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The local (co)homology theorems for equivariant bordism

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We generalize the completion theorem for equivariant MU_G–module spectra for finite extensions of a torus to compact Lie groups using the splitting of global functors proved by Schwede. This proves a conjecture of Greenlees and May.

55N91, 55P91, 55Q91, 57R85

1 Overview

1.1 Introduction

A completion theorem establishes a close relationship between equivariant cohomology theory and its nonequivariant counterpart. It takes various forms, but in favourable cases it states that

$$(E_G^*)^\wedge_{J_G} \cong E^*(BG),$$

where E is a G–spectrum, E_G^* is the associated equivariant cohomology theory and J_G is the augmentation ideal (Definition 3.6).

The first such theorem is the Atiyah–Segal completion theorem for complex K-theory [4]. This is especially favourable because the coefficient ring $\boldsymbol{KU}_G^* = R(G)[v, v^{-1}]$ is well understood, and in particular it is Noetherian, and so in this case we can view the theorem as the calculation of the cohomology of the classifying space. The good behaviour for all groups permits one to make good use of naturality in the group, and indeed [4] uses this to give a proof for all compact Lie groups G. Previous partial results were proved by Atiyah and Hirzebruch in [2; 3]. The result raised the question of what other theories enjoy a completion theorem, and the case of equivariant complex cobordism was considered soon afterwards, with Löffler giving a proof in the abelian case [16]. The fact that the coefficient ring $\boldsymbol{MU}_G^*$ is not known explicitly means that this cannot be viewed as a computation of the cohomology of the classifying space. The fact that the coefficient ring is unknown and not Noetherian was an obstacle to extensions. Despite the algebraic complexity of the coefficients, Segal made the remarkable conjecture that stable cohomotopy should satisfy the completion theorem, and this was proved for finite groups by Carlsson [6], building on important earlier work (see eg Adams, Gunawardena and Miller [1], Carlsson [5], Laitinen [13], Lin [14], Lin, Davis, Mahowald and Adams [15], Ravenel [20], Segal and Stretch [24] and Stretch [25]).

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In this case, the conclusion can only be viewed as a calculation of the cohomotopy of classifying spaces in degrees 0 and below, but the structural content in positive degrees is equally striking. In the course of understanding this, there was a focus on understanding completion in various ways. From a homotopical point of view this led to the connection between completion and local cohomology and the definition of local homology (see Greenlees and May [9]), which is the derived version of completion. More precisely, for a Noetherian ring R and an ideal I, the local homology (resp. cohomology) groups $H_*^I(R;M)$ (resp. $H_I^*(R;M)$) of an R–module M calculate the left (resp. right) derived functors of completion (resp. I–power torsion); see [9] (resp. Hartshorne [12]). This derived approach led to a new proof of the Atiyah–Segal completion theorem and also its counterpart in homology; see Greenlees [8]. This in turn reopened the question of the completion theorem and local cohomology theorem for $\boldsymbol{MU}$, but now with the challenges shifted from the formal behaviour to the algebraic behaviour: the formal structure of the proof of the local cohomology theorem for $\boldsymbol{KU}$ applies precisely for $\boldsymbol{MU}$, but the difficulty is that, since $\boldsymbol{MU}_G^*$ is not Noetherian, it is not clear that the relevant ideals are finitely generated. Accordingly, Greenlees and May [10] isolated the formal argument and observed that if one could find a "sufficiently large" finitely generated ideal (Definition 3.8) of $\boldsymbol{MU}_G^*$, the local cohomology and completion theorems would hold for $\boldsymbol{MU}$. They went on to codify and use the structure of $\boldsymbol{MU}$ as a global spectrum to define and apply multiplicative norm maps, and hence construct "sufficiently large" finitely generated ideals in the case when the identity component of G is a torus. This led to the proof of local cohomology and completion theorems for $\boldsymbol{MU}$ for these groups. Much more recently, Schwede has studied global spectra more systematically [21], and in particular used the global structure of $\boldsymbol{MU}$ in a more sophisticated way to show that tautological unitary Euler classes are regular and give rise to various splittings [22].

