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Stable maps to Looijenga pairs

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A log Calabi–Yau surface with maximal boundary, or Looijenga pair, is a pair (Y, D) with Y a smooth rational projective complex surface and $D = D_1 + \cdots + D_l \in |-K_Y|$ an anticanonical singular nodal curve. Under some natural conditions on the pair, we propose a series of correspondences relating five different classes of enumerative invariants attached to (Y, D):

(1) the log Gromov–Witten theory of the pair (Y, D),
(2) the Gromov–Witten theory of the total space of $\bigoplus_i \mathcal{O}_Y(-D_i)$,
(3) the open Gromov–Witten theory of special Lagrangians in a Calabi–Yau 3–fold determined by (Y, D),
(4) the Donaldson–Thomas theory of a symmetric quiver specified by (Y, D), and
(5) a class of BPS invariants considered in different contexts by Klemm and Pandharipande, Ionel and Parker, and Labastida, Mariño, Ooguri and Vafa.

We furthermore provide a complete closed-form solution to the calculation of all these invariants.

14J33, 14J81, 14N35, 16G20, 53D45

1 Introduction

1.1 Looijenga pairs

A log Calabi–Yau surface with maximal boundary, or Looijenga pair, is a pair $Y(D) := (Y, D)$ consisting of a smooth rational projective complex surface Y and an anticanonical singular nodal curve $D = D_1 + \cdots + D_l \in |-K_Y|$. A prototypical example of Looijenga pair is given by $(Y, D) = (\mathbb{P}^2, D_1 + D_2)$ for D_1 a line and D_2 a conic not tangent to D_1.

Looijenga pairs [79] were first systematically studied in relation with resolutions and deformations of elliptic surface singularities and with degenerations of K3 surfaces; see Friedman and Scattone [41]. More recently, Looijenga pairs have played an important role as two-dimensional examples for mirror symmetry; see Barrott [9], Bousseau [13], Gross, Hacking and Keel [53], Hacking and Keating [60], Mandel [81] and Yu [114; 115] and, for the theory of cluster varieties, Gross, Hacking and Keel [52], Mandel [82] and Zhou [117]. These new developments have had in return nontrivial applications to the classical geometry of Looijenga pairs; see Engel [38], Friedman [40] and Gross, Hacking and Keel [53; 54].

1.2 Summary of the main results

In this paper we develop a series of correspondences relating different enumerative invariants associated to a given Looijenga pair. We start off by giving a very succinct summary of the main objects we will consider, and the main statements we shall prove.

1.2.1 Geometries Let $(Y, D = D_1 + \cdots + D_l)$ be a Looijenga pair with $l \geq 2$. In this paper we will construct four different geometries out of (Y, D):

- the log Calabi–Yau surface geometry $Y(D)$;
- the local Calabi–Yau $(l+2)$–fold geometry $E_{Y(D)} := \mathrm{Tot}\big(\mathcal{O}_Y(-D_1) \oplus \cdots \oplus \mathcal{O}_Y(-D_l)\big)$;
- a noncompact Calabi–Yau threefold geometry canonically equipped with a disjoint union of $l-1$ Lagrangians,
$$Y^{\mathrm{op}}(D) := \big(\mathrm{Tot}\big(\mathcal{O}(-D_l) \to Y \setminus (D_1 \cup \cdots \cup D_{l-1})\big), L_1 \sqcup \cdots \sqcup L_{l-1}\big),$$
where L_i are fibred over real curves in D_i;
- for $l = 2$, a noncommutative geometry given by a symmetric quiver $\mathsf{Q}(Y(D))$ made from the combinatorial data of the divisors D_i and their intersections.

1.2.2 Enumerative theories Our main focus will be on the enumerative geometry of curves in these geometries. More precisely, to a Looijenga pair $Y(D)$ satisfying some natural positivity conditions, we shall associate several classes of a priori different enumerative invariants:

- **log GW** All genus log GW invariants of $Y(D)$, counting curves in the surface Y with maximal tangency conditions along the divisors D_i.
- **local GW** Genus-zero local GW invariants of the CY $(l+2)$–fold $E_{Y(D)}$.

- **open GW** All genus open GW invariants counting open Riemann surfaces in the CY3–fold $Y^{\mathrm{op}}(D)$ with $l-1$ boundary components mapping to $L_1 \sqcup \cdots \sqcup L_{l-1}$.
- **local BPS** Genus-zero local BPS invariants of $E_{Y(D)}$, in the form of Gopakumar–Vafa/Klemm–Pandharipande/Ionel–Parker (GV/KP/IP) BPS invariants.
- **open BPS** All genus open BPS invariants of $Y^{\mathrm{op}}(D)$, in the form of Labastida–Mariño–Ooguri–Vafa (LMOV) BPS invariants.
- **quiver DT** If $l = 2$, Donaldson–Thomas (DT) invariants of $\mathsf{Q}(Y(D))$.

1.2.3 Correspondences Under some positivity conditions on (Y, D), we will prove that the invariants above essentially coincide with one another. In particular, we shall show

(i) an equality between log GW and local GW in genus zero (Theorem 1.4),

(ii) an equality between log GW and open GW in all genera (Theorem 1.5),

(iii) an equality between local BPS and open BPS in genus zero for all l,

(iv) an equality between local BPS and quiver DT for $l = 2$, ie when the local geometry $E_{Y(D)}$ is CY_4 (Theorem 1.6).

The equality (i) establishes for log CY surfaces a version of a conjecture of van Garrel, Graber and Ruddat about log and local GW invariants [43], while (ii) and (iv) are new. Equality (iii) follows from (i)–(ii) after a BPS-type change of variables.

1.2.4 Integrality Furthermore, we shall prove that the enumerative invariants of Looijenga pairs considered in this paper obey strong integrality constraints (Theorem 1.7), reflecting the conjectured integrality of the open BPS and local BPS counts. This shows the existence of novel integral structures underlying the higher-genus log GW theory of $Y(D)$. Restricting to genus zero, we will obtain as a corollary an algebrogeometric proof of the conjectured integrality of the genus-zero Gopakumar–Vafa invariants of the CY $(l+2)$–fold $E_{Y(D)}$. In particular, for $l = 2$, this proves for CY4 local surfaces an integrality conjecture of Klemm and Pandharipande [68, Conjecture 0].

1.2.5 Solutions Moreover, we will completely solve the enumerative counts for these geometries (Theorems 1.4 and 1.5), by finding explicit closed-form, nonrecursive expressions for the generating series of the invariants associated to our Looijenga pairs.

The rest of the introduction is organised as follows:

- Section 1.3 sets the stage by giving a self-contained account of the enumerative theories we shall consider.

- Section 1.4 illustrates the geometric picture underpinning the web of correspondences explored in the paper. We spell out the enumerative relations (i)–(iv) in the broadest generality where we believe them to hold, and describe in detail the geometric heuristics which led us to (i) in Section 1.4.1 (Conjecture 1.1), to (ii) in Section 1.4.2 (Conjectures 1.2 and 1.3), and to (iii)–(iv) in Section 1.4.3.
- Section 1.5 puts these ideas on a rigorous footing. We first place a natural positivity condition on the irreducible components D_i by requiring them to be all smooth and nef; depending on the context, we often supplement this with a mild condition of "quasi-tameness", whose rationale is justified in Sections 1.5.1 and 1.5.2. The statements of the proof of the correspondences, the integrality results, and the full solutions for our enumerative counts are spelled out in Theorems 1.4–1.7.
- Section 1.6 surveys the implications of our results for related work, with emphasis on the possible sheaf-theoretic interpretations of the BPS invariants we consider.

1.3 Enumerative problems

1.3.1 Higher-genus log Gromov–Witten invariants Log Gromov–Witten theory, which was developed by Abramovich and Chen [25; 1] and Gross and Siebert [58], provides a deformation-invariant way to count curves with prescribed tangency conditions along a normal crossings divisor, by virtual intersection theory on moduli spaces of stable log maps. For $Y(D)$ a Looijenga pair where D has $l \geq 2$ irreducible components, we consider rational curves in Y with given degree $d \in \mathrm{H}_2(Y, \mathbb{Z})$ that meet each component D_j in one point of maximal tangency $d \cdot D_j$ and pass through $l-1$ given points in Y. Counting such curves is an enumerative problem of expected dimension 0 and we denote by $N^{\log}_{0,d}(Y(D))$ the corresponding log Gromov–Witten invariants.

For $g \geq 0$, the expected dimension of the moduli space of genus g curves in Y with given degree $d \in \mathrm{H}_2(Y, \mathbb{Z})$ that meet each component D_j in one point of maximal tangency $d \cdot D_j$ and pass through $l-1$ given points in Y, is g. On the other hand, assigning to every stable log map $f: C \to Y(D)$ the vector space $\mathrm{H}^0(C, \omega_C)$ of sections of the dualising sheaf of the domain curve defines a rank g vector bundle over the moduli space, called the Hodge bundle, and we denote by λ_g its top Chern class. We define log Gromov–Witten invariants $N^{\log}_{g,d}(Y(D))$ by integration of $(-1)^g \lambda_g$ over the virtual fundamental class of the moduli space. For genus $g = 0$, $N^{\log}_{0,d}(Y(D))$ recovers the naive count of rational curves but for $g > 0$, the log Gromov–Witten invariants $N^{\log}_{g,d}(Y(D))$ no longer have an obvious interpretation in terms of naive enumeration of curves. Fixing the degree d and summing over all genera, we define generating series

$$\mathsf{N}^{\log}_d(Y(D))(\hbar) := \frac{1}{\left(2\sin\left(\frac{1}{2}\hbar\right)\right)^{l-2}} \sum_{g \geqslant 0} N^{\log}_{g,d}(Y(D)) \hbar^{2g-2+l}. \tag{1-1}$$

The term $\left(2\sin\left(\frac{1}{2}\hbar\right)\right)^{2-l}$ is natural from the point of view of the q–refined scattering diagrams of Section 4. It is accounted for in the correspondence with the open invariants.

1.3.2 Local Gromov–Witten invariants To a Looijenga pair $Y(D) = (Y, D = D_1 + \cdots + D_l)$, we associate the $(l+2)$–dimensional noncompact Calabi–Yau variety $E_{Y(D)} := \mathrm{Tot}\big(\bigoplus_{i=1}^{l}(\mathcal{O}_Y(-D_i))\big)$. We view Y in $E_{Y(D)}$ via the inclusion given by the zero section. We refer to $E_{Y(D)}$ as the local geometry attached to $Y(D)$. If each component D_i is nef, then for every $d \in \mathrm{H}_2(Y, \mathbb{Z})$ intersecting D_i generically, the moduli space of genus-zero stable maps to $E_{Y(D)}$ of degree d is compact: every stable map to $E_{Y(D)}$ of class d factors through the zero section Y. Thus, it makes sense to consider the local genus-zero Gromov–Witten invariants $N^{\mathrm{loc}}_{0,d}(Y(D))$, which define virtual counts of rational curves in $E_{Y(D)}$ passing through $l-1$ given points in Y.

1.3.3 Higher-genus open Gromov–Witten invariants Let X be a semiprojective toric Calabi–Yau 3–fold, ie a toric Calabi–Yau 3–fold which admits a presentation as the GIT quotient of a vector space by a torus action; see Hausel and Sturmfels [61]. We will be concerned with a class of Lagrangian submanifolds of X considered by Aganagic and Vafa [7], which we simply refer to as toric Lagrangians: symplectically, these are singular fibres of the Harvey–Lawson fibration associated to X. A toric Lagrangian is diffeomorphic to $\mathbb{R}^2 \times S^1$, and so its first homology group is isomorphic to $\mathbb{Z}$.

We fix $L = L_1 \cup \cdots \cup L_s$ a disjoint union of toric Lagrangians L_i in X. In informal terms, the open Gromov–Witten theory of $(X, L = L_1 \cup \cdots \cup L_s)$ should be a virtual count of maps to X from open Riemann surfaces of fixed genus, relative homology degree, and boundary winding data around $S^1 \hookrightarrow L$. A precise definition of such counts in the algebraic category has been given by Li, Liu, Liu and Zhou [77; 76] using relative Gromov–Witten theory and virtual localisation. These invariants depend on the choice of a framing f of L, which is a choice of integer f_i for each connected component L_i of L. Given partitions $\mu_1, \dots, \mu_s$ of lengths $\ell(\mu_1), \dots, \ell(\mu_s)$, we denote by $O_{g;\beta;(\mu_1,\dots,\mu_s)}(X, L, \mathsf{f})$ the invariants defined in [77; 76], which are informally open Gromov–Witten invariants counting connected genus g Riemann surfaces of class $\beta \in \mathrm{H}_2(X, L, \mathbb{Z})$ with, for every $1 \le i \le s$, $\ell(\mu_i)$ boundary components wrapping L_i with winding numbers given by the parts of μ_i. We package the open Gromov–Witten invariants $O_{g,\beta,\mu_1,\dots,\mu_s}(X, L, \mathsf{f})$ into formal generating functions

$$\mathsf{O}_{\beta;\vec{\mu}}(X, L, \mathsf{f})(\hbar) := \sum_{g \ge 0} \hbar^{2g-2+\ell(\vec{\mu})} O_{g;\beta;\vec{\mu}}(X, L, \mathsf{f}), \tag{1-2}$$

where $\ell(\vec{\mu}) = \sum_{i=1}^{s} \ell(\mu_i)$. We simply denote by $O_{g;\beta}(X, L, \mathsf{f})$ and $\mathsf{O}_\beta(X, L, \mathsf{f})(\hbar)$ the s–holed open Gromov–Witten invariants obtained when each partition μ_i consists of a single part (whose value is then determined by the class $\beta \in \mathrm{H}_2(X, L, \mathbb{Z})$).

1.3.4 Quiver DT invariants Let Q be a quiver with an ordered set Q_0 of n vertices $v_1, \dots v_n \in \mathsf{Q}_0$ and a set of oriented edges $\mathsf{Q}_1 = \{\alpha : v_i \to v_j\}$. We let $\mathbb{N}\mathsf{Q}_0$ be the free abelian semigroup generated by Q_0, and, for $\mathsf{d} = \sum d_i v_i$ and $\mathsf{e} = \sum e_i v_i \in \mathbb{N}\mathsf{Q}_0$, we write $E_{\mathsf{Q}}(\mathsf{d}, \mathsf{e})$ for the Euler form

$$E_{\mathsf{Q}}(\mathsf{d}, \mathsf{e}) := \sum_{i=1}^{n} d_i e_i - \sum_{\alpha : v_i \to v_j} d_i e_j. \tag{1-3}$$

Assume that Q is symmetric; that is, for every i and j, the number of oriented edges from v_i to v_j is equal to the number of oriented edges from v_j to v_i. The Euler form is then a symmetric bilinear form. The motivic DT invariants $\mathrm{DT}_{\mathsf{d};i}(\mathsf{Q})$ of Q are defined by the equality

$$\sum_{\mathsf{d}\in\mathbb{N}^n} \frac{(-q^{1/2})^{E_Q(\mathsf{d},\mathsf{d})}\mathsf{x}^{\mathsf{d}}}{\prod_{i=1}^n (q;q)_{d_i}} = \prod_{\mathsf{d}\neq 0}\prod_{i\in\mathbb{Z}}\prod_{k\geq 0}\left(1-(-1)^i\mathsf{x}^{\mathsf{d}}q^{-k-(i+1)/2}\right)^{-\mathrm{DT}_{\mathsf{d};i}(\mathsf{Q})}, \tag{1-4}$$

where $\mathsf{x}^{\mathsf{d}} = \prod_{i=1}^n x_i^{d_i}$; see Efimov [34], Kontsevich and Soibelman [70] and Reineke [107]. In other words, the motivic DT invariants are defined by taking the plethystic logarithm of the generating series of Poincaré rational functions of the stacks of representations of Q. The numerical DT invariants $\mathrm{DT}_{\mathsf{d}}^{\mathrm{num}}(\mathsf{Q})$ are defined by

$$\mathrm{DT}_{\mathsf{d}}^{\mathrm{num}}(\mathsf{Q}) := \sum_{i\in\mathbb{Z}} (-1)^i\,\mathrm{DT}_{\mathsf{d},i}(\mathsf{Q}). \tag{1-5}$$

According to Efimov [34], the numerical DT invariants $\mathrm{DT}_{\mathsf{d}}^{\mathrm{num}}(\mathsf{Q})$ are nonnegative integers.

1.3.5 Open/closed BPS invariants Gromov–Witten invariants define virtual counts of curves and are in general rational numbers, but they are well-known to exhibit hidden integrality properties in terms of underlying BPS counts. The original physics definition, due to Gopakumar and Vafa [48; 47] in the classical context of closed Gromov–Witten invariants of Calabi–Yau 3–folds, predicted the form of these counts in terms of degeneracies of BPS particles in four/five dimensions arising from type IIA/M-theory as D2/M2–branes wrapping 2–cycles in the compactification. A longstanding effort has been made on multiple fronts to make the physics definition rigorous either using the associated cohomologies of sheaves (see Katz [64] and Maulik and Toda [90]), stable pairs (see Pandharipande and Thomas [101]), and direct symplectic methods (see Ionel and Parker [62]). In this paper, we will consider BPS invariants for genus-zero Gromov–Witten invariants of Calabi–Yau 4–folds and higher-genus open Gromov–Witten invariants of toric Calabi–Yau 3–folds. As an immediate corollary we obtain a new definition of all genus BPS invariants of Looijenga pairs (1-21).

Let $Y(D) = (Y, D = D_1 + D_2)$ be a 2–component Looijenga pair. The corresponding local geometry $E_{Y(D)}$ is a noncompact Calabi–Yau 4–fold. Following Greene, Morrison and Plesser [50, Appendix B] and Klemm and Pandharipande in [68, Section 1.1], we define BPS invariants $\mathrm{KP}_d(E_{Y(D)})$ in terms of the local genus-zero Gromov–Witten invariants $N_{0,d}^{\mathrm{loc}}(Y(D))$ by the formula

$$\mathrm{KP}_d(E_{Y(D)}) = \sum_{k|d} \frac{\mu(k)}{k^2} N_{d/k}^{\mathrm{loc}}(Y(D)). \tag{1-6}$$

Let X be a toric Calabi–Yau 3–fold, $L = L_1 \cup \cdots \cup L_s$ a disjoint union of toric Lagrangian branes and f a choice of framing. Following Labastida and Mariño [73], Labastida, Mariño and Vafa [74], Mariño and Vafa [88] and Ooguri and Vafa [100], we define the Labastida–Mariño–Ooguri–Vafa (LMOV)

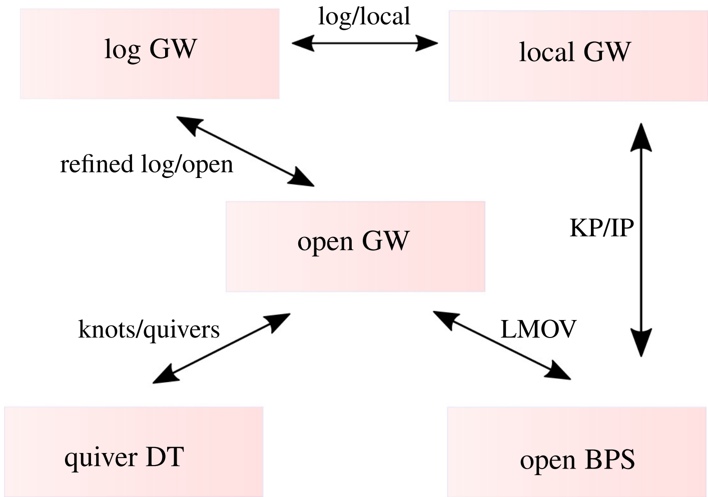

Figure 1: Enumerative invariants of $Y(D)$ and their mutual relations.

generating function of BPS invariants $\Omega_d(X,L,f)(q) \in \mathbb{Q}(q^{1/2})$ in terms of the s–holed higher-genus open Gromov–Witten generating series $\mathsf{O}_\beta(X,L,\mathsf{f})(\hbar)$ by the formula

$$\Omega_\beta(X,L,\mathsf{f})(q) = [1]_q^2 \left(\prod_{i=1}^{s} \frac{w_i}{[w_i]_q}\right) \sum_{k|\beta} \frac{\mu(k)}{k} \mathsf{O}_{\beta/k}(X,L,\mathsf{f})(-\mathrm{i}k\log q), \tag{1-7}$$

where $w_1,\dots,w_s$ are the winding numbers around the Lagrangians $L_1,\dots,L_s$ of the boundary components of an s–holed Riemann surface with relative homology class β, and where $[n]_q := q^{n/2} - q^{-n/2}$ are the q–integers, defined for all integers n.

1.4 The web of correspondences: geometric motivation

The enumerative theories of the previous section have superficially distant flavours, but they will turn out to be in close and often surprising relation to each other (Figure 1). We start by explaining the general geometric motivation behind the web of relations below, deferring rigorous statements for the case of Looijenga pairs to Section 1.5.

1.4.1 From log to local invariants Let $(Y, D = D_1 + \cdots + D_l)$ be a log smooth pair of maximal boundary; unless specified at this stage we do not restrict to Y being a surface, and neither do we impose the condition that (Y,D) be log Calabi–Yau, nor any positivity conditions on D_j. We will say that a curve class $d \in \mathrm{H}_2(Y,\mathbb{Z})$ is *D–convex* if $d \cdot D_i > 0$ for all i, and for every decomposition $d = [C_1] + \cdots + [C_m] \in \mathrm{H}_2(Y,\mathbb{Z})$, with each C_j an effective curve, we have $C_j \cdot D_i \geq 0$ for all i and j.

We begin by introducing some intermediate geometries built from $Y(D)$: for $\mathsf{m} = 1,\dots,l+1$, let

$$Y^{(\mathsf{m})} := \mathrm{Tot}\left(\bigoplus_{k\geq \mathsf{m}} \mathcal{O}_Y(-D_k)\right), \tag{1-8}$$

and $D^{(\mathsf{m})}$ be the preimage $\pi^{-1}\big(\bigcup_{k<\mathsf{m}} D_k\big)$ by the projection $\pi\colon Y^{(\mathsf{m})}\to Y$. Note that, by definition, $Y^{(1)}(D^{(1)})=E_{Y(D)}$ and $Y^{(l+1)}(D^{(l+1)})=Y(D)$: the geometries $Y^{(\mathsf{m})}(D^{(\mathsf{m})})$ for $1<\mathsf{m}\le l$ consist of intermediate setups where a log condition is imposed on $\{D_k\}_{k<\mathsf{m}}$, and a local one on $\{D_k\}_{k\ge\mathsf{m}}$. For d a $D^{(\mathsf{m})}$–convex curve class, we denote by $N^{\log}_{0,d}(Y^{(\mathsf{m})}(D^{(\mathsf{m})}))$ a genus-zero maximal tangency log GW invariant of class d of $Y^{(\mathsf{m})}(D^{(\mathsf{m})})$ with a choice of point and ψ–class insertions; see Section 4.1. $D^{(\mathsf{m})}$–convexity ensures that this is well-defined, despite $Y^{(\mathsf{m})}(D^{(\mathsf{m})})$ not being proper for $\mathsf{m}\le l$.

Assume first that $l=1$, ie that D is a smooth divisor. In van Garrel, Graber and Ruddat [43], the genus-zero local Gromov–Witten invariants of $E_{Y(D)}$ were related to the genus-zero maximal tangency Gromov–Witten theory of (Y,D) by the *stationary log/local correspondence*,

$$N^{\mathrm{loc}}_{0,d}(Y(D))=\frac{(-1)^{d\cdot D-1}}{d\cdot D}N^{\log}_{0,d}(Y(D)). \tag{1-9}$$

The argument of [43] is geometric, and it gives a stronger statement at the level of virtual fundamental classes: $E_{Y(D)}$ is degenerated to $Y\times\mathbb{A}^1$ glued along $D\times\mathbb{A}^1$ to a line bundle over the projective bundle $\mathbb{P}(\mathcal{O}_D\oplus\mathcal{O}_D(-D))$. This degeneration moves genus-zero stable maps in $E_{Y(D)}$ to genus-zero stable maps splitting along both components of the central fibre: the degeneration formula then states that $N^{\mathrm{loc}}_{0,d}(Y(D))$ equals the weighted sum over splitting type of the product of invariants associated to each component, and a careful analysis shows that only one term is nonzero, leading to (1-9). In [43, Conjecture 6.4], a conjectural cycle-level log-local correspondence was also proposed for simple normal crossing pairs: we propose here a slight variation of its restriction to stationary invariants and anticanonical D in the following conjecture.

Conjecture 1.1 (the stationary log/local correspondence for maximal log CY pairs) *Suppose that* $(Y,D=D_1+\cdots+D_l)$ *is a log smooth log Calabi–Yau pair of maximal boundary*, d *a* D*–convex curve class*, *and* $1\le\mathsf{n}<\mathsf{m}\le l+1$. *Then*

$$N^{\log}_{0,d}(Y^{(\mathsf{m})}(D^{(\mathsf{m})}))=\left(\prod_{i=\mathsf{n}}^{\mathsf{m}-1}(-1)^{d\cdot D_i+1}d\cdot D_i\right)N^{\log}_{0,d}(Y^{(\mathsf{n})}(D^{(\mathsf{n})})). \tag{1-10}$$

In particular, *when* $(\mathsf{n},\mathsf{m})=(1,l+1)$,

$$N^{\log}_{0,d}(Y(D))=\left(\prod_{i=1}^{l}(-1)^{d\cdot D_i+1}d\cdot D_i\right)N^{\mathrm{loc}}_{0,d}(Y(D)). \tag{1-11}$$

When all D_j are nef and $(\mathsf{n},\mathsf{m})=(1,l+1)$, this gives the numerical version of [43, Conjecture 6.4] for point insertions and anticanonical D. When $\mathsf{m}-\mathsf{n}=1$, (1-10) is an extension of the main result of [43] to the noncompact case.

The extent to which the argument of [43] generalises to the case of simple normal crossings pairs of Conjecture 1.1 is a somewhat thorny issue. In particular, the cycle-level conjecture of [43, Conjecture 6.4] is known to fail in the nonstationary sector for general l, as recently observed in a non-log Calabi–Yau example by Nabijou and Ranganathan [96]. At the same time, there is a nontrivial body of evidence that a generalisation of the stationary sector equality (1-9) (ie with descendent point insertions only) might hold for simple normal crossings log Calabi–Yau pairs $Y(D)$; see Bousseau, Brini and van Garrel [18] for a proof for toric orbifold pairs. It is therefore an open question to find the exact boundaries of validity of the stationary log-local correspondence, and in this paper we chart a conceptual pathway to delineate them for the (special, but central) case of log Calabi–Yau pairs of Conjecture 1.1, as follows.

At a geometric level, the degeneration of [43] can be generalised to a birational modification of one where the generic fibre is $E_{Y(D)}$, and the special fibre is obtained by gluing, for each $j = 1, \dots, l$, $Y \times (\mathbb{A}^1)^l$ along $D_j \times (\mathbb{A}^1)^l$ to a rank l vector bundle over $\mathbb{P}(\mathcal{O}_{D_j} \oplus \mathcal{O}_{D_j}(-D_j))$. After an (explicit) birational modification this gives a log smooth family: we describe the details of the degeneration for the case of surfaces in Section 5.1. When $l > 1$, instead of the degeneration formula the decomposition formula [2] applies, expressing $N^{\mathrm{loc}}_{0,d}(Y(D))$ as a weighted sum of terms, indexed by tropical curves $h\colon \Gamma \to \Delta$, where Δ is the dual intersection complex of the central fibre:

$$N^{\mathrm{loc}}_{0,d}(Y(D)) = \sum_{h\colon \Gamma \to \Delta} \frac{m_h}{|\mathrm{Aut}(h)|} N^{\mathrm{loc},h}_{0,d}(Y(D)). \tag{1-12}$$

The geometric picture above, and the ensuing decomposition formula (1-12), provides a rather general and geometrically motivated blueprint to measure the deviation, or lack thereof, of the local invariants from their expected relation to maximal tangency log invariants in (1-11). As a proof-of-concept step, and as we shall describe in detail in Section 5.1, in this paper we show how this framework bears fruit in the context of Looijenga pairs:[1] here correction terms indexed by nonmaximal tangency tropical curves turn out, remarkably, to *all* individually vanish, whilst the maximal tangency tropical contribution exactly returns the right-hand side of (1-11).

1.4.2 From log to open invariants Let $Y(D)$ be a log Calabi–Yau surface. By (1-8), the complement $Y^{(l)} \setminus D^{(l)}$ is isomorphic to the total space of $\mathcal{O}(-D_l) \to Y \setminus (D_1 \cup \cdots \cup D_{l-1})$; since D is anticanonical, this is a noncompact Calabi–Yau threefold. We propose that the log invariants $N^{\log}_{0,d}(Y(D))$ can be precisely related to open Gromov–Witten invariants of $Y^{(l)} \setminus D^{(l)}$ with boundary in fixed disjoint Lagrangians L_k, with $k < l$, near the divisor $D^{(l)}$. These Lagrangians should have a specific structure as described in [8, Section 7], namely they should be fibred over Lagrangians L'_k in $\pi^{-1}(D_k)$ with fibres Lagrangians in the normal bundle $(N_{\pi^{-1}(D_k)/Y^{(\mathrm{m})}})|_{L'_k}$. Writing $L := \bigcup_{k<l} L_k$ and $Y^{\mathrm{op}}(D) := (Y^{(l)} \setminus D^{(l)}, L)$, there is a natural isomorphism $\iota\colon \mathrm{H}^{\mathrm{rel}}_2(Y^{\mathrm{op}}(D), \mathbb{Z}) \to \mathrm{H}_2(Y, \mathbb{Z})$ induced by the embedding $Y^{(l)} \setminus D^{(l)} \hookrightarrow Y^{(l)}$ and the identification of winding degrees along L_k with contact orders along D_k; see Proposition 6.6 for details.

[1] It is an intriguing question, and one well beyond the scope of this paper, to test how this philosophy generalises to log Calabi–Yau varieties of any dimension, and to revisit the non-log Calabi–Yau, nonstationary negative result of [96] in this light.

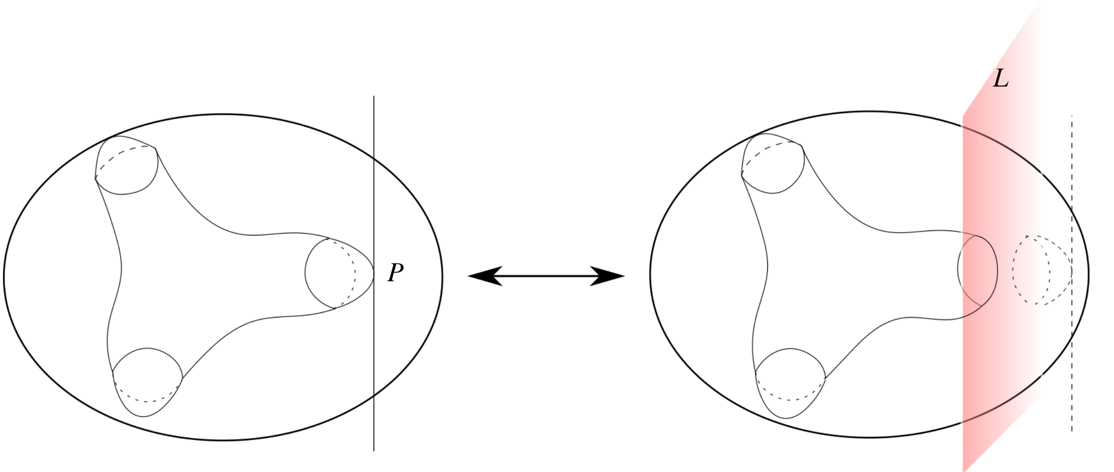

Figure 2: Exchanging log and open conditions.

Suppose now that there is a well-posed definition[2] of genus-zero open GW counts $O_{0;d}(Y^{\mathrm{op}}(D))$ as in Georgieva [45] and Solomon and Tukachinsky [109]. In such a scenario, we expect a close relationship between these and the log invariant $N^{\log}_{0,d}(Y(D))$.

Conjecture 1.2 (log-open correspondence for surfaces) *Let $Y(D)$ be a log Calabi–Yau surface with maximal boundary and d a D–convex curve class. Then*

$$O_{0;\iota^{-1}(d)}(Y^{\mathrm{op}}(D)) = \left(\prod_{k=1}^{l} \frac{(-1)^{d\cdot D_k-1}}{d\cdot D_k}\right) N^{\log}_{0,d}(Y(D)). \tag{1-13}$$

There is an intuitive symplectic heuristics behind Conjecture 1.2: removing a tubular neighbourhood of $D^{(l)}$ turns pseudoholomorphic log curves in $Y^{(l)}$ with prescribed tangencies along $D^{(l)}$ into pseudoholomorphic open Riemann surfaces with boundaries in L, with winding numbers determined by the tangencies; see Figure 2. The relative factor $\prod_{k<l}(-1)^{d\cdot D_k-1}(d\cdot D_k)^{-1}$ at the level of GW counts in Conjecture 1.2 can be understood by looking at the simplest example where $Y=\mathbb{P}^1\times\mathbb{A}^1$, $D_1=\{0\}\times\mathbb{A}^1$ and $D_2=\{\infty\}\times\mathbb{A}^1$, where $0,\infty\in\mathbb{P}^1$. For the curve class d times the class of $\mathbb{P}^1$ we have $N^{\log}_{0,d}(Y(D))=1$, as there exists a unique degree d cover of $\mathbb{P}^1$ fully ramified over two points, and the order d automorphism group of this cover is killed by the point condition. By Conjecture 1.1, and in particular (1-10) with $\mathsf{m}=2$, we deduce that $N^{\log}_{0,d}(Y^{(2)}(D^{(2)}))=(-1)^{d-1}/d$; on the other hand, the open geometry $Y^{\mathrm{op}}(D)$ is $\mathbb{C}^3$ with a singular Harvey–Lawson Lagrangian L of framing zero (see Construction 6.4): the degree d multicovers of the unique embedded disk [65, Theorem 7.2] contribute $O_{0;\iota^{-1}(d)}(Y^{\mathrm{op}}(D))=1/d^2$, from which the relative factor in (1-13) is recovered.

[2]An example of this situation (see Construction 6.4) is when, up to deformation, both Y and the divisors D_k $(k<l)$ are toric, implying that $Y^{\mathrm{op}}(D)$ is a toric Calabi–Yau threefold geometry equipped with framed toric Lagrangians L_k: in this case the open GW invariants were introduced in Section 1.3.3.

Much as in Conjecture 1.1, the invariants in Conjecture 1.2 live in different dimensions: (1-13) relates log invariants of the log CY surface $Y(D)$ to open invariants of special Lagrangians in a Calabi–Yau threefold. Note that combining Conjectures 1.1 and 1.2 further gives a surprising conjectural relation

$$O_{0;\iota^{-1}(d)}(Y^{\mathrm{op}}(D)) = N^{\mathrm{loc}}_{0,d}(Y(D)), \tag{1-14}$$

which equates the GW invariants of the CY3 open geometry $Y^{\mathrm{op}}(D)$ with the local GW invariants of the CY $(l+2)$ variety $E_{Y(D)}$.[3]

We also expect a precise uplift of this picture to higher-genus invariants. For a single irreducible divisor, an all-genus version of the log-local correspondence of [43] was described in [19, Theorems 1.1–1.2]. Its generalisation to a log-open correspondence in higher genus for a completely general pair is likely to take an unwieldy form, but we expect it to be particularly simple for a maximal boundary log Calabi–Yau surface. Indeed, in the degeneration to the normal cone along D_l, only multiple covers of a $\mathbb{P}^1$–fibre in $\mathbb{P}(\mathcal{O}_{D_l} \oplus \mathcal{O}_{D_l}(-D_l))$ will contribute. The resulting combination of the multiplicity $d \cdot D_l$ in the degeneration formula with the higher-genus multiple cover contribution

$$\frac{(-1)^{d\cdot D_l+1}}{(d\cdot D_l)[d\cdot D_l]_q}$$

leads us to predict a precise, and tantalisingly simple q–analogue of Conjecture 1.2.

Conjecture 1.3 (the all-genus log-open correspondence for surfaces) *Let $Y(D)$ be a log CY surface with maximal boundary and d a D–convex curve class. With notation as in Sections 1.3.1 and 1.3.3, we have*

$$\mathsf{O}_{\iota^{-1}(d)}(Y^{\mathrm{op}}(D))(-\mathrm{i}\log q) = [1]_q^{l-2}\,\frac{(-1)^{d\cdot D_l+1}}{[d\cdot D_l]_q}\prod_{k=1}^{l-1}\frac{(-1)^{d\cdot D_k+1}}{d\cdot D_k}\,\mathsf{N}^{\log}_d(Y(D))(-\mathrm{i}\log q). \tag{1-15}$$

The factor $[1]_q^{l-2}$ corresponds to the relative normalisation of the higher-genus generating functions in (1-1) and (1-2). The allusive hints of this section will be put on a rigorous footing in Section 1.5.2.

1.4.3 Quivers and BPS invariants Given $Y(D = D_1 + D_2)$ a 2–component Looijenga pair, the virtual count of curves in the noncompact Calabi–Yau 4–fold $E_{Y(D)}$ (see Klemm and Pandharipande [68]) is expected to be expressible in terms of sheaf counting (see Cao, Maulik and Toda [23; 24]). More precisely, it is expected that the BPS invariants of $E_{Y(D)}$ are extracted from a DT_4 virtual fundamental

[3]The relation (1-14) is in tune with physics expectations from type IIA string theory compactification on $\mathbb{R}^{1,1}\times X$, where X is a Calabi–Yau fourfold: the low energy effective theory is a $\mathcal{N}=(2,2)$ QFT, whose effective holomorphic superpotential is computed by the genus-zero Gromov–Witten invariants of X. Now precisely the same type of holomorphic F–terms can be engineered by considering D4–branes wrapping special Lagrangians on a Calabi–Yau 3–fold [100]: the superpotential here is a generating function of holomorphic disk counts with boundary on the Lagrangian three-cycle. It was suggested by Mayr [91] (see also [4]) that there exist cases where 2d superpotentials can be engineered in both ways, resulting in an identity between local genus-zero invariants of CY 4–folds and disk invariants of CY 3–folds: the equality in (1-14) asserts just that.

class associated to the moduli space of one-dimensional coherent sheaves on $E_{Y(D)}$. As coherent sheaves are often very closely related to modules over quivers, it might be tempting to ask if curve counting in $E_{Y(D)}$ (and, via the arguments of the previous section, the log/open GW theory of $Y(D)$) can be described in terms of some quiver DT theory.

This is more than a suggestive speculation. Consider for example $Y = \mathbb{P}^2$ and $D = D_1 + D_2$ the union of a line D_1 and a conic D_2, so that $E_{Y(D)}$ is the total space of $\mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$. Let $\mathcal{M}^{\mathrm{Higgs}}_{\mathbb{P}^1}(d,n)$ be the moduli space of rank-d, degree-n $\mathcal{O}(1)$–twisted Higgs bundles $\mathcal{O}_{\mathbb{P}^1}^{\oplus d} \to \mathcal{O}_{\mathbb{P}^1}^{\oplus d} \otimes \mathcal{O}_{\mathbb{P}^1}(1)$ on $\mathbb{P}^1$. The total space of $\mathcal{O}_{\mathbb{P}^1}(1)$ is the complement of a point in $\mathbb{P}^2$, and as $\mathbb{P}^1$ has normal bundle $\mathcal{O}(1)$ in $\mathbb{P}^2$, $\mathcal{M}^{\mathrm{Higgs}}_{\mathbb{P}^1}(d,n)$ sits as an open part of the moduli space of one-dimensional coherent sheaves on $E_{Y(D)}$. At the same time, as $\mathcal{O}(1)$ has two sections on $\mathbb{P}^1$, $\mathcal{M}^{\mathrm{Higgs}}_{\mathbb{P}^1}(d,n)$ is isomorphic to the moduli space of representations of the quiver with one vertex and two loops. Strikingly, we remark here that this is reflected into a completely unexpected identity for the corresponding invariants: the Klemm–Pandharipande BPS invariants of $E_{Y(D)}$ computed in [68, Section 3.2] simultaneously coincide (up to sign) with the DT invariants of the 2–loop quiver computed in Reineke [107, Theorem 4.2], as well as with the top Betti numbers[4] $\mathfrak{B}^{\mathrm{Higgs}}_d(\mathbb{P}^1) := \dim \mathrm{H}^{\mathrm{top}}(\mathcal{M}^{\mathrm{Higgs}}_{\mathbb{P}^1}(d,n), \mathbb{Q})$ of the moduli spaces of $\mathcal{O}(1)$–twisted Higgs bundles on the line considered in Rayan [106, Section 5]:

$$
\begin{aligned}
(1\text{-}16)\qquad |\mathrm{KP}_d(\mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2))| &= \mathfrak{B}^{\mathrm{Higgs}}_d(\mathbb{P}^1) = \mathrm{DT}^{\mathrm{num}}_d(\text{2–loop quiver}) \\
&= (1,1,1,2,5,13,35,100,300,925,2915,9386,\dots)_d.
\end{aligned}
$$

From a sheafy point of view, this raises the question how the definition of Calabi–Yau 4–fold invariants from the moduli space of coherent sheaves [23; 24] interacts with the quiver description, and whether such a startling coincidence is an isolated example — or not.

An upshot of Conjectures 1.1 and 1.2 is a surprising Gromov–Witten-theoretic take on this question: for $l = 2$ and when $Y^{\mathrm{op}}(D)$ is an open geometry given by toric Lagrangians in a toric CY3, the quiver can be reconstructed systematically from the geometry of $Y(D)$ via a version of the "branes–quivers" correspondence introduced in Ekholm, Kucharski and Longhi [36; 35], Kucharski, Reineke, Stošić and Sułkowski [72] and Panfil, Stošić and Sułkowski [102]. According to the open GW/quiver dictionary of [35], the quiver nodes are identified with basic (in the sense of [36; 35]) embedded holomorphic disks with boundary on L, edges and self-edges correspond to linking and self-linking numbers of the latter, and the DT invariants of the quiver return (up to signs) the genus-zero LMOV count of holomorphic disks obtained as "boundstates" of the basic ones [36, Section 4].

Now, by the $q \to 1$ limit of (1-7), the genus-zero LMOV and GW invariants of $Y^{\mathrm{op}}(D)$ are related to each other by the *same* BPS change of variables relating KP invariants and local GW invariants of $E_{Y(D)}$ in (1-6). Then a direct consequence of the conjectural open $=$ local GW equality (1-14) is that the KP invariants of the local CY4–fold $E_{Y(D)}$ coincide with the LMOV invariants of the open CY3 geometry $Y^{\mathrm{op}}(D)$ —

[4] The degree-independence of these Betti numbers, at least for $(d,n) = 1$, is explained in [106, Section 5].

which by the branes-quivers correspondence above are in turn DT invariants of a symmetric quiver! In particular, for the example above of $Y = \mathbb{P}^2$ and $D = D_1 + D_2$ the union of a line and a conic, we shall find the open geometry $Y^{\mathrm{op}}(D)$ to be three-dimensional affine space with a single toric Lagrangian at framing one (see Construction 6.4) — and as expected, in this case the quiver construction in [102, Section 5.1] returns exactly the quiver with one vertex and two loops we had found in (1-16). In general, this connection leads to some nontrivial implications for the Gopakumar–Vafa/Donaldson–Thomas theory of CY4 local surfaces from log Gromov–Witten theory, which we describe precisely in Sections 1.6.1 and 1.6.3.

1.5 The web of correspondences: results

In order to state our results, we introduce some notions of positivity for Looijenga pairs. A Looijenga pair $Y(D) = (Y, D = D_1 + \cdots + D_l)$ is *nef* if each irreducible component D_i of D is smooth and nef: note that the condition that the components D_i are smooth implies in particular that $l \geq 2$, and nefness entails that a generic stable map to Y is D–convex, which implies that the corresponding local Gromov–Witten invariants are well-defined.

A nef Looijenga pair $Y(D)$ is *tame* if either $l > 2$ or $D_i^2 > 0$ for all i, and *quasi-tame* if the associated local geometry $E_{Y(D)}$ is deformation equivalent to the local geometry $E_{Y'(D')}$ associated to a tame Looijenga pair $Y'(D')$: we explain the relevance of these two properties in Section 1.5.1. As we will show in Section 2, there are 18 smooth deformation types of nef Looijenga pairs in total, 11 of which are tame and 15 are quasi-tame. In particular, a nef Looijenga pair $Y(D)$ is uniquely determined by Y and the self-intersection numbers D_i^2, and we sometimes use the notation $Y(D_1^2, \ldots, D_l^2)$ for $Y(D)$; see Table 1. We state our results in a slightly discursive form below, including pointers to their precise versions in the main body of the text.

1.5.1 The stationary log-local correspondence Our first result establishes the stationary log/local correspondence of Conjecture 1.1 in the form given by (1-11).

Theorem 1.4 (Theorem 5.1, Lemma 3.1, Theorem 3.3, Theorem 3.5, Proposition 3.6) *For every nef Looijenga pair $Y(D)$, the genus-zero log invariants $N^{\mathrm{log}}_{0,d}(Y(D))$ and the genus-zero local invariants $N^{\mathrm{loc}}_{0,d}(Y(D))$ are related by*

$$N^{\mathrm{loc}}_{0,d}(Y(D)) = \left(\prod_{j=1}^{l} \frac{(-1)^{d \cdot D_j - 1}}{d \cdot D_j} \right) N^{\mathrm{log}}_{0,d}(Y(D)). \tag{1-17}$$

Moreover, we provide a closed-form solution to the calculation of both sets of invariants in (1-17).

As explained in Section 1.4.1, the key idea to prove Theorem 1.4 is by a degeneration argument, illustrated in Section 5.1 for $l = 2$: we follow the general strategy of [43] to deduce log-local relations from a degeneration to the normal cone, and we solve in our case of interest the difficulties of the normal-crossings situation through a detailed study of the tropical curves contributing in the decomposition formula of Abramovich, Chen, Gross and Siebert [2] for log Gromov–Witten invariants. For $l > 2$, and more generally when $Y(D)$ is tame, it turns out to be more convenient to structure the proof so that an uplift to the all-genus story, absent in other approaches, is immediate. The notion of tameness is first shown to be synonymous of finite scattering, and for tame pairs we compute closed-form solutions for the log Gromov–Witten invariants using tropical geometry, more precisely two-dimensional scattering diagrams; see Gross [51], Gross, Hacking and Keel [53], Gross, Pandharipande and Siebert [56] and Mandel [84]. The statement of the theorem for tame cases follows by subsequently comparing with a closed-form solution of the local theory via Givental-style mirror theorems: the proof follows from a general statement valid for local invariants of toric Fano varieties in any dimension twisted by a sum of concave line bundles (Lemma 3.1), and the notion of tameness is shown to coincide here with the vanishing of quantum corrections to the mirror map. For non-quasi-tame cases, we use a blowup formula which allows to restrict to the case of highest Picard number; the proof of the equality (1-17) in this case, in Theorem 3.3, requires a highly nontrivial mirror map calculation.

1.5.2 The all-genus log-open correspondence A notable property of the scattering approach to Theorem 1.4 for $l > 2$ (and, in general, for tame Looijenga pairs) is that it can be bootstrapped to obtain all-genus results for the log invariants through the q–deformed version of the two-dimensional scattering diagrams of Gross [51], Gross, Hacking and Keel [53], Gross, Pandharipande and Siebert [56] and Mandel [84] and the general connection between higher-genus log invariants of surfaces with λ_g–insertion and q–refined tropical geometry studied in Bousseau [12; 14]. This is key to establishing the following version of Conjectures 1.2 and 1.3.

Theorem 1.5 (Theorems 4.5, 4.9, 4.10 and 6.7) *For every quasi-tame Looijenga pair $Y(D)$ distinct from $\mathrm{dP}_3(0,0,0)$, there exists a triple $Y^{\mathrm{op}}(D) = (X, L, \mathsf{f})$, geometrically related to $Y(D)$ by Construction 6.4, where X is a semiprojective toric Calabi–Yau 3–fold, $L = L_1 \cup \dots \cup L_{l-1}$ is a disjoint union of $l-1$ toric Lagrangians in X, f is a framing for L, and there exists an isomorphism $\iota\colon \mathrm{H}_2(X, L, \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}_2(Y, \mathbb{Z})$ such that*

(1-18) $$O_{0;\iota^{-1}(d)}(Y^{\mathrm{op}}(D)) = N^{\mathrm{loc}}_{0,d}(Y(D)) = \prod_{i=1}^{l} \frac{(-1)^{d\cdot D_i+1}}{d\cdot D_i} N^{\log}_{0,d}(Y(D)).$$

Furthermore, if $Y(D)$ is tame,

(1-19) $$\mathsf{O}_{\iota^{-1}(d)}(Y^{\mathrm{op}}(D))(-\mathrm{i}\log q) = [1]_q^{l-2} \frac{(-1)^{d\cdot D_l+1}}{[d\cdot D_l]_q} \prod_{i=1}^{l-1} \frac{(-1)^{d\cdot D_i+1}}{d\cdot D_i} \mathsf{N}^{\log}_{d}(Y(D))(-\mathrm{i}\log q).$$

Moreover, we provide a closed-form solution to the calculation of the invariants in (1-18)–(1-19).

