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Julien Marché

Christopher-Lloyd Simon

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Valuations on the character variety: Newton polytopes and residual Poisson bracket

JULIEN MARCHÉ
CHRISTOPHER-LLOYD SIMON

We study the space of measured laminations ML on a closed surface from the valuative point of view. We introduce and study a notion of Newton polytope for an algebraic function on the character variety. We prove, for instance, that trace functions have unit coefficients at the extremal points of their Newton polytope. Then we provide a definition of tangent space at a valuation and show how the Goldman Poisson bracket on the character variety induces a symplectic structure on this valuative model for ML. Finally, we identify this symplectic space with previous constructions due to Thurston and Bonahon.

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Introduction

The algebra of functions on the character variety Let S be a closed oriented surface of genus $g \geq 2$. Its character variety X is the quotient of the space $\mathrm{Hom}(\pi_1(S), \mathrm{SL}_2(\mathbb{C}))$ by the equivalence relation identifying ρ_1 and ρ_2 if and only if $\operatorname{tr} \rho_1(\gamma) = \operatorname{tr} \rho_2(\gamma)$ for all $\gamma \in \pi_1(S)$. By construction, it is an affine variety whose ring of functions $\mathbb{C}[X]$ is generated by the trace functions $t_\gamma \colon \rho \mapsto \operatorname{tr} \rho(\gamma)$ for $\gamma \in \pi_1(S)$. The function t_γ only depends on the conjugacy class of γ up to inversion, that is, on the free homotopy class of the corresponding unoriented loop.

These trace functions are not algebraically independent: the famous identity

$$\mathrm{tr}(AB) + \mathrm{tr}(AB^{-1}) = \mathrm{tr}(A)\,\mathrm{tr}(B)$$

for $A, B \in \mathrm{SL}_2(\mathbb{C})$ implies, for instance, that if α and β represent simple loops intersecting once, then

$$t_\alpha t_\beta = t_\gamma + t_\delta,$$

where γ and δ are elements in $\pi_1(S)$ representing the simple curves obtained by smoothing the intersection between α and β in the two possible ways.

This phenomenon generalizes as follows. Given a multiloop α, that is, a multiset $\{\alpha_1, \dots, \alpha_n\}$ of nontrivial loops $\alpha_i \in \pi_1(S)$, the function $t_\alpha = t_{\alpha_1} t_{\alpha_2} \cdots t_{\alpha_n}$ can be uniquely decomposed as a linear combination

$$t_\alpha = \sum m_\mu t_\mu, \tag{1}$$

where each μ is a multicurve, that is, a (possibly empty) multiloop represented by pairwise disjoint, simple, nontrivial loops. This means that the set MC of multicurves indexes a linear basis for the algebra of characters $\mathbb{C}[X]$, which is privileged from the topological viewpoint; it is also invariant under the (algebraic) automorphism group of $\mathbb{C}[X]$, as we proved in [12].

It is an old problem to understand the algebraic structure of $\mathbb{C}[X]$, whose study was initiated by Fricke and Vogt in the late 19th century, and revisited in the seventies by the work of Procesi, Horowitz and Magnus among others; see Magnus [11] for a review. One approach is to investigate the coefficients m_μ of the functions t_α.

In this article, we define the Newton set $\Delta(t_\alpha) \subset \mathrm{MC}$ of t_α, in analogy with the extremal points of the ordinary Newton polytope of a polynomial, as follows.

Definition (Newton set) For $f = \sum m_\mu t_\mu$ decomposed in the basis of multicurves, we define its *support* as $\mathrm{Supp}(f) = \{\mu \in \mathrm{MC} \mid m_\mu \neq 0\}$.

We say that $\mu \in \mathrm{Supp}(f)$ is *extremal* in f if there exists a multicurve ξ such that $i(\xi, \mu) > i(\xi, \nu)$ for all $\nu \in \mathrm{Supp}(f)$ distinct from μ.

The *Newton set* $\Delta(f)$ is the set of extremal multicurves in f.

In this definition, $i(\cdot, \cdot)$ denotes the geometric intersection number, and standard properties of measured laminations imply that ξ can be replaced by a simple curve or a measured lamination. Our first result is the following.

Theorem A (trace functions are unitary) *For every multiloop $\alpha = \{\alpha_1, \dots, \alpha_n\}$, the function t_α is unitary in the sense that $m_\mu = \pm 1$ for all $\mu \in \Delta(t_\alpha)$.*

To introduce our next result, recall that the algebra of functions $\mathbb{C}[X]$ carries a natural Poisson bracket stemming from the Atiyah–Bott–Weil–Petersson–Goldman symplectic structure on X. Following Goldman [7], for $\alpha, \beta \in \pi_1(S)$ it is given by the formula

$$\{t_\alpha, t_\beta\} = \sum_{p \in \alpha \cap \beta} \epsilon_p (t_{\alpha_p \beta_p} - t_{\alpha_p \beta_p^{-1}}), \tag{2}$$

where the sum ranges over all intersection points p between transverse representatives for $\alpha \cup \beta$, and ϵ_p is the sign of such an intersection, while α_p and β_p denote the homotopy classes of α and β based at p.

Our second result interprets the coefficients of $\{f, g\}$ at the extremal multicurves of fg in terms of Thurston's PL–symplectic structure on the space ML of measured laminations in S.

Theorem B (extremal structure constants for the Poisson bracket) *Let μ and ν be two multicurves. For $\xi \in \Delta(t_\mu t_\nu)$ we set $E_\xi = \{\lambda \in \mathrm{ML} \mid i(\xi, \lambda) = i(\mu, \lambda) + i(\nu, \lambda)\}$. These closed subsets of* ML *form a piecewise linear partition of* ML *with disjoint interiors.*

For Thurston's symplectic structure, the Poisson bracket $\{i_\mu, i_\nu\}$ of the length functions defined by $i_\mu(\lambda) = i(\mu, \lambda)$ is equal to the coefficient of t_ξ in $\{t_\mu, t_\nu\}$ almost everywhere in E_ξ.

Let us illustrate the theorem with the following example. The curves shown in Figure 1 satisfy $t_\alpha t_\beta = t_{c_1} t_{c_3} + t_{c_2} t_{c_4} - t_\gamma - t_\delta$ and $\{t_\alpha, t_\beta\} = 2t_\delta - 2t_\gamma$, so we find that $\Delta(t_\alpha t_\beta) = \{c_1 \cup c_3, c_2 \cup c_4, \gamma, \delta\}$, whereas $\Delta(\{t_\alpha, t_\beta\}) = \{\gamma, \delta\}$.

The Newton set of $t_\alpha t_\beta$ decomposes ML into 4 domains, where $i(\alpha \cup \beta, \lambda)$ is equal to the intersection of λ with $c_1 \cup c_3$ or $c_2 \cup c_4$ or γ or δ, respectively. In the interior of these domains, $\{t_\alpha, t_\beta\}$ takes the values 0, 0, -2 and 2, respectively.

Strong relations between the symplectic structures on X and ML had already been observed, for instance in Papadopoulos and Penner [17] or Sözen and Bonahon [22]. Theorem B can be related to a formula for $\{i_\mu, i_\nu\}$ obtained in Bonahon [1, Proposition 6] by degenerating Wolpert's "cosine formula". However, our approach is algebraic in the sense that it uses valuations instead of Teichmüller theory.

Beyond these two results, the purpose of this article is to investigate the space of measured laminations from the valuative viewpoint, in particular its symplectic structure. This study was motivated by a new

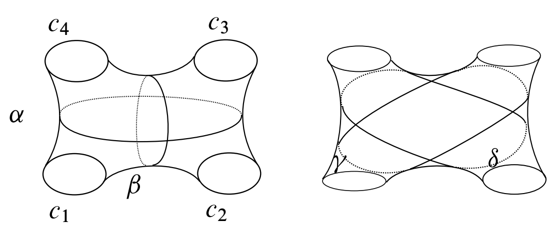

Figure 1: Product and Poisson bracket in a sphere with four punctures.

characterization of valuations associated to measured laminations that we obtained in [12]. We devote the remaining part of this introduction to an overview of our motivations, as well as the intermediate results that we obtained while revisiting the theory of measured laminations from the valuative viewpoint, since we believe they are of independent interest. We take this as an opportunity to recall general ideas for the benefit of a wide audience.

The Newton polytope A leading analogy in this article is to think of the collection (t_μ) as a monomial basis in a polynomial algebra, keeping in mind that it is not stable under multiplication.

Consider the degree $\deg_d$ defined for $d \in \mathbb{R}^n$ on the algebra $\mathbb{C}[t_1, \dots, t_n]$ by

$$\deg_d\Big(\sum_\mu m_\mu t^\mu\Big) = \max\{\langle \mu, d\rangle \mid m_\mu \neq 0\},$$

where $t^\mu = t_1^{\mu_1} \cdots t_n^{\mu_n}$, and $\langle\cdot,\cdot\rangle$ stands for the usual scalar product. This degree is (the opposite of) a monomial valuation. For $P \in \mathbb{C}[t_1, \dots, t_n]$, a monomial t^μ is an extremal point of its usual Newton polytope $\Delta(P)$ if $m_\mu \neq 0$ and for some $d \in \mathbb{R}^n$ the maximum defining $\deg_d$ is attained uniquely at t^μ.

Our starting point is to replace the degree $\deg_d$ by the valuation associated to a measured lamination λ in S. For us a valuation will be a map $v\colon \mathbb{C}[X] \to \{-\infty\} \cup \mathbb{R}$ satisfying $v(fg) = v(f) + v(g)$ and $v(f+g) \leq \max\{v(f), v(g)\}$ for all $f, g \in \mathbb{C}[X]$. We choose this convention, which is opposite to the usual one, to avoid crowding too many signs. In the general language of valuations (see for instance Vaquié [25]), our valuations are centered at infinity on the affine variety X as they take nonnegative values on the ring $\mathbb{C}[X]$ of characters.

In a groundbreaking series of articles starting with [14], Morgan and Shalen showed that the character variety X can be compactified using valuations, in the spirit of the Riemann–Zariski compactification. In particular, the space of measured laminations, viewed as Thurston's compactification of Teichmüller space, can be embedded in the space of valuations on $\mathbb{C}[X]$ with values in an archimedean group. However, this embedding used a degeneration process and is not completely explicit: if v is the valuation associated to a lamination λ, we clearly have $v(t_\mu) = i(\lambda, \mu)$, but it was not clear what $v(f)$ should be for a general element $f \in \mathbb{C}[X]$.

In our previous article [12], we showed that the space of measured laminations ML can be identified with the space of *simple* valuations $v\colon \mathbb{C}[X] \to \{-\infty\} \cup \mathbb{R}_{\geq 0}$. The word simple means monomial with respect to the multicurve basis in the sense that the following holds:

$$v\Big(\sum m_\mu t_\mu\Big) = \max\{v(t_\mu) \mid m_\mu \neq 0\}. \tag{3}$$

This justifies our definition for the Newton set of $f = \sum m_\mu t_\mu$ as the set of $\mu \in \mathrm{Supp}(f)$ such that the maximum in (3) is attained uniquely at t_μ for some $v \in \mathrm{ML}$.

For a concrete example, consider the particular case of a multiloop α contained in an incompressible pair of pants $P \subset S$. The subsurface P contains only three simple curves, its boundary components, and they do not intersect each other. Denoting by t_1, t_2, t_3 the trace functions along these components, we have $t_\alpha \in \mathbb{Z}[t_1, t_2, t_3]$. This polynomial is often called the Fricke polynomial and has been much studied; see [11, Section 2.2]. Now any valuation associated to a measured lamination on S restricts to a monomial valuation on $\mathbb{C}[t_1, t_2, t_3]$, and we find that our Newton set corresponds to the extremal points of the usual Newton polytope. Even for such $\alpha \subset P$, it is not easy to determine $\Delta(t_\alpha)$ from the $\alpha_i \in \pi_1(S)$, and the unitarity property is not an obvious one.

It is worth noticing that we only talk about the Newton set and not about the Newton polytope, as we do not know any reasonable notion of convexity in ML. However, we can define the dual Newton polytope of a function $f \in \mathbb{C}[X]$ as $\Delta^*(f) = \{v \in \mathrm{ML} \mid v(f) \leq 1\}$. Moreover, we could define the poset of faces of $\Delta(f)$ using the order structure. Its combinatorics may be a promising land of investigation, but we did not go further in that direction.

Symplectic and combinatorial volumes of dual polytopes This paragraph only serves motivational purposes and does not claim new results; it may be skipped harmlessly.

Thurston's symplectic form on ML provides a notion of volume; thus we may ask for the topological meaning of the volume $\mathrm{Vol}\,\Delta^*(t_\alpha)$ when α is a multiloop.

When α is a filling multiloop, a celebrated theorem of M Mirzakhani [13], extended by Rafi and Souto [21], estimates the number of elements in its orbit under the modular group $\mathrm{Mod}(S)$ as a bound on their complexity tends to infinity. More precisely, fix another filling multiloop β, and denote by $m_g > 0$ the volume of the moduli space of hyperbolic metrics on S for the Weil–Petersson form. The theorem claims the following:

$$\lim_{r\to\infty} \frac{\mathrm{Card}\{\varphi \in \mathrm{Mod}(S) \mid i(\beta, \varphi(\alpha)) \leq r\}}{r^{6g-6}} = \frac{\mathrm{Vol}\,\Delta^*(t_\beta)\,\mathrm{Vol}\,\Delta^*(t_\alpha)}{m_g}.$$

The identification between measured laminations and simple valuations implies, using equation (3), that the Newton dual polytope $\Delta^*(f)$ of $f \in \mathbb{C}[X]$ equals the intersection of $\Delta^*(t_\mu)$ for $\mu \in \Delta(f)$. These "elementary cones" $\Delta^*(t_\mu) = \{v \in \mathrm{ML} \mid v(t_\mu) \leq 1\}$ are described by explicit sets of linear inequalities in any PL chart of ML, and the volume of their intersection is computable. This yields a constructive procedure to compute Mirzakhani's constant $\mathrm{Vol}\,\Delta^*(t_\alpha)$, and shows that it depends only on $\Delta(t_\alpha)$. It also shows that these volumes are rational.