More precisely, he proves that for every n we have a short sequence

$$0 \to \boldsymbol{MU}_{U(n)}^{*-2n} \xrightarrow{e_{U(n)}(\nu_n)} \boldsymbol{MU}_{U(n)}^{*} \xrightarrow{\mathrm{res}_{U(n-1)}^{U(n)}} \boldsymbol{MU}_{U(n-1)}^{*} \to 0$$

which is split exact, and, denoting by $p_k\colon U(k)\times U(n-k)\to U(k)$ the projection to the first factor, the composite

$$\boldsymbol{MU}_{U(k)}^{*} \xrightarrow{p_k^*} \boldsymbol{MU}_{U(k)\times U(n-k)}^{*} \xrightarrow{\mathrm{tr}_{U(k)\times U(n-k)}^{U(n)}} \boldsymbol{MU}_{U(n)}^{*}$$

is split injective when restricted to the kernel of the restriction map

$$\boldsymbol{MU}_{U(k)}^{*} \to \boldsymbol{MU}_{U(k-1)}^{*}.$$

These two facts together imply that the augmentation ideal $J_{U(n)}$ can be explicitly described as generated by the elements $s_{U(k)}^{U(n)}(e_{U(k)}(\nu_k))$ for $k=1,\dots n$ (Corollary 4.1), where $s_{U(k)}^{U(n)}\colon \boldsymbol{MU}_{U(k)}^{*} \to \boldsymbol{MU}_{U(n)}^{*}$ is a section of $\mathrm{res}_{U(k)}^{U(n)}$. Our contribution is to show that, for every compact Lie group G that embeds in $U(n)$, the finitely generated ideal $\mathrm{res}_G^{U(n)}(J_{U(n)}) \subset \boldsymbol{MU}_G^*$ is "sufficiently large" (Corollary 5.2). This is a consequence of Schwede's results as we will see in Section 5. Working in the highly structured category of G–equivariant $\boldsymbol{MU}_G$–modules guarantees that we can define (Definition 3.3) a homotopical version of the stable Koszul complex for the ideal $I = \mathrm{res}_G^{U(n)}(J_{U(n)}) \subset \boldsymbol{MU}_G^*$, which we denote by $K_\infty(I)$, and for an $\boldsymbol{MU}_G$–module M we set

$$\Gamma_I(M) = K_\infty(I) \wedge_R M$$

(Definition 3.4). By a formal argument, we can then construct a morphism

$$\kappa\colon EG_+ \wedge M \to \Gamma_I(M)$$

(Construction 3.7) of G–equivariant $\boldsymbol{MU}_G$–modules.

Theorem 1.1 *Let G be a compact Lie group with a faithful representation of dimension n, and M an $\boldsymbol{MU}_G$–module. Then the canonical map*

$$\kappa\colon EG_+ \wedge M \to \Gamma_I(M)$$

is an equivalence of G–equivariant $\boldsymbol{MU}_G$–module spectra.

This proves [10, Conjecture 1.4]. As a corollary, we obtain a local cohomology theorem which can be interpreted as a "derived completion theorem" for every compact Lie group.

Corollary 1.2 *Let G be a compact Lie group with a faithful representation of dimension n, M an $\boldsymbol{MU}_G$–module, X a based G–space and $I = \mathrm{res}_G^{U(n)}(J_{U(n)}) \subset \boldsymbol{MU}_G^*$. Then there are spectral sequences*

$$H_I^*(M_*^G(X)) \Rightarrow M_*^G(EG_+ \wedge X) \quad \text{and} \quad H_*^I(M_G^*(X)) \Rightarrow M_G^*(EG_+ \wedge X).$$

Since I has n generators, the local cohomology and homology are concentrated in degrees $\leq n$.

Organization

We start with a preliminary section where we introduce the notation and some basic facts of equivariant and global orthogonal spectra. In Section 3 we review the classical statement. This section is only needed to recall basic constructions and state the main theorem that we will prove in Section 5. In Section 4 we review Schwede's splitting [22, Theorem 1.4, page 5] and his corollary that ensures the regularity of certain Euler classes [22, Corollary 3.2, page 10]. Finally, in Section 5 we prove that the augmentation ideal $J_{U(n)}$ is *sufficiently large*. This will imply the completion theorem [10, Theorem 1.3, page 514] for $U(n)$ and for any compact Lie group G.

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2 Notation, conventions and facts

2.1 Spaces

By a *space* we mean a *compactly generated space* as introduced in [18]. We will denote by $\mathcal{T}$ (resp. $\mathcal{T}_*$) the category of spaces (resp. pointed spaces) with continuous maps (resp. based maps). For a compact Lie group G, $\mathcal{T}_G$ denotes the category of G–spaces and G–equivariant maps. (Equivariant) mapping spaces and (based) homotopy classes of (based) maps are defined as usual, and are denoted by $\mathrm{map}(\cdot\,,\cdot)$ and $[\cdot\,,\cdot]$, respectively.

2.1.1 Universal spaces A *family* of subgroups of a group G is a collection of subgroups closed under conjugation and taking subgroups. When G is a compact Lie group, a universal G–space for a family $\mathcal{F}$ of closed subgroups is a G–CW–complex $E\mathcal{F}$ such that:

- All isotropy groups of $E\mathcal{F}$ belong to the family $\mathcal{F}$.
- For every $H \in \mathcal{F}$ the space $E\mathcal{F}^H$ is weakly contractible.