The open geometry $Y^{\mathrm{op}}(D)$ is constructed following the ideas of Section 1.4.2; see Section 6.2 for full details. Key to the proof of Theorem 1.5 is the fact that quasi-tame Looijenga pairs can always be deformed to pairs for which the both surface Y and the divisors D_i with $i < l$ are toric: as we shall explain in Section 6.2, the corresponding open geometry $Y^{\mathrm{op}}(D)$ is given by suitable Aganagic–Vafa (singular Harvey–Lawson) Lagrangian branes in a toric Calabi–Yau threefold, whose open Gromov–Witten theory can be compactly encoded through the topological vertex.[5] Conjecture 1.3 then predicts a completely unexpected relation between the q–scattering and topological vertex formalisms, which Theorem 1.5 establishes for tame pairs. The combinatorics underlying the resulting comparison of invariants is in general extremely nontrivial: for $l = 2$, it can be shown to be equivalent to Jackson's q–analogue of the Pfaff–Saalschütz summation for the ${}_3\phi_2$ generalised q–hypergeometric function.

We furthermore conjecture that the higher-genus log-open correspondence of Theorem 1.5 extends to all quasi-tame pairs. The scattering diagrams become substantially more complicated in the nontame cases, and (1-18) translates into an intricate novel set of q–binomial identities: see Conjecture B.3 for explicit examples.[6] The log-local correspondence of Theorem 1.4 establishes their limit for $q \to 1$.

1.5.3 BPS invariants and quiver DT invariants As anticipated in Section 1.4.3, the log/open correspondence of Theorem 1.5 can be leveraged to produce a novel correspondence between log/local Gromov–Witten invariants and quiver DT theory.

Theorem 1.6 (Theorem 7.3) *Let $Y(D) = (Y, D_1 + D_2)$ be a 2–component quasi-tame Looijenga pair. Then there exists a symmetric quiver $\mathsf{Q}(Y(D))$ with $\chi(Y) - 1$ vertices and a lattice isomorphism $\kappa\colon \mathbb{Z}(\mathsf{Q}(Y(D)))_0 \xrightarrow{\sim} \mathrm{H}_2(Y, \mathbb{Z})$ such that*

$$\mathrm{DT}_d^{\mathrm{num}}(\mathsf{Q}(Y(D))) = \Big|\mathrm{KP}_{\kappa(d)}(E_{Y(D)}) + \sum_i \alpha_i \delta_{d,v_i}\Big|, \tag{1-20}$$

with $\alpha_i \in \{-1, 0, 1\}$. In particular, $\mathrm{KP}_d(E_{Y(D)}) \in \mathbb{Z}$.

A symplectic proof of the integrality of genus-zero BPS invariants for projective Calabi–Yau 4–folds, although likely adaptable to the noncompact setting, was given by Ionel–Parker in [62]. In Theorem 1.6, the integrality for the local Calabi–Yau 4–folds $E_{Y(D)}$ follows from the identification of the BPS invariants with DT invariants of a symmetric quiver.[7] We construct the symmetric quiver $\mathsf{Q}(Y(D))$ by combining the log-open correspondence given by Theorem 1.5 with a correspondence previously established by Panfil and Sułkowski [103] between toric Calabi–Yau 3–folds with "strip geometries" and symmetric quivers; see also Kimura, Panfil, Sugimoto and Sułkowski [67].

Theorem 1.5 associates to a Looijenga pair $Y(D)$ satisfying Property O the toric Calabi–Yau 3–fold geometry $Y^{\mathrm{op}}(D)$. Denote by $\Omega_d(Y(D))(q) := \Omega_{\iota^{-1}(d)}(Y^{\mathrm{op}}(D))(q)$ the open BPS invariants defined

[5] A conceptual explanation for the exclusion of $dP_3(0,0,0)$ from the statement of Theorem 1.5 is given by the notion of Property O, which we introduce in Definition 6.3.

[6] After the first version of this paper appeared on the arXiv, we received a combinatorial proof of Conjecture B.3 from C Krattenthaler [71].

[7] The equality modulo the integral shift by $\sum_i \alpha_i \delta_{d,v_i}$ in (1-20) can be traded to an actual equality of absolute values at the price of considering a larger disconnected quiver $\widetilde{\mathsf{Q}}$, and a corresponding epimorphism $\widetilde{\kappa}\colon \mathbb{Z}(\widetilde{\mathsf{Q}}(Y(D)))_0 \to \mathrm{H}_2(Y, \mathbb{Z})$; see [103].

in (1-7). In general, for any Looijenga pair we can define

$$\Omega_d(Y(D))(q) := [1]_q^2\left(\prod_{i=1}^{l} \frac{1}{[d\cdot D_i]_q}\right)\sum_{k|d} \frac{(-1)^{d/k\cdot D+l}\mu(k)}{[k]_q^{2-l}k^{2-l}} \mathrm{N}^{\log}_{d/k}(Y(D))(-ik\log q). \tag{1-21}$$

When $Y(D)$ is tame and satisfies Property O, the equivalence of the definitions (1-7) and (1-21) is a rephrasing of the log-open correspondence of Theorem 1.5 at the level of BPS invariants.

A priori, $\Omega_d(Y(D))(q)\in\mathbb{Q}(q^{1/2})$. By a direct arithmetic argument, we prove the following integrality result, which in particular establishes the existence of an integral BPS structure underlying the higher-genus log Gromov–Witten theory of $Y(D)$.

Theorem 1.7 (Theorem 8.1) *Let $Y(D)$ be a quasi-tame Looijenga pair. Then*

$$\Omega_d(Y(D))(q)\in q^{-\frac{1}{2}\mathrm{g}_{Y(D)}(d)}\mathbb{Z}[q]$$

for an integral quadratic polynomial $\mathrm{g}_{Y(D)}(d)$.

1.5.4 Orbifolds In the present paper, we mainly focus on the study of the finitely many deformation families of nef Looijenga pairs (Y,D) with Y smooth. Nevertheless, most of our techniques and results should extend to the more general setting where we allow Y to have orbifold singularities at the intersection of the divisors: the log Gromov–Witten theory is then well-defined since $Y(D)$ is log smooth, and the local Gromov–Witten theory makes sense by viewing Y and $E_{Y(D)}$ as smooth Deligne–Mumford stacks. There are infinitely many examples of nef/tame/quasi-tame Looijenga pairs in the orbifold sense. Deferring a treatment of more general examples to our companion note [17], we content ourselves here to show in Section 9 that the log-local, log-open and Gromov–Witten/quiver correspondences still hold for the infinite family of examples obtained by taking $Y=\mathbb{P}_{(1,1,n)}$, the weighted projective plane with weights $(1,1,n)$, and $D=D_1+D_2$ with D_1 a line passing through the orbifold point and D_2 a smooth member of the linear system given by the sum of the two other toric divisors.

1.6 The web of correspondences: implications

The results of the previous section subsume and were motivated by several disconnected strands of development in the study of the enumerative invariants in Sections 1.3.1–1.3.5. We briefly describe here how they relate to and impact ongoing progress in some allied contexts.

1.6.1 BPS structures in log/local GW theory The relation of log GW invariants to BPS invariants in Theorem 1.7 echoes very similar[8] statements relating log GW theory to DT and LMOV invariants in Bousseau [15; 14], and in particular it partly demystifies the interpretation of log GW partition functions

[8] A nontrivial difference is that here the log Gromov–Witten invariants are *not* interpreted as BPS invariants themselves, unlike in [15; 14], but rather are related to them via (1-21).

as related to some putative open curve counting theory on a Calabi–Yau 3–fold in [14, Section 9] by realising the open BPS count in terms of actual, explicit special Lagrangians in a toric Calabi–Yau threefold. Aside from its conceptual appeal, its power is revealed by some of its immediate consequences: the Klemm–Pandharipande conjectural integrality [68, Conjecture 0] for local CY4 surfaces follows as a zero-effort corollary of the log-open correspondence of Theorem 1.5 by constructing the associated quiver in Theorem 1.6, identifying the KP invariants of the local surface with its DT invariants, and applying Efimov's theorem [34].

We note that this chain of connections opens the way to a proof of the Calabi–Yau 4–fold Gromov–Witten/Donaldson–Thomas correspondence [23; 24], which is an open conjecture even for the simplest local surfaces. The analysis of the underlying integrality of the q–scattering calculation in Theorem 1.7 furthermore gives, in the limit $q \to 1$, an algebrogeometric version of symplectic results of Ionel and Parker [62] for Calabi–Yau vector bundles on toric surfaces; and away from this limit, it provides a refined integrality statement whose enumerative salience for the local theory is hitherto unknown, and worthy of further study: see Section 1.6.3.

1.6.2 The general log-open correspondence for surfaces Throughout the heuristic description of the motivation for Conjectures 1.1–1.3, we have been mindful not to impose any nefness condition on the divisors D_i: the only request we made was for the genus-zero obstruction theory of the local theory to be encoded by a genuine obstruction bundle over the untwisted moduli space. This was taken into account by the condition of D–convexity for the stable maps: restricting to D–convex maps widens the horizon of the log/local correspondence of [43] to a vast spectrum of cases which were not accounted for in previous studies of the correspondence. And indeed, in the broadest generality where the open invariants can be defined in the algebraic category, the methods proposed here extend straightforwardly to treat the cases when one or more of the irreducible components D_i have negative self-intersection: Conjectures 1.1–1.3 hold with flying colours in these cases as well, with all $l > 2$ anticanonical pairs satisfying Property O that remarkably enjoy the same salient properties of the *tame* nef pairs, such as finite scattering, closed-form resummation of the topological vertex, and triviality of the mirror map; their detailed study will appear in work-in-progress of Brini, van Garrel and Schüler.

The discussion of Section 1.4.2 also opens the door to pushing the log/open correspondence beyond the maximal contact setting: it is tempting to see how the maximal tangency condition could be removed from Conjecture 1.2, with the splitting of contact orders amongst multiple points on the same divisor being translated to windings of multiple boundary disks ending on the same Lagrangian. The multicovering factor of (1-19) would then be naturally given by a product of individual contact orders/disk windings — an expectation that the reader can verify to be fulfilled in the basic example presented there of $(Y, D) = (\mathbb{P}^1 \times \mathbb{A}^1, \mathbb{A}^1 \cup \mathbb{A}^1)$. More generally, the link to the topological vertex and open GW invariants of arbitrary topology raises a fascinating question how much the topological vertex knows of the log theory of the surface — and how it can be effectively used in the construction of (quantum) SYZ mirrors.

1.6.3 Relation to the Cao–Maulik–Toda conjecture Another direction towards a geometric understanding of the integrality of KP invariants is provided by sheaf-counting theories for Calabi–Yau 4–folds, which were originally introduced by Borisov and Joyce [11] (see also Cao and Leung [22]) and have recently been given an algebraic construction by Oh and Thomas [98]. More precisely, Cao, Maulik and Toda have conjectured in [24] (resp. in [23]) explicit relations between genus-zero KP invariants and stable pair invariants (resp. counts of one-dimensional coherent sheaves) on Calabi–Yau 4–folds. Recently, Cao, Kool and Monavari [21] have checked the conjecture of [24] for low-degree classes on local toric surfaces; their proof hinges on the solution of the Gromov–Witten/Klemm–Pandharipande side given by Theorems 1.4 and 1.6 in this paper.

The results of Theorems 1.6 and 1.7 also raise a host of new questions. First and foremost, it would be extremely interesting to find for local toric surfaces a direct connection between the symmetric quivers appearing in Theorem 7.3 and the moduli spaces of coherent sheaves appearing in the conjectures of [24; 23]. Furthermore, since for $l=2$ we have $\mathrm{KP}_d(E_{Y(D)}) = \Omega_d(Y(D))$, a fascinating direction would be to find an interpretation of the q–refined invariants $\Omega_d(Y(D))(q)$ in terms of the Calabi–Yau 4–fold $E_{Y(D)}$. A natural suggestion is that $\Omega_d(Y(D))(q)$ should take the form of some appropriately refined Donaldson–Thomas invariants of $E_{Y(D)}$. As the topic of refined DT theory of Calabi–Yau 4–folds is still in its infancy, we leave the question open for now.

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2 Nef Looijenga pairs

We start off by establishing some general facts about the classical geometry of nef log Calabi–Yau (CY) surfaces. We first proceed to classify them in the smooth case, recall some basics of their birational geometry and the construction of toric models, and describe the structure of their pseudoeffective cone in preparation for the study of curve counts in them. We then end by introducing the notions of (quasi)tameness.

2.1 Classification

We start by giving the following definition.

Definition 2.1 An *l–component log CY surface with maximal boundary*, or *l–component Looijenga pair*, is a pair $Y(D) := (Y, D = D_1 + \cdots + D_l)$ consisting of a smooth rational projective surface Y and a singular nodal anticanonical divisor D that admits a decomposition $D = D_1 + \cdots + D_l$. We say that an l–component log CY surface is *nef* if $l \geq 2$ and each D_i is a smooth, irreducible and nef rational curve.

Examples of log CY surfaces arise when Y is a projective toric surface and D is the complement of the maximal torus orbit in Y; we call these pairs *toric*. By definition, if $Y(D)$ is nef, Y is a weak Fano surface together with a choice of distribution of the anticanonical degree amongst components D_i preserving the condition that $D_i \cdot C \geq 0$ for any effective curve C and all $i = 1, \dots, l$ with $l \geq 2$. We classify these by recalling some results of di Rocco [33]; see also [27, Section 2] and [28; 29].

Let $\mathrm{dP_r}$ be the surface obtained from blowing up $r \geq 1$ general points in $\mathbb{P}^2$. The Picard group of $\mathrm{dP_r}$ is generated by the hyperplane class H and the classes E_i of the exceptional divisors. The anticanonical class is $-K_{\mathrm{dP_r}} = 3H - \sum_{i=1}^{r} E_i$. Recall that a *line class* on $\mathrm{dP_r}$ is $l \in \mathrm{Pic}(\mathrm{dP_r})$ such that $l^2 = -1$ and $-K_{\mathrm{dP_r}} \cdot l = 1$; for $r \leq 5$ and up to permutation of the E_i, they are given by E_i, $H - E_1 - E_2$ or $2H - \sum_{i=1}^{5} E_i$. Furthermore, for $n \geq 0$, denote by $\mathbb{F}_\mathrm{n}$ the n^{th} Hirzebruch surface. Its Picard group is of rank 2 generated by the sections C_{-n} (resp. C_n), with self-intersections $-n$ (resp. n), and by the fibre class f, subject to the relation that $C_{-n} + nf = C_n$. Note that $\mathbb{F}_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_1 \simeq \mathrm{dP}_1$ is the blowup of $\mathbb{P}^2$ in one point.

Lemma 2.1 [33] *Assume that $1 \leq r \leq 5$ and let $D \in \mathrm{Pic}(\mathrm{dP_r})$. Then D is nef if and only if*

(i) *for $r = 1$, $D \cdot l \geq 0$ for all line classes l and $D \cdot (H - E_1) \geq 0$,*

(ii) *for $5 \geq r \geq 2$, $D \cdot l \geq 0$ for all line classes l.*

2.1.1 $l = 2$ Let's start by setting $l = 2$. With the sole exception of $\mathrm{dP}_4(H, 2H - E_1 - E_2 - E_3 - E_4)$ versus $\mathrm{dP}_4(H - E_1, 2H - E_2 - E_3 - E_4)$, the next proposition shows that up to deformation and permutation of the factors, and assuming that D_1 and D_2 are nef, $Y(D)$ is determined by Y and the self-intersections $\{D_1^2, D_2^2\}$. We will consequently employ the shorthand notation $Y(D) \leftrightarrow Y(D_1^2, D_2^2)$ to indicate this, making precise which one is meant in the case of $\mathrm{dP}_4(0,1)$.

Proposition 2.2 *Let $Y(D = D_1 + D_2)$ be a 2–component nef log CY surface. Then up to deformation and interchange of D_1 and D_2, $Y(D_1, D_2)$ is one of the following, abbreviated by $Y(D_1^2, D_2^2)$ except in cases (4) and (5):*

(1) $\mathbb{P}^2(1,4)$,

(2) $\mathrm{dP_r}(1, 4-r)$ *for* $1 \leq r \leq 3$,

(3) $\mathrm{dP_r}(0, 5-r)$ *for* $1 \leq r \leq 3$,

(4) $\mathrm{dP}_4(H, 2H - E_1 - E_2 - E_3 - E_4)$,

(5) $\mathrm{dP}_4(H - E_1, 2H - E_2 - E_3 - E_4)$,

(6) $\mathrm{dP}_5(0, 0)$,

(7) $\mathbb{F}_0(0, 4)$,

(8) $\mathbb{F}_0(2, 2)$.

Proof A minimal model of Y is given by $\mathbb{P}^2$, $\mathbb{F}_0$ or $\mathbb{F}_n$ for $n \geq 2$. By assumption $-K_Y = D_1 + D_2$ is nef, ruling out $\mathbb{F}_n$ for $n > 2$. If $Y = \mathbb{F}_0$, then the stipulated decompositions of $-K_{\mathbb{F}_0}$ are immediate. If $\mathbb{F}_2$ is a minimal model of Y, then $Y = \mathbb{F}_2$. In this case, the only possible decomposition of $-K_{\mathbb{F}_2}$ into nef divisors is as $D_1 = C_{-2} + 2f = C_2$ and $D_2 = C_2$. The resulting pair $\mathbb{F}_2(2, 2)$ is deformation equivalent to $\mathbb{F}_0(2, 2)$; see the proof of Proposition 2.6.

Assume now that $\mathbb{P}^2$ is a minimal model of Y. If $Y = \mathbb{P}^2$, we are done. Otherwise, up to deformation, we may assume that $Y = \mathrm{dP}_r$. Since $-K_Y$ is nef, $r \leq 9$. As D_1 and D_2 are nef, they are of the form $dH - \sum_{i=1}^r a_i E_i$ for $d \geq 1$ and $a_i \geq 0$. Applying Lemma 2.1, we find that the only nef decompositions are as follows:

- either $D_1 = H$, $D_2 = 2H - \sum_{i=1}^r E_i$ for $r \leq 4$,
- or $D_1 = H - E_j$, $D_2 = 2H - \sum_{i \neq j}^r E_i$ for $r \leq 5$.

They are all basepoint-free by [33] (see [27, Lemma 2.7]) and hence a general member will be smooth by Bertini. □

2.1.2 $l = 3$ Next, we classify the surfaces with $l = 3$ nef components. The shorthand notation $Y(D_1^2, D_2^2, D_3^2)$ is employed as in the previous section.

Proposition 2.3 *Let $Y(D = D_1 + D_2 + D_3)$ be a 3–component log CY surface with Y smooth and D_1, D_2 and D_3 nef. Then up to deformation and permutation of D_1, D_2 and D_3, $Y(D_1^2, D_2^2, D_3^2)$ is one of the following*:

(1) $\mathbb{P}^2(1, 1, 1)$,

(2) $\mathrm{dP}_1(1, 1, 0)$,

(3) $\mathrm{dP}_2(1, 0, 0)$,

(4) $\mathrm{dP}_3(0, 0, 0)$,

(5) $\mathbb{F}_0(2, 0, 0)$.

Proof A minimal model of Y is given by $\mathbb{P}^2$, $\mathbb{F}_0$ or $\mathbb{F}_n$ for $n \geq 2$. By assumption $-K_Y = D_1 + D_2 + D_3$ is nef, ruling out $\mathbb{F}_n$ for $n \geq 2$. For $\mathbb{P}^2$, the only possibility is to choose D_1, D_2, D_3 in class H. For $\mathbb{F}_0$, it is to choose $D_1 = H_1 + H_2$ the diagonal and $D_2 = H_1$, $D_3 = H_2$. Necessarily, all other surfaces are given by iterated blowups of the minimal models, keeping the divisors nef, leading to the list. As in the previous proposition, they are all basepoint-free and thus a general member will be smooth. □

2.1.3 $\boldsymbol{l \geq 4}$ For $l = 4$, a minimal model for Y is $\mathbb{F}_0$, for which the only possibility is given by D being its toric boundary. There are no other cases preserving nefness of the divisors. For 5 components or more, there are no surfaces keeping each divisor nef.

2.2 Toric models

We consider two basic operations on log CY surfaces $Y(D)$.

- Let $\widetilde{Y}$ be the blowup of Y at a node of D and let $\widetilde{D}$ be the preimage of D in $\widetilde{Y}$. Then the log CY surface $(\widetilde{Y}, \widetilde{D})$ is said to be a *corner blowup* of $Y(D)$.
- Let $\widetilde{Y}$ be the blowup of Y at a smooth point of D. Let $\widetilde{D}$ be the strict transform of D in $\widetilde{Y}$. Then the log CY surface $(\widetilde{Y}, \widetilde{D})$ is said to be an *interior blowup* of $Y(D)$.

A corner blowup does not change the complement $Y \setminus D$, whereas an interior blowup does; accordingly corner blowups do not change log Gromov–Witten invariants [3].

Definition 2.2 Let $\pi : Y(D) \longrightarrow \overline{Y}(\overline{D})$ be a sequence of interior blowups between log CY surfaces such that $\overline{Y}(\overline{D})$ is toric. Then π is said to be a *toric model* of $Y(D)$.

We will describe toric models by giving the fan of $(\overline{Y}, \overline{D})$ with *focus–focus singularities* on its rays. A focus–focus singularity on the ray corresponding to a toric divisor F encodes blowing up F at a smooth point. Each focus–focus singularity produces a wall and interactions of them create a scattering diagram $\operatorname{Scatt}(Y(D))$, as we discuss in Section 4.2.

Proposition 2.4 [53, Proposition 1.3] *Let $Y(D)$ be a log CY surface. Then there exist log CY surfaces $\widetilde{Y}(\widetilde{D})$ and $\overline{Y}(\overline{D})$, with the latter toric, and a diagram*

$$\begin{array}{ccc} & \widetilde{Y}(\widetilde{D}) & \\ {}^{\varphi}\swarrow & & \searrow^{\pi} \\ Y(D) & & \overline{Y}(\overline{D}) \end{array} \tag{2-1}$$

such that φ is a sequence of corner blowups and π is a toric model.

The diagrams as in (2-1) are far from unique, and they are related by cluster mutations [52]. Because of the invariance of log Gromov–Witten invariants by corner blowups, we can calculate the log Gromov–Witten invariants of $Y(D)$ on the scattering diagram $\operatorname{Scatt}(Y(D))$ associated to the toric model π.

2.3 The effective cone of curves

Given $Y(D)$ a nef log CY surface and $d \in A_1(Y)$, it will be convenient for the discussion in the foregoing sections to determine numerical conditions for d to be an element of the pseudoeffective cone. If $Y = \mathbb{F}_n$, $\mathrm{NE}(Y)$ is just the monoid generated by C_{-n} and f, so let us assume that $Y = \mathrm{dP_r}$. We will write a curve class d as $d_0\big(H - \sum_{i=1}^r E_i\big) + \sum_{i=1}^r d_i E_i$. If $\rho(Y) \geq 2$, then the extremal rays of the effective cone $\mathrm{NE}(Y)$ of Y are generated by extremal classes D with $D^2 \leq 0$, and in the case of del Pezzo surfaces these are the line and fibre classes described above. Using the classification [27, Examples 2.3 and 2.11], up to permutation of the E_i and E_j, we find the following lists of generators of extremal rays of $\mathrm{NE}(Y)$:

- If $r = 1$,

$$E_1, \quad H - E_1. \tag{2-2}$$

- If $2 \leq r \leq 4$,

$$E_i, \quad H - E_i, \quad H - E_i - E_j \text{ for } i \neq j. \tag{2-3}$$

- If $r = 5$,

$$E_i, \quad H - E_i, \quad H - E_i - E_j \text{ for } i \neq j, \quad 2H - \sum_{i=1}^{5} E_i. \tag{2-4}$$

Note that the effective cone is closed since it is generated by finitely many elements. The following proposition can be specialised to the del Pezzo surfaces dP_r for $r \leq 5$ by setting the corresponding d_i to 0 and removing the superfluous equations such as the last one, which only holds for $r = 5$.

Proposition 2.5 *A class $d = d_0\big(H - \sum_{i=1}^5 E_i\big) + \sum_{i=1}^5 d_i E_i$ of dP_5 is effective if and only if*

$$d_0 \geq 0, \quad d_i \geq 0, \quad d_i + d_j + d_k \geq d_0, \quad d_i + d_j + d_k + d_l \geq 2d_0, \quad 2d_i + \sum_{j \neq i} d_j \geq 3d_0, \tag{2-5}$$

where the i, j, k, l are always pairwise distinct.

The statement follows from the explicit description of the effective cone as generated by extremal rays. A direct calculation using the Polymake package in Macaulay2 computes the halfspaces defining the cone, yielding the above inequalities for the effective curves.

2.4 Tame and quasi-tame Looijenga pairs

The computation of curve-counting invariants of nef Looijenga pairs is strongly affected by the number l of smooth irreducible components of D and the positivity of D_i for $i = 1, \dots, l$. We spell this out with the following definition, whose significance will be worked out in Sections 3.1 and 4.2.

Let $Y(D)$ be a nef Looijenga pair and let

$$E_{Y(D)} := \mathrm{Tot}\Big(\bigoplus_{i=1}^{l} \mathcal{O}_Y(-D_i)\Big) \tag{2-6}$$

be the total space of the direct sum of the dual line bundles to D_i for $i = 1, \dots, l$.

Definition 2.3 We call a nef log CY surface $(Y, D = D_1 + \cdots + D_l)$ *tame* if either $l > 2$ or $D_i^2 > 0$ for all i. A nef log CY surface $Y(D)$ is *quasi-tame* if $E_{Y(D)}$ is deformation equivalent to $E_{Y'(D')}$, with $Y'(D')$ tame.

We will use the abbreviated notation $E_{Y(D_1^2,D_2^2)}$ for the local Calabi–Yau fourfold $E_{Y(D_1+D_2)}$ associated by (2-6) to a 2–component log CY surface $Y(D_1^2, D_2^2)$ in the classification of Proposition 2.2. Quasitame pairs are classified by the following proposition.

Proposition 2.6 *The following varieties are deformation-equivalent*:

1. $E_{\mathbb{F}_0(0,4)}$, $E_{\mathbb{F}_0(2,2)}$ *and* $E_{\mathbb{F}_2(2,2)}$;
2. $E_{\mathrm{dP}_r(1,4-r)}$ *and* $E_{\mathrm{dP}_r(0,5-r)}$, *where* $1 \le r \le 4$.

Proof For the first part of the proposition, denote by H_1 and H_2 the two generators of the Picard group of $\mathbb{F}_0$ corresponding to the pullbacks of a point in $\mathbb{P}^1$ along $\mathrm{proj}_{1,2}\colon \mathbb{F}_0 \to \mathbb{P}^1$. The Euler sequence on $\mathbb{P}^1$, pulled back to $\mathbb{F}_0$ along proj_1 and tensored by $\mathcal{O}(-H_2)$, yields

$$0 \to \mathcal{O}(-2H_1 - H_2) \to \mathcal{O}(-H_1 - H_2) \oplus \mathcal{O}(-H_1 - H_2) \to \mathcal{O}(-H_2) \to 0. \tag{2-7}$$

This determines a family with general fibre the total space of $\mathcal{O}(-H_1 - H_2) \oplus \mathcal{O}(-H_1 - H_2)$ and special fibre the total space of $\mathcal{O}(-H_2) \oplus \mathcal{O}(-2H_1 - H_2)$, hence a deformation between $E_{\mathbb{F}_0(0,4)}$ and $E_{\mathbb{F}_0(2,2)}$.

Next, consider again the Euler sequence over $\mathbb{P}^1$,

$$0 \to \mathcal{O}(-2) \to \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathcal{O} \to 0, \tag{2-8}$$

and the associated deformation of the total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ into the total space of $\mathcal{O} \oplus \mathcal{O}(-2)$. Taking the projectivisation of this family yields a deformation between $\mathbb{F}_0$ and $\mathbb{F}_2$. In this deformation, $-H_1 - H_2$ specialises to $-C_2$. Taking twice the associated line bundles yields the deformation between $E_{\mathbb{F}_0(2,2)}$ and $E_{\mathbb{F}_2(2,2)}$.

To prove the second part, assume first that $r = 1$. We start with the relative (dual) Euler sequence for the fibration $\mathrm{dP}_1 \to \mathbb{P}^1$ with distinct sections with image H and E_1

$$0 \to \mathcal{O} \to \mathcal{O}(H) \oplus \mathcal{O}(E_1) \to \mathcal{O}(H + E_1) \to 0. \tag{2-9}$$

We tensor it with $\mathcal{O}(-2H)$ to obtain

$$0 \to \mathcal{O}(-2H) \to \mathcal{O}(-H) \oplus \mathcal{O}(-2H + E_1) \to \mathcal{O}(-H + E_1) \to 0. \tag{2-10}$$

This determines a family with general fibre the total space of $\mathcal{O}(-H) \oplus \mathcal{O}(-2H + E_1)$ and special fibre the total space of $\mathcal{O}(-2H) \oplus \mathcal{O}(-H + E_1)$, hence a deformation between $E_{\mathrm{dP}_1(1,3)}$ and $E_{\mathrm{dP}_1(0,4)}$.

$Y(D)$	l	K_Y^2	D_1	D_2	D_3	D_4	tame	quasi-tame
$\mathbb{P}^2(1,4)$	2	9	H	$2H$	–	–	✓	✓
$\mathbb{F}_0(2,2)$	2	8	H_1+H_2	H_1+H_2	–	–	✓	✓
$\mathbb{F}_0(0,4)$	2	8	H_1	H_1+2H_2	–	–	✗	✓
$\mathrm{dP}_1(1,3)$	2	8	H	$2H-E_1$	–	–	✓	✓
$\mathrm{dP}_1(0,4)$	2	8	$H-E_1$	$2H$	–	–	✗	✓
$\mathrm{dP}_2(1,2)$	2	7	H	$2H-E_1-E_2$	–	–	✓	✓
$\mathrm{dP}_2(0,3)$	2	7	$H-E_1$	$2H-E_2$	–	–	✗	✓
$\mathrm{dP}_3(1,1)$	2	6	H	$2H-E_1-E_2-E_3$	–	–	✓	✓
$\mathrm{dP}_3(0,2)$	2	6	$H-E_1$	$2H-E_2-E_3$	–	–	✗	✓
$\mathrm{dP}_4(1,0)$	2	5	H	$2H-E_1-E_2-E_3-E_4$	–	–	✗	✗
$\mathrm{dP}_4(0,1)$	2	5	$H-E_1$	$2H-E_2-E_3-E_4$	–	–	✗	✗
$\mathrm{dP}_5(0,0)$	2	4	$H-E_1$	$2H-E_2-E_3-E_4-E_5$	–	–	✗	✗
$\mathbb{P}^2(1,1,1)$	3	9	H	H	H	–	✓	✓
$\mathbb{F}_0(2,0,0)$	3	8	H_1+H_2	H_1	H_2	–	✓	✓
$\mathrm{dP}_1(1,1,0)$	3	8	H	H	$H-E_1$	–	✓	✓
$\mathrm{dP}_2(1,0,0)$	3	7	H	$H-E_1$	$H-E_2$	–	✓	✓
$\mathrm{dP}_3(0,0,0)$	3	6	$H-E_1$	$H-E_2$	$H-E_3$	–	✓	✓
$\mathbb{F}_0(0,0,0,0)$	4	8	H_1	H_2	H_1	H_2	✓	✓

Table 1: Classification of smooth nef Looijenga pairs.

Dually, we have

$$0 \to \mathrm{H}^0(\mathcal{O}(H-E_1)) \to \mathrm{H}^0(\mathcal{O}(H)) \oplus \mathrm{H}^0(\mathcal{O}(2H-E_1)) \to \mathrm{H}^0(\mathcal{O}(2H)), \tag{2-11}$$

and a section of $\mathcal{O}(2H-E_1)$ in the general fibre gives a section of $\mathcal{O}(2H)$ in the special fibre. Hence we have a divisor $\mathcal{D}$ in the family in class $2H-E_1$ for the general fibre and of class $2H$ for the special fibre. Blowing up a general point of $\mathcal{D}$ in the family gives a deformation between $E_{\mathrm{dP}_2(1,2)}$ and $E_{\mathrm{dP}_2(0,3)}$. Iterating the process, we obtain the desired deformations. □

We summarise the discussion of this section in Table 1. There are 18 smooth deformation types of nef Looijenga pairs in total, 11 of which are tame and 15 of which are quasi-tame. The three non-quasi-tame cases occur when Y is a del Pezzo surface of degree 5 or less.

3 Local Gromov–Witten theory

3.1 1–Pointed local Gromov–Witten invariants

In this section, we provide general formulas for the Gromov–Witten invariants with point insertions of toric Fano varieties in any dimension twisted by a sum of concave line bundles. For the remainder of this section, let Y be an n–dimensional smooth projective variety of Picard rank r, let $D = D_1 + \cdots + D_l \in A_{n-1}(Y)$ with $D \in |-K_Y|$ and each D_i smooth and irreducible, and let d be a D–convex curve class.

Let $E_{Y(D)} := \mathrm{Tot}\big(\bigoplus_{i=1}^l (\mathcal{O}_Y(-D_i))\big)$ be as in (2-6) and let $\pi_Y\colon E_{Y(D)} \to Y$ be the natural projection. Since d is D–convex, the moduli space $\overline{\mathrm{M}}_{0,m}(E_{Y(D)}, d)$ of genus-zero m–marked stable maps $[f: C \to E_{Y(D)}]$ with $f_*([C]) = d \in \mathrm{H}_2(Y, \mathbb{Z})$ is scheme-theoretically the moduli stack $\overline{\mathrm{M}}_{0,m}(Y, d)$ of stable maps to the base Y, as every stable map to the total space factors through the zero section $Y \hookrightarrow E_{Y(D)}$. In particular, $\overline{\mathrm{M}}_{0,m}(E_{Y(D)}, d)$ is proper. Consider the universal curve $\pi: \mathcal{C} \to \overline{\mathrm{M}}_{0,m}(Y, d)$, and denote by $f:\mathcal{C} \to Y$ the universal stable map. Then $\mathrm{H}^0(C, f^*\mathcal{O}_Y(-D_i)) = 0$ and we have obstruction bundles $\mathrm{Ob}_{D_j} := R^1\pi_* f^*\mathcal{O}_Y(-D_j)$, of rank $d \cdot D_j - 1$ with fibre $\mathrm{H}^1(C, f^*\mathcal{O}_Y(-D_j))$ over a stable map $[f: C \to Y]$. The virtual fundamental class on $\overline{\mathrm{M}}_{0,m}(E_{Y(D)}, d)$ is defined by intersecting the virtual fundamental class on $\overline{\mathrm{M}}_{0,m}(Y, d)$ with the top Chern class of $\bigoplus_j \mathrm{Ob}_{D_j}$:

$$[\overline{\mathrm{M}}_{0,m}(E_{Y(D)}, d)]^{\mathrm{vir}} := c_{\mathrm{top}}(\mathrm{Ob}_{D_1}) \cap \cdots \cap c_{\mathrm{top}}(\mathrm{Ob}_{D_l}) \cap [\overline{\mathrm{M}}_{0,m}(Y, d)]^{\mathrm{vir}} \in \mathrm{H}_{m+l-1}(\overline{\mathrm{M}}_{0,m}(Y, d), \mathbb{Q}). \tag{3-1}$$

There are tautological classes $\psi_i := c_1(L_i)$, where L_i is the i^{th} tautological line bundle on $\overline{\mathrm{M}}_{0,m}(Y, d)$ whose fibre at $[f: (C, x_1, \ldots, x_m) \to Y]$ is the cotangent line of C at x_i, and we denote by ev_i the evaluation maps at the i^{th} marked point. For an effective D–convex curve class $d \in \mathrm{H}_2(Y, \mathbb{Z})$, genus-zero local Gromov–Witten invariants of $E_{Y(D)}$ with point insertions on the base are defined as

$$N_{0,d}^{\mathrm{loc}}(Y(D)) := \int_{[\overline{\mathrm{M}}_{0,l-1}(E_{Y(D)}, d)]^{\mathrm{vir}}} \prod_{j=1}^{l-1} \mathrm{ev}_j^*(\pi_Y^*[\mathrm{pt}_Y]), \tag{3-2}$$

$$N_{0,d}^{\mathrm{loc},\psi}(Y(D)) := \int_{[\overline{\mathrm{M}}_{0,1}(E_{Y(D)}, d)]^{\mathrm{vir}}} \mathrm{ev}_1^*(\pi_Y^*[\mathrm{pt}_Y]) \cup \psi_1^{l-2}, \tag{3-3}$$

which we think of as the virtual counts of curves through $l-1$ points (resp. 1–point with a ψ–condition) on the zero section of the vector bundle $E_{Y(D)}$.

Since D is anticanonical, $E_{Y(D)}$ is a noncompact Calabi–Yau $(n+l)$–fold. The case $n + l = 3$ has been the main focus in the study of local mirror symmetry, and as such it has been abundantly studied in the literature [26]. It turns out that the lesser studied situation when $n + l > 3$ has a host of simplifications, often leading to closed-form expressions for (3-2)–(3-3). We start by fixing some notation which will be of further use throughout this section. Let $T \simeq (\mathbb{C}^\star)^l \circlearrowright E_{Y(D)}$ be the fibrewise torus action and denote by $\lambda_i \in \mathrm{H}(BT)$, with $i = 1, \ldots, l$, its equivariant parameters. Let $\{\phi_\alpha\}_\alpha$ be a graded $\mathbb{C}$–basis for the nonequivariant cohomology of the image of the zero section $Y \hookrightarrow E_{Y(D)}$ with $\deg\phi_\alpha \le \deg\phi_{\alpha+1}$; in particular, $\phi_1 = \mathbf{1}_{\mathrm{H}(Y)}$. Its elements have canonical lifts $\phi_\alpha \to \varphi_\alpha$ to T–equivariant cohomology forming a $\mathbb{C}(\lambda_1, \ldots, \lambda_l)$ basis for $\mathrm{H}_T(E_{Y(D)})$. The latter is furthermore endowed with a perfect pairing

$$\eta_{E_{Y(D)}}(\varphi_\alpha, \varphi_\beta) := \int_Y \frac{\phi_\alpha \cup \phi_\beta}{\bigcup_i \mathrm{e}_T(\mathcal{O}_Y(-D_i))}, \tag{3-4}$$

with e_T denoting the T–equivariant Euler class. In what follows, we will indicate by $\eta^{-1}_{E_{Y(D)}}(\varphi_\alpha, \varphi_\beta)$ the inverse of the Gram matrix (3-4).

Let now $\tau \in \mathrm{H}_T(E_{Y(D)})$. The J–function of $E_{Y(D)}$ is the formal power series

$$J_{\mathrm{big}}^{E_{Y(D)}}(\tau, z) := z + \tau + \sum_{d \in \mathrm{NE}(Y)} \sum_{n \in \mathbb{Z}^+} \sum_{\alpha,\beta} \frac{\eta^{-1}_{E_{Y(D)}}(\varphi_\alpha, \varphi_\beta)}{n!} \left\langle \tau, \dots, \tau, \frac{\varphi_\alpha}{z-\psi} \right\rangle_{0,n+1,d}^{E_{Y(D)}} \varphi_\beta, \tag{3-5}$$

where we employed the usual correlator notation for GW invariants,

$$\langle \tau_1 \psi_1^{k_1}, \dots, \tau_n \psi_n^{k_n} \rangle_{0,n,d}^{E_{Y(D)}} := \int_{[\overline{\mathrm{M}}_{0,m}(E_{Y(D)},d)]^{\mathrm{vir}}} \prod_i \mathrm{ev}_i^*(\tau_i) \psi_i^{k_i}. \tag{3-6}$$

Restriction to $t = \sum_{i=1}^{r+1} t_i \varphi_i$ and use of the divisor axiom gives the small J–function

$$J_{\mathrm{small}}^{E_{Y(D)}}(t, z) := z \mathrm{e}^{\sum t_i \varphi_i / z} \left(1 + \sum_{d \in \mathrm{NE}(Y)} \sum_{\alpha,\beta} \eta^{-1}(\varphi_\alpha, \varphi_\beta)_{E_{Y(D)}} \mathrm{e}^{t(d)} \left\langle \frac{\varphi_\alpha}{z(z-\psi_1)} \right\rangle_{0,1,d}^{E_{Y(D)}} \varphi_\beta \right). \tag{3-7}$$

Lemma 3.1 *Suppose that Y is a toric Fano variety and either $n+l=4$ and D_i is ample for all i, or $n+l>4$ and D_i is nef for all i. Let $\mathcal{T} := \{T_i \in A_{\dim Y-1}(Y)\}_{i=1}^{n+r}$ be the collection of its prime toric divisors, and $\bigsqcup_{i=1}^m S_i = \{1, \dots, n+r\}$ a length-m partition of $n+r$ such that $D_i := \sum_{j \in S_i} T_j$. For an effective curve class $d \in \mathrm{NE}(Y)$, write $\mathsf{d}_i := d \cdot D_i$ and $\mathsf{t}_i := d \cdot T_i$ for its intersection multiplicities with the nef divisors D_i and the toric divisors T_i, respectively. Then*

$$N_{0,d}^{\mathrm{loc},\psi}(Y(D_1 + \dots + D_l)) = \frac{(-1)^{\sum_{i=1}^l (\mathsf{d}_i - 1)}}{\prod_{i=1}^l \mathsf{d}_i} \prod_{i=1}^l \binom{\mathsf{d}_i}{\{\mathsf{t}_j\}_{j \in S_i}}, \tag{3-8}$$

where

$$\binom{k}{\{i_j\}_{j=1}^m} = \frac{k!}{\prod_{j=1}^m i_j!}$$

is the multinomial coefficient.

Proof By (3-3) and (3-7), we have

$$N_{0,d}^{\mathrm{loc},\psi}(Y(D)) = \sum_\beta \eta(\varphi_{\overline{\alpha}}, \varphi_\beta)_{E_{Y(D)}} [z^{1-l} \mathrm{e}^{t(d) + \sum t_i \varphi_i / z} \varphi^\beta] J_{\mathrm{small}}^{E_{Y(D)}}(t, z), \tag{3-9}$$

where $\overline{\alpha}$ is defined by $\varphi_{\overline{\alpha}} = [\mathrm{pt}]$. From (3-4), we have $\eta(\varphi_{\overline{\alpha}}, \varphi_\beta)_{E_{Y(D)}} = \delta_{\overline{\alpha} 1} \prod_{i=1}^l \lambda_i^{-1}$, hence

$$N_{0,d}^{\mathrm{loc},\psi}(Y(D)) = \frac{1}{\prod_{i=1}^l \lambda_i} [z^{1-l} \mathrm{e}^{t(d) + \sum t_i \varphi_i / z} \mathbf{1}_{H_T(Y)}] J_{\mathrm{small}}^{E_{Y(D)}}(t, z). \tag{3-10}$$

The right-hand side can be computed by Givental-style toric mirror theorems. Let $\theta_a := T_a^\vee \in \mathrm{H}^2(Y)$ be the Poincaré dual class of the a^{th} toric divisor of Y, let $\kappa_i := c_1(\mathcal{O}(-D_i))$ be the T–equivariant Chern class of D_i, and let y_i with $i = 1, \dots, r+1$ be variables in a formal disk around the origin. Writing

$(x)_n := \Gamma(x+n)/\Gamma(x)$ for the Pochhammer symbol of (x,n) with $n \in \mathbb{Z}$, the I–function of $E_{Y(D)}$ is the $\mathrm{H}_T(E_{Y(D)})$–valued Laurent series

$$I^{E_{Y(D)}}(y,z) := z + \prod_i y_i^{\varphi_i/z} \sum_{0 \neq d \in \mathrm{NE}(Y)} \prod_i y_i^{d_i} z^{1-l} \frac{\prod_i \kappa_i (\kappa_i/z+1)_{\mathsf{d}_i - 1}}{\prod_a (\theta_a/z+1)_{\mathsf{t}_a}}, \tag{3-11}$$

and its mirror map is their formal $\mathcal{O}(z^0)$ coefficient,

$$\tilde{t}^i_{E_{Y(D)}}(y) := [z^0 \varphi_i] I^{E_{Y(D)}}(y,z). \tag{3-12}$$

Then [46; 31; 30]

$$J^{E_{Y(D)}}_{\mathrm{small}}(\tilde{t}_{E_{Y(D)}(D)}(y), z) = I^{E_{Y(D)}}(y,z). \tag{3-13}$$

Inspecting (3-11) shows that if either $n+l > 4$, or $n+l = 4$ and D_i is ample, the mirror map does not receive quantum corrections:

$$\tilde{t}^i_{E_{Y(D)}}(y) = \log y_i. \tag{3-14}$$

Therefore, under the assumptions of the lemma,

$$\begin{aligned} N^{\mathrm{loc},\psi}_{0,d}(Y(D)) &= \frac{1}{\prod_{i=1}^l \lambda_i} \Big[z^{1-l} \prod_i y_i^{d_i} \prod_i y_i^{\varphi_i/z} \mathbf{1}_{H_T(Y(D))} \Big] I^{E_{Y(D)}}(y,z) \\ &= \frac{1}{\prod_{i=1}^l \lambda_i} \Big[z^{1-l} \prod_i y_i^{d_i} \Big] I^{E_{Y(D)}}(y,z) \Big|_{\varphi_\alpha \to 0}. \end{aligned} \tag{3-15}$$

The claim then follows by substituting $\theta_a|_{\varphi_\alpha \to 0} = 0$ and $\kappa_i|_{\varphi_\alpha \to 0} = -\lambda_i$ into (3-11). □

3.1.1 Quasitame Looijenga pairs Let us now go back to the case of log CY surfaces and specialise the discussion in the previous section to $Y(D)$ a tame Looijenga pair. The key observation in the proof of Lemma 3.1 was that no contributions to the $\mathcal{O}(z^0)$ Laurent coefficient of the I–function could possibly come from any stable maps in any degrees, which is automatic for $n+l > 4$, and requires that $\mathsf{d}_i > 0$ when $n+l = 4$. We can in fact partly relax the condition that D_i is ample by just requiring by *fiat* that no curves with $\mathsf{d}_i = d \cdot D_i = 0$ contribute to the mirror map. A direct calculation from (3-11) shows that in the case of nef log CY surfaces with Y a Fano surface, this relaxed assumption coincides with $Y(D)$ being tame as in Definition 2.1. Since, by Proposition 2.2, Y is toric for all tame cases, Lemma 3.1 computes (3-3) for all of them.

Example 3.1 Let $Y(D) = \mathbb{P}^2(1,4)$. Then Lemma 3.1 gives for the degree-d local invariants of the projective plane

$$N^{\mathrm{loc},\psi}_{0,d}(Y(D)) = N^{\mathrm{loc}}_{0,d}(Y(D)) = \frac{(-1)^d}{2d^2}\binom{2d}{d}. \tag{3-16}$$

This recovers a direct localisation calculation by Klemm and Pandharipande in [68, Proposition 2].

Example 3.2 Let $Y(D) = \mathbb{F}_0(2,2)$ and write $d = d_1 H_1 + d_2 H_2$. Lemma 3.1 yields

$$N^{\mathrm{loc},\psi}_{0,d}(Y(D)) = N^{\mathrm{loc}}_{0,d}(Y(D)) = \frac{1}{(d_1+d_2)^2}\binom{d_1+d_2}{d_1}^2 \tag{3-17}$$

as in [68, Proposition 3].

Moreover, if $Y(D)$ is a quasi-tame Looijenga pair, the Calabi–Yau vector bundle $E_{Y(D)}$ is deformation equivalent to $E_{Y'(D')}$ for some tame Looijenga pair by definition. It therefore carries the same local Gromov–Witten theory, and the calculation of $N^{\mathrm{loc},\psi}_{0,d}(Y(D)) = N^{\mathrm{loc},\psi}_{0,d}(Y'(D'))$ from Lemma 3.1 extends immediately to these cases as well.

3.1.2 Nonquasi-tame Looijenga pairs

Lemma 3.1 cannot be immediately extended to non-quasi-tame pairs $Y(D)$, as Y is not toric and $E_{Y(D)}$ does not deform to $E_{Y'(D')}$ for tame $Y'(D')$. We will proceed by exhibiting a closed-form solution for the case of lowest anticanonical degree $Y(D) = \mathrm{dP}_5(0,0)$. This recovers all other cases with $l = 2$ by blowing down, as per the following.

