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Closed geodesics with prescribed intersection numbers

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Let (Σ, g) be a closed oriented negatively curved surface, and fix a simple closed geodesic $\gamma_\star$. We give the asymptotic growth as $L \to +\infty$ of the number of primitive closed geodesics of length less than L intersecting $\gamma_\star$ exactly n times, where n is fixed positive integer. This is done by introducing a dynamical scattering operator associated to the surface with boundary obtained by cutting Σ along $\gamma_\star$ and by using the theory of Pollicott–Ruelle resonances for open systems.

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1 Introduction

Let (Σ, g) be a closed oriented connected negatively curved Riemannian surface, and denote by $\mathcal{P}$ the set of its oriented primitive closed geodesics. For $L > 0$ define

$$N(L) = \#\{\gamma \in \mathcal{P} : \ell(\gamma) \leqslant L\},$$

where, for $\gamma \in \mathcal{P}$, we denote by $\ell(\gamma)$ its length. Then a classical result obtained by Margulis [31] states that

$$N(L) \sim \frac{e^{hL}}{hL} \quad \text{as } L \to \infty,$$

where $h > 0$ is the topological entropy of the geodesic flow of (Σ, g).

Our purpose here is to provide a similar asymptotic result for closed geodesics satisfying certain intersection constraints. Namely, let $\gamma_\star$ be a simple closed geodesic of (Σ, g). For any $\gamma \in \mathcal{P}$, we denote by $i(\gamma, \gamma_\star)$ the geometric intersection number between γ and $\gamma_\star$ (see Section 2.1), and we set

$$N(n, L) = \#\{\gamma \in \mathcal{P} : \ell(\gamma) \leqslant L \text{ and } i(\gamma, \gamma_\star) = n\}.$$

We first state a result assuming $\gamma_\star$ is not separating, in the sense that $\Sigma \setminus \gamma_\star$ is connected.

Theorem 1 *Assume that $\gamma_\star$ is not separating. Then there are $c_\star > 0$ and $h_\star \in]0, h[$ such that, for any $n \geqslant 1$,*

$$N(n, L) \sim \frac{(c_\star L)^n}{n!} \frac{e^{h_\star L}}{h_\star L} \quad \text{as } L \to \infty. \tag{1-1}$$

The number $h_\star$ in the above statement is the topological entropy of the geodesic flow (φ_t) of (Σ, g) when restricted to the trapped set

$$K_\star = \overline{\{(x, v) \in S\Sigma : \pi(\varphi_t(x, v)) \in \Sigma \setminus \gamma_\star \text{ for } t \in \mathbb{R}\}},$$

where the closure is taken in $S\Sigma$ and $\pi: S\Sigma \to \Sigma$ is the natural projection. Also, we provide in Section 7 a description of the constant $c_\star$ in terms of the Pollicott–Ruelle resonant states of the geodesic flow of the compact surface with boundary $\Sigma_\star$ obtained by cutting Σ along $\gamma_\star$.

By using a classical large deviation result by Kifer [25] and Bonahon's intersection form [6], one is able to show that a typical closed geodesic γ satisfies $i(\gamma, \gamma_\star) \approx I_\star \ell(\gamma)$ for some $I_\star > 0$ not depending on γ (see Proposition 8.1 for a precise statement). In particular, Theorem 1 is a statement about very uncommon closed geodesics.

The asymptotics (1-1) for $n = 0$ is well known and follows from the work of Dal'bo [12] and from the growth rate of periodic orbits of axiom A flows obtained by Parry and Pollicott [35] (see Section 2.5). However, to the best of our knowledge, the result is new for $n > 0$. Note that it would be tempting to sum the right-hand side of (1-1) over n in order to recover the asymptotic growth of $N(L)$ — for example, one could hope that $h_\star + c_\star = h$ — but if L is fixed, the left-hand side of (1-1) vanishes whenever n is large enough, and it is very unlikely that such an equality holds.

If $\gamma_\star$ is separating then $i(\gamma, \gamma_\star)$ is even, and we have the following result:

Theorem 2 *Suppose that $\gamma_\star$ separates Σ in two surfaces, Σ_1 and Σ_2. Let $h_j \in \,]0, h[$ denote the entropy of the open system $(\Sigma_j, g|_{\Sigma_j})$ and set $h_\star = \max(h_1, h_2)$. Then there is $c_\star > 0$ such that, for each $n \geqslant 1$, as $L \to +\infty$,*

$$N(2n, L) \sim \begin{cases} \dfrac{(c_\star L)^n}{n!}\dfrac{e^{h_\star L}}{h_\star L} & \text{if } h_1 \neq h_2, \\[2ex] 2\dfrac{(c_\star L^2)^n}{(2n)!}\dfrac{e^{h_\star L}}{h_\star L} & \text{if} h_1 = h_2. \end{cases}$$

As before, the number h_j is defined as the topological entropy of the geodesic flow restricted to the trapped set

$$K_j = \overline{\{(x, v) \in S\Sigma : \pi(\varphi_t(x, v)) \in \Sigma_j \setminus \gamma_\star \text{ for } t \in \mathbb{R}\}},$$

where the closure is taken in $S\Sigma$.

We also have an equidistribution result, as follows. Set

$$\partial_\star = \{(x, v) \in S\Sigma : x \in \gamma_\star\} \quad \text{and} \quad \Gamma = S\gamma_\star \cup \{z \in \partial_\star : \varphi_t(z) \in S\Sigma \setminus \partial_\star \text{ for } t > 0\},$$

where $S\gamma_\star = \{(x, v) \in \partial_\star : v \in T_x\gamma_\star\}$. We define the scattering map $S: \partial_\star \setminus \Gamma \to \partial_\star$ by

$$S(z) = \varphi_{\ell(z)}(z), \quad \ell(z) = \inf\{t > 0 : \varphi_t(z) \in \partial_\star\} \quad \text{for } z \in \partial_\star \setminus \Gamma.$$

For any $n \in \mathbb{N}_{\geqslant 1}$ we set

$$\Gamma_n = \partial_\star \setminus \{z \in \partial_\star \setminus \Gamma : S^k(z) \in \partial_\star \setminus \Gamma \text{ for } k = 1, \dots, n-1\},$$

which is a closed set of Lebesgue measure zero, and

$$\ell_n(z) = \ell(z) + \cdots + \ell(S^{n-1}(z)) \quad \text{for } z \in \partial_\star \setminus \Gamma_n.$$

Theorem 3 *Assume that $\gamma_\star$ is not separating and let $n \geqslant 1$. For any $f \in C^\infty(\partial_\star)$, the limit*

$$\lim_{L\to+\infty} \frac{1}{N(n,L)} \sum_{\substack{\gamma\in\mathcal{P}\\ i(\gamma,\gamma_\star)=n}} \frac{1}{\# I_\star(\gamma)} \sum_{z\in I_\star(\gamma)} f(z)$$

exists, where, for any $\gamma \in \mathcal{P}$, the set $I_\star(\gamma) = \{(x,v) \in S\gamma : x \in \gamma_\star\}$ consists of the incidence vectors of γ along $\gamma_\star$. This formula defines a probability measure μ_n on $\partial_\star$, whose support is contained in Γ_n.

Of course, a similar statement holds even if $\gamma_\star$ is separating, though we will not explicitly state it here. As for $c_\star$, we will provide a full description of μ_n in terms of the Pollicott–Ruelle resonant states of the geodesic flow of $(\Sigma_\star, g)$ for the resonance $h_\star$ in Section 7. Here, as before, $\Sigma_\star$ is the compact surface with boundary obtained by cutting Σ along $\gamma_\star$ (see Section 2.5).

Strategy of proof

A key ingredient used in the proof of Theorems 1, 2 and 3 is the scattering operator $\mathcal{S}(s)\colon C^\infty(\partial_\star) \to C^\infty(\partial_\star \setminus \Gamma)$, which is defined by

$$\mathcal{S}(s) f(z) = f(S(z)) e^{-s\ell(z)} \quad \text{for } z \in \partial_\star \setminus \Gamma \text{ and } s \in \mathbb{C}.$$

As a first step (which is of independent interest; see the corollary on page 714), we prove that, for any $\chi \in C_c^\infty(\partial_\star \setminus S\gamma_\star)$, the family $s \mapsto \chi\mathcal{S}(s)\chi$ extends to a meromorphic family of operators $\mathcal{S}(s)\colon C^\infty(\partial_\star) \to \mathcal{D}'(\partial_\star)$ on the whole complex plane (here $\mathcal{D}'(\partial_\star)$ denotes the space of distributions on $\partial_\star$), whose poles are contained in the set of Pollicott–Ruelle resonances of the geodesic flow of the surface with boundary $(\Sigma_\star, g)$; see Section 2.6 for the definition of those resonances. In this context, the existence of such resonances follows from the work of Dyatlov and Guillarmou [15], and we relate $\mathcal{S}(s)$ with the resolvent of the geodesic flow (see Proposition 3.2). By using the microlocal structure of the resolvent of the geodesic flow provided by [15], we are moreover able to prove that the composition $(\chi\mathcal{S}(s)\chi)^n$ is well defined for any $n \geqslant 1$, as well as its superflat trace (meaning that we also look at the action of $\mathcal{S}(s)$ on differential forms, see Section 3.4), which reads

$$\operatorname{tr}_{\mathrm{s}}^\flat[(\chi\mathcal{S}(s)\chi)^n] = n \sum_{i(\gamma,\gamma_\star)=n} \frac{\ell^\#(\gamma)}{\ell(\gamma)} e^{-s\ell(\gamma)} \prod_{z\in I_\star(\gamma)} \chi^2(z), \tag{1-2}$$

where the products runs over all closed geodesics (not necessarily primitive) γ with $i(\gamma,\gamma_\star) = n$, and $\ell^\#(\gamma)$ is the primitive length of γ. This formula will be obtained by using the Atiyah–Bott trace formula [3], though our scattering map S has singularities that we have to deal with. Furthermore, using a priori bounds on the growth of $N(n,L)$ (obtained in Section 4 by purely geometric techniques coming from the theory of CAT(-1) spaces), we prove that $s \mapsto \operatorname{tr}_{\mathrm{s}}^\flat[(\chi\mathcal{S}(s)\chi)^n]$ has a pole of order n at $s = h_\star$ provided that χ has enough support. For this step, we crucially use the fact that the asymptotics for $N(0,L)$ is already known, although we could recover it by using the modern techniques introduced in [15] without going through the scattering maps. Finally, letting the support of $1-\chi$ be very close to $S\gamma_\star$, and estimating the growth of geodesics having n intersections with $\gamma_\star$ with at least one small angle, we are able to derive Theorems 1 and 2 from a classical Tauberian theorem of Delange [14].

Related works

As mentioned before, the case $n=0$ follows from work of Parry and Pollicott [35] which is based on important contributions of Bowen [9; 10], as the geodesic flow on $(\Sigma_\star, g)$ can be seen as an axiom A flow; see Lemma 2.5 below and [15, Section 6.1]. For counting results on noncompact Riemann surfaces, see also the works of Sarnak [43], Guillopé [21] or Lalley [27]. We refer to the work of Paulin, Pollicott and Schapira [37] for counting results in more general settings.

We also mention a result by Pollicott [39] which says that, if (Σ, g) is of constant curvature -1 and if $\gamma_\star$ is not separating,

$$\frac{1}{N(L)} \sum_{\substack{\gamma\in\mathcal{P} \\ \ell(\gamma)\leqslant L}} i(\gamma,\gamma_\star) \sim I_\star L \tag{1-3}$$

for some $I_\star > 0$. Roughly speaking, this means that the average intersection number between $\gamma_\star$ and closed geodesics of length not greater than L is about $I_\star L$. We will show that this result also holds in our context (see Section 8.2).

Lalley [26], Pollicott [40] and Anantharaman [1] investigated the asymptotic growth of the number of closed geodesics satisfying some homological constraints (see also Phillips and Sarnak [38] and Katsuda and Sunada [24] for the constant curvature case). They showed that, for any homology class $\xi \in H_1(\Sigma, \mathbb{Z})$,

$$\#\{\gamma\in\mathcal{P} : \ell(\gamma)\leqslant L \text{ and } [\gamma]=\xi\} \sim Ce^{hL}/L^{\mathrm{g}+1}$$

for some $C>0$ independent of ξ, where g is the genus of Σ and $h>0$ is the topological entropy of the geodesic flow of (Σ, g). Such asymptotics are obtained by studying L–functions associated to some characters of $H_1(\Sigma,\mathbb{Z})$. However, our problem is very different in nature; indeed, fixing a constraint in homology boils down to fixing *algebraic* intersection numbers, whereas here we are interested in *geometric* intersection numbers. In particular, L–functions are not well suited for this situation.

In the context of hyperbolic surfaces (ie surfaces with constant negative curvature -1), Mirzakhani [32; 33] computed the asymptotic growth of closed geodesics with prescribed self-intersection numbers. Namely, for any $k\in\mathbb{N}$,

$$\#\{\gamma\in\mathcal{P} : \ell(\gamma)\leqslant L \text{ and } i(\gamma,\gamma)=k\} \sim c_k L^{6(\mathrm{g}-1)},$$

where $i(\gamma,\gamma)$ denotes the self-intersection number of γ; see also Erlandsson and Souto [17].

Note that our scattering map S defined above shares some similarities with the Sinai billiard map [44]. Similarly to the map S, which is not defined on the singularity set Γ, the billiard map is not continuous near some singular set consisting in grazing trajectories. In particular, it is plausible that recent functional analytic techniques developed by Baladi, Demers and Liverani [5] (see also Baladi and Demers [4]), as the Sinai billiard map could be used to define an intrinsic spectrum of resonances for the transfer operator associated to S (without going through the resolvent of the geodesic flow of $S\Sigma_\star$).

We finally mention that the techniques presented herein allow one to obtain the asymptotic growth of closed geodesics for which *several* intersection numbers (with a family pairwise disjoint simple closed curves) are prescribed. However, such an extension requires more work, and for simplicity we will focus here on the case where we are given only one simple geodesic. The aforementioned generalization will be the subject of subsequent work.

Organization of the paper

The paper is organized as follows. In Section 2 we introduce some geometric and dynamical tools. In Section 3 we introduce the dynamical scattering operator, which is a central object in this paper, and we compute its flat trace. In Section 4 we prove a priori bounds on $N(n, L)$. In Section 5 we use a Tauberian argument to estimate certain quantities. In Section 6 we prove Theorems 1 and 2. In Section 7 we prove Theorem 3. Finally, in Section 8 we show that a typical closed geodesic γ satisfies $i(\gamma, \gamma_\star) \approx I_\star \ell(\gamma)$ for some $I_\star > 0$.

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2 Geometric preliminaries

We recall here some classical geometric and dynamical notions, and introduce the Pollicott–Ruelle resonances that will arise in our situation. Throughout the article, (Σ, g) will denote a closed connected oriented Riemannian surface of negative curvature.

2.1 Geometric intersection numbers

For any two loops $\alpha, \beta : \mathbb{R}/\mathbb{Z} \to \Sigma$, the *geometric intersection number* between α and β is defined by

$$i(\alpha, \beta) = \inf_{\alpha' \sim \alpha, \beta' \sim \beta} |\alpha \cap \beta|,$$

where the infimum runs over all loops α' and β' freely homotopic to α and β, respectively, and

$$|\alpha \cap \beta| = \{(\tau, \tau') \in (\mathbb{R}/\mathbb{Z})^2 : \alpha(\tau) = \beta(\tau')\}.$$

It is well known that, in every nontrivial free homotopy class of loops $\boldsymbol{c}$, there is a unique oriented closed geodesic $\gamma_{\boldsymbol{c}} \in \boldsymbol{c}$ which minimizes the length among curves in $\boldsymbol{c}$. In fact, closed geodesics also minimize intersection numbers, as follows:

Lemma 2.1 *Let γ_1 and γ_2 be any two nontrivial oriented closed geodesics, and assume that γ_1 (resp. γ_2) is not freely homotopic to a power of γ_2 (resp. γ_1). Then*

$$i(\gamma_1, \gamma_2) = |\gamma_1 \cap \gamma_2|.$$

The above result is rather classical, but for the reader's convenience we provide a proof in Appendix A.

2.2 Structural equations

Here we recall some classical facts from [45, Section 7.2] about geometry of surfaces. Denote by $M = S\Sigma = \{(x, v) \in T\Sigma : \|v\|_g = 1\}$ the unit tangent bundle of Σ, and by X the geodesic vector field on M, that is, the generator of the geodesic flow $\varphi = (\varphi_t)_{t\in\mathbb{R}}$ of (Σ, g), acting on M. The Liouville one-form α on M is defined by

$$\langle \alpha(z), \eta\rangle = \langle \mathrm{d}_{(x,v)}\pi(\eta), v\rangle \quad \text{for } z = (x, v) \in M \text{ and } \eta \in T_{(x,v)}M,$$

where $\pi : M \to \Sigma$ is the natural projection. Then α is a contact form (that is, $\alpha \wedge \mathrm{d}\alpha$ is a volume form on M) and it turns out that X is the Reeb vector field associated to α, meaning that

$$\iota_X \alpha = 1 \quad \text{and} \quad \iota_X\, \mathrm{d}\alpha = 0,$$

where ι denotes the interior product.

We also set $\beta = R^*_{\pi/2}\alpha$, where, for $\theta \in \mathbb{R}$, $R_\theta : M \to M$ is the rotation of angle θ in the fibers. Finally we denote by ψ the connection one-form, defined as the unique one-form on M satisfying

$$\iota_V \psi = 1, \quad \mathrm{d}\alpha = \psi \wedge \beta \quad \text{and} \quad \mathrm{d}\beta = -\psi \wedge \alpha,$$

where V is the vertical vector field, that is, the vector field generating $(R_\theta)_{\theta\in\mathbb{R}}$. Then (α, β, ψ) is a global frame of T^*M, and we denote by H the unique vector field on M such that (X, H, V) is the dual frame of (α, β, ψ). We then have the commutation relations

$$[V, X] = H, \quad [V, H] = -X \quad \text{and} \quad [X, H] = (\kappa \circ \pi)V,$$

where κ is the Gauss curvature of (Σ, g).

2.3 The Anosov property

It is known, by the work of Anosov [2], that the flow (φ_t) is hyperbolic. That is, for any $z \in M$ there is a $\mathrm{d}\varphi_t$–invariant splitting

$$T_zM = \mathbb{R}X(z) \oplus E_s(z) \oplus E_u(z)$$

which depends continuously on z, and has the property that, for any norm $\|\cdot\|$ on TM, there exist $C, \nu > 0$ such that

$$\|\mathrm{d}\varphi_t(z)v\| \leqslant Ce^{-\nu t}\|v\| \quad \text{for } v \in E_s(z),\ t \geqslant 0 \text{ and } z \in M,$$

and

$$\|\mathrm{d}\varphi_{-t}(z)v\| \leqslant Ce^{-\nu t}\|v\| \quad \text{for } v \in E_u(z),\ t \geqslant 0 \text{ and } z \in M.$$

In fact, $E_s(z) \oplus E_u(z) = \ker \alpha(z)$ and there exist two continuous functions $r_\pm \colon M \to \mathbb{R}$ such that $\pm r_\pm > 0$ and

$$E_s(z) = \mathbb{R}(H(z) + r_- V(z)) \quad \text{and} \quad E_u(z) = \mathbb{R}(H(z) + r_+ V(z)) \quad \text{for } z \in M.$$

Moreover, the functions $r_\pm$ are differentiable along the flow direction, and they satisfy the Riccati equation

$$\pm X r_\pm + r_\pm^2 + \kappa \circ \pi = 0,$$

where κ is the curvature of Σ.

We will denote by $T^*M = E_0^* \oplus E_s^* \oplus E_u^*$ the splitting defined by

$$E_0^*(E_u \oplus E_s) = 0, \quad E_s^*(E_s \oplus E_0) = 0, \quad E_u^*(E_u \oplus E_0) = 0.$$

(Here the bundle $\mathbb{R}X$ is denoted by E_0.) Then we have $E_0^* = \mathbb{R}\alpha$ and

$$E_s^* = \mathbb{R}(r_-\beta - \psi), \quad E_u^* = \mathbb{R}(r_+\beta - \psi). \tag{2-1}$$

Note that this decomposition does not coincide with the usual dual decomposition, but it is motivated by the fact that covectors in E_s^* (resp. E_u^*) are exponentially contracted in the future (resp. in the past) by the symplectic lift Φ_t of φ_t, which is defined by

$$\Phi_t(z, \xi) = (\varphi_t(z), \mathrm{d}\varphi_t(z)^{-\top} \cdot \xi) \quad \text{for } (z, \xi) \in T^*M \text{ and } t \in \mathbb{R}, \tag{2-2}$$

where ${}^{-\top}$ denotes the inverse transpose. We have the following lemma:

Lemma 2.2 [13, Section 3.2] *If $t \neq 0$, we have $\iota_V \Phi_t(\beta) \neq 0$ and $\iota_H \Phi_t(\psi) \neq 0$.*

2.4 A nice system of coordinates

In what follows, we write

$$\partial_\star = \{(x, v) \in M : x \in \gamma_\star\} = S\Sigma|_{\gamma_\star}.$$

Lemma 2.3 *There exists a tubular neighborhood U of $\partial_\star$ in M, and coordinates (τ, ρ, θ) on U with*

$$U \simeq (\mathbb{R}/\ell_\star\mathbb{Z})_\tau \times (-\delta, \delta)_\rho \times (\mathbb{R}/2\pi\mathbb{Z})_\theta,$$

where $\ell_\star$ is the length of $\gamma_\star$, and such that

$$|\rho(z)| = \mathrm{dist}_g(\pi(z), \gamma_\star) \quad \text{and} \quad S_z\Sigma = \{(\tau(z), \rho(z), \theta) : \theta \in \mathbb{R}/2\pi\mathbb{Z}\} \quad \text{for } z \in U.$$

Moreover, in these coordinates, on $\{\rho = 0\}$,

$$X = \cos(\theta)\partial_\tau + \sin(\theta)\partial_\rho, \quad H = -\sin(\theta)\partial_\tau + \cos(\theta)\partial_\rho, \quad V = \partial_\theta,$$

and

$$\alpha = \cos(\theta)\,\mathrm{d}\tau + \sin(\theta)\,\mathrm{d}\rho, \quad \beta = -\sin(\theta)\,\mathrm{d}\tau + \cos(\theta)\,\mathrm{d}\rho, \quad \psi = \mathrm{d}\theta.$$

Proof For $\tau \in \mathbb{R}/\ell_\star\mathbb{Z}$ we set $(x_\tau, v_\tau) = \varphi_\tau(\gamma_\star(0), \dot\gamma_\star(0))$. We now define, for $\delta > 0$ small enough,

$$\Psi(\tau,\rho,\theta) = R_{\theta-\pi/2}\varphi_\rho(x_\tau, \nu(x_\tau)) \quad \text{for } (\tau,\rho,\theta) \in \mathbb{R}/\ell_\star\mathbb{Z} \times (-\delta,\delta) \times \mathbb{R}/2\pi\mathbb{Z},$$

where $R_\eta : S\Sigma \to S\Sigma$ is the rotation of angle η and $\nu(x_\tau) = R_{\pi/2} v_\tau$. Then $\mathrm{d}\Psi(\tau,0,\theta)$ is injective for any τ and θ. Indeed, $\partial_\tau(\pi\circ\Psi)(\tau,0,\theta) = v_\tau$ and $\partial_\rho(\pi\circ\Psi)(\tau,0,\theta) = \nu(x_\tau)$. Thus $\mathrm{d}\Psi(\tau,0,\theta) : \mathbb{R}\partial_\tau \oplus \mathbb{R}\partial_\rho \to T\Sigma$ is injective. Moreover, $\partial_\theta(\pi\circ\Psi)(\tau,0,\theta) = 0$ and $\partial_\theta\Psi(\tau,0,\theta) = V(\Psi(\tau,0,\theta)) \neq 0$. Thus $\mathrm{d}\Psi(\tau,0,\theta)$ is injective for any τ and θ, and furthermore, if $\delta > 0$ is small enough, $\Psi : U \to M$ is an immersion. In particular, since $(\tau,\theta) \mapsto \Psi(\tau,0,\theta)$ is clearly injective, we obtain that $\Psi|_U$ is a diffeomorphism onto its image provided that δ is chosen small enough.

Because $V = \partial_\theta$ and $\iota_V\alpha = \iota_V\beta = 0$, we may write $\alpha(\tau,0,\theta) = a(\tau,\theta)\,\mathrm{d}\tau + b(\tau,\theta)\,\mathrm{d}\rho$ and $\beta(\tau,0,\theta) = a'(\tau,\theta)\,\mathrm{d}\tau + b'(\tau,\theta)\,\mathrm{d}\rho$ for some smooth functions a, a', b and b'. Now, since $\mathrm{d}\alpha = \psi\wedge\beta$, we obtain $\mathcal{L}_V\alpha = \iota_V\,\mathrm{d}\alpha = \beta$, and similarly $\mathcal{L}_V\beta = -\alpha$. Thus, $a' = \partial_\theta a$, $b' = \partial_\theta b$ and

$$\partial_\theta^2 a + a = 0, \quad \partial_\theta^2 b + b = 0.$$

In consequence, $a(\tau,\theta) = a_1(\tau)\cos\theta + a_2(\tau)\sin\theta$ and $b(\tau,\theta) = b_1(\tau)\cos\theta + b_2(\tau)\sin\theta$ for some smooth functions a_1, a_2, b_1 and b_2. Moreover, by definition of the coordinates (τ,ρ,θ), one has

$$X(\tau,0,0) = \partial_\tau \quad \text{and} \quad X\left(\tau,0,\tfrac{1}{2}\pi\right) = \partial_\rho. \tag{2-3}$$

Therefore $a_1 = b_2 = 1$ and $a_2 = b_1 = 0$. We thus get the desired formulae for α and β. Now, writing $\psi = a''\,\mathrm{d}\tau + b''\,\mathrm{d}\rho + \mathrm{d}\theta$ and using $\mathcal{L}_V\psi = 0$, we obtain $\partial_\theta a'' = \partial_\theta b'' = 0$. As $\iota_X\psi = 0$ we obtain $a'' = b'' = 0$ by (2-3). The formulae for X, H and V follow. □

Remark 2.4 If $\tilde\partial = \{\rho = 0\}$, then, for any $z = (\tau,0,\theta) \in \partial$,

$$T_z\tilde\partial = \mathbb{R}V(z) \oplus \mathbb{R}(\cos(\theta)X(z) - \sin(\theta)H(z)) \quad \text{and} \quad N_z^*\tilde\partial = \mathbb{R}(\sin(\theta)\alpha(z) + \cos(\theta)\beta(z)).$$

2.5 Cutting the surface along $\gamma_\star$

As mentioned in the introduction, we may see $\Sigma \setminus \gamma_\star$ as the interior of a compact surface $\Sigma_\star$ with boundary consisting of two copies of $\gamma_\star$. By gluing two copies of the annulus U obtained in the preceding subsection on each component of the boundary of $\Sigma_\star$, we construct a slightly larger surface $\Sigma_\delta \supset \Sigma_\star$ whose boundary is identified with the boundary of U (see Figure 1).