A different motivation is that this Newton set, as the usual one, could have applications to the problem of counting solutions of algebraic equations in X. We wonder for instance if it helps estimating the number of solutions to a system of $6g-6$ equations $t_{\gamma_i} = x_i$, where $\gamma_1, \dots, \gamma_{6g-6} \in \pi_1(S)$ and $x_1, \dots, x_{6g-6} \in \mathbb{C}$. This could have interesting applications to three-dimensional topology, for instance to evaluate the number of points in the $\mathrm{SL}_2(\mathbb{C})$–character variety of $\pi_1(M)$ for a 3–manifold M from a Heegaard decomposition.

Measured laminations as valuations In this article we study measured laminations using the tools of valuation theory. There are two well-known invariants for an archimedean valuation v: its *rational rank*, defined as the dimension of the $\mathbb{Q}$–vector space generated by the group Λ_v of its values (that is, differences of lengths for the corresponding measured lamination), and the *transcendence degree* of its residue field k_v. These invariants are related by the celebrated *Abhyankar inequality*, $\operatorname{rat}\operatorname{rk}(v) + \operatorname{tr}\deg(k_v) \leq 6g-6$. Here we will show the following.

Proposition A (characterizing strict valuations) *For a valuation v associated to a measured lamination λ, the following properties are equivalent*:

(i) *Distinct multicurves μ and ν have distinct lengths*: $i(\lambda,\mu) \neq i(\lambda,\nu)$.

(ii) *The residue field of $\mathbb{C}(X)$ at v has transcendence degree* 0, *or* $k_v = \mathbb{C}$.

(iii) *The $\mathbb{Q}$–vector space generated by the set of lengths $i(\lambda,\mu)$ for $\mu \in \mathrm{MC}$ has dimension* $6g-6$.

The first property implies that v defines a total order on the set of multicurves, so the max in equation (3) will always be strict, which is why they deserve to be called *strict valuations*. They played a prominent role in our previous article, where we showed that almost all valuations are strict (in the measure-theoretical sense). They will be equally important in this paper, as property (ii) enables us to define the residual value at v of a function $f \in \mathbb{C}(X)$ satisfying $v(f) \leq 0$. Combined with property (iii), it shows that strict valuations are *Abhyankar* in the sense that his inequality is an equality. We wonder whether any measured lamination gives rise to an Abhyankar valuation.

We have not come across strict valuations in the literature. Instead we encounter *maximal measured laminations*, which are those whose support cannot be enlarged. In this article, we characterize the valuations associated to maximal laminations as being *acute*: for any $\alpha,\beta \in \pi_1(S)\setminus\{1\}$ we never have $v(t_\alpha t_\beta) = v(t_{\alpha\beta}) = v(t_{\alpha\beta^{-1}})$, so that these quantities are the lengths for the edges of an acute isosceles triangle. We will show that a valuation v_λ is acute if and only if any time we smooth a self-intersection of a multiloop which is taut (minimally intersecting in its homotopy class), the two resulting multiloops have distinct λ–lengths. This property plays a crucial role in the proof of the unitarity theorem. We also show that any strict valuation is acute, and wonder if the reciprocal statement is true.

Tangent spaces and Thurston's symplectic structure The space of measured laminations is a PL–manifold but does not carry any sensible smooth structure (for which intersection numbers have smooth variations), so there is no symplectic structure in the usual sense. However, Thurston showed that most points (maximal laminations) have a well-defined tangent space endowed with a nondegenerate skew-symmetric form; see [19, Chapter 3].

In this article we propose a straightforward notion for the tangent space $T_v\mathrm{ML}$ at a valuation, and show that when v is strict, it coincides with the space $\mathrm{Hom}(\Lambda_v,\mathbb{R})$, which has dimension $\operatorname{rat}\operatorname{rk}(v) = 6g-6$. Then we show how the Goldman Poisson bracket induces a "residual Poisson bracket" at any strict valuation v, thus endowing $T_v\mathrm{ML}$ with a symplectic structure. For future reference we shall name this

topology/geometry	dynamics	algebra
measured foliation measured geodesic laminations	action of $\pi_1(S)$ on a real tree	simple valuation
length function	translation length	trace function
filling/aperiodic lamination	free action	positive valuation
maximal lamination	trivalent tree	acute valuation
?	?	strict valuation

Table 1

model after Goldman. This uses the crucial fact that given $f, g \in \mathbb{C}[X]$, we have $v(\{f, g\}) \le v(fg)$ for all $v \in \mathrm{ML}$. This property amounts to the inverse inclusion of the dual polytopes $\Delta^*(\{f, g\}) \supset \Delta^*(fg)$.

Finally, we provide precise identifications between this symplectic vector space and two other existing models in the literature, which we now pass under review. In the work of Morgan and Shalen, the key notion relating measured laminations and valuations is the action of $\pi_1(S)$ on real trees. We may represent this dynamical point of view as lying between the two others as in Table 1, which the reader may use as a dictionary.

For future reference, we name the symplectic vector spaces appearing naturally from each of those approaches after Thurston, Bonahon and Goldman, respectively.

Goldman's model It is given by the residual Poisson bracket on $T_v\mathrm{ML}$, which we introduced briefly. It will be described with more detail in the body of the paper.

Thurston's model One can associate to a maximal measured lamination λ a ramified 2–fold covering $S' \to S$, known as the orientation cover of the lamination. The group $H^1(S', \mathbb{R})$ splits into a symmetric and antisymmetric part with respect to the involution of the covering $S' \to S$. The space $H^1(S', \mathbb{R})^-$ with the cup-product form is the geometric model for $T_\lambda\mathrm{ML}$.

Bonahon's model If we consider a trivalent real tree T with a free and minimal action of $\pi_1(S)$, we can consider the space of functions $c\colon V(T)^2 \to \mathbb{R}$ on the set of pairs of trivalent vertices of T which satisfy

(i) $c(x, y) = c(y, x)$,

(ii) $c(x, y) = c(x, z) + c(z, y)$ if z belongs to the geodesic joining x to y,

(iii) $c(\alpha x, \alpha y) = c(x, y)$ for all $\alpha \in \pi_1(S)$.

Again, this space has a natural antisymmetric form related to the cyclic orientation of T at every trivalent vertex. It is equivalent to the space of transverse cocycles introduced by Bonahon; see [2, page 240]. The identification between Thurston's and Bonahon's models is well known but all proofs we encountered use auxiliary structures like train tracks. At the end of the article, we provide "invariant" proofs for the following result.

Theorem C (symplectomorphisms) *There are natural isomorphisms of symplectic vector spaces between the models of Thurston, Bonahon and Goldman.*

In particular, we provide a new construction of independent interest, reminiscent of Milnor's join construction, which, starting from a trivalent real tree, produces a space homotopically equivalent to the covering S'. We may wonder which of these three symplectic identifications persist for more general actions of Fuchsian groups on real trees.

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1 Background

1.1 Algebra of functions on the character variety

Let S be a closed connected and oriented surface of genus $g \geq 1$. We denote by X the character variety of S, which is the algebraic quotient of its representation variety $\mathrm{Hom}(\pi_1(S), \mathrm{SL}_2(\mathbb{C}))$ by the conjugacy action of $\mathrm{SL}_2(\mathbb{C})$, defined as the spectrum of the algebra of invariant functions:

$$\mathbb{C}[X] = \mathbb{C}[\mathrm{Hom}(\pi_1(S), \mathrm{SL}_2(\mathbb{C}))]^{\mathrm{SL}_2(\mathbb{C})}.$$

A celebrated result of Procesi presents generators and relations for this algebra (which holds for any finitely generated group). It appears in the form presented here in [3, Proposition 9.1]. For $\alpha \in \pi_1(S)$, we denote by $t_\alpha \in \mathbb{C}[X]$ the trace function given by $t_\alpha([\rho]) = \mathrm{tr}\,\rho(\alpha)$.

Theorem 1 (Procesi) *The algebra $\mathbb{C}[X]$ is generated by the t_α for $\alpha \in \pi_1(S)$. The ideal of relations is generated by $t_1 - 2$ and $t_\alpha t_\beta - t_{\alpha\beta} - t_{\alpha\beta^{-1}}$ for all $\alpha, \beta \in \pi_1(S)$.*

Definition 2 A *multiloop* in S is a class of continuous maps $f : \Gamma \to S$ from compact 1–dimensional manifolds Γ to S which are not homotopic to a constant on any component. We consider it modulo the relation declaring f equivalent to $f' : \Gamma' \to S$ when there is a homeomorphism $\phi : \Gamma \to \Gamma'$ such that $f' \circ \phi$ is homotopic to f. We allow the empty multiloop ($\Gamma = \varnothing$).

A *multicurve* is a multiloop which is represented by an embedding. We denote by MC the set of multicurves.

A multiloop amounts to a finite multiset $\{\alpha_1, \dots, \alpha_n\}$ of nontrivial conjugacy classes in $\pi_1(S)$ considered up to inversion: we define $t_\alpha = \prod_{i=1}^n t_{\alpha_i}$, in particular $t_\varnothing = 1$. The components of a multicurve must be noncontractible, simple and pairwise disjoint.

Applying the trace relation recursively to reduce the number of self intersections in multiloops, one may deduce part of the following theorem [20]. The linear independence requires more work.

Theorem 3 *The family $(t_\mu)_{\mu \in \mathrm{MC}}$ forms a linear basis of the algebra $\mathbb{C}[X]$.*

1.2 Deriving the Poisson algebra from the Kauffman algebra

The multiplication and the Poisson bracket on $\mathbb{C}[X]$ appear naturally as byproducts of the Kauffman algebra $K(S,R)$ over some ring R containing an invertible element A. Recall that a banded link in an oriented 3–manifold is the image by a tame embedding of a finite union of oriented annuli.

As an R–module, the Kauffman algebra is the quotient of the free module over isotopy classes of banded links L in $S\times[0,1]$, by the submodule generated by Kauffman's local skein relations

$$[\bigcirc\cup L]=(-A^2-A^{-2})[L]\quad\text{and}\quad [L_\times]=A[L_+]+A^{-1}[L_-],$$

where $L_\times, L_+, L_-$ are banded links differing in a ball as shown in Figure 2.

The product is given by stacking two banded links one above the other. Precisely,

$$[L_0][L_1]=[\Phi_0(L_0)\cup\Phi_1(L_1)],\quad\text{where } \Phi_i(x,t)=\bigl(x,\tfrac{1}{2}(t+i)\bigr).$$

Any multicurve μ on S can be seen as a banded link $[\mu]$ in $S\times[0,1]$ by considering a tubular neighborhood $S\times\{\frac{1}{2}\}$, often called its blackboard framing.

We sum up what we need to know about skein algebras in the following theorem.

Theorem 4 *Using the previous notation*:

(i) *The module $K(S,R)$ is a free R–module generated by multicurves.*

(ii) *The algebra $K(S,\mathbb{C})$ with $A=-1$ is commutative, and there is an isomorphism $K(S,\mathbb{C})\to\mathbb{C}[X]$ defined by sending the blackboard framing $[\mu]$ of a multicurve $\mu\in\mathrm{MC}$ to $(-1)^{|\mu|}t_\mu$, where $|\mu|$ denotes the number of components of μ.*

(iii) *The map sending a multicurve to its blackboard framing yields an isomorphism of $\mathbb{C}[A^{\pm1}]$–modules $K(S,\mathbb{C})\otimes\mathbb{C}[A^{\pm1}]\simeq K(S,\mathbb{C}[A^{\pm1}])$. In this setting, we have*

$$\{f,g\}=\frac{1}{2}\frac{d}{dA}\bigl[fg-gf\bigr]_{A=-1}.$$

These algebras were introduced independently by Przytycki and Turaev. The assertions in part (i), in part (ii) and the isomorphism of part (iii) are [20, Fact 4.1, Fact 2.7 and Theorem 2.8]. Part (ii) is also proved in [4]. Finally, the last formula appears in [5].

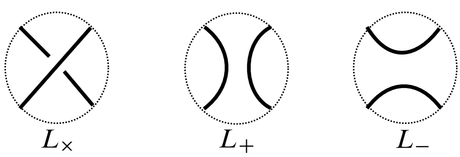


Figure 2: Local skein relation.

Let us explain Theorem 4(i) more precisely. Given a diagram D for a banded link $L \subset S \times [0,1]$, we denote by C its set of crossings. For any map $\sigma\colon C \to \{\pm 1\}$, let $w_\sigma = \sum_c \sigma(c)$ and consider the diagram D_σ obtained after smoothing each crossing $c \in C$ according to the sign $\sigma(c)$, and removing the n_σ trivial components which appear in the result. The following formula holds in $K(S,R)$:

$$[L] = \sum_{\sigma\colon C \to \{\pm 1\}} (-A^2 - A^{-2})^{n_\sigma} A^{w_\sigma} [D_\sigma], \tag{4}$$

which, after grouping terms corresponding to a same diagram $[D_\sigma]$, yields the decomposition of $[L]$ in the basis of multicurves. This formula sheds light on the product of two multicurves μ, ν: intuitively, the product is obtained by taking the union $\mu \cup \nu$ and summing over all possible smoothings.

By Theorem 4(ii), we deduce that the algebra $\mathbb{C}[X]$ has a linear basis indexed by trace functions of multicurves. At $A = -1$, the class of $[L]$ does not change if we change a crossing. Hence, we can replace the notion of banded link with the simpler notion of multiloop that we defined previously.