Proposition 3.2 (blowup formula for local GW invariants) *Let $Y(D)$ be an l–component log CY surface. Let $\pi : Y'(D') \to Y(D)$ be the l–component log CY surface obtained by an interior blowup at a general point of D with exceptional divisor E. Let d be a curve class of $Y(D)$ and let $d' := \pi^* d$. Then*

$$N^{\mathrm{loc}}_{0,d}(Y(D)) = N^{\mathrm{loc}}_{d'}(Y'(D')) \quad \text{and} \quad N^{\mathrm{loc},\psi}_{0,d}(Y(D)) = N^{\mathrm{loc},\psi}_{d'}(Y'(D')). \tag{3-18}$$

Proof By [87, Proposition 5.14],

$$\bar{\pi}_*[\overline{\mathrm{M}}_{0,m}(Y',d')]^{\mathrm{vir}} = [\overline{\mathrm{M}}_{0,m}(Y,d)]^{\mathrm{vir}}, \tag{3-19}$$

where $\bar{\pi}$ is the morphism between the moduli spaces induced by π. Since $E \cdot d' = 0$,

$$\bar{\pi}_*[\overline{\mathrm{M}}_{0,m}(E_{Y'(D')},d')]^{\mathrm{vir}} = [\overline{\mathrm{M}}_{0,m}(E_{Y(D)},d)]^{\mathrm{vir}}. \tag{3-20}$$

□

Theorem 3.3 *With notation as in Proposition 2.5, we have*

$$\begin{aligned} N^{\mathrm{loc}}_{0,d}(\mathrm{dP}_5(0,0)) = {} & \\ \sum_{j_1,\dots,j_4=0}^{\infty} \Bigg[& \frac{(-1)^{d_1+d_2+d_3+d_4+d_5}(d_1+d_2+d_3+d_4+d_5-3d_0+j_1+j_2-1)!}{j_1!\,j_2!\,j_3!\,j_4!\,(d_1+d_2+d_4-2d_0+j_1)!\,(-d_1+d_0-j_1-j_2)!\,(-d_3+d_5+j_4)!} \\ & \times \frac{(d_1+d_4-d_0+j_1+j_3-1)!\,(d_1+d_5-d_0+j_2+j_4-1)!\,(d_4+d_5-d_0+j_3+j_4-1)!}{(d_1+d_3+d_5-2d_0+j_2)!\,(-d_4+d_0-j_1-j_3)!\,(-d_2+d_4+j_3)!\,(-d_5+d_0-j_2-j_4)!} \\ & \times \frac{1}{(d_2+d_3-d_0-j_3-j_4)!\,((d_1+d_4+d_5-2d_0+j_1+j_2+j_3+j_4-1)!)^2}\Bigg]. \end{aligned} \tag{3-21}$$

Sketch of the proof The strategy of the proof runs by deforming dP_5 to the blowup of $\mathbb{F}_0$ at four toric points, which is only weak Fano but allows to work torically along the lines of Lemma 3.1 at the price of extending the pseudoeffective cone by four generators of self-intersection -2. These contribute nontrivially to the mirror map, alongside curves with zero intersections with the boundary divisors $D_1 = H - E_1$ and $D_2 = 2H - \sum_{i\neq 1} E_i$. However, the mirror map turns out to be algebraic, and furthermore it remarkably has a closed-form rational inverse, leading to the final result (3-21). Full details are given in Appendix A. □

Remark 3.4 The final expression (3-21) is significantly more involved than (3-8), to which it reduces when blowing down to the quasi-tame del Pezzo cases dP_k for $k \le 3$ by setting $d_i = d_0$ for all $i \ge k+1$ using (3-18), since then only the summand with $j_i = 0$ for all i survives. It is also noteworthy that, while the summands in (3-21) are not symmetric under permutation of the degrees $\{d_2, d_3, d_4, d_5\}$, the final sum is highly nonobviously warranted to be S_4–invariant since the left-hand side is,[9] and we verified this explicitly in low degrees. The BPS invariants arising from (3-21) should also be integers, and we checked this is indeed the case for a large sample of nonprimitive classes with multicovers of order up to 11.

3.2 Multipointed local GW invariants

The primary multipoint invariants (3-2) of nef Looijenga pairs with $l > 2$ can be reconstructed from the descendent single-insertion invariants (3-3). We shall show how this arises by combining the associativity of the quantum product with the vanishing of quantum corrections for particular classes.

3.2.1 $l = 3$ It suffices to compute the invariants for the case of maximal Picard rank, $dP_3(0,0,0)$, from which the other $l = 3$ cases can be recovered by blowing down.

Theorem 3.5 *With notation as in Proposition 2.5, we have*

$$N^{\mathrm{loc}}_{0,d}(dP_3(0,0,0)) = (d_0^2 - d_1 d_0 - d_2 d_0 - d_3 d_0 + d_1 d_2 + d_1 d_3 + d_2 d_3) N^{\mathrm{loc},\psi}_{0,d}(dP_3(0,0,0)), \tag{3-22}$$

$$N^{\mathrm{loc},\psi}_{0,d}(dP_3(0,0,0)) = \frac{(-1)^{d_1+d_2+d_3+1}(d_1-1)!(d_2-1)!(d_3-1)!}{(d_1+d_2-d_0)!(d_1+d_3-d_0)!(d_2+d_3-d_0)!(d_0-d_1)!(d_0-d_2)!(d_0-d_3)!}. \tag{3-23}$$

Proof In the notation of the proof of Lemma 3.1, for $i, j = 2, \dots, 5$ the components of the small J–function of $E_{dP_3(0,0,0)}$ satisfy the quantum differential equations

$$z \nabla_{\varphi_i} \nabla_{\varphi_j} J^{E_{dP_3(0,0,0)}}_{\mathrm{small}}(t,z) = \nabla_{\varphi_i \star_t \varphi_j} J^{E_{dP_3(0,0,0)}}_{\mathrm{small}}(t,z), \tag{3-24}$$

where $\alpha \star_t \beta$ denotes the small quantum cohomology product, and the cohomology classes

$$\{\varphi_1 = \mathbf{1}_{H_T(E_{dP_3(0,0,0)})}, \dots, \varphi_5\}$$

[9] There is an obvious S_5 symmetry under permutation of the exceptional classes E_i in Y, which is reduced to an S_4 symmetry in the degrees (d_2, d_3, d_4, d_5) in $E_{Y(D)}$ by the splitting $D_1 = H - E_1$, $D_2 = 2H - E_2 - E_3 - E_4 - E_5$.

are denoted as in the proof of Lemma 3.1. We take $\{\varphi_i\}_{i=2}^5$ to be the basis elements of $\mathrm{H}_T(E_{\mathrm{dP}_3(0,0,0)})$ given by lifts to T–equivariant cohomology of the integral Kähler classes dual to $\{C_i \in \mathrm{H}_2(\mathrm{dP}_3,\mathbb{Z})\}_i$ with $C_{i+1}=E_i$ for $i=1,2,3$, and $C_5=H-E_1-E_2-E_3$, and an effective curve will be written $d=d_0C_5+\sum_{i=1}^3 d_iC_{i+2}$. From the proof of Lemma 3.1, the small J–function in the tame setting equates the I–function,

$$J_{\mathrm{small}}^{E_{\mathrm{dP}_3(0,0,0)}}(t,z)= \sum_{d_i>0} \mathrm{e}^{\sum_{i=0}^3 t_{i+2}d_i}\left[\frac{(-1)^{d_1+d_2+d_3}(\phi_2-\lambda_1)(\phi_3-\lambda_2)}{z^2\left(\frac{z+\phi_2+\phi_3-\phi_5}{z}\right)_{-d_0+d_1+d_2}\left(\frac{z+\phi_2+\phi_4-\phi_5}{z}\right)_{-d_0+d_1+d_3}} \frac{(\phi_4-\lambda_3)\left(\frac{z-\lambda_1+\phi_2}{z}\right)_{d_1-1}\left(\frac{z-\lambda_2+\phi_3}{z}\right)_{d_2-1}\left(\frac{z-\lambda_3+\phi_4}{z}\right)_{d_3-1}}{\left(\frac{z+\phi_3+\phi_4-\phi_5}{z}\right)_{-d_0+d_2+d_3}\left(\frac{z-\phi_2+\phi_5}{z}\right)_{d_0-d_1}\left(\frac{z-\phi_3+\phi_5}{z}\right)_{d_0-d_2}\left(\frac{z-\phi_4+\phi_5}{z}\right)_{d_0-d_3}}\right]. \tag{3-25}$$

By (3-7), the small quantum product can be computed from the $\mathcal{O}(z^{-1})$ formal Taylor coefficient of (3-25) as

$$\varphi_i \star_t \varphi_j=\sum_\alpha \varphi_\alpha[z^{-1}\varphi_\alpha]\partial^2_{t_it_j}J_{\mathrm{small}}^{E_{\mathrm{dP}_3(0,0,0)}}(t,z). \tag{3-26}$$

Inspection of (3-25) shows that the right-hand side receives quantum corrections of the form $1/n^2$ from curves with either $d_i=\delta_{ij}n$ or $d_i=(1-\delta_{ij})n$ and $j\neq 0$, $n\in\mathbb{N}^+$, with vanishing contributions in all other degrees. This implies that

$$(\partial^2_{t_5}-\partial_{t_2}\partial_{t_5}-\partial_{t_3}\partial_{t_5}-\partial_{t_4}\partial_{t_5}+\partial_{t_2}\partial_{t_3}+\partial_{t_3}\partial_{t_4}+\partial_{t_2}\partial_{t_4})[z^{-1}]J_{\mathrm{small}}^{E_{\mathrm{dP}_3(0,0,0)}}(t,z)=0, \tag{3-27}$$

which amounts to

$$\varphi_5\star_t\varphi_5-\sum_{j=2}^4\varphi_5\star_t\varphi_i+\sum_{j>i=2}^5\varphi_i\star_t\varphi_j=\varphi_5\cup\varphi_5-\sum_{j=2}^4\varphi_5\cup\varphi_i+\sum_{j>i=2}^4\varphi_i\cup\varphi_j. \tag{3-28}$$

It is immediate to verify that the right-hand side is the Poincaré dual of the point class. Therefore, from (3-24),

$$N^{\mathrm{loc}}_{0,d}(\mathrm{dP}_3(0,0,0))=(d_0^2-d_1d_0-d_2d_0-d_3d_0+d_1d_2+d_1d_3+d_2d_3)N^{\mathrm{loc},\psi}_{0,d}(\mathrm{dP}_3(0,0,0)), \tag{3-29}$$

and the second equation in the statement follows by Lemma 3.1. □

3.2.2 $l=4$ In this case D is the toric boundary, and the invariants $N^{\mathrm{loc}}_{0,d}(\mathbb{F}_0(0,0,0,0))$ were computed in [18] by a strategy similar to that of Theorem 3.5. The final result is the following proposition.

Proposition 3.6 [18, Theorem 3.1, Corollary 6.4]

$$N^{\mathrm{loc}}_{0,d}(\mathbb{F}_0(0,0,0,0))=d_1^2d_2^2N^{\mathrm{loc},\psi}_{0,d}(\mathbb{F}_0(0,0,0,0))=1. \tag{3-30}$$

$Y(D)$	$N^{\mathrm{loc},\psi}_{0,d}$	$N^{\mathrm{loc}}_{0,d}/N^{\mathrm{loc},\psi}_{0,d}$
$\mathbb{P}^2(1,4)$	$\frac{1}{2d^2}\binom{2d}{d}$	1
$\mathbb{F}_0(2,2)$ $\mathbb{F}_0(0,4)$	$\left(\frac{1}{d_1+d_2}\binom{d_1+d_2}{d_1}\right)^2$	1
$\mathrm{dP}_1(1,1)$ $\mathrm{dP}_2(0,4)$	$\frac{(-1)^{d_1}}{d_0(d_1+d_0)}\binom{d_0}{d_1}\binom{d_1+d_0}{d_0}$	1
$\mathrm{dP}_2(1,2)$ $\mathrm{dP}_2(0,3)$	$\frac{(-1)^{d_0+d_1+d_2}}{d_0(d_1+d_2)}\binom{d_0}{d_1}\binom{d_0}{d_2}\binom{d_1+d_2}{d_0}$	1
$\mathrm{dP}_3(1,1)$ $\mathrm{dP}_3(0,2)$	$\frac{(-1)^{d_1+d_2+d_3}(d_0-1)!(d_1+d_2+d_3-d_0-1)!}{(d_0-d_1)!(d_0-d_2)!(d_0-d_3)!(d_1+d_2-d_0)!(d_1+d_3-d_0)!(d_2+d_3-d_0)!}$	1
$\mathrm{dP}_4(1,0)$ $\mathrm{dP}_4(0,1)$	(3-21)$\big\|_{d_5\to d_0}$	1
$\mathrm{dP}_5(0,0)$	(3-21)	1
$\mathbb{P}^2(1,1,1)$	$\frac{(-1)^{d+1}}{d^3}$	d^2
$\mathbb{F}_0(2,0,0)$	$-\frac{1}{d_1d_2(d_1+d_2)}\binom{d_1+d_2}{d_2}$	d_1d_2
$\mathrm{dP}_1(1,1,0)$	$\frac{(-1)^{d_1+1}}{d_0^2d_1}\binom{d_0}{d_1}$	d_1d_0
$\mathrm{dP}_2(1,0,0)$	$\frac{(-1)^{d_0+d_1+d_2+1}}{d_0d_1d_2}\binom{d_0}{d_1}\binom{d_1}{d_0-d_2}$	d_1d_2
$\mathrm{dP}_3(0,0,0)$	$\frac{(-1)^{d_1+d_2+d_3+1}}{d_1d_2d_3}\binom{d_1}{d_0-d_2}\binom{d_2}{d_0-d_3}\binom{d_3}{d_0-d_1}$	$d_0^2-(d_1+d_2+d_3)d_0$ $+d_1d_2+d_1d_3+d_2d_3$
$\mathbb{F}_0(0,0,0,0)$	1	$d_1^2d_2^2$

Table 2: Local Gromov–Witten invariants of nef Looijenga pairs.

This concludes the calculation of local invariants with point insertions for nef Looijenga pairs. We collate the results in Table 2.

4 Log Gromov–Witten theory

4.1 Log Gromov–Witten invariants of maximal tangency

Let $Y(D)$ be an l–component log CY surface with maximal boundary. We endow Y with the divisorial log structure coming from D. The log structure is used to impose tangency conditions along the components D_j of D. In this paper we will be looking at genus g stable maps into Y of class $d\in \mathrm{H}_2(Y,\mathbb{Z})$

that meet each component D_j in one point of maximal tangency $d \cdot D_j$. The appropriate moduli space $\overline{\mathrm{M}}^{\log}_{g,m}(Y(D), d)$ of maximally tangent basic stable log maps was constructed in all generality in [58; 25; 1].

There are tautological classes $\psi_i := c_1(L_i)$ for L_i the i^{th} tautological line bundle on $\overline{\mathrm{M}}^{\log}_{g,m}(Y(D), d)$ whose fibre at $[f : (C, x_1, \dots, x_m) \to Y]$ is the cotangent line of C at x_i. Let ev_i be the evaluation map at the i^{th} marked point, and for $\pi : \mathcal{C} \to \overline{\mathrm{M}}^{\log}_{g,m}(Y(D), d)$ the universal curve with relative dualising sheaf ω_π, denote by $\mathbb{E} := \pi_* \omega_\pi$ the Hodge bundle, which is a rank g vector bundle on $\overline{\mathrm{M}}^{\log}_{g,m}(Y(D), d)$. The g^{th} *lambda class* is its top Chern class $\lambda_g := c_g(\mathbb{E})$.

We will be concerned with the virtual log GW count of genus g curves in Y of degree d meeting D_j in one point of maximal tangency $d \cdot D_j$, passing through $l-1$ general points of Y and with insertion λ_g,

$$N^{\log}_{g,d}(Y(D)) := \int_{[\overline{\mathrm{M}}^{\log}_{g,l-1}(Y(D),d)]^{\mathrm{vir}}} (-1)^g \lambda_g \prod_{j=1}^{l-1} \mathrm{ev}_j^*([\mathrm{pt}]). \tag{4-1}$$

Furthermore, we will denote by $N^{\log,\psi}_{0,d}(Y(D))$ the genus-zero log GW invariants of maximal tangency passing through one general point of Y with psi class to the power $l-2$,

$$N^{\log,\psi}_{0,d}(Y(D)) := \int_{[\overline{\mathrm{M}}^{\log}_{0,1}(Y(D),d)]^{\mathrm{vir}}} \mathrm{ev}_1^*([\mathrm{pt}]) \cup \psi_1^{l-2}. \tag{4-2}$$

It will be useful in the following to define all-genus generating functions for the logarithmic invariants of $Y(D)$ at fixed degree,

$$\mathsf{N}^{\log}_d(Y(D))(\hbar) := \frac{1}{\left(2\sin\left(\frac{1}{2}\hbar\right)\right)^{l-2}} \sum_{g \geqslant 0} N^{\log}_{g,d}(Y(D)) \hbar^{2g-2+l}. \tag{4-3}$$

In the setting of Proposition 2.4, it follows from [3] that $N^{\log}_{g,d}(Y(D))$ (resp. $N^{\log,\psi}_{0,d}(Y(D))$) equals the log GW invariant of $(\widetilde{Y}, \widetilde{D})$, of class $\varphi^* d$, with maximal tangency along each of the strict transforms of D_i, not meeting the other boundary components and meeting $l-1$ general points of $\widetilde{Y}$ (resp. one point with psi class to the power of $l-2$). The above numbers are deformation invariant in log smooth families [85].

4.2 Scattering diagrams

Our main tool for the calculation of (4-1)–(4-2) will be their associated quantum scattering diagrams and quantum broken lines [83; 14; 13; 32; 16]. In the classical limit, in dimension 2 this is treated in [56; 53; 51] and in full generality in [57; 55]. The quantum scattering diagram consists of an affine integral manifold B and a collection of walls $\mathfrak{d}$ with wall-crossing functions $f_{\mathfrak{d}}$. The latter are functions on open subsets of the mirror.

Let $\pi : (\widetilde{Y}, \widetilde{D}) \longrightarrow (\overline{Y}, \overline{D})$ be a toric model as in Proposition 2.4 with s interior blowups. Up to deformation, we may assume that the blowup points are disjoint. Note that $s = \chi_{\mathrm{top}}(Y \setminus D) = \chi_{\mathrm{top}}(\widetilde{Y} \setminus \widetilde{D})$ is an invariant of the interior. We construct an affine integral manifold B from π as follows. First, we start with the fan of $(\overline{Y}, \overline{D})$. Then, for every interior blowup, we add a focus–focus singularity in the direction of the corresponding ray. In practice, we introduce cuts connecting the singularities to infinity and we use charts to identify the complements of the cuts with an open subset of $\mathbb{R}^2$.

Let $\delta_1, \dots, \delta_s$ denote the focus–focus singularities and $B(\mathbb{Z})$ be the set of integral points of $B \setminus \{\delta_1, \dots, \delta_s\}$. In the limit where the singularities are sent to infinity, $B(\mathbb{Z})$ can be identified with the integral points of the fan of $(\overline{Y}, \overline{D})$. The singularity δ_j corresponds to an interior blowup on a toric divisor $D(\delta_j)$ of $(\overline{Y}, \overline{D})$ with exceptional divisor $\mathcal{E}_j$. Viewing the ray of the fan of $(\overline{Y}, \overline{D})$ corresponding to $D(\delta_j)$ as going from $(0,0)$ to infinity, denote by ρ_j its primitive direction.

Each δ_j creates a quantum wall $\mathfrak{d}_j$ propagating into the direction $-\rho_j$ and decorated with the wall-crossing function $f_{\mathfrak{d}_j} := 1 + t_j z^{\rho_j}$, where $t_j = t^{[\mathcal{E}_j]}$ is a formal variable keeping track of the exceptional divisor and z^{ρ_j} is the tangent monomial $x^a y^b$ if $\rho_j = (a,b)$. Note that the wall also propagates into the ρ_j direction (decorated with $1 + t_j z^{-\rho_j}$), but that part of the scattering diagram is not relevant to us.

When two walls meet, this creates scattering: up to perturbation, we may assume that at most two walls $\mathfrak{d}_j$ and $\mathfrak{d}_k$ come together at one point, which in the following is taken to be the origin for simplicity. We refer to [56] for the general case and only describe the explicit result in the two cases relevant to us:

- **Simple scattering** $(\det(\rho_j, \rho_k) = \pm 1)$ The scattering algorithm draws an additional quantum wall $\mathfrak{d}$ in the direction $-\rho_j - \rho_k$ decorated with the function $1 + t_j t_k z^{\rho_j + \rho_k}$.
- **Infinite scattering** $(\det(\rho_j, \rho_k) = \pm 2)$ The algorithm creates a central quantum wall $\mathfrak{d}$ in the direction $-\rho_j - \rho_k$ decorated with the function

$$\prod_{\ell=-\frac{1}{2}(\operatorname{ind}(\rho_j+\rho_k)-1)}^{\frac{1}{2}(\operatorname{ind}(\rho_j+\rho_k)-1)} (1 - q^{-\frac{1}{2}+\ell} t_j t_k z^{\rho_j+\rho_k})^{-1} (1 - q^{\frac{1}{2}+\ell} t_j t_k z^{\rho_j+\rho_k})^{-1}, \tag{4-4}$$

  where $\operatorname{ind}(\rho_j + \rho_k)$ is the index of $\rho_j + \rho_k$. We then add quantum walls $\mathfrak{d}_1, \dots, \mathfrak{d}_n, \dots$ in the directions $-(n+1)\rho_j - n\rho_k$ decorated with functions

$$1 + t_j^{n+1} t_k^n z^{(n+1)\rho_j + n\rho_k}, \tag{4-5}$$

  for $n \geq 0$, as well as quantum walls ${}_1\mathfrak{d}, \dots, {}_n\mathfrak{d}, \dots$ in the directions $-n\rho_j - (n+1)\rho_k$ decorated with functions

$$1 + t_j^n t_k^{n+1} z^{n\rho_j + (n+1)\rho_k} \tag{4-6}$$

  for $n \geq 0$.

The classical scattering algorithm is recovered in the classical limit $q^{\frac{1}{2}} = 1$. Only the central quantum wall in the case $\det(\rho_j, \rho_k) = \pm 2$ is different from its classical version, for which the wall-crossing function specialises to $(1 - t_j t_k z^{\rho_j+\rho_k})^{-2\operatorname{ind}(\rho_j+\rho_k)}$.

If $\mathfrak{u}$ and $\mathfrak{u}'$ are adjacent chambers of B separated by the quantum wall $\mathfrak{d}$ decorated with $f_{\mathfrak{d}}$, we can define a *quantum wall-crossing transformation* $\theta_{\mathfrak{d}}$ from $\mathfrak{u}$ to $\mathfrak{u}'$ as follows. Denote by $n_{\mathfrak{d}/\mathfrak{u}}$ the primitive orthogonal vector pointing from $\mathfrak{d}$ into $\mathfrak{u}$. Let m be such that $\langle n_{\mathfrak{d}/\mathfrak{u}}, m\rangle \geq 0$. For a polynomial a in the variables t_j, consider an expression az^m, which we think of as a function on $\mathfrak{u}$. Then, writing

$$f_{\mathfrak{d}} = \sum_{r\geq 0} c_r z^{r\rho_{\mathfrak{d}}}, \tag{4-7}$$

where $-\rho_{\mathfrak{d}}$ is the primitive direction of $\mathfrak{d}$,

$$\theta_{\mathfrak{d}} \colon az^m \mapsto az^m \prod_{\ell=-\frac{1}{2}(\langle n_{\mathfrak{d}/\mathfrak{u}}, m\rangle - 1)}^{\frac{1}{2}(\langle n_{\mathfrak{d}/\mathfrak{u}}, m\rangle - 1)} \Bigl(\sum_{r\geq 0} c_r q^{r\ell} z^{r\rho_{\mathfrak{d}}}\Bigr). \tag{4-8}$$

Note that in the classical limit $q^{\frac{1}{2}} = 1$, we recover the formula for the classical wall-crossing transformation, which is $\theta_{\mathfrak{d}}^{\mathrm{cl}} \colon az^m \mapsto f_{\mathfrak{d}}^{\langle n_{\mathfrak{d}/\mathfrak{u}}, m\rangle} az^m$. Writing $\theta_{\mathfrak{d}}(az^m) = \sum_i a_i z^{m_i}$, any summand $a_i z^{m_i}$ is called *a result of quantum transport of* az^m from $\mathfrak{u}$ to $\mathfrak{u}'$.

The final object we will need is the algebra of quantum broken lines associated to the scattering diagram, which we describe in the generality needed here; see [55] for full details in the classical limit. Let $B_0 := B \setminus \{\delta_1, \dots, \delta_s, \mathfrak{d}_j \cap \mathfrak{d}_k \mid \text{for all } j,k\}$. Let z^m be an *asymptotic monomial*, in our case this means that $m = (a,b) \neq (0,0)$, and let $p \in B$. Then a *quantum broken line* β with asymptotic monomial z^m and endpoint p consists of

(1) a directed piecewise straight path in B_0 of rational slopes, coming from infinity in the direction $-m$, bending only at quantum walls and ending at p;

(2) a labelling of the initial ray by L_1 and the successive line segments in order by $L_2, \dots, L_s$, where p is the endpoint of L_s;

(3) if $L_i \cap L_{i+1} \in \mathfrak{d}_i$, then, iteratively defined from 1 to s, the assignment of a monomial $a_i z^{m_i}$, where
   - $a_1 z^{m_1} = z^m$,
   - $a_{i+1} z^{m_{i+1}}$ is a result of the quantum transport of $a_i z^{m_i}$ across $\mathfrak{d}_i$,
   - L_i is directed in the direction $-m_i$.

Note that if $n_{\mathfrak{d}_i/L_i}$ is the primitive orthogonal vector to $\mathfrak{d}_i$ pointing into the half-plane containing L_i, then, as L_i is directed in the direction $-m_i$, we have $\langle n_{\mathfrak{d}_i/L_i}, m_i\rangle \geq 0$, and so the quantum transport of $a_i z^{m_i}$ across $\mathfrak{d}_i$ is indeed well-defined. We call $a_{\mathrm{end}} z^{m_{\mathrm{end}}} = a_s z^{m_s}$ the *end monomial* of β and a_{end} the *end coefficient* of β.

If z^m is an asymptotic monomial, the *theta function* ϑ_m is the sum of the end monomials of all broken lines with asymptotic monomial z^m and ending at p. Note that a priori ϑ_m depends on p, but it is one of the main results of [55] that it is constant in chambers and transforms from chamber to chamber according to the wall-crossing transformations.

We first describe the classical algebra of theta functions, ie we set $q^{\frac{1}{2}} = 1$. For A an element in the algebra of theta functions, we denote by $\langle A, \vartheta_m \rangle$ the coefficient of ϑ_m in A; note that $\langle A, \vartheta_m \rangle$ is a polynomial in the t_j. Then the identity component $\langle \vartheta_{m^1} \cdot \vartheta_{m^2}, \vartheta_0 \rangle$ is given as the sum of products of end coefficients $a^1_{\mathrm{end}} a^2_{\mathrm{end}}$ over all broken lines β_1 with asymptotic monomial z^{m^1} and β_2 with asymptotic monomial z^{m^2} such that $m^1_{\mathrm{end}} = -m^2_{\mathrm{end}}$. The identity component $\langle \vartheta_{m^1} \cdot \vartheta_{m^2} \cdot \vartheta_{m^3}, \vartheta_0 \rangle$ is given as the sum of products of end coefficients $a^1_{\mathrm{end}} a^2_{\mathrm{end}} a^3_{\mathrm{end}}$ over all broken lines β_1, β_2, β_3, with asymptotic monomials z^{m^1}, z^{m^2}, z^{m^3} and such that $m^1_{\mathrm{end}} + m^2_{\mathrm{end}} + m^3_{\mathrm{end}} = 0$.

For $(Y(D = D_1 + \cdots + D_l))$, consider the scattering diagram associated to a toric model π coming from a diagram as in Proposition 2.4,

(4-9)
$$\begin{array}{ccc} & \widetilde{Y}(\widetilde{D}) & \\ {}^{\varphi}\swarrow & & \searrow^{\pi} \\ Y(D) & & \overline{Y}(\overline{D}) \end{array}$$

Then the proper transform and pushforward of D_j is a toric divisor in $\overline{Y}$ corresponding to a ray in B. Up to reordering the indices, we assume that the ray corresponding to D_j is directed by ρ_j.

Proposition 4.1 [84] *Let $Y(D)$ be an l–component log Calabi–Yau surface of maximal boundary. Let $d \in \mathrm{H}_2(Y, \mathbb{Z})$ be an effective curve class and write $e_j := d \cdot \varphi_* \mathcal{E}_j$ for $j = 1, \ldots, s$, where φ is as in (4-9).*

- *Assume that $l = 2$. Set $m^1 = (d \cdot D_1)\rho_1$ and $m^2 = (d \cdot D_2)\rho_2$. Then $N^{\log}_{0,d}(Y(D))$ is the coefficient of $\prod_{j=1}^s t_j^{e_j}$ in $\langle \vartheta_{m^1} \cdot \vartheta_{m^2}, \vartheta_0 \rangle$.*
- *Assume that $l = 3$. Set $m^1 = (d \cdot D_1)\rho_1$, $m^2 = (d \cdot D_2)\rho_2$ and $m^3 = (d \cdot D_3)\rho_3$. Then $N^{\log,\psi}_{0,d}(Y(D))$ is the coefficient of $\prod_{j=1}^s t_j^{e_j}$ in $\langle \vartheta_{m^1} \cdot \vartheta_{m^2} \cdot \vartheta_{m^3}, \vartheta_0 \rangle$.*

We return to the algebra of quantum theta functions. For every m^1, m^2 and $p \in B_0$, denote by C_{m^1,m^2} the polynomial in the variables t_j with coefficients in $\mathbb{Z}[q^{\pm\frac{1}{2}}]$ given as the sum of products of end coefficients $a^1_{\mathrm{end}} a^2_{\mathrm{end}}$ over all quantum broken lines β_1 with asymptotic monomial z^{m^1} and β_2 with asymptotic monomial z^{m^2}, with common endpoint p and such that $m^1_{\mathrm{end}} = -m^2_{\mathrm{end}}$. The polynomial C_{m^1,m^2} is independent of the choice of $p \in B_0$.

Proposition 4.2 *Let $Y(D)$ be an l–component log Calabi–Yau surface of maximal boundary. Let $d \in \mathrm{H}_2(Y, \mathbb{Z})$ be an effective curve class and write $e_j := d \cdot \varphi_* \mathcal{E}_j$ for $j = 1, \ldots, s$, where φ is as in (4-9).*

- *Assume that $l = 2$. Set $m^1 = (d \cdot D_1)\rho_1$ and $m^2 = (d \cdot D_2)\rho_2$. Then after the change of variables $q = e^{i\hbar}$, the series*

(4-10)
$$\mathsf{N}^{\log}_d(Y(D))(\hbar) = \sum_{g \geqslant 0} N^{\log}_{g,d}(Y(D)) \hbar^{2g}$$

is the $\hbar$–expansion of the q–polynomial which is the coefficient of $\prod_{j=1}^s t_j^{e_j}$ in C_{m^1,m^2}.

- *Assume that $l = 3$. Set $m^1 = (d \cdot D_1)\rho_1$, $m^2 = (d \cdot D_2)\rho_2$ and $m^3 = (d \cdot D_3)\rho_3$. Then after the change of variables $q = e^{i\hbar}$, the series*

$$\mathsf{N}_d^{\log}(Y(D))(\hbar) = \frac{1}{2\sin\left(\frac{1}{2}\hbar\right)} \sum_{g \geqslant 0} N_{g,d}^{\log}(Y(D))\hbar^{2g+1} \tag{4-11}$$

is the $\hbar$–expansion of the q–polynomial obtained as the sum over all quantum broken lines β_1 with asymptotic monomial z^{m_1}, β_2 with asymptotic monomial z^{m_2}, and β_3 with asymptotic monomial z^{m_3}, with common endpoint and such that $m_{\text{end}}^1 + m_{\text{end}}^2 + m_{\text{end}}^3 = 0$, of

$$\frac{[|\det(m_{\text{end}}^1, m_{\text{end}}^2)|]_q}{[1]_q} a_{\text{end}}^1 a_{\text{end}}^2 a_{\text{end}}^3. \tag{4-12}$$

Here $a_{\text{end}}^i z^{m_{\text{end}}^i}$ are the end monomials of the broken lines β_i and the q–integers $[\,\cdot\,]_q$ are defined in (4-18) below.

Proof We only give a sketch of the proof as it is an adaptation of the proof of the Frobenius structure conjecture of [84], which in the setting relevant to us is stated in Proposition 4.1 above.

Recall first the geometric argument of the proof of [84]. The starting point is to consider the degeneration of [56] of $Y(D)$ to a toric situation: using toric transversality in the cluster setting, the curves do not fall into the codimension-one strata of D and one may apply the degeneration formula, expressing $N_{0,d}^{\log,\psi}(Y(D))$ in terms of log GW invariants of the central fibre, which can in turn be computed via the toric tropical correspondence theorem [97; 85]. In the scattering diagram, the tropical curves correspond to constellations of broken lines, and the product of the end coefficients equals the product of the multiplicity of the tropical curve with the terms coming from the degeneration formula.

To see how this is modified to obtain higher-genus invariants, the study of the degeneration of [56] is done using the techniques introduced in [14], and then the result follows from the toric tropical correspondence theorem for higher-genus log Gromov–Witten invariants with λ_g–insertion proven in [12]. The toric transversality of the log maps in the degeneration is a consequence of the vanishing result of [12, Lemma 8].

To further encompass two-pointed insertions, one can see that tropically, by [85], a ψ–class corresponds to a marked 3–valent vertex with multiplicity 1. In the case of 2–pointed invariants, one can carry out the same degeneration as above, therefore leading to the same tropical curves in the fan of the central fibre. The one difference is that previously one 3–valent vertex corresponded to a point with a ψ–class, and hence carried multiplicity 1, whereas in the case of 2–pointed invariants this vertex is no longer marked and carries its Block–Göttsche [10] multiplicity

$$\frac{[|\det(m_{\text{end}}^1, m_{\text{end}}^2)|]_q}{[1]_q},$$

and instead there are two marked 2–valent vertices elsewhere (which do not carry any multiplicity). □

4.2.1 Binomials and q–binomial coefficients In our applications of Propositions 4.1 and 4.2, we will mostly consider (quantum) broken lines bending along (quantum) walls $\mathfrak{f}_{\mathfrak{d}}$ decorated by a function of the form

$$f_{\mathfrak{d}} = 1 + t z^{\rho_{\mathfrak{d}}}, \tag{4-13}$$

where $-\rho_{\mathfrak{d}}$ is the primitive direction of $\mathfrak{d}$. By the binomial theorem, we have

$$f_{\mathfrak{d}}^{\langle n,m\rangle} = (1 + t z^{\rho_{\mathfrak{d}}})^{\langle n,m\rangle} = \sum_{k=0}^{\langle n,m\rangle} \binom{\langle n,m\rangle}{k} t^k z^{k\rho_{\mathfrak{d}}}. \tag{4-14}$$

Therefore, each application of transport across such a wall will produce a binomial coefficient, and so our genus-zero log Gromov–Witten invariants will be product of binomial coefficients. By the q–binomial theorem, we have

$$\prod_{\ell=-\frac{1}{2}(\langle n,m\rangle-1)}^{\frac{1}{2}(\langle n,m\rangle-1)} \left(1 + t q^{\ell} z^{\rho_{\mathfrak{d}}}\right) = \sum_{k=0}^{\langle n,m\rangle} \begin{bmatrix} \langle n,m\rangle \\ k \end{bmatrix}_q t^k z^{k\rho_{\mathfrak{d}}}, \tag{4-15}$$

where the q–binomial coefficients

$$\begin{bmatrix} N \\ k \end{bmatrix}_q := \frac{[N]_q!}{[k]_q![(N-k)]_q!} \tag{4-16}$$

are defined in terms of the q–factorials

$$[n]_q! := \prod_{j=1}^{n} [j]_q, \tag{4-17}$$

where the q–integers are

$$[n]_q := q^{n/2} - q^{-n/2}. \tag{4-18}$$

It follows that the formulas for the higher-genus log Gromov–Witten invariants $\mathsf{N}_d^{\log}(Y(D))(\hbar)$ will be obtained by replacing binomial coefficients by q–binomial coefficients in the formulas for the genus-zero invariant $N_{0,d}^{\log}(Y(D))$.

4.3 Log Gromov–Witten invariants under interior blowup

Proposition 4.3 (blowup formula for log GW invariants) *Let $Y(D)$ be an l–component log CY surface with maximal boundary. Let $\pi: Y'(D') \to Y(D)$ be the l–component log CY surface with maximal boundary obtained by an interior blowup at a general point of D with exceptional divisor E. Let d be a curve class of $Y(D)$ and let $d' := \pi^* d$. Then*

$$N_{g,d}^{\log}(Y(D)) = N_{g,d'}^{\log}(Y'(D')), \tag{4-19}$$

$$N_{0,d}^{\log,\psi}(Y(D)) = N_{0,d'}^{\log,\psi}(Y'(D')). \tag{4-20}$$

Proof Let D_j be the irreducible component of D containing the point that we blow up. We consider the degeneration of $Y(D)$ to the normal cone of D_j: the fibre over any point of $\mathbb{A}^1 - \{0\}$ is $Y(D)$ and the special fibre over $\{0\}$ has two irreducible components, which are isomorphic to $Y(D)$ and a $\mathbb{P}^1$–bundle $\mathbb{P}_j$ over D_j, and are glued together along a copy of D_j. Let $\mathcal{D}_j$ be the closure of $D_j \times (\mathbb{A}^1 - \{0\})$ in the total space of the degeneration. After blowing up a section of $\mathcal{D}_j \to \mathbb{A}^1$, we obtain a family with fibre $Y'(D')$ over any point of $\mathbb{A}^1 - \{0\}$, and special fibre over $\{0\}$ given by the union of two irreducible components, which are isomorphic to $Y(D)$ and to the blowup $\widetilde{\mathbb{P}}_j$ of $\mathbb{P}_j$ at one point. We compare the invariants $N_{g,d}^{\log}$ and $N_{0,d}^{\log,\psi}$ of $Y(D)$ and $Y'(D')$ using this degeneration. Following the general strategy of [14, Section 5], using in particular the vanishing result of [12, Lemma 8] to guarantee toric transversality of the log maps in the degeneration, we obtain that the invariants of $Y(D)$ and $Y'(D')$ only differ by a multiplicative factor coming from multiple covers of a fibre of $\widetilde{\mathbb{P}}_j \to D_j$. By deformation invariance, we can assume that this fibre is a smooth $\mathbb{P}^1$–fibre, with trivial normal bundle in $\widetilde{\mathbb{P}}_j$. Therefore, the correction factor is an integral over a moduli space of stable log maps to $\mathbb{P}^1$ with extra insertion of the class $e(\mathrm{H}^1(C, \mathcal{O}_C)) = (-1)^g \lambda_g$. Because our genus g invariants already contain an insertion of λ_g and $\lambda_g^2 = 0$ for $g > 0$ by Mumford's relation [95], the correction factor only receives contributions from genus zero. The genus-zero corrections involves degree $d \cdot D_j$ stable log maps to $(\mathbb{P}^1, \{0\} \cup \{\infty\})$, fully ramified over 0 and ∞. The corresponding moduli space is a point with an automorphism group of order $d \cdot D_j$ and so contributes $1/(d \cdot D_j)$. Because of the extra $(d \cdot D_j)$ multiplicity factor in the degeneration formula, the total multiplicative correction factor is 1. □

As a consequence of Proposition 4.3, if we calculate $N_{g,d}^{\log}(Y(D))$ and $N_{0,d}^{\log,\psi}(Y(D))$ for all g and d, then we will know the invariants for all interior blowdowns of $Y(D)$. Therefore it is enough to calculate the invariant for the cases of highest Picard rank in Propositions 2.2 and 2.3. In the following section, we calculate the higher-genus log invariants $\mathsf{N}_d^{\log}(Y(D))(\hbar)$ for all tame Looijenga pairs: using Proposition 4.3, it is enough to consider the pairs $\mathrm{dP}_3(1,1)$, $\mathrm{dP}_3(0,0,0)$ and $\mathbb{F}_0(0,0,0,0)$, which are treated in Theorems 4.5, 4.9 and 4.10.

For nontame pairs, the genus-zero invariants can be obtained by combining the log-local correspondence of Theorem 5.1 and (3-21) in Theorem 3.3 giving the local invariants. For quasi-tame pairs we furthermore make the following general conjecture for the higher-genus invariants $\mathsf{N}_d^{\log}(Y(D))(\hbar)$.

Conjecture 4.4 *Let $Y(D)$ and $Y'(D')$ be nef 2–component log CY surfaces with maximal boundary such that the corresponding local geometries $E_{Y(D)}$ and $E_{Y'(D')}$ are deformation equivalent. Then, under suitable identification of d, we have*

$$\left(\prod_{j=1}^{l=2} [d \cdot D'_j]_q\right) \mathsf{N}_d^{\log}(Y(D))(\hbar) = \left(\prod_{j=1}^{l=2} [d \cdot D_j]_q\right) \mathsf{N}_d^{\log}(Y'(D'))(\hbar). \tag{4-21}$$

Conjecture 4.4 holds in the genus-zero, ie $q^{\frac{1}{2}} = 1$, limit, as a corollary of the log-local correspondence given by Theorem 5.1 and of the deformation invariance of local Gromov–Witten invariants. In higher genus, Conjecture 4.4 translates to conjectural, new nontrivial q–binomial identities: see eg Conjecture B.3 for the cases of $\mathrm{dP}_1(0,4)$ and $\mathbb{F}_0(0,4)$.

4.4 Toric models: $l = 2$

Extending [15, Section 5] we find toric models for all $l = 2$ nef log Calabi–Yau surfaces except for $\mathbb{F}_0(2,2)$, which we leave to the reader as an exercise. For each toric model, we draw the corresponding fans with focus–focus singularities. By [40, Lemma 2.10], a log Calabi–Yau surface with maximal boundary $(\overline{Y}, \overline{D})$ is toric if the sequence of self-intersection numbers of irreducible components of $\overline{D}$ is realised as the sequence of self-intersection numbers of toric divisors on a toric surface. Once we have the toric models, we calculate the part of the scattering diagram relevant to us, and by Proposition 4.1 the relevant structural coefficients for the multiplication of theta functions yield the maximal tangency log Gromov–Witten invariants.

4.4.1 Tame pairs: simple scattering By Proposition 4.3, it suffices to consider the case $Y(D) = \mathrm{dP}_3(1,1)$. Start with $\mathbb{P}^2(1,4)$. The anticanonical decomposition of D is given by D_1 a line and D_2 a smooth conic not tangent to D_1. For notational convenience, in what follows we will identify D_1 and D_2 (resp. F_1 and F_2) with their strict transforms (resp. pushforwards) under blowups (resp. blowdowns).

Denote by pt one of the intersection points of D_1 and D_2 and by L the line tangent to D_2 at pt. We blow up pt, leading to the exceptional divisor F_1. We further blow up the intersection of F_1 with D_2 and write F_2 for the exceptional divisor. Denote the resulting log Calabi–Yau surface with maximal boundary by $(\widetilde{\mathbb{P}^2(1,4)}, \widetilde{D})$, where $\widetilde{D}$ is the strict transform of D.

The toric model $(\overline{\mathbb{P}^2(1,4)}, \overline{D})$ is given by blowing down the strict transform of L, so that $\overline{\mathbb{P}^2(1,4)} = \mathbb{F}_2$ and $\overline{D} = D_1 \cup F_1 \cup F_2 \cup D_2$, with F_1 the (-2)–curve of $\mathbb{F}_2$, D_2 a section of self-intersection 2, and D_1 and F_2 linearly equivalent to fibre classes. Labelling the toric boundary divisors with their self-intersections, we obtain the diagram at the left of Figure 3.

To obtain the toric model for $\mathrm{dP}_3(1,1)$, we need to blow up a nontoric point on F_2 (thus reproducing L), and three nontoric points on D_2. Tropically, this amounts to introducing a focus–focus singularity on the ray of F_2 and three on the ray of D_2 as in Figure 3 to the right. Walls emanate out of these focus–focus singularities. While they propagate into two directions, for our calculations only one direction matters (the other ray being close to infinity and thus noninteracting). We perturb the focus–focus singularities on D_2 horizontally.

The cone of curves is generated by $H - E_i - E_j$ for $1 \le i < j \le r$ and the E_i. In particular, any curve class $d \in \mathrm{H}_2(\mathrm{dP}_3, \mathbb{Z})$ can be written as $d = d_0(H - E_1 - E_2 - E_3) + d_1 E_1 + d_2 E_2 + d_3 E_3$.

Theorem 4.5 *Putting $q = \mathrm{e}^{\mathrm{i}\hbar}$, we have*

$$\mathrm{N}_d^{\log}(\mathrm{dP}_3(1,1))(\hbar) = \begin{bmatrix} d_3 \\ d_0 - d_1 \end{bmatrix}_q \begin{bmatrix} d_3 \\ d_0 - d_2 \end{bmatrix}_q \begin{bmatrix} d_0 \\ d_3 \end{bmatrix}_q \begin{bmatrix} d_1 + d_2 + d_3 - d_0 \\ d_3 \end{bmatrix}_q. \tag{4-22}$$

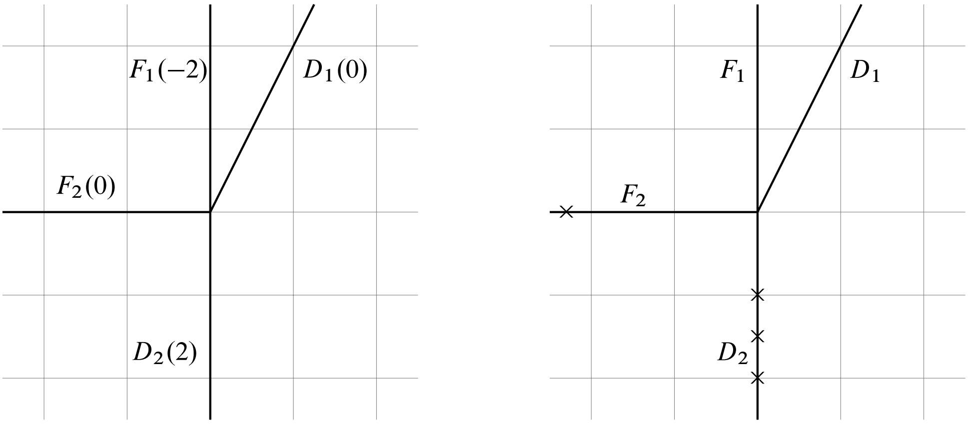


Figure 3: Left: the toric model of $\mathbb{P}^2(1,4)$. Right: the toric model of $\mathrm{dP}_3(1,1)$.

Proof Write $t = z^{[L]}$ and let $t_i = z^{[E_i]}$. Since $D_1 = H$ and $D_2 = 2H - E_1 - E_2 - E_3$, we have the following intersection multiplicities:

$$d \cdot D_1 = d_0, \quad d \cdot D_2 = d_1 + d_2 + d_3 - d_0 \quad \text{and} \quad d \cdot E_i = d_0 - d_i.$$

All of the scattering is simple. The initial wall-crossing functions are drawn in Figure 4, and all successive functions are easily obtained. We have two broken lines, one coming from the D_1–direction with attaching monomial $(xy^2)^{d\cdot D_1}$ and one coming from the D_2–direction with attaching monomial $(y^{-1})^{d\cdot D_2}$. Provided we choose our endpoint p to be sufficiently far into the x–direction, Figure 4 contains all the relevant walls. We start from the broken line coming from the D_2 direction and summarise the wall-crossing functions attached to the walls it meets:

① $1 + tx^{-1}$, ② $1 + tt_3x^{-1}y^{-1}$, ③ $1 + tt_2x^{-1}y^{-1}$,
④ $1 + tt_1x^{-1}y^{-1}$, ⑤ $1 + t^2t_1t_2t_3x^{-2}y^{-3}$.