Lemma 2.5 *The surface Σ_δ has strictly convex boundary, in the sense that the second fundamental form of the boundary $\partial\Sigma_\delta$ with respect to its outward normal pointing vector is strictly negative.*

Proof In the coordinates (τ,ρ) given by Lemma 2.3, the metric g has the form

$$\mathrm{d}\rho^2 + f(\tau,\rho)\,\mathrm{d}\tau^2 \tag{2-4}$$

for some $f > 0$ satisfying $\partial_\rho f(\tau,0) = 0$. Indeed, if ∇ is the Levi-Civita connection, one has

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\langle\partial_\rho,\partial_\tau\rangle = \langle\nabla_{\partial_\rho}\partial_\rho,\partial_\tau\rangle + \langle\partial_\rho,\nabla_{\partial_\rho}\partial_\tau\rangle = \langle\partial_\rho,\nabla_{\partial_\tau}\partial_\rho\rangle = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\langle\partial_\rho,\partial_\rho\rangle = 0,$$

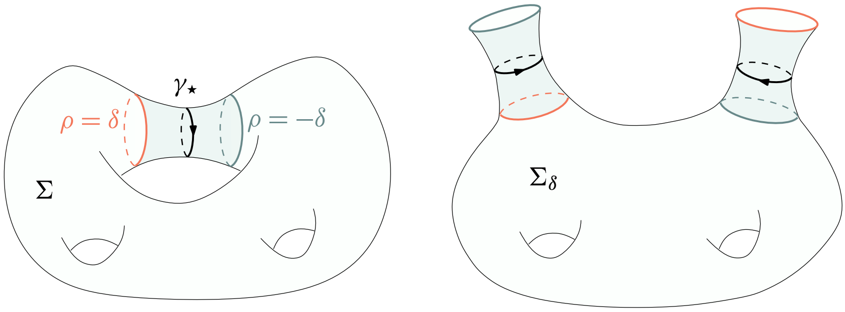

Figure 1: The surfaces Σ (on the left) and Σ_δ (on the right) in the case where $\gamma_\star$ is not separating. In Σ, the darker region corresponds to the neighborhood $\pi(U)$ of $\gamma_\star$.

since $\nabla_{\partial_\rho}\partial_\rho = 0$ (indeed, $\rho \mapsto (\tau,\rho)$ is a geodesic curve). Thus $\langle \partial_\tau, \partial_\rho\rangle = \langle \partial_\tau, \partial_\rho\rangle|_{\rho=0} = 0$. In particular, g has the form (2-4) with $f(\tau,\rho) = \langle \partial_\tau, \partial_\tau\rangle$, and we have $\partial_\rho f(\tau,0) = \partial_\rho\langle \partial_\tau, \partial_\tau\rangle = 2\partial_\tau\langle \partial_\rho, \partial_\tau\rangle|_{\rho=0} = 0$ (indeed, since $\tau \mapsto (\tau, 0)$ is a geodesic curve, $\nabla_{\partial_\tau}\partial_\tau = 0$ on $\{\rho = 0\}$). In those coordinates, the scalar curvature reads

$$\kappa(\tau,\rho) = \frac{-\partial_\rho^2 f(\tau,\rho)}{f(\tau,\rho)}.$$

As $\kappa < 0$, we get $\partial_\rho^2 f > 0$, which gives $\pm\partial_\rho f > 0$ on $\{\pm\rho > 0\}$. The second fundamental form of $\partial\Sigma_\delta$ with respect to ∂_ρ is defined by

$$\langle \nabla_{\partial_\tau}\partial_\tau, \partial_\rho\rangle = -\tfrac{1}{2}\partial_\rho f(\tau,\rho),$$

which concludes the proof, since ∂_ρ is outward pointing (resp. inward pointing) on $\{\rho = \delta\}$ (resp. $\{\rho = -\delta\}$). ◻

Lemma 2.6 *In the coordinates given by Lemma 2.3,*

$$\pm X^2\rho > 0 \quad \text{on } \{\pm\rho > 0\}.$$

Proof Since, in the coordinates (τ,ρ), the metric g has the form (2-4), the Christoffel symbols of g are given by

$$\Gamma^\rho_{\rho\rho} = \Gamma^\rho_{\tau\rho} = 0 \quad \text{and} \quad \Gamma^\rho_{\tau\tau} = -\tfrac{1}{2}\partial_\rho f.$$

In particular, if $t \mapsto (\tau(t), \rho(t))$ is a geodesic path,

$$\ddot\rho(t) - \tfrac{1}{2}\partial_\rho f(\tau(t),\rho(t)) = 0.$$

Because $\partial_\rho f(\tau,0) = 0$ and $-\partial_\rho^2 f/f = \kappa < 0$, we obtain that $\pm\partial_\rho f > 0$ whenever $\pm\rho > 0$. ◻

2.6 The resolvent of the geodesic flow for open systems

In what follows, we denote by $\Omega^\bullet(M_\delta)$ the set of differential forms on M_δ and by $\Omega^\bullet_c(M_\delta)$ the elements of $\Omega^\bullet(M_\delta)$ whose support is contained in the interior of M_δ. Here $M_\delta = S\Sigma_\delta$ is the unit tangent bundle of Σ_δ. The set of currents on M_δ, denoted by $\mathcal{D}'^\bullet(M_\delta)$, is defined as the topological dual of $\Omega_c^\bullet(M_\delta)$. Note that we have an inclusion $\Omega^\bullet(M_\delta) \hookrightarrow \mathcal{D}'^\bullet(M_\delta)$ via the pairing

$$\langle u, v\rangle = \int_{M_\delta} u \wedge v \quad \text{for } u, v \in \Omega^\bullet(M_\delta).$$

The geodesic flow φ on M induces a flow on $M_\delta = S\Sigma_\delta$, which we still denote by φ. We set

$$\partial_\pm M_\delta = \{(x,v) \in \partial M_\delta : \pm\langle v, \nu_\delta(x)\rangle > 0\} \quad \text{and} \quad \partial_0 M_\delta = \{(x,v) \in \partial M_\delta : \pm\langle v, \nu_\delta(x)\rangle = 0\},$$

where $\nu_\delta(x)$ is the unit vector orthogonal to $\partial\Sigma_\delta$, based at x, and pointing outward. Next, define

$$\ell_{\pm,\delta}(z) = \inf\{t > 0 : \varphi_{\pm t}(z) \in \partial M_\delta\} \quad \text{for } z \in \operatorname{int}(M_\delta) \cup \partial_\mp M_\delta,$$

and $\ell_{\pm,\delta}(z) = 0$ for $z \in \partial_\pm M_\delta \cup \partial_0 M_\delta$, where $\operatorname{int}(M_\delta)$ denotes the interior of M_δ. The numbers $\ell_{\pm,\delta}(z)$ are the first exit times of z in the future and in the past. We also set

$$\Gamma_{\pm,\delta} = \{z \in M_\delta : \ell_\mp(z) = +\infty\} \quad \text{and} \quad K_\delta = \Gamma_\delta^+ \cap \Gamma_\delta^-,$$

and we define the operators $R_{\pm,\delta}(s)$ by

$$R_{\pm,\delta}(s)\omega(z) = \pm\int_0^{\ell_{\mp,\delta}(z)} \varphi_{\mp t}^*\omega(z)e^{-ts}\,\mathrm{d}t \quad \text{for } z \in M_\delta \text{ and } \omega \in \Omega_c^\bullet(M_\delta), \tag{2-5}$$

which are well defined as operators from $\Omega_c^\bullet(M_\delta)$ to $C\big(M_\delta, \bigwedge^\bullet T^*M_\delta\big)$ whenever $\operatorname{Re}(s) \gg 1$, where $C\big(M_\delta, \bigwedge^\bullet T^*M_\delta\big)$ denotes the space of continuous differential forms on M_δ. Note that our convention of $R_{\pm,\delta}(s)$ differs from that of [18]. The operator $R_{+,\delta}(s)$ (resp. $R_{-,\delta}(s)$) is the resolvent of $\mathcal{L}_X$ in the future (resp. in the past) for the spectral parameter s. More precisely,

$$(\mathcal{L}_X \pm s)R_{\pm,\delta}(s) = \operatorname{Id}_{\Omega_c^\bullet(M_\delta)}, \tag{2-6}$$

and for any $(u, v) \in \Omega_c^\bullet(M_\delta \setminus \Gamma_{-,\delta}) \times \Omega_c^\bullet(M_\delta \setminus \Gamma_{+,\delta})$,

$$\int_{M_\delta} (R_{+,\delta}(s)u) \wedge v = -\int_{M_\delta} u \wedge R_{-,\delta}(s)v. \tag{2-7}$$

Indeed, for such u and v, there is $L > 0$ such that

$$\operatorname{supp}(u) \subset \{\ell_{+,\delta} \leqslant L\} \quad \text{and} \quad \operatorname{supp}(v) \subset \{\ell_{-,\delta} \leqslant L\}. \tag{2-8}$$

In particular, the forms $R_{+,\delta}(s)u$ and $R_{-,\delta}(s)v$ are smooth up to the boundary of M_δ. Indeed, (2-8) implies that, for any $z \in M_\delta$ and $t \in [0, \ell_{-,\delta}(z)]$,

$$\varphi_{-t}^*u(z) \neq 0 \implies t \leqslant L.$$

Therefore, for any $z \in M_\delta$,

$$R_{+,\delta}(s)u(z) = \int_0^{\ell_{-,\delta}(z)} \varphi_{-t}^*u(z)e^{-ts}\,\mathrm{d}t = \int_0^{\min(\ell_{-,\delta}(z), L+1)} \varphi_{-t}^*u(z)e^{-ts}\,\mathrm{d}t,$$

and thus $R_{+,\delta}u$ is smooth, since $\varphi_{-t}^*u(z) = 0$ if $L \leqslant t \leqslant \ell_{-,\delta}(z)$. Similarly, $R_{-,\delta}(s)v$ is smooth. Finally, note that $\operatorname{supp}(R_{+,\delta}(s)u) \cap \partial M_\delta \subset \partial_+ M_\delta$ and $\operatorname{supp}(R_{-,\delta}(s)v) \cap \partial M_\delta \subset \partial_- M_\delta$. In particular, Stokes' formula and (2-6) imply (2-7).

Because the boundary of Σ_δ is strictly convex, it follows from [15, Proposition 6.1] that the family of operators $R_\pm(s)$ extends to a meromorphic family of operators

$$R_{\pm,\delta}(s) \colon \Omega_c^\bullet(M_\delta) \to \mathcal{D}'^\bullet(M_\delta)$$

satisfying

$$\mathrm{WF}'(R_{\pm,\delta}(s)) \subset \Delta(T^*M_\delta) \cup \Upsilon_{\pm,\delta} \cup (E^*_{\pm,\delta} \times E^*_{\mp,\delta}), \tag{2-9}$$

where $\Delta(T^*M_\delta)$ is the diagonal in $T^*M_\delta \times T^*M_\delta$,

$$\Upsilon_{\pm,\delta} = \{(\Phi_t(z,\xi),(z,\xi)) \in T^*(M_\delta \times M_\delta) : 0 \leqslant \pm t \leqslant \ell_{\pm,\delta}(z) \text{ and } \langle X(z),\xi\rangle = 0\},$$

and where

$$E^*_{+,\delta} = E^*_u|_{\Gamma^+_\delta}, \quad E^*_{-,\delta} = E^*_s|_{\Gamma^-_\delta}.$$

Here, we write

$$\mathrm{WF}'(R_{\pm,\delta}(s)) = \{(z,\xi,z',\xi') \in T^*(M_\delta \times M_\delta) : (z,\xi,z',-\xi') \in \mathrm{WF}(R_{\pm,\delta}(s))\},$$

where WF is the classical Hörmander wavefront set [23, Section 8]. In fact, by (2-9) we mean that $s \mapsto R_\pm(s)$ is meromorphic as a map $\mathbb{C} \to \mathcal{D}'_{\mathbf{\Gamma}'_\pm}(M_\delta \times M_\delta)$ — we identify $R_\pm(s)$ and its Schwartz kernel — where $\mathbf{\Gamma}_\pm$ is given by the right-hand side of (2-9), $\mathbf{\Gamma}'_\pm = \{(z,\xi,z',-\xi') : (z,\xi,z',-\xi') \in \mathbf{\Gamma}_\pm\}$, and where

$$\mathcal{D}'_{\mathbf{\Gamma}'_\pm}(M_\delta \times M_\delta) = \{R \in \mathcal{D}'(M_\delta \times M_\delta) : \mathrm{WF}(R) \subset \mathbf{\Gamma}'_\pm\}$$

is endowed with its natural topology; see [23, Definition 8.2.2].

Near any $s_0 \in \mathbb{C}$, we have the expansion

$$R_{\pm,\delta}(s) = Y_{\pm,\delta}(s) + \sum_{j=1}^{J(s_0)} \frac{(X \pm s_0)^{j-1}\Pi_{\pm,\delta}(s_0)}{(s-s_0)^j},$$

where $Y_{\pm,\delta}(s)$ is holomorphic near $s = s_0$ and $\Pi_{\pm,\delta}(s_0)$ is a finite-rank projector satisfying

$$\mathrm{WF}'(\Pi_{\pm,\delta}(s_0)) \subset E^*_{\pm,\delta} \times E^*_{\mp,\delta} \quad \text{and} \quad \mathrm{supp}(\Pi_{\pm,\delta}(s_0)) \subset \Gamma^\pm_\delta \times \Gamma^\mp_\delta,$$

where we identified $\Pi_{\pm,\delta}(s_0)$ and its Schwartz kernel.

2.7 Restriction of the resolvent on the geodesic boundary

For any $\varepsilon > 0$, define the open sets

$$A_{\pm,\varepsilon} = \{\ell_{\pm,\delta} > \varepsilon\} \cap \{\ell_{\mp,\delta} > 0\} \subset \mathrm{int}(M_\delta),$$

and notice that, if ε is small, $M_{\delta/2} \subset A_{\pm,\varepsilon}$. Then we have diffeomorphisms $\varphi_{\pm\varepsilon} \colon A_{\pm,\varepsilon} \to A_{\mp,\varepsilon}$, which induce maps

$$\varphi^*_{\pm\varepsilon} \colon \mathcal{D}'^\bullet(A_{\mp,\varepsilon}) \to \mathcal{D}'^\bullet(A_{\pm,\varepsilon}).$$

Using a slight abuse of notation, we will still denote by $\varphi_{\pm\varepsilon}^* : \mathcal{D}'^{\bullet}(M_\delta) \to \mathcal{D}'^{\bullet}(A_{\pm,\varepsilon})$ the composition of $\varphi_{\pm\varepsilon}^*$ with the inclusion $\mathcal{D}'^{\bullet}(M_\delta) \hookrightarrow \mathcal{D}'^{\bullet}(A_{\mp,\varepsilon})$, which is given by the restriction. Let

$$\partial = \partial(S\Sigma_\star) = \{(x,v) \in M_\delta : x \in \gamma_\star \sqcup \gamma_\star\}$$

and $\partial_0 = S\gamma_\star \sqcup S\gamma_\star \subset \partial$.

Lemma 2.7 *For any $\varepsilon > 0$ small enough, we have*

$$\mathrm{WF}(\varphi_{\mp\varepsilon}^* R_{\pm,\delta}(s)) \cap N^*(\partial \times \partial) = \varnothing,$$

where

$$N^*(\partial \times \partial) = \{(z',\xi',z,\xi) \in T^*(M_\delta \times M_\delta) : \langle \xi', T_{z'}\partial \rangle = \langle \xi, T_z \partial \rangle = 0\}.$$

Proof We prove the statement for $R_{+,\delta}(s)$. By (2-9) and multiplicativity of wavefront sets (see [23, Theorem 8.2.14]),

$$\mathrm{WF}'(\varphi_{-\varepsilon}^* R_{+,\delta}(s)) \subset \Delta_\varepsilon \cup \Upsilon_{+,\delta}^\varepsilon \cup (E_{+,\delta}^* \times E_{-,\delta}^*), \tag{2-10}$$

where

$$\Delta_\varepsilon = \{(\Phi_\varepsilon(z,\xi),(z,\xi)) : (z,\xi) \in T^*M_\delta\}$$

and

$$\Upsilon_{+,\delta}^\varepsilon = \{(\Phi_t(z,\xi),(z,\xi)) : \varepsilon \leqslant t \leqslant \ell_{+,\delta}(z), \langle X(z), \xi \rangle = 0\}.$$

Now assume that there is $\Xi = (z',\xi',z,\xi)$ lying in

$$N^*(\partial \times \partial) \cap (\Delta_\varepsilon \cup \Upsilon_{+,\delta}^\varepsilon \cup (E_{+,\delta}^* \times E_{-,\delta}^*)).$$

If $\Xi \in \Delta_\varepsilon$, then necessarily $z, z' \in \partial_0$, because $\varphi_\varepsilon(\partial \setminus \partial_0) \cap \partial = \varnothing$ whenever $\varepsilon > 0$ is smaller than the injectivity radius of the manifold.[1] We thus have $\xi \in N_z^*\partial = \mathbb{R}\beta(z)$ by Remark 2.4; now $\Phi_\varepsilon(\beta(z))$ does not lie in $\mathbb{R}\beta(\varphi_\varepsilon(z))$ by Lemma 2.2, and therefore $\xi = 0$.

If $\Xi \in \Upsilon_{+,\delta}^\varepsilon$, then there is $T \geqslant \varepsilon$ such that $\Phi_T(z,\xi) = (z',\xi')$ with $\langle \xi, X(z) \rangle = 0$. However, by Remark 2.4, if $(z,\xi) \in N_z^*\partial$ and $\langle \xi, X(z) \rangle = 0$, then $z \in \partial_0$. Thus by what precedes, $\xi = 0$.

Finally, (2-1) and Remark 2.4 imply that $N^*\partial \cap E_{\pm,\delta}^* \subset \{0\}$. Thus we have shown that

$$\mathrm{WF}'(\varphi_{-\varepsilon}^* R_{+,\delta}(s)) \cap N^*(\partial \times \partial) = \varnothing,$$

which is equivalent to the conclusion of the lemma.[2] □

Remark 2.8 This estimate together with [23, Theorem 8.2.4] imply that the operator $\iota^* \iota_X \varphi_{\mp\varepsilon}^* R_{+,\delta}(s) \iota_*$ is well defined and satisfies

$$\mathrm{WF}(\iota^* \iota_X \varphi_{\mp\varepsilon}^* R_{+,\delta}(s) \iota_*) \subset \mathrm{d}(\iota \times \iota)^\top \, \mathrm{WF}(\varphi_{\mp\varepsilon}^* R_{+,\delta}(s)),$$

[1] Let $x \in \partial\Sigma$. If $(x,v) \in \partial \setminus \partial_0$ satisfies that $(y,w) = \varphi_\varepsilon(x,v) \in \partial$, then the exponential map at x is not injective on the closed ball $B(0,\varepsilon) \subset T_x\Sigma$ of radius ε, since $\pi(\varphi_{\varepsilon'}(x,v')) = y$ for some $v' \in S_x\Sigma$ tangent to $\partial\Sigma$ and some $\varepsilon' \in [0,\varepsilon]$. This follows from the fact that $\partial\Sigma$ is totally geodesic.

[2] Since the set $\{(z,\xi,z',\xi') : (z,\xi,z',-\xi') \in N^*(\partial \times \partial)\}$ coincides with $N^*(\partial \times \partial)$, we may use WF or WF′ interchangeably.

where $\iota\colon \partial \hookrightarrow M_\delta$ and $\iota\times\iota\colon \partial\times\partial \hookrightarrow M_\delta\times M_\delta$ are the inclusions. Indeed, the Schwartz kernel of $\iota^*\iota_X\varphi^*_{\mp\varepsilon}R_{+,\delta}(s)\iota_*$ coincides with the pullback by $\iota\times\iota$ of the kernel of $\iota_X\varphi^*_{\mp\varepsilon}R_{+,\delta}(s)$. It also follows from [23, Theorem 8.2.14] that the operator $\iota^*\iota_X\varphi^*_{\mp\varepsilon}R_{+,\delta}(s)$ maps

$$\mathcal{D}'^{\bullet}_{N^*\partial}(M_\delta) \to \mathcal{D}'^{\bullet}(\partial)$$

continuously.

Here the pushforward $\iota_*\colon \Omega^\bullet(\partial) \to \mathcal{D}'^{\bullet+1}(M_\delta)$ is defined as follows. If $u\in\Omega^k(\partial)$, we define the current $\iota_* u \in \mathcal{D}'^{k+1}(M_\delta)$ by

$$\langle \iota_* u, v\rangle = \int_\partial u\wedge \iota^* v, \qquad v\in\Omega^{n-k-1}(M_\delta).$$

3 The scattering operator

In this section we introduce the dynamical scattering operator $\mathcal{S}_\pm(s)$ associated to our problem. By relating the scattering operator to the resolvent described above, we are able to compute its wavefront set. In consequence, the composition $(\chi\mathcal{S}_\pm(s))^n$ is well defined for $\chi\in C^\infty_c(\partial\setminus\partial_0)$, and we give a formula for its flat trace.

For each $x\in\partial\Sigma_\star$, let $\nu(x)$ be the normal outward pointing vector to the boundary of $\Sigma_\star$, and set

$$\partial_\pm = \{(x,v)\in\partial : \pm\langle \nu(x), v\rangle_g > 0\}.$$

3.1 First definitions

We define the exit times in the future and in the past by

$$\ell_\pm(z) = \inf\{t>0 : \varphi_{\pm t}(z)\in\partial\} \quad \text{for } z\in M\setminus(\partial_\pm\cup\partial_0),$$

and we declare that $\ell_\pm(z)=\infty$ whenever $z\in\partial_\pm\cup\partial_0$. Then we set

$$\Gamma_\pm = \{z\in M : \ell_\mp(z) = +\infty\}.$$

The set Γ_+ (resp. Γ_-) is the set of points of M which are trapped in the past (resp. in the future). The scattering map $S_\pm\colon \partial_\mp\setminus\Gamma_\mp \to \partial_\pm\setminus\Gamma_\pm$ is defined by

$$S_\pm(z) = \varphi_{\pm\ell_\pm(z)}(z) \quad \text{for } z\in\partial_\mp\setminus\Gamma_\mp,$$

and satisfies $S_\pm\circ S_\mp = \mathrm{Id}_{\partial_\pm\setminus\Gamma_\pm}$. For $s\in\mathbb{C}$, the scattering operator

$$\mathcal{S}_\pm(s)\colon \Omega^\bullet_c(\partial_\mp\setminus\Gamma_\mp) \to \Omega^\bullet_c(\partial_\pm\setminus\Gamma_\pm)$$

is given by

$$\mathcal{S}_\pm(s)\omega = (S^*_\mp\omega)e^{-s\ell_\mp(\cdot)} \quad \text{for } \omega\in\Omega^\bullet_c(\partial_\mp\setminus\Gamma_\mp).$$

Remark 3.1 If $\mathrm{Re}(s)$ is large enough, $\mathcal{S}_\pm(s)$ extends as a map

$$C^0\big(\partial, \textstyle\bigwedge^\bullet T^*\partial\big) \to C^0\big(\partial, \textstyle\bigwedge^\bullet T^*\partial\big),$$

where $C^0\big(\partial, \bigwedge^\bullet T^*\partial\big)$ is the space of continuous forms on ∂, by declaring that

$$\mathcal{S}_\pm(s)\omega(z) = S_\mp^*\omega(z)e^{-s\ell_\mp(z)} \quad \text{if } z \in \partial_\pm \setminus \Gamma_\pm$$

and $\mathcal{S}_\pm(s)\omega(z) = 0$ otherwise. Indeed, by Lemma 3.8 and (3-16), there is $C > 0$ such that

$$\|S_\mp^*\omega(z)\| \leqslant Ce^{C\ell_\mp(z)}\|\omega\|_\infty \quad \text{for } z \in \partial_\pm \setminus \Gamma_\pm \text{ and } \omega \in \Omega^\bullet(M),$$

where $\|\omega\|_\infty$ is the uniform norm on $C^0\big(M, \bigwedge^\bullet T^*M\big)$.

3.2 The scattering operator via the resolvent

In this section we will see that $\mathcal{S}_\pm(s)$ can be computed in terms of the resolvent. More precisely, we have the following result:

Proposition 3.2 *For any* $\mathrm{Re}(s)$ *large enough*,

$$\mathcal{S}_\pm(s) = (-1)^N e^{\pm\varepsilon s}\iota^*\iota_X\varphi_{\mp\varepsilon}^* R_{\pm,\delta}(s)\iota_*$$

as maps $\Omega_c^\bullet(\partial \setminus \partial_0) \to \mathcal{D}'^\bullet(\partial)$, *where* $N : \Omega^\bullet(\partial) \to \mathbb{N}$ *is the degree operator. That is*, $N(w) = k$ *if* w *is a* k*–form.*

As a consequence of this proposition, Remark 2.8 and the continuity of the pullback [23, Theorem 8.2.4],

$$(\iota \times \iota)^* : \mathcal{D}'^\bullet_{\mathbf{\Gamma}_{\pm,\varepsilon}}(M_\delta \times M_\delta) \to \mathcal{D}'^\bullet(\partial \times \partial),$$

where $\mathbf{\Gamma}_{\pm,\varepsilon}$ is the right-hand side of (2-10), we get:

Corollary *The scattering operator* $s \mapsto \mathcal{S}_\pm(s) : \Omega^\bullet(\partial \setminus \partial_0) \to \mathcal{D}'^\bullet(\partial)$ *extends as a meromorphic family of* $s \in \mathbb{C}$ *with poles of finite rank*, *with poles contained in the set of Pollicott–Ruelle resonances of* $\mathcal{L}_X$, *that is*, *the set of poles of* $s \mapsto R_{\pm,\delta}(s)$.