We denote by $\widetilde{E}\mathcal{F}$ the reduced mapping cone of the collapse map $E\mathcal{F}_+ \to S^0$ which sends $E\mathcal{F}$ to the nonbasepoint of S^0. Any two such universal G–spaces are G–homotopy equivalent; hence we will refer to $E\mathcal{F}$ as the universal space for the family $\mathcal{F}$. Note that $E\{e\} = EG$.

2.2 Algebra

For a graded commutative ring A and a finitely generated ideal $I = (a_1, \dots, a_n)$ of A, we let $K^\bullet_\infty(I)$ be the graded cochain complex

$$\bigotimes_{i=1,\dots,n} (A \to A[1/a_i]),$$

where A and $A[1/a_i]$ sit in homological degrees 0 and 1, respectively, and the tensor product is over the ring A. If N is a graded A–module, the local cohomology groups are defined as

$$H^{s,t}_I(A; N) = H^{s,t}(K^\bullet_\infty(I) \otimes N).$$

When A is Noetherian, the functor $H^*_I(A; \cdot)$ calculates the right derived functors of the torsion functor

$$\Gamma_I(N) = \{n \in N \text{ such that } I^k n = 0 \text{ for some } k\}.$$

The main references for the theory of local cohomology are [11; 12]. Dually, we let the local homology groups be

$$H^I_{s,t}(A; N) = H_{s,t}\big(\mathrm{Hom}(\widetilde{K}^\bullet_\infty(I), N)\big),$$

where $\widetilde{K}^\bullet_\infty(I)$ is an A–free chain complex quasi-isomorphic to $K^\bullet_\infty(I)$; see [9]. When A is Noetherian and N is free or finitely generated, then the functor $H^I_k(A; \cdot)$ calculates the left derived functors of the I–adic completion functor. In particular, under these assumptions

$$H^I_k(A; N) \cong \begin{cases} N^\wedge_I & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

2.3 Spectra

A spectrum will be an orthogonal spectrum, and we will denote by Sp the category of orthogonal spectra as defined in [21, Definition 3.1.3, page 230]. The category Sp is closed symmetric monoidal with respect to the smash product $\cdot \wedge \cdot$. We will denote by $\mathrm{map}(\cdot\,,\cdot)$ the right adjoint of the smash product. Sp is also cotensored over $\mathcal{T}_*$, ie for every based space A and every spectrum X a mapping spectrum between these two is defined. We will also use the notation $\mathrm{map}(\cdot\,,\cdot)$ in this case. The definitions in the equivariant case are similar.

2.4 Global and equivariant stable homotopy categories

We let $\mathcal{GH}$ and $\mathcal{SH}_G$ denote respectively the *global stable homotopy category* and the *G–equivariant stable homotopy category* for a compact Lie group G. The first can be realized as a localization of the category of orthogonal spectra at the class of *global equivalences* [21, Definition 4.1.3, page 352] as constructed in [21, Theorem 4.3.18, page 400], and the second as a localization at the class of *π_*–isomorphisms* of the category of G–orthogonal spectra as constructed in [17, Theorem 4.2, page 47]. Hence,

$$\mathcal{SH}_G \cong \mathrm{Sp}_G[(\pi_*\text{–isos})^{-1}] \quad \text{and} \quad \mathcal{GH} \cong \mathrm{Sp}[(\text{global equivalences})^{-1}].$$

Both global equivalences and π_*–isomorphisms are weak equivalences of stable model structures; see [21, Theorem 4.3.17, page 398] for the global case and [17, Theorem 4.2, page 47] for the equivariant one. This implies that both categories $\mathcal{GH}$ and $\mathcal{SH}_G$ come with a preferred structure of triangulated categories, and we denote by Σ the *shift functor* in both cases. The derived smash product of Sp (resp. Sp_G) endows the category $\mathcal{GH}$ (resp. $\mathcal{SH}_G$) with a closed symmetric monoidal structure. For every compact Lie group G there is a forgetful functor $(-)_G\colon \mathcal{GH} \to \mathcal{SH}_G$ obtained from the point–set level functor of endowing a global spectrum with the trivial G–action. This functor is strong symmetric monoidal and exact; see [21, Theorem 4.5.24, page 450].