Crossing these walls leads to $y^{d_0-d_1-d_2-d_3}$ mapping to

$$\begin{bmatrix} d_1+d_2+d_3-d_0 \\ k \end{bmatrix}_q \begin{bmatrix} d_1+d_2+d_3-d_0-k \\ k_1 \end{bmatrix}_q \begin{bmatrix} d_1+d_2+d_3-d_0-k \\ k_2 \end{bmatrix}_q \cdot \begin{bmatrix} d_1+d_2+d_3-d_0-k \\ k_3 \end{bmatrix}_q \begin{bmatrix} 2d_1+2d_2+2d_3-2d_0-3k-k_1-k_2-k_3 \\ k_4 \end{bmatrix}_q \cdot t^{k+k_1+k_2+k_3+2k_4}\, t_3^{k_1+k_4}\, t_2^{k_2+k_4}\, t_1^{k_3+k_4}\, x^{-k-k_1-k_2-k_3-2k_4}\, y^{d_0-d_1-d_2-d_3-k_1-k_2-k_3-3k_4}.$$

The intersection multiplicities with the divisors impose the following conditions:

$$k + k_1 + k_2 + k_3 + 2k_4 = d_0, \quad k_1 + k_4 = d_0 - d_3, \quad k_2 + k_4 = d_0 - d_2, \quad k_3 + k_4 = d_0 - d_1. \tag{4-23}$$

Choose as indeterminate k. For the coefficient to be nonzero, $0 \le k \le d_1 + d_2 + d_3 - d_0$. Then

$$\begin{aligned} k_4 &= k + 2d_0 - d_1 - d_2 - d_3, & k_1 &= d_1 + d_2 - d_0 - k, \\ k_2 &= d_1 + d_3 - d_0 - k, & k_3 &= d_2 + d_3 - d_0 - k. \end{aligned} \tag{4-24}$$

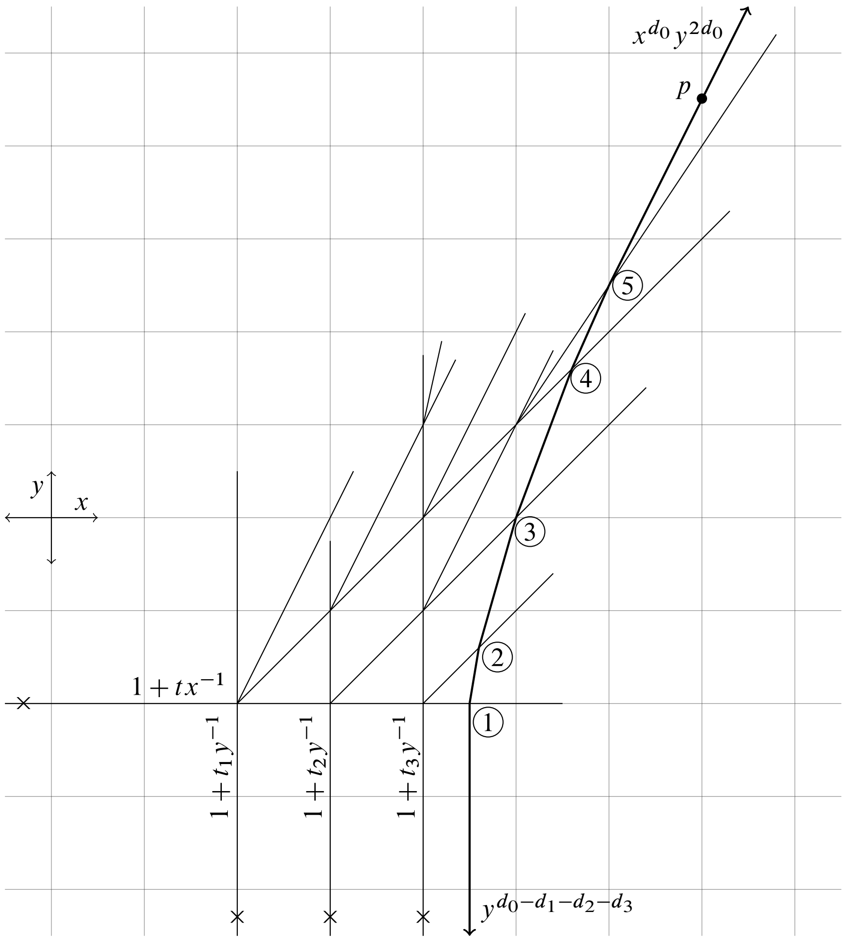


Figure 4: $\operatorname{Scatt} dP_3(1,1)$.

Hence the sum of the coefficients of the broken lines is

$$\sum_{k=0}^{d_1+d_2+d_3-d_0}\left(\begin{bmatrix} d_1+d_2+d_3-d_0 \\ k \end{bmatrix}_q \begin{bmatrix} d_1+d_2+d_3-d_0-k \\ d_3 \end{bmatrix}_q \begin{bmatrix} d_1+d_2+d_3-d_0-k \\ d_2 \end{bmatrix}_q \cdot \begin{bmatrix} d_1+d_2+d_3-d_0-k \\ d_1 \end{bmatrix}_q \begin{bmatrix} 2d_1+2d_2+2d_3-2d_0-3k-k_1-k_2-k_3 \\ k_4 \end{bmatrix}_q\right)$$

$$= \sum_{k=0}^{k(d_0,d_1,d_2,d_3)}\left(\begin{bmatrix} d_1+d_2+d_3-d_0 \\ k \end{bmatrix}_q \begin{bmatrix} d_1+d_2+d_3-d_0-k \\ d_3 \end{bmatrix}_q \begin{bmatrix} d_1+d_2+d_3-d_0-k \\ d_2 \end{bmatrix}_q \cdot \begin{bmatrix} d_1+d_2+d_3-d_0-k \\ d_1 \end{bmatrix}_q \begin{bmatrix} d_0 \\ k+2d_0-d_1-d_2-d_3 \end{bmatrix}_q\right),$$

where $k(d_0,d_1,d_2,d_3) := \min\{d_0, d_1+d_2-d_0, d_1+d_3-d_0, d_2+d_3-d_0\}$.

Therefore, we obtain

(4-25) $$\mathsf{N}_d^{\log}(\mathrm{dP}_3(1,1))(\hbar) = \sum_{k\geqslant 0}\Bigg(\begin{bmatrix} d_1+d_2+d_3-d_0 \\ k\end{bmatrix}_q \begin{bmatrix} d_1+d_2+d_3-d_0-k \\ d_3\end{bmatrix}_q \begin{bmatrix} d_1+d_2+d_3-d_0-k \\ d_2\end{bmatrix}_q \cdot \begin{bmatrix} d_1+d_2+d_3-d_0-k \\ d_1\end{bmatrix}_q \begin{bmatrix} d_0 \\ k+2d_0-d_1-d_2-d_3\end{bmatrix}_q\Bigg).$$

Writing the q–binomial coefficients in terms of q–factorials, and changing the indexing variable

$$k \mapsto k - d_0 + \tfrac{1}{2}(d_1+d_2+d_3),$$

we have

(4-26) $$\frac{[d_0]_q![d_1+d_2+d_3-d_0]_q!}{[d_1]_q![d_2]_q![d_3]_q!} \cdot \sum_k \frac{\big[\frac{1}{2}(d_1+d_2+d_3)-k\big]_q!}{\Big(\big[\frac12(d_1+d_2-d_3)-k\big]_q!\big[\frac12(d_1+d_3-d_2)-k\big]_q!\big[\frac12(d_2+d_3-d_1)-k\big]_q! \cdot \big[k-d_0+\frac12(d_1+d_2+d_3)\big]_q!\big[k+d_0-\frac12(d_1+d_2+d_3)\big]_q!\Big)}.$$

We re-sum this explicitly using the q–Pfaff–Saalschütz identity[10] in the form given in [116, Equation (1q)]:

(4-27) $$\sum_k \frac{[a+b+c-k]_q!}{[a-k]_q![b-k]_q![c-k]_q![k-m]_q![k+m]_q!} = \begin{bmatrix} a+b \\ a+m\end{bmatrix}_q \begin{bmatrix} a+c \\ c+m\end{bmatrix}_q \begin{bmatrix} b+c \\ b+m\end{bmatrix}_q.$$

Therefore, specialising (4-27) to $a+b=d_1$, $b+c=d_3$, $a+c=d_2$, $a+m=d_0-d_3$, $b+m=d_0-d_2$ and $c+m=d_0-d_1$, we have

(4-28) $$\mathsf{N}_d^{\log}(\mathrm{dP}_3(1,1))(\hbar) = \frac{[d_0]_q![d_1+d_2+d_3-d_0]_q!}{[d_1]_q![d_2]_q![d_3]_q!}\begin{bmatrix} d_1 \\ d_0-d_3\end{bmatrix}_q \begin{bmatrix} d_2 \\ d_0-d_1\end{bmatrix}_q \begin{bmatrix} d_3 \\ d_0-d_2\end{bmatrix}_q,$$

which after elementary simplifications gives (4-22). □

Remark 4.6 It follows from the above proof that Theorem 4.5 is in fact equivalent to the q–Pfaff–Saalschütz identity. In genus zero, Theorem 5.1 applied to $\mathrm{dP}_3(1,1)$ gives a geometric proof of Theorem 4.5. Thus, we obtain a new geometric, albeit quite indirect, proof of the classical ($q=1$) Pfaff–Saalschütz identity.

4.4.2 Nontame pairs: infinite scattering Figure 19 gives the toric model of $\mathbb{F}_0(0,4)$. For the other nontame pairs, let $1 \le r \le 5$. Then $\mathrm{dP_r}(0,5-r)$ is obtained from $\mathbb{P}^2(1,4)$ by blowing up the first point on the line D_1 and the remaining $r-1$ points on the conic D_2. Hence we obtain the toric model of $\mathrm{dP_r}(0,5-r)$ by adding 1 focus–focus singularity on the ray D_1 and $r-1$ focus–focus singularities on the ray D_2, as in Figure 5. The singularities on the ray of D_2 can be perturbed horizontally.

[10] Unlike [44; 116], we are using q–factorials and q–binomial coefficients symmetric under $q \mapsto q^{-1}$. This explains the absence in the above expression of the power $q^{n^2-k^2}$, which is present in [116, Equation (1q)].

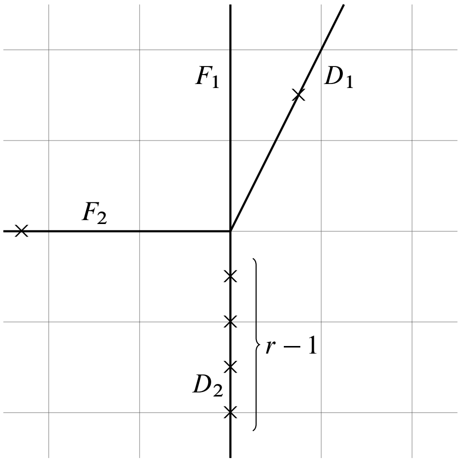


Figure 5: $\mathrm{dP_r}(0, 5-r)$.

Write a curve class $d \in \mathrm{H}_2(\mathrm{dP}_3(0,2), \mathbb{Z})$ as $d = d_0(H - E_1 - E_2 - E_3) + d_1 E_1 + d_2 E_2 + d_3 E_3$. As $D_1 = H - E_1$ and $D_2 = 2H - E_2 - E_3$, we get that $d \cdot D_1 = d_1$ and $d \cdot D_2 = d_2 + d_3$. As $E_{\mathrm{dP}_3(0,2)}$ is deformation equivalent to $E_{\mathrm{dP}_3(1,1)}$ by Proposition 2.6, Conjecture 4.4 and Theorem 4.5 give the following conjecture.

Conjecture 4.7 *The generating function* $\mathsf{N}_d^{\log}(\mathrm{dP}_3(0,2))(\hbar)$ *equals*

$$\frac{[d_1]_q [d_2+d_3]_q}{[d_0]_q [d_1+d_2+d_3-d_0]_q} \begin{bmatrix} d_3 \\ d_0 - d_1 \end{bmatrix}_q \begin{bmatrix} d_3 \\ d_0 - d_2 \end{bmatrix}_q \begin{bmatrix} d_0 \\ d_3 \end{bmatrix}_q \begin{bmatrix} d_1+d_2+d_3-d_0 \\ d_3 \end{bmatrix}_q, \tag{4-29}$$

where $q = \mathrm{e}^{\mathrm{i}\hbar}$.

Theorem 5.1 in Section 5.1 implies that Conjecture 4.7 holds in the classical limit $q^{\frac{1}{2}} = 1$. Direct scattering computation for $\mathrm{dP}_r(0, 5-r)$ with $r > 1$ are particularly daunting owing to the presence of infinite scattering, and in particular the final formulas take the shape of somewhat intricate multiple q–sums, which Conjecture 4.7 predicts should take a remarkably simple q–binomial form. We exemplify this for the blowdown geometries $\mathrm{dP}_1(0,4)$ and $\mathbb{F}_0(0,4)$ in Section B. For these cases, the specialisation of Conjecture 4.7 reduces to nontrivial, and apparently novel, conjectural q–binomial identities; see eg Conjecture B.3.

4.5 Toric models: $l = 3$

For $l = 3$, recall from (4-2) (resp. (4-1)) that $N_{0,d}^{\log,\psi}(Y(D))$ (resp. $N_{0,d}^{\log}(Y(D))$) is the genus-zero log Gromov–Witten invariant of maximal tangency passing through one point with psi-class (resp. passing through 2 points). By Proposition 4.3, it is enough to treat $\mathrm{dP}_3(0,0,0)$ as the other cases are obtained from it by interior blowdowns. We leave the description of the other toric models as an exercise to the reader.

Via Proposition 4.1, the invariant $N_{0,d}^{\log,\psi}(Y(D))$ is calculated from the scattering diagram as a structural coefficient of the product of three theta functions. For each constellation of three broken lines, the union of these corresponds to a tropical curve in the degeneration encoded by the scattering diagram. It is counted with multiplicity given by the product of the coefficients of the final monomials of the broken lines. Using Proposition 4.2, one can compute the generating series $\mathsf{N}_d^{\log}(Y(D))(\hbar)$ of higher-genus 2–point log Gromov–Witten invariants. The relevant tropical curves are identical to those entering the computation of $N_{0,d}^{\log,\psi}(Y(D))$. The difference is in the weighting of the tropical curves. For the ψ class, the trivalent vertex at the endpoint of the broken lines carry weight 1. For the 2–point invariants, we consider quantum broken lines, and the trivalent vertex is counted with Block–Göttsche multiplicity.

Let $Y(D) = \mathrm{dP}_3(0,0,0)$. We take D_1 in class $H-E_3$, D_2 in class $H-E_2$, D_3 in class $H-E_1$ and $d = d_0(H-E_1-E_2-E_3)+d_1E_1+d_2E_2+d_3E_3$. Then $d\cdot D_1 = d_3$, $d\cdot D_2=d_2$, $d\cdot D_3 = d_1$, $d\cdot E_1 = d_0-d_1$, $d\cdot E_2 = d_0-d_2$ and $d\cdot E_3 = d_0-d_3$. The calculations of Figure 6 give for the broken line ① the contribution

$$\binom{d_1}{d_0-d_2} t_2^{d_0-d_2} x^{-d_1} y^{d_2-d_0}, \tag{4-30}$$

for the broken line ② the contribution

$$\binom{d_2}{d_0-d_3} t_3^{d_0-d_3} x^{d_0-d_3} y^{d_0-d_2-d_3}, \tag{4-31}$$

and from the broken line ③ the contribution

$$\binom{d_3}{d_0-d_1} t_1^{d_0-d_1} x^{d_3+d_1-d_0} y^{d_3}. \tag{4-32}$$

Taken together, we obtain the following result.

Theorem 4.8 *We have*

$$N_{0,d}^{\log,\psi}(\mathrm{dP}_3(0,0,0)) = \binom{d_1}{d_0-d_2}\binom{d_2}{d_0-d_3}\binom{d_3}{d_0-d_1}. \tag{4-33}$$

For the 2–point invariant, the tropical multiplicity at p is

$$\left|\det\begin{pmatrix} d_1 & d_1+d_3-d_0 \\ -d_2+d_0 & d_3\end{pmatrix}\right| = |d_1d_3+d_1d_2+d_2d_3-d_0d_2-d_0d_1-d_0d_3+d_0^2|. \tag{4-34}$$

For the invariant to be nonzero, the curve class needs to lie in the effective cone determined (see Proposition 2.5) by

$$d_0\ge 0,\quad d_i\ge 0,\quad d_1+d_2+d_3\ge d_0. \tag{4-35}$$

Also, for the binomial coefficients to be nonzero, the curve class needs to satisfy the equations

$$0\le d_0-d_2\le d_1,\quad 0\le d_0-d_3\le d_2,\quad 0\le d_0-d_1\le d_3. \tag{4-36}$$

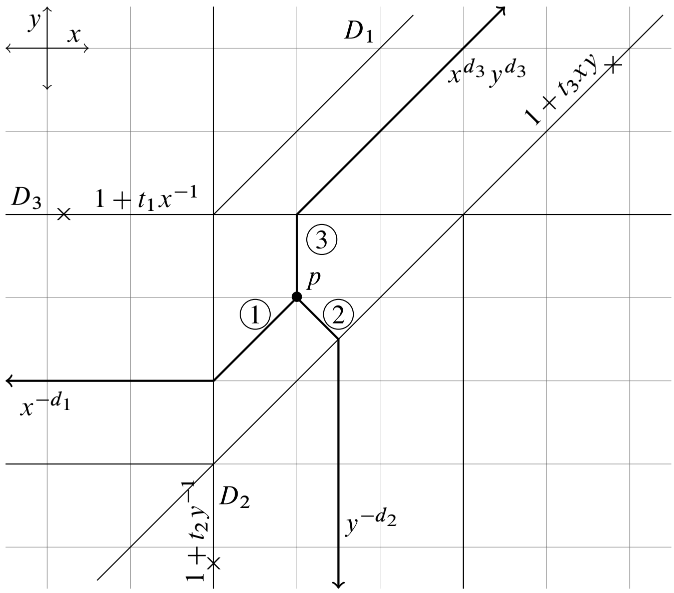


Figure 6: $\mathrm{dP}_3(0,0,0)$.

These inequalities determine a cone. Using the *Polyhedra* package of Macaulay2, in the basis

$$(H-E_1-E_2-E_3, E_1, E_2, E_3)$$

we find extremal rays generated by

$$(1,1,1,0), \quad (1,1,0,1), \quad (1,0,1,1), \quad (2,1,1,1). \tag{4-37}$$

Using this as a new basis, we find that the quadratic form in (4-34) is given by

$$xy+xz+yz+w(x+y+z)+w^2, \tag{4-38}$$

which is always positive in the cone. Therefore, we have proven the following result.

Theorem 4.9 *The generating function* $\mathsf{N}_d^{\log}(\mathrm{dP}_3(0,0,0))(\hbar)$ *equals*

$$\frac{[d_0^2-d_1(d_0-d_2)-d_2(d_0-d_3)-d_3(d_0-d_1)]_q}{[1]_q}\begin{bmatrix} d_1 \\ d_0-d_2\end{bmatrix}_q\begin{bmatrix} d_2 \\ d_0-d_3\end{bmatrix}_q\begin{bmatrix} d_3 \\ d_0-d_1\end{bmatrix}_q, \tag{4-39}$$

where $q=\mathrm{e}^{\mathrm{i}\hbar}$.

4.6 Toric models: $l=4$

There is only one 4–component log Calabi–Yau surface with maximal boundary, namely the toric surface $\mathbb{F}_0(0,0,0,0)$. For $d=d_1H_1+d_2H_2$, through tropical correspondence [92; 97; 85; 86], we calculated in [18] that

$$N_{0,d}^{\log,\psi}(\mathbb{F}_0(0,0,0,0))=1 \quad \text{and} \quad N_{0,d}^{\log}(\mathbb{F}_0(0,0,0,0))=d_1^2d_2^2. \tag{4-40}$$

To obtain the higher-genus invariant, we replace the tropical multiplicities by the Block–Göttsche multiplicities [10]. Applying [12] we obtain the following result.

Theorem 4.10 *We have*

$$\mathsf{N}_d^{\log}(\mathbb{F}_0(0,0,0,0))(\hbar) = \frac{[d_1 d_2]_q^2}{[1]_q^2}. \tag{4-41}$$

5 Log-local correspondence

In this section, we prove the following log-local correspondence theorem.

Theorem 5.1 *For every nef Looijenga pair $Y(D)$, the genus-zero log invariants $N_{0,d}^{\log}(Y(D))$ and the genus-zero local invariants $N_{0,d}^{\mathrm{loc}}(Y(D))$ are related by*

$$N_{0,d}^{\mathrm{loc}}(Y(D)) = \left(\prod_{j=1}^{l} \frac{(-1)^{d\cdot D_j - 1}}{(d\cdot D_j)}\right) N_{0,d}^{\log}(Y(D)). \tag{5-1}$$

The proof will be divided into two parts. In Section 5.1, we prove the result for $l = 2$ by a degeneration to the normal cone argument. In Section 5.2, we prove the result for $l = 3$ and $l = 4$ by direct comparison of the local results of Section 3 with the log results of Section 4.

5.1 Log-local for 2 components

For convenience in the following proof, we state separately the case $l = 2$ of Theorem 5.1.

Theorem 5.2 *For every 2–component nef Looijenga pair $Y(D)$, we have*

$$N_{0,d}^{\mathrm{loc}}(Y(D)) = \left(\prod_{j=1}^{l} \frac{(-1)^{d\cdot D_j - 1}}{(d\cdot D_j)}\right) N_{0,d}^{\log}(Y(D)). \tag{5-2}$$

The proof of Theorem 5.2 takes the remainder of Section 5.1, and is a degeneration argument in log Gromov–Witten theory.

5.1.1 Construction of the degeneration

We first construct the relevant degeneration for a general l–component nef Looijenga pair $Y(D) = (Y, D_1 + \cdots + D_l)$.

Let $\overline{\nu}_{\overline{\mathcal{Y}}} \colon \overline{\mathcal{Y}} \to \mathbb{A}^1$ be the degeneration of Y to the normal cone of D, obtained by blowing up $D \times \{0\}$ in $Y \times \mathbb{A}^1$. Irreducible components of the special fibre $\overline{\mathcal{Y}}_0 := \overline{\nu}_{\overline{\mathcal{Y}}}^{-1}(0)$ are Y and, for every $1 \le j \le l$, $\overline{\mathbb{P}}_j := \mathbb{P}(\mathcal{O} \oplus N_{D_j|Y})$, where $N_{D_j|Y}$ is the normal bundle to D_j in Y. For every double point $p \in D_j \cap D_{j'}$ of D, a local description of $\overline{\mathcal{Y}}_0$ is given by Figure 7, left. In particular, we have a point p^∂ in $\overline{\mathcal{Y}}_0$ where

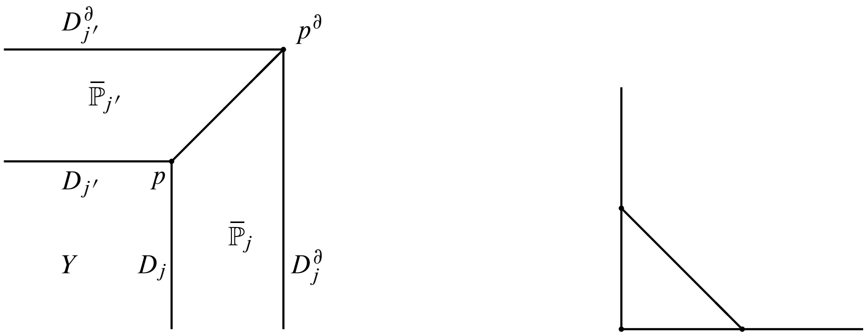


Figure 7: Left: local description of $\overline{\mathcal{Y}}_0$. Right: toric polyhedral decomposition of $\mathbb{R}^2_{\geq 0}$ describing locally $\overline{\mathcal{Y}}_0$ (fan picture).

the total space $\overline{\mathcal{Y}}$ is singular. This can be seen as follows. Locally near a double point $p \in D_{j'} \cap D_j$, the degeneration to the normal cone admits a toric description, whose fan is given by the closure of the cone over the polyhedral decomposition of $\mathbb{R}^2_{\geq 0}$ in Figure 7, right. The point p^{∂} corresponds to the 3–dimensional cone obtained by taking the closure of the cone over the unbounded region of $\mathbb{R}^2_{\geq 0}$ in Figure 7, right. This cone is generated by four rays, so is not simplicial and so p^{∂} is a singular point. More precisely, p^{∂} is an ordinary double point in $\overline{\mathcal{Y}}$. Every singular point of $\overline{\mathcal{Y}}$ is of the form p^{∂} for p a double point of D.

We resolve the singularities of $\overline{\mathcal{Y}}$ by blowing up the ordinary double points p^{∂}, and we obtain a new degeneration $\nu_{\mathcal{Y}} \colon \mathcal{Y} \to \mathbb{A}^1$. The total space $\mathcal{Y}$ is now smooth and the special fibre $\mathcal{Y}_0 := \nu_{\mathcal{Y}}^{-1}(0)$ is a normal crossings divisor on $\mathcal{Y}$. We view $\mathcal{Y}$ as a log scheme for the divisorial log structure defined by $\mathcal{Y}_0 \subset \mathcal{Y}$. Viewing $\mathbb{A}^1$ as a log scheme for the divisorial log structure defined by $\{0\} \subset \mathbb{A}^1$, the morphism $\nu_{\mathcal{Y}} \colon \mathcal{Y} \to \mathbb{A}^1$ can naturally be viewed as a log smooth log morphism.

Irreducible components of $\mathcal{Y}_0$ consist of Y, the strict transform $\mathbb{P}_j$ of the $\overline{\mathbb{P}}_j$ for every $1 \leq j \leq l$, and for every double point p of D the exceptional divisor $\mathbb{S}_p \simeq \mathbb{P}^1 \times \mathbb{P}^1$ created by the blowup of p^{∂}. Locally near a double point $p \in D_j \cap D_{j'}$, irreducible components of $\mathcal{Y}_0$ are glued together as in Figure 8, left. Locally near p, the total space $\mathcal{Y}$ admits a toric description whose fan is the closure of the cone over the polyhedral decomposition of $\mathbb{R}^2_{\geq 0}$ given in Figure 8, right. We remark that the log structure that we consider on $\mathcal{Y}$ is only partially compatible with this local toric description: one needs to remove from the toric boundary the horizontal toric divisors in order to obtain the divisorial log structure defined by the special fibre.

For every $1 \leq j \leq l$, let $\mathcal{D}_j$ be the closure in $\mathcal{Y}$ of the divisor $D_j \times (\mathbb{A}^1 - \{0\}) \subset Y \times (\mathbb{A}^1 - \{0\})$. We have

$$\mathcal{O}_{\mathcal{Y}}(-\mathcal{D}_j)|_{\mathcal{Y}_0} = \mathcal{O}_{\mathcal{Y}_0}\Big(-\Big(D_j^{\partial} \cup \bigcup_{p \in D_j} D_{j,p}^{\partial}\Big)\Big), \tag{5-3}$$

where the union is taken over the double points p of D contained in D_j.

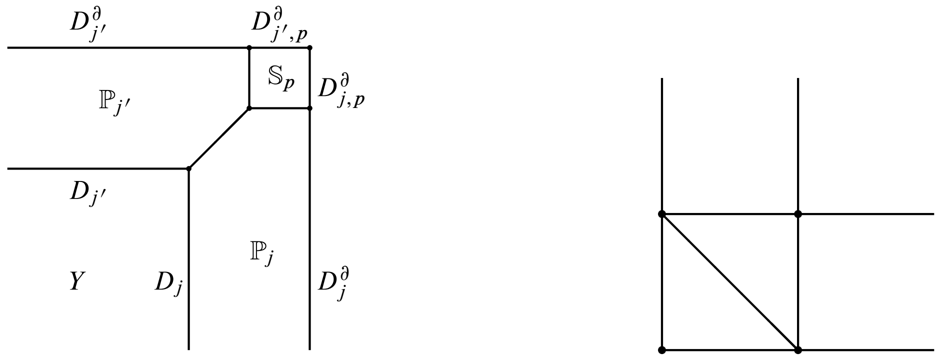


Figure 8: Left: local description of $\mathcal{Y}_0$. Right: toric polyhedral decomposition of $\mathbb{R}^2_{\geq 0}$ describing locally $\mathcal{Y}_0$ (fan picture).

We define $\mathcal{V} := \mathrm{Tot}\big(\bigoplus_{j=1}^{l} \mathcal{O}_{\mathcal{Y}}(-\mathcal{D}_j)\big)$ and denote by $\pi_{\mathcal{V}}\colon \mathcal{V} \to \mathcal{Y}$ and $\nu_{\mathcal{V}}\colon \mathcal{V} \to \mathbb{A}^1$ the natural projections. We also denote by $\mathcal{V}_0 := \nu_{\mathcal{V}}^{-1}(0)$ the special fibre and by $\pi_{\mathcal{V}_0}\colon \mathcal{V}_0 \to \mathcal{Y}_0$ the restriction of $\pi_{\mathcal{V}}$ to the special fibre.

The irreducible components of $\mathcal{V}_0$ are

$$\mathcal{V}_{0,Y} := \mathrm{Tot}(\mathcal{O}_Y^{\oplus l}), \tag{5-4}$$

$$\mathcal{V}_{0,j} := \mathrm{Tot}(\mathcal{O}_{\mathbb{P}_j}(-D^{\partial}_j) \oplus \mathcal{O}_{\mathbb{P}_j}^{\oplus(l-1)}) \quad \text{for every } 1 \leq j \leq l, \tag{5-5}$$

and, for every double point $p \in D_j \cap D_{j'}$ of D,

$$\mathcal{V}_{0,p} := \mathrm{Tot}(\mathcal{O}_{\mathbb{S}_p}(-D^{\partial}_{j,p}) \oplus \mathcal{O}_{\mathbb{S}_p}(-D^{\partial}_{j',p}) \oplus \mathcal{O}_{\mathbb{S}_p}^{\oplus(l-2)}). \tag{5-6}$$

We view $\mathcal{V}$ as a log scheme for the divisorial log structure defined by $\mathcal{V}_0 \subset \mathcal{V}$, and then $\nu_{\mathcal{V}}\colon \mathcal{V} \to \mathbb{A}^1$ is naturally a log smooth log morphism. We remark that the log structure on $\mathcal{V}$ is the pullback of the log structure on $\mathcal{Y}$, ie the log morphism $\pi_{\mathcal{V}}\colon \mathcal{V} \to \mathcal{Y}$ is strict. In particular, $\mathcal{V}$ and $\mathcal{Y}$ have identical tropicalisations.

For every $1 \leq j \leq l$, we consider the projectivisation $\mathbb{P}(\mathcal{O}_{\mathcal{Y}}(-\mathcal{D}_j) \oplus \mathcal{O}_{\mathcal{Y}})$ of $\mathcal{O}_{\mathcal{Y}}(-\mathcal{D}_j)$ and the corresponding fibrewise compactification

$$\boldsymbol{P} := \mathbb{P}(\mathcal{O}_{\mathcal{Y}}(-\mathcal{D}_1) \oplus \mathcal{O}_{\mathcal{Y}}) \times_{\mathcal{Y}} \cdots \times_{\mathcal{Y}} \mathbb{P}(\mathcal{O}_{\mathcal{Y}}(-\mathcal{D}_l) \oplus \mathcal{O}_{\mathcal{Y}}) \tag{5-7}$$

of $\mathcal{V}$. We denote by $\pi_{\boldsymbol{P}}\colon \boldsymbol{P} \to \mathcal{Y}$ and $\nu_{\boldsymbol{P}}\colon \boldsymbol{P} \to \mathbb{A}^1$ the natural projections. We also denote by $\boldsymbol{P}_0 := \nu_{\boldsymbol{P}}^{-1}(0)$ the special fibre and by $\pi_{\boldsymbol{P}_0}\colon \boldsymbol{P}_0 \to \mathcal{Y}_0$ the restriction of $\pi_{\boldsymbol{P}}$ to the special fibre. We denote by $\boldsymbol{P}_{0,Y}$, $\boldsymbol{P}_{0,j}$ and $\boldsymbol{P}_{0,p}$ the irreducible components of $\boldsymbol{P}_0$ obtained by compactification of $\mathcal{V}_{0,Y}$, $\mathcal{V}_{0,j}$ and $\mathcal{V}_{0,p}$.

We view $\boldsymbol{P}$ as a log scheme for the divisorial log structure defined by $\boldsymbol{P}_0 \subset \boldsymbol{P}$, and then $\nu_{\boldsymbol{P}}\colon \boldsymbol{P} \to \mathbb{A}^1$ is naturally a log smooth log morphism. We remark that the log structure on $\boldsymbol{P}$ is the pullback of the log structure on $\mathcal{Y}$, ie the log morphism $\pi_{\boldsymbol{P}}\colon \boldsymbol{P} \to \mathcal{Y}$ is strict. In particular, $\boldsymbol{P}$ and $\mathcal{Y}$ have identical tropicalisations.

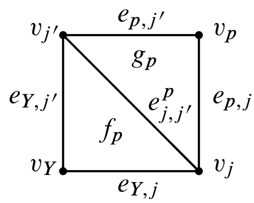


Figure 9: Local description of Δ.

Let Δ be the polyhedral complex obtained by taking the fibre over 1 of the tropicalisation of $\nu_{\boldsymbol{P}} : \boldsymbol{P} \to \mathbb{A}^1$. Combinatorially, Δ is the dual intersection complex of the special fibre $\mathcal{V}_0$; see Figure 9. Vertices of Δ consist of

- v_Y corresponding to the irreducible component $\boldsymbol{P}_{0,\mathcal{Y}}$,
- v_j corresponding to the irreducible component $\boldsymbol{P}_{0,j}$ for every $1 \le j \le l$,
- v_p corresponding to the irreducible component $\boldsymbol{P}_{0,p}$ for every double point p of D.

Edges of Δ consist of

- $e_{Y,j}$ connecting v_Y and v_j for every $1 \le j \le l$, corresponding to the divisor $\boldsymbol{P}_{0,Y} \cap \boldsymbol{P}_{0,j}$,
- $e^p_{j,j'}$ connecting v_j and $v_{j'}$ for every double point $p \in D_j \cap D_{j'}$ of D, corresponding to the component of the divisor $\boldsymbol{P}_{0,j} \cap \boldsymbol{P}_{0,j'}$ containing p,
- $e_{p,j}$ connecting v_p and v_j for every double point $p \in D_j \cap D_{j'}$ of D, corresponding to the divisor $\boldsymbol{P}_{0,j} \cap \boldsymbol{P}_{0,p}$, and $e_{p,j'}$ connecting v_p and $v_{j'}$, corresponding to the divisor $\boldsymbol{P}_{0,j'} \cap \boldsymbol{P}_{0,p}$.

Faces of Δ consist of

- a triangle f_p of sides $e_{Y,j}$, $e_{Y,j'}$, $e^p_{j,j'}$ for every double point $p \in D_j \cap D_{j'}$ of D, corresponding to the triple intersection $\boldsymbol{P}_{0,Y} \cap \boldsymbol{P}_{0,j} \cap \boldsymbol{P}_{0,j'}$,
- a triangle g_p of sides $e^p_{j,j'}$, $e_{p,j}$, $e_{p,j'}$ for every double point $p \in D_j \cap D_{j'}$ of D, corresponding to the triple intersection $\boldsymbol{P}_{0,p} \cap \boldsymbol{P}_{0,j} \cap \boldsymbol{P}_{0,j'}$.

As we are assuming that the components of D form a cycle, the boundary $\partial\Delta$ of Δ can be described as

$$\partial\Delta = \bigcup_{1 \le j \le l} (\partial\Delta)_j, \tag{5-8}$$

where for every $1 \le j \le l$,

$$(\partial\Delta)_j := \bigcup_{p \in D_j \cap D_{j'}} e_{p,j}. \tag{5-9}$$

We view $\boldsymbol{P}_0$ as a log scheme by restriction of the log structure on $\boldsymbol{P}$. We denote by $\nu_{\boldsymbol{P}_0} : \boldsymbol{P}_0 \to \mathrm{pt}_{\mathbb{N}}$ the corresponding log smooth log morphism to the standard log point. We view the curve class d as a class on $\boldsymbol{P}_0$ via the embedding $\mathcal{Y}_0 \to \boldsymbol{P}_0$ induced by the zero section of $\mathcal{V}_0$. Let $\overline{\mathrm{M}}_{0,m}(Y^{\mathrm{loc}}(D), d)$ be the moduli space of genus-zero class d stable log maps to $\nu_{\boldsymbol{P}_0}\colon \boldsymbol{P}_0 \to \mathrm{pt}_{\mathbb{N}}$ with m marked points with contact order 0 with $\boldsymbol{P}_{0,Y}$. Let $[\overline{\mathrm{M}}_{0,m}(\boldsymbol{P}_0,d)]^{\mathrm{vir}}$ be the corresponding virtual fundamental class, of dimension $l-1+m$. Using the nefness of the divisors D_j, the condition $d\cdot D_j>0$ for every $1\le j\le l$, and the deformation invariance of log Gromov–Witten invariants, we have

$$N_{0,d}^{\mathrm{loc}}(Y(D)) = \int_{[\overline{\mathrm{M}}_{0,l-1}(\boldsymbol{P}_0,d)]^{\mathrm{vir}}} \prod_{k=1}^{l-1} \mathrm{ev}_k^*(\pi_{\boldsymbol{P}_0}^*[\mathrm{pt}_Y]), \tag{5-10}$$

where ev_k is the evaluation at the k^{th} interior marked point and $[\mathrm{pt}_Y]$ is the class of a point on $Y\subset\mathcal{Y}_0$.

5.1.2 Degeneration formula According to the decomposition formula of Abramovich, Chen, Gross and Siebert [2], we have

$$[\overline{\mathrm{M}}_{0,l-1}(\boldsymbol{P}_0,d)]^{\mathrm{vir}} = \sum_{h\colon\Gamma\to\Delta} \frac{m_h}{|\mathrm{Aut}(h)|}[\overline{\mathrm{M}}_{0,l-1}^h(\boldsymbol{P}_0,d)]^{\mathrm{vir}}. \tag{5-11}$$

The sum is over the genus-zero rigid decorated parametrised tropical curves $h\colon\Gamma\to\Delta$, where Γ has $l-1$ unbounded edges, all contracted by h to v_Y, and the sum of classes attached to the vertices of Γ is d. The moduli space $\overline{\mathrm{M}}_{0,l-1}^h(\boldsymbol{P}_0,d)$ parametrises genus-zero class d stable log maps to $\nu_{\boldsymbol{P}_0}\colon\boldsymbol{P}_0\to\mathrm{pt}_{\mathbb{N}}$ marked by h.

Therefore, we have

$$N_{0,d}^{\mathrm{loc}}(Y(D)) = \sum_{h\colon\Gamma\to\Delta}\frac{m_h}{|\mathrm{Aut}(h)|}N_{0,d}^{\mathrm{loc},h}(Y(D)), \tag{5-12}$$

where

$$N_{0,d}^{\mathrm{loc},h}(Y(D)) := \int_{[\overline{\mathrm{M}}_{0,l-1}^h(\boldsymbol{P}_0,d)]^{\mathrm{vir}}} \prod_{k=1}^{l-1} \mathrm{ev}_k^*(\pi_{\boldsymbol{P}_0}^*[\mathrm{pt}_Y]), \tag{5-13}$$

and ev_k is the evaluation at the k^{th} marked point. Thus, for every $h\colon\Gamma\to\Delta$, we have to compute $N_d^{\mathrm{loc},h}(Y(D))$.

Let Δ^h be a polyhedral complex obtained by refining the polyhedral decomposition of Δ and containing the $h(\Gamma)$ in its one-skeleton, ie such that, for every vertex V of Γ, $h(V)$ is a vertex of Δ^h, and for every edge E of Γ, $h(E)$ is an edge of Δ^h. We denote by $\mathcal{Y}_0^h$, $\mathcal{V}_0^h$ and $\boldsymbol{P}_0^h$ the corresponding log modifications of $\mathcal{Y}_0$, $\mathcal{V}_0$ and $\boldsymbol{P}_0$. Let $\overline{\mathrm{M}}_{0,l-1}^h(\boldsymbol{P}_0^h,d)$ the moduli space of stable log maps to $\boldsymbol{P}_0^h$ marked by h. By the invariance of log Gromov–Witten invariants under log modification [3], we have

$$N_{0,d}^{\mathrm{loc},h}(Y(D)) := \int_{[\overline{\mathrm{M}}_{0,l-1}^h(\boldsymbol{P}_0^h,d)]^{\mathrm{vir}}} \prod_{k=1}^{l-1} \mathrm{ev}_k^*(\pi_{\boldsymbol{P}_0^h}^*[\mathrm{pt}_Y]). \tag{5-14}$$

For every vertex V of Γ, let $\boldsymbol{P}_V$ be the irreducible component of $\boldsymbol{P}_0^h$ corresponding to the vertex $h(V)$ of Δ^h. We view $\boldsymbol{P}_V$ as a log scheme for the divisorial log structure defined by the divisor $\partial\boldsymbol{P}_V$, which is the union of intersection divisors with the other irreducible components of $\boldsymbol{P}_0^h$. Similarly, we define the component Y_V of $\mathcal{Y}_0^h$ and ∂Y_V, so that $\boldsymbol{P}_V$ is a $(\mathbb{P}^1)^l$–bundle over Y_V.

If $h(V) \in \Delta^h - \partial\Delta^h$, then $\boldsymbol{P}_V$ is the trivial $(\mathbb{P}^1)^l$–bundle over Y_V. If furthermore, $h(V) \notin \bigcup_{j=1}^l e_{Y,j}$, then $(Y_V, \partial Y_V)$ is a toric variety with its toric boundary.

If $h(V) \in e_{p,j} - v_p$ for some $p \in D_j \cap D_{j'}$, let $D_{j,V}^{\partial}$ be the irreducible component of $\mathcal{D}_j \cap \mathcal{Y}_0^h$ contained in Y_V. Then $\boldsymbol{P}_V$ is the fibrewise product over Y_V of the $\mathbb{P}^1$–bundle

$$\mathbb{P}\big(\mathcal{O}_{Y_V}(-D_{j,V}^{\partial}) \oplus \mathcal{O}_{Y_V}\big) \tag{5-15}$$

with the trivial $(\mathbb{P}^1)^{l-1}$–bundle. Moreover, $(Y_V, \partial Y_V \cup D_{j,V}^{\partial})$ is a toric variety with its toric boundary.

If $h(V) = v_p$ for some $p \in D_j \cap D_{j'}$, then, still denoting by $\mathbb{S}_p$, $D_{j,p}^{\partial}$ and $D_{j,p}^{\partial}$ the strict transforms in $\mathcal{Y}_0^h$ of $\mathbb{S}_p$, $D_{j,p}^{\partial}$, and $D_{j,p}^{\partial}$, $\boldsymbol{P}_V$ is the fibrewise product over $Y_V = \mathbb{S}_p$ of the $\mathbb{P}^1$–bundle

$$\mathbb{P}\big(\mathcal{O}_{\mathbb{S}_p}(-D_{j,p}^{\partial}) \oplus \mathcal{O}_{\mathbb{S}_p}\big), \tag{5-16}$$

of the $\mathbb{P}^1$–bundle

$$\mathbb{P}\big(\mathcal{O}_{\mathbb{S}_p}(-D_{j',p}^{\partial}) \oplus \mathcal{O}_{\mathbb{S}_p}\big), \tag{5-17}$$

and of the trivial $(\mathbb{P}^1)^{l-2}$–bundle. Moreover, $(Y_V, \partial Y_V \cup D_{j,p}^{\partial} \cup D_{j',p}^{\partial})$ is a toric variety with its toric boundary.

For every vertex V of Γ, let M_V be the moduli space of genus-zero stable log maps to $\boldsymbol{P}_{0,V}^h$, with class given by the class decoration of V, and contact orders specified by the local behaviour of h around V. Our goal is to compute the invariant $N_d^{\mathrm{loc},h}(Y(D))$ in terms of the virtual classes $[M_V]^{\mathrm{vir}}$. We are in a particularly favourable situation: we consider curves of genus zero and the dual intersection complex Δ^h has dimension 2. In such case, the degeneration formula in log Gromov–Witten theory has a particular simple form, as described in Section 6.5.2 of [105]; see also Section 4 of [104] for the corresponding discussion in the language of exploded manifolds.

We choose a flow on Γ such that unbounded edges are incoming and such that every vertex has at most one outgoing edge. Such flow exists as Γ has genus zero and then there is exactly one vertex, which we denote by V_0, without outgoing incident edge, and which we call the *sink* of the flow. All vertices distinct from V_0 have exactly one outgoing edge. In fact, for every vertex V of Γ, we can find such flow with sink $V_0 = V$.

For every edge E of Γ, we denote by $\boldsymbol{P}_E$ the stratum of $\boldsymbol{P}_0^h$ dual to E. The stratum $\boldsymbol{P}_E$ is a divisor if E is bounded and is the irreducible component $\mathbb{P}_V$ if E is unbounded and incident to the vertex V. For every E, $\mathbb{P}_E$ is a $(\mathbb{P}^1)^l$–bundle over a stratum Y_E of $\mathcal{Y}_0^h$, and we denote by $\pi_E\colon \boldsymbol{P}_E \to Y_E$ the corresponding projection.

For every edge E incident to a vertex V, we have the evaluation map

$$\mathrm{ev}_{V,E}\colon M_V \to \boldsymbol{P}_E. \tag{5-18}$$

For every vertex V distinct from V_0, let $\mathcal{E}_{\mathrm{in}}(V)$ be the set of incoming incident edges to V, and let E_V be the outgoing incident edge to V. The virtual class $[M_V]^{\mathrm{virt}}$ defines a map

$$\eta_V\colon \prod_{E\in\mathcal{E}_{\mathrm{in}}(V)} \mathrm{H}^*(\boldsymbol{P}_E) \to \mathrm{H}^*(\boldsymbol{P}_{E_V}) \tag{5-19}$$

by

$$\eta_V\Big(\prod_{E\in\mathcal{E}_{\mathrm{in}}(V_0)} \alpha_E\Big) := (\mathrm{ev}_{V,E_V})_*\Big(\Big(\prod_{E\in\mathcal{E}_{\mathrm{in}}(V)} \mathrm{ev}^*_{V,E}\,\alpha_E\Big)\cap [M_V]^{\mathrm{virt}}\Big). \tag{5-20}$$

Note that if $\mathcal{E}_{\mathrm{in}}(V)$ is empty, then η_V is a map of the form

$$\eta_V\colon \mathbb{Q} \to \mathrm{H}^*(\boldsymbol{P}_{E_V}). \tag{5-21}$$

Denote by $\mathcal{E}_{\mathrm{in}}(V_0)$ the set of incoming incident edges to V_0. The virtual class $[M_{V_0}]^{\mathrm{virt}}$ defines a map

$$\eta_{V_0}\colon \prod_{E\in\mathcal{E}_{\mathrm{in}}(V_0)} \mathrm{H}^*(\boldsymbol{P}_E) \to \mathbb{Q} \tag{5-22}$$

by

$$\eta_{V_0}\Big(\prod_{E\in\mathcal{E}_{\mathrm{in}}(V_0)} \alpha_E\Big) := \int_{[M_{V_0}]^{\mathrm{virt}}} \prod_{E\in\mathcal{E}_{\mathrm{in}}(V_0)} \mathrm{ev}^*_{V_0,E}\,\alpha_E. \tag{5-23}$$

Denote by $\mathcal{E}_\infty(\Gamma)$ the set of unbounded edges of Γ. Composing the maps η_V and η_{V_0}, we obtain a map

$$\eta_h\colon \prod_{E\in\mathcal{E}_\infty(\Gamma)} \mathrm{H}^*(\boldsymbol{P}_E) \to \mathbb{Q}. \tag{5-24}$$

For every edge E of Γ, let $\mathrm{pt}_E \in \mathrm{H}^2(Y_E)$ be the class of a point on Y_E. We consider the class $\pi_E^*\mathrm{pt}_E \in \mathrm{H}^2(\boldsymbol{P}_E)$. The degeneration formula is then

$$N_d^{\mathrm{loc},h}(Y(D)) = \eta_h\Big(\prod_{E\in\mathcal{E}_\infty(\Gamma)} \pi_E^*\mathrm{pt}_E\Big). \tag{5-25}$$

We define a rigid genus-zero parametrised tropical curve $\overline{h}\colon\overline{\Gamma}\to\Delta$ as follows. Let $\overline{\Gamma}$ be the star-shaped graph consisting of vertices V_j for $0\le j\le l$, and edges E_j connecting V_0 and V_j for $1\le j\le l$. We assign the length $1/(d\cdot D_j)$ to the edge E_j. Let $\overline{h}\colon\overline{\Gamma}\to\Delta$ be the piecewise linear map such that $\overline{h}(V_0)=v_Y$ and $\overline{h}(V_j)=v_j$ for $1\le j\le l$. In particular, we have $\overline{h}(E_j)=e_{Y,j}$ for $1\le j\le j$. As $e_{Y,j}$ has integral length 1, we deduce that E_j has weight $d\cdot D_j$. Finally, curve classes decoration of the vertices are given by: $d_{V_0}=d$ and d_{V_j} is equal to $(d\cdot D_j)$ times the class of a $\mathbb{P}^1$–fibre of $\mathbb{P}_j$ for $1\le j\le l$.