Before proving Proposition 3.2, we start with an intermediate result:

Lemma 3.3 *We have* $\mathcal{S}_\pm(s) = (-1)^N e^{\pm\varepsilon s}\iota^*\iota_X\varphi_{\mp\varepsilon}^* R_{\pm,\delta}(s)\iota_*$ *as maps*

$$\Omega_c^\bullet(\partial_\mp \setminus \Gamma_\mp) \to \mathcal{D}'^\bullet(\partial_\pm \setminus \Gamma_\pm).$$

Remark 3.4 (i) Proposition 3.2 is not a direct consequence of Lemma 3.3. Indeed, the operator $\mathcal{Q}_{\varepsilon,\pm}(s) = (-1)^N e^{\pm\varepsilon s}\iota^*\iota_X\varphi_{\mp\varepsilon}^* R_{\pm,\delta}(s)\iota_*$ could hide some singularities near $\Gamma_\pm$; Proposition 3.2 tells us that this is not the case, at least far from ∂_0.

(ii) A consequence of Proposition 3.2 is that $\mathcal{Q}_{\varepsilon,\pm}(s)$ is identically zero on $\partial_\pm$ (in the sense that $\mathcal{Q}_{\varepsilon,\pm}(s)u = 0$ whenever $\mathrm{supp}(u) \subset \partial_\pm$), as is the case for $\mathcal{S}_\pm(s)$. This can be seen directly from using the fact that

$$\mathrm{supp}(\varphi_{\mp\varepsilon}^* R_{\pm,\delta}(s)\iota_* u) \subset \{\varphi_t(z) : z \in \mathrm{supp}(u) \text{ and } \varepsilon \leqslant \pm t \leqslant \ell_{\pm,\delta}(z)\}.$$

Proof Let $u \in \Omega_c^\bullet(\partial_- \setminus \Gamma_-)$, and $U' \subset \partial_-$ be a neighborhood of $\operatorname{supp} u$ such that $\overline{U}'$ does not intersect ∂_0. Let $\varepsilon > 0$ be small enough that

$$z \in \partial_- \implies \ell_+(z) > \varepsilon.$$

The existence of such an ε follows from the fact that, for each $x \in \partial\Sigma$, the exponential map $\exp_x : T_x\Sigma \to \Sigma$ is injective on $B(0,\varepsilon) \subset T_x\Sigma$ whenever $\varepsilon > 0$ is small enough (independent of x). Note also that, for every $z \in \partial_-$,

$$\pi(\varphi_t(z)) \in \Sigma_\delta \setminus \Sigma_\star \quad \text{for } -\ell_{-,\delta}(z) < t < 0,$$

by Lemma 2.6. Next, let us set

$$U = \{(t,z) \in \mathbb{R} \times U' : -\ell_{-,\delta}(z) < t < \varepsilon\}.$$

Then U is diffeomorphic to a tubular neighborhood of U' in M_δ via $(t,z) \mapsto \varphi_t(z)$.[3] Let $\chi \in C^\infty(\mathbb{R})$ be such that $\chi \equiv 1$ near $]-\infty, 0]$ and $\chi \equiv 0$ on $]\frac{1}{2}\varepsilon, +\infty[$. Set, in the above coordinates,

$$\psi(t,z) = \chi(t)e^{-ts}u(z) \in \textstyle\bigwedge^\bullet T^*_{(t,z)}M_\delta,$$

where we see $u(z)$ as a form in $T^*_{(t,z)}M$ by declaring $\iota_{\partial_t}u(z) = 0$. We extend ψ by 0 on M, and we set

$$\phi = \psi - R_{+,\delta}(s)(\mathcal{L}_X + s)\psi.$$

Then ϕ is smooth by (2-5), since $\operatorname{supp}\psi \cap \Gamma_- = \varnothing$. Moreover $(\mathcal{L}_X + s)\phi = 0$, and we have

$$\phi|_{\partial_-} = u \quad \text{and} \quad \phi|_{\partial_+} = \mathcal{S}_+(s)u,$$

where $\mathcal{S}_+(s) = \mathcal{S}_+(s)|_{\Omega_c^\bullet(\partial_- \setminus \Gamma_-)}$. Let $h \in \Omega_c^\bullet(M_\delta \setminus \Gamma_{+,\delta})$, so that $R_{-,\delta}(s)h$ is smooth (see the discussion following (2-7)). We have, by (2-6) and (2-7),

$$\begin{aligned}
\int_{M_\delta} \phi \wedge h &= \int_{M_\delta} \psi \wedge h - \int_{M_\delta} R_{+,\delta}(s)(\mathcal{L}_X + s)\psi \wedge h = \int_{M_\delta} \psi \wedge h + \int_{M_\delta} (\mathcal{L}_X + s)\psi \wedge R_{-,\delta}(s)h \\
&= \int_{M_\delta} \psi \wedge h - \int_{M_\delta} \psi \wedge (\mathcal{L}_X - s)R_{-,\delta}(s)h + \int_{\partial M_\delta} \iota_X(\psi \wedge R_{-,\delta}(s)h) \\
&= \int_{\partial M_\delta} \iota_X(\psi \wedge R_{-,\delta}(s)h) = (-1)^{\deg\psi} \int_{\partial_{-,\delta}} \psi \wedge \iota_X R_{-,\delta}(s)h,
\end{aligned}$$

since $\iota_X\psi = 0$ and ψ has no support near $\partial_{+,\delta}$. Now we let $\Phi\colon \partial_- \to \partial_{-,\delta}$ be defined by $\Phi(z) = \varphi_{-\ell_{-,\delta}(z)}(z)$. Assume that the support of h does not intersect U. Then a change of variable gives

$$\Phi^*(\iota_X R_{-,\delta}(s)h)|_{\partial_{-,\delta}} = \iota_X R_{-,\delta}(s)he^{-s\ell_{-,\delta}(\cdot)}.$$

As we have $\Phi^*(\psi|_{\partial_{-,\delta}}) = (\psi|_{\partial_-})e^{+s\ell_{-,\delta}(\cdot)} = ue^{+s\ell_{-,\delta}(\cdot)}$ by definition of ψ, we obtain

$$\int_{M_\delta} \phi \wedge h = (-1)^{\deg u} \int_{\partial_-} u \wedge \iota^*(\iota_X R_{-,\delta}(s)h). \tag{3-1}$$

Now because $(\mathcal{L}_X - s)R_{-,\delta}(s)h = h$, we get $(\mathcal{L}_X - s)R_{-,\delta}(s)h = 0$ near U, and thus $\varphi_\varepsilon^* R_{-,\delta}(s)h = e^{\varepsilon s}R_{-,\delta}(s)h$ near U. Let $v \in \Omega_c^\bullet(\partial_+ \setminus \Gamma_+)$. Then $\overline{U} \cap \operatorname{supp}(v) = \varnothing$ (because $\operatorname{supp}(v) \subset \partial_+ \setminus \Gamma_+$). As

[3] The map $G : (t,z) \mapsto \varphi_t(z)$ is clearly smooth on U. By Lemma 2.6, $t \mapsto \rho(\varphi_t(z))$ is strictly increasing for $z \in \partial_-$. Therefore, by uniqueness of the integral curves of X, we see that G is injective. The inverse of G is given by $G^{-1}(z') = (t(z'), z(z'))$, where $t(z') = \inf\{t \geqslant 0 : \varphi_t(z') \in \partial\}$ and $z(z') = \varphi_{-t(z')}(z')$, which is smooth on $G(U)$ by the implicit function theorem.

$\mathrm{WF}(\iota_* v) \subset N^*\partial$, we may find $h_n \in \Omega_c^\bullet(M_\delta \setminus \Gamma_{+,\delta})$, for $n \in \mathbb{N}$, such that $h_n \to \iota_* v$ in $\mathcal{D}'^\bullet_{N^*\partial}(M_\delta)$, and with the property that $\mathrm{supp}(h_n) \cap \overline{U} = \varnothing$.[4] Then applying (3-1) to $h = h_n$ and letting $n \to \infty$ yields[5]

$$\int_{\partial_+} (\mathcal{S}_+(s)u) \wedge v = (-1)^{\deg u} e^{-\varepsilon s} \int_{\partial_-} u \wedge \iota^* \iota_X \varphi_\varepsilon^* R_{-,\delta}(s) \iota_* v,$$

because $\phi|_{\partial_+} = \mathcal{S}_+(s)u$. Since $\int_{\partial_+} \mathcal{S}_+(s)u \wedge v = \int_{\partial_-} u \wedge \mathcal{S}_-(s)v$, we obtain

$$\mathcal{S}_-(s) = (-1)^{\deg u} e^{-\varepsilon s} \iota^* \iota_X \varphi_\varepsilon^* R_{-,\delta}(s) \iota_*$$

as maps $\Omega_c^\bullet(\partial_+ \setminus \Gamma_+) \to \Omega_c^\bullet(\partial_- \setminus \Gamma_-)$. We can replace X by $-X$ to obtain the desired formula for $\mathcal{S}_+(s)$. □

Proof of Proposition 3.2 Let $u \in \Omega^\bullet(\partial \setminus \partial_0)$ and write $u = u(\tau, \theta) \in T^*_{(\tau,\theta)}\partial$. Let $\chi \in C_c^\infty(\mathbb{R}, [0,1])$ be such that $\int_{\mathbb{R}} \chi = 1$, $\chi(0) \neq 0$, $\chi \equiv 0$ on $\mathbb{R} \setminus \left]-\frac{1}{2}\delta, \frac{1}{2}\delta\right[$ and $\chi > 0$ on $\left]-\frac{1}{2}\delta, \frac{1}{2}\delta\right[$. For $n \in \mathbb{N}_{\geqslant 1}$ we set $\chi_n = n\chi(n\,\cdot)$, so that χ_n converges to the Dirac measure on $\mathbb{R}$ as $n \to +\infty$. We define $u_n \in \Omega_c^\bullet(M_\delta)$ in the (τ, ρ, θ) coordinates by

$$u_n = \chi_n(\rho) u(\tau, \theta) \wedge \mathrm{d}\rho.$$

Then $u_n \to (-1)^N \iota_* u$ in $\mathcal{D}'_{N^*\partial}(M_\delta)$, since $\partial = \{\rho = 0\}$. In particular, setting

$$f_n = \iota^* \varphi_{-\varepsilon}^* \iota_X R_{+,\delta}(s) u_n \quad \text{for } n \geqslant 1,$$

Remark 2.8 gives that $f_n \to (-1)^N \iota^* \varphi_{-\varepsilon}^* \iota_X R_{+,\delta}(s) \iota_* u$ in $\mathcal{D}'^\bullet(\partial)$. Moreover, if $\mathrm{Re}(s)$ is large enough, then for any $n \in \mathbb{N}$, we have $(-1)^N \iota^* \varphi_{-\varepsilon}^* \iota_X R_{+,\delta}(s) u_n \in C^0\big(M_\delta, \bigwedge^\bullet T^* M_\delta\big)$ and thus $f_n \in C^0\big(\partial, \bigwedge^\bullet T^*\partial\big)$. Then we claim that $f_n \to \mathcal{S}_+(s)u$ is in $\mathcal{D}'^\bullet(\partial \setminus \partial_0)$ when $n \to +\infty$, where we recall that

$$\mathcal{S}_+(s)u(z) = \begin{cases} S_-^* u(z) e^{-s\ell_-(z)} & \text{if } z \in \partial_+ \setminus \Gamma_+, \\ 0 & \text{if not.} \end{cases}$$

Let $F = \left\{|\rho| \leqslant \frac{1}{2}\delta\right\}$. Since the neighborhood $\left\{|\rho| < \frac{1}{2}\delta\right\}$ is strictly convex, there exists $L > 0$ such that, for any $z \in F$ and $T > 0$ with $\varphi_{-T}(z) \in F$, we have

$$\varphi_{-t}(z) \notin F \quad \text{for all } t \in]0, T[\quad \Longrightarrow \quad T \geqslant L. \tag{3-2}$$

Next, take $z \in \partial_+ \setminus \Gamma_+$. Then the set $\{t \in [\varepsilon, \ell_{-,\delta}(z)] : \varphi_{-t}(z) \in F\}$ is a finite union of closed intervals, say

$$\{t \geqslant \varepsilon : \varphi_{-t}(z) \in F\} = \bigcup_{k=0}^{K(z)} [a_k(z), b_k(z)],$$

with $a_k(z) \leqslant b_k(z) \leqslant +\infty$ and $b_k(z) < a_{k+1}(z)$ for every k. We set $\rho(t) = \rho(\varphi_{-t}(z))$ for any $t \geqslant 0$, and we take any smooth norm $\|\cdot\|$ on $\bigwedge^\bullet T^* M_\delta$. Note that $u_n = \chi_n(\rho) u_1$. Moreover, if $z \in M_\delta$ and $t < \ell_{-\delta}(z)$, we have

$$\|\varphi_{-t}^* u_1(z)\| \leqslant C \|u_1(\varphi_{-t} z)\| \exp(C|t|) \tag{3-3}$$

[4]For example, we may take $h_n(\rho, \tau, \theta) = \chi_n(\rho) v(\tau, \theta) \wedge \mathrm{d}\rho$, where $\chi_n \in C_c^\infty(]-\delta, \delta[)$ converges to the Dirac measure.

[5]Here we use that $\iota^* \iota_X \varphi_\varepsilon^* R_{-,\delta}(s) h_n \to \iota^* \iota_X \varphi_\varepsilon^* R_{-,\delta}(s) \iota_* v$ in $\mathcal{D}'^\bullet(\partial)$ as $n \to \infty$ by Remark 2.8, since $h_n \to \iota_* v$ in $\mathcal{D}'^\bullet_{N^*\partial}(M_\delta)$.

for some $C > 0$. Let $\theta_0 > 0$ small and $h \in C^\infty(M_\delta, [0,1])$ such that $h = 1$ on $\operatorname{supp} u_1$ and

(3-4) $$h(\tau, \rho, \theta) = 0 \quad \text{when } \operatorname{dist}(\theta, \pi\mathbb{Z}) < \theta_0.$$

(Such an h exists if θ_0 is small enough, since $u \in \Omega^\bullet(\partial \setminus \partial_0)$.) Then there is $c = c(\theta_0) > 0$ such that $|X\rho| \geqslant c$ on $\operatorname{supp} h$, by Lemma 2.3. In particular, if $\operatorname{Re}(s) > C$, then, by (3-3) and (3-4),

$$\begin{aligned} \|f_n(z)\| &\leqslant \int_\varepsilon^{\ell_{-,\delta}(z)} (\chi_n \circ \rho)(\varphi_{-t}(z)) \|\varphi_{-t}^*(\iota_X u_1)(z)\| e^{-ts}\,\mathrm{d}t \\ &\leqslant C\|u\|_\infty \sum_{k=0}^{K(z)} e^{(C-s)a_k(z)} \int_{a_k(z)}^{b_k(z)} \chi_n(\rho(t)) h(\varphi_{-t}(z))\,\mathrm{d}t \\ &\leqslant Cc^{-1}\|u\|_\infty \sum_{k=0}^{K(z)} e^{(C-s)a_k(z)} \int_{a_k(z)}^{b_k(z)} \chi_n(\rho(t)) |X\rho(\varphi_{-t}(z))|\,\mathrm{d}t. \end{aligned}$$

Of course, for $t < \ell_{-,\delta}(z)$, we have $X\rho(\varphi_{-t}(z)) = \rho'(t)$. Moreover, by Lemma 2.6, $\pm X^2\rho > 0$ if $\pm\rho > 0$. Thus we may separate each interval $[a_k(z), b_k(z)]$ into two subintervals on which $|\rho'| > 0$, and change variables to get

$$\int_{a_k(z)}^{b_k(z)} \chi_n(\rho(t))|\rho'(t)|\,\mathrm{d}t \leqslant 2\int_{\mathbb{R}} \chi_n(\rho)\,\mathrm{d}\rho \leqslant 2.$$

By (3-2), $a_k(z) \geqslant kL$ for any k. Therefore we obtain

(3-5) $$\|f_n(z)\| \leqslant \frac{2\|u\|_\infty}{1 - e^{(C-\operatorname{Re}(s))L}} \quad \text{for } z \in \partial_+ \setminus \Gamma_+ \text{ and } n \geqslant 1.$$

Moreover, if $z \in \partial_-$, we have that $t \mapsto \rho(\varphi_{-t}(z))$ is strictly increasing for any $z \in \partial_-$ by Lemma 2.6. Thus we may reproduce the argument made above to obtain that (3-5) also holds for $z \in \partial_-$. Finally, it is shown in [18, Section 2.4] that $\operatorname{Leb}(\Gamma_+ \cap \partial_+) = 0$.[6] In particular, since each f_n is a continuous, (3-5) holds for any $z \in \overline{(\partial_+ \cup \partial_-) \setminus \Gamma_+} = \partial$.

Next, let $v \in \Omega^\bullet(\partial)$. By Lemma 2.6, the set $\{\varphi_{-t}(z) : t \geqslant \varepsilon\}$ is included in $\{\rho \geqslant \rho(\varphi_{-\varepsilon}(z))\}$ for any $z \in \partial_-$. In particular, as $\operatorname{supp}(u_n) \to \partial$ when $n \to \infty$, we have $f_n(z) \to 0$ for $z \in \partial_-$. By dominated convergence we get, as $n \to \infty$,

$$\int_{\partial_-} f_n \wedge v \to 0.$$

Next, let $\eta > 0$, and $\chi_\pm \in C_c^\infty(\partial_\pm \setminus \Gamma_\pm)$ such that

(3-6) $$\chi_- \equiv 1 \quad \text{on } \operatorname{supp}(\chi_+ \circ S_+) \qquad \text{and} \qquad \operatorname{vol}(\operatorname{supp}(1 - \chi_+)) < \eta.$$

Such functions exist, as $\operatorname{Leb}(\Gamma_+ \cap \partial) = 0$. We have

$$\int_{\partial_+} f_n \wedge v = \int_{\partial_+} \chi_+ f_n \wedge v + \int_{\partial_+} (1 - \chi_+) f_n \wedge v.$$

[6] Actually, Section 2.4 of [18] says that $\operatorname{Leb}(\Gamma_{+,\delta} \cap \partial_{+,\delta}) = 0$. However, $J_\delta : z \mapsto \varphi_{\ell_{+,\delta}(z)}(z)$ realizes a local diffeomorphism $\partial_+ \to J_\delta(\partial_{+,\delta})$, and we have $J_\delta(\Gamma_+) \subset \Gamma_{+,\delta}$.

Note that $f_n = \tilde{f}_n$ on $\operatorname{supp}\chi_+$, where $\tilde{f}_n$ is defined exactly as f_n, replacing u by $\tilde{u} = \chi_- u \in \Omega^\bullet(\partial_- \setminus \Gamma_-)$. By Lemma 3.3, $\mathcal{Q}_{\varepsilon,+}(s)\tilde{u} = \mathcal{S}_+(s)\tilde{u}$, and since $\tilde{f}_n \to \mathcal{Q}_{\varepsilon,+}(s)\tilde{u}$, we have

$$\int_{\partial_+} \chi_+ f_n \wedge v = \int_{\partial_+} \chi_+ \tilde{f}_n \wedge v \to \int_{\partial_+} \chi_+ \mathcal{S}_+(s)\tilde{u} \wedge v = \int_{\partial_+} \chi_+ \mathcal{S}_+(s) u \wedge v,$$

where we used that $\mathcal{S}_+(s)u = \mathcal{S}_+(s)\tilde{u}$ on $\operatorname{supp}\chi_+$. On the other hand, as the forms f_n are uniformly bounded by (3-5) and the discussion below, there is $C > 0$ such that, for any $n \geqslant 1$,

$$\left| \int_{\partial_+} (1-\chi_+)\mathcal{S}_+(s)u \wedge v \right| < C\eta \quad \text{and} \quad \left| \int_{\partial_+} (1-\chi_+) f_n \wedge v \right| < C\eta,$$

where we used the second part of (3-6). Summarizing the above facts, we obtain that, for $n \geqslant 1$ big enough,

$$\left| \int_{\partial} f_n \wedge v - \int_{\partial} \mathcal{S}_+(s)u \wedge v \right| \leqslant 4C\eta.$$

Thus, $f_n \to \mathcal{S}_+(s)u$ in $\mathcal{D}'^\bullet(\partial)$. □

3.3 Composing the scattering maps

Recall that ∂ has two connected components $\partial^{(1)}$ and $\partial^{(2)}$ that we can identify in a natural way. We denote by $\psi : \partial \to \partial$ the map exchanging those components via this identification (in particular, $\psi(\partial_\pm) = \partial_\mp$), and we set

$$\widetilde{\mathcal{S}}_\pm(s) = \psi^* \circ \mathcal{S}_\pm(s).$$

Also we denote by $\Psi = T^*\partial \to T^*\partial$ the symplectic lift of ψ to $T^*\partial$; that is,

$$\Psi(z,\xi) = (\psi(z), \mathrm{d}\psi_z^{-\top}\xi) \quad \text{for } (z,\xi) \in T^*\partial.$$

Lemma 3.5 *Let $\chi \in C_c^\infty(\partial \setminus \partial_0)$. Then for any $n \geqslant 1$, the composition $(\chi\widetilde{\mathcal{S}}_\pm(s)\chi)^n$, which is well defined from $C^0\big(\partial, \bigwedge^\bullet T^*\partial\big)$ to $C^0\big(\partial, \bigwedge^\bullet T^*\partial\big)$ for* Re(s) *large and holomorphic with respect to s by Remark 3.1, admits a meromorphic continuation as a family of operators $\Omega^\bullet(\partial) \to \mathcal{D}'^\bullet(\partial)$.*

Proof We prove the lemma for $\mathcal{S}_+(s)$. First, assume that $n = 2$. According to [23, Theorem 8.2.14], it suffices to show that $A_1 \cap B_1 = \varnothing$, where for $n \geqslant 1$ we set

(3-7)
$$A_n = \{(z,\xi) : (z',0,z,\xi) \in \mathrm{WF}'\big((\chi\widetilde{\mathcal{S}}_\pm(s))^n\big) \text{ for some } z' \in \partial\},$$
$$B_n = \{(z,\xi) : (z,\xi,z',0) \in \mathrm{WF}\big((\chi\widetilde{\mathcal{S}}_\pm(s))^n\big) \text{ for some } z' \in \partial\}.$$

By Proposition 3.2 and Remark 2.8,

(3-8)
$$\mathrm{WF}'(\chi\mathcal{S}_+(s)\chi)|_{\operatorname{supp}(\chi\times\chi)} \subset \mathrm{d}(\iota\times\iota)^\top(\Delta_\varepsilon \cup \Upsilon^\varepsilon_{+,\delta} \cup (E^*_{+,\delta} \times E^*_{\mp,\delta})),$$

where Δ_ε and $\Upsilon^\varepsilon_{+,\delta}$ are defined as in the proof of Lemma 2.7. Note that in the coordinates of Lemma 2.3, $\iota(z) = (\tau, 0, \theta) \in \partial$ for any $z = (\tau,\theta) \in \partial$, and thus

$$\mathrm{d}\iota^\top(z,\eta) = \eta_\tau\,\mathrm{d}\tau + \eta_\theta\,\mathrm{d}\theta \quad \text{for } \eta = \eta_\tau\,\mathrm{d}\tau + \eta_\rho\,\mathrm{d}\rho + \eta_\theta\,\mathrm{d}\theta \in T_z^*M.$$

As χ is supported far from ∂_0, we have $(\varphi_\varepsilon(z'), z') \notin \partial\times\partial$ for any $z' \in \operatorname{supp}\chi$ (see for example Lemma 2.6), and, for any $\eta \in T^*_{z'}M_\delta$ such that $\langle X(z'), \eta\rangle = 0$, we have

$$\mathrm{d}\iota^\top(z', \eta) = 0 \implies \eta = 0 \tag{3-9}$$

by Lemma 2.3, since $\partial_0 = \{(\tau, 0, \theta) : \theta \in \pi\mathbb{Z}\}$. This implies that A_1 is contained in $E^*_{-,\partial}$, while B_1 is contained in $\Psi(E^*_{+,\partial})$ where $E^*_{+,\partial} = (\mathrm{d}\iota)^\top(E^*_{+,\delta})$. Now we claim that $\Psi(E^*_{+,\partial}) \cap E^*_{-,\partial} \subset \{0\}$ far from ∂_0. By Lemma 2.3 and Section 2.3, for any $z = (\tau, 0, \theta) \in \partial^{(j)} \cap \Gamma_\pm$,

$$E^*_{+,\partial}(z) = \mathbb{R}(\mathrm{d}\iota)^\top_z(r_+(z)\beta(z) - \psi(z)) = \mathbb{R}(-\sin(\theta) r_+(z)\,\mathrm{d}\tau - \mathrm{d}\theta),$$

since $\iota(\tau, \theta) = (\tau, 0, \theta)$. Then $r_+(\psi(z)) \neq r_-(z)$ for all z. Indeed, the contrary would mean that $E_s(z') \cap E_u(z') \neq \{0\}$ for some $z' \in M$ (represented by both z and $\psi(z)$ in M_δ), which is not possible. Now we have $\sin(\theta) \neq 0$ for $z \notin \partial_0$. As a consequence, (3-7) is true, since $\operatorname{supp}\chi \cap \partial_0 = \varnothing$. This concludes the case $n = 2$, and by [23, Theorem 8.2.14] we also have the bound

$$\mathrm{WF}'\big((\chi\widetilde{\mathcal{S}}_+(s)\chi)^2\big) \subset \big(\mathrm{WF}'(\chi\widetilde{\mathcal{S}}_+(s)\chi) \circ \mathrm{WF}'(\chi\widetilde{\mathcal{S}}_+(s)\chi)\big) \cup (B_1 \times \underline{0}) \cup (\underline{0} \times A_1),$$

where $\underline{0}$ denotes the zero section in $T^*\partial$, with $A_1 \subset E^*_{-,\partial}$ and $B_1 \subset \Psi(E^*_{+,\partial})$, and where, for any conical subsets $\Upsilon_1, \Upsilon_2 \subset T^*(M \times M)$, we write

$$\Upsilon_1 \circ \Upsilon_2 = \{(x_1, \xi_1, x_2, \xi_2) : (x_1, \xi_1, y, \eta) \in \Upsilon_1 \text{ and } (y, \eta, x_2, \xi_2) \in \Upsilon_2 \text{ for some } (y, \eta)\}.$$

Note that, if we set

$$E^*_{s,\partial_\pm} = \mathrm{d}\iota^\top(E^*_s|_{\partial_\pm}) \quad\text{and}\quad E^*_{u,\partial_\pm} = \mathrm{d}\iota^\top(E^*_u|_{\partial_\pm}),$$

we have $A_1 \subset E^*_{s,\partial_-}$ and $B_1 \subset \Psi(E^*_{u,\partial_+}) = E^*_{u,\partial_-}$.