Homotopy groups are defined for equivariant spectra and for global spectra as usual [21, page 232]. In both cases, for a fixed X and k, the system of homotopy groups $\{\pi_k^H(X)\}_{H\subset G}$ has a lot of additional structure. For G–spectra, $\{\pi_k^H(X)\}_{H\subset G}$ is a *Mackey functor* [21, Definition 3.4.15, page 319]; for global spectra, $\{\pi_k^G(X)\}_{G \text{ compact Lie}}$ is a *global functor* [21, Definition 4.2.2, page 369].

If we fix a compact Lie group G and a G–spectrum X, the functor

$$\pi_k^G\colon \mathcal{SH}_G \to \mathsf{Ab}$$

is corepresented by the pair $(\Sigma^k\mathbb{S}, \mathrm{id})$, ie we have a natural isomorphism

$$\mathcal{SH}_G(\Sigma^k\mathbb{S}, X) \cong \pi_k^G(X). \tag{1}$$

A similar statement holds in the global setting; refer to [21, Theorem 4.4.3, page 412]. Finally, for a G–spectrum X, we adopt the convention

$$X_*^G = \pi_*^G(X), \quad X_G^* = X_{-*}^G.$$

2.5 Global complex cobordism

Since our work concerns the complex bordism ring, we recollect here some facts about this theory. We will write $\boldsymbol{MU}$ for the global Thom ring spectrum defined in [21, Example 6.1.53] which is a model for the homotopical equivariant bordism $\boldsymbol{MU}_G$ introduced by tom Dieck in [7] for every compact Lie group G. The global theory $\boldsymbol{MU}$ has the structure of an *ultracommutative ring spectrum* in the sense of [21, Definition 5.1.1, page 463]; this assures that, for every compact Lie group G, the category of $\boldsymbol{MU}_G$–modules is symmetric monoidal. For every compact Lie group G and for every unitary representation V, a *Thom class* $\sigma_G(V) \in \boldsymbol{MU}_G^{2n}(S^V)$ is defined, where $n = \dim_{\mathbb{C}}(V)$. The Euler class $e_G(V) \in \boldsymbol{MU}_G^{2n}$ is by definition the image of the Thom class along the fixed point inclusion $S^0 \to S^V$. If V has nontrivial G–fixed points, then the previous inclusion is G–nullhomotopic, and hence $e_G(V) = 0$ if $V^G \neq \{0\}$. On the other hand, tom Dieck showed that if $V^G = \{0\}$ then the Euler class $e_G(V)$ is a nonzero element in $\boldsymbol{MU}_G^{2n}$ [7, Corollary 3.2, page 352].

The theory $\boldsymbol{MU}$ has an equivariant Thom isomorphism for every unitary representation V

$$\boldsymbol{MU}_G^0(S^{k+V}) \cong \boldsymbol{MU}_G^{-k-2n} \tag{2}$$

given by RO(G)–graded multiplication with the Thom class $\sigma_G(V)$, where $n = \dim_{\mathbb{C}}(V)$. This isomorphism takes the multiplication by the class a_n defined in Section 4 to multiplication by the Euler class of the representation V.

3 Classical statement

We recall some basic constructions that can be found in [10]. To make sense of them, we need to work with highly structured equivariant ring spectra known as E_∞ ring G–spectra or commutative $\mathbb{S}_G$–algebras. In particular, all the constructions below are well defined for $R = \boldsymbol{MU}_G$. We refer to [10, page 511] for a more detailed explanation and bibliography.

Construction 3.1 Let $R \in \mathcal{SH}_G$ be a G–ring spectrum as explained above. By (1), every element of R_n^G specifies by adjunction a morphism $\alpha\colon \mathbb{S} \to \Sigma^{-n} R$ in $\mathcal{SH}_G$. We let

$$\tilde{\alpha}\colon R \to \Sigma^{-n} R$$

be the composition

$$R \xrightarrow{\alpha \wedge R} \Sigma^{-n} R \wedge R \cong \Sigma^{-n}(R \wedge R) \xrightarrow{\Sigma^{-n}\mu} \Sigma^{-n} R,$$

where $\mu\colon R \wedge R \to R$ is the multiplication of R. This defines a morphism in $\mathcal{SH}_G$, and we let

$$R[1/\alpha] := \text{telescope}(R \xrightarrow{\tilde{\alpha}} \Sigma^{-n} R \xrightarrow{\Sigma^{-n}\tilde{\alpha}} \Sigma^{-2n} R \to \cdots) \tag{3}$$

be the *mapping telescope* of the iterates of $\tilde{\alpha}$.

Remark 3.2 The *mapping telescope* in Construction 3.1 models the sequential homotopy colimit in $\mathcal{SH}_G$. For a discussion of homotopy colimits in $\mathcal{SH}_G$, refer to [19, Appendix C, page 160].