The Kauffman algebra is not completely necessary for our purposes. However, we find it conceptually useful for the following reasons. It transforms the trace relation into a local relation whose sign is more convenient (for instance while performing successive diagrammatic computations), and a better understanding of the product in terms of smoothings. It also provides a simple reason as to why the Goldman bracket actually satisfies the Jacobi relation: this comes from Theorem 4(iii) and the obvious associativity of multiplication in the Kauffman algebra. Finally, in the context of this article, it provides an alternative formula for the Poisson bracket which enlightens Theorem B. Indeed, a smoothing σ which is extremal for $[\alpha][\beta]$ in $K(S,R)$ is also extremal for the Poisson bracket $\{t_\alpha, t_\beta\}$ in $\mathbb{C}[X]$. Its coefficient in the former is ± 1 by Theorem A, and we will interpret its coefficient $\pm w_\sigma$ in the latter as a residual Poisson bracket.

2 Measured laminations and simple valuations

2.1 Simple valuations

It is well known that a measured lamination λ on S is characterized by the length $i(\lambda,\gamma)$ it assigns to every simple curve γ. This "functional" point of view can be extended to define a map $v_\lambda\colon \mathbb{C}[X] \to \{-\infty\} \cup \mathbb{R}_{\geq 0}$ satisfying $v(0) = -\infty$ and for all $f = \sum m_\mu t_\mu$ decomposed in the multicurve basis,

$$v_\lambda(f) = \max\{i(\lambda,\mu) \mid m_\mu \neq 0\}, \tag{5}$$

where $i(\lambda,\mu) = i(\lambda,\mu_1) + \dots + i(\lambda,\mu_n)$ for a multicurve μ with components $\mu_1, \dots, \mu_n$. By [12, Proposition 1.2], equation (5) is coherent with the fact that for any $\alpha \in \pi_1(S)$, not necessarily simple, we actually have $v_\lambda(t_\alpha) = i(\lambda,\alpha)$. Let us recall [12, Definition 1.1].

Definition 5 A simple valuation on $\mathbb{C}[X]$ is a map $v\colon \mathbb{C}[X] \to \{-\infty\} \cup \mathbb{R}_{\geq 0}$ satisfying:

(i) $v(f) = -\infty$ if and only if $f = 0$.

(ii) $v(fg) = v(f) + v(g)$ for all $f, g \in \mathbb{C}[X]$.

(iii) If $f = \sum m_\mu t_\mu$ then $v(f) = \max\{v(t_\mu) \mid m_\mu \neq 0\}$.

The following characterization was of fundamental importance in [12]: it yields a homeomorphism between the space of simple valuations and ML, both topologies being defined by simple convergence for the evaluations of multicurves.

Theorem 6 (Marché–Simon) *The simple valuations on $\mathbb{C}[X]$ are precisely the v_λ for $\lambda \in \mathrm{ML}$.*

In this paper we only consider simple valuations, so we write $v \in \mathrm{ML}$ and $\lambda \in \mathrm{ML}$ interchangeably.

The maximality condition of Definition 5 implies that for any $f, g \in \mathbb{C}[X]$, we have $v(f+g) \leq \max\{v(f), v(g)\}$, with equality if $v(f) \neq v(g)$. Given a multiloop α with a self-intersection p, the two smoothings at p give multiloops α_+ and α_- and the trace relation reads $t_\alpha = \pm t_{\alpha_+} \pm t_{\alpha_-}$. Hence any valuation v satisfies $v(t_\alpha) \leq \max\{v(t_{\alpha_+}), v(t_{\alpha_-})\}$. The following lemma was proven by Dylan Thurston in [23], and removes the condition $v(t_{\alpha_+}) \neq v(t_{\alpha_-})$ for the equality to hold. We provide an independent proof in Section 5, which relies on the geometry of real trees.

Lemma 7 (smoothing lemma) *Let α be a taut multiloop, having a self-intersection p with smoothings α_+ and α_-. For any $v \in \mathrm{ML}$ we have $v(t_\alpha) = \max\{v(t_{\alpha_+}), v(t_{\alpha_-})\}$.*

Still, it will prove useful to consider valuations v for which we always have $v(t_{\alpha_+}) \neq v(t_{\alpha_-})$. This holds over subsets of full measure in ML, as we now explain.

2.2 Acute valuations

We say that a simple valuation $v = v_\lambda \in \mathrm{ML}$ is *positive* if $v(f) > 0$ for all nonconstant $f \in \mathbb{C}[X]$. It is equivalent to saying that $i(\lambda, \alpha) > 0$ for all $\alpha \in \pi_1(S)$, or $i(\lambda, \mu) > 0$ for all simple curves μ. Such measured laminations are called filling or aperiodic in the literature.

We now introduce the notion of acute valuation, which will happen to be equivalent to the notion of maximal measured geodesic lamination, as we will show in Proposition 27.

Definition 8 A *simple* valuation $v \in \mathrm{ML}$ is called acute if it is positive and for any nontrivial $\alpha, \beta \in \pi_1(S)$, we do not have $v(t_{\alpha\beta}) = v(t_\alpha t_\beta) = v(t_{\alpha\beta^{-1}})$.

Lemma 9 (unique smoothing) *A positive simple valuation $v_\lambda \in \mathrm{ML}$ is acute if and only if for every taut multiloop α, and smoothings $\alpha_\pm$ at a self-intersection, we have*

$$i(\lambda, \alpha_+) \neq i(\lambda, \alpha_-).$$

This justifies the terminology: $v \in \mathrm{ML}$ is acute when for every such a multiloop α, we have either $v(\alpha) = v(\alpha_+) > v(\alpha_-)$ or $v(\alpha) = v(\alpha_-) > v(\alpha_+)$, so the values $v(\alpha), v(\alpha_+), v(\alpha_-)$ are the lengths of an acute isosceles triangle with one shortest edge corresponding to either $v(\alpha_-)$ or $v(\alpha_+)$.

Proof Suppose $v \in \mathrm{ML}$ is acute. By decomposing α into connected components, we observe that the smoothing concerns at most two of them, and the proof reduces to the following cases.

(i) Either α is a single loop, self-intersecting at p. Denote by $\gamma, \delta \in \pi_1(S, p)$ the elements such that α is homotopic to $\gamma\delta$. The tautness assumption implies that γ and δ are nontrivial. Depending on the combinatorics of the intersection, one smoothing is homotopic to $\gamma\delta^{-1}$ and the other to the union $\gamma \cup \delta$. If $v(t_{\gamma\delta^{-1}}) = v(t_\gamma t_\delta)$ then, from the acute property, $v(t_{\gamma\delta})$ differs from them, which contradicts the smoothing lemma (Lemma 7).

(ii) Otherwise the multiloop α has two components intersecting at p. We denote by $\gamma, \delta \in \pi_1(S, p)$ the (nontrivial) homotopy classes of the two components. Again, α_+ and α_- are homotopic to $\gamma\delta$ and $\gamma\delta^{-1}$; the reasoning is the same.

Conversely, suppose $\alpha, \beta \in \pi_1(S)$ are nontrivial. If they are powers of a same element, say $\alpha = \gamma^n$ and $\beta = \gamma^m$, then $v(t_{\alpha\beta}) = |n+m| v(t_\gamma)$ and $v(t_{\alpha\beta^{-1}}) = |n-m| v(t_\gamma)$. As $v(t_\gamma) > 0$, the equality $v(t_{\alpha\beta}) = v(t_\alpha t_\beta) = v(t_{\alpha\beta^{-1}})$ implies $mn = 0$, which is impossible.

Consider a hyperbolic structure on S, so that α and β act on $\widetilde{S} \simeq \mathbb{H}^2$ by hyperbolic translations along distinct axes A_α and A_β, respectively.

(i) If $A_\alpha \cap A_\beta = \{p\}$, then p projects to a point on $\alpha \cap \beta$. The smoothings at p are $\alpha\beta$ and $\alpha\beta^{-1}$. The assumption $i(\lambda, \alpha\beta) \neq i(\lambda, \alpha\beta^{-1})$ says that v satisfies the condition $v(t_\alpha t_\beta) \neq v(t_{\alpha\beta^{-1}})$, ensuring that of Definition 8.

(ii) If $A_\alpha \cap A_\beta = \varnothing$, then up to replacing β with β^{-1}, we may assume the axes point in the same direction. Now, the axes of $\alpha\beta$ and $\beta\alpha$ intersect at a point p. This point projects to a self-intersection of $\alpha\beta$ which, after smoothing, gives alternatively $\alpha \cup \beta$ and $\alpha\beta^{-1}$. The assumption $i(\lambda, \alpha \cup \beta) \neq i(\lambda, \alpha\beta^{-1})$ says that v satisfies the condition $v(t_\alpha t_\beta) \neq v(t_{\alpha\beta^{-1}})$, ensuring that of Definition 8. □

2.3 Strict valuations

A simple valuation v can be extended to $\mathbb{C}(X)$ by $v(f/g) = v(f) - v(g)$. We define its valuation ring $\mathcal{O}_v = \{f \in \mathbb{C}(X) \mid v(f) \leq 0\}$, which has a unique maximal ideal $\mathcal{M}_v = \{f \in \mathbb{C}(X) \mid v(f) < 0\}$ and residue field $k_v = \mathcal{O}_v / \mathcal{M}_v$.

Lemma 10 *A simple valuation $v = v_\lambda$ satisfies $k_v = \mathbb{C}$ if and only if for all distinct multicurves μ, ν we have $i(\lambda, \mu) \neq i(\lambda, \nu)$.*

Following [12], we will refer to them as *strict* valuations. We showed in [12, Lemma 3.4] that the set of nonstrict valuations has zero measure in ML.

Proof Suppose that $k_v = \mathbb{C}$ and consider two distinct multicurves μ and ν. If $v(t_\mu) = v(t_\nu)$ then $t_\mu/t_\nu \in \mathcal{O}_v \setminus \mathcal{M}_v$, so there exists $\lambda \in \mathbb{C}^*$ such that $t_\mu/t_\nu - \lambda \in \mathcal{M}_v$ thus $v(t_\mu/t_\nu - \lambda) < 0$. But this implies $v(t_\mu - \lambda t_\nu) < v(t_\nu)$, which contradicts the third condition in Definition 5.

Conversely, suppose that v takes distinct values on distinct multicurves and pick $f = P/Q \in \mathcal{O}_v \setminus \mathcal{M}_v$. Then $v(P) = v(Q)$, so the decompositions of P and Q in the basis of multicurves must be of the form $P = at_\mu + P'$ and $Q = bt_\mu + Q'$ with $a, b \in \mathbb{C}^*$ and $v(P'), v(Q') < v(t_\mu)$. This gives

$$f = \frac{at_\mu + P'}{bt_\mu + Q'} = \frac{a + P'/t_\mu}{b + Q'/t_\mu} = \frac{a}{b} \mod \mathcal{M}_v. \qquad \square$$

For a simple valuation $v = v_\lambda$, the set of values $\Lambda_v^+ = v(\mathbb{C}[X] \setminus \{0\})$ coincides with

$$\Lambda_v^+ = \{i(\lambda, \mu) \mid \mu \in \mathrm{MC}\}$$

by condition (iii) in Definition 5, and has the structure of an abelian semigroup by condition (ii) in Definition 5. Its associated group is $\Lambda_v = v(\mathbb{C}(X)^*)$ and consists of differences of λ–lengths.

When v is strict, the map $\mu \mapsto i(\lambda, \mu)$ is a bijection between MC and Λ_v^+. It is enlightening to think about the semigroup structure on multicurves obtained by pulling back the addition in Λ_v^+ in the following way. Let μ and ν be two multicurves, viewed as elements of $K(S, \mathbb{C})$. All smoothings of $\mu \cup \nu$ are multicurves ξ with $i(\lambda, \xi) \le i(\lambda, \mu) + i(\lambda, \nu)$ and equality holds for exactly one of them corresponding to the "sum of μ and ν with respect to v".

We define the *rational rank* of v to be $\mathrm{rat\,rk}(v) = \dim_{\mathbb{Q}} \Lambda_v \otimes \mathbb{Q}$. It satisfies the following Abhyankar inequality (see [16])

$$\mathrm{rat\,rk}(v) + \mathrm{tr\,deg}(k_v) \le \dim X,$$

from which we deduce that if a simple valuation has maximal rational rank, that is $\mathrm{rat\,rk}(v) = \dim X$, then it is strict.

Proof of Proposition A By Lemma 10, we know that the first two properties of the proposition are equivalent. The Abhyankar inequality gives the implication $\mathrm{rat\,rk}(v) = 6g-6 \implies \mathrm{tr\,deg}(k_v) = 0$. The reverse implication will follow from the results of the remaining sections. Precisely, given a strict valuation v, we will define a tangent space $T_v\mathrm{ML}$ whose dimension is $\mathrm{rat\,rk}(v)$. Then, we will show successively that this tangent space is isomorphic to the Bonahon and Thurston models. It is well known that the latter has dimension $6g-6$, proving the last step of the proposition. $\square$

3 Newton polytopes of trace functions

This section relies on the following lemma, whose proof is postponed to Section 5.

Lemma 11 *The set of acute valuations has full measure in* ML.

Definition 12 Let $v \in \mathrm{ML}$ be any simple valuation and $f \in \mathbb{C}[X]$ any function decomposed as $\sum m_\mu t_\mu$ in the multicurve basis.

- The multicurve $\mu \in \mathrm{Supp}(f)$ is *v–extremal* in f if $v(t_\nu) < v(t_\mu)$ for every other $\nu \in \mathrm{Supp}(f)$.
- The multicurve μ is *extremal* in f if it is v–extremal in f for some v.
- The *Newton set* of f is the subset $\Delta(f) \subset \mathrm{MC}$ of extremal curves in f.
- The function f is *unitary* if $m_\mu = \pm 1$ for any extremal multicurve in f.

Observe that if v is strict, then μ is v–extremal in f if and only if $v(f) = v(t_\mu)$. Moreover, the density of strict valuations in ML implies that a multicurve is extremal in f if and only if it is v–extremal in f for some strict v.