We proceed by induction, assuming that, for some $n \geqslant 2$, the composition $(\chi\widetilde{\mathcal{S}}_\pm(s))^n$ is well defined with the bound

$$\mathrm{WF}'\big((\chi\widetilde{\mathcal{S}}_+(s))^n\big) \subset \big(\mathrm{WF}'(\chi\widetilde{\mathcal{S}}_+(s)\chi)^{n-1} \circ \mathrm{WF}'(\chi\widetilde{\mathcal{S}}_+(s)\chi)\big) \cup (B_{n-1} \times \underline{0}) \cup (\underline{0} \times A_1), \tag{3-10}$$

and that $A_{n-1} \subset E^*_{s,\partial_-}$ and $B_{n-1} \subset E^*_{u,\partial_-}$. This formula implies that the set A_n is included in

$$\big\{(z,\xi) \in T^*\partial : (z', 0, z'', \eta) \in \mathrm{WF}'\big((\chi\widetilde{\mathcal{S}}_+(s)\chi)^{n-1}\big) \text{ and } (z'', \eta, z, \xi) \in \mathrm{WF}'(\chi\widetilde{\mathcal{S}}_+(s)\chi) \text{ for some } z', z'' \in \partial\big\} \cup A_1.$$

We have $A_{n-1} \subset E^*_{s,\partial_-}$, and note that $\Psi(E^*_{+,\partial}) \subset E^*_{u,\partial_-}$ and $E^*_{u,\partial_-} \cap E^*_{s,\partial_-} = \{0\}$. Moreover, as mentioned above, $\varphi_\varepsilon(z') \notin \partial$ whenever $z' \in \operatorname{supp}(\chi)$. Thus we obtain, by (3-8),

$$A_n \subset \{(z, \xi) : (z'', \eta, z, \xi) \in \mathrm{d}(\iota \times \iota)^\top(\Upsilon^\varepsilon_{+,\delta}) \text{ for some } \eta \in \Psi(E^*_{s,\partial_-})\} \cup A_1.$$

Now suppose $(z'', \eta, z, \xi) \in \mathrm{d}(\iota \times \iota)^\top(\Upsilon^\varepsilon_{+,\delta})$ with $z'', z \in \operatorname{supp}\chi$. Note that $\Psi(E^*_{s,\partial_-}) = E^*_{s,\partial_+}$ and thus, if $\eta \in \Psi(E^*_{s,\partial_-}) \cap \mathrm{d}\iota(z'')^\top \ker X(z'')$, then $\eta = \mathrm{d}\iota(z'')^\top\tilde{\eta}$ for some $\tilde{\eta} \in E^*_s(z'')$ by (3-9). Since E^*_s is preserved by Φ_{-t}, we obtain $(z, \xi) \in \mathrm{d}\iota^\top(E^*_s)$. In particular, this yields $A_n \subset E^*_{s,\partial_-}$. Reversing the roles of $(\chi\widetilde{\mathcal{S}}_+(s))^{n-1}$ and $\chi\widetilde{\mathcal{S}}_+(s)$ in (3-10), we get that B_n is included in

$$\{(z,\xi)\in T^*\partial : (z,\xi,z',-\eta)\in \mathrm{WF}(\chi\widetilde{\mathcal{S}}_+(s)\chi) \text{ and } (z',\eta,z'',0)\in \mathrm{WF}\big((\chi\widetilde{\mathcal{S}}_+(s)\chi)^{n-1}\big) \text{ for some } z',z''\in\partial\} \cup B_1.$$

Proceeding as above, one gets $B_n \subset E^*_{u,\partial_-}$. Finally, $B_n \cap A_1 = \varnothing$, since $E^*_{u,\partial_-} \cap E^*_{s,\partial_-}$ on $\operatorname{supp}\chi$ by (3-9). As a consequence, the composition $(\chi\widetilde{\mathcal{S}}_+(s)\chi)^{n+1} = (\chi\widetilde{\mathcal{S}}_+(s)\chi)^n \circ (\chi\widetilde{\mathcal{S}}_+(s)\chi)$ is well defined by [23, Theorem 8.2.14], and (3-10) holds with n replaced by $n+1$. □

Remark 3.6 Using (3-10) inductively, one can actually show that $\mathrm{WF}'\big((\chi\widetilde{\mathcal{S}}_+(s)\chi)^n\big)$ is contained in $d(\hat\iota\times\hat\iota)^\top\widetilde{\mathbf{\Gamma}}_{\varepsilon,+}$, where

$$\widetilde{\mathbf{\Gamma}}_{\varepsilon,+} = \{(\widehat{\Phi}_t(z,\xi),(z,\xi)) : z,\hat\varphi_t(z)\in S\Sigma|_{\gamma_\star}\cap\hat\iota(\operatorname{supp}\chi),\ \langle X(z),\xi\rangle = 0,\ t\geqslant\varepsilon\} \cup (E_u^*\times E_s^*)|_{\operatorname{supp}(\chi\times\chi)}.$$

Here (and only here), in order to avoid confusion, we denote by $\hat\varphi$ (resp. $\widehat{\Phi}_t$) the complete geodesic flow on $M = S\Sigma$ (resp. the symplectic lift of the geodesic flow on T^*M), and by $\hat\iota\colon \partial \to S\Sigma|_{\gamma_\star} \hookrightarrow M$ the identification of both components of ∂.

3.4 The flat trace of the scattering operator

Let $A\colon \Omega^\bullet(\partial)\to\mathcal{D}'^\bullet(\partial)$ be an operator such that $\mathrm{WF}'(A)\cap\Delta(T^*\partial) = \varnothing$, where $\Delta(T^*\partial)$ is the diagonal in $T^*(\partial\times\partial)$. Then by [23, Theorem 8.2.4], the pullback $\iota_\Delta^* K_A$ is well defined, where $\iota_\Delta : z\mapsto (z,z)$ is the diagonal inclusion and $K_A\in\mathcal{D}'^3(\partial\times\partial)$ is the Schwartz kernel of A, defined by

$$\int_\partial Au\wedge v = \int_{\partial\times\partial} K_A\wedge\pi_1^*u\wedge\pi_2^*v \quad \text{for } u,v\in\Omega^\bullet(\partial),$$

where $\pi_j : \partial\times\partial\to\partial$ is the projection on the j^{th} factor (for $j=1,2$). We then define the (super)flat trace of A by

$$-\mathrm{tr}_\mathrm{s}^\flat A = \langle \iota_\Delta^* K_A, 1\rangle.$$

In fact, one can show that

$$-\mathrm{tr}_\mathrm{s}^\flat(A) = \sum_{k=0}^{2}(-1)^k\,\mathrm{tr}^\flat(A_k), \tag{3-11}$$

where $\mathrm{tr}^\flat$ is the transversal trace of Atiyah and Bott [3] and A_k is the operator

$$A_k : C^\infty\big(\partial,\textstyle\bigwedge^k T^*\partial\big)\to\mathcal{D}'\big(\partial,\textstyle\bigwedge^k T^*\partial\big)$$

induced by A on the space of k–forms (see also [16, Section 2.4] for an introduction to the flat trace).

The purpose of this section is to compute the flat trace of $\mathcal{S}_\pm(s)$. In what follows, for any closed geodesic $\gamma\colon \mathbb{R}/\ell\mathbb{Z}\to\Sigma$, we will write

$$I_\star(\gamma) = \{z\in S\Sigma|_{\gamma_\star} : z = (\gamma(\tau),\dot\gamma(\tau)) \text{ for some } \tau\in\mathbb{R}/\ell\mathbb{Z}\}$$

for the set of incidence vectors of γ along $\gamma_\star$, and

$$I_{\star,\pm}(\gamma) = p_\star^{-1}(I_\star(\gamma))\cap\partial_\mp,$$

where $p_\star\colon S\Sigma_\star\to S\Sigma$ is the natural projection.

Proposition 3.7 *Let $\chi \in C_c^\infty(\partial \setminus \partial_0)$. For any $n \geqslant 1$, the operator $(\chi \widetilde{\mathcal{S}}_\pm(s))^n$ has a well-defined flat trace, and for* $\operatorname{Re}(s)$ *big enough,*

$$\operatorname{tr}_s^\flat\big((\chi \widetilde{\mathcal{S}}_\pm(s)\chi)^n\big) = n \sum_{i(\gamma,\gamma_\star)=n} \frac{\ell^\#(\gamma)}{\ell(\gamma)} e^{-s\ell(\gamma)} \Bigg(\prod_{z \in I_{\star,\pm}(\gamma)} \chi^2(z)\Bigg)^{\ell(\gamma)/\ell^\#(\gamma)}, \tag{3-12}$$

where the sum runs over all (not necessarily primitive) closed geodesics γ of (Σ, g) such that $i(\gamma,\gamma_\star) = n$. Here $\ell(\gamma)$ is the length of γ and $\ell^\#(\gamma)$ its primitive length.

This formula should be compared with the formula

$$\operatorname{tr}_s^\flat((\chi f^* \chi)^n) = \sum_{\gamma \in \operatorname{Per}_n(f)} m^\#(\gamma) \operatorname{sgn}(\det(1 - P_\gamma)) \Bigg(\prod_{z \in \gamma} \chi^2(z)\Bigg)^{n/m^\#(\gamma)},$$

which is valid for any smooth Anosov diffeomorphism $f : Z \to Z$ of a closed manifold Z and $\chi \in C^\infty(Z)$. Here $f^*: C^\infty(Z) \to C^\infty(Z)$ is the pullback operator, $\operatorname{Per}_n(f)$ is the set of n–periodic orbits of f, $m^\#(\gamma)$ is the minimal period of γ and P_γ is the linearized Poincaré map of γ (that is, $P_\gamma = \mathrm{d}f(z)$ for $z \in \gamma$). Note that the above sum is finite, unlike the sum in (3-12). This is due to the fact that $S_\pm$ is singular at $\Gamma_\pm$, which allows $S_\pm$ to have an infinite number of n–periodic points.

Proof The proof that the intersection

$$\mathrm{WF}'\big((\chi \widetilde{\mathcal{S}}_\pm(s)\chi)^n\big) \cap \Delta(T^*\partial) \tag{3-13}$$

is empty follows from the estimate in Remark 3.6, since $E_u^* \cap E_s^* = \{0\}$ and $\mathrm{d}\hat\imath(z)^\top : \ker X(\hat\imath(z)) \to T_z^*\partial$ is injective for any $z \in \operatorname{supp}(\chi)$.

For any $n \geqslant 1$, we define the set $\widetilde{\Gamma}_\pm^n \subset \partial$ by

$$\complement \widetilde{\Gamma}_\pm^n = \{z \in \partial : (\widetilde{S}_\pm)^k(z) \text{ is well defined for } k = 1, \dots, n\},$$

where $\widetilde{S} = \psi \circ S$. Equivalently,

$$\widetilde{\Gamma}_\pm^1 = \Gamma_\pm \quad \text{and} \quad \widetilde{\Gamma}_\pm^{n+1} = \widetilde{\Gamma}_\pm^n \cap (\widetilde{S}_\mp)^n(\Gamma_\pm \setminus \widetilde{\Gamma}_\mp^n)$$

for $n \geqslant 1$. Also, we set

$$\tilde\ell_{\pm,n}(z) = \ell_\pm(z) + \ell_\pm(\widetilde{S}_\pm(z)) + \cdots + \ell_\pm(\widetilde{S}_\pm^{n-1}(z)) \quad \text{for } z \in \complement\widetilde{\Gamma}_\pm^n, \tag{3-14}$$

where $\ell_\pm(z) = \inf\{t > 0 : \varphi_{\pm t}(z) \in \partial\}$, with the convention that $\tilde\ell_{\pm,n}(z) = +\infty$ if $z \in \widetilde{\Gamma}_\pm^n$. We will need the following:

Lemma 3.8 *Let $n \geqslant 1$. For any $k \geqslant 1$, there exists $C_{k,n} > 0$ such that*

$$\|\mathrm{d}^k \ell_{\pm,n}(z)\| \leqslant C_{k,n} \exp(C_{k,n}\ell_{\pm,n}(z)) \quad \text{for } z \in \complement\widetilde{\Gamma}_\pm^n.$$

Proof By induction on n, using (3-14) and the fact that $S_\pm(\complement\widetilde{\Gamma}_\pm^n) = \complement\widetilde{\Gamma}_\pm^{n-1}$, we see that the lemma reduces to proving the estimate

$$\|\mathrm{d}^k \ell_\pm(z)\| \leqslant C_k \exp(C_k \ell_\pm(z)) \quad \text{for } z \in \complement\widetilde{\Gamma}_\pm^1. \tag{3-15}$$

In what follows, C_k is a constant depending only on k, which may change at each line. First, notice that $\|\mathrm{d}^k\varphi_t(z)\| \leqslant C_k e^{C_k|t|}$ for any $t\in\mathbb{R}$ and $z\in M_\delta$ such that $\varphi_t(z)\in M_\delta$, for some constant C_k; see for example [8, Proposition A.4.1]. Moreover,

$$\mathrm{d}S_\pm(z) = \mathrm{d}[\varphi_{\ell_\pm(z)}](z) + X(S_\pm(z))\,\mathrm{d}\ell_\pm(z) \quad \text{for } z\in \complement\widetilde{\Gamma}^1_\pm.$$

By induction we obtain that, for any k,

$$\|\mathrm{d}^k S_\pm(z)\| \leqslant C_k \exp(C_k\ell_\pm(z)) + C_k\sum_{j=1}^{k}\|\mathrm{d}^j\ell_\pm(z)\|^{m_j} \quad \text{with } m_j\in\mathbb{N} \text{ for } j=1,\dots,k \tag{3-16}$$

for any $z\in\complement\widetilde{\Gamma}^1_\pm$. Let (τ,ρ,θ) be the coordinates defined near ∂ given by Lemma 2.3. Then $\rho(S_\pm(z))=0$ for $z\in\widetilde{\Gamma}^\pm_1$, and thus

$$(X\rho)(S_\pm(z))\,\mathrm{d}\ell_\pm(z) = -\mathrm{d}\rho(S_\pm(z))\circ\mathrm{d}[\varphi_{\ell_\pm(z)}](z) \quad \text{for } z\in\complement\widetilde{\Gamma}^1_\pm. \tag{3-17}$$

Let $z\notin\widetilde{\Gamma}^\pm_1$; Lemma 2.3 gives

$$(X\rho)(S_\pm(z)) = \sin\bigl(\theta(S_\pm(z))\bigr). \tag{3-18}$$

Set $z'=S_\pm(z)$, and write $(\tau(t),\rho(t))=\pi(\varphi_{\mp t}(z'))$, so that $\rho(0)=0$. By the proof of Lemma 2.6, $t\mapsto|\rho(t)|$ is strictly increasing (indeed $z\notin\widetilde{\Gamma}^\pm_1$ and thus $\dot\rho(0)=\pm X\rho(z')\neq 0$), and whenever $|\rho(t)|\leqslant\frac12\delta$,

$$\ddot\rho(t) = G(\tau(t),\rho(t)) \tag{3-19}$$

for some smooth function $G\in C^\infty\bigl((\mathbb{R}/\ell_\star\mathbb{Z})_\tau\times\bigl[-\frac12\delta,\frac12\delta\bigr]_\rho\bigr)$ satisfying $G(\tau,0)=0$ and $\partial_\rho G(\tau,\rho)>0$. If $D=\sup|\partial_\rho G|$, we have $|G(\tau,\rho)|\leqslant D|\rho|$ and thus $|\ddot\rho(t)|\leqslant D|\rho(t)|$, with $\rho(0)=\ddot\rho(0)=0$ and $\dot\rho(0)=\pm X\rho(S_\pm(z))$. By comparing the solution of (3-19) with the solutions of $\ddot y(t)=Dy(t)$, we obtain

$$|\rho(t)|\leqslant|X\rho(z')|\operatorname{sh}(Dt).$$

In particular, $|\rho(t)|<\frac12\delta$ whenever $|X\rho(S_\pm(z))|\operatorname{sh}(Dt)<\frac12\delta$, and thus $\operatorname{sh}(D\ell_\mp(z'))\geqslant\frac12\delta|X\rho(z')|$. By (3-18), we conclude that there is $C>0$ such that

$$\bigl|\sin\bigl(\theta(S_\pm(z))\bigr)\bigr|\geqslant C\exp(-C\ell_\pm(z)) \quad \text{for } z\in\complement\widetilde{\Gamma}^1_\pm. \tag{3-20}$$

We therefore obtain, for any $z\in\widetilde{\Gamma}^\pm_1$,

$$\|\mathrm{d}\ell_\pm(z)\|\leqslant C^{-1}\exp(C\ell_\pm(z))\|\mathrm{d}\rho(S_\pm(z))\|\cdot\|\mathrm{d}[\varphi_{\ell_\pm(z)}](z)\|\leqslant Ce^{C\ell_\pm(z)}.$$

Now, repeatedly using (3-16), (3-17) and (3-20), we obtain (3-15) by induction on k. □

Consider $\tilde\chi\in C^\infty(\mathbb{R},[0,1])$ such that $\tilde\chi\equiv 1$ on $]-\infty,1]$ and $\tilde\chi\equiv 0$ on $[2,+\infty[$, and set $\tilde\chi_L(z)=\tilde\chi(\ell_{\pm,n}(z)-L)$ for $z\in\partial$. Then $\tilde\chi_L\in C^\infty_c(\partial\setminus\widetilde{\Gamma}^n_\pm)$, and by (3-11) we see that the Atiyah–Bott trace formula [3, Corollary 5.4] reads in our case

$$\langle\iota^*_\Delta K_{\chi,\pm,n}(s),\tilde\chi_L\rangle = \sum_{(\widetilde{S}_\mp)^n(z)=z} e^{-s\ell_{\pm,n}(z)}\tilde\chi_L(z)\prod_{k=0}^{n-1}\chi^2((\widetilde{S}_\mp)^k(z)), \tag{3-21}$$

where $K_{\chi,\pm,n}(s)$ is the Schwartz kernel of $(\chi\mathcal{S}_\pm(s))^n$. Indeed, a simple computation (for example in the spirit of [16, Appendix B][7]) shows that, for any diffeomorphism $f:\partial\to\partial$ with isolated nondegenerate fixed points,

$$\operatorname{tr}^\flat(F_k)=\sum_{f(z)=z}\frac{\operatorname{tr}\bigwedge^k \mathrm{d}f(z)}{|\det(1-\mathrm{d}f(z))|}, \tag{3-22}$$

where $F_k:\Omega^k(\partial)\to\Omega^k(\partial)$ is defined by $F_k\omega=f^*\omega$ and $\bigwedge^k \mathrm{d}f(z)$ is the map induced by $\mathrm{d}f(z)$ on $\bigwedge^k T_z^*\partial$. Since $\sum_k(-1)^k\operatorname{tr}\big(\bigwedge^k \mathrm{d}f(z)\big)=\det(1-\mathrm{d}f(z))$, it holds that

$$\operatorname{tr}_{\mathrm{s}}^\flat(F)=\sum_k(-1)^{k+1}\operatorname{tr}^\flat(F_k)=-\sum_{f(z)=z}\operatorname{sgn}\det(1-\mathrm{d}f(z)). \tag{3-23}$$

Now note that $\tilde\chi_L(\chi\widetilde{\mathcal{S}}_\pm(s)\chi)^n$ is by definition the operator given by

$$\omega\mapsto\tilde\chi_L(\cdot)\bigg(\prod_{k=1}^n(\chi\circ(\widetilde{S}_\mp)^k)(\chi\circ(\widetilde{S}_\mp)^{k-1})\bigg)e^{-s\ell_{\pm,n}(\cdot)}(\widetilde{S}_\mp)^{n*}w. \tag{3-24}$$

Moreover, $\operatorname{sgn}\det(1-\mathrm{d}(\widetilde{S}_\mp)^n(z))=-1$ for any z such that $(\widetilde{S}_\mp)^n(z)=z$. Indeed, for such a z, $\mathrm{d}(\widetilde{S}_\mp)^n(z)$ is conjugated to the linearized Poincaré map

$$P_z=\mathrm{d}(\varphi_{\ell_{\pm,n}(z)})(z)|_{E^u(z)\oplus E^s(z)},$$

which satisfies $\det(1-P_z)<0$ as the matrix of P_z in the decomposition $E^u(z)\oplus E^s(z)$ reads $\left(\begin{smallmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{smallmatrix}\right)$ for some $\lambda>1$ (since φ_t preserves the volume form $\alpha\wedge\mathrm{d}\alpha$). Finally, by (3-13), the pairing in the left-hand side of (3-21) is well defined; moreover, the proof of (3-22) can be revisited for the operator (3-24) thanks to the introduction of our cutoff functions $\tilde\chi_L$ and χ, yielding (3-21).

As $L\to+\infty$, the right-hand side of (3-21) converges to

$$n\sum_{i(\gamma,\gamma_\star)=n}\frac{\ell^\#(\gamma)}{\ell(\gamma)}e^{-s\ell(\gamma)}\bigg(\prod_{z\in I_{\star,\pm}(\gamma)}\chi^2(z)\bigg)^{\ell(\gamma)/\ell^\#(\gamma)},$$

since for any closed geodesic $\gamma:\mathbb{R}/\mathbb{Z}\to\Sigma$ such that $i(\gamma,\gamma_\star)=n$,

$$\#\{z\in\partial : z=(\gamma(\tau),\gamma'(\tau))\text{ for some }\tau\}=n\frac{\ell^\#(\gamma)}{\ell(\gamma)}.$$

Note that the sum converges whenever $\operatorname{Re}(s)$ is large enough by Margulis' asymptotic formula, given in the introduction. It remains to see that $\langle i_\Delta^*K_{\chi,\pm,n}(s),1-\tilde\chi_L\rangle\to 0$ as $L\to+\infty$. Note that Lemma 3.8 gives

$$\|\mathrm{d}^k\tilde\chi_L\|\leqslant C_ke^{C_kL}. \tag{3-25}$$

By Remark 3.1, if $s_0>0$ is large enough, one has $\mathcal{S}_\pm(s_0):\Omega^\bullet(\partial)\to C^0\big(\partial,\bigwedge^\bullet T^*\partial\big)$. Also, for any $s\in\mathbb{C}$ with $\operatorname{Re}(s)>0$,

$$\mathcal{S}_\pm(s_0+s)w=(\mathcal{S}_\pm(s_0)w)e^{-s\ell_\pm(\cdot)}\quad\text{for }w\in\Omega^\bullet(\partial). \tag{3-26}$$

[7] Actually, in the aforementioned reference, the authors deal with flows, but the diffeomorphism case is even simpler.

Let $N \in \mathbb{N}$ such that $\iota_\Delta^* K_{\chi,\pm,n}(s_0)$ extends as a continuous linear form on $C^N(\partial)$. Then applying Lemma 3.8, we see that if $\mathrm{Re}(s)$ is large enough, the function $\exp(-s\ell_{\pm,n}(\cdot))$ lies in $C^N(\partial)$. Thus, the product $e^{-s\ell_{\pm,n}(\cdot)}\iota_\Delta^* K_{\chi,\pm,n}(s_0)$ is well defined and by (3-25) we have

$$\begin{aligned}|\langle e^{-s\ell_{\pm,n}(\cdot)}\iota_\Delta^* K_{\chi,\pm,n}(s_0), (1-\tilde{\chi}_L)\rangle| &= |\langle \iota_\Delta^* K_{\chi,\pm,n}(s_0), (1-\tilde{\chi}_L)e^{-s\ell_{\pm,n}(\cdot)}\rangle| \\ &\leqslant C\|(1-\tilde{\chi}_L)e^{-s\ell_{\pm,n}(\cdot)}\|_{C^N(\partial)} \leqslant C_N e^{(C_N-\mathrm{Re}(s))L},\end{aligned}$$

since $\ell_{\pm,n} \geqslant L$ on $\mathrm{supp}(1-\tilde{\chi}_L)$. Therefore, to obtain that $\langle i_\Delta^* K_{\chi,\pm,n}(s_0+s), 1-\tilde{\chi}_L\rangle \to 0$ as $L \to +\infty$, it suffices to show that

$$e^{-s\ell_{\pm,n}(\cdot)}\iota_\Delta^* K_{\chi,\pm,n}(s_0) = \iota_\Delta^* K_{\chi,\pm,n}(s_0+s).$$

This equality is a consequence of (3-26) and Lemma B.1, since we can take s arbitrarily large. □

Recall from Remark 3.6 that $s \mapsto (\chi\widetilde{\mathcal{S}}_\pm(s)\chi)^n$ admits a meromorphic continuation in $\mathcal{D}'^3_{\Gamma'_{\varepsilon,\pm}}(\partial\times\partial)$, where $\Gamma'_{\varepsilon,\pm}$ does not intersect the conormal to the diagonal in $\partial\times\partial$. In particular:

Corollary *The function $s \mapsto \eta_{\pm,\chi,n}(s)$ defined for $\mathrm{Re}(s) \gg 1$ by the right-hand side of (3-12) extends to a meromorphic function on the whole complex plane.*

To prove Theorem 1, we wish to use a standard Tauberian argument near the first pole of $\eta_{\pm,\chi,n}$ to obtain the growth of $N(n,L)$. Indeed, it is known (see Section 5) that $s \mapsto R_{\pm,\delta}(s)$ has a simple pole at $s = h_\star$. However, since $\eta_{\pm,\chi,n}$ is given by the trace of the n^{th} self-composition of the restriction of $R_{\pm,\delta}$ to ∂, it is not clear a priori that $\eta_{\pm,\chi,n}$ will have a singularity at $s = h_\star$. In the next section we obtain some a priori bounds on $N(n,L)$; this will imply that $\eta_{\pm,\chi,n}$ indeed has a pole at $s = h_\star$, of order n.