Definition 3.3 Let R and α be as above and let $I = (\alpha_1, \dots, \alpha_n)$ be an ideal in R_*^G. We define

$$K_\infty(\alpha) := \mathrm{fib}(R \to R[1/\alpha]) \quad \text{and} \quad K_\infty(I) := K(\alpha_1) \wedge_R \cdots \wedge_R K(\alpha_n).$$

Definition 3.4 Let M be an R–module and $I \subset R_*^G$ be a finitely generated ideal. Then we define

$$\Gamma_I(M) = K_\infty(I) \wedge_R M \quad \text{and} \quad (M)_I^\wedge = \mathrm{map}_R(K_\infty(I), M).$$

Remark 3.5 There is a spectral sequence of local cohomology

$$H_I^*(R_*^G; M_*^G) \Rightarrow \Gamma_I(M)_*^G,$$

and there is a spectral sequence of local homology

$$H_*^I(R_*^G; M_*^G) \Rightarrow ((M)_I^\wedge)_G^*.$$

Note that when $M = R$ we obtain the spectral sequence that computes $K_\infty(I)_*^G$; see [10].

Definition 3.6 Let G be a compact Lie group and R be an orthogonal G–spectrum. The augmentation ideal J_G of R at G is the kernel of

$$\mathrm{res}_1^G \colon R_*^G \to R_*.$$

Construction 3.7 By construction of $R[1/\alpha]$, if $\alpha \in J_G$ then

$$\mathrm{res}_1^G R[1/\alpha] \simeq 0.$$

Hence, applying the restriction to the fibre sequence

$$\Gamma_\alpha R \to R \to R[1/\alpha],$$

we obtain a fibre sequence in which the third term is contractible. This implies, by the long exact sequence in homotopy groups induced by a fibre sequence of spectra, that the canonical map

$$\mathrm{res}_1^G(\Gamma_\alpha R) = \Gamma_{\mathrm{res}_1^G(\alpha)} \mathrm{res}_1^G R \xrightarrow{\simeq} \mathrm{res}_1^G R$$

is an equivalence. The same argument applies for an ideal $I \subset R_*^G$, giving an equivalence

$$\mathrm{res}_1^G(\Gamma_I R) = \Gamma_{\mathrm{res}_1^G(I)} \mathrm{res}_1^G R \xrightarrow{\simeq} \mathrm{res}_1^G R.$$

Smashing the above morphism with the universal G–space EG_+, we obtain an equivalence

$$EG_+ \wedge \Gamma_I R \to EG_+ \wedge R$$

in $\mathcal{SH}_G$. Inverting this and composing with the collapse map $EG_+ \wedge \Gamma_I R \xrightarrow{\mathrm{coll} \wedge \Gamma_I R} S^0 \wedge \Gamma_I R \cong \Gamma_I R$, we obtain a zigzag

$$\begin{tikzcd} EG_+ \wedge R & EG_+ \wedge \Gamma_I R \arrow[l, "\simeq"'] \arrow[r] & \Gamma_I R, \arrow[from=1-1, to=1-3, bend left, "\kappa"] \end{tikzcd}$$

which defines a morphism of R–modules in $\mathcal{SH}_G$.

We now turn our attention to $\boldsymbol{MU}$ (Section 2.5) and we recall what it means for an ideal of the ring $\boldsymbol{MU}_G^*$ to be "sufficiently large". This is the key property to assure that the E^2–page of a specific spectral sequence that appears in the proof of Theorem 3.10 is zero.

Definition 3.8 [10, Definition 2.4, page 517] An ideal $I \subset \boldsymbol{MU}_G^*$ is sufficiently large at H if there exists a nonzero complex representation V of H such that $V^H = \{0\}$ and the Euler class $e_H(V) \in \boldsymbol{MU}_H^{2n}$ is in the radical $\sqrt{\mathrm{res}_H^G(I)}$, where $n = \dim_{\mathbb{C}}(V)$. The ideal I is sufficiently large if it is sufficiently large at all closed subgroups $H \neq 1$ of G.

Remark 3.9 Being sufficiently large is transitive with respect to subgroup inclusion, ie if $I \subset \boldsymbol{MU}_*^G$ is sufficiently large then so is $\mathrm{res}_H^G(I) \subset \boldsymbol{MU}_H^*$.

Theorem 3.10 [10, Theorem 2.5, page 518] *Let G be a compact Lie group. Then, for any sufficiently large finitely generated ideal $I \subset J_G$,*

$$\kappa\colon EG_+ \wedge \boldsymbol{MU}_G \to \Gamma_I \boldsymbol{MU}_G$$

is an equivalence in $\mathcal{SH}_G$. Therefore,

$$EG_+ \wedge M \to \Gamma_I(M) \quad \text{and} \quad (M)_I^\wedge \to \mathrm{map}(EG_+, M)$$

are equivalences for any $\boldsymbol{MU}_G$–module M.