3.1 Trace functions are unitary

Theorem 13 (unitarity) *If α is a multiloop in S, then t_α is unitary.*

Proof Let v be a strict acute valuation and μ be the unique multicurve such that $v(t_\alpha) = v(t_\mu)$. We must prove that $m_\mu = \pm 1$. We proceed by induction on the number of intersections of α. If there are none, then the result is obvious. Otherwise, put α in taut position and consider its smoothings at an intersection. Lemma 7 and the assumption that v is acute imply that $v(t_{\alpha+}) \neq v(t_{\alpha-})$. One can suppose that $v(t_\alpha) = v(t_{\alpha+}) > v(t_{\alpha-})$. The coefficient of t_μ in t_α is the same as in $\pm t_{\alpha+}$, so the induction hypothesis yields the result. □

Remark If we represent a taut multiloop as the projection of a banded link L in $S \times [0,1]$, we may decompose it in the basis of multicurves $\mu \in K(S, \mathbb{Z}[A^{\pm}])$ with blackboard framing. Then, the coefficient of μ in L is equal to $A^{n^+ - n^-}$, where $n^{\pm}$ counts the number of $\pm$–resolutions performed while transforming L into μ. At $A = -1$, we find the sign $(-1)^s$ for the extremal coefficient, where s is the number of self-intersections of α. The proof is the same, using the skein relation inductively.

Remark We know from [24] that MC indexes another basis (t'_μ) of $\mathbb{C}[X]$ for which the multiplicative structure constants are positive. The change of basis from (t_μ) to (t'_μ) is triangular, in the sense that if $\mu = \{\mu_1, \dots, \mu_k\}$ as a multiset, then t'_μ is a polynomial in the t_{μ_j} with leading monomial $\pm t_{\mu_1} \cdots t_{\mu_k}$. In this basis, the analogous notion of Newton set will be the same (that is, indexed by the same multicurves), and its extremal coefficients will be 1.

Corollary 14 *Any strict valuation is acute.*

Proof Let v be a strict valuation and consider a taut multiloop α. Suppose $v(t_{\alpha+}) = v(t_{\alpha-})$. Then $t_{\alpha+}$ and $t_{\alpha-}$ must have the same v–extremal multicurve μ. This defines an open condition on $v \in \mathrm{ML}$, namely that $v(t_\mu) > v(t_\nu)$ for all $\nu \in \Delta(t_{\alpha-} t_{\alpha+}) \setminus \{\mu\}$. But simple acute valuations are dense in ML so the same will hold for some acute valuation, contradicting Lemma 9. The conclusion follows from the converse part of that lemma. □

3.2 Extremal multicurves of $t_\mu t_\nu$ and $\{t_\mu, t_\nu\}$

Let μ and ν be multicurves in S and consider a taut immersion $\mu \cup \nu$ for their union. Note that such an immersion is unique up to isotopy and permutations of parallel strands. This follows from the methods and results of [8], specifically Theorem 2.1 and the discussion following Example 2.4.

We define the embedding $L_\mu(\nu)$ obtained by smoothing all intersections of $\mu \cup \nu$ with a left turn as we travel along a segment of μ and meet a segment of ν. Smoothing all intersections with a right turn would yield $L_\nu(\mu)$.

This is the product considered by Luo in [10]; in particular his Lemma 8.1 shows that $L_\mu(\nu)$ is a multicurve (it has no trivial components) and his Theorem 2.1 describes several of its properties.

Proposition 15 *Let μ and ν be multicurves. The multicurves $L_\mu(\nu)$ and $L_\nu(\mu)$ are extremal for the product $t_\mu t_\nu$, and if $i(\mu, \nu) > 0$ then they are distinct.*

Proof If $i(\mu, \nu) = 0$ then $L_\mu(\nu) = \mu \cup \nu = L_\nu(\mu)$ and the statement follows.

Now suppose $i(\mu, \nu) > 0$. We first observe that among all smoothings of the union $\mu \cup \nu$, those which maximize v_μ are precisely $L_\mu(\nu)$ and $L_\nu(\mu)$. Indeed, we know from [10, Theorem 2.1(iii)] that $i(\mu, L_\mu(\nu)) = i(\mu, \nu) = i(\mu, L_\nu(\mu))$, but any other smoothing ξ is made of segments of μ and ν which somewhere alternate between a left turn and right turn, thus forming a bigon with μ so that $i(\mu, \xi) < i(\mu, \nu)$. The fact that $L_\mu(\nu) \neq L_\nu(\mu)$ can be obtained from [10, Corollary 8.2], which proves $i(L_\mu(\nu), L_\nu(\mu)) = 2i(\mu, \nu)$.

We deduce from the preceding discussion and the multiplication formula (4) that the distinct multicurves $L_\mu(\nu)$ and $L_\nu(\mu)$ both appear in the decomposition of $t_\mu t_\nu$, and are the only two maximizers of v_μ. The condition $v_\lambda(L_\mu(\nu)) = v_\lambda(L_\nu(\mu))$ defines on $\lambda \in \mathrm{ML}$ a codimension-1 PL–subset; see [12, Lemma 1.6] for a proof. Hence a slight perturbation of the valuation v_μ off that subset in one direction or the other shows that $L_\mu(\nu)$ and $L_\nu(\mu)$ are indeed extremal terms in the product. □

Corollary 16 *If μ and ν are multicurves such that $i(\mu, \nu) > 0$, then $L_\mu(\nu)$ and $L_\nu(\mu)$ are extremal in the Poisson bracket $\{t_\mu, t_\nu\}$, and their coefficients in the basis of multicurves are equal to $\pm i(\mu, \nu)$.*

Proof We may deduce this using Theorem 4, which derives the Poisson bracket from the commutator in the skein algebra, but let us detail the computation without referring to the skein product.

For this, apply the Goldman formula (2) to the multiloops α and β, and for each $p \in \alpha \cap \beta$, decompose the terms $t_{\alpha_p \beta_p}$ and $t_{\alpha_p \beta_p^{-1}}$ in the basis of multicurves $\xi \in \mathrm{MC}$, to find

$$\{t_\alpha, t_\beta\} = \sum_\xi w_\xi t_\xi = \sum_\xi \Bigl(\sum_{\sigma_\xi} \prod_p \sigma_\xi(p) \Bigr) t_\xi,$$

where $w_\xi = \sum_{\sigma_\xi} \prod_p \sigma_\xi(p)$ is the sum over the smoothings $\sigma_\xi\colon \alpha \cap \beta \to \{\pm 1\}$ of $\alpha \cup \beta$ yielding the multiloop ξ.

Now suppose that $\alpha = \mu$ and $\beta = \nu$ are multicurves with $i(\mu, \nu) > 0$, and consider the multicurves ξ indexing the sum that are obtained by smoothing all intersections of $\mu \cup \nu$. Reasoning as in the proof of Proposition 15, we find that $L_\mu(\nu)$ and $L_\nu(\mu)$ both index a term corresponding to a unique smoothing map σ_ξ which is constant, equal to 1 or -1. □

Remark In the next section, we will prove that extremal coefficients of $\{t_\mu, t_\nu\}$ which are also extremal for $t_\mu t_\nu$ are values of the Thurston Poisson bracket $\{i_\mu, i_\nu\}_\lambda$ for $\lambda \in \mathrm{ML}$, as announced in Theorem B. Our approach consists in reinterpreting the Thurston Poisson bracket $\{i_\mu, i_\nu\}_\lambda$ as a residual value $\{t_\mu, t_\nu\}_v$ of the Goldman Poisson bracket at $v = v_\lambda$.

The previous corollary shows that the (residual) Poisson bracket of multicurves determines their intersection number by the formula

$$i(\mu, \nu) = \max\{\{i_\mu, i_\nu\}_\lambda \mid \lambda \in \mathrm{ML}\}.$$

4 Residual Poisson structure on ML

4.1 Tangent space

Recall that ML embeds in the space of real functions on $\mathbb{C}[X]^* = \mathbb{C}[X] \setminus \{0\}$. We thus define its tangent space at v as the set of maps

$$\phi = \frac{d}{ds}\Big|_{s=0} v_s\colon \mathbb{C}[X]^* \to \mathbb{R},$$

where v_s is a family of simple valuations depending on a parameter $s \in [0, \epsilon[$ starting at $v_0 = v$, such that the map $s \mapsto v_s(t_\gamma)$ is differentiable for every curve γ.

Observe that the pair $(v, \phi)\colon \mathbb{C}[X]^* \to [0, +\infty) \times \mathbb{R}$ satisfies all the axioms in Definition 5 of simple valuations provided the maximum is taken with respect to the lexicographic ordering. When v is a strict valuation, the lexicographic ordering depends only on the first coordinate and everything becomes much easier. As we only deal with the strict case, we consider straight away the following as a definition.

Definition 17 Let $v \in \mathrm{ML}$ be a strict valuation. We define $T_v\mathrm{ML}$ to be the set of group homomorphisms $\phi\colon \mathbb{C}(X)^* \to \mathbb{R}$ satisfying the property that, for any function $f \in \mathbb{C}[X]$ decomposed as $f = \sum m_\mu t_\mu$ in the linear basis of multicurves,

$$\phi(f) = \phi(t_\nu), \quad \text{where } \nu \text{ is } v\text{–extremal in } f. \tag{6}$$

We will refer to this definition of the tangent space as the Goldman model. In this section, we define a symplectic structure on it, and will relate it to the models of Thurston and Bonahon introduced later on.

Proposition 18 *For any strict valuation we have a sequence of natural isomorphisms*

$$T_v\mathrm{ML} = \mathrm{Hom}(\Lambda_v^+, \mathbb{R}) = \mathrm{Hom}(\Lambda_v, \mathbb{R}) = \mathrm{Hom}(\Lambda_v \otimes \mathbb{Q}, \mathbb{R}) = \mathrm{Hom}(\mathbb{C}(X)^*/\mathcal{O}_v^\times, \mathbb{R}),$$

where Hom *is understood first as the space of semigroup homomorphisms, and then as the space of group homomorphisms. In particular,* $T_v\mathrm{ML}$ *has dimension* $\operatorname{rat\,rk}(v)$ (*which is* $\le \dim X$).

Proof Recall that the map $\mu \mapsto v(t_\mu)$ is a bijection between the set of multicurves and Λ_v^+. We have $v(t_\mu) + v(t_\nu) = v(t_\xi)$, where ξ is the v–extremal multicurve in $t_\mu t_\nu$. Given $\phi \in T_v\mathrm{ML}$, the map $v(t_\mu) \mapsto \phi(t_\mu)$ is by construction a homomorphism of semigroups $\Lambda_v^+ \to \mathbb{R}$, and this construction can easily be reversed, giving the isomorphism $T_v\mathrm{ML} = \mathrm{Hom}(\Lambda_v^+, \mathbb{R})$. The remaining isomorphisms are purely formal, noticing that $\mathcal{O}_v^\times$ is the kernel of the group homomorphism $v\colon \mathbb{C}(X)^* \to \mathbb{R}$. □

Definition 19 For $f \in \mathbb{C}(X)$, we define the *differential* of the map $v \mapsto v(f)$ at v by

$$d_v \log f \colon T_v\mathrm{ML} \to \mathbb{R}, \quad d_v \log f(\phi) = \phi(f).$$

We introduced the log to make the formula $d_v \log(fg) = d_v \log f + d_v \log g$ look more natural.

By Proposition 18, the elements $d_v \log f$ span $T_v^*\mathrm{ML}$. More precisely, we obtain a basis by letting f range over a family of multicurves whose v–lengths form a basis of $\Lambda_v \otimes \mathbb{Q}$.

4.2 Residual Poisson structure

Proposition 20 *For all* $f, g \in \mathbb{C}[X]$ *and* $v \in \mathrm{ML}$, *we have* $v(\{f, g\}) \le v(f) + v(g)$.

Proof By linearity of the Poisson bracket, it is sufficient to prove the inequality for $f = t_\mu$ and $g = t_\nu$, where μ and ν are multicurves. Then, by the Leibnitz formula, it is sufficient to prove it for curves μ and ν. Suppose that μ and ν are in taut position and apply Goldman's formula (2). It is sufficient to prove that for any $p \in \mu \cap \nu$ we have $v(t_{\mu_p \nu_p} - t_{\mu_p \nu_p^{-1}}) \le v(t_\mu t_\nu)$, but this is a consequence of the smoothing lemma (Lemma 7). □

Given a strict valuation $v \in \mathrm{ML}$, the preceding proposition allows us to define the residual Poisson bracket at v in the following way.

Definition 21 For $f, g \in \mathbb{C}[X]$ and $v \in \mathrm{ML}$ strict, we define $\{f, g\}_v \in k_v = \mathbb{C}$ by

$$\{f, g\}_v = \frac{\{f, g\}}{fg} \mod \mathcal{M}_v.$$

Proposition 22 *There is an element* $\pi_v \in \Lambda^2 T_v\mathrm{ML}$ *representing this Poisson structure, in the sense that for any* $f, g \in \mathbb{C}[X]$, *we have*

$$\{f, g\}_v = \langle \pi_v, d_v \log(f) \wedge d_v \log(g) \rangle.$$

Proof Let us fix f and consider the map $F\colon \mathbb{C}[X] \to \mathbb{C}$ defined by $F(g) = \{f, g\}_v$. By the Leibnitz identity, this map satisfies $F(g_1 g_2) = F(g_1) + F(g_2)$ and thus extends to an element of $\mathrm{Hom}(\mathbb{C}(X)^*, \mathbb{C})$, and we must first show that it vanishes on $\mathcal{O}_v^\times$. Any $g \in \mathcal{O}_v^\times$ can be written $g = \alpha + h$ with $\alpha \in \mathbb{C}^*$ and $v(h) < 0$. We compute

$$F(g) = \frac{\{f, \alpha + h\}}{f(\alpha + h)} = \frac{\{f, h\}}{f(\alpha + h)}.$$

Since $v(h) < 0$ we have $v(f(\alpha+h)) = v(f) + v(\alpha+h) = v(f)$, and with Proposition 20, $v(\{f, h\}) < v(f)$; thus $F(g) \in \mathcal{M}_v$, and the claim is proved. What we have shown implies that there exists an element $\phi_f \in T_v\mathrm{ML}$ such that $F(g) = \langle \phi_f, d_v \log g\rangle$. As the Poisson bracket is antisymmetric, the same is true with the variables interchanged, and the conclusion follows. □

5 Actions of $\pi_1(S)$ on real trees

A real tree is a metric space T such that any two points $x, y \in T$ are joined by a unique injective segment. Recall that S is a closed oriented surface of genus $g \geq 2$. We consider real trees with an action of $\pi_1(S)$ that is *minimal*, in the sense that the only subtrees $T' \subset T$ satisfying $\gamma T' \subset T'$ for all $\gamma \in \pi_1(S)$ are $\varnothing$ and T.