4 A priori bounds on the growth of geodesics with fixed intersection number with $\gamma_\star$

The purpose of this section is to get a priori bounds on $N(1,L)$ — and $N(2,L)$ in the case where $\gamma_\star$ is separating — using Parry and Pollicott's bound for axiom A flows [35].

Choose some point $x_\star \in \gamma_\star$. Let g be the genus of Σ and $(a_1, b_1, \dots, a_{\mathrm{g}}, b_{\mathrm{g}})$ be a basis of generators of Σ, so that the fundamental group of Σ is the finitely presented group given by

$$\pi_1(\Sigma) = \langle a_1, b_1, \dots, a_{\mathrm{g}}, b_{\mathrm{g}}, [a_1,b_1]\cdots[a_{\mathrm{g}},b_{\mathrm{g}}] = 1\rangle, \tag{4-1}$$

where we set $\pi_1(\Sigma) = \pi_1(\Sigma, x_\star)$ for some choice of $x_\star \in \gamma_\star$ (see Figure 2 for the case where $\gamma_\star$ is not separating, and Figure 4 otherwise).

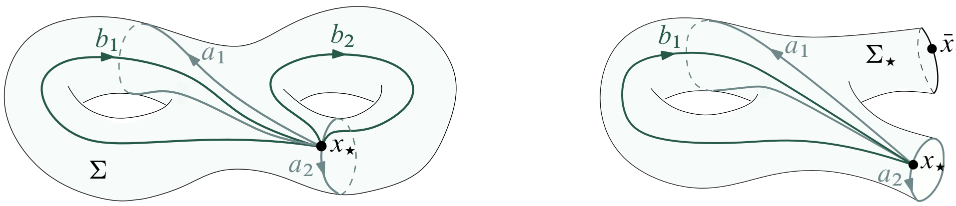

Figure 2: The generators $a_1, b_1, \ldots, a_g, b_g$ of $\pi_1(\Sigma)$ (on the left) and the generators $a_1, b_1, \ldots, a_g$ of $\pi_1(\Sigma_\star)$ (on the right) when $g = 2$. Here $\gamma_\star$ is assumed to be not separating and is represented by a_2 in $\pi_1(\Sigma)$.

4.1 The case $\gamma_\star$ is not separating

Up to applying a diffeomorphism to Σ, we may assume that $\gamma_\star$ is represented by $a_g \in \pi_1(\Sigma)$. The cut surface $\Sigma_\star$ is a topological surface of genus $g-1$ with 2 punctures, and the fundamental group[8] $\pi_1(\Sigma_\star) = \pi_1(\Sigma_\star, x_\star)$ is the free group given by $\langle a_1, b_1, \ldots, a_g \rangle$, which follows from the fact that $\Sigma_\star$ is homotopically equivalent to a connected sum of $2g-1$ circles. We refer to Figure 2 for a picture of the generators and the choice of $x_\star$. By the presentation of $\pi_1(\Sigma)$ given above, we have

$$b_g a_g b_g^{-1} = a'_g \quad \text{where } a'_g = [a_1, b_1] \cdots [a_{g-1}, b_{g-1}] a_g, \tag{4-2}$$

and note that a'_g also defines an element of $\pi_1(\Sigma_\star)$.

Lemma 4.1 *The map $q_\star\colon \Sigma_\star \to \Sigma$ given by the identification of the boundary components of $\Sigma_\star$ induces a map $q_{\star,*}\colon \pi_1(\Sigma_\star) \to \pi_1(\Sigma)$, which is injective.*

Proof Let $\langle a_g \rangle$ (resp. $\langle a'_g \rangle$) be the infinite cyclic subgroup of $\pi_1(\Sigma_\star)$ generated by a_g (resp. a'_g). Then by (4-1) and (4-2), the group $\pi_1(\Sigma)$ is the HNN[9] extension $\pi_1(\Sigma_\star)*_\phi$ of $\pi_1(\Sigma_\star)$ with respect to the isomorphism $\phi\colon \langle a'_g \rangle \to \langle a_g \rangle$ given by $\phi(a'_g) = a_g$, that is, $\pi_1(\Sigma_\star)*_\phi$ is the finitely presented group defined by

$$\pi_1(\Sigma_\star)*_\phi = \langle a_1, b_1, \ldots, a_g, t : t^{-1} a'_g t = a_g \rangle;$$

see [30, Section IV.2]. Now the map $q_{\star,*}\colon \pi_1(\Sigma_\star) \to \pi_1(\Sigma)$ coincides with the natural map $\pi_1(\Sigma_\star) \to \pi_1(\Sigma_\star)*_\phi$, and this map is injective by [30, Theorem IV.2.1]. □

We may see the cut surface $\Sigma_\star$ as the convex core of a complete, noncompact, negatively curved surface, with funnels. Indeed, by Lemma 4.1, the group $\pi_1(\Sigma_\star)$ can be thought of as a subgroup of $\pi_1(\Sigma)$, and the convex core of the infinite surface $\Sigma_\star^e = \pi_1(\Sigma_\star)\backslash\widetilde{\Sigma}$ is canonically isometric to $\Sigma_\star$ (here $\widetilde{\Sigma}$ is a universal cover of Σ). Another way to obtain this is by gluing two arbitrary funnels as follows. Recall that near each connected component of the boundary $\partial\Sigma_\star \subset \Sigma_\delta$ we have coordinates

[8] Here, in order not to burden the notation, we still denote by $x_\star \in \Sigma_\star$ a lift of $x_\star \in \Sigma$ by the natural map $q_\star\colon \Sigma_\star \to \Sigma$; see Figure 2.

[9] HNN refers to the authors Graham Higman, Bernhard Neumann and Hanna Neumann [22].

$(\tau,\rho)\in\mathbb{R}/\ell_\star\mathbb{Z}_\tau\times[-\delta,\delta]_\rho$ given by Lemma 2.3, for which $\partial\Sigma_\star=\{\rho=0\}$ and $\partial\Sigma_\delta=\{\rho=\delta\}$. In those coordinates, the metric has the form $\mathrm{d}\rho^2+f(\tau,\rho)\,\mathrm{d}\tau^2$ for some smooth function f satisfying $\partial_\rho f(\tau,0)=0$ and $\kappa(\tau,\rho)=-\partial_\rho^2 f(\tau,\rho)/f(\tau,\rho)$. Then we arbitrarily extend f to a smooth function on $(\mathbb{R}/\ell_\star\mathbb{Z})_\tau\times[-\delta,+\infty[$ so that, for some constants $c,C>0$,

$$c\leqslant\frac{\partial_\rho^2 f}{f}\leqslant C.$$

By gluing the funnels $(\mathbb{R}/\ell_\star\mathbb{Z})\times[0,\infty[$ and $\Sigma_\star$ along the corresponding connected components, we obtain a complete negatively curved surface $\Sigma_\star^e$, whose metric in the funnels is given by $\mathrm{d}\rho^2+f(\tau,\rho)\,\mathrm{d}\tau^2$. We will again denote by (φ_t) the geodesic flow on the unit tangent bundle $S\Sigma_\star^e$ of $\Sigma_\star^e$.

Let $\widetilde{\Sigma}_\star$ denote the universal cover of $\Sigma_\star^e$ and let $\tilde{x}_\star\in\widetilde{\Sigma}_\star$ be such that $\pi(\tilde{x}_\star)=x_\star$, where $\pi:\widetilde{\Sigma}_\star\to\Sigma_\star^e$ is the natural projection. Then $\pi_1(\Sigma_\star^e,x_\star)=\pi_1(\Sigma_\star)$ acts on $\widetilde{\Sigma}_\star$ by deck transformations so that $\Sigma_\star^e\simeq\pi_1(\Sigma_\star)\backslash\widetilde{\Sigma}_\star$. Moreover, Lemma 2.6 implies that the recurrent set of the geodesic flow on $S\Sigma_\star^e$ is compact and included in $S\Sigma_\star$; thus $\pi_1(\Sigma_\star)$ is convex–cocompact in the sense of [12]. The aforementioned lemma also implies that every closed geodesic in $\Sigma_\star^e$ which is not contained in $\partial\Sigma_\star$ is actually contained in the interior of $\Sigma_\star$.

It is well known that there is a one-to-one correspondence between oriented closed geodesics on $\Sigma_\star^e$ (all of them belonging to $\Sigma_\star$) and the set of free homotopy classes of loops in $\Sigma_\star^e$. The latter set is itself in one-to-one correspondence with the set of conjugacy classes of $\pi_1(\Sigma_\star)$. We set

$$\ell_\star(w)=\mathrm{dist}(\tilde{x}_\star,w\tilde{x}_\star)\quad\text{for }w\in\pi_1(\Sigma_\star),$$

where the distance comes from the metric π^*g on $\widetilde{\Sigma}_\star$. For any $w\in\pi_1(\Sigma_\star)$, we denote by $[w]$ the associated conjugacy class of $\pi_1(\Sigma_\star)$. Note that if $\gamma_{[w]}$ denotes the unique geodesic in the free homotopy class of w (which is represented by the conjugacy class $[w]$), we have $\ell(\gamma_{[w]})\leqslant\ell_\star(w)$. We also denote by

(4-3) $$\mathrm{wl}(w)=\min\{n\geqslant 0: w=\alpha_1\cdots\alpha_n\text{ with }\alpha_j\in\mathscr{L}_{\mathrm{g}}\setminus\{b_{\mathrm{g}},b_{\mathrm{g}}^{-1}\}\}$$

the word length of an element $w\in\pi_1(\Sigma_\star)$, where $\mathscr{L}_{\mathrm{g}}=\bigcup_{k=1}^{\mathrm{g}}\{a_k,a_k^{-1},b_k,b_k^{-1}\}$. We will say that a word $\alpha_1\cdots\alpha_k$ with $\alpha_j\in\mathscr{L}_{\mathrm{g}}$ is reduced if $\alpha_j\neq(\alpha_{j+1})^{-1}$ for any $j=1,\dots,k-1$. As $\pi_1(\Sigma_\star)$ is free, for each $w\in\pi_1(\Sigma_\star)$, there is exactly one reduced word $\alpha_1\cdots\alpha_n$ such that $n=\mathrm{wl}(w)$; see [30, page 4]. It follows from the Milnor–Švarc lemma [11, Proposition I.8.19] that, for some constant $D>0$,

(4-4) $$\frac{1}{D}\,\mathrm{wl}(w)-D\leqslant\ell_\star(w)\leqslant D\,\mathrm{wl}(w)+D\quad\text{for }w\in\pi_1(\Sigma_\star).$$

Also, as $\pi_1(\Sigma_\star)$ is convex cocompact, we have the classical orbital counting (see [42, paragraphe 1.F and corollaire 2])

(4-5) $$\#\{w\in\pi_1(\Sigma_\star):\ell_\star(w)\leqslant L\}\sim Ae^{h_\star L}\quad\text{as }L\to\infty$$

for some $A>0$, where $h_\star>0$ is the topological entropy of the geodesic flow of $(\Sigma_\star^e,g)$ restricted to the trapped set

$$K_\star^e=\{(x,v)\in S\Sigma_\star^e:\varphi_t(x,v)\in S\Sigma_\star\text{ for }t\in\mathbb{R}\}.$$

In fact, $h_\star > 0$ also coincides with the entropy of the geodesic flow of (Σ, g) restricted to the trapped set $K_\star$ mentioned in the introduction,

$$K_\star = \overline{\{(x,v) \in S\Sigma : \pi(\varphi_t(x,v)) \in \Sigma \setminus \gamma_\star \text{ for } t \in \mathbb{R}\}},$$

where the closure is taken in $S\Sigma$ and $K_\star^e = p_\star^{-1}(K_\star)$, where $p_\star\colon S\Sigma_\star \to S\Sigma$ is the natural map given by the identification of both components of $\partial S\Sigma_\star$.

4.1.1 Lower bound In this section we will prove:

Proposition 4.2 *If $\gamma_\star$ is not separating, then there is $C > 0$ such that, for any L large enough,*

$$N(1,L) \geqslant C\frac{e^{h_\star L}}{L}.$$

Note that Theorem 1 actually gives $N(1,L) \sim c_\star e^{h_\star L}$, so Proposition 4.2 is not sharp. We could obtain a better bound with the methods presented in Section 4.2, which deals with the separating case; however, Proposition 4.2 will be sufficient for our purposes (see Remarks 5.2, 5.3 and 5.4).

Lemma 4.3 *Take $w, w' \in \pi_1(\Sigma_\star)$. Then $[wb_{\mathrm{g}}] = [w'b_{\mathrm{g}}]$ as conjugacy classes of $\pi_1(\Sigma)$ if and only if $w = a_{\mathrm{g}}^n w' a_{\mathrm{g}}'^{-n}$ in $\pi_1(\Sigma_\star)$ for some $n \in \mathbb{Z}$.*

Proof If $w = a_{\mathrm{g}}^n w' b_{\mathrm{g}} a_{\mathrm{g}}^{-n} b_{\mathrm{g}}^{-1}$, then clearly wb_{g} and $w'b_{\mathrm{g}}$ are conjugate in $\pi_1(\Sigma, x_\star)$. Reciprocally, assume that $[wb_{\mathrm{g}}] = [w'b_{\mathrm{g}}]$. We may find smooth paths γ and γ' representing respectively the elements wb_{g} and $w'b_{\mathrm{g}}$, with $i(\gamma,\gamma_\star) = i(\gamma',\gamma_\star) = 1$ and such that the intersections $\gamma \cap \gamma_\star$ and $\gamma' \cap \gamma_\star$ are transverse. As $[wb_{\mathrm{g}}] = [w'b_{\mathrm{g}}]$, the loops γ and γ' lie in the same free homotopy class. Thus there is a smooth homotopy $H\colon [0,1] \times \mathbb{R}/\mathbb{Z} \to \Sigma$ such that $H(0,\cdot) = \gamma$ and $H(1,\cdot) = \gamma'$. We may assume that H is transverse to $\gamma_\star$ (see for example [20, Corollary, page 73]) in the sense that

$$\mathrm{d}H(s,\tau)(T_{(s,\tau)}([0,1]\times\mathbb{R}/\mathbb{Z})) + T_{H(s,\tau)}\gamma_\star = T_{H(s,\tau)}\Sigma \quad \text{for } H(s,\tau) \in \gamma_\star.$$

In particular, $H^{-1}(\gamma_\star)$ is a smooth submanifold of $[0,1]\times\mathbb{R}/\mathbb{Z}$. As γ and γ' intersect $\gamma_\star$ transversally exactly once, $H^{-1}(\gamma_\star) \cap (\{j\}\times\mathbb{R}/\mathbb{Z}) = \{j\}\times\{[0]\}$ for $j = 0,1$ (here $[0]$ is sent to $x_\star$ by both γ and γ'). Thus, necessarily, there exists an embedding $F\colon [0,1] \to [0,1]\times\mathbb{R}/\mathbb{Z}$ such that $\mathrm{Im}(F) \subset H^{-1}(\gamma_\star)$ and $F(j) = (j,[0])$ for $j = 0,1$ (see Figure 3). Write $F = (S,T)$, and define

$$\widetilde{H}(s,t) = H\big(S(s), [T(s)+t]\big) \quad \text{for } (s,t) \in [0,1]\times[0,1].$$

It is immediate to check that $\widetilde{H}$ realizes a homotopy between γ and γ', and we have $\widetilde{H}(s,0) = H(F(s)) \in \gamma_\star$ for any $s \in [0,1]$. For any s, let us denote by c_s the path $[0,1] \ni u \mapsto \widetilde{H}(su,0)$ which links $x_\star$ to $H\big(S(s),[T(s)]\big)$ within $\gamma_\star$. The continuous family of paths $s \mapsto \gamma_s$, where γ_s is given by the concatenation $c_s^{-1}\widetilde{H}(s,\cdot)c_s$, realizes a continuous interpolation between $\gamma_0 = \gamma$ and $\gamma_1 = c_1^{-1}\gamma' c_1$. As $S(1) = 1$ and $T(1) = [0]$ we have $c_1(0) = c_1(1) = x_\star$, and since $c_1(u) \in \gamma_\star$ for each $u \in [0,1]$ we get $c_1 = a_{\mathrm{g}}^{-n}$ for some $n \in \mathbb{Z}$. This yields $wb_{\mathrm{g}} = a_{\mathrm{g}}^n w' b_{\mathrm{g}} a_{\mathrm{g}}^{-n}$ in $\pi_1(\Sigma)$, and thus $w = a_{\mathrm{g}}^n w' a_{\mathrm{g}}'^{-n}$, where the equality stands in $\pi_1(\Sigma)$. By Lemma 4.1, this equality actually holds in $\pi_1(\Sigma_\star)$. $\square$

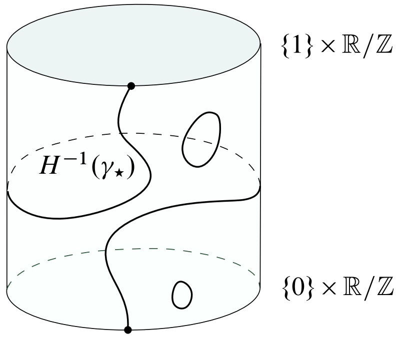

Figure 3: Proof of Lemma 4.3. The path linking $(0,[0]) \in \{0\} \times \mathbb{R}/\mathbb{Z}$ to $(1,[0])$ is the image of F.

Proof of Proposition 4.2 In what follows, C is a constant that may change at each line. For any $w \in \pi_1(\Sigma_\star)$ and $n \in \mathbb{Z}$, by (4-4),

$$\ell_\star(a_g^n w a_g'^{-n}) \geqslant \frac{1}{D} \operatorname{wl}(a_g^n w a_g'^{-n}) - D. \tag{4-6}$$

Let w' be the unique reduced word such that $w' = w a_g'^{-n}$. Then write $w' = a_g^{-k} w''$ for some w'', where $|k|$ is maximal, and note that necessarily $|k| \leqslant \operatorname{wl}(w) + 1$, since $a_g' = [a_1, b_1] \cdots [a_{g-1}, b_{g-1}] a_g$. Then

$$\operatorname{wl}(a_g^n w a_g'^{-n}) = |n| - |k| + \operatorname{wl}(w'') = |n| - 2|k| + \operatorname{wl}(w') \geqslant |n| - 2(\operatorname{wl}(w)+1) + \operatorname{wl}(w').$$

Now the triangle inequality for wl gives $(4(g-1)+1)|n| = \operatorname{wl}(a_g'^{-n}) \leqslant \operatorname{wl}(w') + \operatorname{wl}(w^{-1})$, and thus we obtain $\operatorname{wl}(a_g^n w a_g'^{-n}) \geqslant C|n| - C\operatorname{wl}(w) - C$ for each n. Injecting this in (4-6) yields (for some different C)

$$\ell_\star(a_g^n w a_g'^{-n}) \geqslant C|n| - C\operatorname{wl}(w) - C \quad \text{for } n \in \mathbb{Z}.$$

In particular, for any L and w such that $\ell_\star(w) \leqslant L$, by (4-4),

$$|\{n \in \mathbb{Z} : \ell_\star(a_g^n w a_g'^{-n}) \leqslant L\}| \leqslant CL + C. \tag{4-7}$$

Now, for $w \in \pi_1(\Sigma_\star)$ set $\mathcal{C}_w = \{a_g^n w a_g'^{-n} : n \in \mathbb{Z}\} \subset \pi_1(\Sigma_\star)$, and denote by $\mathscr{C}$ the set $\{\mathcal{C}_w : w \in \pi_1(\Sigma_\star)\}$. For $\mathcal{C} \in \mathscr{C}$, we set $\ell_\star(\mathcal{C}) = \inf_{w \in \mathcal{C}} \ell_\star(w)$. Then by Lemma 4.3, we have a well-defined and injective map

$$\{\mathcal{C} \in \mathscr{C} : \ell_\star(\mathcal{C}) \leqslant L\} \to \{\gamma \in \mathcal{P}_1 : \ell(\gamma) \leqslant L + C\}, \quad \mathcal{C}_w \mapsto [w b_g],$$

where $\mathcal{P}_1$ denotes the set of primitive geodesics γ such that $i(\gamma, \gamma_\star) = 1$.[10] In particular we get, with (4-7) and (4-5),

$$N(1,L) \geqslant |\{\mathcal{C} \in \mathscr{C} : \ell_\star(\mathcal{C}) \leqslant L - C\}| \geqslant \frac{1}{CL+C} \sum_{\substack{\mathcal{C} \in \mathscr{C} \\ \ell_\star(\mathcal{C}) \leqslant L - C}} |\{w \in \mathcal{C} : \ell_\star(w) \leqslant L - C\}| \tag{4-8}$$

$$= \frac{1}{CL+C} |\{w \in \pi_1(\Sigma_\star) : \ell_\star(w) \leqslant L - C\}| \geqslant \frac{1}{CL+C} \exp(h_\star(L-C)),$$

where the equality comes from the fact that $\pi_1(\Sigma_\star)$ is the disjoint union of the subsets $\mathcal{C}$ with $\mathcal{C} \in \mathscr{C}$. □

[10] Each class $[w b_g]$ defines a geodesic in $\mathcal{P}_1$. Indeed, it follows from Lemma 2.1 that $i([w b_g], \gamma_\star) \leqslant 1$. On the other hand, the absolute value of the algebraic intersection number between $w b_g$ and a_g is 1, and this implies that there is at least one intersection point between $[w b_g]$ and $\gamma_\star$, since the algebraic intersection number is preserved by free homotopies.

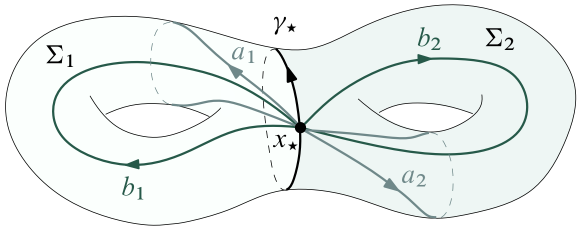

Figure 4: The generators $a_1, b_1, \dots, a_g, b_g$ of $\pi_1(\Sigma)$. Here $\gamma_\star$ is assumed to be separating and $g_1 = g_2 = 1$.

4.1.2 Upper bound Each $\gamma \in \mathcal{P}_1$ with $\ell(\gamma) \leqslant L$ lies in the free homotopy class of $w' b_g^{\pm 1}$ for some $w' \in \pi_1(\Sigma_\star, x_\star)$ and $\ell_\star(w) \leqslant L + C$. In particular, (4-5) gives the bound

$$N(1, L) \leqslant C \exp(h_\star L)$$

for large L. Now let $\gamma \in \mathcal{P}_2$ with $\ell(\gamma) \leqslant L$. Then we may find a deformation of the loop γ into a loop γ' which is represented by the conjugacy class of $w b_g^{\pm 1} w' b_g^{\pm 1}$ in $\pi_1(\Sigma)$ for some $w, w' \in \pi_1(\Sigma_\star)$. This deformation can be made so that $\ell_\star(w) + \ell_\star(w') \leqslant L + C$. Thus,

$$N(2, L) \leqslant C \sum_{\substack{w, w' \in \pi_1(\Sigma_\star) \\ \ell_\star(w) + \ell_\star(w') \leqslant L + C}} 1 \leqslant \sum_{k=0}^{L+C} C \exp(h_\star k) C \exp(h_\star (L + C - k)) \leqslant C' L \exp(h_\star L).$$

Iterating this process, we finally get, for large L,

$$N(n, L) \leqslant C L^{n-1} \exp(h_\star L).$$

4.2 The case $\gamma_\star$ is separating

In this section we assume $\gamma_\star$ is separating, and we write $\Sigma \setminus \gamma_\star = \Sigma_1 \sqcup \Sigma_2$, where the surfaces Σ_j are connected. Up to applying a diffeomorphism to Σ, we may assume that $\gamma_\star$ represents the class

$$[a_1, b_1] \cdots [a_{g_1}, b_{g_1}] = [a_g, b_g]^{-1} \cdots [a_{g_1+1}, b_{g_1+1}]^{-1} \in \pi_1(\Sigma) \tag{4-9}$$

(see Figure 4). Here g_1 is the genus of the surface Σ_1, and the genus g_2 of Σ_2 satisfies $g_1 + g_2 = g$.

We set $\pi_1(\Sigma) = \pi_1(\Sigma, x_\star)$ and $\pi_1(\Sigma_j) = \pi_1(\Sigma_j, x_\star)$ for $j = 1, 2$ (we see Σ_j as a compact surface with boundary $\gamma_\star$ so that $x_\star$ lives on both surfaces). Then $\pi_1(\Sigma_1)$ and $\pi_1(\Sigma_2)$ are the free groups generated by $a_1, b_1, \dots, a_{g_1}, b_{g_1}$ and $a_{g_1+1}, b_{g_1+1}, \dots, a_g, b_g$, respectively, and we denote by $w_{\star,1}$ and $w_{\star,2}$ the two natural words given by (4-9) representing $\gamma_\star$ in $\pi_1(\Sigma_1)$ and $\pi_1(\Sigma_2)$, respectively. Note that we have a well-defined map

$$\pi_1(\Sigma_1) \times \pi_1(\Sigma_2) \to \pi_1(\Sigma), \quad (w_1, w_2) \mapsto w_2 w_1,$$

given by the composition of two curves.