Proof Here, we only give a sketch of the argument, following the main reference. The point is that, if $I \subset \boldsymbol{MU}_G^*$ is sufficiently large, then $\mathrm{res}_H^G I \subset \boldsymbol{MU}_H^*$ is also sufficiently large. Moreover, since every descending sequence of compact Lie groups stabilizes, we can use induction and assume that the theorem holds for any proper closed subgroup of G. Passing to the cofibre of the map κ, it is enough to show that

$$\pi_*^G(\widetilde{E}G \wedge \Gamma_I \boldsymbol{MU}_G) = 0.$$

We then let $\mathcal{P}$ be the family of proper subgroups of G and let $E\mathcal{P}$ be the universal space associated to $\mathcal{P}$. Since

$$\widetilde{E}\mathcal{P} \wedge S^0 \to \widetilde{E}\mathcal{P} \wedge \widetilde{E}G$$

is an equivalence, it suffices to show that $\widetilde{E}\mathcal{P} \wedge K(I)$ is contractible. Let $\mathcal{U}$ be a complete complex G–universe and define $\mathcal{U}^\perp$ to be the orthogonal complement of the G–fixed points $\mathcal{U}^G$ in $\mathcal{U}$. Then,

$$\mathrm{colim}_{V \in \mathcal{U}^\perp} S^V$$

is a model for $\widetilde{E}\mathcal{P}$. We can then compute

$$\begin{aligned}\pi_*^G(\widetilde{E}\mathcal{P} \wedge \Gamma_I \boldsymbol{MU}_G) = \pi_*^G((\mathrm{colim}_{V \in \mathcal{U}^\perp} S^V) \wedge \Gamma_I \boldsymbol{MU}_G) &\cong \mathrm{colim}_{V \in \mathcal{U}^\perp} \pi_*^G(S^V \wedge \Gamma_I \boldsymbol{MU}_G) \\ &\cong \mathrm{colim}_{V \in \mathcal{U}^\perp} \pi_{*-|V|}^G(\Gamma_I \boldsymbol{MU}_G) \cong \pi_*^G(\Gamma_I \boldsymbol{MU}_G)[\{e_G(V)^{-1}\}_{V \in \mathcal{U}^\perp}].\end{aligned}$$

Localizing the spectral sequence in Remark 3.5,

$$H_I^*(\boldsymbol{MU}_*^G) \Rightarrow \pi_*^G(\Gamma_I \boldsymbol{MU}_G),$$

away from the Euler classes, we obtain another spectral sequence

$$H_I^*(MU_*^G)[\{e_G(V)^{-1}\}_{V\in\mathcal{U}^\perp}] \Rightarrow \pi_*^G(\Gamma_I MU_G)[\{e_G(V)^{-1}\}_{V\in\mathcal{U}^\perp}].$$

Since local cohomology of a ring at an ideal becomes zero when localized by inverting an element in that ideal, we obtain that the E^2–term of the spectral sequence is zero for I sufficiently large. This proves the claim. □

Remark 3.11 As stated in [10, Theorem 2.5, page 518], the previous theorem holds more generally for all commutative $\mathbb{S}_G$–algebras (or E_∞ ring G–spectra) which are equivariantly complex oriented and have natural Thom isomorphisms for unitary G–representations. For example, the theorem holds for equivariant K–theory.

The paper [10] proceeds by constructing a sufficiently large subideal of the augmentation ideal J_G whenever G is a finite group or a finite extension of a torus using "norm maps" [10, Section 3].

Here is where our approach differs from the classical one. In fact, we do not make use of norm maps, and the strategy to construct a sufficiently large subideal of J_G splits in two steps:

Step 1 We use Schwede's splitting (4) to prove that $J_{U(n)}$ is generated by "Euler classes". Thanks to this, we prove that $J_{U(n)}$ is sufficiently large (Proposition 5.1).

Step 2 Using the fact that any compact Lie group embeds into a unitary group $U(N)$ for N sufficiently large, we conclude that $\mathrm{res}_G^{U(N)} J_{U(N)}$ is a sufficiently large subideal of J_G for any compact Lie group G by Remark 3.9.