The action of an element $\alpha \in \pi_1(S)$ on T either fixes a point and is called elliptic; otherwise it is a hyperbolic translation along an axis A_α with positive translation length $l(\alpha) = \min\{d(x, \alpha x) \mid x \in T\}$, and $d(x, \alpha x) = l(\alpha)$ if and only if $x \in A_\alpha$.

We face the following alternative. If all elements of $\pi_1(S)$ act elliptically, then they have a common fixed point; the minimality assumption implies that T is reduced to a point. If at least one element of $\pi_1(S)$ acts hyperbolically, then the union of all translation axes forms an invariant subtree (see [18]), which equals T by the minimality assumption.

An action of $\pi_1(S)$ is *free* when only the trivial element of $\pi_1(S)$ has a fixed point, or equivalently when $l(\alpha) > 0$ for all nontrivial $\alpha \in \pi_1(S)$. It is *small* when the stabilizer of any nontrivial segment in T is cyclic. This condition appears naturally in the following important results.

Theorem 23 (Culler and Morgan) *For real trees T_1 and T_2 with small minimal actions of $\pi_1(S)$, there exists an equivariant isometry $\Phi\colon T_1 \to T_2$ if and only if $l_1(\alpha) = l_2(\alpha)$ for all $\alpha \in \pi_1(S)$.*

Theorem 24 (Thurston, Skora) *To any measured lamination $\lambda \in \mathrm{ML}$ one can associate a "dual tree" T_λ together with a small minimal action of $\pi_1(S)$ on T_λ such that $l(\alpha) = 2i(\lambda, \alpha)$ for all $\alpha \in \pi_1(S)$. Conversely, any tree with a small and minimal action of $\pi_1(S)$ is produced in this way.*

Let us briefly outline the construction of the dual tree to a measured lamination, in the case where λ is filling (or equivalently when the simple valuation v_λ is positive).

First represent the filling measured lamination λ on S by a measured geodesic lamination for some fixed hyperbolic metric, and lift it in $\widetilde{S}$ to obtain a $\pi_1(S)$–invariant measured geodesic lamination $\widetilde{\lambda}$. Following [15, Section 2.3], the tree T_λ is the quotient of $\widetilde{S}$ by the equivalence relation whose classes are given either by the closure of a connected component of $\widetilde{S} \setminus \widetilde{\lambda}$ or else by a leaf of $\widetilde{\lambda}$ which is not contained in the previous classes. The quotient map $f : \widetilde{S} \to T_\lambda$ is clearly $\pi_1(S)$–equivariant.

To describe the complement of a point $x \in T_\lambda$, consider its preimage $f^{-1}(\{x\})$. If it consists of a geodesic leaf of $\widetilde{\lambda}$, then $T_\lambda \setminus \{x\}$ has two connected components. Otherwise it is isometric to the closure of an ideal hyperbolic polygon with k sides, so $T_\lambda \setminus \{x\}$ has $k > 2$ components, and x is called a *branch point* of T. In any case the connected components of $T_\lambda \setminus \{x\}$ have a cyclic orientation which is $\pi_1(S)$–invariant. These local cyclic orientations match together to give a global cyclic orientation on the Gromov boundary of T_λ. See [26] for more details.

The map $f : \widetilde{S} \to T_\lambda$ is not proper, so does not extend to the Gromov boundary. A nontrivial element $\alpha \in \pi_1(S)$ acts on $\widetilde{S} \simeq \mathbb{H}^2$ by hyperbolic translation along an axis which is transverse to λ, and thus crosses every leaf at most once. Hence the projection f maps it bijectively to a geodesic in T which, by equivariance, coincides with the axis A_α. Hence we can associate to the attractive and repulsive points of α in $\partial\mathbb{H}^2 = \partial\pi_1(S)$ the corresponding endpoints of A_α in ∂T. This partially defined map between the Gromov boundaries of $\pi_1(S)$ and T is $\pi_1(S)$–equivariant, orientation-preserving and independent of the initial hyperbolic metric.

We recall the following proposition from [6], which we will use repeatedly.

Proposition 25 *Let γ and δ be two hyperbolic isometries acting on a real tree T with axes A_γ and A_δ. Then one of the following holds.*

- (i) *If $A_\gamma \cap A_\delta = \varnothing$ then $l(\gamma\delta) = l(\gamma) + l(\delta) + 2D$ where D is the distance between A_γ and A_δ.*
- (ii) *If $A_\gamma \cap A_\delta \neq \varnothing$, we denote by $D \in [0, +\infty]$ the length of the intersection.*
  - (a) *If $D > 0$ and the translation directions of γ and δ on $A_\gamma \cap A_\delta$ coincide, or if $D = 0$, then $l(\gamma\delta) = l(\gamma) + l(\delta)$.*
  - (b) *If $D > 0$ and the translation directions of γ and δ on $A_\gamma \cap A_\delta$ are opposite, then we have $l(\gamma\delta) < l(\gamma) + l(\delta)$.*

Corollary 26 *Let γ and δ be two hyperbolic isometries acting on a real tree T with axes A_γ and A_δ. When the segment $A_\gamma \cap A_\delta$ has positive length, we may compare the translation directions of γ and δ: let $\operatorname{cosign}(\gamma, \delta) = \pm 1$ be $+1$ if they coincide and -1 if they differ. One of the following holds:*

$(\gamma \cup \delta)$	$l(\gamma) + l(\delta) < l(\gamma\delta) = l(\gamma\delta^{-1})$	*if $A_\gamma \cap A_\delta = \varnothing$.*
(equil)	$l(\gamma\delta) = l(\gamma) + l(\delta) = l(\gamma\delta^{-1})$	*if $A_\gamma \cap A_\delta$ is reduced to a point.*
$(\gamma\delta^{-1})$	$l(\gamma\delta^{-1}) < l(\gamma) + l(\delta) = l(\gamma\delta)$	*if $l(A_\gamma \cap A_\delta) > 0$ and $\operatorname{cosign}(\gamma, \delta) = 1$.*
$(\gamma\delta)$	$l(\gamma\delta) < l(\gamma) + l(\delta) = l(\gamma\delta^{-1})$	*if $l(A_\gamma \cap A_\delta) > 0$ and $\operatorname{cosign}(\gamma, \delta) = -1$.*

To illustrate how we will apply this corollary, let us propose a new proof of the smoothing lemma, which does not rely on the equivalence between measured laminations and simple valuations (that we showed in [12] using the smoothing lemma).

Notice that the equivalence between measured laminations λ and simple valuations v recovers the smoothing lemma, because for a taut multiloop α we have for obvious geometric reasons $i(\lambda,\alpha) \geq \max\{i(\lambda,\alpha_+), i(\lambda,\alpha_-)\}$, and as $t_\alpha = \pm t_{\alpha_+} \pm t_{\alpha_-}$ we have $v_\lambda(t_\alpha) \leq \max\{v_\lambda(t_{\alpha_+}), v_\lambda(t_{\alpha_-})\}$.

Proof of the smoothing lemma (Lemma 7) Let us represent our measured lamination λ by an action of $\pi_1(S)$ on a tree T. Fix a hyperbolic metric on S to identify $\widetilde{S} \simeq \mathbb{H}^2$.

Consider a taut multiloop α with a self-intersection point p, which may either be a self-intersection of a single component or an intersection between two components. We wish to prove that $i(\lambda,\alpha) = \max\{i(\lambda,\alpha_+), i(\lambda,\alpha_-)\}$.

Suppose first that p is the intersection point between two components which we write as $\gamma, \delta \in \pi_1(S,p)$. Lift γ and δ in $\widetilde{S} \simeq \mathbb{H}^2$ starting from $\tilde{p}$ to obtain geodesics $\tilde{\gamma}$ and $\tilde{\delta}$ which intersect only at $\tilde{p}$ and transversely at $\tilde{p}$. Consequently, their endpoints are linked in $\partial\mathbb{H}^2$ with respect to the cyclic orientation. As the same holds for the endpoints of A_γ and A_δ in ∂T, we must have $A_\gamma \cap A_\delta \neq \varnothing$, so we are not in case $(\gamma \cup \delta)$ of Corollary 26, whence $l(\gamma) + l(\delta) = \max\{l(\gamma\delta), l(\gamma\delta^{-1})\}$.

Suppose now that p is the self-intersection point of a single component of α which we may decompose as $\gamma\delta$ for $\gamma, \delta \in \pi_1(S,p)$. Lift γ and δ in $\mathbb{H}^2$ starting from $\tilde{p}$ to obtain quasigeodesics $\tilde{\gamma}$ and $\tilde{\delta}$. They intersect only at $\tilde{p}$ because, using the monodromy homomorphism associated to the developing map, another intersection point would imply an equality of the form $\gamma^m = \delta^n$ for some $m, n > 0$, which is impossible. The projection map $\widetilde{S} \to S$ is a local diffeomorphism, so the germs of arcs $(\tilde{\gamma} \cup \tilde{\delta}, \tilde{p})$ and $(\gamma \cup \delta, p)$ are topologically equivalent. Hence, up to inversion and exchange of γ and δ, the endpoints of $\tilde{\gamma}$ and $\tilde{\delta}$ in $\partial\mathbb{H}^2$ have cyclic order $(\gamma_+, \gamma_-, \delta_+, \delta_-)$ as shown in Figure 3. Consequently we are not in case $(\gamma\delta)$ of Corollary 26, whence $l(\gamma\delta) = \max\{l(\gamma) + l(\delta), l(\gamma\delta^{-1})\}$. □

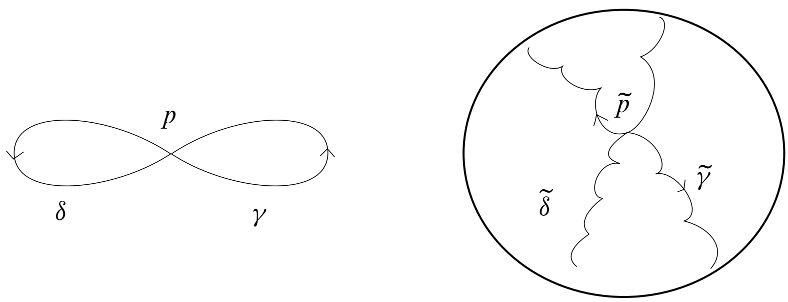


Figure 3: Configuration of axes at a self-crossing.

5.1 Trivalent real trees

Recall that a point x in a real tree is a *branch point* if $T \setminus \{x\}$ has at least three connected components. We will denote by $V(T)$ the set of branch points of T. A real tree is *trivalent* if any branch point disconnects it into three connected components.

A measured geodesic lamination λ is called maximal if there is no measured geodesic lamination whose support is strictly bigger; or equivalently if the regions in its complement $S \setminus \lambda$ are isometric to the interiors of ideal hyperbolic triangles.

Proposition 27 *Let T be a real tree with a free minimal action of $\pi_1(S)$, associated to a filling measured lamination λ. Denote by v the associated positive valuation.*

The following are equivalent:

(i) *v is acute.*

(ii) *T is trivalent.*

(iii) *λ is maximal.*

Proof (1) $\Longleftrightarrow$ (2) Suppose T is trivalent. Let α and β be nontrivial elements in $\pi_1(S)$ and consider their translation axes $A_\alpha, A_\beta \subset T$. From Corollary 26, we find that $l(\alpha\beta) = l(\alpha\beta^{-1}) = l(\alpha) + l(\beta)$ holds only when A_α and A_β meet in exactly one point, which is forbidden by the trivalence assumption. Thus v is acute.

Conversely, suppose T is not trivalent. Consider a branch point $x \in T$ with valency $k > 3$. We denote by $C_1, \dots, C_k$ the components of $T \setminus \{x\}$. They decompose the Gromov boundary of T into disjoint open subsets $\partial C_1, \dots, \partial C_k$. It is known that the set of pairs of ends of axes A_γ for $\gamma \in \pi_1(S)$ form a dense subset of $\partial T \times \partial T$. One proof consists in considering the sequence of fixed points for the elements $\alpha\beta^n$: the attractive points converge to the image by α of the attractive point of β and the repulsive points to the repulsive point of β. By minimality, the set of repulsive points of all β's is dense in ∂T, and again by minimality the images of a given attractive point by all α's is dense in ∂T. Thus we can find two axes A_α and A_β whose ends are respectively in $\partial C_1 \times \partial C_3$ and $\partial C_2 \times \partial C_4$. These two axes meet exactly at x, and Corollary 26 implies that $l(\alpha\beta) = l(\alpha\beta^{-1}) = l(\alpha) + l(\beta)$, showing that v is not acute.

(3) $\Longleftrightarrow$ (2) Recall from the construction of the dual tree T to a filling measured geodesic lamination $\lambda \subset S$ that the valency of a branch point in T is equal to the number of sides of the corresponding hyperbolic ideal polygon in $\widetilde{S} \setminus \widetilde{\lambda}$. Hence T is trivalent if and only if λ is maximal. □

It is well known that the set of maximal laminations has a full measure in ML; see [9, Lemma 2.3]. Hence Proposition 27 implies the following corollary.

Corollary 28 *The set of acute simple valuations has a full measure in* ML.

5.2 Bonahon cycles

Let T be a trivalent real tree with a free and minimal action of $\pi = \pi_1(S)$. To define its tangent space in the "moduli space" of such objects, imagine the combinatorial structure as being fixed while the distance function undergoes an infinitesimal deformation. Restricting attention to the variation of the distance between branch points, we obtain a symmetric map $c\colon V(T)^2 \to \mathbb{R}$, which is $\pi_1(S)$–invariant and satisfies $c(x,y) = c(x,z) + c(z,y)$ whenever z belongs to the geodesic joining x to y. We will refer to these maps as Bonahon cocycles and introduce them formally using a dual approach.