Lemma 4.4 *For $j=1,2$, the map $q_{j,*}\colon \pi_1(\Sigma_j)\to\pi_1(\Sigma)$ induced by the inclusion $\Sigma_j\hookrightarrow\Sigma$ is injective.*

Proof For $j=1,2$ let $\langle w_{\star,j}\rangle$ be the infinite cyclic group of $\pi_1(\Sigma_j)$ generated by $w_{\star,j}$, and let $\phi\colon\langle w_{\star,1}\rangle\to\langle w_{\star,2}\rangle$ be the isomorphism given by $\phi(w_{\star,1})=w_{\star,2}$. By (4-1), the group $\pi_1(\Sigma)$ is the free product with amalgamation $\pi_1(\Sigma_1)*_\phi\pi_1(\Sigma_2)$, that is, the finitely presented group given by

$$\pi_1(\Sigma_1)*_\phi\pi_1(\Sigma_2)=\{a_1,b_1,\dots,a_g,b_g:w_{\star,1}=\phi(w_{\star,1})\};$$

see [30, Section IV.2]. With this representation, the map $q_{j,*}$ coincides with the natural map $\pi_1(\Sigma_j)\to\pi_1(\Sigma_1)*_\phi\pi_1(\Sigma_2)$, which is injective by [30, Theorem IV.2.6]. □

For any $w\in\pi_1(\Sigma)$, we will denote by $[w]$ its conjugacy class and by γ_w the unique geodesic of Σ such that γ_w is isotopic to any curve in w (in fact we will often identify $[w]$ and γ_w). Let $(\widetilde\Sigma,\tilde g)$ be the universal cover of (Σ,g), and choose $\tilde x_\star\in\widetilde\Sigma$ some lift of $x_\star$. Then $\pi_1(\Sigma)$ acts as deck transformations on $\widetilde\Sigma$ and we will write

$$\ell_\star(w)=\mathrm{dist}_{\widetilde\Sigma}(\tilde x_\star,w\tilde x_\star)\quad\text{for } w\in\pi_1(\Sigma).$$

As in the preceding subsection, we have the orbital counting

$$\#\{w_j\in\pi_1(\Sigma_j):\ell_\star(w_j)\leqslant L\}\sim A_je^{h_jL}\quad\text{as } L\to\infty \text{ for } j=1,2 \tag{4-10}$$

for some $A_1,A_2>0$, where $h_j>0$ is the topological entropy of the geodesic flow restricted to the trapped set

$$K_j=\overline{\{(x,v)\in S\Sigma_j^\circ:\varphi_t(x,v)\in S\Sigma_j^\circ \text{ for } t\in\mathbb{R}\}},$$

where $\Sigma_j^\circ=\Sigma_j\setminus\partial\Sigma_j$ for $j=1,2$.

4.2.1 Lower bound Unlike the case where $\gamma_\star$ is not separating, we will need a better lower bound. Namely, we prove here the following result:

Proposition 4.5 *Assume that $\gamma_\star$ is separating and that $h_1=h_2=h_\star$. Then there is $C>0$ such that, for L large enough,*

$$N(2,L)\geqslant\frac{CLe^{h_\star L}}{\log(L)^4}. \tag{4-11}$$

If $h_1\neq h_2$ we have, for L large enough and $h_\star=\max(h_1,h_2)$,

$$N(2,L)\geqslant\frac{Ce^{h_\star L}}{\log(L)^2}. \tag{4-12}$$

Note that Theorem 2 gives $N(2,L)\sim CLe^{h_\star L}$ if $h_1=h_2$ and $N(2,L)\sim Ce^{h_\star L}$ if $h_1\neq h_2$. In particular, Proposition 4.5 gives a bound which is sharp up to a logarithmic loss, whereas in Proposition 4.2, we had a linear loss. Indeed, obtaining a sharper bound is important here, because a linear defect would not be sufficient to obtain Theorem 2 in the case $h_1=h_2$ — at least with our methods. If $h_1\neq h_2$, a linear loss would nevertheless be sufficient, but our proof of (4-11) actually gives (4-12) without too much effort. We refer to Remarks 5.2, 5.3 and 5.4 for a more detailed discussion about the importance of (4-11).

The strategy to prove Proposition 4.5 is the following. We wish to construct enough closed geodesics intersecting $\gamma_\star$ exactly twice by considering conjugacy classes of the form $[w_2w_1]$ where $w_j \in \pi_1(\Sigma_j)$ for $j = 1, 2$. Lemma 4.6 will tell us that, if w_j is not a power of $w_{\star,j}$ for $j = 1, 2$, then the closed geodesic representing $[w_2w_1]$ indeed intersects $\gamma_\star$ exactly twice. Next, in Lemma 4.7, we describe the injectivity defect of the map $(w_1, w_2) \mapsto [w_2w_1]$. Finally, in Proposition 4.8, we show that this injectivity defect is not too harmful in the sense that there are not too many $w_j, w'_j \in \pi_1(\Sigma_j)$ such that $[w_2w_1] = [w'_2w'_1]$. This will allow us to obtain the desired bound with a logarithmic loss.

Lemma 4.6 *For two elements $w_j \in \pi_1(\Sigma_j)$ for $j = 1, 2$, we have $i(\gamma_{w_2w_1}, \gamma_\star) = 2$ except if $w_j = w_{\star,j}^k$ in $\pi_1(\Sigma_j)$ for some $k \in \mathbb{Z}$ and $j \in \{1, 2\}$, in which case $i(\gamma_{w_2w_1}, \gamma_\star) = 0$.*

Proof Let $\gamma\colon \mathbb{R}/\mathbb{Z} \to \Sigma$ be a smooth curve in the free homotopy class of w_2w_1 such that

$$\{\tau \in \mathbb{R}/\mathbb{Z} : \gamma(\tau) \in \gamma_\star\} = \{\tau_1, \tau_2\} \quad \text{for some } \tau_1 \neq \tau_2 \in \mathbb{R}/\mathbb{Z}.$$

We may also choose γ so that $\gamma|_{[\tau_1,\tau_2]}$ (resp. $\gamma|_{[\tau_2,\tau_1]}$) is homotopic to some representative $\gamma_1\colon [0,1] \to \Sigma_1$ of w_1 (resp. some representative $\gamma_2\colon [0,1] \to \Sigma_2$ of w_2) relative to $\gamma_\star$, meaning that there is a homotopy between $\gamma|_{[\tau_1,\tau_2]}$ and γ_1 with endpoints (not necessarily fixed) in $\gamma_\star$. Here $[\tau_1, \tau_2] \subset \mathbb{R}/\mathbb{Z}$ is the interval linking τ_1 and τ_2 in the counterclockwise direction.

As $\gamma_{w_2w_1}$ minimizes the quantity $i(\gamma, \gamma_\star)$ for $\gamma \in [\gamma_{w_2w_1}]$ (see Lemma 2.1) we have either $i(\gamma_{w_2w_1}, \gamma_\star) = 0$ or $i(\gamma_{w_2w_1}, \gamma_\star) = 2$. If $i(\gamma_{w_2w_1}, \gamma_\star) = 0$, then there exists a homotopy $H\colon [0,1] \times \mathbb{R}/\mathbb{Z} \to \Sigma$ such that $H(0,\cdot) = \gamma$ and $H(1,\cdot) = \gamma$, so that $H(1,\tau) \notin \gamma_\star$ for any τ. As in the proof of Lemma 4.3, we may assume that H is transverse to $\gamma_\star$, in the sense that

$$\mathrm{d}H(s,\tau)(T_{(s,\tau)}([0,1] \times \mathbb{R}/\mathbb{Z})) + T_{H(s,\tau)}\gamma_\star = T_{H(s,\tau)}\Sigma \quad \text{for } H(s,\tau) \in \gamma_\star,$$

so that the preimage

$$H^{-1}(\gamma_\star) \subset [0,1] \times \mathbb{R}/\mathbb{Z}$$

is an embedded submanifold of $[0,1] \times \mathbb{R}/\mathbb{Z}$ (see Figure 5). As $H^{-1}(\gamma_\star) \cap \{s = 0\} = \{\tau_1, \tau_2\}$ and $H^{-1}(\gamma_\star) \cap \{s = 1\} = \varnothing$, it follows that there is an embedding $F\colon [0,1] \to [0,1] \times \mathbb{R}/\mathbb{Z}$ such that $F(0) = (0, \tau_1)$, $F(1) = (0, \tau_2)$ and

$$F(t) \in H^{-1}(\gamma_\star) \quad \text{for } t \in [0,1].$$

As F is an embedding, F is homotopic (by a homotopy which preserves the endpoints) either to $J_{[\tau_1,\tau_2]}$ or to $J_{[\tau_2,\tau_1]}$, where $J_{[\tau,\tau']}\colon [0,1] \to [0,1] \times \mathbb{R}/\mathbb{Z}$ is the natural map that sends $[0,1]$ to $\{0\} \times [\tau, \tau']$. We may assume without loss of generality that $F \sim J_{[\tau_1,\tau_2]}$. In particular, writing $F = (S, T)$, the map T is homotopic to $I_{[\tau_1,\tau_2]} = p_2 \circ J_{[\tau_1,\tau_2]}$, where $p_2\colon [0,1] \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is the projection over the second factor. This means that there is $G\colon [0,1] \times [0,1] \to \mathbb{R}/\mathbb{Z}$ such that, for any $s, t \in [0,1]$,

$$G(s,0) = \tau_1, \quad G(s,1) = \tau_2, \quad G(0,t) = \tau_1 + t(\tau_2 - \tau_1) \quad \text{and} \quad G(1,t) = T(t).$$

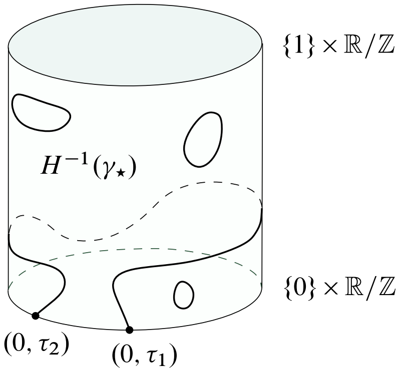

Figure 5: Proof of Lemma 4.6. The path linking $(0,\tau_1)$ to $(0,\tau_2)$ is the image of F.

Now we set $\widetilde{H}(s,t) = H(sS(t), G(s,t))$ for $s,t\in[0,1]$. Then

$$\widetilde{H}(0,t)=\gamma(\tau_1+t(\tau_2-\tau_1)) \quad\text{and}\quad \widetilde{H}(1,t)=(H\circ F)(t) \qquad \text{for } t\in[0,1],$$
$$\widetilde{H}(s,0)=H(0,\tau_1)=x_1 \qquad\text{and}\quad \widetilde{H}(s,1)=H(0,\tau_2)=x_2 \quad \text{for } s\in[0,1].$$

We conclude that $t\mapsto \gamma|_{[\tau_1,\tau_2]}(\tau_1+t(\tau_2-\tau_1))$, and thus γ_1, is homotopic (relative to $\gamma_\star$) to some curve contained in $\gamma_\star$. Thus $w_1=w_\star^k$, for some $k\in\mathbb{Z}$, in $\pi_1(\Sigma)$. As the inclusion $\pi_1(\Sigma_j)\to\pi_1(\Sigma)$ is injective by Lemma 4.4, the lemma follows. □

Now, we need to understand when the geodesics given by $[w_2w_1]$ and $[w_2'w_1']$ are the same. This is the purpose of the following:

Lemma 4.7 *Take $w_j, w_j'\in\pi_1(\Sigma_j)$ for $j=1,2$ such that $i(\gamma_{[w_2w_1]},\gamma_\star)=2$. Then $[w_2w_1]=[w_2'w_1']$ as conjugacy classes of $\pi_1(\Sigma)$ if and only if there are $p,q\in\mathbb{Z}$ such that*

$$w_2=w_{\star,2}^p w_2' w_{\star,2}^q \quad\textit{and}\quad w_1=w_{\star,1}^{-q}w_1'w_{\star,1}^{-p}. \tag{4-13}$$

Proof Again, let $\gamma:\mathbb{R}/\mathbb{Z}\to\Sigma$ be a smooth curve intersecting $\gamma_\star$ transversely such that

$$\{\tau\in\mathbb{R}/\mathbb{Z}:\gamma(\tau)\in\gamma_\star\}=\{\tau_1,\tau_2\} \quad \text{for some } \tau_1\neq\tau_2\in\mathbb{R}/\mathbb{Z},$$

with $\gamma([\tau_1,\tau_2])\subset\Sigma_1$ and $\gamma([\tau_2,\tau_1])\subset\Sigma_2$. Let $x_j=\gamma(\tau_j)$ for $j=1,2$, and chose arbitrary paths c_j contained in $\gamma_\star$ linking x_j to $x_\star$. Note that all the preceding choices can be made so that the curve $\gamma_1=c_2\gamma|_{[\tau_1,\tau_2]}c_1^{-1}$ (resp. $\gamma_2=c_1\gamma|_{[\tau_2,\tau_1]}c_2^{-1}$) represents $w_\star^p w_1 w_\star^q$ (resp. $w_\star^{-q}w_2w_\star^{-p}$) for some $p,q\in\mathbb{Z}$. We may proceed in the same way to obtain γ', τ_1', τ_2', c_1', c_2', p' and q' so that the same properties hold with w_1 and w_2 replaced by w_1' and w_2'. By hypothesis, γ is freely homotopic to γ'. Thus we may find a smooth map $H:[0,1]\times\mathbb{R}/\mathbb{Z}\to\Sigma$ such that $H(0,\cdot)=\gamma$ and $H(1,\cdot)=\gamma'$. As in Lemma 4.6, H may be chosen to be transverse to $\gamma_\star$, so that

$$H^{-1}(\gamma_\star)\subset[0,1]\times\mathbb{R}/\mathbb{Z}$$

is a finite union of smooth embedded submanifolds of $[0,1]\times\mathbb{R}/\mathbb{Z}$. Let $(x,\rho)\colon\Sigma\to\mathbb{R}/\mathbb{Z}\times(-\varepsilon,\varepsilon)$ be coordinates near $\gamma_\star$ such that $\{\rho=0\}=\gamma_\star$ and $|\rho|=\operatorname{dist}(\gamma_\star,\cdot)$, and such that $\{(-1)^{j-1}\rho\geqslant 0\}\subset\Sigma_j$. As $H^{-1}(\gamma_\star)\cap\{s=0\}=\{\tau_1,\tau_2\}$ and $H^{-1}(\gamma_\star)\cap\{s=1\}=\{\tau_1',\tau_2'\}$, we have two smooth embeddings $F_1,F_2\colon[0,1]\to[0,1]\times\mathbb{R}/\mathbb{Z}$ such that $F_j([0,1])\subset H^{-1}(\gamma_\star)$ and $F_j(0)=(0,\tau_j)$ for $j=1,2$, with $F_j(1)=\tau_1'$ or τ_2' (indeed we have $i(\gamma,\gamma_\star)=2$ and thus there is a path in $H^{-1}(\gamma_\star)$ linking $\{s=0\}$ to $\{s=1\}$, since otherwise we could proceed as in the proof of Lemma 4.6 to obtain that $i(\gamma,\gamma_\star)=0$). In fact, $F_1(1)=(1,\tau_1')$ and $F_2(1)=(1,\tau_2')$, which we shall prove later. Set $F_j=(S_j,T_j)$ and

$$\widetilde{H}(s,t)=H\big((1-t)S_1(s)+tS_2(s),T_1(s)+t(T_2(s)-T_1(s))\big)\quad\text{for } s,t\in[0,1].$$

Then

$$\begin{aligned}\widetilde{H}(0,t)&=\gamma(\tau_1+t(\tau_2-\tau_1)) &&\text{and}\quad \widetilde{H}(1,t)=\gamma'(\tau_1'+t(\tau_2'-\tau_1'))&&\text{for } t\in[0,1],\\ \widetilde{H}(s,0)&=H(S_1(s),T_1(s)) &&\text{and}\quad \widetilde{H}(s,1)=H(S_2(s),T_2(s))&&\text{for } s\in[0,1].\end{aligned}$$

For $j=1,2$, let $c_j(s)$, $s\in[0,1]$ be paths, contained in $\gamma_\star$ depending continuously on s and linking $T_j(s)$ to $x_\star$, such that $c_j(0)=c_j$. Then the construction of $\widetilde{H}$ shows that

$$c_2(0)\gamma|_{[\tau_1,\tau_2]}c_1(0)^{-1}\sim c_2(1)\gamma'|_{[\tau_1',\tau_2']}c_1(1)^{-1},$$

and reversing the role of τ_1 and τ_2 in the constructions made above,

$$c_1(0)\gamma|_{[\tau_2,\tau_1]}c_2(0)^{-1}\sim c_1(1)\gamma'|_{[\tau_2',\tau_1']}c_2(1)^{-1}.$$

Thus we obtain

$$w_\star^p w_1 w_\star^q=c_2(1)c_2'^{-1}w_\star^{p'}w_1'w_\star^{q'}c_1'c_1(1)^{-1}\quad\text{and}\quad w_\star^{-q}w_2w_\star^{-p}=c_1(1)c_1'^{-1}w_\star^{-q'}w_2w_\star^{-p'}c_2'c_2(1)^{-1},$$

which is the conclusion of Lemma 4.7 as the paths $c_1(1)c_1'^{-1}$ and $c_2(1)c_2'^{-1}$ are contained in $\gamma_\star$ (and, again, the inclusions $\pi_1(\Sigma_j)\to\pi_1(\Sigma)$ for $j=1,2$ are injective).

Thus it remains to show that $F_j(1)=(1,\tau_j')$ for $j=1,2$. We extend ρ into a smooth function $\rho\colon\Sigma\to\mathbb{R}$ such that $(-1)^{j-1}\rho>0$ on $\Sigma_j\setminus\gamma_\star$. There exists a continuous path $G\colon[0,1]\to([0,1]\times\mathbb{R}/\mathbb{Z})\setminus H^{-1}(\gamma_\star)$ such that

$$G(0)\in\{0\}\times\,]\tau_1,\tau_2[\quad\text{and}\quad G(1)\in\{1\}\times(\mathbb{R}/\mathbb{Z}\setminus\{\tau_1',\tau_2'\}).$$

(Indeed, otherwise it would mean that there is a continuous path in $[0,1]\times\mathbb{R}/\mathbb{Z}$ linking $(0,\tau_1)$ to $(0,\tau_2)$, which would imply, as in Lemma 4.6, that $i(\gamma,\gamma_\star)=0$.) In particular, $\rho\circ H\circ G>0$ since $\rho(H(0,\tau))>0$ for $\tau\in\,]\tau_1,\tau_2[$. Thus necessarily $G(1)\in\{1\}\times\,]\tau_1',\tau_2'[$, since $\rho(H(1,\tau))<0$ for $\tau\in\,]\tau_2',\tau_1'[$. Now, as $\operatorname{Im}(F_1)\cap\operatorname{Im}(F_2)=\varnothing$ (again, if the intersection was not empty we could find a path linking $(0,\tau_1)$ to $(0,\tau_2)$), we have that $G(1)$ lies in $]T_1(1),T_2(1)[$. Since $(\rho\circ H\circ G)(1)>0$, it follows that $T_1(1)=\tau_1'$ and $T_2(1)=\tau_2'$. □

The above lemma motivates the next result:

Proposition 4.8 *There is a constant $C > 0$ such that the following holds. For any $w \in \pi_1(\Sigma_j)$ such that w is not a power of $w_{\star,j}$, there are $p_w, q_w \in \mathbb{Z}$ such that if $w' = w_{\star,j}^{p_w} w w_{\star,j}^{q_w}$,*

$$\ell_\star(w_{\star,j}^{p} w' w_{\star,j}^{q}) \geqslant (|p|+|q|)\ell(\gamma_\star) + \ell_\star(w') - C \quad \textit{for } p, q \in \mathbb{Z}. \tag{4-14}$$

In what follows, for any $x, y \in \widetilde{\Sigma}$ we will denote by $[x, y]$ the unique geodesic segment joining x and y. Before starting the proof of Proposition 4.8, we state a classical result valid in negatively curved spaces:

Lemma 4.9 *For each $\delta > 0$ there exists a constant $C > 0$ such that the following holds. For any sequence of geodesic segments $[x_0, x_1], [x_1, x_2], [x_2, x_3]$ in $\widetilde{\Sigma}$ such that $\operatorname{dist}(x_1, x_2) \geqslant \delta$ and such that the angle between $[x_{j-1}, x_j]$ and $[x_j, x_{j+1}]$ is equal to $\pm\frac{1}{2}\pi$ for $j = 1, 2$,*

$$\operatorname{dist}(x_0, x_3) \geqslant \operatorname{dist}(x_0, x_1) + \operatorname{dist}(x_1, x_2) + \operatorname{dist}(x_2, x_3) - C. \tag{4-15}$$

We will need the following intermediate result:

Fact 4.10 *For any $\varepsilon > 0$ there is $C > 0$ such that, for any pairwise distinct points $x, y, z \in \widetilde{\Sigma}$ such that the absolute value of the angle (taken in $]-\pi, \pi]$) between $[x, y]$ and $[y, z]$ is not smaller than ε, we have*

$$\operatorname{dist}(x, z) \geqslant \operatorname{dist}(x, y) + \operatorname{dist}(y, z) - C.$$

Proof We prove the result by comparing $\widetilde{\Sigma}$ with a model space of constant curvature, as follows. Let $a = \operatorname{dist}(x, y)$, $b = \operatorname{dist}(y, z)$, $c = \operatorname{dist}(x, z)$ and $\gamma = \angle([x, y], [y, z])$. Let $\widetilde{\Sigma}_k$ be a simply connected complete Riemannian surface with constant curvature $-k^2 < 0$ such that $\kappa \leqslant -k^2$ everywhere for some $k > 0$ (recall that κ is the curvature of Σ). Consider any points $\bar{x}, \bar{y}, \bar{z} \in \widetilde{\Sigma}_k$ such that

$$\operatorname{dist}_k(\bar{x}, \bar{y}) = a, \quad \operatorname{dist}_k(\bar{y}, \bar{z}) = b \quad \text{and} \quad \angle([\bar{x}, \bar{y}], [\bar{y}, \bar{z}]) = \gamma,$$

where dist_k is the distance in $\widetilde{\Sigma}_k$, and set $\bar{c} = \operatorname{dist}_k(x, z)$. Then by a classical trigonometric formula for spaces of constant negative curvature (see [11, I.2.7]),

$$\operatorname{ch}(k\bar{c}) = \operatorname{ch}(ka)\operatorname{ch}(kb) - \operatorname{sh}(ka)\operatorname{sh}(kb)\cos(\gamma).$$

As $\gamma \in]-\pi, \pi] \setminus]-\varepsilon, \varepsilon[$, we have $\cos(\gamma) \leqslant 1 - \eta$ for some $\eta \in]0, 1[$ depending on ε. Thus

$$\operatorname{ch}(k\bar{c}) \geqslant \eta \operatorname{ch}(ka)\operatorname{ch}(kb).$$

Using $\frac{1}{2}\exp(t) \leqslant \operatorname{ch}(t) \leqslant \exp(t)$ for $t \geqslant 0$, one gets

$$\bar{c} \geqslant a + b + \frac{\log\left(\frac{1}{4}\eta\right)}{k}.$$

As the scalar curvature of $\widetilde{\Sigma}$ is everywhere not greater than $-k^2$, the space $\widetilde{\Sigma}$ is a CAT$(-k^2)$ space; see [11, Theorem II.4.1]. In particular, by comparison, one obtains $c \geqslant \bar{c}$ (see [11, Proposition II.1.7]), which concludes the proof. □

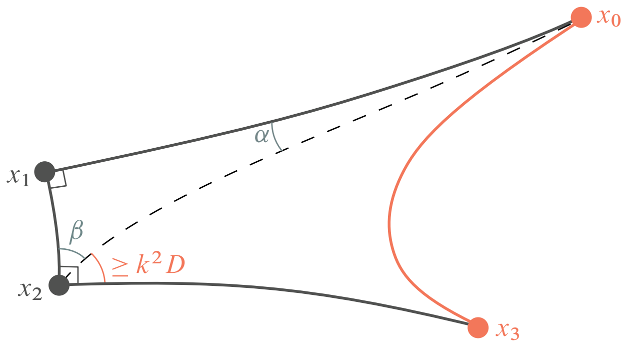

Figure 6: Proof of Lemma 4.9.