Lemma 5.3 *We have*

$$N_{0,d}^{\mathrm{loc},\overline{h}}(Y(D)) = \Big(\prod_{j=1}^{l} \frac{(-1)^{d\cdot D_j-1}}{(d\cdot D_j)^2}\Big) N_{0,d}^{\log}(Y(D)). \tag{5-26}$$

Proof We choose the flow on Γ with sink V_0. Applying the degeneration formula gives immediately the result, using the fact that the normal bundle in $\boldsymbol{P}_0$ to a $\mathbb{P}^1$–fibre of $\mathbb{P}_j$ is $\mathcal{O}(-1) \oplus \mathcal{O}^{\oplus(l+1)}$ and so the corresponding multicover contribution is

$$\frac{(-1)^{d \cdot D_j - 1}}{(d \cdot D_j)^2}, \tag{5-27}$$

by [20, Proof of Theorem 5.1]. □

Theorem 5.4 *Assume $l = 2$. Let $h\colon \Gamma \to \Delta$ be a rigid decorated parametrised tropical curve as above with $N^{\mathrm{loc},h}_{0,d}(Y(D)) \neq 0$. Then $h = \bar{h}$.*

Theorem 5.4 follows from a judicious analysis of the possible topologies of contributing tropical curves, which we perform in Appendix C. Theorem 5.2 then follows from the combination of Theorem 5.4, Lemma 5.3, and the decomposition formula using that $|\mathrm{Aut}(\bar{h})| = 1$ and $m_{\bar{h}} = \prod_{j=1}^{l} (d \cdot D_j)$.

5.2 The log-local correspondence for 3 and 4 components

We end the proof of Theorem 5.1 for $l = 3$ and $l = 4$. For $l = 3$, it is enough to treat the case of $\mathrm{dP}_3(0,0,0)$, as all the other 3–component cases are obtained from it by blowup, and the result is preserved under blowup by combination of Propositions 3.2 and 4.3. The result for $\mathrm{dP}_3(0,0,0)$ follows by comparing the local result given by Theorem 3.5 with the log result given by Theorem 4.9.

For $l = 4$, the result follows by comparing the local result given by (3-30) and the log result given by (4-41).

6 Open Gromov–Witten theory

In this section we relate the quantised scattering calculations of Section 4 to the higher-genus open Gromov–Witten theory of Aganagic–Vafa A–branes. We first give in Section 6.1 an overview of the framework of [76] to cast open toric Gromov–Witten theory within the realm of formal relative invariants, and recall the topological vertex formalism of Aganagic, Klemm, Mariño and Vafa. Our treatment throughout this section, while self-contained, will keep the level of detail to the necessary minimum, and we refer the reader to [76; 39] for further details. The reader who is familiar with this material may wish to skip to Section 6.2, where the stable log counts of Section 4 are related to open Gromov–Witten theory, with the main statement condensed in Theorem 6.7, and proved in Section 6.3.

In the following, for a partition $\lambda \vdash d$ of $d \in \mathbb{N}$ we write $|\lambda| = d$ for the order of λ, $\ell_\lambda = r$ for the cardinality of the partitioning set, $\kappa_\lambda := \sum_{i=1}^{\ell_\lambda} \lambda_i(\lambda_i - 2i + 1)$ for its second Casimir invariant, and let $m_j(\lambda) := \#\{\lambda_i \mid \lambda_i = j\}_{i=1}^{\ell_\lambda}$ and $z_\lambda := \prod_j m_j(\lambda)!\, j^{m_j(\lambda)}$. We furthermore denote by $\mathcal{P}$ the set of partitions, and $\mathcal{P}^d$ the set of partitions of order d. We will extensively need, particularly in the proof of Theorem 6.7, some classical results on principally specialised shifted symmetric functions, for which notation and necessary basic results are collected in Appendix D.

6.1 Toric special Lagrangians

Let X be a smooth complex toric threefold with $K_X \simeq \mathcal{O}_X$. If the affinisation morphism to $\mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$ is projective, X can be realised as a symplectic quotient $\mathbb{C}^{r+3} /\!\!/ G$, where $G \simeq U(1)^r$ acts on the affine coordinates $\{z_i\}_{i=1}^{r+3}$ of $\mathbb{C}^{r+3} = \mathrm{Spec}\mathbb{C}[z_1, \dots, z_{r+3}]$ by

$$(t_1, \dots, t_r) \cdot (z_1, \dots, z_{r+3}) = \left(\prod_{i=1}^r t_i^{w_1^{(i)}} \cdot z_1, \dots, \prod_{i=1}^r t_i^{w_{r+3}^{(i)}} \cdot z_{r+3} \right),$$

where $w_j^{(i)} \in \mathbb{Z}$ for $i = 1, \dots, r$ and $j = 1, \dots, r+3$ are the weights of the G–action [61]. This is a Hamiltonian action with respect to the canonical Kähler form on $\mathbb{C}^{r+3}$,

$$\omega := \frac{\mathrm{i}}{2} \sum_{i=1}^{r+3} \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i, \tag{6-1}$$

with moment map

$$\tilde{\mu}(z_1, \dots, z_{r+3}) = \left(\sum_{i=1}^{r+3} w_i^{(1)} |z_i|^2, \dots, \sum_{i=1}^{r+3} w_i^{(r)} |z_i|^2 \right).$$

If $(t_1, \dots, t_k) \in \mathrm{H}^{1,1}(X; \mathbb{R}) \simeq (\mathfrak{u}(1)^r)^\star$ is a Kähler class, then X is the geometric quotient

$$X = \tilde{\mu}^{-1}(t_1, \dots, t_r)/G, \tag{6-2}$$

with symplectic structure given by the Marsden–Weinstein reduction ω_t of (6-1) onto the quotient (6-2), where $[\omega_t] = (t_1, \dots, t_r) \in \mathrm{H}^{1,1}(X; \mathbb{R})$.

We will be concerned with a class of special Lagrangian submanifolds $L = L_{\hat{w},c}$ of (X, ω_t) constructed by Aganagic and Vafa [7], which are invariant under the natural Hamiltonian torus action on X. They are defined by

$$\sum_{i=1}^{r+3} \hat{w}_i^1 |z_i|^2 = c, \quad \sum_{i=1}^{r+3} \hat{w}_i^2 |z_i|^2 = 0, \quad \sum_{i=1}^{r+3} \arg z_i = 0, \tag{6-3}$$

with $\hat{w}_i^a \in \mathbb{Z}$, $\sum_{i=1}^{r+3} \hat{w}_i^a = 0$ and $c \in \mathbb{R}$. These Lagrangians have the topology of $\mathbb{R}^2 \times S^1$, and they intersect a unique torus fixed curve C_L along an S^1: we say that L is an *inner* (resp. *outer*) brane if $C_L \simeq \mathbb{P}^1$ (resp. $\mathbb{C}$). Throughout the foregoing discussion we will assume that L is always an outer brane.

Let $T \simeq (\mathbb{C}^\star)^2$ be the algebraic subtorus of $(\mathbb{C}^\star)^3 \subset X$ acting trivially on K_X, and $T_\mathbb{R} \simeq U(1)^2$ be its maximal compact subgroup. Then by construction any toric Lagrangian L is preserved by $T_\mathbb{R}$, which acts on $\mathbb{C} \times S^1$ by scaling $(\lambda_1, \lambda_2) \cdot (w, \theta) \to (\lambda_1 w, \lambda_2 \theta)$. Writing $\mu_T : X \to \mathbb{R}^2 \simeq (\mathfrak{u}(1)^2)^*$ for the moment map of the $T_\mathbb{R}$–action, the union of the 1–dimensional $(\mathbb{C}^\star)^3$ orbit closures of X is mapped by μ_T to a planar trivalent metric graph Γ_X whose sets of vertices $(\Gamma_X)_0$, compact edges $(\Gamma_X)_1^{\mathrm{cp}}$ and noncompact edges $(\Gamma_X)_1^{\mathrm{nc}}$ correspond to T–fixed points, T–invariant proper curves, and T–invariant affine lines in X respectively. Since the moment map is an integral quadratic form, the tangent directions of the edges have rational slopes in $\mathbb{R}^2$: we can explicitly keep track of this information by regarding Γ_X as a topological graph[11] decorated by the assignment to each vertex $v \in (\Gamma_X)_0$ of primitive integral lattice vectors $\mathfrak{p}_v^e \in \mathbb{Z}^2$, representing the directions of the edges e emanating from $v \in (\Gamma_X)_0$. The graph Γ_X is determined bijectively by the weights $w_j^{(i)}$, and knowing it suffices to reconstruct X.

Remark 6.1 Let $\Sigma(X)$ be the fan of X. As $K_X \simeq \mathcal{O}_X$, $\Sigma(X)$ can be described as a cone in $\mathbb{R}^3$ over a polyhedral decomposition of an integral polygon P in $\mathbb{R}^2 \times \{1\} \subset \mathbb{R}^3$. The graph Γ_X can be obtained as the dual graph of the polyhedral decomposition of P taking orientations to be outgoing at every vertex. Conversely, one can recover (the $\mathrm{SL}(2,\mathbb{Z})$–equivalence class of) $P \subset \mathbb{R}^2$ and its decomposition as the dual polygon of Γ_X, and then $\Sigma(X)$ as the cone in $\mathbb{R}^3$ over $P \subset \mathbb{R}^2 \times \{1\}$.

If L is a toric outer Lagrangian, its image under μ_T is a point $\mu_T(L)$ lying on the noncompact edge $\mu_T(C)$ representing the curve it is incident to. Write $e_L := \mu_T(C)$, v for its adjacent vertex, and e'_L for the first edge met by moving clockwise from e_L with respect to the orientation determined by the plane containing Γ_X.

Definition 6.1 A *framing* of L is the choice of an integral vector f such that $\mathsf{p}_v^{e_L} \wedge \mathsf{p}_v^{e'_L} = \mathsf{p}_v^{e_L} \wedge \mathsf{f}$; equivalently, $\mathsf{f} = \mathsf{p}_v^{e'_L} - f\,\mathsf{p}_v^{e_L}$ for some $f \in \mathbb{Z}$. We say that L is canonically framed if $f = 0$, ie $\mathsf{f} = \mathsf{p}_v^{e'_L}$.

Remark 6.2 By construction, since $\mathsf{f} \wedge \mathsf{p}_v^{e_L} > 0$, a framing at an outer vertex is always pointing in the clockwise direction.

Definition 6.2 We call (X, L, f) a *toric Lagrangian triple* if

- X is a semiprojective toric CY3 variety,
- $L = \bigsqcup_i L_{\hat{w}_i, c_i}$ is a disjoint union of Aganagic–Vafa special Lagrangian submanifolds of X, and
- f is the datum of a framing choice for each connected component of L.

We will write $\Gamma_{(X,L,\mathsf{f})}$ for the graph obtained from Γ_X by the extra decoration of an integral vector incident to the edge e_L representing the toric outer Lagrangian L at framing f; see Figure 10.

Example 6.1 Let $w^{(1)} = (1,1,-1,-1)$, $\hat{w}^{(2)} = (1,0,-1,0)$ and $\hat{w}^{(3)} = (0,1,-1,0)$. For any $t \neq 0$, the corresponding toric variety X is the resolved conifold $\mathrm{Tot}\big(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)\big)$, with $\int_{0_*\mathbb{P}^1} \omega_t = t$. The compact edge e_5 corresponds to the $\mathbb{P}^1$ given by the zero section of X. The edges e_i for $i = 1,2,3,4$ correspond to the T–invariant $\mathbb{A}^1$–fibres above the points $[1:0]$ and $[0:1]$ of the $\mathbb{P}^1$ base. The weights $\hat{w}^{(i)}$ furthermore determine a toric Lagrangian, whose image in the toric graph lies in e_1, and is depicted in Figure 10 at framing $\mathsf{f} = \mathsf{p}_{v_1}^{e_5} - \mathsf{p}_{v_1}^{e_1}$.

[11] In doing so we forget the metric information about Γ_X which stems from a choice of a Kähler structure on X: this is inconsequential for the definition of the invariants in the next section. We thus make a slight abuse of notation, by indicating the decorated topological graph obtained by forgetting the information about the lengths of the edges by the same symbol Γ_X.

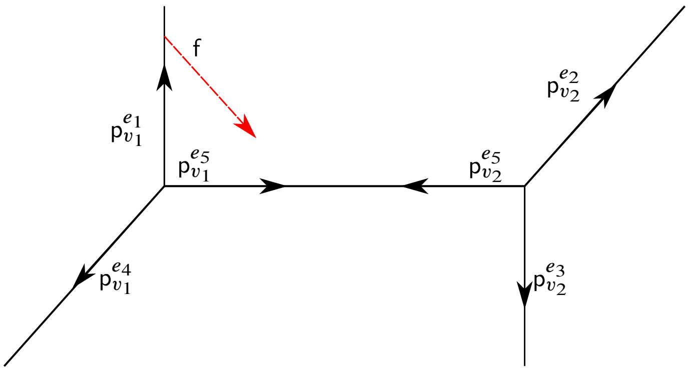


Figure 10: The toric Calabi–Yau graph $\Gamma_{(X,L,\mathsf{f})}$ of the resolved conifold with an outer Lagrangian at framing $\mathsf{f} = \mathsf{f}_{\mathrm{can}} - \mathsf{p}_{v_1}^{e_1}$, ie $f = 1$.

6.1.1 Open Gromov–Witten invariants In informal terms, the open Gromov–Witten theory of $(X, L = L_1 \cup \cdots \cup L_s)$ for toric Lagrangians L_i with $i = 1, \dots, s$ is a virtual count of maps to X from open Riemann surfaces of fixed genus, relative homology degree, and boundary winding data around $S^1 \hookrightarrow L$. This raises two orders of problems when trying to define these counts in the algebraic category, as the boundary conditions for the curve counts are imposed in odd real dimension, and the target geometry is noncompact. A strategy to address both issues simultaneously for framed outer toric Lagrangians, and which we will follow for the purposes for the paper, was put forward by Li, Liu, Liu and Zhou [76], which we briefly review below. The main idea in [76] is to replace the toric Lagrangian triple (X, L, f) by a formal relative Calabi–Yau pair $(\hat{X}, \hat{D})$, where $\hat{X}$ is obtained as the formal neighbourhood along a partial compactification, specified by L and the framing f, of the toric 1–skeleton of X, and $\hat{D} = \hat{D}_1 + \cdots + \hat{D}_s$ is a formal divisor[12] in the partial compactification $\hat{X}$ with $K_{\hat{X}} + \hat{D} = 0$, the aim being to trade the theory of open stable maps with prescribed windings along the boundary circles on L by a theory of relative stable maps with prescribed ramification profile above torus fixed points in $\hat{X}$, as previously suggested in [77]. The resulting moduli space $\overline{\mathcal{M}}^{\mathrm{rel}}_{g;\beta;\mu_1,\dots,\mu_s}(\hat{X}, \hat{D})$ of degree β stable maps from $\ell(\mu_1) + \cdots + \ell(\mu_s)$–pointed, arithmetic genus g nodal curves with ramification profile μ_i above $\hat{D}_i$ at the punctures is a formal Deligne–Mumford stack carrying a perfect obstruction theory $[\mathcal{T}^1 \to \mathcal{T}^2]$ of virtual dimension $\ell(\mu_1) + \cdots + \ell(\mu_s)$. While the moduli space is not itself proper, it inherits a $T \simeq (\mathbb{C}^\star)^2$ action from $\hat{X}$ with compact fixed loci, and open Gromov–Witten invariants

$$O_{g;\beta;(\mu_1,\dots,\mu_s)}(X, L, \mathsf{f}) := \frac{1}{|\mathrm{Aut}(\vec{\mu})|} \int_{[\overline{\mathcal{M}}^{\mathrm{rel}}_{g;\beta;\mu_1,\dots,\mu_s}(\hat{X},\hat{D})]^{\mathrm{vir},T}} \frac{\mathrm{e}_T(\mathcal{T}^{1,m})}{\mathrm{e}_T(\mathcal{T}^{2,m})}, \tag{6-4}$$

where the $\mathcal{T}^{i,m}$ for $i = 1, 2$ denote the moving parts of the obstruction theory, and are defined in a standard manner by T–virtual localisation [49]. It is a central result of [76] that the Calabi–Yau condition on T entails that $O_{g,\beta,(\mu_1,\dots,\mu_s)}(X, L, \mathsf{f})$ are nonequivariantly well-defined rational numbers: the invariants

[12] See [76, Section 5] for the details of the relevant construction.

however do depend on the framings f_i specified to construct the formal relative Calabi–Yau $(\widehat{X}, \widehat{D})$, in keeping with expectations from large N duality [100].

It will be helpful to package the open Gromov–Witten invariants $O_{g,\beta,\mu_1,\dots,\mu_s}(X, L, \mathsf{f})$ into formal generating functions. Let $\mathsf{x}^{(i)} = (\mathsf{x}_1^{(i)}, \mathsf{x}_2^{(i)}, \dots)$ for $i = 1, \dots, s$ be formal variables and for a partition μ define $\mathsf{x}_\mu^{(i)} := \prod_{j=1}^{\ell(\mu)} \mathsf{x}_{\mu_j}^{(i)}$. We further abbreviate $\vec{\mathsf{x}}_{\vec{\mu}} = (\mathsf{x}_{\mu_1}^{(1)}, \dots, \mathsf{x}_{\mu_s}^{(s)})$, $\vec{\mu} = (\mu_1, \dots, \mu_s)$, $|\vec{\mu}| = \sum_i |\mu_i|$ and $\ell(\vec{\mu}) = \sum_i \ell(\mu_i)$, and define the connected generating functions

$$
\begin{aligned}
\mathsf{O}_{\beta;\vec{\mu}}(X, L, \mathsf{f})(\hbar) &:= \sum_g \hbar^{2g-2+\ell(\vec{\mu})} O_{g;\beta;\vec{\mu}}(X, L, \mathsf{f}), \\
\mathcal{O}_{\vec{\mu}}(X, L, \mathsf{f})(Q, \hbar) &:= \sum_\beta \mathsf{O}_{\beta,\vec{\mu}}(X, L, \mathsf{f})(\hbar) Q^\beta, \\
\mathfrak{O}(X, L, \mathsf{f})(Q, \hbar, \mathsf{x}) &:= \sum_{\vec{\mu} \in (\mathcal{P})^s} \mathcal{O}_{\vec{\mu}}(X, L, \mathsf{f})(Q, \hbar) \vec{\mathsf{x}}_{\vec{\mu}},
\end{aligned} \tag{6-5}
$$

as well as generating functions of disconnected invariants in the winding number and representation bases

$$
\begin{aligned}
\mathfrak{Z}(X, L, \mathsf{f})(Q, \hbar, \mathsf{x}) &:= \exp\left(\mathfrak{O}(X, L, \mathsf{f})(Q, \hbar, \mathsf{x})\right) \\
&=: \sum_{\vec{\mu} \in (\mathcal{P})^s} \mathcal{Z}_{\vec{\mu}}(X, L, \mathsf{f})(Q, \hbar) \vec{\mathsf{x}}_{\vec{\mu}} \\
&=: \sum_{\vec{\mu} \in (\mathcal{P})^s} \sum_{\vec{\nu} \in (\mathcal{P})^s} \prod_{i=1}^s \frac{\chi_{\nu_i}(\mu_i)}{z_{\mu_i}} \mathcal{W}_{\vec{\nu}}(X, L, \mathsf{f})(Q, \hbar) \vec{\mathsf{x}}_{\vec{\mu}}.
\end{aligned} \tag{6-6}
$$

Here $\chi_\nu(\mu)$ denotes the irreducible character of $S_{|\nu|}$ evaluated on the conjugacy class labelled by μ. When $\mathsf{x} = 0$, (6-6) reduces to the ordinary generating function of disconnected Gromov–Witten invariants of X.

6.1.2 The topological vertex

The invariants (6-6) can be computed algorithmically to all genera using the topological vertex of Aganagic, Klemm, Mariño and Vafa [6]. We can succinctly condense this into the following three statements:

(1) Let $X = \mathbb{C}^3$, $L = \bigcup_{i=1}^3 L_i$, and $L_i = L_{\widehat{w}^{(i)},c}$ with $\widehat{w}_j^{(i)} = \delta_{i,j} - \delta_{i,j+1 \bmod 3}$, $i = 1, \dots, 3$ be the outer Lagrangians of $\mathbb{C}^3$ as in Figure 11, and fix framing vectors f_i for each of them. Then

$$
\mathcal{W}_{\vec{\mu}}(\mathbb{C}^3, L, \mathsf{f}) = \prod_{i=1}^3 q^{f_i \kappa(\mu_i)/2} (-1)^{f_i |\mu_i|} \mathcal{W}_{\vec{\mu}}(\mathbb{C}^3, L, \mathsf{f}_{\mathrm{can}}), \tag{6-7}
$$

where $q = \mathrm{e}^{\mathrm{i}\hbar}$.

(2) Let $(X^{(1)}, L^{(1)}, \mathsf{f}^{(1)})$ and $(X^{(2)}, L^{(2)}, \mathsf{f}^{(2)})$ be smooth toric Calabi–Yau 3–folds with framed outer toric Lagrangians $L^{(i)} = \bigcup_{j=1}^{s_i} L_j^{(i)}$. Suppose that there exist noncompact edges $\widetilde{e}_i \in (\Gamma_{(X^{(i)},L^{(i)},\mathsf{f}^{(i)})})_1^{\mathrm{nc}}$

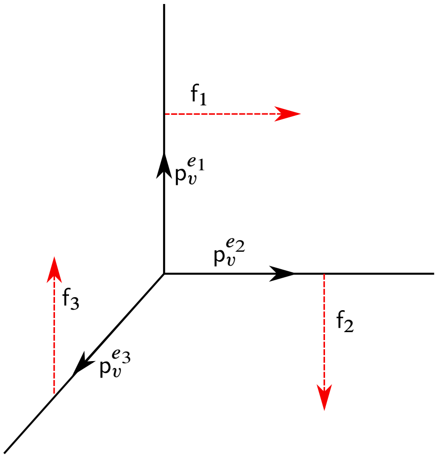


Figure 11: The framed vertex $(\mathbb{C}^3, L_1 \cup L_2 \cup L_3)$, depicted with framings $f_1 = p_v^{e_2}$, $f_2 = p_v^{e_2} + p_v^{e_3}$ and $f_3 = p_v^{e_1}$.

emanating from vertices $\tilde{v}_i \in (\Gamma_{(X^{(i)}, L^{(i)}, f^{(i)})})_0$ such that $\mu_T(L_{s_i}^{(i)}) \cap \tilde{e}_i \neq \varnothing$, and that moreover, $p_{\tilde{v}_1}^{\tilde{e}_1} = -p_{\tilde{v}_2}^{\tilde{e}_2}$ and $f_{s_1}^{(1)} = f_{s_2}^{(2)}$; see Figure 12. We can construct a planar trivalent graph $\Gamma_{X_1 \cup_{e_{12}} X_2}$ decorated with triples of primitive integer vectors at every vertex by considering the disconnected union of $\Gamma_{X^{(1)}}$ and $\Gamma_{X^{(2)}}$, deleting $\tilde{e}_1$ and $\tilde{e}_2$, and adding a compact edge e_{12} connecting $\tilde{v}_1$ to $\tilde{v}_2$. A toric Calabi–Yau 3–graph reconstructs uniquely a smooth toric CY3 with a T action isomorphic to the T–equivariant formal neighbourhood of the configuration of rational curves specified by the edges, and we call X the threefold determined by the glueing procedure such that $\Gamma_X = \Gamma_{X_1 \cup_{e_{12}} X_2}$. In the same vein, the collection of framed Lagrangians $L^{(i)}$ on X_i determine framed outer Lagrangians $L = \bigcup_{i=1}^{s_1+s_2-2} L_i$ on X: we have canonical projection maps $\pi_i : \Gamma_X \to \Gamma_{X^{(i)}}$, and we place an outer Lagrangian brane at framing f_j on all noncompact edges e such that $\pi_i(e) \cap \mu_T(L_j^{(i)}) \neq \varnothing$ for some j. Write

$$\vec{\mu} = (\mu_1^{(1)}, \dots, \mu_{s_1-1}^{(1)}, \mu_1^{(2)}, \dots, \mu_{s_2-1}^{(2)}),$$
$$\vec{\mu}_{12}^{(1)} = (\mu_1^{(1)}, \dots, \mu_{s_1-1}^{(1)}, \nu_{12}) \quad \text{and} \quad \vec{\mu_{12}}^{(2)} = (\mu_1^{(2)}, \dots, \mu_{s_2-1}^{(2)}, \nu_{12}^T).$$

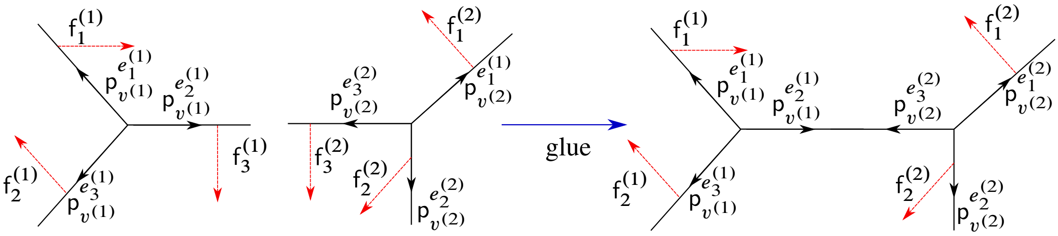


Figure 12: The glueing procedure for the topological vertex. In the notation of the text, we have $s_1 = s_2 = 3$, $\tilde{e}_1 = e_2^{(1)}$, $\tilde{e}_2 = e_3^{(2)}$, $\tilde{e}'_1 = e_3^{(1)}$ and $\tilde{e}'_2 = e_1^{(2)}$.

Then the following glueing formula holds:

(6-8) $$\mathcal{W}_{\vec{\mu}}(X, L, \mathsf{f})(Q, \hbar) = \sum_{\nu_{12}\in\mathcal{P}} (-Q_{\beta_{12}})^{|\nu_{12}|} q^{f_{12}\kappa(\nu_{12})/2} (-1)^{f_{12}|\nu_{12}|} \mathcal{W}_{\vec{\mu}^{(1)}_{12}}(X^{(1)}, L^{(1)}, \mathsf{f}^{(1)})(Q, \hbar) \cdot \mathcal{W}_{\vec{\mu}^{(2)}_{12}}(X^{(2)}, L^{(2)}, \mathsf{f}^{(2)})(Q, \hbar).$$

Here $f_{12} = \det(\mathsf{p}_{\tilde{e}'_1}, \mathsf{p}_{\tilde{e}'_2})$, where $\tilde{e}'_i \in (\Gamma_{(X,L,\mathsf{f})})_1$ is the first edge met when moving counterclockwise from $\tilde{e}_i$, and $Q_{\beta_{12}}$ is the exponentiated Kähler parameter associated to the homology class $\beta_{12} = [\mu_T^{-1}(e_{12})] \in \mathrm{H}_2(X, \mathbb{Z})$. The glueing formula (6-8) originally proposed by [6] is derived in [76] as a consequence of Li's degeneration formula for relative Gromov–Witten theory [75].

(3) The glueing formula (6-8) allows us to recursively compute open Gromov–Witten invariants of any toric Lagrangian triple (X, L, f) starting from those of the framed vertex, ie affine 3–space with framed toric Lagrangians incident to each coordinate line. The framing transformation (6-7) further reduces the problem to the knowledge of the open Gromov–Witten invariants of $(\mathbb{C}^3, L = L_1 \cup L_2 \cup L_3, \mathsf{f}^{\mathrm{can}})$ in canonical framing $\mathsf{f}_i = \mathsf{f}_i^{\mathrm{can}} := \mathsf{p}_{i+1} \bmod 3$. This is given by

(6-9) $$\mathcal{W}_{\mu_1,\mu_2,\mu_3}(\mathbb{C}^3, L, \mathsf{f}^{\mathrm{can}})(\hbar) = q^{\kappa(\mu_1)/2} \sum_{\delta\in\mathcal{P}} s_{\mu_1^t/\delta}(q^{\rho+\mu_3}) s_{\mu_2/\delta}(q^{\rho+\mu_3^t}) s_{\mu_3}(q^{\rho}),$$

where the shifted skew Schur function $s_{\alpha/\beta}(q^{\rho+\gamma})$ is defined in (D-12). The formula (6-9) follows from an explicit evaluation of formal relative Gromov–Witten invariants in terms of descendent triple Hodge integrals. It was first proved in [78; 76] when $\mu_3 = \varnothing$, and the general case was established in [89].

An immediate consequence of (6-8) and (6-9) is that if $i: (X, L, \mathsf{f}) \hookrightarrow (X', L', \mathsf{f}')$ is an embedding of toric Lagrangian triples corresponding to an embedding of graphs $i_\#: \Gamma_{(X,L,\mathsf{f})} \hookrightarrow \Gamma_{(X',L',\mathsf{f}')}$, where $\Gamma_{(X',L',\mathsf{f}')}$ is obtained from $\Gamma_{(X,L,\mathsf{f})}$ by addition of a single vertex v_2 and glueing along a compact edge e_{12} to a vertex $v_1 \in (\Gamma_{(X,L,\mathsf{f})})_0$ by the above procedure, then

(6-10) $$\mathcal{W}_{\vec{\mu}}(X, L, \mathsf{f})(Q, \hbar) = \mathcal{W}_{\vec{\mu}}(X', L', \mathsf{f}')(Q, \hbar)|_{Q_{\beta_{12}}=0}.$$

6.2 The higher-genus log-open principle

In this section we associate certain toric Lagrangian triples to the geometry of Looijenga pair, under an additional condition given by the following definition.

Definition 6.3 Let $Y(D = D_1 + \cdots + D_l)$ be a nef Looijenga. We say that it satisfies *Property O* if $E_{Y(D)}$ deforms to $E_{Y'(D')}$ for a Looijenga pair $Y'(D' = D'_1 + \cdots + D'_l)$ such that

- Y' is a toric surface,
- D'_i is a prime toric divisor for all $i = 1, \dots, l-1$, and
- any nontrivial effective curve in Y' is D'_l–convex.

Example 6.2 Denote by $Y'(D'_1, D'_2)$ the toric surface whose fan is given by Figure 14, with $D'_1 = H - E_3$ and the class of D'_2 corresponding to the sum of the other rays. $Y'(D'_1, D'_2)$ is obtained from $\mathbb{P}^2(1,4) = \mathbb{P}^2(D_1, D_2)$ by blowing up a smooth point on D_1 and two infinitesimally close points on D_2. Moving the latter two apart (while staying on D_2) determines a deformation to $\mathrm{dP}_3(0,2)$. Given nefness of D'_2, it follows that $\mathrm{dP}_3(0,2)$ satisfies Property O. By Proposition 2.6, $\mathrm{dP}_3(1,1)$ also satisfies Property O. The property holds after blowing down (-1)–curves, including for $\mathbb{F}_0(0,4)$. Applying Proposition 2.6 it thus also holds for $\mathbb{F}_0(2,2)$ and $\mathbb{F}_2(2,2)$.

Example 6.3 Consider now $\mathrm{dP}_4(D_1, D_2)$ with $D_1^2 = 0$. Deforming $\mathrm{dP}_4(D_1, D_2)$ to a smooth toric surface with D_1 a toric divisor leads to the fan of Figure 14 with an additional ray in the lower half-plane. Up to deformation, there are two ways of doing so: by adding a ray either between $H - E_1 - E_2$ and E_2, or between E_2 and $E_1 - E_2$. Either way, this creates a curve C with $C \cdot (-K - D_1) < 0$ and therefore $\mathrm{dP}_4(0,1)$ does not satisfy Property O. The same argument applies to $\mathrm{dP}_5(0,0)$.

Example 6.4 When $l > 2$, Property O is always satisfied for all surfaces except for $\mathrm{dP}_3(0,0,0)$. The only way of deforming dP_3 to a toric surface with D_1 and D_2 toric is to take the fan of Figure 15 and add a ray in the lower-left quadrant. But this creates a curve C with $C \cdot (-K - D_1 - D_2) < 0$, and hence $\mathrm{dP}_3(0,0,0)$ does not satisfy Property O.

From Table 1, Property O coincides with quasi-tameness of $Y(D)$, with the sole exception of $\mathrm{dP}_3(0,0,0)$.

We make some informal comments about the geometric transition from stable log maps to open maps, which inform the construction of the open geometries below. This discussion is motivated by [8, Section 7] and in particular a natural generalisation of [8, Conjecture 7.3]. That description applied to our setting makes clear the structure of the toric Lagrangians. Denote by $(Y, D = D_1 + \cdots + D_l)$ a possibly noncompact log Calabi–Yau variety. For a maximally tangent stable log map to (Y, D), the expectation is that maximal tangency d_j with D_j can be replaced by an open boundary condition of winding number d_j with a special Lagrangian L_j near D_j. The special Lagrangian needs to have the property that it bounds a holomorphic disk $\mathcal{D}$ in the normal bundle to D_j; see [8, Section 7]. This property dictates how to compactify $Y \setminus D_j$: in a toric limit, $\mathcal{D}$ is simply the disk used to compactify the edge the framing lies on. If d is a D_j–convex curve class, then we can alternatively remove the maximal tangency condition by twisting the geometry by $\mathcal{O}_Y(-D_j)$. D_j–convexity then guarantees that no maps move into the fibre direction. To obtain the Calabi–Yau threefold geometry from a surface, we adopt the convention of twisting by the last divisor D_l.

In the toric limits of Construction 6.4, the choice of framing corresponds to a choice of compactification. If an outer edge e has framing f, then (see [76, Section 3.2]) the normal bundle of the compactification C of e is $\mathcal{O}(f) \oplus \mathcal{O}(-1-f)$. In our setting, one line bundle is the normal bundle $\mathcal{O}_C(C^2)$ of the curve C in the surface and the other is the normal bundle $\mathcal{O}_Y(-D_l)|_C$ of the curve in the fibre direction. In Construction 6.4, it follows from our conventions that if the framing points to the interior of the polytope, then the normal bundle of C in the surface is $\mathcal{O}(f)$, and if the framing points to the outside of the polytope, then the normal bundle of C in the surface is $\mathcal{O}(-1-f)$. In particular, from a Looijenga pair $Y(D_1,\dots,D_l)$ satisfying Property O, we construct a dual Aganagic–Vafa open Gromov–Witten geometry via the following construction.

Construction 6.4 Let $Y(D_1,\dots,D_l)$ be a Looijenga pair satisfying Property O for $Y'(D'_1,\dots,D'_l)$. Denote by $\Delta_{Y'}$ the polytope of Y' polarised by $-K_{Y'}$. We assume that $\Delta_{Y'\setminus\bigcup_{j\neq l}D'_j}$ is 2–dimensional or, equivalently, that D'_l is not toric, implying $l<4$. Denote by e_j the edge of $\Delta_{Y'}$ corresponding to D'_j for $1\le j\le l-1$ and denote by $e_l,\dots,e_{l+r}$ the remaining edges. Up to reordering, we may assume that the e_i are oriented clockwise. We construct a toric Lagrangian triple $Y^{\mathrm{op}}(D) := (X, L, \mathsf{f})$ as follows. In $\Delta_{Y'}$ remove the edge e_1 and replace it by a framing f_1 on e_{l+r} parallel to e_1. By Definition 6.1 and Remark 6.2, there is a unique way to do so, and f_1 points into the interior of $\Delta_{Y'}$. Denote the resulting graph by Δ^1. If $l=2$, add outer edges to Δ^1 so that each vertex satisfies the balancing condition and denote the resulting toric Calabi–Yau graph by Γ. If $l=3$, in Δ^1 remove the edge e_2 and replace it by a framing on e_3 parallel to e_2. Denote the resulting graph by Δ^2. Add edges to Δ^2 so that each vertex satisfies the balancing condition, and denote the resulting toric Calabi–Yau graph by Γ.

The graph Γ in Construction 6.4 gives the discriminant locus of the SYZ fibration of the toric Calabi–Yau threefold $X=\mathrm{Tot}(K_{Y'\setminus\bigcup_{j\neq l}D'_j})$. The base of the fibration is an $\mathbb{R}$–bundle over the polyhedron $\Delta_{Y'\setminus\bigcup_{j\neq l}D'_j}$. The framings determine toric special Lagrangians L_j, and the added outer edges correspond to the toric fibres of $\mathcal{O}(-D'_l)$. As is readily seen from the fan, f_1 (resp. $-1-f_2$) is the degree of the normal bundle of the divisor in Y' corresponding to e_{l+r} (resp. to e_3). The framing keeps track of the compactification of $Y'\setminus\bigcup_{j\neq l}D'_j$.

Remark 6.3 Tangency with more than one point can be incorporated by having parallel framings on different outer edges.

Remark 6.4 If $\Delta_{Y'\setminus D'_1\cup D'_2}$ is not 2–dimensional, then we blow up Y in a smooth point of D such that the resulting $\widetilde{Y}(\widetilde{D})$ satisfies Property O. We construct $\widetilde{Y}^{\mathrm{op}}(\widetilde{D})$, and recover the open invariants of $Y(D)$ by considering the curve classes that do not meet the exceptional divisor. In particular, for $l>3$ we stipulate that Construction 6.4 can be extended through suitable flopping of $(-1,-1)$–curves in the toric Calabi–Yau 3–fold geometry. We leave a precise formulation to future work, and develop the sole example relevant to our paper to illustrate this.

Example 6.5 We adapt the construction to the only nef Looijenga pair with 4 boundary components $\mathbb{F}_0(H_1,H_2,H'_1,H'_2)$. Since $\Delta_{\mathbb{F}_0\setminus(H_1\cup H_2\cup H'_1)}$ is 1–dimensional, we start by blowing up a smooth point of H'_1. In a toric deformation, we obtain $\mathrm{dP}_2(0,0,-1,0)$ with divisors $D'_1=H_1$, $D'_2=H_2$, $D'_3=H'_1-E$ and $D'_4=H'_2$. We assume that the corresponding $e_1,\dots,e_5$ are ordered clockwise in Δ_{dP_2}. Start with the graph Δ^2 from Construction 6.4. Balance the vertices and flop the inner edge. On the inner edge, add

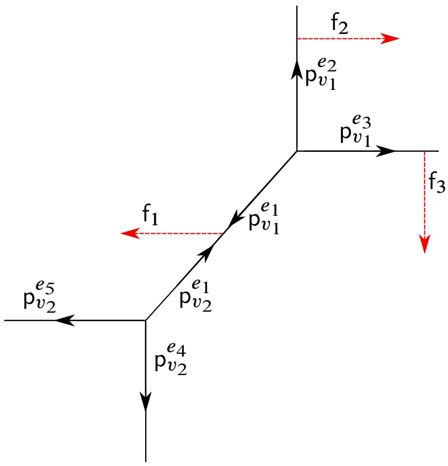


Figure 13: The toric CY3 graph of $\mathrm{dP}_2^{\mathrm{op}}(0,0,-1,0)$.

a framing parallel to e_3. The result is the graph of Figure 13, with the notational shift $\mathsf{f}_2 \leftrightarrow D_1'$, $\mathsf{f}_3 \leftrightarrow D_2'$, $\mathsf{f}_1 \leftrightarrow D_3'$. To obtain the graph for $\mathbb{F}_0(0,0,0,0)$, we remove the two outer edges that have no framing. The result is Figure 16.

Lemma 6.5 *Let $Y(D_1,\dots,D_l)$ and $Y'(D_1',\dots,D_l')$ be as in Construction 6.4. Then $\mathrm{H}_2(Y,\mathbb{Z}) = \mathrm{H}_2(Y',\mathbb{Z})$ is generated by the divisors corresponding to $e_3,\dots,e_{l+r}$.*

Proof In the fan of Y', define an ordering of the 2–dimensional cones by letting σ_i be the cone corresponding to $e_i \cap e_{i+1}$ when $1 \le i < l+r$ and σ_{l+r} be the cone corresponding to $e_{l+r} \cap e_1$. Define cones $\tau_i := \sigma_i \cap \bigcap_{j \in J_i} \sigma_j$, where J_i is the set of $j > i$ such that $\sigma_i \cap \sigma_j$ is 1–dimensional. Then $\tau_1 = \{0\}$, τ_i is the ray corresponding to e_{i+1} for $2 \le i < l+r$ and $\tau_{l+r} = \sigma_{l+r}$. By [42, Section 5.2, Theorem], these cones generate $\mathrm{H}_*(Y',\mathbb{Z})$; hence the divisors corresponding to $e_3,\dots,e_{l+r}$ generate $\mathrm{H}_2(Y',\mathbb{Z})$. □

Note that $\mathrm{H}_2^{\mathrm{rel}}(Y^{\mathrm{op}}(D),\mathbb{Z})$ is generated by the curve classes $[e]$ corresponding to inner edges e and by the relative disk classes $[\mathcal{D}_e]$ corresponding to outer edges e with framings. By the corresponding short exact sequence, the latter can be identified with $[S^1] \in \mathrm{H}_1(S^1,\mathbb{Z})$, where $L \supset S^1 = \partial\mathcal{D}_e$ and the degrees in the $[S^1]$ keep track of the winding numbers. By construction, the e thus described are edges of $\Delta_{Y'}$.

Definition 6.5 Let $Y(D_1,\dots,D_l)$ and $Y'(D_1',\dots,D_l')$ be as in Construction 6.4. Define

$$\iota\colon \mathrm{H}_2^{\mathrm{rel}}(Y^{\mathrm{op}}(D),\mathbb{Z}) \to \mathrm{H}_2(Y,\mathbb{Z}) \tag{6-11}$$

by sending $[e]$ to the divisor corresponding to e in Y.

Proposition 6.6 *The morphism ι is an isomorphism.*

Proof This is a direct consequence of Lemma 6.5. □

Example 6.6 We continue with Example 6.5. Following Figure 13, denote by e_i the edge with framing f_i for $i=1,2,3$. Generalising Definition 6.5, we define $\iota\colon \mathrm{H}_2^{\mathrm{rel}}(\mathrm{dP}_2^{\mathrm{op}}(0,0,-1,0),\mathbb{Z}) \xrightarrow{\sim} \mathrm{H}_2(\mathrm{dP}_2,\mathbb{Z})$ by

$$\iota[e_1]=[D_2']=[H_2],\quad \iota[e_2]=[D_4'-E]=[H_2-E],\quad \iota[e_3]=[D_3']=[H_1-E], \tag{6-12}$$

which yields an isomorphism.

Theorem 6.7 (the higher-genus log-open principle) *Suppose $Y(D)$ satisfies Property O. Then*

$$O_{0;\iota^{-1}(d)}(Y^{\mathrm{op}}(D)) = N^{\mathrm{loc}}_{0,d}(Y(D)) = \prod_{i=1}^{l} \frac{(-1)^{d\cdot D_i+1}}{d\cdot D_i} N^{\log}_{0,d}(Y(D)). \tag{6-13}$$

Moreover, if $Y(D)$ is tame,

$$\mathsf{O}_{\iota^{-1}(d)}(Y^{\mathrm{op}}(D))(-\mathrm{i}\log q) = [1]_q^{l-2}\frac{(-1)^{d\cdot D_l+1}}{[d\cdot D_l]_q}\prod_{i=1}^{l-1}\frac{(-1)^{d\cdot D_i+1}}{d\cdot D_i}\mathsf{N}^{\log}_d(Y(D))(-\mathrm{i}\log q). \tag{6-14}$$

Remark 6.8 As is evident from Example 6.2, $Y^{\mathrm{op}}(D)$ depends on the toric model and hence is not unique. However, it can be checked directly for the examples of Table 1 that if $(X^{(1)},L^{(1)},\mathsf{f}^{(1)})$ and $(X^{(2)},L^{(2)},\mathsf{f}^{(2)})$ correspond to two such choices, then there exists a $\varpi\colon \mathrm{H}_2^{\mathrm{rel}}(X^{(1)},L^{(1)},\mathbb{Z}) \xrightarrow{\sim} \mathrm{H}_2^{\mathrm{rel}}(X^{(2)},L^{(2)},\mathbb{Z})$ such that $\iota^{(1)}=\iota^{(2)}\circ\varpi$.

6.3 Proof of Theorem 6.7

In order to work our way to a general $Y(D)$ satisfying Property O, we first show that if $\pi: Y'(D')\to Y(D)$ is an interior blowup, Construction 6.4 implies that the higher-genus open GW invariants $\mathsf{O}_{\iota^{-1}(d)}(Y^{\mathrm{op}}(D))$ satisfy the same blowup formula (4-19) of the log invariants on the right-hand side of (6-14).

Proposition 6.9 (blowup formula for open GW invariants) *Let $\pi:\widetilde{Y}(\widetilde{D})\to Y(D)$ be an interior blowup of Looijenga pairs with both $\widetilde{Y}(\widetilde{D})$ and $Y(D)$ satisfying Property O, and denote by π^*_{op} the monomorphism of abelian groups defined by*

$$\begin{array}{ccc} \mathrm{H}_2(Y(D),\mathbb{Z}) & \xhookrightarrow{\pi^*} & \mathrm{H}_2(\widetilde{Y}(\widetilde{D}),\mathbb{Z}) \\ \downarrow{\scriptstyle \iota^{-1}} & & \downarrow{\scriptstyle \iota^{-1}} \\ \mathrm{H}_2^{\mathrm{rel}}(Y^{\mathrm{op}}(D),\mathbb{Z}) & \xhookrightarrow{\pi^*_{\mathrm{op}}} & \mathrm{H}_2^{\mathrm{rel}}(\widetilde{Y}^{\mathrm{op}}(\widetilde{D}),\mathbb{Z}) \end{array} \tag{6-15}$$

Then $\mathsf{O}_j(Y^{\mathrm{op}}(D)) = \mathsf{O}_{\pi^*_{\mathrm{op}}j}(\widetilde{Y}^{\mathrm{op}}(\widetilde{D}))$ for all $j\in \mathrm{H}_2^{\mathrm{rel}}(Y^{\mathrm{op}}(D),\mathbb{Z})$.

Sketch of the proof We provide an overview here and leave the details to the reader. The claim is proved by noting that Construction 6.4 implies the following: if $Y(D)$ is obtained from $\widetilde{Y}(\widetilde{D})$ by contraction of a (-1)–curve, then $Y^{\mathrm{op}}(D)$ is an open embedding into a flop of $\widetilde{Y}^{\mathrm{op}}(\widetilde{D})$ along a $(-1,-1)$–curve. The resulting nontrivial equality of open Gromov–Witten invariants under restriction on the image of π^*_{op} is then a combination of the invariance of open Gromov–Witten invariants under "forgetting an edge" in (6-10) and the flop invariance of the topological vertex [69]. ◻

By the previous proposition it then suffices to prove Theorem 6.7 for the pairs $Y(D)$ of highest Picard number for each value of $l = 2, 3, 4$, as all other pairs are recovered from these by blowing-down. We show this from a direct use of the topological vertex to determine the left-hand side of (6-14). The reader is referred to Appendix D for notation and basic results for shifted power sums $p_\alpha(q^{\rho+\gamma})$ and shifted skew Schur functions $s_{\alpha/\beta}(q^{\rho+\gamma})$ in the principal stable specialisation. The notation $\{\alpha,\beta\}_Q$ indicates the symmetric pairing on $\mathcal{P}$ of (D-15).

6.3.1 $l = 2$: holomorphic disks The classification of Propositions 2.2 and 2.3, the deformation equivalences in Proposition 2.6 and the definition of Property O in Definition 6.3 together imply that if $Y(D = D_1 + \cdots + D_l)$ and $Y(D' = D'_1 + \cdots + D'_l)$ are l–component Looijenga pairs both satisfying Property O, then there is a toric model for both with resulting $Y^{\mathrm{op}}(D) = Y^{\mathrm{op}}(D')$: in other words a model $Y^{\mathrm{op}}(D)$ for the open geometry only depends on Y and the number of irreducible components of D. Since 2–component log CY surfaces with maximal boundary come in pairs $Y(D)$ and $Y(D')$ from Table 1, throughout this section we will simplify notation and write $\Upsilon(Y) := Y^{\mathrm{op}}(D) = Y^{\mathrm{op}}(D')$ for the toric Lagrangian triple they share.

By Proposition 6.9, it suffices to consider the case of highest Picard rank $Y = \mathrm{dP}_3$. If $Y(D)$ is either $\mathrm{dP}_3(1,1)$ or $\mathrm{dP}_3(0,2)$, a toric model for Y is given by the toric surface Y' described by the fan of Figure 14, and in particular $D' = H - E_3$ is a toric divisor. Therefore $Y(D)$ satisfies Property O and, by Remark 6.1, $\Upsilon(\mathrm{dP}_3)$ is described by the toric CY3 graph of Figure 14. With conventions as in Figure 14, let $C_1 = \mu_T^{-1}(e_2)$, $C_2 = \mu_T^{-1}(e_5)$ and $C_3 = \mu_T^{-1}(e_7)$, and for a relative 2–homology class $j \in \mathrm{H}_2(\Upsilon(\mathrm{dP}_3), \mathbb{Z})$, write $j = j_0[S^1] + \sum_{i=1}^3 j_i[C_i]$.