Definition 29 We define the space $\mathcal{B}(T)$ as the real vector space generated by pairs (x,y) of elements in $V(T)$ subject to the following relations:

(i) $(x,y) = (y,x)$ for all $x, y \in V(T)$.

(ii) $(x,y) = (x,z) + (z,y)$ if z belongs to the geodesic joining x to y.

The group $\pi = \pi_1(S)$ acts linearly on $\mathcal{B}(T)$ by $g(x,y) = (gx, gy)$, and Bonahon cocycles are the elements of $\mathrm{Hom}_\pi(\mathcal{B}(T), \mathbb{R}) = \mathrm{Hom}(\mathcal{B}(T)_\pi, \mathbb{R})$, where $\mathcal{B}(T)_\pi$ is the space of coinvariants.

Proposition 30 *There is a unique alternating bilinear form* $\cdot$ *on* $\mathcal{B}(T)$ *such that for all pairs* (x,y) *and* (z,t) *in* $V(T)^2$ *we have*:

(i) $(x,y)\cdot(z,t) = 0$ *if the geodesics from* x *to* y *and from* z *to* t *are disjoint.*

(ii) $(x,z)\cdot(z,y) = \frac{1}{2}\epsilon$ *if* z *belongs to the geodesic from* x *to* y*, where* $\epsilon = \pm 1$ *is the cyclic order of the components* (h_x, h, h_y) *of* $T \setminus \{z\}$ *such that* $x \in h_x$ *and* $y \in h_y$.

Proof The intersection of (x,y) and (z,t) is either empty or has the form (a,b). Decomposing (x,y) and (z,t) into segments involving a and b as in Figure 4, we are reduced, by bilinearity and antisymmetry, to cases (i) or (ii). This proves both uniqueness and existence. □

It is an amusing exercise to show that this pairing is nondegenerate. Instead we will deduce it from Poincaré duality in Thurston's model in Section 6. Indeed, we are interested in the space $\mathcal{B}(T)_\pi$ endowed with the following pairing obtained by averaging the previous one, whose nondegeneracy will thus follow from standard arguments in cohomology.

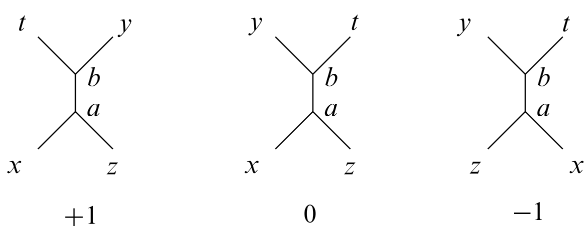


Figure 4

Proposition 31 *The following sum is finite, and it defines an alternating bilinear pairing on* $\mathcal{B}(T)_\pi$*:*

$$(x,y)\cdot_\pi(z,t)=\sum_{g\in\pi_1(S)}(x,y)\cdot g(z,t).$$

Proof We only have to prove finiteness of the sum. For that, we view T as the dual tree to a maximal measured lamination λ on S. The vertices x,y,z,t correspond to ideal triangles in $\widetilde{S}\simeq\mathbb{H}^2$: choose x_0,y_0,z_0,t_0 in each one of them. Since $\pi_1(S)$ acts properly on $\mathbb{H}^2$, the geodesics $[x_0,y_0]$ and $g[z_0,t_0]$ are disjoint for all but a finite number of $g\in\pi_1(S)$. When they are disjoint, their projections in the tree are disjoint or meet as in the middle case of Figure 4, so their intersection vanishes. □

We shall prove in Section 6 that $\mathcal{B}(T)_\pi$ is the antisymmetric part of $H_1(\widetilde{S},\mathbb{R})$, where $\widetilde{S}$ is the orientation covering of the measured lamination λ, thus recovering Thurston's original point of view on the tangent space $T_\lambda\mathrm{ML}$.

5.3 The symplectomorphism theorem

Fix a strict valuation $v\in\mathrm{ML}$, and recall it identifies the set of multicurves with Λ_v^+. Let T be a real tree with a free and minimal action of $\pi_1(S)$ representing v, so that $l(\alpha)=2v(t_\alpha)$ for all $\alpha\in\pi_1(S)$.

Lemma 32 *The distance between two branch points in* T *belongs to* Λ_v*.*

Proof This lemma can be deduced from repeated applications of Proposition 25. For instance, the distance between two disjoint axes A_γ and A_δ can be written $D=\frac{1}{2}(l(\gamma\delta)-l(\gamma)-l(\delta))$ and hence belongs to Λ_v. Instead, we may prove it as a direct consequence of a more conceptual construction for T_v using Bass–Serre theory: we refer to formula (3) in [16, Section 4.1]. □

Given $\phi\in T_v\mathrm{ML}=\mathrm{Hom}(\Lambda_v,\mathbb{R})$, we define a corresponding $c_\phi\in\mathrm{Hom}_\pi(\mathcal{B}(T),\mathbb{R})$ by setting $c_\phi(x,y)=\frac{1}{2}\phi(d(x,y))$, where d is the distance in T. As d is $\pi_1(S)$–invariant, c is also, and the identity $c(x,z)=c(x,y)+c(y,z)$ for y between x and z follows from the triangular equality satisfied by d. In other words, there is a well-defined map

$$\Psi\colon T_v\mathrm{ML}\to\mathrm{Hom}_\pi(\mathcal{B}(T),\mathbb{R}),\qquad \phi\mapsto c_\phi. \tag{7}$$

Proposition 33 *The map* Ψ *induces an isomorphism* $T_v\mathrm{ML}\simeq\mathrm{Hom}_\pi(\mathcal{B}(T),\mathbb{R})$*.*

Proof The linearity of Ψ is obvious. We first prove injectivity: suppose $c_\phi=0$. For any nontrivial $\alpha\in\pi_1(S)$, choose a branch point x on its axis A_α so that the translation length satisfies $l(\alpha)=2v(t_\alpha)=d(x,\alpha x)$. As $c_\phi(x,\alpha x)=\frac{1}{2}\phi(d(x,\alpha x))$ we get $\phi(v(t_\alpha))=0$, but Λ_v is generated by the $v(t_\alpha)$ for $\alpha\in\pi_1(S)$, so $\phi=0$.

This suggests the construction of the inverse, but this time we think of ϕ as a map $\phi\colon \mathbb{C}(X)^*/\mathcal{O}_v^\times \to \mathbb{R}$. Given $c \in \mathrm{Hom}_\pi(\mathcal{B}(T), \mathbb{R})$, we define $\phi(t_\alpha) = c(x, \alpha x)$ for any simple curve α, where x is any branch point in A_α (by additivity of c, this does not depend on the branch point). We extend ϕ to any multicurve by linearity. Finally for any $f \in \mathbb{C}[X]^*$ we set $\phi(f) = \phi(t_\mu)$, where μ is the v–extremal multicurve in f. The point is to show that ϕ indeed belongs to $T_v\mathrm{ML}$: as it satisfies equation (6) by construction, it remains to prove that it is multiplicative.

We first show that the defining property $\phi(t_\gamma) = c(x, \gamma x)$ extends to all loops $\gamma \in \pi_1(S)$ by induction on the number of self-intersections. Suppose γ has $n > 0$ intersections. Let p be one of them and denote by α and β the two elements of $\pi_1(S, p)$ such that $\gamma = \alpha\beta$. Since v is acute, we have either $v(t_{\alpha\beta^{-1}}) < v(t_\alpha) + v(t_\beta) = v(t_{\alpha\beta})$ or $v(t_\alpha) + v(t_\beta) < v(t_{\alpha\beta^{-1}}) = v(t_{\alpha\beta})$, and we apply either case (2)(i) or case (1) of [18, Proposition 1.6] (which are unmodified in [6]).

In the first case, the axes A_α and A_β intersect along a segment xy such that both isometries push x in the direction of y, and we have $x \in A_{\alpha\beta}$. If $l(\beta) \ge d(x, y)$ then $l(\alpha\beta) = d(x, \alpha\beta x) = d(x, y) + d(y, \alpha y) + d(\alpha y, \alpha\beta x)$, whence $c(x, \alpha\beta x) = c(x, y) + c(y, \alpha y) + c(y, \beta x) = c(y, \alpha y) + c(x, \beta x)$, with $x \in A_\beta$ and $y \in A_\alpha$. If $l(\beta) \le d(x, y)$ then $d(x, \alpha\beta x) = d(x, \beta x) + d(\beta x, \alpha\beta x)$ whence $c(x, \alpha\beta x) = c(x, \beta x) + c(z, \alpha z)$ with $z = \beta x \in A_\alpha$. Each time, the induction hypothesis applies, showing that both definitions of $\phi(t_\gamma)$ coincide.

In the second case, the axes A_α and A_β are disjoint: let xy be the geodesic joining them, and note that x also belongs to the axes of $\alpha\beta$ and $\alpha\beta^{-1}$. By the induction hypothesis, $\phi(t_{\alpha\beta^{-1}})$ is equal to $c(x, \alpha\beta^{-1}x)$. Then $d(x, \alpha\beta^{-1}x) = d(x, \alpha\beta x)$, whence $c(x, \alpha\beta^{-1}x) = c(x, \alpha\beta x)$ and $2\phi(t_{\alpha\beta}) = c(x, \alpha\beta x)$ as claimed.

To finish the proof, we must consider $f, g \in \mathbb{C}[X]$ and show that $v(fg) = v(f) + v(g)$. If μ and ν are the v–extremal multicurves of f and g, then the v–extremal multicurve of fg is that of $t_\mu t_\nu$, denoted by ξ. We must show that $\phi(t_\mu t_\nu) = \phi(t_\xi) = \phi(t_\mu) + \phi(t_\nu)$. Let us prove more generally that if $\alpha = \alpha_1 \cup \cdots \cup \alpha_n$ is a multiloop then $\phi(t_\alpha) = \phi(t_{\alpha_1}) + \cdots + \phi(t_{\alpha_n})$, reasoning by induction on the self-intersection number of α.

If the components α_j are disjoint, we may replace each one of them by its v–extremal smoothing, which remain disjoint, and the result follows from the definition of ϕ. Hence suppose that α_1 and α_2 intersect at p. Up to changing the orientation of α_2, we can suppose that $v(t_{\alpha_1\alpha_2}) = v(t_{\alpha_1}) + v(t_{\alpha_2})$. The computation in the first case at the beginning of the proof shows that $\phi(t_{\alpha_1\alpha_2}) = \phi(t_{\alpha_1}) + \phi(t_{\alpha_2})$. We also have $\phi(t_{\alpha_1\alpha_2} t_{\alpha_3} \cdots t_{\alpha_n}) = \phi(t_{\alpha_1\alpha_2}) + \phi(t_{\alpha_3}) + \cdots + \phi(t_{\alpha_n})$ by the induction hypothesis. □

Theorem 34 *The isomorphism $\Psi^*\colon \mathcal{B}(T)_\pi \to T_v^*\mathrm{ML}$ preserves the symplectic form.*

Explicitly, $\Psi^*(x, \alpha x) = d_v \log t_\alpha$ for all $\alpha \in \pi_1(S)$ and any branch point $x \in A_\alpha$. Indeed for all $\phi \in T_v\mathrm{ML}$, equation (7) and Definition 19 yield

$$\Psi(\phi)(x, \alpha x) = \tfrac{1}{2}\phi(d(x, \alpha x)) = \phi(t_\alpha) = d_v \log(t_\alpha)(\phi).$$

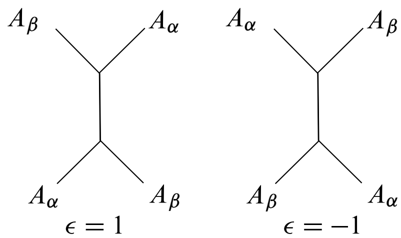


Figure 5: Sign rule for the axes.

Proof Let $\alpha, \beta \in \pi_1(S)$ represent two simple curves in S. We must prove that for $x \in A_\alpha$ and $y \in A_\beta$, $\{t_\alpha, t_\beta\}_v = \langle \pi_v, d_v \log t_\alpha \wedge d_v \log t_\beta \rangle$ equals $(x, \alpha x) \cdot_\pi (y, \beta y)$. If $i(\alpha, \beta) = 0$ then both quantities are null. Otherwise, put $\alpha \cup \beta$ in taut position.

We first compute the sum defining $\{t_\alpha, t_\beta\}_v$, in which every intersection $p \in \alpha \cap \beta$ contributes to a term $\epsilon_p(t_{\alpha_p \beta_p} - t_{\alpha_p \beta_p^{-1}}) t_\alpha^{-1} t_\beta^{-1} \bmod \mathcal{M}_v$. The set $\alpha \cap \beta$ is in bijection with pairs of intersecting lifts $(\widetilde{\alpha}, \widetilde{\beta}) \subset \widetilde{S} \times \widetilde{S}$ modulo the diagonal action of $\pi_1(S)$. These lifts correspond bijectively to axes of the form $(A_{\widetilde{\alpha}}, A_{\widetilde{\beta}})$ in T through the equivariant map $f: \widetilde{S} \to T$ which preserves the cyclic orientations on the boundaries. Fixing representatives $\alpha, \beta \in \pi_1(S)$, every such pair is represented by some (A_α, gA_β) for a unique $g \in \langle \alpha \rangle \backslash \pi / \langle \beta \rangle$. Using again Proposition 25, we can rewrite

$$\{t_\alpha, t_\beta\}_v = \sum_{g \in \langle \alpha \rangle \backslash \pi / \langle \beta \rangle} \epsilon(A_\alpha, gA_\beta) = \sum_{g \in \langle \alpha \rangle \backslash \pi / \langle \beta \rangle} \epsilon(A_\alpha, A_{g\beta g^{-1}}), \tag{8}$$

where $\epsilon(A_\alpha, A_\beta) = \pm 1$ if A_α and A_β are as in Figure 5, and $\epsilon(A_\alpha, A_\beta) = 0$ in any other configuration. Notice that this formula does not depend on the orientations of the axes, but on their cyclic orders at the branch points of the tree.