Proof of Lemma 4.9 Let x_0, x_1, x_2 and x_3 be as in the statement. For $j = 0, 1, 2$ we set $d_j = \operatorname{dist}(x_j, x_{j+1})$. We first assume one of the numbers d_0 or d_2 is not greater than δ, say $d_0 \leqslant \delta$. Then Fact 4.10 (applied with $x = x_1$, $y = x_2$ and $z = x_3$) yields $\operatorname{dist}(x_1, x_3) \geqslant d_1 + d_2 - C$, and thus

$$\operatorname{dist}(x_0, x_3) \geqslant \operatorname{dist}(x_1, x_3) - \operatorname{dist}(x_0, x_1) \geqslant d_1 + d_2 + C - d_0 \geqslant d_0 + d_1 + d_2 + C - 2\delta.$$

Therefore we may assume that $d_0, d_2 \geqslant \delta$. Applying Fact 4.10 for the points x_0, x_1 and x_2 yields

$$\operatorname{dist}(x_0, x_2) \geqslant d_0 + d_1 - C. \tag{4-16}$$

For any pairwise distinct $x, y, z \in \widetilde{\Sigma}$, we denote by $\Delta(x, y, z)$ the triangle generated by x, y and z. Then as $d_0, d_1 \geqslant \delta$, the triangle $\Delta(x_0, x_1, x_2)$ contains some triangle $\Delta(x, y, z)$ with a right angle at y and $\operatorname{dist}(x, y) = \operatorname{dist}(y, z) = \delta$ (namely, $y = x_1$, $x \in [x_1, x_0]$ and $z \in [x_1, x_2]$). Clearly the area $|\Delta(x, y, z)|$ of $\Delta(x, y, z)$ is bounded from below by some constant $D > 0$ depending only on $\delta > 0$ (indeed, it suffices to verify this property for x, y and z lying in a compact set given by a finite union of fundamental domains of Σ). Therefore, $|\Delta(x_0, x_1, x_2)| \geqslant D$. Let α and β be the angles of $\Delta(x_0, x_1, x_2)$ at x_0 and x_2, respectively (see Figure 6). Let $\tilde{\mu}_g$ bet the Riemannian measure of $\widetilde{\Sigma}$, and $\tilde{\kappa}$ its scalar curvature. Then, by the Gauss–Bonnet formula [29, Theorem 9.3],

$$\int_{\Delta(x_0, x_1, x_2)} \tilde{\kappa} \, d\tilde{\mu}_g + \tfrac{1}{2}\pi + (\pi - \alpha) + (\pi - \beta) = 2\pi.$$

This gives

$$\beta \leqslant \tfrac{1}{2}\pi - \alpha - k^2 |\Delta(x_0, x_1, x_2)| \leqslant \tfrac{1}{2}\pi - k^2 D.$$

Therefore the angle between $[x_0, x_2]$ and $[x_2, x_3]$ is not smaller than $k^2 D$. In particular, we may apply Fact 4.10 to get $\operatorname{dist}(x_0, x_3) \geqslant \operatorname{dist}(x_0, x_2) + d_2 - C$ for some C depending only on $k^2 D$. Combining this with (4-16), we conclude the proof. □

Proof of Proposition 4.8 We fix $j \in \{1, 2\}$ and write $w_\star = w_{\star, j}$ for simplicity. Let $w \in \pi_1(\Sigma_j)$ be such that $w \neq w_\star^k$ for any k. Then w is not the trivial element, and thus it is hyperbolic. Recall that $(\widetilde{\Sigma}, \tilde{g})$ is

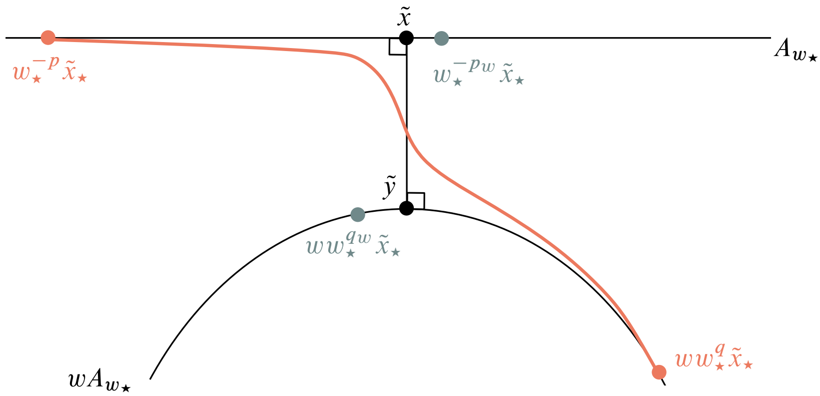

Figure 7: Proof of Proposition 4.8.

the universal cover of (Σ, g) and that $\pi_1(\Sigma)$ acts by deck transformations on $\widetilde{\Sigma}$. For any $u \in \pi_1(\Sigma) \setminus \{1\}$, we denote by

$$u_\pm = \lim_{k \to +\infty} u^{\pm k}(z)$$

the two distinct fixed points of u in the boundary at infinity $\partial_\infty \widetilde{\Sigma}$ of $\widetilde{\Sigma}$ (here z denotes any point in $\widetilde{\Sigma}$). We also denote by A_u the translation axis of u, that is, the unique complete geodesic of $(\widetilde{\Sigma}, \tilde{g})$ converging towards u_+ (resp. u_-) in the future (resp. in the past). Note that $A_{ww_\star w^{-1}} = wA_{w_\star}$. As the conjugacy classes $[ww_\star w^{-1}]$ and $[w_\star]$ both represent the geodesic $\gamma_\star$, we have either $A_{w_\star} = wA_{w_\star}$ or $A_{w_\star} \cap wA_{w_\star} = \varnothing$. Since w is not a power of $w_\star$, we necessarily have $A_{w_\star} \cap wA_{w_\star} = \varnothing$. Write $\gamma_\star = \{\varphi_s(z_\star) : s \in [0, \ell(\gamma_\star)]\}$ for some $z_\star = (x_\star, v_\star) \in M$. By hyperbolicity of the geodesic flow, there is $\delta > 0$ such that the following holds. For any $z \in M$ such that $\inf_{s \in \mathbb{R}} \operatorname{dist}_M(z, \varphi_s(z_\star)) \leqslant \delta$,

$$\varphi_{\ell(\gamma_\star)}(z) = z \implies z = \varphi_s(z_\star) \quad \text{for some } s \in \mathbb{R}. \tag{4-17}$$

As $\ell([ww_\star w^{-1}]) = \ell([w_\star]) = \ell(\gamma_\star)$, we obtain

$$\operatorname{dist}(A_{w_\star}, wA_{w_\star}) \geqslant \delta. \tag{4-18}$$

Let $\tilde{x} \in A_{w_\star}$ and $\tilde{y} \in wA_{w_\star}$ be the unique points such that $\operatorname{dist}(\tilde{x}, \tilde{y}) = \operatorname{dist}(A_{w_\star}, wA_{w_\star})$, and take $p, q \in \mathbb{Z}$. Then $\operatorname{dist}(\tilde{x}, \tilde{y}) \geqslant \delta$ by (4-18), and thus we may apply Lemma 4.9 with the sequence of geodesic segments $[w_\star^{-p}\tilde{x}_\star, \tilde{x}]$, $[\tilde{x}, \tilde{y}]$, $[\tilde{y}, ww_\star^q \tilde{x}_\star]$ to obtain

$$\operatorname{dist}(ww_\star^q \tilde{x}_\star, w_\star^{-p}\tilde{x}_\star) \geqslant \operatorname{dist}(ww_\star^q \tilde{x}_\star, \tilde{y}) + \operatorname{dist}(\tilde{y}, \tilde{x}) + \operatorname{dist}(\tilde{x}, w_\star^{-p}\tilde{x}_\star) - C$$

for some $C > 0$ independent of w, p and q (see Figure 7). Next, let $p_w, q_w \in \mathbb{Z}$ such that

$$\operatorname{dist}(\tilde{x}, w_\star^{-p_w}\tilde{x}_\star) < \ell(\gamma_\star) \quad \text{and} \quad \operatorname{dist}(\tilde{y}, ww_\star^{q_w}\tilde{x}_\star) < \ell(\gamma_\star).$$

Then, for any $p, q \in \mathbb{Z}$,

$$\operatorname{dist}(\tilde{x}, w_\star^{-p}\tilde{x}_\star) \geqslant |p - p_w|\ell(\gamma_\star) - \ell(\gamma_\star) \quad \text{and} \quad \operatorname{dist}(\tilde{y}, ww_\star^q\tilde{x}_\star) \geqslant |q - q_w|\ell(\gamma_\star) - \ell(\gamma_\star),$$

which yields

$$\operatorname{dist}(w_\star^p w w_\star^q \tilde{x}_\star, \tilde{x}_\star) \geqslant (|p - p_w| + |q - q_w|)\ell(\gamma_\star) + \operatorname{dist}(\tilde{x}, \tilde{y}) - C - 2\ell(\gamma_\star).$$

Finally, we note that

$$\operatorname{dist}(\tilde{x}, \tilde{y}) \geqslant \operatorname{dist}(w w_\star^{q_w} \tilde{x}_\star, w_\star^{-p_w} \tilde{x}_\star) - 2\ell(\gamma_\star) = \ell_\star(w_\star^{p_w} w w_\star^{q_w}) - 2\ell(\gamma_\star). \qquad \square$$

Building on Lemmata 4.6 and 4.7 and Proposition 4.8, we prove Proposition 4.5:

Proof of Proposition 4.5 In what follows, C is a positive constant independent of L that may change at each line. First, assume that $h_1 = h_2 = h_\star$. For $j = 1, 2$ we denote by $\langle w_{\star,j} \rangle = \{w_{\star,j}^n : n \in \mathbb{Z}\}$ the infinite cyclic subgroup of $\pi_1(\Sigma_j)$ generated by $w_{\star,j}$, and we set $\pi_1(\Sigma_j)_\star = \pi_1(\Sigma_j) \setminus \langle w_{\star,j} \rangle$. Since $\ell_\star(w_{\star,j}^n) = |n|\ell(\gamma_\star)$, there is C such that, for any large L,

$$C^{-1} e^{h_\star L} \leqslant N_{\star,j}(L) \leqslant C e^{h_\star L} \tag{4-19}$$

by (4-10), where $N_{\star,j}(L) = \#\{w \in \pi_1(\Sigma_j)_\star : \ell_\star(w) \leqslant L\}$. For $w \in \pi_1(\Sigma_j)_\star$, we set

$$\mathcal{C}_w = \{w_\star^p w w_\star^q : p, q \in \mathbb{Z}\} \subset \pi_1(\Sigma_j)_\star,$$

and we define $\mathscr{C}_j = \{\mathcal{C}_w : w \in \pi_1(\Sigma_j)_\star\}$. Note that the elements $\mathcal{C} \in \mathscr{C}_j$ are pairwise disjoint, and thus we have a partition $\bigsqcup_{\mathcal{C} \in \mathscr{C}_j} \mathcal{C}$ of $\pi_1(\Sigma_j)_\star$. We also write

$$\ell_\star(\mathcal{C}) = \inf\{\ell_\star(w) : w \in \mathcal{C}\} \quad \text{for } \mathcal{C} \in \mathscr{C}_j \text{ with } j = 1, 2.$$

Then Proposition 4.8 yields

$$\#\{w \in \mathcal{C} : \ell_\star(w) \leqslant L\} \leqslant C(L - \ell_\star(\mathcal{C}) + C)^2$$

for any $\mathcal{C} \in \mathscr{C}_j$ such that $\ell_\star(\mathcal{C}) \leqslant L$. Thus

$$N_{\star,j}(L) = \sum_{\substack{\mathcal{C} \in \mathscr{C}_j \\ \ell_\star(\mathcal{C}) \leqslant L}} \#\{w \in \mathcal{C} : \ell_\star(w) \leqslant L\} \leqslant C \sum_{\substack{\mathcal{C} \in \mathscr{C}_j \\ \ell_\star(\mathcal{C}) \leqslant L}} (L - \ell_\star(\mathcal{C}) + C)^2.$$

Let $\beta > 0$ be a large number. Then

$$\sum_{\substack{\mathcal{C} \in \mathscr{C}_j \\ \ell_\star(\mathcal{C}) \leqslant L - \beta \log L}} (L - \ell_\star(\mathcal{C}) + C)^2 \leqslant (L + C)^2 \, \#\{\mathcal{C} \in \mathscr{C}_j : \ell_\star(\mathcal{C}) \leqslant L - \beta \log L\}. \tag{4-20}$$

However, using (4-19), we obtain

$$\#\{\mathcal{C} \in \mathscr{C}_j : \ell_\star(\mathcal{C}) \leqslant L - \beta \log L\} \leqslant N_{\star,j}(L - \beta \log L) \leqslant C L^{-h_\star \beta} e^{h_\star L}.$$

In particular, if $h_\star \beta > 2$, and if $A_\beta(L)$ denotes the left-hand side of (4-20), we have the bound $A_\beta(L) \ll N_{\star,j}(L)$ as $L \to \infty$. Thus, for large L,

$$C^{-1} N_{\star,j}(L) \leqslant \sum_{\substack{\mathcal{C} \in \mathscr{C}_j \\ \ell_\star(\mathcal{C}) \in [L - \beta \log L, L]}} (L - \ell_\star(\mathcal{C}) + C)^2 \leqslant (\beta \log L + C)^2 \, \#\{\mathcal{C} \in \mathscr{C}_j : \varepsilon L \leqslant \ell(\mathcal{C}) \leqslant L\},$$

where $\varepsilon > 0$ is any small number. This finally yields, for any large L,

$$\#\{\mathcal{C} \in \mathscr{C}_j : \varepsilon L \leqslant \ell(\mathcal{C}) \leqslant L\} \geqslant \frac{C^{-1} e^{h_\star L}}{(\beta \log L + C)^2}. \tag{4-21}$$

For any $\mathcal{C} \in \mathscr{C}_j$, we choose some $w_{\mathcal{C}} \in \mathcal{C}$ such that $\ell_\star(w_{\mathcal{C}}) = \ell_\star(\mathcal{C})$. Then Lemmata 4.6 and 4.7 imply that we have a well-defined and injective map

$$\mathscr{C}_1 \times \mathscr{C}_2 \to \{\gamma \in \mathcal{P} : i(\gamma, \gamma_\star) = 2\}, \quad (\mathcal{C}_1, \mathcal{C}_2) \mapsto [w_{\mathcal{C}_2} w_{\mathcal{C}_1}] \equiv \gamma_{w_{\mathcal{C}_2} w_{\mathcal{C}_1}}.$$

Obviously, $\ell(\gamma_{w_2 w_1}) \leqslant \ell_\star(w_1) + \ell_\star(w_2)$ for any w_1 and w_2, and thus we get, for large L,

$$\begin{aligned} N(2, L) &\geqslant \#\{(\mathcal{C}_1, \mathcal{C}_2) \in \mathscr{C}_1 \times \mathscr{C}_2 : \ell_\star(\mathcal{C}_1) + \ell_\star(\mathcal{C}_2) \leqslant L \text{ and } \ell_\star(\mathcal{C}_1), \ell_\star(\mathcal{C}_2) \geqslant \varepsilon L\} \\ &\geqslant \sum_{\substack{\mathcal{C}_1 \in \mathscr{C}_1 \\ \varepsilon L \leqslant \ell_\star(\mathcal{C}_1) \leqslant L}} \#\{\mathcal{C}_2 \in \mathscr{C}_2 : \varepsilon L \leqslant \ell_\star(\mathcal{C}_2) \leqslant L - \ell_\star(\mathcal{C}_1)\} \geqslant \sum_{\substack{\mathcal{C}_1 \in \mathscr{C}_1 \\ \varepsilon L \leqslant \ell_\star(\mathcal{C}_1) \leqslant L}} \frac{C^{-1} e^{h_\star (L - \ell_\star(\mathcal{C}_1))}}{\big(\beta \log(L - \ell_\star(\mathcal{C}_1)) + C\big)^2}. \end{aligned}$$

For simplicity, in what follows we will use the notation $f(\ell) = C^{-1} e^{h_\star \ell} / (\beta \log(\ell) + C)^2$ and $N(\mathscr{C}_1, L) = \#\{\mathcal{C} \in \mathscr{C}_j : \varepsilon L \leqslant \ell(\mathcal{C}) \leqslant L\}$. Fix some large number $\mu > 0$. Note that, if μ is large enough, there is $C > 0$ (depending on μ) such that, for any large ℓ,

$$f(\ell + \mu) - f(\ell) \geqslant C^{-1} f(\ell). \tag{4-22}$$

There holds

$$\begin{aligned} N(2, L) &\geqslant C^{-1} \sum_{k \in [\varepsilon L/\mu, L/\mu]} \big(N(\mathscr{C}_1, k\mu) - N(\mathscr{C}_1, (k-1)\mu)\big) f(L - (k-1)\mu) \\ &\geqslant C^{-1} \sum_{k \in [\varepsilon L/\mu + 1, L/\mu - 1]} N(\mathscr{C}_1, k\mu) \big(f(L - (k-1)\mu) - f(L - k\mu)\big) \\ &\qquad - N(\mathscr{C}_1, \varepsilon L + \mu) f(L - \varepsilon L), \end{aligned} \tag{4-23}$$

where we used an Abel transformation in the last inequality. Next, note that by (4-19), one has $N(\mathscr{C}_1, L) \leqslant N_{\star,1}(L) \leqslant C e^{h_\star L}$. This yields

$$N(\mathscr{C}_1, \varepsilon L + \mu) f(L - \varepsilon L) = \mathcal{O}(e^{h_\star L}) \tag{4-24}$$

as $L \to \infty$. On the other hand, (4-22) gives, for any large L,

$$\begin{aligned} &\sum_{k \in [\varepsilon L/\mu + 1, L/\mu - 1]} N(\mathscr{C}_1, k\mu) \big(f(L - (k-1)\mu) - f(L - k\mu)\big) \\ &\qquad \geqslant \sum_{k \in [\varepsilon L/\mu + 1, L/\mu - 1]} N(\mathscr{C}_1, k\mu) f(L - k\mu) \\ &\qquad \geqslant C^{-1} \sum_{k \in [\varepsilon L/\mu + 1, L/\mu - 1]} \frac{e^{h_\star k\mu}}{(\beta \log(k\mu) + C)^2} \frac{e^{h_\star (L - k\mu)}}{(\beta \log(L - k\mu) + C)^2} \geqslant \frac{C^{-1} L e^{h_\star L} (1 - \varepsilon)}{2\mu (\log(L) + C)^4}. \end{aligned}$$

We conclude the proof of Proposition 4.5 for the case $h_1 = h_2$ by combining this last estimate with (4-23) and (4-24).

If $h_1 \neq h_2$, say $h_1 > h_2$ (the case $h_1 < h_2$ is identical), one is able to obtain the desired bound by considering, for example, the injective map $\mathscr{C}_1 \to \{\gamma \in \mathcal{P} : i(\gamma, \gamma_\star) = 2\}$ given by $\mathcal{C} \mapsto [a_g w_{\mathcal{C}}]$ and by using (4-21). □

4.2.2 Upper bound Clearly, each $\gamma \in \mathcal{P}_2$ with $\ell(\gamma) \leqslant L$ may be represented by the conjugacy class of $w_1 w_2$ for some $w_j \in \pi_1(\Sigma_j)$ with $\ell_\star(w_1) + \ell_\star(w_2) \leqslant L + C$. Therefore, (4-5) implies

$$\begin{aligned} N(2, L) &\leqslant \#\{(w_1, w_2) \in \pi_1(\Sigma_1) \times \pi_1(\Sigma_2) : \ell_\star(w_1) + \ell_\star(w_2) \leqslant L + C\} \\ &\leqslant \sum_{k=0}^{L+C} C \exp(h_1 k) \exp(h_2(L - k + C)), \end{aligned}$$

which gives, for large L, if $h_\star = \max(h_1, h_2)$,

$$N(2, L) \leqslant \begin{cases} CL \exp(h_\star L) & \text{if } h_1 = h_2, \\ C \exp(h_\star L) & \text{if } h_1 \neq h_2. \end{cases}$$

Iterating this process we obtain (with C depending on n)

$$N(2n, L) \leqslant \begin{cases} CL^{2n-1} \exp(h_\star L) & \text{if } h_1 = h_2, \\ CL^{n-1} \exp(h_\star L) & \text{if } h_1 \neq h_2. \end{cases}$$

4.3 Relative growth of closed geodesics with a small intersection angle

For $x = \gamma_\star(\tau) \in \mathrm{Im}(\gamma_\star)$, we let $v_\star(x) = \dot\gamma_\star(\tau)$. For any $\eta > 0$ small, we consider the number $N(n, \eta, L) = \#\mathcal{P}_{\eta,n}(L)$, where $\mathcal{P}_{\eta,n}(L)$ is the set of closed geodesics $\gamma : \mathbb{R}/\ell(\gamma)\mathbb{Z} \to \Sigma$ of length not greater than L, intersecting $\gamma_\star$ exactly n times, and such that there is $t \in \mathbb{R}/\ell(\gamma)\mathbb{Z}$ with $\gamma(t) \in \mathrm{Im}(\gamma_\star)$ and

$$\mathrm{angle}\big(\dot\gamma(t), v_\star(\gamma(t))\big) < \eta \quad \text{or} \quad \mathrm{angle}\big(\dot\gamma(t), -v_\star(\gamma(t))\big) < \eta.$$

The purpose of this section is to prove the following estimate:

Lemma 4.11 *Let $n \geqslant 1$. For any $\varepsilon, L_0 > 0$, there exists $\eta_0 > 0$ such that, for any $\eta \in]0, \eta_0[$ and any large L,*

$$N(1, \eta, L) \leqslant 4N(1, L - L_0) \quad \textit{and} \quad N(n, \eta, L) \leqslant \varepsilon L^{n-1} \exp(h_\star L) \tag{4-25}$$

if $\gamma_\star$ is not separating, and

$$N(2, \eta, L) \leqslant 4N(2, L - L_0) \qquad \textit{and} \qquad N(2n, \eta, L) \leqslant \begin{cases} \varepsilon L^{2n-1} \exp(h_\star L) & \textit{if } h_1 = h_2, \\ \varepsilon L^{n-1} \exp(h_\star L) & \textit{if } h_1 \neq h_2, \end{cases} \tag{4-26}$$

if $\gamma_\star$ is separating.

Proof We first prove the lemma when $\gamma_\star$ is assumed not separating. Let $\gamma : [0, \ell(\gamma)] \to \Sigma$ be an element of $\mathcal{P}_{\eta,n}(L)$ parametrized by arc length. Let $0 \leqslant t_1 < t_2 < \cdots < t_n < \ell(\gamma)$ be such that $\gamma(t_j) \in \mathrm{Im}(\gamma_\star)$. For every $j = 1, \dots, n$, we choose a path c_j contained in $\mathrm{Im}(\gamma_\star)$ of length not greater than $\ell(\gamma_\star)$ that links

$x_j = \gamma(t_j)$ to $x_\star$. Recall that we have a map $q_\star\colon \Sigma_\star \to \Sigma$ given by the identification of the boundary components of $\Sigma_\star$. Write $q_\star^{-1}(x_\star) = \{x_\star, \bar{x}_\star\}$, where we chose some $x_\star \in \Sigma_\star$ with $q_\star(x_\star) = x_\star$, as in Section 4.1. Then γ is freely homotopic to the composition

$$w_1 w_2 \cdots w_n, \quad \text{where } w_j = c_{j+1}\gamma|_{[t_j, t_{j+1}]} c_j^{-1} \in \pi_1(\Sigma) \text{ for } j = 1, \dots, n,$$

with the convention that $t_{n+1} = \ell(\gamma)$ and $c_{n+1} = c_1$. Note also that

$$\ell_\star(w_j) \leqslant |t_{j+1} - t_j| + 2\ell(\gamma_\star).$$

In fact, the elements w_j actually define elements of the space $\pi_1(\Sigma_\star, \{x_\star, \bar{x}_\star\})$, that is, the space of equivalence classes of paths $c\colon [0,1] \to \Sigma_\star$ with $c(0), c(1) \in \{x_\star, \bar{x}_\star\}$, where two paths are equivalent if they are homotopic via a homotopy preserving the endpoints. The space $\pi_1(\Sigma_\star, \{x_\star, \bar{x}_\star\})$ is not a group (we may not be able to concatenate two paths); however, we have a natural map $\pi_1(\Sigma_\star, \{x_\star, \bar{x}_\star\}) \to \pi_1(\Sigma)$. In particular, for any $u_1, \dots, u_n \in \pi_1(\Sigma_\star, \{x_\star, \bar{x}_\star\})$, the composition $u_n \cdots u_1$ is well defined in $\pi_1(\Sigma)$. For any $u \in \pi_1(\Sigma_\star, \{x_\star, \bar{x}_\star\})$, we will denote by $\ell_\star(u)$ the infimum of the lengths of curves in the equivalence class u.