4 Schwede's splitting

We recall that a *global functor* F associates to every compact Lie group G an abelian group $F(G)$, and this association is contravariantly functorial with respect to continuous group homomorphisms. Moreover, for every closed subgroup inclusion $H < G$, a transfer map $\mathrm{tr}_H^G\colon F(H) \to F(G)$ is defined. This data needs to satisfy some relations that can be found in [21, page 373]. In [22, Theorem 1.4, page 5], Schwede proves that, for any global functor F, the restriction homomorphism

$$\mathrm{res}^{U(n)}_{U(n-1)}\colon F(U(n)) \to F(U(n-1))$$

is a split epimorphism. He then deduces a splitting of global functors when evaluated on the unitary group $U(n)$. Explicitly, the splitting takes the form

$$F(U(n)) \cong F(e) \oplus \bigoplus_{k=1,\dots,n} \mathrm{Ker}\big(\mathrm{res}^{U(k)}_{U(k-1)}\colon F(U(k)) \to F(U(k-1))\big). \tag{4}$$

The most important application of the splitting for us is when the global functor comes from the homotopy groups of an orthogonal spectrum. In fact, for every global stable homotopy type X, that is, an object in $\mathcal{GH}$, we have a global functor

$$\underline{\pi}_*(X)(G) = \pi_*^G(X).$$

The splitting then tells us that, for every $k \leq n$, the group $\pi_*^{U(k)}(X)$ is a natural summand of $\pi_*^{U(n)}(X)$. In this case, a more explicit description of the right-hand side of the splitting is available. In fact, let ν_n be the tautological representation of $U(n)$ and let

$$a_n \in \pi_0^{U(n)}(\Sigma^\infty S^{\nu_n})$$

be the Euler class of ν_n, ie the element represented by the inclusion $S^0 \to S^{\nu_n}$. Then the short sequence

$$0 \to \pi_{*+\nu_n}^{U(n)}(X) \xrightarrow{a_n} \pi_*^{U(n)}(X) \xrightarrow{\mathrm{res}^{U(n)}_{U(n-1)}} \pi_*^{U(n-1)}(X) \to 0$$

is exact [22, Corollary 3.1, page 10].

When $X = \boldsymbol{MU}$, the equivariant Thom isomorphism identifies the previous short exact sequence with the following short exact sequence:

$$0 \to \boldsymbol{MU}_{U(n)}^{*-2n} \xrightarrow{e_{U(n)}(\nu_n)} \boldsymbol{MU}_{U(n)}^{*} \xrightarrow{\mathrm{res}^{U(n)}_{U(n-1)}} \boldsymbol{MU}_{U(n-1)}^{*} \to 0.$$

Moreover, we have the following corollary:

Corollary 4.1 *Let $J_{U(n)}$ be the augmentation ideal of $\boldsymbol{MU}_*^{U(n)}$* (see Definition 3.6), *and let $s_{U(k)}^{U(n)}$ be a section of* $\mathrm{res}_{U(k)}^{U(n)}$ (see [22, Construction 1.3, page 4]). *Then*

$$J_{U(n)} = \big(s_{U(k)}^{U(n)}(e_{U(k)}(\nu_k)) \mid k = 1, \dots, n\big),$$

and, in particular,

$$\mathrm{res}_{U(k)}^{U(n)} J_{U(n)} = J_{U(k)}$$

for all $k \leq n$.

Proof This is just the combination of the splitting (4) and the short exact sequence above. □

Remark 4.2 Since the forgetful functor $(-)_G : \mathcal{GH} \to \mathcal{SH}_G$ is strong symmetric monoidal and exact, the global splitting (4) at the unitary group translates in a splitting in $\mathcal{SH}_G$.

Remark 4.3 Schwede [23, Definition 1.1] gives an explicit construction of the sections $s_{U(k)}^{U(n)}$, and he shows that the resulting elements

$$s_{U(1)}^{U(n)} e_{U(1)}(\nu_1), \quad s_{U(2)}^{U(n)} e_{U(2)}(\nu_2), \quad \dots, \quad s_{U(n-1)}^{U(n)} e_{U(n-1)}(\nu_{n-1})$$

are "genuine equivariant Chern classes". In particular, they map to the classical Chern classes under the bundling map

$$\boldsymbol{MU}_{U(n)}^* \to \boldsymbol{MU}^*(BU(n)),$$

and they have similar naturality properties [23, Theorem 1.3].