To end the proof, we fix $x \in A_\alpha$ and $y \in A_\beta$ to compare formula (8) with $\sum_{g \in \pi} (x, \alpha x) \cdot (gy, g\beta y)$. Grouping them depending on the class of g in $\langle \alpha \rangle \backslash \pi / \langle \beta \rangle$, we are reduced to the following equality, which is easily checked:

$$\epsilon(A_\alpha, A_\beta) = \sum_{m,n \in \mathbb{Z}} (\alpha^n x, \alpha^{n+1} x) \cdot (\beta^m y, \beta^{m+1} y). \qquad \square$$

6 Identifying the symplectic tangent models

Following [19, Section 3.2], we recall Thurston's description for the tangent space to ML at a maximal measured lamination λ. We start with an orientation covering $p: S' \to S$, which is a ramified covering of degree 2 with one ramification point in each triangle of the complement $S \setminus \lambda$, and such that the preimage $p^{-1}(\lambda)$ is naturally cooriented (meaning that its normal bundle is oriented). By the Gauss–Bonnet theorem, the set R of ramification points has $4g - 4$ elements and the monodromy of the covering is a homomorphism $\rho: \pi_1(S \setminus R) \to \{\pm 1\}$, which is nontrivial around each ramification point. For later purposes, it will be useful to consider the orbifold S^o where ramification points are thought as conical singularities of order 2.

Let $H_1(S',\mathbb{R})^{\pm}$ be the symmetric and antisymmetric part of $H_1(S',\mathbb{R})$ with respect to the involution of the covering: they are orthogonal for the intersection form. Hence (half) the intersection form restricted to $H_1(S',\mathbb{R})^-$ is nondegenerate. We shall refer to this symplectic space as Thurston's model for T^*_λML. We can avoid introducing the covering by considering instead the homology group $H_1(S^o,\mathbb{R}^-)$ with coefficients in the $\pi_1(S^o)$–module $\mathbb{R}$ together with the action given by $\gamma\,.\,x=\rho(\gamma)x$. The twisted intersection product $H_1(S^o,\mathbb{R}^-)\times H_1(S^o,\mathbb{R}^-)\to H_0(S^o,\mathbb{R})=\mathbb{R}$ coincides with the previous definition for Thurston's model. We will stick to this point of view in the sequel.

Let T be a trivalent real tree endowed with a free minimal action of $\pi_1(S)$, which is dual to a measured geodesic lamination λ. In the next section we first recover a model for S^o which depends only on T: our space will be an infinite dimensional CW–complex homotopic to S^o. As a consequence, its fundamental group is canonically attached to T and its homology will be easy to compute from T. We will use it extensively to prove that the Bonahon model $\mathcal{B}(T)_\pi$ and Thurston model $H_1(S^o,\mathbb{R}^-)$ are naturally isomorphic symplectic vector spaces.

6.1 A homotopical construction of the orbifold tree

6.1.1 Idea of the construction We first construct a space corresponding to the tree T with an orbifold singularity of order 2 at every branch point. As the topology of T induced by the metric is not given by a cell structure, our first task is to build a cellular model of T.

To motivate our construction, let us begin with the following analogy: suppose we wish to replace the real line $\mathbb{R}$, with its usual topology, by a CW–complex whose 0–cells consist of the set $\mathbb{Q}$ of rationals with the discrete topology. We may first add a 1–cell between every pair of distinct 0–cells to make the space connected. This creates a 1–cycle for every triple of distinct rational points, so we attach a 2–cell to each of those in order to make the space simply connected. Now every 4–tuple of rationals form the vertices of a 2–cycle, to which we attach a 3–cell, and so on. In the limit, we obtain Milnor's join construction $E\mathbb{Q}$, which is a space homotopic to $\mathbb{R}$ endowed with a free and proper action of $\mathbb{Q}$.

We shall play a similar game, replacing $\mathbb{R}$ by the real tree T, and $\mathbb{Q}$ by its set of branch points $V(T)$. We first attach a 1–cell to every pair of distinct branch points. However, we close the triangle (x,y,z) only if $x,y,z\in V(T)$ belong to a same geodesic in T. Then we go on similarly in higher dimensions, so that our space will resemble $E\mathbb{Q}$ in restriction to any geodesic of T. At this stage, we have a space on which $\pi_1(S)$ acts freely and properly. As it is contractible, its quotient by $\pi_1(S)$ is homotopic to S. Next comes the orbifold singularities: in homotopy theory, this is represented by a $K(\mathbb{Z}/2,1)$–space, that is, $\mathbb{RP}^\infty$. It remains to blow up the preceding construction at every branch point and insert an infinite-dimensional projective space. This construction may look complicated but we shall do it in one shot and few lines below.

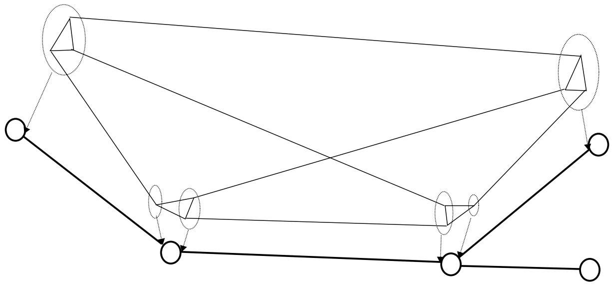


Figure 6: Attaching a 3–cell in T^o.

6.1.2 Formal construction A *half-edge* of T is a pair (x,h) consisting of a branch point x of T and a connected component h of $T \setminus \{x\}$; we sometimes just write h, as it determines x. Let us construct a CW–complex T^o whose 0–skeleton is the set of half-edges of T. First, we attach a 1–cell denoted by (h,k) between every pair of half-edges incident to the same branch point $x \in V(T)$. Now at every branch point x, the incident half-edges h_1, h_2, h_3 form a triangle homeomorphic to $\mathbb{RP}^1$, along which we attach a copy of $\mathbb{RP}^\infty$. For the moment, T^o is a disjoint union of infinite projective spaces indexed by the set of branched points $V(T)$; we call it the *orbifold part*.

Now, we add a *connecting part*, as suggested in Figure 6. Fix $\epsilon > 0$ small enough, say $\frac{1}{3}$. Consider a finite set W of branch points $\{x_0, \dots, x_n\}$ aligned on a geodesic of T, and denote by h_i and k_i the half-edges incident to x_i containing (a nonempty) part of that geodesic. The n–cell

$$\Delta_W = \left\{ (r_x)_{x \in W} \in [0, 1-\epsilon]^W \;\middle|\; \sum_{x \in W} r_x = 1 \right\}$$

is a truncated simplex, and there is an obvious inclusion $\Delta_{W'} \subset \Delta_W$ when $W' \subset W$. The face of Δ_W truncated at x_i corresponds to the set $\Delta_W^{x_i}$ of families (r_x) satisfying $r_{x_i} = 1-\epsilon$. We attach $\Delta_W^{x_i}$ to the orbifold part of T^o through the map $W \setminus \{x_i\} \to \{h_i, k_i\}$ sending the branch point x_j to the half-edge based at x_i which contains x_j, as in Figure 6. The 1–cells $\Delta_{\{x,y\}}$ will be called edges and denoted by (x,y).

As promised, the action of $\pi_1(S)$ on T^o is now proper, so that we may form the quotient $\Sigma^o = T^o/\pi_1(S)$. The following lemma shows that Σ^o and S^o are homotopic. Interestingly, the proof consists in constructing an equivariant map $F: T^o \to \widetilde{S}^o$, which plays the role of a (nonexistent) retraction for the map $f: \widetilde{S} \to T$.

Lemma 35 *Let $\widetilde{S}^o$ be the covering of the orbifold S^o corresponding to the kernel of the natural map $\pi_1(S^o) \to \pi_1(S)$. There exists a $\pi_1(S)$–equivariant map $F: T^o \to \widetilde{S}^o$ which induces a homotopy equivalence between Σ^o and S^o.*

Proof To define F, represent T as the dual tree to a measured geodesic lamination λ, and consider the collection of circles inscribed in each triangle of the complement $S \setminus \lambda$: they lift to a collection of

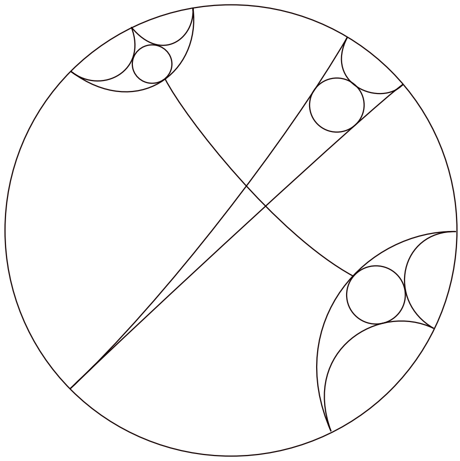


Figure 7: Lifting a geodesic to $\mathbb{H}^2$ (done with Geogebra).

circles C_x in $\widetilde{S} \simeq \mathbb{H}^2$ indexed by $x \in V(T)$. Moreover, the half-edges incident to x correspond bijectively to the three intersection points of C_x with the leaves of the lamination; see Figure 7.

The covering $\widetilde{S}^o$ is obtained from $\mathbb{H}^2$ by drilling out the interior of C_x and gluing back a copy of $\mathbb{RP}^\infty$ along $\mathbb{RP}^1 \simeq C_x$ for every $x \in V(T)$. By construction, the orbifold S^o is homotopic to the quotient $\widetilde{S}^o/\pi_1(S)$.

We now proceed to the construction of an equivariant map $F\colon T^o \to \widetilde{S}^o$. There is already an identification between the orbifold parts of both spaces, so that we are left to define the map F on the connecting part.

For every pair $(x, y) \in V(T)^2$, we must define a path $F(x, y)$ in $\widetilde{S}^o$ connecting the points of C_x and C_y identified to the endpoints h_x, h_y of (x, y) in T^o. A first guess would be to consider the geodesic path γ between the points h_x and h_y. This path actually projects to the geodesic joining x to y in T. However, it may intersect a forbidden circle C_z, in which case it enters its circumscribed ideal triangle Δ_z by one side and leaves it by another. Call p_z the ideal vertex at the intersection of these two sides. We can homotope γ inside Δ_z to a path avoiding C_z which stays on the side containing p_z; see Figure 7.

Moreover, we can choose those paths in such a way that F is $\pi_1(S)$–equivariant. Let us now consider a triple of points x, z, y lying on a geodesic of T in that order. We have defined $F(x, z)$, $F(z, y)$ and $F(x, y)$: it is not hard to see that the region enclosed by the three arcs and the boundary of C_z does not contain any other circle, hence it can be filled by a triangle: this extends F to the 2–skeleton of T^o. This procedure can be continued to define an equivariant map $F\colon T^o \to \widetilde{S}^o$, which induces a map $\overline{F}\colon \Sigma^o \to S^o$.

We would like to show that $\overline{F}$ is a homotopy equivalence. The space $\widetilde{S}^o$ is Eilenberg–Mac Lane, and Lemma 36 below shows that so is T^o, hence it is sufficient to prove that $\overline{F}$ induces an isomorphism between fundamental groups. Behold the following commutative diagram, and observe that the five lemma reduces the statement to showing that F_* is an isomorphism:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_1(T^o) & \longrightarrow & \pi_1(\Sigma^o) & \longrightarrow & \pi_1(S) & \longrightarrow & 0 \\ & & \downarrow F_* & & \downarrow \bar{F}_* & & \| & & \\ 0 & \longrightarrow & \pi_1(\widetilde{S}^o) & \longrightarrow & \pi_1(S^o) & \longrightarrow & \pi_1(S) & \longrightarrow & 0 \end{array}$$

This last statement is clear from the fact that $\pi_1(T^o)$ and $\pi_1(\widetilde{S}^o)$ are both isomorphic to a free product of copies of $\mathbb{Z}/2\mathbb{Z}$ indexed by $V(T)$; see again Lemma 36. □

6.2 Homology of T^o

The homology of T^o can be computed from its finite subcomplexes, which are easy to understand thanks to the following lemma. For a finite set $W \subset V(T)$, let $T^o_{(W)}$ be the union of cells involving W only: a cell belongs to $T^o_{(W)}$ when all its 0–faces are of the form (x, h) for $x \in W$. We define T^o_W to be the subcomplex of $T^o_{(W)}$ whose connecting part reduces to the 1–cells (x, y) for $x, y \in W$ such that there is no other element in W on the geodesic joining them. In more intuitive terms, T^o_W is a collection of $\mathbb{RP}^\infty$ indexed by W, connected in a tree-like fashion given by the embedding of W in T.

Lemma 36 *For all finite $W \subset V(T)$, the cell complex $T^o_{(W)}$ retracts by deformation on T^o_W.*

Proof We define the retraction by induction on the maximal dimension of the truncated simplices $\Delta_U \subset T^o_{(W)}$. Let $U = \{x_0, \dots, x_n\}$ correspond to one of them, it is the intersection of W with a geodesic in T. We retract Δ_U by deformation onto the union of $\Delta_{U'}$ for $U' \subset U$ ranging over all subsets which do not contain both x_0 and x_n. This procedure stops when $U = \{x, y\}$ and x and y are closest neighbors in W. □

We define a 1–cochain $\rho \in C^1(T^o, \{\pm 1\})$ sending every 1–cell of T^o to -1. It is a cocycle because the 2–cells of T^o, being either hexagons (orbifold part) or squares (contained in some Δ_W for W of cardinality 3), have an even number of 1–faces. The geometric idea underlying this definition is that any half-edge stands for a local coorientation of the lamination $\widetilde{\lambda}$, say pointing to the closest singular point. Following an edge e in T^o (transverse to $\widetilde{\lambda}$), we arrive at the other end with the opposite coorientation, giving $\rho(e) = -1$.