We will compute generating functions of higher-genus 1–holed open Gromov–Witten invariants of $\Upsilon(\mathrm{dP}_3)$ in class j, using the theory of the topological vertex. For simplicity, we'll employ the shorthand notation $\mathsf{O}_{j_1,j_2,j_3;j_0}(\Upsilon(\mathrm{dP}_3))$ (resp. $\mathcal{O}_{j_0}(\Upsilon(\mathrm{dP}_3))$) to denote the generating function $\mathsf{O}_{\beta;\mu}(\Upsilon(\mathrm{dP}_3))$ (resp. $\mathcal{O}_\beta(\Upsilon(\mathrm{dP}_3))$) in (6-6) with $\beta = \sum_{i=1}^3 j_i[C_i]$ and $\mu = (j_0)$ a 1–row partition of length j_0. From

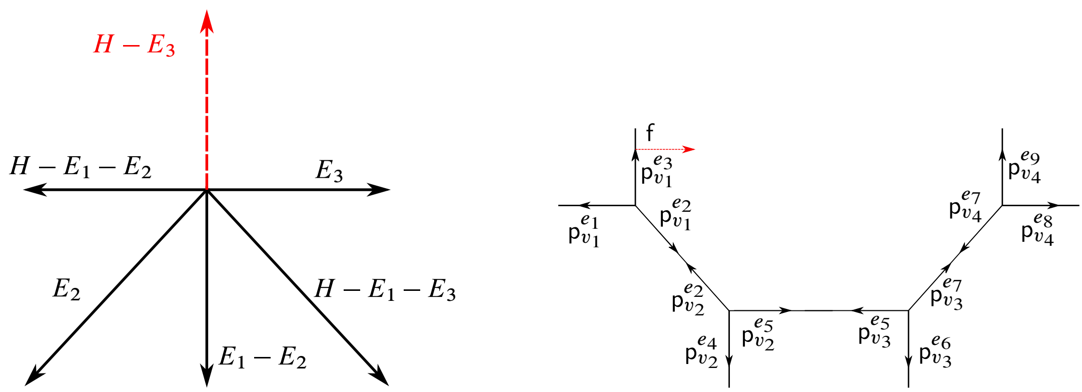


Figure 14: $\Upsilon(\mathrm{dP}_3) = \mathrm{dP}_3^{\mathrm{op}}(0,2) = \mathrm{dP}_3^{\mathrm{op}}(1,1)$ from the blowup of the plane at three nongeneric toric points.

(6-5) and (6-6), we have

$$\text{(6-16)}\quad \mathcal{O}_{j_0}(\Upsilon(\mathrm{dP}_3)) = \frac{\mathcal{Z}_{(j_0)}(\Upsilon(\mathrm{dP}_3))}{\mathcal{Z}_{\varnothing}(\Upsilon(\mathrm{dP}_3))} = \sum_{\nu\in\mathcal{P}} \frac{\chi_\nu((j_0))}{z_{(j_0)}} \frac{\mathcal{W}_\nu(\Upsilon(\mathrm{dP}_3))}{\mathcal{W}_\varnothing(\Upsilon(\mathrm{dP}_3))} = \sum_{s=0}^{j_0-1} \frac{(-1)^s}{j_0} \frac{\mathcal{W}_{(j_0-s,1^s)}(\Upsilon(\mathrm{dP}_3))}{\mathcal{W}_\varnothing(\Upsilon(\mathrm{dP}_3))},$$

where we have used the Murnaghan–Nakayama rule [112, Corollary 7.17.5]

$$\text{(6-17)}\qquad \chi_\nu((j_0)) = \begin{cases} (-1)^s & \text{if } \nu = (j_0-s, 1^s), \\ 0 & \text{else.} \end{cases}$$

The framing f in Figure 14 is shifted by one unit $f=-1$ from the canonical choice $\mathsf{f}_{\mathrm{can}} = \mathsf{p}_{v_1}^{e_2}$. From (6-7), (6-8) and (6-9), we then have, for any $\alpha\in\mathcal{P}$, that

$$\begin{aligned}\text{(6-18)}\quad & \mathcal{W}_\alpha(\Upsilon(\mathrm{dP}_3))(Q,\hbar) \\ &= (-1)^{|\alpha|} q^{-\frac{1}{2}\kappa(\alpha)} \\ &\quad\cdot \sum_{\lambda,\mu,\nu,\delta,\epsilon\in\mathcal{P}} s_{\lambda^t}(-Q_1 q^{\rho+\alpha}) s_\alpha(q^\rho) s_{\lambda/\delta}(q^\rho) s_{\mu/\delta}(q^\rho) Q_2^{|\mu|} s_{\mu/\epsilon}(q^\rho) s_{\nu/\epsilon}(q^\rho) s_{\nu^t}(-Q_3 q^\rho) \\ &= \frac{(-1)^{|\alpha|} s_{\alpha^t}(q^\rho)\{\alpha,\varnothing\}_{Q_1}\{\alpha,\varnothing\}_{Q_1Q_2}\{\varnothing,\varnothing\}_{Q_3}\{\varnothing,\varnothing\}_{Q_2Q_3}}{\{\varnothing,\varnothing\}_{Q_2}\{\alpha,\varnothing\}_{Q_1Q_2Q_3}},\end{aligned}$$

where we have used (D-13) and, repeatedly, (D-15) to express the sums over partitions in terms of Cauchy products. Then, specialising to $\alpha = (j_0-s, 1^s)$ a hook partition with j_0 boxes and $s+1$ rows, and using (D-11) and (D-18), we have

$$\begin{aligned}\text{(6-19)}\quad \frac{\mathcal{W}(\Upsilon(\mathrm{dP}_3))_{(j_0-s,1^s)}}{\mathcal{W}(\Upsilon(\mathrm{dP}_3))_\varnothing} &= \frac{(-1)^{j_0} s_{(s+1,1^{j_0-s-1})}(q^\rho)\{(j_0-s,1^s),\varnothing\}_{Q_1}\{(j_0-s,1^s),\varnothing\}_{Q_1Q_2}}{\{(j_0-s,1^s),\varnothing\}_{Q_1Q_2Q_3}} \\ &= \frac{(-1)^{j_0} q^{-\frac{1}{2}\binom{j_0}{2}+\frac{1}{2}j_0 s}}{[j_0]_q[j_0-s-1]_q![s]_q!} \frac{\prod_{k=0}^{j_0-1}(1-q^k Q_1 q^{-s})\prod_{l=0}^{j_0-1}(1-q^l Q_1Q_2 q^{-s})}{\prod_{m=0}^{j_0-1}(1-q^m Q_1Q_2Q_3 q^{-s})}.\end{aligned}$$

Replacing this into (6-16) we get

$$\begin{aligned}\text{(6-20)}\quad & \mathcal{O}_{j_0}(\Upsilon(\mathrm{dP}_3))(Q,\hbar) \\ &= \sum_{s=0}^{j_0-1} \frac{(-1)^s}{j_0} \frac{\mathcal{W}_{(j_0-s,1^s)}(\Upsilon(\mathrm{dP}_3))(Q,\hbar)}{\mathcal{W}_\varnothing(\Upsilon(\mathrm{dP}_3))(Q,\hbar)} \\ &= \frac{(-1)^{j_0} q^{-\frac{1}{2}\binom{j_0}{2}}}{j_0[j_0]_q!} \sum_{j_1,j_2,j_3=0}^{\infty} \Bigg(q^{\frac{1}{2}j_1(j_0-1)} \begin{bmatrix} j_0 \\ j_1-j_2 \end{bmatrix}_q \begin{bmatrix} j_0 \\ j_2-j_3 \end{bmatrix}_q \begin{bmatrix} j_0+j_3-1 \\ j_3 \end{bmatrix}_q \\ &\qquad\qquad \cdot (-1)^{j_1+j_3} Q_1^{j_1} Q_2^{j_2} Q_3^{j_3} \sum_{s=0}^{j_0-1} \begin{bmatrix} j_0-1 \\ s \end{bmatrix}_q (-q^{-j_1})^s q^{\frac{1}{2}j_0 s} \Bigg) \\ &= \frac{(-1)^{j_0}}{j_0[j_0]_q!} \sum_{j_1,j_2,j_3}^{\infty} \begin{bmatrix} j_0 \\ j_1-j_2 \end{bmatrix}_q \begin{bmatrix} j_0 \\ j_2-j_3 \end{bmatrix}_q \begin{bmatrix} j_0+j_3-1 \\ j_3 \end{bmatrix}_q (-1)^{j_1+j_3} Q_1^{j_1} Q_2^{j_2} Q_3^{j_3} \frac{[j_1-1]_q!}{[j_1-j_0]_q!},\end{aligned}$$

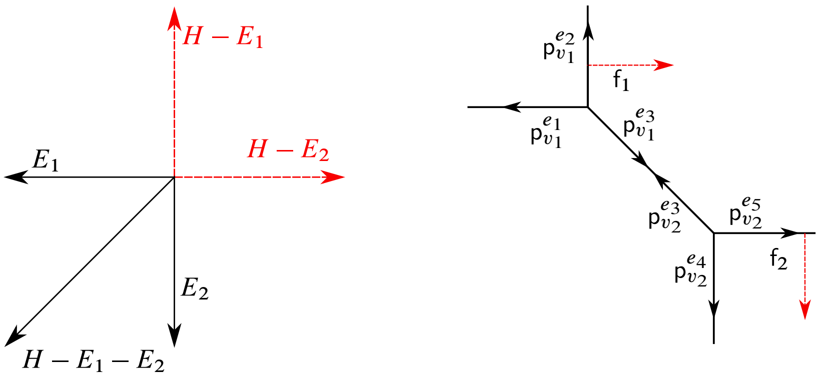


Figure 15: $\mathrm{dP}_2^{\mathrm{op}}(1,0,0)$ from dP_2 and $D_1 = H - E_1$, $D_2 = H - E_2$.

where the q–binomial theorem has been used to expand the products in (6-19) and to perform the summation over s in the last line. Isolating the $\mathcal{O}(Q_1^{j_1} Q_2^{j_2} Q_2^{j_3})$ coefficient yields

$$\mathsf{O}_{j_1,j_2,j_3;j_0}(\Upsilon(\mathrm{dP}_3))(\hbar) = \frac{(-1)^{j_1+j_0+j_3}[j_0]_q}{j_0[j_1]_q[j_0+j_3]_q} \begin{bmatrix} j_0 \\ j_1-j_2 \end{bmatrix}_q \begin{bmatrix} j_0 \\ j_2-j_3 \end{bmatrix}_q \begin{bmatrix} j_0+j_3 \\ j_3 \end{bmatrix}_q \begin{bmatrix} j_1 \\ j_0 \end{bmatrix}_q. \tag{6-21}$$

From Figure 14, the lattice isomorphism $\iota\colon \mathrm{H}_2^{\mathrm{rel}}(\Upsilon(\mathrm{dP}_3),\mathbb{Z}) \to \mathrm{H}_2(\mathrm{dP}_3,\mathbb{Z})$ in this case reads

$$\iota[S^1] = [H-E_1-E_2], \quad \iota[C_1] = [E_2], \quad \iota[C_2] = [E_1-E_2], \quad \iota[C_3] = [H-E_1-E_3], \tag{6-22}$$

and the change of variables relating the curve degrees (d_0,d_1,d_2,d_3) in $\mathrm{H}_2(\mathrm{dP}_3,\mathbb{Z})$ and the relative homology variables $(j_0; j_1, j_2, j_3)$ in $\mathrm{H}_2^{\mathrm{rel}}(\Upsilon(\mathrm{dP}_3),\mathbb{Z})$ is therefore

$$d_0 \to j_0 + j_3, \quad d_1 \to j_2, \quad d_2 \to j_1 - j_2 + j_3, \quad d_3 \to j_0. \tag{6-23}$$

Combining the change of variables (6-23) and the log result of (4-22) in Theorem 4.5 returns (6-21), establishing (6-14) for $Y(D) = \mathrm{dP}_3(1,1)$. Furthermore, taking the genus-zero limit $q \to 1$ and using Theorem 5.1, Lemma 3.1 and Proposition 2.6 implies (6-13), completing the proof of Theorem 6.7 for $Y(D) = \mathrm{dP}_3(D_1^2, D_2^2)$. Use of Propositions 6.9 and 4.3 then concludes the proof of Theorem 6.7 for any $Y(D)$ satisfying Property O with $l = 2$.

6.3.2 $l = 3$: holomorphic annuli The 3–component Looijenga pair of highest Picard rank satisfying Property O is $Y(D) = \mathrm{dP}_2(1,0,0)$. Taking $D_1 = H - E_1$, $D_2 = H - E_2$ we have that Y, D_1 and D_2 are toric, and $\mathrm{dP}_2^{\mathrm{op}}(1,0,0)$ is described by the toric CY3 graph on the left in Figure 15. Write $C = \mu_T^{-1}(e_3)$, and for a relative 2–homology class $j \in \mathrm{H}_2^{\mathrm{rel}}(\mathrm{dP}_2^{\mathrm{op}}(1,0,0),\mathbb{Z})$ write $j = j_1[\mathcal{D}_1] + j_2[\mathcal{D}_2] + j_C[C]$, where $[\mathcal{D}_i]$ are integral generators of the first homology of the outer Lagrangians incident to edges adjacent to the vertices v_i for $i = 1,2$ in Figure 15. As in the previous section, we will write $\mathsf{O}_{j_C;j_1,j_2}(\mathrm{dP}_2^{\mathrm{op}}(1,0,0))$ (resp. $\mathcal{O}_{j_1,j_2}(\mathrm{dP}_2^{\mathrm{op}}(1,0,0))$) for the generating function $\mathsf{O}_{\beta,\vec{\mu}}(\mathrm{dP}_2^{\mathrm{op}}(1,0,0))$ (resp. $\mathcal{O}_{\vec{\mu}}(\mathrm{dP}_2^{\mathrm{op}}(1,0,0))$), with $\beta = j_C[C]$ and $\vec{\mu} = ((j_1), (j_2))$ a pair of 1–row partitions of length (j_1, j_2). From (6-5), (6-6) and (6-17), we have

(6-24) $$\mathcal{O}_{j_1,j_2}(\mathrm{dP}_2^{\mathrm{op}}(1,0,0))(Q,\hbar) = \frac{\mathcal{Z}_{j_1,j_2}(\mathrm{dP}_2^{\mathrm{op}}(1,0,0))}{Z_{\varnothing,\varnothing}(\mathrm{dP}_2^{\mathrm{op}}(1,0,0))} - \frac{\mathcal{Z}_{(j_1),\varnothing}(\mathrm{dP}_2^{\mathrm{op}}(1,0,0))\mathcal{Z}_{\varnothing,(j_2)}(\mathrm{dP}_2^{\mathrm{op}}(1,0,0))}{\mathcal{Z}_{\varnothing,\varnothing}(\mathrm{dP}_2^{\mathrm{op}}(1,0,0))^2}$$
$$= \sum_{s_1=0}^{j_1-1}\sum_{s_2=0}^{j_2-1} \frac{(-1)^{s_1+s_2}}{j_1 j_2}\Bigg[\frac{\mathcal{W}_{(j_1-s_1,1^{s_1}),(j_2-s_2,1^{s_2})}(\mathrm{dP}_2^{\mathrm{op}}(1,0,0))}{\mathcal{W}_{\varnothing,\varnothing}(\mathrm{dP}_2^{\mathrm{op}}(1,0,0))} - \frac{\mathcal{W}_{(j_1-s_1,1^{s_1}),\varnothing}(\mathrm{dP}_2^{\mathrm{op}}(1,0,0))\mathcal{W}_{\varnothing,(j_2-s_2,1^{s_2})}(\mathrm{dP}_2^{\mathrm{op}}(1,0,0))}{\mathcal{W}_{\varnothing,\varnothing}(\mathrm{dP}_2^{\mathrm{op}}(1,0,0))^2}\Bigg].$$

The framings f_1 and f_2 in Figure 15 are, respectively, shifted by one unit $f=-1$ from the canonical choice $\mathsf{f}_{\mathrm{can}} = \mathsf{p}_{v_1}^{e_3}$, and equal to the canonical framing $\mathsf{f}_2 = \mathsf{p}_{v_2}^{e_4}$. Then (6-7), (6-8) and (6-9) give

(6-25) $$\mathcal{W}_{\alpha\beta}(\mathrm{dP}_2^{\mathrm{op}}(1,0,0))(Q,\hbar) = (-1)^{|\alpha|} q^{-\kappa(\alpha)/2} \sum_{\mu,\delta,\in\mathcal{P}} s_{\mu^t}(-q^{\rho+\alpha}Q) s_\alpha(q^\rho) s_{\mu/\delta}(q^\rho) s_{\beta/\delta}(q^\rho)$$
$$= (-1)^{|\alpha|} s_{\alpha^t}(q^\rho)\{\alpha,\varnothing\}_Q \sum_{\delta\in\mathcal{P}} s_{\beta^t/\delta^t}(-q^{-\rho}) s_{\delta^t}(-q^{\rho+\alpha}Q)$$
$$= (-1)^{|\alpha|} s_{\alpha^t}(q^\rho)\{\alpha,\varnothing\}_Q s_{\beta^t}(-q^{-\rho}, -q^{\rho+\alpha}Q),$$

where we have used (D-14), (D-7) and (D-15) to perform the summations over partitions. Then, restricting to $\alpha = (j_1 - s_1, 1^{s_1})$ and $\beta = (j_2 - s_2, 1^{s_2})$,

(6-26) $$\mathcal{O}_{j_1,j_2}(\mathrm{dP}_2^{\mathrm{op}}(1,0,0))(Q,\hbar)$$
$$= \sum_{s_1=0}^{j_1-1} \frac{(-1)^{j_1+s_1+j_2+1}}{j_1 j_2} s_{(s_1+1,1^{j_1-s_1-1})}(q^\rho) \prod_{k=0}^{j_1-1}(1-q^k Q q^{-s_1}) \times [p_{(j_2)}(-Qq^{\rho+(j_1-s_1,1^{s_1})}, -q^{-\rho}) - p_{(j_2)}(-Qq^\rho, -q^{-\rho})]$$
$$= \sum_{s_1=0}^{j_1-1} \frac{(-1)^{j_1+s_1+j_2+1}}{j_1 j_2} s_{(s_1+1,1^{j_1-s_1-1})}(q^\rho) \prod_{k=0}^{j_1-1}(1-q^k Q q^{-s_1}) \times [p_{(j_2)}(-Qq^{\rho+(j_1-s_1,1^{s_1})}) - p_{(j_2)}(-Qq^\rho)]$$
$$= \sum_{s_1=0}^{j_1-1} \frac{(-1)^{j_1+s_1+j_2+1}}{j_1 j_2} s_{(s_1+1,1^{j_1-s_1-1})}(q^\rho) \prod_{k=0}^{j_1-1}(1-q^k Q q^{-s_1})(-Qq^{-s_1-\frac{1}{2}})^{j_2}[q^{j_2 j_1}-1]$$
$$= \frac{(-1)^{j_1+1} Q^{j_2} [j_1 j_2]_q}{j_1 j_2 [j_2+m]_q} \sum_{m=0}^{j_1} \begin{bmatrix} j_1 \\ m \end{bmatrix}_q \begin{bmatrix} j_2+m \\ j_1 \end{bmatrix}_q (-Q)^m,$$

where in the first equality we have used (D-3) and (6-17), in the second the fact that for a 1–row partition $\alpha = (d)$, $p_{(d)}(x_1,\dots,x_n,\dots;y_1,\dots,y_n,\dots) = p_{(d)}(x_1,\dots,x_n,\dots) + p_{(d)}(y_1,\dots,y_n,\dots)$, and in the third equality the fact that the difference of infinite power sums in the term in square brackets telescopes to just two terms; the final calculations are repeated applications of the q–binomial theorem. Extracting the $\mathcal{O}(Q^{j_C})$ coefficient, we get

$$\mathsf{O}_{j_C;j_1,j_2}(\mathrm{dP}_2^{\mathrm{op}}(1,0,0))(\hbar) = \frac{(-1)^{j_1+1+j_C+j_2}[j_1 j_2]_q}{j_1 j_2 [j_C]_q} \begin{bmatrix} j_1 \\ j_C - j_2 \end{bmatrix}_q \begin{bmatrix} j_C \\ j_1 \end{bmatrix}_q . \tag{6-27}$$

From Figure 15, the homomorphism of homology groups $\iota\colon \mathrm{H}_2^{\mathrm{rel}}(\mathrm{dP}_2^{\mathrm{op}}(1,0,0),\mathbb{Z}) \to \mathrm{H}_2(\mathrm{dP}_2,\mathbb{Z})$ is given by

$$\iota[D_1] = [E_1], \quad \iota[D_2] = [E_2], \quad \iota[C] = [H - E_1 - E_2], \tag{6-28}$$

and the resulting map of curve degrees is

$$d_0 \to j_C, \quad d_1 \to j_1, \quad d_2 \to j_2. \tag{6-29}$$

Together with the log results given by (4-39) in Theorem 4.9 and the blowup formulas of Propositions 4.3 and 6.9 for the log and open invariants, this proves Theorem 6.7 for $l = 3$.

6.3.3 $l = 4$: holomorphic pairs of pants According to Example 6.5, for the only 4–component case $Y(D) = \mathbb{F}_0(0,0,0,0)$, we have that $Y^{\mathrm{op}}(D)$ is given by the 3–dimensional affine space with Aganagic–Vafa A–branes $L^{(i)}$ for $i = 1,2,3$ at framing shifted by -1, 0, and -1 ending on the three legs of the vertex, as in Figure 16. We will be concerned with counts of 3–holed open Gromov–Witten invariants of $\mathbb{F}_0^{\mathrm{op}}(0,0,0,0)$, with winding numbers (j_1, j_1, j_2); see Example 6.6.

The connected generating function, by (6-5) and (6-6), is

$$\begin{aligned} \mathsf{O}_{j_1,j_1,j_2}&(\mathbb{F}_0^{\mathrm{op}}(0,0,0,0))(\hbar) \\ &= \mathcal{Z}_{(j_1)(j_1)(j_2)}(\mathbb{F}_0^{\mathrm{op}}(0,0,0,0)) - \mathcal{Z}_{(j_1)(j_1)\varnothing}(\mathbb{F}_0^{\mathrm{op}}(0,0,0,0))\mathcal{Z}_{\varnothing\varnothing(j_2)}(\mathbb{F}_0^{\mathrm{op}}(0,0,0,0)) \\ &\quad - \mathcal{Z}_{(j_1)\varnothing(j_2)}(\mathbb{F}_0^{\mathrm{op}}(0,0,0,0))\mathcal{Z}_{\varnothing(j_1)\varnothing}(\mathbb{F}_0^{\mathrm{op}}(0,0,0,0)) \\ &\quad - \mathcal{Z}_{\varnothing(j_1)(j_2)}(\mathbb{F}_0^{\mathrm{op}}(0,0,0,0))\mathcal{Z}_{(j_1)\varnothing\varnothing}(\mathbb{F}_0^{\mathrm{op}}(0,0,0,0)) \\ &\quad + 2\mathcal{Z}_{(j_1)\varnothing\varnothing}(\mathbb{F}_0^{\mathrm{op}}(0,0,0,0))\mathcal{Z}_{\varnothing(j_1)\varnothing}(\mathbb{F}_0^{\mathrm{op}}(0,0,0,0))\mathcal{Z}_{\varnothing\varnothing(j_2)}(\mathbb{F}_0^{\mathrm{op}}(0,0,0,0)), \end{aligned} \tag{6-30}$$

where, by (6-17),

$$\begin{aligned} \mathcal{Z}_{(j_1),(j_1),(j_2)}&(\mathbb{F}_0^{\mathrm{op}}(0,0,0,0)) \\ &= \sum_{s_0,s_1,s_2} \frac{(-1)^{s_0+s_1+s_2}}{j_1^2 j_2} \mathcal{W}(\mathbb{F}_0^{\mathrm{op}}(0,0,0,0))_{(j_1-s_0,1^{s_0}),(j_1-s_1,1^{s_1}),(j_2-s_2,1^{s_2})} \end{aligned} \tag{6-31}$$

and, from (6-7) and (6-9),

$$\mathcal{W}_{\alpha\beta\gamma}(\mathbb{F}_0^{\mathrm{op}}(0,0,0,0)) = (-1)^{|\alpha|+|\gamma|} \sum_{\delta} s_{\alpha^t/\delta}(q^{\rho+\gamma}) s_{\beta/\delta}(q^{\rho+\gamma^t}) s_{\gamma^t}(q^{\rho}). \tag{6-32}$$

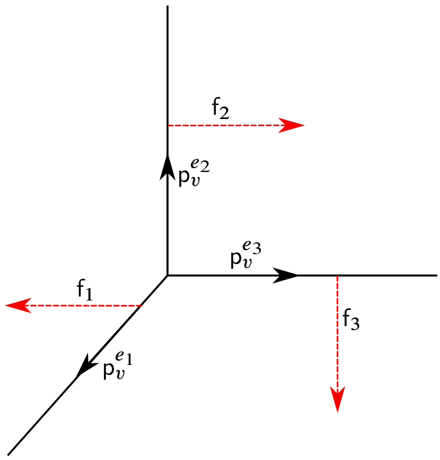


Figure 16: The toric CY3 graph of $\mathbb{F}_0^{\mathrm{op}}(0,0,0,0)$.

Elementary manipulations from (6-30)–(6-32) lead to

$$\begin{aligned}(6\text{-}33)\quad & \mathsf{O}_{j_1,j_1,j_2}(\mathbb{F}_0^{\mathrm{op}}(0,0,0,0))(\hbar) \\ &= \sum_{s_0,s_1,s_2} \frac{(-1)^{s_0+s_1+s_2+j_1+j_2}}{j_1^2 j_2} \Big[\big(s_{\alpha^t}(q^{\rho+\gamma}) - s_{\alpha^t}(q^\rho)\big)\big(s_\beta(q^{\rho+\gamma^t}) - s_\beta(q^\rho)\big) s_{\gamma^t}(q^\rho) \\ &\qquad + \sum_{\delta\neq\varnothing} \big(s_{\alpha^t/\delta}(q^{\rho+\gamma}) s_{\beta/\delta}(q^{\rho+\gamma^t}) - s_{\alpha^t/\delta}(q^\rho) s_{\beta/\delta}(q^\rho)\big) s_{\gamma^t}(q^\rho)\Big]\end{aligned}$$

with

$$\alpha = (j_1 - s_0, 1^{s_0}), \quad \beta = (j_1 - s_1, 1^{s_1}), \quad \gamma = (j_2 - s_2, 1^{s_2}).$$

The part of the summation in the middle line, after carrying out the sums over s_0, s_1 and s_2 using (D-3) and (6-17), is equal to

$$\begin{aligned}(6\text{-}34)\quad & \sum_{s_2=0}^{j_2-1} \frac{(-1)^{s_2+1+j_2}}{j_1^2 j_2} \big(p_{(j_1)}(q^{\rho+(j_2-s_2,1^{s_2})}) - p_{(j_1)}(q^\rho)\big)\big(p_{(j_1)}(q^{\rho+(s_2+1,1^{j_2-s_2-1})}) - p_{(j_1)}(q^\rho)\big) \\ &\qquad \times s_{(s_2+1,1^{j_2-s_2-1})}(q^\rho) \\ &= \sum_{s_2=0}^{j_2-1} \frac{(-1)^{s_2}}{j_1^2 j_2} (q^{j_1(j_2-s_2-1/2)} - q^{j_1(-s_2-1/2)})(q^{j_1(s_2+1/2)} - q^{j_1(s_2+1/2-j_2)}) s_{(j_2-s_2,1^{s_2})}(q^\rho) \\ &= \frac{1}{j_1^2 j_2} \frac{[j_1 j_2]_q^2}{[j_2]_q},\end{aligned}$$

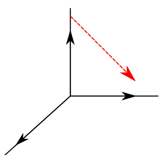
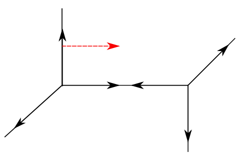
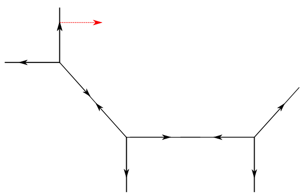
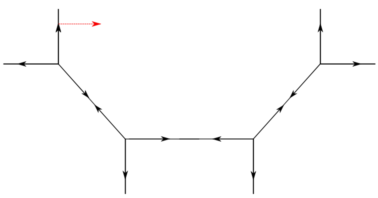
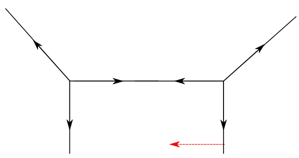
while the part in the last line is equal to zero. Indeed, when $\delta = \alpha^t$, we have $s_{\beta/\delta}(x) = \delta_{\beta\alpha^t}$ (since $|\alpha| = |\beta| = j_1$ in our case, $\alpha^t \preceq \beta$ implies $\alpha^t = \beta$), so the terms appearing in the difference in the second row of (6-33) are either individually zero or cancel out each other. When $\delta \neq \alpha^t$, we can use Lemma D.1 to expand $s_{\alpha^t/\delta}(x)$ in terms of ordinary Schur functions $s_\lambda(x)$ with $|\lambda| = |\alpha| - |\delta|$: it is easy to see that in the sum over s_0 the contribution labelled by each such Young diagram appears exactly twice and weighted with opposite signs. Therefore,

$$\mathsf{O}_{j_1,j_1,j_2}(\mathbb{F}_0^{\mathrm{op}}(0,0,0,0))(\hbar) = \frac{1}{j_1^2 j_2} \frac{[j_1 j_2]_q^2}{[j_2]_q}. \tag{6-35}$$

By construction from Examples 6.5 and 6.6,

$$d_1 \to j_2 \quad \text{and} \quad d_2 \to j_1, \tag{6-36}$$

and comparing with (4-41) gives (6-14), which concludes the proof of Theorem 6.7. □

$Y(D)$	$\Gamma_{Y^{\mathrm{op}}(D)}$
$\mathbb{P}^2(1,4)$	
$\mathrm{dP}_1(1,3)$ $\mathrm{dP}_1(0,4)$	
$\mathrm{dP}_2(1,2)$ $\mathrm{dP}_2(0,3)$	
$\mathrm{dP}_3(1,1)$ $\mathrm{dP}_3(0,2)$	
$\mathbb{F}_0(2,2)$ $\mathbb{F}_0(0,4)$	

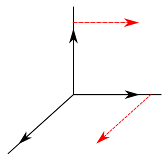
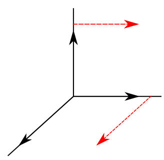
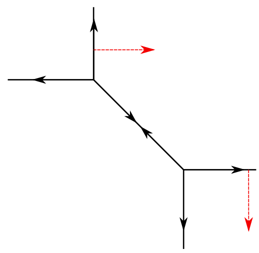
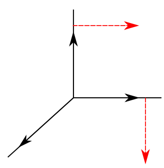
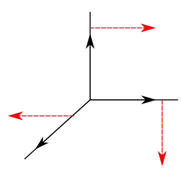
$Y(D)$	$\Gamma_{Y^{\mathrm{op}}(D)}$
$\mathbb{P}^2(1,1,1)$	
$\mathrm{dP}_1(1,1,0)$	
$\mathrm{dP}_2(1,0,0)$	
$\mathbb{F}_0(2,0,0)$	
$\mathbb{F}_0(0,0,0,0)$	

Table 3: $Y^{\mathrm{op}}(D)$ for l–component Looijenga pairs satisfying Property O.

7 KP and quiver DT invariants

7.1 Klemm–Pandharipande invariants of CY4–folds

Let Z be a smooth projective complex Calabi–Yau variety of dimension four and $d \in \mathrm{H}_2(Z,\mathbb{Z})$. Since $\mathrm{vdim}\overline{\mathcal{M}}_{g,n}(Z,d) = 1-g+n$, the only nonvanishing genus-zero primary Gromov–Witten invariants of Z without divisor insertions are[13]

$$\mathrm{GW}_{0,d;\gamma}(Z) := \int_{[\overline{\mathcal{M}}_{0,1}(Z,d)]^{\mathrm{vir}}} \mathrm{ev}_1^\star \gamma \quad \text{for } \gamma \in \mathrm{H}^4(Z,\mathbb{Z}). \tag{7-1}$$

The same considerations apply to the case of Z the Calabi–Yau total space of a rank $(4-r)$ concave vector bundle on an r–dimensional smooth projective variety. It was proposed by Greene, Morrison and Plesser in [50, Appendix B] and further elaborated upon by Klemm and Pandharipande in [68, Section 1.1] that a higher-dimensional version of the Aspinwall–Morrison should conjecturally produce integral invariants $\mathrm{KP}_{0,d}(Z)$, virtually enumerating rational degree-d curves incident to the Poincaré dual cycle of γ,

$$\mathrm{GW}_{0,d;\gamma}(Z) = \sum_{k|d} \frac{\mathrm{KP}_{0,d/k;\gamma}(Z)}{k^2}. \tag{7-2}$$

Conjecture 7.1 (Klemm–Pandharipande) $\mathrm{KP}_{0,d;\gamma}(Z) \in \mathbb{Z}$.

A symplectic proof of Conjecture 7.1 for projective Z, although likely adaptable to the noncompact setting, was given by Ionel and Parker in [62].

Our main focus will be on Z a noncompact CY4 local surface, ie $r=2$. In this case there is a single generator $\gamma = [\mathrm{pt}]$ for the fourth cohomology of Z, given by the Poincaré dual of the point class on the zero section, and we will henceforth use the simplified notation $\mathrm{KP}_{0,d}(Z) := \mathrm{KP}_{0,d;[\mathrm{pt}]}(Z)$.

7.2 Quiver Donaldson–Thomas theory

Let Q be a quiver with an ordered set Q_0 of n vertices $v_1,\dots v_n \in \mathsf{Q}_0$ and a set of oriented edges $\mathsf{Q}_1 = \{\alpha : v_i \to v_j\}$. We let $\mathbb{N}\mathsf{Q}_0$ be the free abelian semigroup generated by Q_0, and for $\mathsf{d} = \sum d_i v_i$ and $\mathsf{e} = \sum e_i v_i \in \mathbb{N}\mathsf{Q}_0$, we write $E_{\mathsf{Q}}(\mathsf{d},\mathsf{e})$ for the Euler form

$$E_{\mathsf{Q}}(\mathsf{d},\mathsf{e}) := \sum_{i=1}^n d_i e_i - \sum_{\alpha : v_i \to v_j} d_i e_j. \tag{7-3}$$

We assume in what follows that Q is symmetric; that is, for every i and j, the number of oriented edges from v_i to v_j is equal to the number of oriented edges from v_j to v_i. The Euler form is then a symmetric

[13]By the same formula, there are nonvanishing elliptic unpointed Gromov–Witten invariants for Z, which will not concern us in this paper. There are no Gromov–Witten invariants for a CY4 in genus $g > 1$.

bilinear form. To C a symmetric bilinear pairing on $\mathbb{Z}^n$, we associate the generalised q–hypergeometric series

$$\Phi_C(q; x_1, \dots, x_n) := \sum_{\mathsf{d}\in\mathbb{N}^n}^{\infty} \frac{(-q^{1/2})^{C(\mathsf{d},\mathsf{d})}\mathsf{x}^{\mathsf{d}}}{\prod_{i=1}^n (q;q)_{d_i}}, \tag{7-4}$$

where $\mathsf{x}^{\mathsf{d}} = \prod_{i=1}^n x_i^{d_i}$. The motivic Donaldson–Thomas partition function associated to the cohomological Hall algebra of Q (without potential) is the generating function [34]

$$P_{\mathsf{Q}}(q; x_1, \dots, x_n) := \Phi_{E_{\mathsf{Q}}}(q; x_1, \dots, x_n), \tag{7-5}$$

and the motivic DT invariants $\mathrm{DT}_{\mathsf{d};i}(\mathsf{Q})$ of Q are the formal Taylor coefficients in the expansion of its plethystic logarithm [34; 70; 107],

$$\begin{aligned} P_{\mathsf{Q}}(q; x_1, \dots, x_n) &= \operatorname{Exp}\biggl(\frac{1}{[1]_q} \sum_{\mathsf{d}\neq 0} \sum_{i\in\mathbb{Z}} \mathrm{DT}_{\mathsf{d};i}(\mathsf{Q}) \mathsf{x}^{\mathsf{d}} (-q^{1/2})^{-i}\biggr) \\ &= \exp\biggl[\sum_{n=1}^{\infty} \frac{1}{n[n]_q} \sum_{\mathsf{d}\neq 0} \sum_{i\in\mathbb{Z}} \mathrm{DT}^{\mathsf{Q}}_{\mathsf{d};i} \mathsf{x}^{n\mathsf{d}} (-q^{1/2})^{-ni}\biggr] \\ &= \prod_{\mathsf{d}\neq 0} \prod_{i\in\mathbb{Z}} \prod_{k\geq 0} \bigl(1 - (-1)^i \mathsf{x}^{\mathsf{d}} q^{-k-(i+1)/2}\bigr)^{-\mathrm{DT}_{\mathsf{d};i}(\mathsf{Q})}. \end{aligned} \tag{7-6}$$

It will be of particular interest for us to consider a suitable semiclassical limit of (7-6)

$$\begin{aligned} y^{(i)}_{\mathsf{Q}}(x_1, \dots, x_n) &:= \lim_{q\to 1} \frac{P_{\mathsf{Q}}(q; x_1, \dots, q^{1/2}x_i, \dots, x_n)}{P_{\mathsf{Q}}(q; x_1, \dots, q^{-1/2}x_i, \dots x_n)} \\ &= \lim_{q\to 1} \operatorname{Exp}\biggl(\sum_{\mathsf{d}\neq 0} \frac{1}{[1]_q} \sum_{i\in\mathbb{Z}} [d_i]_q \mathrm{DT}^{\mathsf{Q}}_{\mathsf{d};i} \mathsf{x}^{\mathsf{d}} (-q^{1/2})^{-i}\biggr) \\ &= \prod_{\mathsf{d}\neq 0} \prod_{i\in\mathbb{Z}} (1 - \mathsf{x}^{\mathsf{d}})^{-|\mathsf{d}|\mathrm{DT}^{\mathrm{num}}_{\mathsf{d}}(\mathsf{Q})}, \end{aligned} \tag{7-7}$$

where

$$\mathrm{DT}^{\mathrm{num}}_{\mathsf{d}}(\mathsf{Q}) := \sum_{i\in\mathbb{Z}} (-1)^i \mathrm{DT}_{\mathsf{d},i}(\mathsf{Q}) \tag{7-8}$$

are the numerical DT invariants. From (7-7), the numerical invariants can be extracted from the logarithmic primitive of $y^{(i)}_Q(\mathsf{x})$ with respect to x_i,

$$\int \frac{\mathrm{d}x_i}{x_i} \log y^{(i)}_Q(\mathsf{x}) =: \sum_{\mathsf{d}\neq 0} A_{\mathsf{d}}(\mathsf{Q}) \mathsf{x}^{\mathsf{d}}, \tag{7-9}$$

as

$$A_{\mathsf{d}}(\mathsf{Q}) = \sum_{k|d} \frac{\mathrm{DT}^{\mathrm{num}}_{\mathsf{d}/\mathsf{k}}(\mathsf{Q})}{k^2}. \tag{7-10}$$

The generating series

$$y_{\mathsf{Q}}(x_1,\dots,x_n) := \prod_{i=1}^{n} y_{\mathsf{Q}}^{(i)}(x_1,\dots,x_n)$$

has an interpretation as a generating function of Euler characteristics of certain noncommutative Hilbert schemes $\mathrm{Hilb}_{\mathsf{d}}(\mathsf{Q})$ attached to the moduli space of semistable representations of the quiver Q [37; 107],

$$y_{\mathsf{Q}}(x_1,\dots,x_n) = \sum_{\mathsf{d}\in\mathbb{Z}\mathsf{Q}_0} \chi(\mathrm{Hilb}_{\mathsf{d}}(\mathsf{Q}))\mathsf{x}^{\mathsf{d}} \in \mathbb{Z}[\![\mathsf{x}]\!]. \tag{7-11}$$

In particular, this implies that $\big(\sum_{i=1}^{n} d_i\big)\mathrm{DT}_{\mathsf{d}}^{\mathrm{num}}(\mathsf{Q}) \in \mathbb{Z}$. More is true [34; 107] by the following theorem.

Theorem 7.2 (Efimov [34]) *The numerical Donaldson–Thomas invariants of a symmetric quiver* Q *without potential are positive integers*, $\mathrm{DT}_{\mathsf{d}}^{\mathrm{num}}(\mathsf{Q}) \in \mathbb{N}$.

7.3 KP integrality from DT theory

The genus-zero log-local and log-open correspondences of Theorem 6.7 imply that KP invariants of toric local surfaces are, up to a sign and possibly an integral shift, numerical DT invariants of a symmetric quiver. Combined with Theorem 7.2 this gives an algebrogeometric proof of Conjecture 7.1 for

$$Z = \mathrm{Tot}\big(\mathcal{O}(-D_1)\oplus\mathcal{O}(-D_2)\to Y\big).$$

Theorem 7.3 *Let $Y(D)$ be a 2–component quasi-tame Looijenga pair. Then there exists a symmetric quiver $\mathsf{Q}(Y(D))$ with $\chi(Y)-1$ vertices and a lattice isomorphism $\kappa\colon \mathbb{Z}(\mathsf{Q}(Y(D)))_0 \xrightarrow{\sim} \mathrm{H}_2(Y,\mathbb{Z})$ such that*

$$\mathrm{DT}_d^{\mathrm{num}}(\mathsf{Q}(Y(D))) = \Big|\mathrm{KP}_{\kappa(d)}(E_{Y(D)}) + \sum_i \alpha_i \delta_{d,v_i}\Big|, \tag{7-12}$$

with $\alpha_i \in \{-1,0,1\}$. In particular, $\mathrm{KP}_d(E_{Y(D)}) \in \mathbb{Z}$.

Proof The statement is a direct consequence of Theorem 6.7 combined with the strips–quivers correspondence of [103], which we briefly review here in our context. Since $Y(D)$ is a 2–component quasi-tame pair, it satisfies Property O by the discussion of Section 6.3. From Lemma D.1 and the proof of Theorem 6.7 (see in particular (6-19)), we have

$$\frac{\mathcal{W}_{(j_0)}(Y^{\mathrm{op}}(D))(Q,\hbar)}{\mathcal{W}_{\varnothing}(Y^{\mathrm{op}}(D))(Q,\hbar)} = \frac{(-1)^{fj_0}q^{(f+1/2)\binom{j_0}{2}}}{[j_0]_q!}\frac{\prod_{i=1}^{r}(\widetilde{Q}_i^{(1)};q)_{j_0}}{\prod_{k=1}^{s}(\widetilde{Q}_k^{(2)};q)_{j_0}}, \tag{7-13}$$

where f is the integral shift of f from canonical framing, (r,s) are nonnegative integers with $r+s+1=\chi(Y)-1$, and $\widetilde{Q}_i=\prod_{m=1}^{r+s} Q_m^{a_{m,i}}$ with $a_{m,i}\in\{-1,0,1\}$ for $i=1,\dots,r+s$. Elementary manipulations and use of the q–binomial theorem (see [103, Section 4.1]) show that

$$\text{(7-14)}\quad \begin{aligned}\psi_{Y(D)}(Q,\hbar,z) &:= \sum_{j_0\geq 0}\frac{\mathcal{W}_{(j_0)}(Y^{\mathrm{op}}(D))(Q,\hbar)}{\mathcal{W}_{\varnothing}(Y^{\mathrm{op}}(D))(Q,\hbar)}z^{j_0}\\ &= \frac{\prod_{i=1}^{r}(\tilde{Q}_i;q)_\infty}{\prod_{k=1}^{s}(\tilde{Q}_{r+k};q)_\infty}\\ &\qquad\cdot\Phi_{C(Y(D))}\big(q^{(r-s-1)/2}z,\tilde{Q}_1^{(1)},\ldots,\tilde{Q}_r^{(1)},q^{1/2}\tilde{Q}_1^{(2)},\ldots,q^{1/2}\tilde{Q}_s^{(2)}\big),\end{aligned}$$

where

$$\text{(7-15)}\qquad C(Y(D)) = \left(\begin{array}{c|ccc|ccc} f+1 & \overbrace{1 & \cdots & 1}^{r} & \overbrace{1 & \cdots & 1}^{s} \\ \hline 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \hline 1 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 1 \end{array}\right)\begin{array}{l} \\ \left.\begin{array}{l}\\ \\ \\ \end{array}\right\} r \\ \left.\begin{array}{l}\\ \\ \\ \end{array}\right\} s\end{array}$$

and moreover, the genus-zero limit of the logarithm of (7-14) is the generating function of disk invariants of $Y^{\mathrm{op}}(D)$ [5],

$$\text{(7-16)}\quad \begin{aligned}\lim_{\hbar\to 0}\hbar\log\psi_{Y(D)}(Q,\hbar,z) &= \lim_{\hbar\to 0}\hbar\mathcal{O}(Y^{\mathrm{op}}(D))(Q,\hbar,\mathsf{x})\big|_{\mathsf{x}_{\vec{\mu}}=z^{j_0}\delta_{\vec{\mu},(j_0)}}\\ &= \sum_{\beta} O_{0;j_1,\ldots,j_{r+s};(j_0)}(Y^{\mathrm{op}}(D))z^{j_0}\prod_{i=1}^{r+s}Q_i^{j_i}.\end{aligned}$$

The matrix C has nonnegative off-diagonal entries, and $\Phi_C(q;x_1,\ldots,x_{r+s+1})$ cannot therefore be immediately interpreted as a motivic quiver DT partition function. However, writing $\mathsf{Q}(Y(D))$ for the symmetric quiver with adjacency matrix $C(Y(D))$, we have [103, Appendix A],

$$\text{(7-17)}\quad \begin{aligned}\Phi_{C(Y(D))}(q;x_1,\ldots,x_{r+s+1}) &= \prod_{\mathsf{d}\neq 0}\prod_{j\in\mathbb{Z}}\prod_{k\geq 0}\big(1-(-1)^j\mathsf{x}^{\mathsf{d}}q^{-k-(j+1)/2}\big)^{-\mathcal{E}_{\mathsf{d};j}^{C(Y(D))}}\\ &= \Phi_{E_{\mathsf{Q}(Y(D))}}(q^{-1};q^{-1/2}x_1,\ldots,q^{-1/2}x_{r+s+1}).\end{aligned}$$

The exponents $\mathcal{E}_{\mathsf{d};j}^{C(Y(D))}$ are then equal to the motivic DT invariants of $\mathsf{Q}(Y(D))$ up to sign. Furthermore, the numerical DT invariants also agree with the absolute value of $\mathcal{E}_{\mathsf{d}}^{C(Y(D)),\mathrm{num}} := \sum_j (-1)^j\mathcal{E}_{\mathsf{d};j}^{C(Y(D))}$ [103, Appendix A],

$$\text{(7-18)}\qquad \mathrm{DT}_{\mathsf{d}}^{\mathrm{num}}(\mathsf{Q}(Y(D))) = |\mathcal{E}_{\mathsf{d}}^{C(Y(D)),\mathrm{num}}|.$$

For $j=(j_0,j_1,\ldots,j_{r+s})$, define now the disk BPS invariants of $Y^{\mathrm{op}}(D)$ by

$$\text{(7-19)}\qquad O_{0,j_1,\ldots,j_{r+s};(j_0)}(Y^{\mathrm{op}}(D)) := \sum_{k|\gcd(j_0,\ldots,j_{r+s})}\frac{1}{k^2}\mathfrak{D}_{j/k}(Y^{\mathrm{op}}(D)).$$

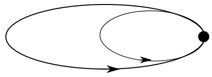
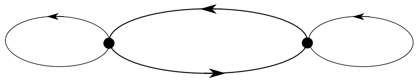
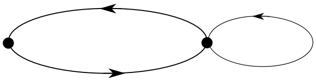
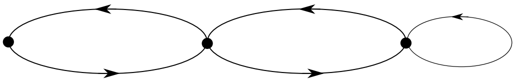
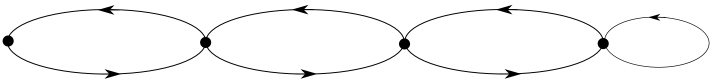
$Y(D)$	$\mathsf{Q}(Y(D))$
$\mathbb{P}^2(1,4)$	
$\mathbb{F}_0(2,2)$ $\mathbb{F}_0(0,4)$	
$\mathrm{dP}_1(1,3)$ $\mathrm{dP}_1(0,4)$	
$\mathrm{dP}_2(1,2)$ $\mathrm{dP}_2(0,3)$	
$\mathrm{dP}_3(1,1)$ $\mathrm{dP}_3(0,2)$	

Table 4: Quivers for 2–component quasi-tame Looijenga pairs.

