’Orsay  
Orsay, France*

SS: *Institute for Advanced Study in Mathematics, Zhejiang University  
Hangzhou, China*

*Department of Mathematics, University of California, Berkeley  
Berkeley, CA, United States*

ronan.conlon@utdallas.edu, alix.deruelle@universite-paris-saclay.fr,  
songsun1987@gmail.com

Proposed: Bruce Kleiner

Seconded: Tobias H Colding, Simon Donaldson

Received: 25 March 2021

Revised: 27 February 2022





# GEOMETRY & TOPOLOGY

msp.org/gt

## MANAGING EDITOR

András I Stipsicz    Alfréd Rényi Institute of Mathematics  
stipsicz@renyi.hu

## BOARD OF EDITORS

Mohammed Abouzaid	Stanford University abouzaid@stanford.edu	Frances Kirwan	University of Oxford frances.kirwan@balliol.oxford.ac.uk
Dan Abramovich	Brown University dan_abramovich@brown.edu	Bruce Kleiner	NYU, Courant Institute bkleiner@cims.nyu.edu
Ian Agol	University of California, Berkeley ianagol@math.berkeley.edu	Urs Lang	ETH Zürich urs.lang@math.ethz.ch
Mark Behrens	University of Notre Dame mbehren1@nd.edu	Marc Levine	Universität Duisburg-Essen marc.levine@uni-due.de
Mladen Bestvina	University of Utah bestvina@math.utah.edu	John Lott	University of California, Berkeley lott@math.berkeley.edu
Martin R Bridson	University of Oxford bridson@maths.ox.ac.uk	Ciprian Manolescu	University of California, Los Angeles cm@math.ucla.edu
Jim Bryan	University of British Columbia jbryan@math.ubc.ca	Haynes Miller	Massachusetts Institute of Technology hrm@math.mit.edu
Dmitri Burago	Pennsylvania State University burago@math.psu.edu	Tomasz Mrowka	Massachusetts Institute of Technology mrowka@math.mit.edu
Tobias H Colding	Massachusetts Institute of Technology colding@math.mit.edu	Aaron Naber	Northwestern University anaber@math.northwestern.edu
Simon Donaldson	Imperial College, London s.donaldson@ic.ac.uk	Walter Neumann	Columbia University neumann@math.columbia.edu
Yasha Eliashberg	Stanford University eliash-gt@math.stanford.edu	Jean-Pierre Otal	Université Paul Sabatier, Toulouse otal@math.univ-toulouse.fr
Benson Farb	University of Chicago farb@math.uchicago.edu	Peter Ozsváth	Princeton University petero@math.princeton.edu
Steve Ferry	Rutgers University sferry@math.rutgers.edu	Leonid Polterovich	Tel Aviv University polterov@post.tau.ac.il
David M Fisher	Rice University davidfisher@rice.edu	Colin Rourke	University of Warwick gt@maths.warwick.ac.uk
Mike Freedman	Microsoft Research michaelf@microsoft.com	Roman Sauer	Karlsruhe Institute of Technology roman.sauer@kit.edu
David Gabai	Princeton University gabai@princeton.edu	Stefan Schwede	Universität Bonn schwede@math.uni-bonn.de
Stavros Garoufalidis	Southern U. of Sci. and Tech., China stavros@mpim-bonn.mpg.de	Natasa Sesum	Rutgers University natasas@math.rutgers.edu
Cameron Gordon	University of Texas gordon@math.utexas.edu	Peter Teichner	Max Planck Institut für Mathematik teichner@mac.com
Lothar Göttsche	Abdus Salam Int. Centre for Th. Physics gottsche@ictp.trieste.it	Richard P Thomas	Imperial College, London richard.thomas@imperial.ac.uk
Jesper Grodal	University of Copenhagen jg@math.ku.dk	Gang Tian	Massachusetts Institute of Technology tian@math.mit.edu
Misha Gromov	IHÉS and NYU, Courant Institute gromov@ihes.fr	Ulrike Tillmann	Oxford University tillmann@maths.ox.ac.uk
Mark Gross	University of Cambridge mgross@dpms.cam.ac.uk	Nathalie Wahl	University of Copenhagen wahl@math.ku.dk
Rob Kirby	University of California, Berkeley kirby@math.berkeley.edu	Anna Wienhard	Universität Heidelberg wienhard@mathi.uni-heidelberg.de

---

See inside back cover or [msp.org/gt](http://msp.org/gt) for submission instructions.


The subscription price for 2024 is US \$805/year for the electronic version, and \$1135/year (+\$70, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Geometry & Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Geometry & Topology (ISSN 1465-3060 printed, 1364-0380 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

---

GT peer review and production are managed by EditFLOW<sup>®</sup> from MSP.

PUBLISHED BY

 **mathematical sciences publishers**  
nonprofit scientific publishing

<http://msp.org/>

© 2024 Mathematical Sciences Publishers

# GEOMETRY & TOPOLOGY

Volume 28

Issue 1 (pages 1–496)

2024

---

Homological invariants of codimension 2 contact submanifolds	1
LAURENT CÔTÉ and FRANÇOIS-SIMON FAUTEUX-CHAPLEAU	
The desingularization of the theta divisor of a cubic threefold as a moduli space	127
AREND BAYER, SJOERD VIKTOR BEENTJES, SOHEYLA FEYZBAKSH, GEORG HEIN, DILETTA MARTINELLI, FATEMEH REZAAE and BENJAMIN SCHMIDT	
Coarse-median preserving automorphisms	161
ELIA FIORAVANTI	
Classification results for expanding and shrinking gradient Kähler–Ricci solitons	267
RONAN J CONLON, ALIX DERUELLE and SONG SUN	
Embedding calculus and smooth structures	353
BEN KNUDSEN and ALEXANDER KUPERS	
Stable maps to Looijenga pairs	393
PIERRICK BOUSSEAU, ANDREA BRINI and MICHEL VAN GARREL	