5 The main result

Proposition 5.1 *$J_{U(n)} \subseteq \boldsymbol{MU}_{U(n)}$ is sufficiently large.*

Proof Let H be a closed subgroup of $U(n)$. We need to show that there exists a nonzero complex H–representation with no nontrivial fixed points, the Euler class of which is in $\sqrt{\operatorname{res}_H^{U(n)}(J_{U(n)})}$. We will actually show that there is no need to take radicals in this case. Consider

$$V = \operatorname{res}_H^{U(n)} \nu_n - (\operatorname{res}_H^{U(n)} \nu_n)^H,$$

and let $k = \dim_{\mathbb{C}}(V)$. Note that $k = 0$ if and only if $H = 1$. We claim that the Euler class $e_H(V)$ is in $\operatorname{res}_H^{U(n)} J_{U(n)}$. If $k = n$, then $V = \operatorname{res}_H^{U(n)} \nu_n$ and

$$e_H(V) = \operatorname{res}_H^{U(n)}(e_{U(n)}\nu_n) \in \operatorname{res}_H^{U(n)} J_{U(n)},$$

and the claim holds.

Now let $k > 0$. We choose an orthonormal basis $(x_1, \dots, x_{n-k})$ of $(\operatorname{res}_H^{U(n)} \nu_n)^H$ and a unitary matrix $g \in U(n)$ that sends the canonical basis of $\mathbb{C}^n$ to any other orthonormal basis that has as last $n-k$ vectors $(x_1, \dots, x_{n-k})$. Then, for any $h \in H$,

$$h^g = \left(\begin{array}{c|c} \tilde{h} & 0 \\ \hline 0 & \mathrm{Id}_{n-k} \end{array}\right),$$

where $h^g = g^{-1}hg$ and $\tilde{h} \in U(k)$.

This implies that V is conjugate to the H^g–representation $\operatorname{res}_{H^g}^{U(k)}(\nu_k)$. Letting $g_\star\colon \boldsymbol{MU}_{H^g}^* \to \boldsymbol{MU}_H^*$ be the conjugation action (see the relations in [21, Definition 3.4.15, page 319]), we pass to Euler classes, obtaining the relation

$$e_H(V) = g_\star\big(e_{H^g}(\operatorname{res}_{H^g}^{U(k)}(\nu_k))\big).$$

We then compute the right-hand side of the last equation:

$$\begin{aligned} g_\star\big(e_{H^g}(\operatorname{res}_{H^g}^{U(k)}(\nu_k))\big) &= g_\star\big(\operatorname{res}_{H^g}^{U(k)}(e_{U(k)}(\nu_k))\big) = g_\star\big(\operatorname{res}_{H^g}^{U(n)}\big(s_{U(k)}^{U(n)}(e_{U(k)}(\nu_k))\big)\big) \\ &= \operatorname{res}_H^{U(n)}\big(s_{U(k)}^{U(n)}(e_{U(k)}(\nu_k))\big). \end{aligned}$$

In the second equality we have used the chosen section $s_{U(k)}^{U(n)}$ (Corollary 4.1), and in the last one the formula

$$g_\star \circ \operatorname{res}_{H^g}^{U(n)} = \operatorname{res}_H^{U(n)}$$

(again, see the relations in [21, Definition 3.4.15, page 319]). By Corollary 4.1, it is clear that $\operatorname{res}_H^{U(n)}\big(s_{U(k)}^{U(n)}(e_{U(k)}(\nu_k))\big) \in \operatorname{res}_H^{U(n)} J_{U(n)}$, and hence we have proved the claim. Since, by construction, $e_H(V)$ is nonzero, we conclude that $J_{U(n)}$ is sufficiently large at H. $\square$

We now let G be any compact Lie group. Since every compact Lie group has a faithful representation, G is isomorphic to a closed subgroup of $U(n)$ where n is the dimension of a chosen faithful representation of G. Then we have the following corollary:

Corollary 5.2 *The ideal $\operatorname{res}_G^{U(n)} J_{U(n)} \subset J_G$ is a sufficiently large finitely generated ideal. Hence, Theorem 3.10 (the completion theorem) holds for any compact Lie group G if we choose $I = \operatorname{res}_G^{U(n)} J_{U(n)}$.*

Proof By transitivity of restrictions, the ideal $\mathrm{res}_G^{U(n)} J_{U(n)}$ is sufficiently large and is contained in J_G. The completion theorem then holds by the argument above. □

Remark 5.3 Proposition 5.1 and Theorem 1.1 hold more generally for all global $\boldsymbol{MU}$–modules.

Remark 5.4 The subideal $J = \mathrm{res}_G^{U(n)} J_{U(n)}$ of J_G is not special. Indeed, if I is any other finitely generated subideal of J_G containing J, then

$$\Gamma_I \boldsymbol{MU}_G \simeq \Gamma_J \boldsymbol{MU}_G \quad \text{and} \quad (\boldsymbol{MU}_G)^\wedge_I \simeq (\boldsymbol{MU}_G)^\wedge_J.$$

In fact, Theorem 3.10 implies that the $\boldsymbol{MU}_G$–modules $K_\infty(I)$ and $\Gamma_I M$ are independent of the choice of I.

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