From (7-13) and (7-16), we have that

$$\mathfrak{D}_{\tau(d)}(Y^{\mathrm{op}}(D)) + \sum_i \alpha_i \delta_{d,v_i} = \mathcal{E}_{\mathsf{d}}^{C(Y(D)),\mathrm{num}},$$

where

$$\alpha_i = \begin{cases} 0 & \text{for } i = 1, \\ -1 & \text{for } i = 2, \dots, r+1, \\ +1 & \text{for } i = r+2, \dots, r+s+1, \end{cases} \tag{7-20}$$

$$\tau(d_1, \dots, d_{r+s+1}) = \left(d_1, \sum_{m=1}^{r+s} a_{m,2} d_{m+1}, \dots, \sum_{m=1}^{r+s} a_{m,r+s} d_{m+1} \right). \tag{7-21}$$

But by (6-13), $O_j(Y^{\mathrm{op(D)}}) = N^{\mathrm{loc}}_{\iota(j)}(Y(D))$, and therefore $\mathfrak{D}_j(Y^{\mathrm{op}}(D)) = \mathrm{KP}_{\iota(j)}(E_{Y(D)})$, from which the claim follows by setting $\kappa := \iota \circ \tau$. ◻

Remark 7.4 Theorem 7.3, combined with Theorem 5.1, resembles previous correspondences identifying log GW invariants to DT invariants of quivers, and in particular [15], but it differs from them in a number of key respects: the quiver DT invariants here are identified with the (absolute value of the) BPS invariants of the local geometry, and therefore imply a finer integrality property of the log invariants via (5-1) and (7-2). Furthermore, unlike in [15], the motivic refinement is not expected to reconstruct the open Gromov–Witten count at higher genus, as the higher orders in $\hbar$ of (7-16) include contributions of open stable maps with more than one boundary component. A separate discussion of the open BPS structure of the higher-genus theory is the subject of the next section.

Example 7.1 Let $Y(D) = \mathbb{P}^2(1,4)$. In this case we have $r = s = 0$, $f = 1$, and $\mathsf{Q}(\mathbb{P}^2(1,4))$ is the 2–loop quiver. Moreover, the identification of dimension vectors with curve degrees is simply the identity, $\kappa = \mathrm{id}$, and the integral shift in (7-12) and (7-20) vanishes, $\alpha_1 = 0$. Then, by Theorem 7.3, the absolute value of the KP invariants of $E_{\mathbb{P}^2(1,4)}$ gives the unrefined DT invariants of $\mathsf{Q}(\mathbb{P}^2(1,4))$. We can in fact check directly that $\mathrm{KP}_d(E_{\mathbb{P}^2(1,4)}) = (-1)^d \mathrm{DT}_d^{\mathrm{num}}(\mathsf{Q}(\mathbb{P}^2(1,4)))$: according to [107, Theorem 3.2],

$$\mathrm{DT}_d^{\mathrm{num}}(\mathsf{Q}(\mathbb{P}^2(1,4))) = \frac{(-1)^d}{d^2} \sum_{k|d} \mu\Big(\frac{d}{k}\Big)(-1)^k \binom{2k-1}{k-1}, \tag{7-22}$$

and the result follows from (3-16) and the equality

$$\frac{1}{2}\binom{2k}{k} = \frac{1}{2}\frac{(2k)!}{(k!)^2} = \frac{1}{2}\frac{2k}{k}\frac{(2k-1)!}{k!(k-1)!} = \binom{2k-1}{k-1}. \tag{7-23}$$

Remark 7.5 (non-quasi-tame pairs) The condition in Theorem 7.3 that $Y(D)$ is a 2–component quasi-tame pair is likely to be necessary. For example, for $Y(D)$ a non-quasi-tame pair, we do not expect that the result of the finite summation (3-21) can be further simplified down to a form akin to (3-8) as a ratio of products of factorials, unlike the case of the hypergeometric summations in the proof of Theorem 4.5. A little experimentation shows that, writing $N_{0,d}^{\mathrm{loc}}(\mathrm{dP}_5(0,0)) = m(d)/n(d)$ with $\gcd(m(d), n(d)) = 1$, the numerator $m(d)$ is divisible by very large primes $\approx 10^7$ for low degrees $d_i \approx 10^1$ with $d_i \neq d_0$ for $i > 0$. This creates a tension with $m(d)$ being a product of factorials with arguments linear in d_i with coefficients $\approx 10^1$, as those would be divisible by at most the largest prime in the range $\approx 10^1 - 10^2$. As generating functions of numerical DT invariants are always generalised hypergeometric functions [102], and their coefficients are therefore always products of ratios of factorials in the degrees, the KP/DT correspondence of Theorem 7.3 is unlikely to extend to the non-quasi-tame setting.

Remark 7.6 ($l > 2$) For l–components pairs with $l > 2$, a correspondence between quivers and $(l-1)$–holed open GW partition functions has received some preliminary investigation in the context of the links-quivers correspondence [72; 36], where open stable maps are considered with the same colouring by symmetric Young diagrams for all the connected components of the boundary. The general case of stable maps with arbitrary windings which is relevant for our purposes may, however, fall outside the remit of the open BPS/quiver DT correspondence. In particular, suppose that Q is a symmetric quiver such that

$$P_{\mathsf{Q}}(\alpha_1 x, \dots, \alpha_r x, \beta_1 y, \dots, \beta_2 y) = \sum_{m,n} x^m y^n \mathcal{W}_{(m),(n)}(X, L_1 \cup L_2, \mathsf{f}_1, \mathsf{f}_2)$$

with $\alpha_i, \beta_i \in \mathbb{C}[q]$ and framed toric special Lagrangians L_1, L_2 in a Calabi–Yau threefold X. The simplest instance is $X = \mathbb{C}^3$ and L_1, L_2 framed toric Lagrangians on different legs: this arises for instance by considering $\mathrm{dP}_1^{\mathrm{op}}(1,1,0)$ and $\mathbb{F}_0^{\mathrm{op}}(2,0,0)$. It is easy to check that the analogue of the semiclassical limit (7-16) for the annulus generating function would be

$$\lim_{q\to 1} D_x^q D_y^q \log P_{\mathsf{Q}}(\alpha_1 x, \dots, \alpha_r x, \beta_1 y, \dots, \beta_s y) = \sum_{j_1, j_2, \beta} x^{j_1} y^{j_2} O_{0;j_1,j_2}(X, L_1 \cup L_2, \mathsf{f}_1, \mathsf{f}_2), \tag{7-24}$$

where D_x^q denotes the q–derivative with respect to x. When $X=\mathbb{C}^3$, a natural guess in line with the disk case would be to take Q a quiver with two vertices, with dimension vectors in bijection with winding numbers along the homology circles in L_1 and L_2. However it is straightforward to verify from (7-6) that for $r+s=2$, the left-hand side of (7-24) does not have a limit as $q\to 1$ unless Q is disconnected, in which case the limit is identically zero, and hence disagrees with the right-hand side. Although this may not necessarily extend to quivers with higher number of vertices and finely tuned identifications of dimension vectors with winding degrees, it does suggest that a suitable generalisation of the correspondence might be required to encompass the counts of annuli as well.

8 Higher-genus BPS invariants

For $Y(D)$ a (not necessarily tame) l–component Looijenga pair satisfying Property O, we define

(8-1) $$\Omega_d(Y(D))(q) := [1]_q^2\left(\prod_{i=1}^{l-1}\frac{d\cdot D_i}{[d\cdot D_i]_q}\right)\sum_{k|d}\frac{\mu(k)}{k}\mathrm{O}_{\iota^{-1}(d/k)}(Y^{\mathrm{op}}(D))(-\mathrm{i}k\log q),$$

and for $Y(D)$ an l–component pair, not necessarily satisfying Property O, we write

(8-2) $$\Omega_d(Y(D))(q):=[1]_q^2\left(\prod_{i=1}^{l}\frac{1}{[d\cdot D_i]_q}\right)\sum_{k|d}(-1)^{d/k\cdot D+l}[k]_q^{l-2}\,k^{l-2}\mu(k)\,\mathrm{N}^{\log}_{d/k}(Y(D))(-\mathrm{i}k\log q).$$

The compatibility of (8-1) and (8-2) when $Y(D)$ satisfies both tameness and Property O follows from Theorem 6.7. From Table 1 and the discussion following Definition 6.3, any quasi-tame l–component Looijenga pair either satisfies Property O, or it is tame, or both: in this setting we will take $\Omega_d(Y(D))(q)$ to be either of the applicable definitions (8-1) or (8-2). We further write simply $\Omega_d(Y(D))$ for the genus-zero limit $\Omega_d(Y(D))(1)$,

(8-3) $$\begin{aligned}\Omega_d(Y(D)) &:= \frac{1}{\prod_{i=1}^{l}(d\cdot D_i)}\sum_{k|d}(-1)^{\sum_{i=1}^{l}d/k\cdot D_i+1}\frac{\mu(k)}{k^{4-2l}}N^{\log}_{0,d/k}(Y(D))\\ &= \sum_{k|d}\frac{\mu(k)}{k^{4-l}}O_{0,\iota^{-1}(d/k)}(Y^{\mathrm{op}}(D))\\ &= \sum_{k|d}\frac{\mu(k)}{k^{4-l}}N^{\mathrm{loc}}_{d/k}(Y(D)).\end{aligned}$$

A priori we can only expect $\Omega_d(Y(D))\in\mathbb{Q}$ and $\Omega_d(Y(D))(q)\in\mathbb{Q}(q^{1/2})$. By (8-2) and (8-3), however, $\Omega_d(Y(D))$ and $\Omega_d(Y(D))(q)$ are amenable to a physical interpretation as Labastida–Mariño–Ooguri–Vafa (LMOV) partition functions [74; 73; 100; 88]. These heuristically count BPS domain walls in an M–theory compactification on $Y^{\mathrm{op}}(D)$ (see in particular [88, equation 2.10]): writing $\Omega_d(Y(D))(q)=\sum_j n_{d,j}(Y(D))q^j$, the LMOV invariants $n_{d,j}(Y(D))$ would compute the net degeneracy of M2–branes with spin j and magnetic and bulk charge specified by d, ending on an M5–brane wrapped around the framed toric Lagrangian L in $Y^{\mathrm{op}}(D)=(X,L,\mathsf{f})$. From the vantage point of string theory, (8-2) (resp. (8-3)) are then expected to be integral Laurent polynomials (resp. integers: for $l=2$, since $\Omega_d(Y(D)) = \mathrm{KP}_d(E_{Y(D)}) = \mathfrak{D}(Y^{\mathrm{op}}(D))$ by (7-2), (7-19), (8-3) and (6-13), this is implied by Theorem 7.3). The next theorem shows that this is indeed the case.

Theorem 8.1 (the higher-genus open BPS property) *Let $Y(D)$ be a quasi-tame Looijenga pair. Then $\Omega_d(Y(D))(q) \in q^{-\frac{1}{2}g_{Y(D)}(d)}\mathbb{Z}[q]$ for an integral quadratic polynomial $g_{Y(D)}(d)$.*

Clearly, from (4-3) and (8-1)–(8-2), we have $\Omega_d(q) = \Omega_d(q^{-1})$, so Theorem 8.1 implies in particular that $\Omega_d(q)$ is a Laurent polynomial truncating at $\mathcal{O}(q^{\pm g_{Y(D)}(d)/2})$.

To prove Theorem 8.1 we shall need the following two lemmas. Let ω_d be a primitive d^{th} root of unity.

Lemma 8.2 (the q–Lucas theorem [99]) *Let $n \geq m$ be nonnegative integers. Then*

$$\begin{bmatrix} n \\ m \end{bmatrix}_{\omega_d} = \omega_d^{\frac{1}{2}m(m-n)} \binom{\lfloor n/d \rfloor}{\lfloor m/d \rfloor} \begin{bmatrix} n - d\lfloor n/d \rfloor \\ m - d\lfloor m/d \rfloor \end{bmatrix}_{\omega_d}. \tag{8-4}$$

In particular, if $d \mid m$ and $d \mid n$,

$$\begin{bmatrix} n \\ m \end{bmatrix}_{\omega_d} = \omega_d^{\frac{1}{2}m(m-n)} \binom{n/d}{m/d}.$$

Proof See eg [108, Theorem 2.2] for a proof. □

Lemma 8.3 *Let $d \mid m \mid n \in \mathbb{Z}^+$. Then*

$$\partial_q \begin{bmatrix} n \\ m \end{bmatrix}_q \bigg|_{q=\omega_d} = 0.$$

Proof For every $i < n$ with $d \nmid i$ we have $\left[\begin{smallmatrix} n \\ i \end{smallmatrix}\right]_{\omega_d} = 0$, since then

$$\begin{bmatrix} n - d\lfloor n/d \rfloor \\ i - d\lfloor i/d \rfloor \end{bmatrix}_{\omega_d} = \begin{bmatrix} 0 \\ i \mod d \end{bmatrix}_{\omega_d} = 0. \tag{8-5}$$

The Cauchy binomial theorem,

$$\sum_{m=0}^{n} t^m q^{\frac{1}{2}m(n+1)} \begin{bmatrix} n \\ m \end{bmatrix}_q = \prod_{i=1}^{n} (1 + tq^i), \tag{8-6}$$

implies that

$$q^{\frac{1}{2}m(n+1)} \begin{bmatrix} n \\ m \end{bmatrix}_q = e_m(q, \dots, q^n), \tag{8-7}$$

where $e_j(x_1, \dots, x_n)$ is the j^{th} elementary symmetric polynomials in n variables. We differentiate (8-7) and evaluate at $q = \omega_d$, where now $d \mid m \mid n$. Write $n = abd$, $m = bd$ for $a, b \in \mathbb{Z}^+$. From (8-6) we find

$$\partial_q \prod_{i=1}^{n} (1+tq^i) = \prod_{i=1}^{n} (1+tq^i) \left(\sum_{j=1}^{n} \frac{jtq^{j-1}}{1+tq^j} \right) = \sum_{i=0}^{n} t^i e_i(q, \dots, q^n) \cdot t \sum_{j=1}^{n} \sum_{k=0}^{\infty} j(-t)^k q^{kj+j-1}. \tag{8-8}$$

Let us now evaluate at $q=\omega_d$ and take the $\mathcal{O}(t^m)$ coefficient on both sides. We have

$$(8\text{-}9)\quad \partial_q e_m(q,\dots,q^n)|_{q=\omega_d} = [t^m]\sum_{i=0}^{n} t^i e_i(\omega_d,\dots,\omega_d^n)\cdot t\sum_{j=1}^{n}\sum_{k=0}^{\infty} j(-t)^k\omega_d^{kj+j-1}$$
$$= [t^{bd}]\sum_{i=0}^{ab} t^{di}\omega_d^{id(n+1)/2}\omega_d^{id(id-n)/2}\binom{ab}{i}\cdot t\sum_{j=1}^{n}\sum_{k=0}^{\infty} j(-t)^k\omega_d^{kj+j-1}$$
$$= \sum_{i=0}^{b-1}\omega_d^{\frac{1}{2}id(d+1)}\binom{ab}{i}\sum_{j=1}^{abd} j(-1)^{bd-1-id}\omega_d^{bdj-1-idj}$$
$$= (-1)^{m+1}\frac{n(n+1)}{2\omega_d}\sum_{i=0}^{b-1}(-1)^i\binom{ab}{i} = \frac{(-1)^{b+m}n(n+1)}{2a\omega_d}\binom{ab}{b},$$

where we have used (8-5) and Lemma 8.2. On the other hand,

$$(8\text{-}10)\quad \frac{\partial}{\partial q}q^{\frac{1}{2}m(n+1)}\begin{bmatrix} n\\ m\end{bmatrix}_q\bigg|_{q=\omega_d} = \frac{m(n+1)}{2\omega_d}\omega_d^{m(m+1)/2}\binom{ab}{b}+\omega_d^{m(n+1)/2}\partial_q\begin{bmatrix} n\\ m\end{bmatrix}_q\bigg|_{q=\omega_d}$$
$$= \frac{m(n+1)}{2\omega_d}(-1)^{b+m}\binom{ab}{b}+\omega_d^{m(n+1)/2}\partial_q\begin{bmatrix} n\\ m\end{bmatrix}_q\bigg|_{q=\omega_d},$$

where in tracking down the last sign factor we have been mindful that $(-1)^{bm}=(-1)^m$ since $b\,|\,m$. The claim then follows by equating (8-9) to (8-10). □

Proof of Theorem 8.1 We break up the proof of the theorem by considering each value of l separately.

• ($l=2$) It suffices to prove the theorem in the case $Y(D)=\mathrm{dP}_3(1,1)$, since $\Omega_d(\mathrm{dP}_3(1,1))=\Omega_d(\mathrm{dP}_3(0,2))$ from (8-1) and the discussion of Section 6.3.1, and all other cases are then recovered from the blowup formulas of Propositions 4.3 and 6.9. Let $\tilde d:=\gcd(d_0,d_1,d_2,d_3)$. We first plug (4-22) into (8-1),

$$(8\text{-}11)\quad \Omega_d(\mathrm{dP}_3(1,1))(q) = [1]_q^2\sum_{k|\tilde d}\mu(k)\frac{(-1)^{(d_1+d_2+d_3)/k}}{[d_0/k]_{q^k}[(d_1+d_2+d_3-d_0)/k]_{q^k}}\Theta_{d/k}(q^k),$$
$$= \frac{[1]_q^2}{[d_0]_q[d_1+d_2+d_3-d_0]_q}\sum_{k|\tilde d}\mu(k)(-1)^{(d_1+d_2+d_3)/k}\Theta_{d/k}(q^k),$$

where

$$(8\text{-}12)\quad \Theta_d(q):=\begin{bmatrix} d_0\\ d_1\end{bmatrix}_q\begin{bmatrix} d_1\\ d_0-d_2\end{bmatrix}_q\begin{bmatrix} d_1+d_2+d_3-d_0\\ d_1\end{bmatrix}_q\begin{bmatrix} d_1\\ d_0-d_3\end{bmatrix}_q.$$

It is immediate to verify that $\Omega_d(\mathrm{dP}_3(1,1))(q)\in q^{-\frac{1}{2}g_{\mathrm{dP}_3(1,1)}(d)}\mathbb{Z}[\![q]\!]$, with

$$(8\text{-}13)\quad g_{\mathrm{dP}_3(1,1)}(d) = 2\,(d_1+d_2+d_3-d_0)\,d_0-d_1^2-d_2^2-d_3^2-d_1-d_2-d_3+2$$

since

$$\begin{bmatrix} n \\ m \end{bmatrix}_q \in q^{-m(n-m)/2}\mathbb{Z}[q], \qquad \frac{1}{[n]_q} \in q^{n/2}\mathbb{Z}[\![q]\!] \quad \text{and} \quad [m]_q \in q^{-m/2}\mathbb{Z}[\![q]\!]$$

as formal Laurent series at $q=0$ with truncating principal part for any positive integers n,m. Furthermore, away from $q=0,\infty$, $\Omega_d(\mathrm{dP}_3(1,1))(q) \in \mathbb{Q}(q^{1/2})$ is a rational function of $q^{1/2}$ with at worst double poles possibly at the zeroes of $[d_0]_q[d_1+d_2+d_3-d_0]_q$, namely $q=\omega^j_{d_0}$ for $j=1,\dots,d_0-1$, and $q=\omega^j_{d_1+d_2+d_3-d_0}$ for $j=1,\dots,d_1+d_2+d_3-d_0-1$. We shall now prove that $\Omega_d(\mathrm{dP}_3(1,1))(q)$ is in fact regular on the unit circle.

First off, upon expanding all q–analogues in (8-11) in cyclotomic polynomials,

$$[n]_q = \prod_{d|n} \Phi_d(q), \tag{8-14}$$

it is straightforward to check that [59, Lemma 5.2]

$$\frac{[\gcd(n,m)]_q}{[n+m]_q}\begin{bmatrix} n+m \\ m \end{bmatrix}_q \in q^{\frac{1}{2}(n+m-nm-\gcd(n,m))}\mathbb{Z}[q], \tag{8-15}$$

which implies that $\Omega_d(\mathrm{dP}_3(1,1))(q)$ is regular on the unit circle outside of $\{\omega^j_{\tilde d}\}_{j=0}^{\tilde d}$, where we recall that $\tilde d := \gcd(d_0,d_1,d_2,d_3)$. Let now

$$\tilde\Omega_d(\mathrm{dP}_3(1,1))(q) := \frac{[d_0]_q[d_1+d_2+d_3-d_0]_q}{[1]_q^2}\Omega_d$$

and $\tilde d_i = d_i/\tilde d$. From Lemma 8.2, we have

$$\begin{aligned}(8\text{-}16)\quad & \Theta_{d/k}(\omega^{kj}_{\tilde d}) \\ &= (-1)^{(d_1+d_2+d_3)j/k}\binom{\tilde d_1\epsilon_{k,j}}{\tilde d_2\epsilon_{k,j}}\binom{\tilde d_1\epsilon_{k,j}}{(\tilde d_0-\tilde d_2)\epsilon_{k,j}}\binom{(\tilde d_1+\tilde d_2+\tilde d_3-\tilde d_0)\epsilon_{k,j}}{\tilde d_1\epsilon_{k,j}}\binom{\tilde d_1\epsilon_{k,j}}{(\tilde d_0-\tilde d_3)\epsilon_{k,j}},\end{aligned}$$

where $\epsilon_{k,j}=\gcd(\tilde d/k, j)$. Then

$$\begin{aligned}(8\text{-}17)\quad & \tilde\Omega_{\tilde d}(\mathrm{dP}_3(1,1))(\omega^j_{\tilde d}) \\ &= \sum_{k|\tilde d}\mu\Big(\frac{\tilde d}{k}\Big)(-1)^{(\tilde d_1+\tilde d_2+\tilde d_3)k(j+1)}\binom{\tilde d_1\epsilon_{\tilde d/k,j}}{\tilde d_2\epsilon_{\tilde d/k,j}}\binom{\tilde d_1\epsilon_{\tilde d/k,j}}{(\tilde d_0-\tilde d_2)\epsilon_{\tilde d/k,j}} \\ &\qquad\cdot\binom{(\tilde d_1+\tilde d_2+\tilde d_3-\tilde d_0)\epsilon_{\tilde d/k,j}}{\tilde d_1\epsilon_{\tilde d/k,j}}\binom{\tilde d_1\epsilon_{\tilde d/k,j}}{(\tilde d_0-\tilde d_3)\epsilon_{\tilde d/k,j}}.\end{aligned}$$

Consider first $\tilde d \neq 1$ and write $\nu_p(n)$ and $\mathrm{rad}(n)$ for the p–adic valuation and the radical of $n\in\mathbb{Z}^+$, respectively. Let $k \mid \tilde d$ and suppose without loss of generality that $\tilde d/k$ has no repeated prime factors, $\tilde d/k = \mathrm{rad}(\tilde d/k)$. Then, for $\omega_{\tilde d}\neq 1$, the following trichotomy holds:

(I) $\tilde{d}/k \nmid j$ and there exists p' prime with $p' \mid \tilde{d}/k$ and $p' \nmid j$.

Let $k' := kp'$. Then $k' \mid \tilde{d}$, $\gcd(k', j) = \gcd(k, j)$, $\mu(\tilde{d}/k') = -\mu(\tilde{d}/k)$. Moreover $(-1)^{k'(j+1)} = (-1)^{k(j+1)}$, which is obvious when p' is odd, and it also holds when $p' = 2$ since in that case j must be odd. Then the contributions from k' and k to the sum (8-17) cancel each other.

(II) $\tilde{d}/k \mid j$ and there exists $p' < \tilde{d}$ such that $p' \mid d$ and $p' \nmid j$.

In this case we have $p' \nmid \tilde{d}/k$, $p' \mid k$. Let $k' := k/p'$. Then as before $\mu(\tilde{d}/k) = -\mu(\tilde{d}/k')$, $(-1)^{k'(j+1)} = (-1)^{k'(j+1)}$ and $\gcd(k', j) = \gcd(k, j)$, and the summand corresponding to k' has opposite sign to the one corresponding to k in (8-17).

(III) $\tilde{d}/k \mid j$ and $\tilde{d}$ has no prime factor $p' \nmid j$.

Suppose for simplicity that $\mathrm{rad}(j)/\mathrm{rad}(\tilde{d})$ is odd, the even case being essentially identical. Then (8-17) is unchanged upon replacing $j =: \prod_{p\mid j} p^{\nu_p(j)}$ with $\prod_{p\mid j,\, p\mid \tilde{d}} p^{\nu_p(j)}$, so we may assume $\mathrm{rad}(j) = \mathrm{rad}(\tilde{d})$. Let p' be such that $\nu_{p'}(\tilde{d}) > \nu_{p'}(j)$ and let $k' := k/p'$. Then once again the contributions of k and k' to (8-17) cancel each other.

All in all, the above shows that $\tilde{\Omega}_d(\mathrm{dP}_3(1,1))$ vanishes at $\omega_{\tilde{d}}^{j}$ for all $\tilde{d} > 1$ and $j = 1, \dots, \tilde{d}-1$. But by Lemma 8.3 these are all double zeroes, and therefore $\Omega_d(\mathrm{dP}_3(1,1))$ is regular therein. Moreover, $\Omega_d(\mathrm{dP}_3(1,1))$ is regular by construction at $q = 1$, where its value is given by replacing all q–expressions in (8-11) by their classical counterparts. Hence $\Omega_d(\mathrm{dP}_3(1,1)) \in \mathbb{Q}[q^{\pm 1/2}]$ is a rational Laurent polynomial; but we also know that $\Omega(\mathrm{dP}_3(1,1))_d \in q^{-\frac{1}{2} g_{\mathrm{dP}_3(1,1)}(d)} \mathbb{Z}[\![q]\!]$ is an integral Laurent series, which thus truncates at $\mathcal{O}(q^{\frac{1}{2} g_{\mathrm{dP}_3(1,1)}(d)})$. The statement of the theorem follows.

- $(l = 3)$ As before, we prove the statement for $Y(D) = \mathrm{dP}_3(0,0,0)$ and recover all 3–component pairs by restriction in the degrees. Let

$$\tilde{d} := \gcd(d_0, d_1, d_2, d_3) \quad \text{and} \quad \hat{d} := d_0^2 - d_0(d_1 + d_2 + d_3) + d_1 d_2 + d_1 d_3 + d_2 d_3.$$

From (4-39),

$$\Omega_d(\mathrm{dP}_3(0,0,0))(q) = [1]_q^2 \sum_{k \mid \tilde{d}} \mu(k) k \frac{(-1)^{(d_0+d_1+d_2)/k+1} [\hat{d}/k^2]_{q^k}}{[d_1/k]_q [d_2/k]_q [d_3/k]_{q^k}} \Xi_{d/k}(q^k), \tag{8-18}$$

where

$$\Xi_d(q) := \begin{bmatrix} d_1 \\ d_0 - d_2 \end{bmatrix}_q \begin{bmatrix} d_2 \\ d_0 - d_3 \end{bmatrix}_q \begin{bmatrix} d_3 \\ d_0 - d_1 \end{bmatrix}_q. \tag{8-19}$$

Outside $q = 0, \infty$, the polynomial $\Omega_d(\mathrm{dP}_3(0,0,0))(q)$ has at worst double poles at $q = \omega_d^j$ only; also it is verified directly that $q^{\frac{1}{2} g_{\mathrm{dP}_3(0,0,0)}(d)} \Omega_d$ has a Taylor expansion at $q = 0$ with integer coefficients, where

$$g_{\mathrm{dP}_3(0,0,0)}(d) = g_{\mathrm{dP}_3(1,1)}(d). \tag{8-20}$$

For $q=1$, the ratios of q–numbers in (8-18) limits to the corresponding classical counterparts, so $\Omega_d(\mathrm{dP}_3(0,0,0))(1)$ is well-defined. Suppose then $q=\omega_{\tilde d}^j \neq 1$. We have that

$$\frac{[\hat d/k^2]_{q^k}}{[d_1/k]_{q^k}[d_2/k]_{q^k}[d_3/k]_{q^k}} = \frac{\hat d}{k d_1 d_2 d_3}\left[\frac{\omega_{\tilde d}^{2j}}{(q-\omega_{\tilde d}^j)^2} + \frac{1}{q-\omega_{\tilde d}^j} + \mathcal{O}(1)\right]. \tag{8-21}$$

This is nearly k–independent, save for the factor of k that cancels the one present in the summand of (8-18). By the same arguments of the previous point, the resulting divisor sum

$$\sum_{k|\tilde d} \mu(k)(-1)^{(d_0+d_1+d_2)/k+1}\,\Xi_{d/k}(q^k)$$

vanishes quadratically at $\omega_{\tilde d}^j$, and therefore $\Omega_d(\mathrm{dP}_3(0,0,0))(q)$ is regular on the unit circle, concluding the proof.

- $(l=4)$ This consists of the single case $Y(D)=\mathbb{F}_0(0,0,0,0)$. Let $\tilde d := \gcd(d_1,d_2)$. We have, from (6-35), that

$$\Omega_d(\mathbb{F}_0(0,0,0,0))(q) = \frac{[1]_q^2}{[d_1]_q^2[d_2]_q^2}\sum_{k|\tilde d}\mu(k)k^2[d_1d_2/k^2]_{q^k}^2. \tag{8-22}$$

In this case we have

$$\mathrm{g}_{\mathbb{F}_0(0,0,0,0)}(d) = 2(d_1d_2-d_1-d_2+1). \tag{8-23}$$

As before, $\Omega_d(\mathbb{F}_0(0,0,0,0))(q)$ is a rational function with an integral Taylor–Laurent expansion at $q=0$, order $\mathrm{g}_{\mathbb{F}_0(0,0,0,0)}(d)/2$ singularities at $q=0,\infty$ and possibly double poles at $q=\omega_{\tilde d}^j$. Expanding (8-22) at $\omega_{\tilde d}^j$ yields

$$\Omega_d(\mathbb{F}_0(0,0,0,0))(q) = \sum_{k|d}\mu(k)\left[\frac{\omega_{\tilde d}^j(\omega_{\tilde d}^j-1)^2}{(q-\omega_{\tilde d}^j)^2} + \frac{2\omega_{\tilde d}^j(\omega_{\tilde d}^j-1)}{q-\omega_{\tilde d}^j}\right] + \mathcal{O}(1), \tag{8-24}$$

which vanishes up to $\mathcal{O}(1)$ since $\sum_{k|d}\mu(k)=0$, hence $\Omega_d(\mathbb{F}_0(0,0,0,0))(q) \in q^{-\frac12 \mathrm{g}_{\mathbb{F}_0(0,0,0,0)}(d)}\mathbb{Z}[q]$. □

9 Orbifolds

In [18], we proposed in the context of toric pairs that the log-local principle should extend to Y a possibly singular $\mathbb{Q}$–factorial projective variety. We expect that this should also hold for nef Looijenga pairs, at least as long as the orbifold singularities are at the intersection of the divisors: the log GW theory is then well-defined since $Y(D)$ is log smooth, and the local GW theory makes sense by viewing Y and $E_{Y(D)}$ as smooth Deligne–Mumford stacks. In particular, introducing singularities gives new infinite lists of examples of nef/quasi-tame/tame Looijenga pairs.

We propose that also Theorems 6.7, 7.3 and 8.1 may extend to the orbifold setting. We present the simplest instance here, and defer a more in-depth discussion, including criteria for the validity of the orbifold versions of Theorems 6.7, 7.3 and 8.1 to [17].

Example 9.1 Let $Y = \mathbb{P}_{(1,1,n)}$ be the weighted projective plane with weights $(1,1,n)$, and $D = D_1 + D_2$ with D_1 a toric line passing through the orbifold point and D_2 a smooth member of the linear system given by the sum of the two other toric divisors. Since $D_1 \sim H/n$, $D_2 \sim (n+1)/nH$ and $H^2 = n$, we have $D_1^2 = 1/n$ and $D_2^2 = (n+1)^2/n$. Therefore $\mathbb{P}_{(1,1,n)}(1/n, (n+1)^2/n)$ is a tame orbifold Looijenga pair.

Local Gromov–Witten invariants of $Y(D)$ can be computed by the orbifold quantum Riemann–Roch theorem of [113]: when restricted to point insertions, it gives (3-8) specialised to the case at hand, and we get

$$N_{0,d}^{\mathrm{loc}}\left(\mathbb{P}_{(1,1,n)}\left(\frac{1}{n}, \frac{(n+1)^2}{n}\right)\right) = \frac{(-1)^{nd}}{(n+1)d^2}\binom{(n+1)d}{d}. \tag{9-1}$$

A toric model and a quantised scattering diagram for $Y(D)$ can be constructed as follows. The fan of $\mathbb{P}_{(1,1,n)}$ has 1–skeleton given by rays generated by $(-1,0)$, $(0,-1)$ and $(1,n)$. We may choose $D_1 = D_{(1,n)}$. Denote by $\widetilde{Y}$ the toric blowup obtained by adding a ray in the direction $(-1,1)$, and denote by E the corresponding divisor. Choose $\widetilde{D} = D_1 + D_2 + E$, where we identify D_1 and D_2 with their proper transforms. Then $\widetilde{Y}(\widetilde{D}) \to Y(D)$ is a corner blowup. The proper transform of $D_{(-1,0)}$ is a (-1)–curve, which we contract $\widetilde{Y}(\widetilde{D}) \to \overline{Y}(\overline{D})$. Then $\overline{Y} \setminus \overline{D}$ has Euler characteristic 0, hence is $(\mathbb{C}^*)^2$, and therefore $\overline{Y}(\overline{D})$ is toric, ie $\widetilde{Y}(\widetilde{D}) \to \overline{Y}(\overline{D})$ is a toric model and we are in the setting of Proposition 2.4. Identifying proper transforms, we have that $\overline{D} = D_1 + D_2 + E$, with D_1 corresponding to the ray $(1,n)$, D_2 to the ray $(0,-1)$ and E to the ray $(-1,1)$. Applying the $\mathrm{SL}(2,\mathbb{Z})$ transformation

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \tag{9-2}$$

we obtain the toric model depicted at left of Figure 17, for which the broken line calculation is straightforward. The result is

$$N_{0,d}^{\log}\left(\mathbb{P}_{(1,1,n)}\left(\frac{1}{n}, \frac{(n+1)^2}{n}\right)\right) = \binom{(n+1)d}{d}, \tag{9-3}$$

$$\mathsf{N}_{d}^{\log}\left(\mathbb{P}_{(1,1,n)}\left(\frac{1}{n}, \frac{(n+1)^2}{n}\right)\right) = \begin{bmatrix} (n+1)d \\ d \end{bmatrix}_q. \tag{9-4}$$

To construct

$$\mathbb{P}^{\mathrm{op}}_{(1,1,n)}\left(\frac{1}{n}, \frac{(n+1)^2}{n}\right),$$

we delete the line D_1. Then $\mathcal{O}(-D_2)$ is trivial on $\mathbb{P}_{(1,1,n)} \setminus D_1 = \mathbb{C}^2$, and $\mathrm{Tot}(K_{\mathbb{P}_{(1,1,n)} \setminus D_1}) = \mathbb{C}^3$, with an outer toric Lagrangian at framing shifted by n. A topological vertex calculation of higher-genus

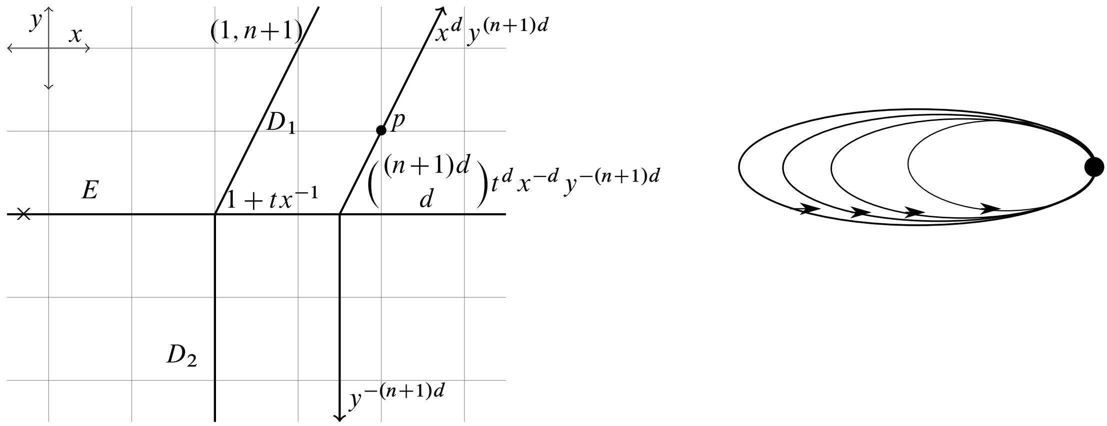


Figure 17: Left: Scatt $\mathbb{P}_{(1,1,n)}$. Right: The quiver for $Y(D) = \mathbb{P}(1,1,3)\left(\frac{1}{3}, \frac{16}{3}\right)$.

1–holed open Gromov–Witten invariants as in Section 6.3.1 shows that

$$\mathsf{O}_d\left(\mathbb{P}^{\mathrm{op}}_{(1,1,n)}\left(\frac{1}{n}, \frac{(n+1)^2}{n}\right)\right) = \frac{(-1)^{nd}}{d\,[(n+1)d]_q}\begin{bmatrix}(n+1)d\\ d\end{bmatrix}_q. \tag{9-5}$$

Equations (9-1), (9-3), (9-4) and (9-5) together imply that Theorems 5.1 and 6.7 extend to this case as well. The arguments in the proof of Theorem 7.3 also apply verbatim, with $\mathsf{Q}(\mathbb{P}_{(1,1,n)}(1/n, (n+1)^2/n))$ the $(n+1)$–loop quiver. An interesting consequence is that the integrality statement of Conjecture 7.1 appears to persist in the orbifold world too. The proof of the higher-genus open BPS property in Theorem 8.1 also carries through to this setting with no substantial modification.

Appendix A Proof of Theorem 3.3

Let Y be the toric surface given by the fan of Figure 18. It is described by the exact sequence

$$0 \to \mathbb{Z}^6 \xrightarrow{\begin{pmatrix} 1&1&0&0&0&-2\\ 0&0&1&0&0&1\\ 1&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&1&0&1&0&0\\ 1&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&1 \end{pmatrix}} \mathbb{Z}^8 \xrightarrow{\begin{pmatrix} 1&1&0&-1&-1&-1&0&1\\ 0&1&1&1&0&-1&-1&-1 \end{pmatrix}} \mathbb{Z}^2 \to 0 \tag{A-1}$$

showing that Y is a GIT quotient

$$\mathbb{C}^8 /\!\!/ (\mathbb{C}^\star)^6 = (\mathbb{C}^8 \setminus \{x_i x_j = 0\}_{(i,j)\neq(1,8),\, j\neq i+1})/(\mathbb{C}^\star)^6,$$

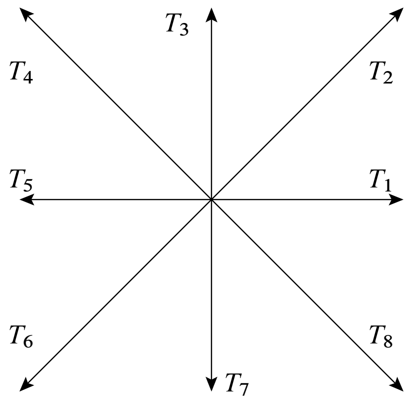


Figure 18: The fan of $\mathrm{Bl}_{4\,\mathrm{pts}}\mathbb{P}^1 \times \mathbb{P}^1$.

with $(\tau_1, \dots, \tau_6) \in (\mathbb{C}^\star)^6$ acting as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{pmatrix} \longrightarrow \begin{pmatrix} \tau_1\tau_2\tau_6^{-2}x_1 \\ \tau_3\tau_6x_2 \\ \tau_1\tau_4x_3 \\ \tau_5x_4 \\ \tau_2\tau_4x_5 \\ \tau_1\tau_3x_6 \\ \tau_4x_7 \\ \tau_5\tau_6x_8 \end{pmatrix}. \tag{A-2}$$

There are dominant birational morphisms $Y \xrightarrow{\pi_1} \mathbb{P}^2$ and $Y \xrightarrow{\pi_2} \mathbb{P}^1 \times \mathbb{P}^1$, obtained by deleting the loci $\{x_i = 0\}_{i \in \{2,4,6,7,8\}}$ and $\{x_{2i} = 0\}$, respectively. Therefore $Y \simeq \mathrm{Bl}_{4\mathrm{pts}}\mathbb{P}^1 \times \mathbb{P}^1$, or equivalently, Y is a five-point toric blowup of $\mathbb{P}^2$, and deforms to dP_5 upon taking the points in general position. From (A-1) and Figure 18, in terms of the hyperplane H and exceptional classes $E_i \in \mathrm{Pic}(\mathrm{dP}_5)$, the toric divisors $T_i := \{x_i = 0\}$ read

$$\begin{aligned} &T_1 = H - E_1 - E_2 - E_4, && T_3 = H - E_1 - E_3 - E_5, && T_5 = E_2 - E_4, && T_7 = E_3 - E_5, \\ &T_2 = E_1, && T_4 = E_4, && T_6 = H - E_2 - E_3, && T_8 = E_5. \end{aligned} \tag{A-3}$$

Under this identification the -2–curve classes T_{2k+1} do not belong to $\mathrm{NE}(\mathrm{dP}_5)$ (see the discussion of Section 2.3); however they do have, by construction, effective representatives in $A_1(Y)$, since they are prime toric divisors.

To write the I–function, we fix the following set of $\frac{1}{2}\mathbb{Z}$–generators of $A_1(Y)$:

$$C_i = \begin{cases} T_{2i} & \text{for } i = 1, \dots, 4, \\ D_{i+4} & \text{for } i = 1, 2, \end{cases} \tag{A-4}$$

where $D_1 = H - E_1 = T_1 + T_3 + 2T_6$ and $D_2 = 2H - E_2 - E_3 - E_4 - E_5 = T_2 + T_4 + T_5 + T_7 - T_8$. We will write φ_i with $(\varphi_i, C_j) = \delta_{ij}$ for their dual basis in cohomology, and denote curve classes in this basis as $d=\sum_i \frac{1}{2}\sigma_i\delta_i C_i$ with $\delta_i\in\mathbb{Z}$, and $\sigma_i=-1$ for $1\le i\le 4$ and $\sigma_i=1$ otherwise. To write the twisted I–function $I^{E_{Y(D)}}$, we need to expand $\theta_a=c_1(\mathcal{O}(T_a))$ and $\kappa_i=c_1(\mathcal{O}(D_i))$ in (3-11), yielding

(A-5)
$$I^{E_{Y(D)}}(y,z)=\sum_{\delta_i\in\mathbb{Z}}\Bigg[\frac{y_1^{-\frac{1}{2}\delta_1}y_2^{-\frac{1}{2}\delta_2}y_3^{-\frac{1}{2}\delta_3}y_4^{-\frac{1}{2}\delta_4}y_5^{\frac{1}{2}\delta_5}y_6^{\frac{1}{2}\delta_6}(-1)^{\delta_5+\delta_6}}{\left(1-\frac{2\varphi_1}{z}\right)_{\delta_1}\left(1-\frac{2\varphi_2}{z}\right)_{\delta_2}\left(1-\frac{2\varphi_3}{z}\right)_{\delta_3}\left(1-\frac{2\varphi_4}{z}\right)_{\delta_4}\left(\frac{z+\varphi_1+\varphi_2+\varphi_5}{z}\right)_{\frac{1}{2}(-\delta_1-\delta_2+\delta_5)}}$$
$$\frac{(2\varphi_6-\lambda_1)(2\varphi_5-\lambda_2)\left(\frac{z+2\varphi_6-\lambda_1}{z}\right)_{\delta_6-1}\left(\frac{z+2\varphi_5-\lambda_2}{z}\right)_{\delta_5-1}}{z\left(\frac{z+\varphi_3+\varphi_4+\varphi_5}{z}\right)_{\frac{1}{2}(-\delta_3-\delta_4+\delta_5)}\left(\frac{z+\varphi_1+\varphi_3+\varphi_6}{z}\right)_{\frac{1}{2}(-\delta_1-\delta_3+\delta_6)}\left(\frac{z+\varphi_2+\varphi_4+\varphi_6}{z}\right)_{\frac{1}{2}(-\delta_2-\delta_4+\delta_6)}}\Bigg],$$

where

(A-6)
$$(a)_n=a(a+1)\cdots(a+n-1)$$

is the Pochhammer symbol. By (3-12), the mirror map is extracted as the formal $\mathcal{O}(z^0)$ Taylor coefficient around $z=\infty$. We find that the sole contributions to the mirror map arise from multiple covers of our chosen generators C_i, that is when $\delta_i=2\sigma_i n$ for some $n\in\mathbb{N}^+$,

(A-7)
$$\tilde{t}^i(y)=\sum_{\delta_i=1}^{\infty}\frac{(2\delta_i-1)!}{(\delta_i!)^2}y^{\delta_i},$$

which is closed-form inverted as

(A-8)
$$y_i(t)=\frac{\exp t^i}{(1-\exp t^i)^2}.$$

Then[14]
$$J_{\text{small}}^{E_{Y(D)}}=I^{E_{Y(D)}}(y(t),z),$$

and from (3-10) and (A-5) we find that whenever $d\ne 2\sigma_i n$ for $n\in\mathbb{N}^+$,

(A-9)
$$N^{\text{loc},\psi}_{\delta_1,\dots,\delta_6}(Y(D))=\frac{1}{\lambda_1\lambda_2}[z^{-1}\mathrm{e}^{\sum t_i\varphi_i/z}\mathbf{1}_{H_T(E_{Y(D)})}]I^{E_{Y(D)}}(y(t),z)$$
$$=[\mathrm{e}^{\sum_i\delta_i t_i}]\sum_{\delta_i'}^{\infty}\mathsf{S}^{[0]}_{\delta_1',\dots,\delta_6'}\prod_{i=1}^{6}y_i(t)^{\frac{1}{2}\sigma_i\delta_i'},$$

[14]To obtain the small J–function, we should include a string-equation induced shift by multiplying the I–function by an overall factor of $\mathrm{e}^{\lambda_1\tilde{t}^5(y)+\lambda_2\tilde{t}^6(y)/z}$, in order to guarantee that the small J–function satisfies its defining property to be the unique family of Lagrangian cone elements with a Laurent expansion of the form $z+t+O(1/z)$ at $z=\infty$. These would result in a correction of the foregoing discussion for degrees $\delta_i=0$ when $i=1,\dots,4$. It is justified to ignore this for our purposes: since $d\cdot D_i=0$ and $\mathcal{O}(-D_i)$ is not a concave line bundle, the corresponding invariants are nonequivariantly ill-defined; and any sensible nonequivariant definition would satisfy automatically the log-local correspondence of Section 5, as the corresponding log invariants are trivially zero.

where

$$\text{(A-10)}\quad \mathsf{S}^{[0]}_{\delta'_1,\dots,\delta'_6} := \frac{(-1)^{\delta'_5+\delta'_6}(\delta'_5-1)!\,(\delta'_6-1)!}{\delta'_1!\,\delta'_2!\,\delta'_3!\,\delta'_4!\,\big(\frac12(\delta'_5-\delta'_1-\delta'_2)\big)!\,\big(\frac12(\delta'_5-\delta'_3-\delta'_4)\big)!\,\big(\frac12(\delta'_6-\delta'_1-\delta'_3)\big)!\,\big(\frac12(\delta'_6-\delta'_2-\delta'_4)\big)!}.$$

The arguments of the factorials in the denominator constrain the range of summation to extend over $\delta_i \neq 0$ alone; in particular the right-hand side is a Taylor series in $(y_1^{-1/2}, y_2^{-1/2}, y_3^{-1/2}, y_4^{-1/2}, y_5^{1/2}, y_6^{1/2})$, convergent in a ball centred at $y_i^{\sigma_i} = 0$.