This cocycle defines a homomorphism $\rho\colon \pi_1(T^o) \to \mathbb{R}$ and we denote by $\mathbb{R}^-$ the vector space $\mathbb{R}$ with the action $\gamma.x = \rho(\gamma)x$. Our first task is to compute the homology of T^o with coefficients in $\mathbb{R}$ and $\mathbb{R}^-$.

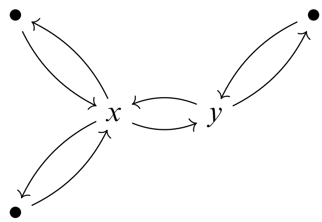
Lemma 37 *We have $H_k(T^o, \mathbb{R}) = 0$ if $k \neq 0$, $H_k(T^o, \mathbb{R}^-) = 0$ if $k \neq 1$, and*

$$H_1(T^o, \mathbb{R}^-) \simeq \mathcal{B}(T).$$

Proof Observe that $T^o = \varinjlim T^o_{(W)}$ as W exhausts the finite subsets of the countable set of branch points $V(T)$ and by Lemma 36, $T^o_{(W)}$ retracts by deformation on T^o_W, thus $H_*(T^o, \mathbb{R}^\pm) = \varinjlim H_*(T^o_W, \mathbb{R}^\pm)$.

We may forget about the cocycle ρ while computing the untwisted real homology, and further retract the space T^o_W onto a wedge of infinite projective spaces. Thus $H_0(T^o_W, \mathbb{R}) = \mathbb{R}$ and $H_k(T^o_W, \mathbb{R}) = 0$ for $k > 0$, so the same holds for T^o.

We now return to the twisted homology of T^o_W. For this we consider the double cover $T'_W \to T^o_W$ corresponding to ρ and compute the untwisted homology of the total space: it splits into the ± 1–eigenspaces of the involution, which coincide with $H_*(T^o_W, \mathbb{R}^\pm)$, respectively. The space T'_W is homotopy equivalent to a graph with vertex set W, and two edges above each edge e of T^o_W connecting its endpoints with opposite orientations, as shown below:



It follows that $H_0(T'_W, \mathbb{R}) = \mathbb{R}$ and $H_k(T'_W, \mathbb{R}) = 0$ if $k > 1$. Moreover $H_1(T'_W, \mathbb{R})$ has a basis formed by the cycles $c(x, y) \in H_1(T'_W, \mathbb{R})$ indexed by the edges (x, y) of T'_W, which consist in making a round trip from x to y, following the arrows. The Galois involution of T'_W exchanges the orientation of $c(x, y)$, so $H_1(T^o_W, \mathbb{R}^-)$ is freely generated by pairs (x, y) where x and y are closest neighbors in W.

Taking the limit as W converges to $V(T)$, we obtain $H_k(T^o, \mathbb{R}^-) = 0$ for $k = 0$ and $k > 1$. If an edge (x, y) gets subdivided into (x, z) and (z, y) as W increases, we have $c(x, y) = c(x, z) + c(z, y)$, which is compatible with the equality $(x, y) = (x, z) + (z, y)$, and provides the desired isomorphism for the inductive limit of $H_1(T^o_W, \mathbb{R}^-)$. □

6.3 Homology of the quotient $\Sigma^o = T^o/\pi$

Let us write $\pi = \pi_1(S)$ for short. The cocycle ρ on T^o is π–invariant, so it induces a homomorphism $\pi_1(\Sigma^o) \to \{\pm 1\}$ that we also denote by ρ. The π–equivariant homotopy equivalence between Σ^o and S^o thus yields a homomorphism $\pi_1(S^o) \to \{\pm 1\}$. By the remark following Lemma 36, this homomorphism is the coorientation monodromy of λ, so its kernel corresponds to the covering $S' \to S^o$. Consequently, we may deduce the homology of S^o with coefficients in $\mathbb{R}^\pm_\rho$ from that of Σ^o with the same coefficients.

The 2–fold covering S' of S^o ramified over R satisfies $\chi(S') = 2\chi(S) - (4g - 4) = 8 - 8g$ by the Riemann–Hurwitz formula. As $H_*(S^o, \mathbb{R}^\pm) = H_*(S', \mathbb{R})^\pm$, we get that $H_*(S^o, \mathbb{R}) = H_*(S, \mathbb{R})$, whereas $H_k(S^o, \mathbb{R}^-) = 0$ if $k \neq 1$ and $\dim H_1(S^o, \mathbb{R}^-) = 6g - 6$.

On the other hand, we can compute $H_*(T^o/\pi, \mathbb{R}^\pm)$ from $H_*(T^o, \mathbb{R}^\pm)$ using the Cartan–Leray spectral sequence. Its second page is $E^2_{p,q} = H_p(\pi, H_q(T^o, \mathbb{R}^\pm))$ and converges to $H_{p+q}(\Sigma^o, \mathbb{R}^\pm)$. Lemma 37 implies that, with both coefficients, the second page has only one line, whence the isomorphisms

$$H_*(\Sigma^o, \mathbb{R}) = H_*(\pi, \mathbb{R}) = H_*(S, \mathbb{R}) \quad \text{and} \quad H_*(\Sigma^o, \mathbb{R}^-) = H_{*-1}(\pi, \mathcal{B}(T)).$$

This yields the proposition that we are after.

Proposition 38 *Given a maximal measured lamination λ with associated covering $S' \to S$ and corresponding tree T, there is a natural isomorphism*

$$H_1(S', \mathbb{R})^- = H_1(\Sigma^o, \mathbb{R}^-) = H_0(\pi, \mathcal{B}(T)) = \mathcal{B}(T)_\pi.$$

We also have $H_k(\pi, \mathcal{B}(T)) = 0$ for $k = 1, 2$. Observe that from Poincaré duality we get $H_2(\pi, \mathcal{B}(T)) = H^0(\pi, \mathcal{B}(T)) = \mathcal{B}(T)^\pi = 0$. It is not surprising that $\mathcal{B}(T)$ has no invariant cycles as π acts freely on $V(T)$. We do not have a similar explanation for the vanishing of $H^1(\pi, \mathcal{B}(T))$.

6.4 Intersection form

In the commutative diagram

$$\begin{array}{ccc} \widetilde{S}' & \xrightarrow{\ \widetilde{p}\ } & \widetilde{S}^o \\ \Big\downarrow \pi & & \Big\downarrow \pi \\ S' & \xrightarrow{\ p\ } & S^o \end{array}$$

the first column is a Galois covering of surfaces with group π. We have the identifications

$$H_1(\widetilde{S}', \mathbb{R})^- = H_1(\widetilde{S}^o, \mathbb{R}^-) = H_1(T^o, \mathbb{R}^-) = \mathcal{B}(T) \quad \text{and} \quad H_1(S', \mathbb{R})^- = H_1(S^o, \mathbb{R}^-) = \mathcal{B}(T)_\pi.$$

Proposition 39 *The isomorphisms $H_1(\widetilde{S}', \mathbb{R})^- = \mathcal{B}(T)$ and $H_1(S', \mathbb{R})^- = \mathcal{B}(T)_\pi$ preserve the symplectic forms.*

Proof Let us begin with the first isomorphism. Recall that we defined an equivariant map $F \colon T^o \to \widetilde{S}^o$: it sends the cell (x, y) to a path $F(x, y)$ joining the orbifold points corresponding to x and y and avoiding all other orbifold points. As the homology of the orbifold part of T^o with coefficients $\mathbb{R}^-$ vanishes identically, these paths actually define cycles in $H_1(\widetilde{S}^o, \mathbb{R}^-)$, which in $H_1(\widetilde{S}', \mathbb{R})$ are represented geometrically by $c(x, y) = \widetilde{p}^{-1}(F(x, y)) \subset \widetilde{S}'$. Notice that these cycles have a natural orientation (given by the coorientation of the lifted lamination $\widetilde{\lambda}'$).

Recalling the definition of the pairing in $\mathcal{B}(T)$ given in Proposition 30, it suffices to compute $c(x, y) \cdot c(z, t)$ in the case where (x, y) and (z, t) are disjoint or consecutive.

In the first case, the cycles $c(x, y)$ and $c(z, t)$ are also disjoint, so their intersection vanishes. In the second case, the cycles $c(x, z)$ and $c(z, y)$ only intersect in a neighborhood of z which looks like the

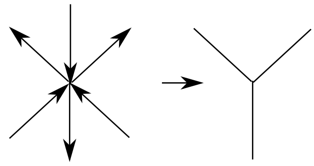


Figure 8: Double covering over a branching point.

right-hand side of Figure 8. The lifted cycles $c(x,z)$ and $c(z,y)$ go straight through the intersection point, oriented as shown. Analyzing the two possible cases, we find that the signs coincide.

Let us now consider the quotient. We showed in Section 6.3 that $H_1(S',\mathbb{R}) = H_1(\widetilde{S}',\mathbb{R})_\pi$. The result follows from the fact that the intersection form on $H_1(S',\mathbb{R})$ coincides with the averaged intersection form on $H_1(S',\mathbb{R})$. □

References

[1] **F Bonahon**, *Earthquakes on Riemann surfaces and on measured geodesic laminations*, Trans. Amer. Math. Soc. 330 (1992) 69–95 MR Zbl

[2] **F Bonahon**, *Shearing hyperbolic surfaces, bending pleated surfaces and Thurston's symplectic form*, Ann. Fac. Sci. Toulouse Math. 5 (1996) 233–297 MR Zbl

[3] **G W Brumfiel**, **H M Hilden**, SL(2) *representations of finitely presented groups*, Contemporary Mathematics 187, Amer. Math. Soc., Providence, RI (1995) MR Zbl

[4] **D Bullock**, *Rings of* $\mathrm{SL}_2(\mathbb{C})$*–characters and the Kauffman bracket skein module*, Comment. Math. Helv. 72 (1997) 521–542 MR Zbl

[5] **D Bullock**, **C Frohman**, **J Kania-Bartoszyńska**, *Understanding the Kauffman bracket skein module*, J. Knot Theory Ramifications 8 (1999) 265–277 MR Zbl

[6] **M J Conder**, **F Paulin**, *The translation length of the product of hyperbolic isometries of* $\mathbb{R}$*–trees* MR Zbl Appendix to M J Conder, "Discrete and free two-generated subgroups of SL_2", J. Algebra 553 (2020) 248–267

[7] **W M Goldman**, *The symplectic nature of fundamental groups of surfaces*, Adv. in Math. 54 (1984) 200–225 MR Zbl

[8] **J Hass**, **P Scott**, *Shortening curves on surfaces*, Topology 33 (1994) 25–43 MR Zbl

[9] **E Lindenstrauss**, **M Mirzakhani**, *Ergodic theory of the space of measured laminations*, Int. Math. Res. Not. 2008 (2008) art. id. rnm126 MR Zbl

[10] **F Luo**, *Simple loops on surfaces and their intersection numbers*, J. Differential Geom. 85 (2010) 73–115 MR Zbl

[11] **W Magnus**, *The uses of* 2 *by* 2 *matrices in combinatorial group theory: a survey*, Results Math. 4 (1981) 171–192 MR Zbl

[12] **J Marché**, **C-L Simon**, *Automorphisms of character varieties*, Ann. H. Lebesgue 4 (2021) 591–603 MR Zbl

[13] **M Mirzakhani**, *Counting mapping class group orbits on hyperbolic surfaces*, preprint (2016) arXiv 1601.03342

[14] **J W Morgan**, **P B Shalen**, *Valuations, trees, and degenerations of hyperbolic structures, I*, Ann. of Math. 120 (1984) 401–476 MR Zbl

[15] **J-P Otal**, *The hyperbolization theorem for fibered* 3*–manifolds*, SMF/AMS Texts and Monographs 7, Amer. Math. Soc., Providence, RI (2001) MR Zbl

[16] **J-P Otal**, *Compactification of spaces of representations after Culler, Morgan and Shalen*, from "Berkovich spaces and applications" (A Ducros, C Favre, J Nicaise, editors), Lecture Notes in Math. 2119, Springer (2015) 367–413 MR Zbl

[17] **A Papadopoulos**, **R C Penner**, *La forme symplectique de Weil–Petersson et le bord de Thurston de l'espace de Teichmüller*, C. R. Acad. Sci. Paris Sér. I Math. 312 (1991) 871–874 MR Zbl

[18] **F Paulin**, *The Gromov topology on* $\mathbb{R}$*–trees*, Topology Appl. 32 (1989) 197–221 MR Zbl

[19] **R C Penner**, **J L Harer**, *Combinatorics of train tracks*, Annals of Mathematics Studies 125, Princeton Univ. Press (1992) MR Zbl

[20] **J H Przytycki**, **A S Sikora**, *On skein algebras and* $\mathrm{Sl}_2(\mathbb{C})$*–character varieties*, Topology 39 (2000) 115–148 MR Zbl

[21] **K Rafi**, **J Souto**, *Geodesic currents and counting problems*, Geom. Funct. Anal. 29 (2019) 871–889 MR Zbl

[22] **Y Sözen**, **F Bonahon**, *The Weil–Petersson and Thurston symplectic forms*, Duke Math. J. 108 (2001) 581–597 MR Zbl

[23] **D Thurston**, *Geometric intersection of curves on surfaces*, preprint (2009) Available at `https://dpthurst.pages.iu.edu/DehnCoordinates.pdf`

[24] **D P Thurston**, *Positive basis for surface skein algebras*, Proc. Natl. Acad. Sci. USA 111 (2014) 9725–9732 MR Zbl

[25] **M Vaquié**, *Valuations*, from "Resolution of singularities" (H Hauser, J Lipman, F Oort, A Quirós, editors), Progr. Math. 181, Birkhäuser, Basel (2000) 539–590 MR Zbl

[26] **M Wolff**, *Connected components of the compactification of representation spaces of surface groups*, Geom. Topol. 15 (2011) 1225–1295 MR Zbl

Sorbonne Université
Paris, France

Laboratoire Paul Painlevé UMR CNRS, Université de Lille
Lille, France

`julien.marche@imj-prg.fr`, `christopher-lloyd.simon@ens-lyon.fr`

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