We first perform the summation over δ'_6 to obtain

$$\text{(A-11)}\quad \sum_{\delta'_6=0}^{\infty} \mathsf{S}^{[0]}_{\delta'_1,\dots,\delta'_6}\, y_6^{\frac12\delta'_6} = \frac{(-1)^{\delta'_2+\delta'_4+\delta'_5}(\delta'_2+\delta'_4-1)!\,(\delta'_5-1)!\left(\dfrac{e^{t_6}}{(e^{t_6}+1)^2}\right)^{\frac12(\delta'_2+\delta'_4)}}{\delta'_1!\,\delta'_2!\,\delta'_3!\,\delta'_4!\,\big(\frac12(-\delta'_1+\delta'_2-\delta'_3+\delta'_4)\big)!\,\big(\frac12(-\delta'_1-\delta'_2+\delta'_5)\big)!\,\big(\frac12(-\delta'_3-\delta'_4+\delta'_5)\big)!} \times {}_2F_1\left(\tfrac12(\delta'_2+\delta'_4), \tfrac12(\delta'_2+\delta'_4+1); \tfrac12(-\delta'_1+\delta'_2-\delta'_3+\delta'_4+2); \frac{4e^{t_6}}{(e^{t_6}+1)^2}\right),$$

where

$$\text{(A-12)}\qquad {}_pF_r(a_1,\dots,a_p;b_1,\dots,b_r;z) := \sum_{k\ge 0} \frac{z^k}{k!}\frac{\prod_{j=1}^{p}(a_j)_k}{\prod_{j=1}^{r}(b_j)_k}$$

is the generalised hypergeometric function. Applying Kummer's quadratic transformation,

$$\text{(A-13)}\qquad {}_2F_1(a,b;a-b+1;z) = (z+1)^{-a}\,{}_2F_1\left(\frac{a}{2},\frac{a+1}{2};a-b+1;\frac{4z}{(z+1)^2}\right),$$

we obtain

$$\text{(A-14)}\qquad N^{\mathrm{loc},\psi}_{\delta_1,\dots,\delta_6}(Y(D)) = [e^{\sum_{i=1}^{5}\delta_i t_i}] \sum_{\delta'_i}^{\infty} \mathsf{S}^{[1]}_{\delta'_1,\dots,\delta'_5,\delta_6} \prod_{i=1}^{5} y_i(t)^{\frac12\sigma_i\delta'_i},$$

where

$$\text{(A-15)}\quad \mathsf{S}^{[1]}_{\delta'_1,\dots,\delta'_5,\delta_6} := \frac{(-1)^{\delta'_2+\delta'_4+\delta'_5}(\delta'_5-1)!\,\big(\frac12(\delta'_1+\delta'_3)+\delta_6-1\big)!\,\big(\frac12(\delta'_2+\delta'_4)+\delta_6-1\big)!}{\delta'_1!\,\delta'_2!\,\delta'_3!\,\delta'_4!\,\big(\frac12(\delta'_1+\delta'_2+\delta'_3+\delta'_4-2)\big)!\,\big(\frac12(-\delta'_1-\delta'_2+\delta'_5)\big)!} \times \frac{1}{\big(\frac12(-\delta'_3-\delta'_4+\delta'_5)\big)!\,\big(-\frac12\delta'_1-\frac12\delta'_3+\delta_6\big)!\,\big(-\frac12\delta'_2-\frac12\delta'_4+\delta_6\big)!}.$$

Performing the same sequence of operations on the sum over δ'_5 yields

$$\text{(A-16)}\qquad N^{\mathrm{loc},\psi}_{\delta_1,\dots,\delta_6}(Y(D)) = [e^{\sum_{i=1}^{4}\delta_i t_i}] \sum_{\delta'_i}^{\infty} \mathsf{S}^{[2]}_{\delta'_1,\dots,\delta'_4,\delta_5,\delta_6} \prod_{i=1}^{4} y_i(t)^{\frac12\sigma_i\delta'_i},$$

where

$$\mathsf{S}^{[2]}_{\delta'_1,\dots,\delta'_4,\delta_5,\delta_6} := \frac{(-1)^{\delta'_1+\delta'_4}\left(\frac{1}{2}(\delta'_1+\delta'_2)+\delta_5-1\right)!\left(\frac{1}{2}(\delta'_3+\delta'_4)+\delta_5-1\right)!\left(\frac{1}{2}(\delta'_1+\delta'_3)+\delta_6-1\right)!}{\delta'_1!\,\delta'_2!\,\delta'_3!\,\delta'_4!\left(\left(\frac{1}{2}(\delta'_1+\delta'_2+\delta'_3+\delta'_4-2)\right)!\right)^2\left(-\frac{1}{2}\delta'_1-\frac{1}{2}\delta'_2+\delta_5\right)!} \times \frac{\left(\frac{1}{2}(\delta'_2+\delta'_4)+\delta_6-1\right)!}{\left(-\frac{1}{2}\delta'_3-\frac{1}{2}\delta'_4+\delta_5\right)!\left(-\frac{1}{2}\delta'_1-\frac{1}{2}\delta'_3+\delta_6\right)!\left(-\frac{1}{2}\delta'_2-\frac{1}{2}\delta'_4+\delta_6\right)!}. \tag{A-17}$$

The final step is to now plug in the mirror maps (A-8) for $i=1,\dots,4$. This gives

$$N^{\mathrm{loc},\psi}_{\delta_1,\dots,\delta_6}(Y(D)) = \sum_{j_1,\dots,j_4=0}^{\infty} \mathsf{S}^{[3]}_{\delta'_1+2j_1,\dots,\delta'_4+2j_4,j_1,\dots,j_4,\delta_5,\delta_6}, \tag{A-18}$$

where

$$\mathsf{S}^{[3]}_{\delta'_1,\dots,\delta'_4,j_1,\dots,j_4,\delta_5,\delta_6} := \mathsf{S}^{[2]}_{\delta'_1,\dots,\delta'_4,\delta_5,\delta_6}\prod_{i=1}^{4}\binom{\delta'_i}{j_i}. \tag{A-19}$$

The change of basis $\{C_1,\dots,C_6\}\to\{H-E_1-\dots-E_5,E_1,\dots E_5\}$ in (A-3) and the corresponding change of variables in the curve degree parameters $\{\delta_1,\dots,\delta_6\}\to\{d_0,\dots,d_5\}$ finally leads to (3-21).

Appendix B Infinite scattering

We compute the invariants of Conjecture 4.7 for the geometries $\mathrm{dP}_1(0,4)$ and $\mathbb{F}_0(0,4)$. This application of our correspondences predicts new relations for q–hypergeometric sums in Conjecture B.3. We provide calculations by picture and leave the details to the reader.

Denote by E the exceptional divisor obtained by blowing up a point on D_1 in $\mathbb{P}^2(1,4)$. We write a curve class $d\in \mathrm{H}_2(\mathrm{dP}_1(0,4),\mathbb{Z})$ as $d=d_0(H-E)+d_1E$. If $d_0=0$ or $d_1=0$, then the moduli space of stable log maps is empty and $\mathsf{N}^{\log}_d(\mathrm{dP}_1(0,4))(\hbar)=0$. If $d_1>d_0$, then there are no irreducible curve classes and $\mathsf{N}^{\log}_d(\mathrm{dP}_1(0,4))(\hbar)=0$. The toric model of $\mathrm{dP}_1(0,4)$ is obtained from the toric model of $\mathbb{P}^2(1,4)$ by adding a focus–focus singularity in the direction of D_1. The opposite primitive vectors in the F_2 and D_1 directions are $\gamma_1=(1,0)$ and $\gamma_2=(-1,-2)$. Since the absolute value of their determinant is 2 and not 1, there is infinite scattering, which is described in Section 4.2. By choosing our broken lines to be sufficiently into the x–direction, we can restrict to walls that lie on the halfspace $x>0$. Then these walls have slope $(n+1)\gamma_1+n\gamma_2=(1,-2n)$ for $n\geq 0$. The wallcrossing functions attached to them are $1+t^{n+1}t_1^n x^{-1}y^{2n}$. The broken line computation is summarised in Figure 20.

Theorem B.1 *Let $d_0>d_1\geq 1$ and $d=d_0(H-E)+d_1E$. Then $\mathsf{N}^{\log}_d(\mathrm{dP}_1(0,4))(\hbar)$ equals*

$$\sum_{m=1}^{d_1}\sum \begin{bmatrix}2d_0\\ k_1\end{bmatrix}_q \begin{bmatrix}2d_0-2(n_1-n_2)k_1\\ k_2\end{bmatrix}_q \cdots \begin{bmatrix}2d_0-2\sum_{j=1}^{m-1}(n_j-n_m)k_j\\ k_m\end{bmatrix}_q \begin{bmatrix}2d_1\\ k_0\end{bmatrix}_q, \tag{B-1}$$

the second summation being over $k_0\geq 0$, $k_1,\dots,k_m>0$ and $n_1>n_2>\dots>n_m>0$ satisfying $k_0+\sum_{j=1}^m k_j=d_1$ and $\sum_{j=1}^m n_jk_j=d_0-d_1$.

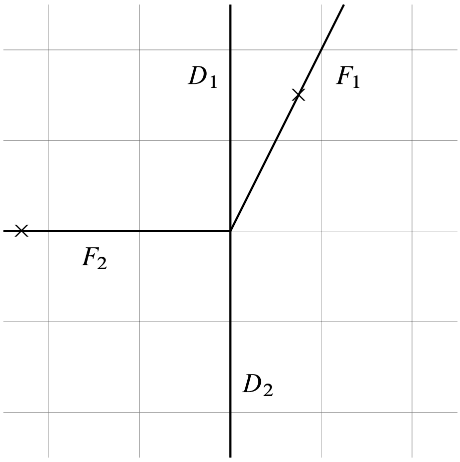


Figure 19: $\mathbb{F}_0(0,4)$.

For the case of $\mathbb{F}_0(0,4)$, let D_1 be a line of bidegree $(1,0)$ and let D_2 be a smooth divisor of bidegree $(1,2)$. Let d be a curve class of bidegree (d_1,d_2). We have $d\cdot D_1 = d_2$ and $d\cdot D_2 = 2d_1+d_2$. Denote by pt_1 (resp. pt_2) their intersection points and by L_1 (resp. L_2) the lines of bidegree $(0,1)$ passing through pt_1 (resp. pt_2). We blow up pt_1 and pt_2, leading to exceptional divisors F_1 and F_2, and blow down the strict transforms of L_1 and L_2. The result is the Hirzebruch surface $\mathbb{F}_2$ with a focus–focus singularity on each of the fibrewise toric divisors, as in Figure 19.

Let d be a curve class of bidegree (d_1,d_2). The opposite primitive vectors in the F_2 and F_1 directions are $\gamma_1=(1,0)$ and $\gamma_2=(-1,-2)$. The absolute value of their determinant is 2, so there is infinite scattering as described in Section 4.2. We choose p to be in the lower left quadrant with coordinate (a,b) for $-1\ll a<0$ and $b\ll 0$. This depends on the degree and ensures that the broken lines are vertical at p. In particular, we can restrict to walls that lie on the halfspace $x<0$. Then these walls have slope $(n-1)\gamma_1+n\gamma_2=(-1,-2n)$ for $n\geq 1$. The wall-crossing functions attached to them are $1+t^{n-1}t_1^n xy^{2n}$. The broken line calculation is summarised in Figure 20.

Theorem B.2 *For $d_1\geq 1$, the generating function $\mathsf{N}^{\log}_{(d_1,d_2)}(\mathbb{F}_0(0,4))(\hbar)$ equals*

$$\sum_{m=1}^{\lfloor\frac12(\sqrt{1+8d_1}-1)\rfloor}\;\sum_{\substack{d_1=\sum_{j=1}^m n_jk_j,\\ k_m,\dots,k_1>0,\\ n_m>\dots>n_1>0}}\begin{bmatrix}d_2+2d_1\\k_m\end{bmatrix}_q\cdots\begin{bmatrix}d_2+2n_i\sum_{j=i}^m k_j+2\sum_{j=1}^{i-1}n_jk_j\\k_i\end{bmatrix}_q$$
$$\cdots\begin{bmatrix}d_2+2n_2\sum_{j=2}^m k_j+2n_1k_1\\k_2\end{bmatrix}_q\begin{bmatrix}d_2+2n_1\sum_{j=1}^m k_j\\k_1\end{bmatrix}_q\begin{bmatrix}d_2\\\sum_{j=1}^m k_j\end{bmatrix}_q.$$

Conjecture 4.4 predicts that the multivariate q–hypergeometric sums of Theorems B.1 and B.2 dramatically simplify to remarkably compact q–binomial expressions. This is expressed by the following new conjectural q–binomial identities.

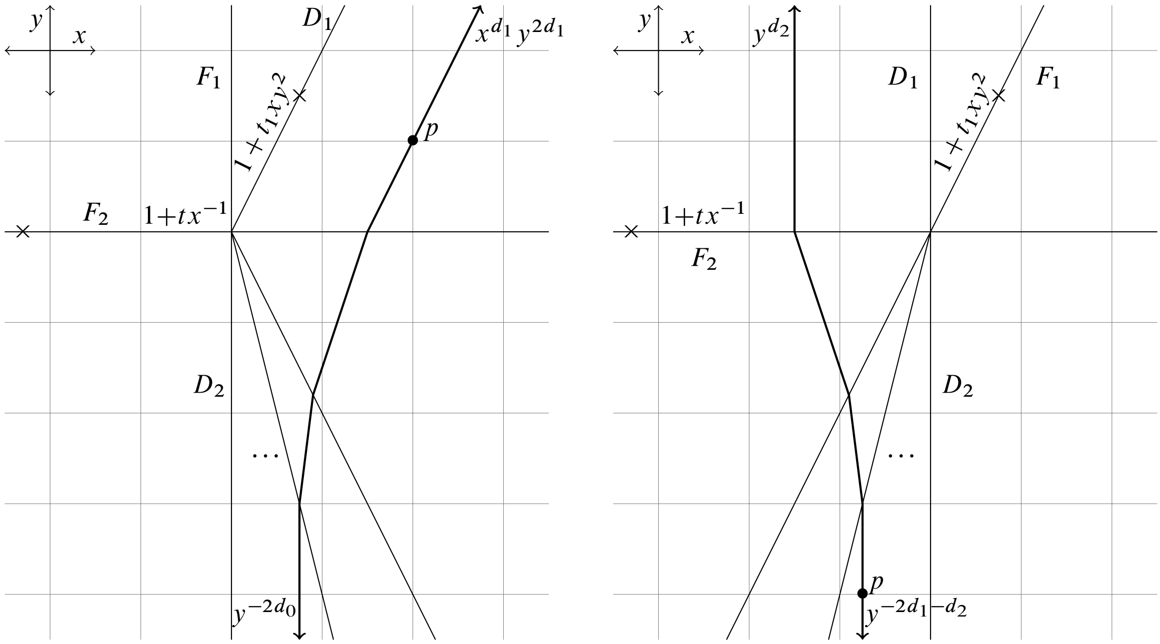


Figure 20: Scattering diagrams of $\mathrm{dP}_1(0,4)$, left, and $\mathbb{F}_0(0,4)$, right.

Conjecture B.3 *The q–hypergeometric sums of Theorems B.1 and B.2 are equal to*

$$\mathsf{N}_d^{\log}(\mathrm{dP}_1(0,4))(\hbar) = \frac{[2d_0]_q}{[d_0]_q}\begin{bmatrix} d_0 \\ d_1 \end{bmatrix}_q \begin{bmatrix} d_0+d_1-1 \\ d_0 \end{bmatrix}_q, \tag{B-2}$$

$$\mathsf{N}_d^{\log}(\mathbb{F}_0(0,4))(\hbar) = \frac{[2d_1+d_2]_q}{[d_2]_q}\begin{bmatrix} d_1+d_2-1 \\ d_1 \end{bmatrix}_q^2, \tag{B-3}$$

where $q = \mathrm{e}^{\mathrm{i}\hbar}$.

A proof of the identities of Conjecture B.3 was communicated to us by C Krattenthaler [71]. Note that the genus-zero log-local correspondence of Theorem 5.1 and the deformation invariance of local Gromov–Witten invariants give an entirely geometric proof of their classical limit at $q=1$.

Appendix C Proof of Theorem 5.4

Recall the notation of Section 5 and let $h\colon \Gamma \to \Delta$ be a rigid decorated parametrised tropical curve with $N_{0,d}^{\mathrm{loc},h}(Y(D)) \neq 0$. Our goal is to prove that $h = \overline{h}$. This will be done by a series of Lemmas constraining further and further the possible shape of h.

Recall that we are considering a degeneration with special fibre $\boldsymbol{P}_0^h$, which is a $(\mathbb{P}^1)^l$–bundle over the special fibre $\mathcal{Y}_0^h$ of a degeneration of the original surface Y. For every vertex V (resp. edge E) of Γ, the corresponding component (resp. node) of a stable log map with tropicalisation h maps to the irreducible component $\boldsymbol{P}_V$ (resp. divisor $\boldsymbol{P}_E$) of the special fibre $\boldsymbol{P}_0^h$. $\boldsymbol{P}_V$ (resp. $\mathbb{P}_E$) is a $(\mathbb{P}^1)^l$–bundle over a component (resp. divisor) Y_V (resp. Y_E) of $\mathcal{Y}_0^h$.

We are considering stable log maps to $\boldsymbol{P}_0^h$ with $l-1>0$ marked points mapping to the interior of $\boldsymbol{P}_{0,Y}$. So irreducible components containing these marked points map to $\boldsymbol{P}_{0,Y}$, and the corresponding vertices of Γ are mapped to v_Y by h. Hence, we can choose a flow on Γ such that unbounded edges are incoming, such that every vertex has at most one outgoing edge, and such that the sink V_0 satisfies $h(V_0)=v_Y$. For every vertex $V \neq V_0$, we denote by E_V the edge outgoing from V. Following the flow, the maps η_V define a cohomology class $\alpha_E \in \mathrm{H}^*(\boldsymbol{P}_E)$ for every edge E of Γ. The degeneration formula can be rewritten as

$$N_{0,d}^{\mathrm{loc},h}(Y(D)) = \eta_{V_0}\biggl(\prod_{E \in \mathcal{E}_{\mathrm{in}}(V_0)} \alpha_E\biggr). \tag{C-1}$$

When used below, "descendant" and "ancestor" always refer to the ordering on the vertices of Γ induced by the flow: a vertex V is "older" than a vertex V' if the flow goes from V to V'.

The proof below consists of three steps. First, in Section C.1, we constrain the form of Γ near the boundary $\partial\Delta$ of the tropicalisation. Then we study the local structure of Γ near the vertex v_Y in Section C.2. Finally, in Section C.3, we combine together the local information obtained near the boundary and near v_Y to obtain global control on Γ.

C.1 Study near the boundary $\partial\Delta$

Recall from (5-8) that the boundary $\partial\Delta$ of Δ is the union of segments $(\partial\Delta)_j$ indexed by $1 \le j \le l$.

Most of the analysis below involves the cohomology classes $\alpha_E \in H^\star(\mathbb{P}_E)$ recursively attached by the flow to the edges E of Γ. Geometrically, the class α_E captures the constraints on the position of the node dual to the edge E imposed by the ability to glue together the curve components corresponding to vertices coming before E in the flow. For every edge E of Γ, we denote by $H_{j,E} \in \mathrm{H}^2(\boldsymbol{P}_E)$ the first Chern class of the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathcal{Y}}(-\mathcal{D}_j)|_{Y_E} \oplus \mathcal{O}_{Y_E})}(1)$. Geometrically, to have α_E positively proportional to $H_{j,E}$ means that the node corresponding to E is constrained to be in the preimage by the natural projection $\boldsymbol{P}_E \to \mathbb{P}(\mathcal{O}_{\mathcal{Y}}(-\mathcal{D}_j)|_{Y_E} \oplus \mathcal{O}_{Y_E})$ of a section of $\mathbb{P}(\mathcal{O}_{\mathcal{Y}}(-\mathcal{D}_j)|_{Y_E} \oplus \mathcal{O}_{Y_E})$.

We will use below the following facts on the classes $H_{j,E}$. We have $H_{j,E}^2 = -c_1(\mathcal{O}_{\mathcal{Y}}(-\mathcal{D}_j)|_{Y_E})H_{j,E}$. If $h(E) \not\subset (\partial\Delta)_j$, then $\mathcal{O}_{\mathcal{Y}}(-\mathcal{D}_j)|_{Y_E} = \mathcal{O}_{Y_E}$ and so $H_{j,E}^2 = 0$. If $h(E) \subset (\partial\Delta)_j$, then $\mathcal{O}_{\mathcal{Y}}(-\mathcal{D}_j)|_{Y_E} = \mathcal{O}(-1)$, and so $H_{j,E}^2 = (\pi_E^* \mathrm{pt}_E) H_{j,E}$. If V is a vertex of Γ with $h(V) \in (\partial\Delta)_j$ and $d_V \cdot D_{j,V}^\partial > 0$ for some $1 \le j \le l$, then, as the line bundle $\mathcal{O}_{\mathcal{Y}}(-\mathcal{D}_j)|_{Y_V} = \mathcal{O}_{Y_V}(-D_{j,V}^\partial)$ has negative degree in restriction to the curve corresponding to V, this curve is constrained to lie in the zero section of $\mathbb{P}(\mathcal{O}_{\mathcal{Y}}(-\mathcal{D}_j)|_{Y_V} \oplus \mathcal{O}_{Y_V})$, and so α_{E_V} is a nonzero multiple of H_{j,E_V}.

Lemma C.1 *Let V be a vertex of Γ with $h(V) \in (\partial\Delta)_j$ for some $1 \le j \le l$. Then we have $d_V \cdot D_{j,V}^\partial > 0$ if and only if there is an edge E of Γ incident to V such that $h(E) \not\subset (\partial\Delta)_j$.*

Figure 21: Left: toric fan of Y_V for $V \in (\partial\Delta)_j - \{v_j\}$ obtained by adding rays in the lower part of the toric fan of $\mathbb{P}^1 \times \mathbb{P}^1$ (in thick). Right: toric fan of Y_V for $V = v_j$ obtained by adding rays in the lower part of the toric fan of $\mathbb{F}_{D_j^2}$ (in thick).

Proof First assume that $h(V) \neq v_j$. Then Y_V can be described as a toric blowup of $\mathbb{P}^1 \times \mathbb{P}^1$, where all the added rays are contained in the lower half-plane of the fan, and where the vertical ray corresponds to $D^{\partial}_{j,V}$; see Figure 21, left. The lower part of the fan gives a local picture of Δ^h near $h(V)$. By definition of the Δ^h, every edge E of Γ incident to V is mapped by h to one of the rays in the lower part of the fan. We have $h(E) \not\subset (\partial\Delta)_j$ if and only if E is contained in one of the rays in the strict lower part of the fan. The result then follows from toric homological balancing.

If $h(V) = v_j$, the argument is similar. Recall that we have $D_j \simeq \mathbb{P}^1$. The key point is that D_j is nef and so $D_j^2 \geq 0$. Therefore, Y_V can be described as a toric blowup of the Hirzebruch surface $\mathbb{F}_{D_j^2}$, where all the added rays are contained in the lower half-part of the fan, and where the vertical ray, with self-intersection D_j^2, corresponds to $D^{\partial}_{j,V}$; see Figure 21, right. The lower part of the fan gives a local picture of Δ^h near $h(V)$. By definition of the Δ^h, every edge E of Γ incident to V is mapped by h to one of the rays in the lower part of the fan. We have $h(E) \not\subset (\partial\Delta)_j$ if and only if $h(E)$ is contained in on of the rays in the strict lower part of the fan. As $D_j^2 \geq 0$, the lower part of the fan is convex and so the result follows from toric homological balancing. □

Lemma C.2 *Let V be a vertex of Γ with $h(V) \in (\partial\Delta)_j$. Assume that there exists an incoming edge E incident to V such that α_E is a nonzero multiple of $H_{j,E}$. Then $h(E_V) \subset (\partial\Delta)_j$ and α_{E_V} is a nonzero multiple of H_{j,E_V}.*

Proof If $h(E_V) \not\subset (\partial\Delta)_j$, then, by Lemma C.1, we have $d_V \cdot D^{\partial}_{j,V} > 0$ and so α_{E_V} is proportional to $H^2_{j,E_V} = 0$. Therefore, $\alpha_{E_V} = 0$, in contradiction with the assumption $N_d^{\mathrm{loc},h}(Y(D)) \neq 0$, and so this does not happen. Hence, we can assume that $h(E_V) \subset (\partial\Delta)_j$. If $d \cdot D^{\partial}_{j,V} > 0$, then α_{E_V} is a multiple of $H^2_{j,E_V} = (\pi^*_{E_V} \mathrm{pt}_{E_V}) H_{j,E_V}$. If $d \cdot D^{\partial}_{j,V} = 0$, then α_{E_V} is a multiple of H_{j,E_V}. □

Lemma C.3 *Let V be a vertex of Γ with $V \neq V_0$ and an incident incoming edge E with α_E a nonzero multiple of $H_{j,E}$. Then α_{E_V} is a nonzero multiple of H_{j,E_V}.*

Proof If $h(V) \in (\partial\Delta)_j$, then the result follows from Lemma C.2. If $h(V) \notin (\partial\Delta)_j$, then the result is clear as the line bundle $\mathcal{O}_{\mathcal{Y}}(-\mathcal{D}_j)|_{Y_V}$ is trivial. □

Lemma C.4 *Let V be a vertex of Γ such that $V \in (\partial\Delta)_j$ and such that there exists an incoming edge E incident to V with $h(E) \not\subset (\partial\Delta)_j$. Then α_{E_V} is a nonzero multiple of H_{j,E_V}, and we have $h(E_V) \not\subset (\partial\Delta)_j$.*

Proof By Lemma C.1, we have $d_V \cdot D^{\partial}_{j,V} > 0$, and so α_{E_V} is a nonzero multiple of H_{j,E_V}. If we had $h(E_V) \subset (\partial\Delta)_j$, then by iterative application of Lemma C.2, all the descendants of V would be mapped by h to $(\partial\Delta)_j$, in contradiction with the fact that the sink V_0 of Γ is mapped by h to v_Y. □

We say that a vertex V of Γ is a source if every bounded edge incident to V is outgoing. As we are assuming that every vertex of Γ has at most one outgoing edge, a source has a unique bounded incident edge. For a vertex V of Γ such that $h(V) \in \Delta - \partial\Delta - \{v_Y\}$, the toric balancing condition holds at $h(V)$. As the toric balancing condition cannot hold at a vertex with a unique incident bounded edge, we deduce that if V is a source of Γ, then either $h(V) = v_Y$ or $h(V) \in \partial\Delta$.

Lemma C.5 *Let V be a vertex of Γ such that V is a source and $h(V) \in \partial\Delta$. Then there exists $1 \leq j \leq l$ such that α_{E_V} is a nonzero multiple of H_{j,E_V}.*

Proof We know that $h(V) \in (\partial\Delta)_j$ for at least one j. Assume first that $h(V) \neq v_p$ for every $p \in D_j \cap D_{j'}$, that is, $h(V) \in (\partial\Delta)_j$ for a unique j. As V is a source, there is a single edge incident to V. By homological toric balancing (see Figure 21, left), this is possible only if $h(E_V)$ is contained in the ray opposite to the ray corresponding to $D^{\partial}_{j,V}$, and in particular we then have $d_V \cdot D^{\partial}_{j,V} > 0$.

It remains to treat the case where $h(V) = v_p$ for some $p \in D_j \cap D_{j'}$. In this case, we have $h(V) = v_p \in (\partial\Delta)_j \cap (\partial\Delta)_{j'}$. By homological toric balancing (see Figure 21, right), we necessarily have $d_V \cdot D^{\partial}_{k,V} > 0$ for some $k \in \{j, j'\}$. □

Lemma C.6 *Let V be a vertex of Γ such that V is a source, $h(V) \in (\partial\Delta)_j$ for some $1 \leq j \leq l$, and $h(V) \neq v_p$ for every $p \in D_j \cap D_{j'}$. Then d_V is a multiple of the class of a $\mathbb{P}^1$–fibre of Y_V and $h(E_V) \not\subset (\partial\Delta)_j$.*

Proof Similar to the proof of Lemma C.5. □

C.2 Study near the centre v_Y

Lemma C.7 *Let E be a bounded edge of Γ such that α_E is not a nonzero multiple of any $H_{j,E}$. Then we have $E = E_V$, where V is a source of Γ with $h(V) = v_Y$.*

Proof For a source V of Γ, we have either $h(V) = v_Y$ or $h(V) \in \partial\Delta$. If one of the source ancestors V of E had $h(V) \in \partial\Delta$, we would have, by combination of Lemma C.5 and Lemma C.3, that α_E is a nonzero multiple of $H_{j,E}$ for some $1 \leq j \leq l$. Therefore, for every source V that is an ancestor of E, we have $h(V) = v_Y$.

Assume by contradiction that there are at least two distinct source ancestors of E. Then there exists a vertex V which is an ancestor of E where at least two distinct source edges meet. As the source edges are emitted by sources mapped to v_Y by h, they can only meet if their images by h are contained in a common half-line in Δ with origin v_Y. If $h(V) \in (\partial\Delta)_j$ for some j, then α_{E_V}, and so α_E by Lemma C.3, would have been a nonzero multiple of $H_{j,E}$ by Lemma C.3. Therefore, $h(V) \in \Delta - \partial\Delta$. On the other hand, we have $h(V) \neq v_Y$. Therefore, the toric balancing condition applies at $h(V)$ and $h(E_V)$ is parallel to the direction of the incoming edges. Moving $h(V)$ along the common direction of all the edges incident to V produces a contradiction with the assumed rigidity of h.

Therefore, E admits a unique ancestor source V. So any other vertex of Γ along the flow from V to E would have to be a 2–valent vertex, in contradiction with the rigidity of h. We conclude that $E = E_V$. □

From now on, we assume that $l = 2$. In this case, Γ has a unique unbounded edge, and we choose the flow such that V_0 is the vertex V of Γ incident to this unbounded edge.

Lemma C.8 *The set of bounded edges of Γ incident to V_0 consists of two elements E_1 and E_2 with $\alpha_{E_1} = \lambda_1 H_{1,E_1}$ and $\alpha_{E_2} = \lambda_2 H_{2,E_2}$, where $\lambda_1, \lambda_2 \in \mathbb{Q} - \{0\}$.*

Proof It follows from Lemma C.7 that, for every bounded edge E incident to V_0, there exists a $1 \le j \le 2$ such that α_E is a nonzero multiple of $H_{j,E}$. As the moduli space M_{V_0} contains two $\mathbb{P}^1$–factors corresponding to the two extra directions $\mathcal{O}_Y^{\oplus 2}$, the condition $N_d^{\mathrm{loc},h}(Y(D)) \neq 0$ implies that for every $1 \le j \le 2$, there exists at least one bounded edge E incident to V_0 with α_E a nonzero multiple of $H_{j,E}$. As $H_1^2 = H_2^2 = 0$ on $\boldsymbol{P}_{V_0}$, for every $1 \le j \le 2$ there is at most one bounded edge incident to V_0 with α_E a nonzero multiple of $H_{j,E}$.

Therefore, we have two cases. Either the set of bounded edges incident to V_0 consists of one edge E with α_E a nonzero multiple of $H_{1,E} H_{2,E}$, or the set of bounded edges incident to V_0 consists of two edges E_1 and E_2 with α_{E_1} a nonzero multiple of $H_{1,E}$ but not of $H_{2,E}$, and α_{E_2} a nonzero multiple of $H_{2,E}$ but not $H_{1,E}$.

Let us show that the first case does not arise. If the set of bounded edges incident to V_0 consists of a single element, then the moduli space M_{V_0} has virtual dimension 2. Indeed, the virtual dimension of M_{V_0} is $0+2$, where 0 is the virtual dimension for rational curves in the log Calabi–Yau surface Y intersecting the boundary divisor D in a single point, and 2 comes from the two extra trivial directions $\mathcal{O}_Y^{\oplus 2}$. But we need to integrate over $[M_{V_0}]^{\mathrm{vir}}$ the pullbacks of the class $H_{1,E} H_{2,E}$ (coming from the bounded edge E incident to V_0) and the pullback of $\pi_{V_0}^* \mathrm{pt}_Y$ (coming from the unbounded edge incident to V_0). Therefore, the integrand is a class of degree at least $3 > 2$, and so this case does not arise if $N_d^{\mathrm{loc},h}(Y(D)) \neq 0$.

Thus, we are in the second case, where the set of bounded edges incident to V_0 consists of two edges E_1 and E_2 with α_{E_1} a nonzero multiple of $H_{1,E}$ but not of $H_{2,E}$, and α_{E_2} a nonzero multiple of $H_{2,E}$ but not $H_{1,E}$. In particular, the moduli space M_{V_0} has virtual dimension 3. Indeed, the virtual dimension of M_{V_0} is $1+2$, where 1 is the virtual dimension for rational curves in the log Calabi–Yau surface Y

intersecting the boundary divisor D in two points, and 2 comes from the two extra trivial directions $\mathcal{O}_Y^{\oplus 2}$. As we need to integrate over $[M_{V_0}]^{\mathrm{vir}}$ the pullbacks of the classes α_{E_1}, α_{E_2} and $\pi_{V_0}^* \mathrm{pt}_Y$, with $\deg \alpha_{E_1} \geq 1$ and $\deg \alpha_{E_2} \geq 1$, the condition $N_d^{\mathrm{loc},h}(Y(D)) \neq 0$ implies that $\deg \alpha_{E_1} = \deg \alpha_{E_2} = 1$ and so the classes α_{E_1} and α_{E_2} are scalar multiples of H_{1,E_1} and H_{2,E_2}, respectively. ◻

C.3 End of the proof

It follows from Lemma C.8 that there exists a unique decomposition

$$\Gamma = \Gamma_1 \cup \Gamma_2,$$

where Γ_1 and Γ_2 are connected subgraphs of Γ such that

(i) V_0 is a vertex of both Γ_1 and Γ_2,

(ii) if V is a vertex of Γ distinct from V_0, then V is a vertex of Γ_1 (resp. Γ_2) if and only if the flow starting at V ends at V_0 along the edge E_1 (resp. E_2).

As Γ is a graph of genus zero, the intersection $\Gamma_1 \cap \Gamma_2$ consists only of the common vertex V_0.

Lemma C.9 *For every $1 \leq j \leq 2$, there exists a unique vertex V_j of Γ such that $V_j \in \Gamma_j$ and $V_j \in \partial\Delta$. Moreover, $V_j \in (\partial\Delta)_j$ and $V_{j'} \notin (\partial\Delta)_{j'}$, where $\{j, j'\} = \{1, 2\}$.*

Proof By symmetry, we can assume $j = 1$ and $j' = 2$. We first remark that if V is a vertex of Γ_1 such that $V \in \partial\Delta$, then $V \in (\partial\Delta)_1$ and $V \notin (\partial\Delta)_2$. Otherwise, there would be a descendant V' of V_1 such that $h(V') \notin (\partial\Delta)_2$ and $h(E_{V'}) \not\subset (\partial\Delta)_2$, and so by Lemma C.1, $\alpha_{E_{V'}}$ would be a nonzero multiple of $H_{2,E_{V'}}$, and so by Lemma C.3, α_{E_1} would be a nonzero multiple of H_{2,E_1}, a contradiction.

As $\alpha_{E_1} = \lambda_1 H_{1,E_1}$ with $\lambda_1 \neq 0$, it follows from Lemma C.1 that there exists a vertex V_1 of Γ_1 such that $V_1 \in (\partial\Delta)_1$ and $h(E_{V_1}) \not\subset (\partial\Delta)_1$. Moreover, there exists a unique vertex with these properties: else, by Lemma C.3, α_{E_1} would be proportional to $H_{1,E_1}^2 = 0$, a contradiction. Our first remark applied to V_1 shows that $V_1 \notin (\partial\Delta)_2$.

It remains to show that if V is a vertex of Γ_1 such that $h(V) \in (\partial\Delta)_1 \setminus (\partial\Delta)_2$, then $V = V_1$. Assume by contradiction that there exists a vertex V of Γ_1 such that $h(V) \in \partial\Delta$ and $V \neq V_1$. Up to replacing V by one of its ancestors, we can assume that no ancestor of V is contained in $\partial\Delta$. There are now two cases. First, if V is a source, then, by Lemma C.6, $h(E_V) \not\subset (\partial\Delta)_1$, and so $V = V_1$ by the uniqueness of V_1, a contradiction. Second, if V is not a source, then there exists an edge E incident to V such that $h(E) \not\subset (\partial\Delta)_1$, and so by Lemma C.4, $h(E_V) \not\subset (\partial\Delta)_1$, and so $V = V_1$ by the uniqueness of V_1, a contradiction again. ◻

We now explain how to conclude the proof of Theorem 5.4, that is, show that $h = \overline{h}$. We say that an edge E of Γ is *radial* if $h(E) \not\subset \partial\Delta$ and the direction of $h(E)$ passes through v_Y. We claim that all edges of Γ_1 are radial. Indeed, let V be a vertex of Γ_1 such that $V \neq V_1$ and $h(V) \neq v_Y$. Then $h(V) \notin \partial\Delta$, so $H_{1,V}^2 = 0$, and so there exists at most one edge E incident to V such that α_E is a nonzero multiple of $H_{1,E}$. On the other hand, edges E such that α_E is not a nonzero multiple of $H_{1,E}$ are radial by Lemma C.7. As $h(V) \notin \partial\Delta$ and $h(V) \neq v_Y$, the toric balancing condition holds for V, and so if all incident edges to E except possibly one are radial, then they are in fact all radial.

As all edges of Γ_1 are radial, every vertex V of Γ_1 satisfies either $V = V_1$ or $h(V) = v_Y$: else, moving V in the radial direction would contradict the assumed rigidity of h. In other words, the graph Γ_1 has a very simple form: a vertex V_1 connected by some vertices V such as $h(V) = v_Y$. On the other hand, as all the edges through V_1 are radial, it follows from toric homological balancing that the curve class d_{V_1} is a multiple of the class of a $\mathbb{P}^1$–fibre of Y_{V_1} and that $h(V_1) = v_1$. In this context, the dimension argument of [43, Lemma 5.4] shows that a nonzero Gromov–Witten invariant is only possible if the curve component corresponding to the vertex V_1 has maximal tangency, that is, if there is a single edge incident to V_1. It follows in particular that V_0 is the single vertex of Γ_1 whose image by h is v_Y. Replacing Γ_1 by Γ_2 in the previous arguments, we finally obtain that $h = \bar{h}$. □

Appendix D Symmetric functions

D.0.1 Partitions and representations of S_n A partition $\lambda \vdash d$ of a nonnegative integer $d \in \mathbb{N}$ is a monotone nonincreasing sequence $\lambda := \{\lambda_i\}_{i=1}^r$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0$ such that $\sum_{i=1}^r \lambda_i = d$; when $d = 0$ we write $\lambda = \varnothing$ for the empty partition. We will often use the shorthand notation

(D-1) $$\{\lambda_1^{n_1}, \dots, \lambda_k^{n_k}\} := \{\overbrace{\lambda_1, \dots, \lambda_1}^{n_1 \text{ times}}, \dots, \overbrace{\lambda_k, \dots, \lambda_k}^{n_k \text{ times}}\}$$

for partitions with repeated entries.

With notation as in the beginning of Section 6, a partition λ is bijectively associated to:

- A Young diagram Y_λ with $m_j(\lambda)$ rows of boxes of length j; there is a natural involution in the space of partitions, $\lambda \to \lambda^t$, given by transposition of the corresponding Young diagram.
- A conjugacy class $C_\lambda \in \mathrm{Conj}(S_{|\lambda|})$ of the symmetric group $S_{|\lambda|}$ with automorphism group of order $|\mathrm{Aut}_{C_\lambda}| = |\lambda|! z_\lambda$, with
$$z_\lambda := \prod_j m_j(\lambda)! j^{m_j(\lambda)}.$$
- An irreducible representation $\rho_\lambda \in \mathrm{Rep}(S_d)$. For $\eta \in \mathrm{Conj}(S_d)$, we write $\chi_\lambda(\eta)$ for the irreducible character $\mathrm{Tr}_{\rho_\lambda}(\eta)$.
- By Schur–Weyl duality, an irreducible representation $R_\lambda \in \mathrm{Rep}(\mathrm{GL_n}(\mathbb{C}))$ for $n \geq \ell_\lambda$.

We will be concerned with two linear bases of the ring of integral symmetric polynomials in n variables, $\Lambda_n := \mathbb{Z}[x_1, \dots, x_n]^{S_n}$, labelled by partitions with $\ell(\lambda) \leq n$. Write $x := (x_1, \dots, x_n)^{S_n} \in \mathbb{C}^n / S_n$ for an orbit x of the adjoint action of $\mathrm{GL}_n(\mathbb{C})$ (equivalently, the Weyl group action on $\mathbb{C}^n$), and g_x for any element of the orbit. We write

(D-2) $$p_\lambda(x) := \prod_i \mathrm{Tr}_{\mathbb{C}^n} g_x^{m_i(\lambda)} \quad \text{and} \quad s_\lambda(x) := \mathrm{Tr}_{R_\lambda}(g_x)$$

for, respectively, the symmetric power function and the Schur function determined by λ; we have $\Lambda_n = \mathrm{span}_{\mathbb{Z}}\{p_\lambda\}_{\{\lambda\in\mathcal{P},\ell_\lambda\le n\}} = \mathrm{span}_{\mathbb{Z}}\{s_\lambda\}_{\{\lambda\in\mathcal{P},\ell\lambda\le n\}}$. These two bases are related as

$$s_\mu(x) = \sum_{|\lambda|=|\mu|} \frac{\chi_\mu(\lambda)}{z_\lambda} p_\lambda(x) \quad \text{and} \quad p_\mu(x) = \sum_{|\lambda|=|\mu|} \chi_\mu(\lambda) s_\lambda(x). \tag{D-3}$$

For λ,μ a pair of partitions, the skew Schur polynomials $s_{\lambda/\mu}(x)$ are defined by

$$s_{\lambda/\mu}(x) = \sum_{\nu\in\mathcal{P}} \mathsf{LR}^\lambda_{\mu\nu} s_\nu(x), \tag{D-4}$$

where $\mathsf{LR}^\lambda_{\mu\nu}$ are the Littlewood–Richardson coefficients $R_\mu \otimes R_\nu =: \bigoplus_{\lambda\vdash(|\mu|+|\nu|)} \mathsf{LR}^\lambda_{\mu\nu} R_\lambda$.

Let $\rho\colon \Lambda_n \to \Lambda_{n+1}$ be the monomorphism of rings defined by $\rho(p_{(i)}(x_1,\dots,x_n)) = p_{(i)}(x_1,\dots,x_{n+1})$. We define the ring of symmetric functions $\Lambda := \varinjlim \Lambda_n$ as the direct limit under these inclusions, and denote by the same symbols p_λ, s_λ and $s_{\lambda/\mu}$ the symmetric functions obtained as the images of the power sums, Schur polynomials and skew Schur polynomials under the direct limit. In the next sections it will be of importance to formally expand the infinite product $\prod_{i,j}(1-x_iy_j) \in \Lambda\otimes_{\mathbb{Z}}\Lambda$ around $(x,y)=(0,0)$, and it is a classical result in the theory of symmetric functions out that this expansion can be cast in multiple ways in terms of an average over partitions of bilinear expressions of linear generators of Λ. In particular, we have the Cauchy identities

$$\sum_{\lambda\in\mathcal{P}} s_\lambda(x)s_\lambda(y) = \prod_{i,j}(1-x_iy_j)^{-1} \quad \text{and} \quad \sum_{\lambda\in\mathcal{P}} s_\lambda(x)s_{\lambda^t}(y) = \prod_{i,j}(1+x_iy_j). \tag{D-5}$$

A skew generalisation of these [80, Section I.5] is

$$\begin{aligned} \sum_{\lambda\in\mathcal{P}} s_{\lambda/\mu}(x)s_{\lambda/\nu}(y) &= \prod_{i,j}(1-x_iy_j)^{-1} \sum_{\eta\in\mathcal{P}} s_{\nu/\eta}(x)s_{\mu/\eta}(y), \\ \sum_{\lambda\in\mathcal{P}} s_{\lambda^t/\mu}(x)s_{\lambda/\nu}(y) &= \prod_{i,j}(1+x_iy_j) \sum_{\eta\in\mathcal{P}} s_{\nu^t/\eta}(x)s_{\mu^t/\eta^t}(y). \end{aligned} \tag{D-6}$$

Another noteworthy sum we will need is [80, Section I.5]

$$\sum_{\delta\in\mathcal{P}} s_{\lambda/\delta}(x)s_{\delta/\nu}(y) = s_{\lambda/\mu}(x,y), \tag{D-7}$$

where $s_{\lambda/\mu}(x,y)$ denotes the skew Schur function in the variables $(x_1,x_2,\dots,x_i,\dots,y_1,y_2,\dots,y_i,\dots)$.

D.0.2 Shifted symmetric functions and the principal stable specialisation From these ingredients and $\mu\in\mathcal{P}$, we define a class of Laurent series of a single variable $q^{1/2}$ obtained by the *principal stable specialisation*

$$\mathfrak{q}\colon \Lambda \to \mathbb{Q}[\![q^{-1/2}]\!], \quad f(x_1,\dots,x_i,\dots) \mapsto f(x_1 = q^{-1+1/2},\dots,x_n = q^{-i+1/2},\dots). \tag{D-8}$$

As is customary in the topological vertex literature, and since $-i+\frac{1}{2}$ is the component of the Weyl vector ρ of A_n with respect to the fundamental weight ω_{n-i}, we use the shorthand notation $f(q^\rho) := f(x_i = q^{-i+1/2})$. For f a power sum or Schur function, $f(q^\rho)$ converges to a rational function of $q^{1/2}$.

In particular,

$$p_{(d_1,\dots,d_n)}(q^\rho) = \prod_{i=1}^{n} \frac{1}{[d_i]_q}, \tag{D-9}$$

and, for Schur functions, Stanley [110; 111] proved the product formula

$$s_\lambda(q^\rho) = \frac{q^{\frac{1}{4}\kappa(\lambda)}}{\prod_{(i,j)\in\lambda}[h(i,j)]}, \tag{D-10}$$

where $h(i,j)$ is the number of squares directly below or to the right of a cell (i,j) (counting (i,j) once) in the Young diagram of λ. For example, when $\lambda=(i-j,1^j)$ is a hook Young diagram with i boxes and $j+1$ rows, this gives

$$s_{(i-j,1^j)}(q^\rho) = \frac{q^{\frac{1}{2}(\binom{i}{2}-ij)}}{[i]_q[i-j-1]_q![j]_q!}. \tag{D-11}$$

More generally, for $\mu\in\mathcal{P}$ we will consider the shifted power, Schur and skew Schur functions,

$$\begin{aligned} p_\lambda(q^{\rho+\mu}) &:= p_\lambda(x_i = q^{-i+\mu_i+1/2}),\\ s_\lambda(q^{\rho+\mu}) &:= s_\lambda(x_i = q^{-i+\mu_i+1/2}),\\ s_{\lambda/\delta}(q^{\rho+\mu}) &:= \sum_{\nu\in\mathcal{P}} \mathsf{LR}^\lambda_{\delta\nu} s_\nu(q^{\rho+\mu}). \end{aligned} \tag{D-12}$$

The identities

$$s_\lambda(q^\rho) = q^{\kappa(\rho)/2} s_{\lambda^t}(q^\rho), \tag{D-13}$$

$$s_{\lambda/\mu}(q^{\rho+\alpha}) = s_{\lambda^t/\mu^t}(-q^{-\rho-\alpha^t}) \tag{D-14}$$

follow easily from (D-11), (D-12) and the fact that Littlewood–Richardson coefficients are invariant under simultaneous transposition of their arguments. Following [63], we introduce the following notation for the Cauchy infinite products (D-5) in the principal stable specialisation:

$$\begin{aligned} \{\alpha,\beta\}_Q := \prod_{i,j\ge 1}(1-Qq^{-i-j+1+\alpha_i+\beta_i}) &= \sum_{\lambda\in\mathcal{P}} s_\lambda(q^{\rho+\alpha}) s_{\lambda^t}(-Qq^{\rho+\beta})\\ &= \Big[\sum_{\lambda\in\mathcal{P}} s_\lambda(q^{\rho+\alpha}) s_\lambda(Qq^{\rho+\beta})\Big]^{-1}. \end{aligned} \tag{D-15}$$

Finally, we will need to specialise expressions involving skew Schur functions and Cauchy products to the case of hook Young diagrams. These can be given closed-form q–factorial expressions, as follows.

Lemma D.1 *We have*

$$\mathsf{LR}^{(i-r,1^r)}_{\beta,\gamma} = \begin{cases} \delta_{i,j+k}(\delta_{r,s+t}+\delta_{r,s+t+1}) & \text{if } \beta=(j-s,1^s) \text{ and } \gamma=(k-t,1^t),\\ 0 & \text{else.}\end{cases} \tag{D-16}$$

Moreover,

$$s_{(i-j,1^j)/\gamma}(q^\rho) = \begin{cases} \dfrac{q^{\frac{1}{4}(i-k-1)(i-2j-k+2l)}}{[i-k+l-j]_q![j-l]_q!} & \text{if } \gamma = (k-l, 1^l), \\ 0 & \text{else}, \end{cases} \tag{D-17}$$

$$\frac{\{(i-j,1^j),\varnothing\}_Q}{\{\varnothing,\varnothing\}_Q} = \prod_{k=0}^{i-1}(1-q^kQq^{-j}) = (Qq^{-j};q)_i. \tag{D-18}$$

The content of the lemma follows from a straightforward application of the Littlewood–Richardson rule in the case of hook partitions $(i-r,1^r)$. The product formula[15] for the hook skew-Schur functions (D-17) follows then immediately from (D-11). Finally, (D-18) follows from a straightforward calculation from (D-15); see [63, Section 3.4] for details.

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