Up to reparametrizing of γ, we may assume that $t_1 = 0$, and either $\angle(v, v_\star) < \eta$ or $\angle(v, -v_\star) < \eta$, where we set $x = \gamma(0)$, $v_\star = v_\star(x)$ and $v = \dot{\gamma}(0)$. We will first assume that $\angle(v, v_\star) < \eta$. Let $L_0 > 0$ be a large number and $\varepsilon > 0$ be small. By continuity of the geodesic flow (φ_t), there is $\eta_0 > 0$ such that, if $\eta < \eta_0$,

$$\operatorname{dist}_M(\varphi_t(v), \varphi_t(v_\star)) \leqslant \varepsilon \quad \text{for } t \in [0, L_0].$$

Let K be a positive integer such that $K \in [L_0/\ell(\gamma_\star) - 1, L_0/\ell(\gamma_\star)]$, so that

$$\operatorname{dist}_\Sigma\big(\pi(\varphi_{K\ell(\gamma_\star)}(v)), x\big) < \varepsilon.$$

Let c_K be a path in Σ of length not greater than ε linking $\pi(\varphi_{K\ell(\gamma_\star)}(v))$ and x. Then, if $\varepsilon > 0$ is small enough,[11]

$$c_1 c_K \gamma|_{[0, K\ell(\gamma_\star)]} c_1^{-1} = a_{\mathrm{g}}^K \quad \text{in } \pi_1(\Sigma).$$

In particular, $w_1 = w_1' a_{\mathrm{g}}^K$ in $\pi_1(\Sigma)$, where $w_1' = c_2 \gamma|_{[K\ell(\gamma_\star), t_2]} c_K^{-1} c_1^{-1}$. Note also that

$$\ell_\star(w_1') \leqslant |t_2 - K\ell(\gamma_\star)| + 2\ell(\gamma_\star) + \varepsilon,$$

where w_1' is seen as an element of $\pi_1(\Sigma_\star, \{x_\star, \bar{x}_\star\})$. Note that if we had assumed $\angle(v, -v_\star) < \eta$, we would have obtained the same factorization with a_{g}^{-K} instead of a_{g}^K. Next, let

$$A_{K,n}(L) = \left\{ (w_1, \dots, w_n) \in \pi_1(\Sigma_\star, \{x_\star, \bar{x}_\star\})^n : \sum_{j=1}^n \ell_\star(w_j) \leqslant L + (2n - K)\ell(\gamma_\star) + \varepsilon \right\},$$

[11] If $\varepsilon > 0$ is small enough, we have the following property. For any $x \in \Sigma$ and $L > 0$, if we are given two paths $c, c'\colon [0, L] \to \Sigma$ such that $c(0) = c'(0) = c(L) = c'(L) = x$ and $\operatorname{dist}_\Sigma(c(t), c'(t)) < \varepsilon$, then c and c' define the same element in $\pi_1(\Sigma, x)$.

and consider the map $\Psi_{K,n,\pm}\colon A_{K,n}(L) \to \mathcal{P}$ given by $(w_1,\dots,w_n) \mapsto [w_1\cdots w_n a_{\mathrm{g}}^{\pm K}]$. Then the discussion above shows that

$$\mathcal{P}_{\eta,n}(L) \subset \mathrm{Im}(\Psi_{K,n,+}) \cup \mathrm{Im}(\Psi_{K,n,-}).$$

In particular, $N(n,\eta,L) \leqslant 2\#A_{K,n}(L)$. Next, we obtain a bound on $A_{K,n}(L)$ as follows. Let $c_\star$ be a path connecting $\bar{x}_\star$ and $x_\star$ in $\Sigma_\star$, so that the image of $c_\star^{-1}$ in $\pi_1(\Sigma)$ is b_{g} (see Figure 2). Then it is not hard to see that, for any $w \in \pi_1(\Sigma_\star, \{x_\star, \bar{x}_\star\})$, there is $u \in \pi_1(\Sigma_\star, x_\star)$ such that w can be written as

$$u, \quad c_\star u, \quad u c_\star^{-1} \quad \text{or} \quad c_\star u c_\star^{-1}$$

(depending on the endpoints of w), with $\ell_\star(u) \leqslant \ell_\star(w) + 2\ell(c_\star)$. This immediately gives

$$\#A_{K,1}(L) \leqslant 4\#\{u \in \pi_1(\Sigma_\star) : \ell_\star(u) \leqslant L\} \leqslant C \exp(h_\star L).$$

As in Section 4.1.2, we obtain, for some $C_n > 0$ depending only n,

$$\#A_{K,n}(L) \leqslant C_n L^{n-1} \exp(h_\star(L - L_0)),$$

where we used that $K\ell(\gamma_\star) \geqslant L_0 - \ell(\gamma_\star)$. This proves the second part of (4-25). For the first part, we proceed as follows. With the notation of the proof of Proposition 4.5, one has well-defined maps

$$\Psi_{K,1,\pm,r}, \Psi_{K,1,\pm,l} : \{\mathcal{C} \in \mathscr{C} : \ell_\star(w) \leqslant L - K\ell(\gamma_\star)\} \to \{\gamma \in \mathcal{P}_1 : \ell(\gamma) \leqslant L + 2C\},$$

given respectively by $\mathcal{C} \mapsto [a_{\mathrm{g}}^{\pm K} w b_{\mathrm{g}}]$ and $\mathcal{C} \mapsto [b_{\mathrm{g}}^{-1} w a_{\mathrm{g}}^{\pm K}]$, where w is any element of $\mathcal{C}$. Next, we remark that the above discussion implies that every $\gamma \in \mathcal{P}_{\eta,1}(L)$ can be written as

$$[a_{\mathrm{g}}^{\pm K} w b_{\mathrm{g}}] \quad \text{or} \quad [b_{\mathrm{g}}^{-1} w a_{\mathrm{g}}^{\pm K}]$$

for some $w \in \pi_1(\Sigma_\star)$ with $\ell_\star(w) \leqslant L - K\ell(\gamma_\star) + C$. Therefore the union of the images of the maps $\Psi_{K,1,\pm,r}$ and $\Psi_{K,1,\pm,l}$ contains $\mathcal{P}_\eta(L + 2C)$, and thus

$$N(1,\eta,L) \leqslant 4\#\{\mathcal{C} \in \mathscr{C} : \ell_\star(w) \leqslant L - K\ell(\gamma_\star) + 2C\} \leqslant 4N(1, L - K\ell(\gamma_\star) + 3C),$$

where we used the first inequality of (4-8). This gives the first part of (4-25).

Next, assume that $\gamma_\star$ is separating. Then, as above, every $\gamma\colon [0,\ell(\gamma)] \to \Sigma$ such that $\gamma \in \mathcal{P}_{2n,\eta}(L)$ can be written as a composition $w_{1,1}w_{1,2}\cdots w_{1,n}w_{2,n}$ for some $w_{k,j} \in \pi_1(\Sigma_k)$ for $k = 1,2$ and $j = 1,2,\dots,n$, with

$$\sum_{j=1}^{n} \ell_\star(w_{2,j}) + \ell_\star(w_{1,j}) \leqslant \ell(\gamma) + 4n\ell(\gamma_\star).$$

Now, if η is small, we may proceed as before to obtain (up to reparametrization of γ) that $w_{1,1} = w_{\star,1}^{\pm K} w_{1,1}'$ or $w_{1,1} = w_{1,1}' w_{\star,1}^{\pm K}$ for some $w_{1,1}' \in \pi_1(\Sigma_1)$ with

$$\ell_\star(w_{1,1}') \leqslant \ell_\star(w_{1,1}) - K\ell(\gamma_\star) + C.$$

Here K is a large number depending on η (ie such that $K \to \infty$ as $\eta \to 0$) and $C > 0$ is a constant independent of γ and K. Thus we get

$$N(2n,\eta,L) \leqslant C\,\#\Big\{(w_{1,1},w_{2,1},\dots,w_{1,n},w_{2,n}) : w_{k,j} \in \pi_1(\Sigma_k),\ \sum_{j=1}^{n} \ell_\star(w_{1,j})+\ell_\star(w_{2,j}) \leqslant L-K\ell(\gamma_\star)+C_n\Big\}.$$

Then we obtain the second part of (4-26) by proceeding as in Section 4.2.2. For the first part of (4-26), we proceed as follows. For $w_j \in \pi_1(\Sigma_j)_\star$, we define

$$\mathcal{C}_{w_1,w_2} = \{(w_1',w_2') : [w_1'w_2'] = [w_1w_2]\}$$

and $\ell_\star(\mathcal{C}_{w_1,w_2}) = \inf\{\ell_\star(w_1') + \ell_\star(w_2') : (w_1',w_2') \in \mathcal{C}_{(w_1,w_2)}\}$. We also introduce the notation $\mathscr{C}_{1,2} = \{\mathcal{C}_{w_1,w_2} : w_j \in \pi_1(\Sigma_j)_\star\}$. By Lemmata 4.6 and 4.7, we have well-defined maps

$$\Psi_{K,1,\pm,r},\Psi_{K,1,\pm,l} : \{\mathcal{C} \in \mathscr{C}_{1,2} : \ell_\star(\mathcal{C}_{w_1,w_2}) \leqslant L-K\ell(\gamma_\star)\} \to \{\gamma \in \mathcal{P}_2 : \ell(\gamma) \leqslant L\}$$

given respectively by $\mathcal{C} \mapsto [w_1 w_{\star,1}^{\pm K} w_2]$ and $\mathcal{C} \mapsto [w_{\star,1}^{\pm K} w_1 w_2]$. By the discussion above, the union of the images of those maps contains $\mathcal{P}_{2,\eta}(L)$. Therefore

$$N(2,\eta,L) \leqslant 4\,\#\{\mathcal{C} \in \mathscr{C}_{1,2} : \ell_\star(\mathcal{C}_{w_1,w_2}) \leqslant L-K\ell(\gamma_\star)\} \leqslant 4N(2,L-K\ell(\gamma_\star)),$$

where we used Lemmata 4.6 and 4.7 again in the last inequality. The first part of (4-26) follows. □

5 A Tauberian argument

The goal of this section is to give an asymptotic growth of the quantity

$$N_\pm(n,\chi,t) = \sum_{\substack{\gamma\in\mathcal{P}\\ i(\gamma_\star,\gamma)=n\\ \ell(\gamma)\leqslant t}} I_{\star,\pm}(\gamma,\chi)$$

as $t \to +\infty$, where $\chi \in C_c^\infty(\partial \setminus \partial_0)$ and $I_{\star,\pm}(\gamma,\chi) = \prod_{z\in I_{\star,\pm}(\gamma)} \chi^2(z)$.

5.1 The case $\gamma_\star$ is not separating

By [15, Theorem 3 and Section 6.2], the zeta function

$$\zeta_{\Sigma_\star}(s) = \prod_{\gamma\in\mathcal{P}_\star} (1-e^{-s\ell(\gamma)})$$

extends meromorphically to the whole complex plane, and moreover we may write

$$\frac{\zeta_{\Sigma_\star}'(s)}{\zeta_{\Sigma_\star}(s)} = \sum_{k=0}^{2} (-1)^k \operatorname{tr}^\flat(e^{\pm\varepsilon s}\varphi_{\mp\varepsilon}^* R_{\pm,\delta}(s)|_{\Omega_c^k(M_\delta)\cap\ker\iota_X}),$$

where the flat trace is computed on M_δ. Here $\mathcal{P}_\star$ denotes the set of primitive closed geodesics of $(\Sigma_\star, g)$. By [12], we may apply [35, Proposition 9] (see also [36, Theorem 9.1]) to obtain that $\zeta_{\Sigma_\star}$ is holomorphic in $\{\mathrm{Re}(s) \geqslant h_\star\}$, except for a simple pole at $s = h_\star$, where $h_\star > 0$ is the topological entropy of the geodesic flow of $(\Sigma_\star, g)$ restricted to its trapped set. Write the Laurent expansion given in Section 2.6 of $R_{\pm,\delta}(s)$ near $s = h_\star$ as

$$R_{\pm,\delta}(s) = Y_{\pm,\delta}(s) + \frac{\Pi_{\pm,\delta}(h_\star)}{s-h_\star} + \sum_{j=2}^{J(h_\star)} \frac{(X \pm h_\star)^{j-1}\Pi_{\pm,\delta}(h_\star)}{(s-h_\star)^j} : \Omega_c^\bullet(M_\delta) \to \mathcal{D}'^\bullet(M_\delta).$$

By [15, (5.8)], we have $\mathrm{tr}^\flat(e^{\pm\varepsilon h_\star}\varphi_{\mp\varepsilon}^*\Pi_{\pm,\delta}(h_\star)) = \mathrm{rank}\,\Pi_{\pm,\delta}(h_\star)$ and

$$\mathrm{tr}^\flat(\varphi_{\mp\varepsilon}^*(X \pm h_\star)^j\,\Pi_{\pm,\delta}(h_\star)) = 0 \quad \text{for } j = 1, \dots, J(h_\star) - 1.$$

We write $\Omega^k = \Omega_c^k(M_\delta)$ and $\Omega_0^k = \Omega^k \cap \ker \iota_X$. Then, by [18, Propositions 2.4 and 4.4], the map $s \mapsto R_{\pm,\delta}(s)|_{\Omega_0^0}$ has no pole in $\{\mathrm{Re}(s) > 0\}$. Since $\Omega_0^2 = \Omega_0^0 \wedge \mathrm{d}\alpha$, and $R_{\pm,\delta}(s)|_{\Omega_0^2} = R_{\pm,\delta}(s)|_{\Omega_0^0} \wedge \mathrm{d}\alpha$ (because $\varphi_t^*\alpha = \alpha$), it follows that $s \mapsto R_{\pm,\delta}(s)|_{\Omega_0^2}$ has no poles in $\{\mathrm{Re}(s) > 0\}$. In particular, the residue of $\zeta'_{\Sigma_\star}(s)/\zeta_{\Sigma_\star}(s)$ at $s = h_\star$ is given by $\mathrm{rank}(\Pi_{\pm,\delta}(h_\star)|_{\Omega_0^1})$, and since $\zeta_{\Sigma_\star}(s)$ has a simple pole at $s = h_\star$, this residue is equal to 1. Therefore,

$$\mathrm{rank}(\Pi_{\pm,\delta}(h_\star)|_{\Omega_0^1}) = 1.$$

In particular, $(X \pm h_\star)^j\,\Pi_{\pm,\delta} = 0$ for each $j = 1, \dots, J(h_\star) - 1$. As $R_{\pm,\delta}(s)$ commutes with ι_X, it preserves the spaces Ω_0^k. Writing $\Omega^k = \Omega_0^k \oplus \alpha \wedge \Omega_0^{k-1}$ we have, for any $w = u + \alpha \wedge v$ with $\iota_X u = 0$ and $\iota_X v = 0$,

$$\Pi_{\pm,\delta}(h_\star)|_{\Omega^2}(u + \alpha \wedge v) = \Pi_{\pm,\delta}(h_\star)|_{\Omega_0^2}(u) + \alpha \wedge \Pi_{\pm,\delta}(h_\star)|_{\Omega_0^1}(v).$$

Thus $\Pi_{\pm,\delta}(h_\star)|_{\Omega^2} = \alpha \wedge \iota_X \Pi_{\pm,\delta}(h_\star)|_{\Omega_0^1}$. By Proposition 3.2 and the fact that $\varphi_{\pm\varepsilon}^*\Pi_{\pm,\delta}(h_\star) = e^{\pm\varepsilon h_\star}\Pi_{\pm,\delta}(h_\star)$, we have, near $s = h_\star$,

$$\chi\widetilde{\mathcal{S}}_\pm(s)\chi = \chi Y_\pm(s)\chi + \frac{\chi\psi^*\iota^*\iota_X\Pi_{\pm,\delta}(h_\star)\iota_*\chi}{s-h_\star}, \tag{5-1}$$

where $s \mapsto Y_\pm(s)$ is holomorphic in a neighborhood of $h_\star$. We write

$$\Pi_{\pm,\partial} = \psi^*\iota^*\iota_X\Pi_{\pm,\delta}(h_\star)\iota_* : \Omega^\bullet(\partial) \to \mathcal{D}'^\bullet(\partial).$$

Then, by what precedes, and since $\iota_X\Pi_{\pm,\delta}(h_\star)|_{\Omega^1} = 0$, we obtain that $\mathrm{rank}(\Pi_{\pm,\partial}) \leqslant 1$. Finally, for any $\chi \in C_c^\infty(\partial \setminus \partial_0)$, we set

$$c_\pm(\chi) = \mathrm{tr}_s^\flat(\chi\Pi_{\pm,\partial}\chi).$$

Lemma 5.1 *Let $\chi \in C_c^\infty(\partial \setminus \partial_0)$ be such that $c_\pm(\chi) > 0$. Then*

$$N_\pm(n, \chi, t) \sim \frac{(c_\pm(\chi)t)^n}{n!}\frac{e^{h_\star t}}{h_\star t} \quad \text{as } t \to +\infty.$$

Proof Because $\chi\Pi_{\pm,\partial}$ is of rank one, it follows that $\mathrm{tr}_s^\flat((\chi\Pi_{\pm,\partial})^n) = c_\pm(\chi)^n$ for any $n \geqslant 1$ (since the flat trace of a finite-rank operator coincides with its usual trace), and thus

$$\mathrm{tr}_s^\flat\big((\chi\widetilde{\mathcal{S}}_\pm(s)\chi)^n\big) = \frac{c_\pm(\chi)^n}{(s-h_\star)^n} + \mathcal{O}((s-h_\star)^{-n+1}) \quad \text{as } s \to h_\star.$$

Note that here we implicitly used the fact that the flat trace of products of the form

(5-2) $$(\chi Y_\pm(s)\chi)^{k_1}(\chi\Pi_{\pm,\partial}\chi)^{\ell_1}(\chi Y_\pm(s)\chi)^{k_2}(\chi\Pi_{\pm,\partial}\chi)^{\ell_2}\cdots$$

makes sense. Indeed, note that both $\mathrm{WF}(\chi\Pi_{\pm,\partial}\chi)$ and $\mathrm{WF}(\chi Y_\pm(s)\chi)$ are contained in $\mathrm{WF}(\chi\widetilde{\mathcal{S}}_\pm(s)\chi)$ by (5-1) and Cauchy's integral formula. Thus we may reproduce the proofs of Lemma 3.5, Remark 3.6 and Proposition 3.7 to obtain that the composition (5-2) is well defined and that its flat trace makes sense. Next, set $\eta_{n,\chi}(s) = \mathrm{tr}_\mathrm{s}^\flat\big((\chi\widetilde{\mathcal{S}}_\pm(s)\chi)^n\big)$ and

$$g_{n,\chi}(t) = \sum_{\substack{\gamma\in\mathcal{P}\\ i(\gamma,\gamma_\star)=n}} \ell^\#(\gamma) \sum_{\substack{k\geqslant 1\\ k\ell(\gamma)\leqslant t}} I_{\star,\pm}(\gamma,\chi)^k \quad \text{for } t\geqslant 0.$$

Now, if $G_{n,\chi}(s) = \int_0^{+\infty} g_{n,\chi}(t)e^{-ts}\,\mathrm{d}t$, a simple computation leads to

$$G_{n,\chi}(s) = \frac{1}{s}\sum_{i(\gamma,\gamma_\star)=n} \ell^\#(\gamma)e^{-s\ell(\gamma)} I_{\star,\pm}(\gamma,\chi)^{\ell(\gamma)/\ell^\#(\gamma)} = -\frac{\eta'_{n,\chi}(s)}{ns},$$

where the last equality comes from Proposition 3.7. Using the expansion

$$\eta'_{n,\chi}(s) = -nc_\pm(\chi)^n(s-h_\star)^{-(n+1)} + \mathcal{O}((s-h_\star)^{-n}) \quad \text{as } s\to h_\star,$$

we obtain

$$G_{n,\chi}(h_\star s) = \frac{c_\pm(\chi)^n}{h_\star^{n+2}(s-1)^{n+1}} + \mathcal{O}((s-h_\star)^{-n}) \quad \text{as } s\to h_\star.$$

Then, applying the Tauberian theorem of Delange [14, théorème III],

$$\frac{1}{h_\star}g_{n,\chi}\Big(\frac{t}{h_\star}\Big) \sim \frac{c_\pm(\chi)^n}{h_\star^{n+2}}\frac{e^t}{n!}t^n \quad \text{as } t\to+\infty,$$

and so

(5-3) $$g_{n,\chi}(t) \sim \frac{(c_\pm(\chi)t)^n}{n!h_\star}\exp(h_\star t).$$

Now note that, if $\mathcal{P}_n$ is the set of primitive closed geodesics γ with $i(\gamma,\gamma_\star)=n$,

$$g_{n,\chi}(t) \leqslant \sum_{\substack{\gamma\in\mathcal{P}_n\\ \ell(\gamma)\leqslant t}} \ell(\gamma)\left\lfloor\frac{t}{\ell(\gamma)}\right\rfloor I_{\star,\pm}(\gamma,\chi) \leqslant tN(n,\chi,t).$$

As a consequence,

(5-4) $$\liminf_{t\to+\infty} N_\pm(n,\chi,t)\frac{n!h_\star t}{(c_\pm(\chi)t)^n e^{h_\star t}} \geqslant 1.$$

For the other bound, we use the a priori bound, obtained in Section 4.1.2,

(5-5) $$N_\pm(n,\chi,t) \leqslant N(n,t) \leqslant \frac{Ct^n}{n!}\frac{e^{h_\star t}}{h_\star t}$$

to deduce that, for any $\sigma>1$,

(5-6) $$\limsup_{t\to+\infty} N_\pm\Big(n,\chi,\frac{t}{\sigma}\Big)\frac{n!}{t^n}\frac{h_\star t}{e^{h_\star t}} = 0.$$

Now we may write

$$N_\pm(n,\chi,t) = N_\pm\Big(n,\chi,\frac{t}{\sigma}\Big) + \sum_{\substack{\gamma\in\mathcal{P}\\ i(\gamma_\star,\gamma)=n\\ t/\sigma\leqslant\ell(\gamma)\leqslant t}} I_{\star,\pm}(\gamma,\chi) \tag{5-7}$$

$$\leqslant N_\pm\Big(n,\chi,\frac{t}{\sigma}\Big) + \frac{\sigma}{t}\sum_{\substack{\gamma\in\mathcal{P}\\ i(\gamma_\star,\gamma)=n\\ t/\sigma\leqslant\ell(\gamma)\leqslant t}} I_{\star,\pm}(\gamma,\chi)\ell(\gamma) \leqslant N_\pm\Big(n,\chi,\frac{t}{\sigma}\Big) + \frac{\sigma}{t} g_{n,\chi}(t),$$

which gives, with (5-3) and (5-6),

$$\limsup_{t\to+\infty} N_\pm(n,\chi,t)\frac{n!}{(c_\pm(\chi)t)^n}\frac{h_\star t}{e^{h_\star t}} \leqslant \sigma.$$

As $\sigma > 1$ is arbitrary, the lemma is proven. ◻

Remark 5.2 If we assume that $c_\pm(\chi) = 0$, then with the notation of the above proof, the map $s \mapsto \eta_{1,\chi}(s)$ has no pole on the line $\{\mathrm{Re}(s) = h_\star\}$. In particular, we may reproduce the arguments of the aforementioned proof, replacing $g_{n,\chi}(t)$ by $g_{n,\chi}(t) + \exp(h_\star t)$, to obtain that $s \mapsto \int_0^\infty (g_{n,\chi}(t) + \exp(h_\star t))\exp(-ts)\,\mathrm{d}t$ has a pole of order 1 at $s = h_\star$, which implies that $g_{n,\chi}(t) + \exp(h_\star t) \sim \exp(h_\star t)$ as $t \to \infty$. This gives $g_{n,\chi}(t) \ll_{t\to\infty} \exp(h_\star t)$, and hence

$$N_\pm(1,\chi,t) \ll \frac{\exp(h_\star t)}{t} \quad \text{as } t \to \infty,$$

where we used the last line of (5-7) and (5-5). Note that this bound is incompatible with the one provided by Proposition 4.2; this will help us to prove that $c_\pm(\chi) > 0$, by showing that $N(1,t)$ can be controlled by $N_\pm(1,\chi,t)$ whenever χ has enough support (see Section 6.1).

5.2 The case $\gamma_\star$ is separating

In this case, Σ_δ consists of two surfaces, $\Sigma_\delta^{(1)}$ and $\Sigma_\delta^{(2)}$. We write $M_\delta = M_\delta^{(1)} \sqcup M_\delta^{(2)}$, where $M_\delta^{(j)} = S\Sigma_\delta^{(j)}$ for $j = 1, 2$, and $\partial = \partial^{(1)} \sqcup \partial^{(2)}$ with $\partial^{(j)} \subset M_\delta^{(j)}$. Note that, if $\widetilde{\mathcal{S}}_\pm^{(j)}(s)$ denotes the restriction of $\widetilde{\mathcal{S}}_\pm(s)$ to $\partial^{(j)}$, we have

$$\widetilde{\mathcal{S}}_\pm^{(1)}(s)\colon \Omega^\bullet(\partial^{(1)}) \to \mathcal{D}'^\bullet(\partial^{(2)}) \quad\text{and}\quad \widetilde{\mathcal{S}}_\pm^{(2)}(s)\colon \Omega^\bullet(\partial^{(2)}) \to \mathcal{D}'^\bullet(\partial^{(1)}).$$

As in Section 5.1,

$$\chi\widetilde{\mathcal{S}}_\pm^{(j)}(s)\chi = \chi Y_\pm^{(j)}(s)\chi + \frac{\chi\Pi_{\pm,\partial}^{(j)}\chi}{s-h_j} \quad \text{as } s \to h_j,$$

with $\mathrm{rank}(\Pi_{\pm,\partial}^{(j)}) = 1$. Here $Y_\pm^{(j)}(s)$ is holomorphic near $s = h_j$ and h_j is the topological entropy of the geodesic flow of $\Sigma_\delta^{(j)}$. As before, fix $\chi \in C_c^\infty(\partial \setminus \partial_0)$.

5.2.1 The case $h_1 \neq h_2$ We may assume $h_1 > h_2$, and we define

$$c_\pm(\chi) = \operatorname{tr}_s^\flat(\chi \widetilde{\mathcal{S}}_\pm^{(2)}(h_1)\chi^2 \Pi_{\pm,\partial}^{(1)}\chi).$$

Because $\Pi_{\pm,\partial}^{(1)}$ is of rank one, $\operatorname{tr}_s^\flat\big((\chi \widetilde{\mathcal{S}}_\pm^{(2)}(h_1)\chi^2 \Pi_{\pm,\partial}^{(1)}\chi)^n\big) = c_\pm(\chi)^n$ for any $n \geqslant 1$, and thus, by cyclicity of the flat trace (indeed the flat trace coincides with the real trace for operators of finite rank), as $s \to h_1$,

$$\begin{aligned}\operatorname{tr}_s^\flat\big((\chi \widetilde{\mathcal{S}}_\pm(s)\chi)^{2n}\big) &= \operatorname{tr}_s^\flat\big((\chi \widetilde{\mathcal{S}}_\pm^{(1)}(s)\chi^2 \widetilde{\mathcal{S}}_\pm^{(2)}(s)\chi)^n + (\chi \widetilde{\mathcal{S}}_\pm^{(2)}(s)\chi^2 \widetilde{\mathcal{S}}_\pm^{(1)}(s)\chi)^n\big) \\ &= \frac{2c_\pm(\chi)^n}{(s-h_1)^n} + \mathcal{O}((s-h_1)^{-n+1}).\end{aligned}$$

Now we may proceed exactly as in Section 5.1 to obtain that, if $c_\pm(\chi) > 0$,

$$N_\pm(2n,\chi,t) \sim \frac{(c_\pm(\chi)t)^n}{n!}\frac{e^{h_\star t}}{h_\star t} \quad \text{as } t \to +\infty.$$

Remark 5.3 (continuation of Remark 5.2) If $h_1 \neq h_2$ and if we assume that $c_\pm(\chi) = 0$, then the map $s \mapsto \operatorname{tr}_s^\flat\big((\chi \widetilde{\mathcal{S}}_\pm(s)\chi)^2\big)$ has no pole on the line $\{\operatorname{Re}(s) = h_\star\}$. As in Remark 5.2, this yields

$$N_\pm(2,\chi,t) \ll \frac{\exp(h_\star t)}{t} \quad \text{as } t \to \infty. \tag{5-8}$$

Again, the bound given in Proposition 4.5 is incompatible with (5-8) — in fact, even a weaker bound (say, a lower bound with a linear loss with respect to Theorem 2) would be incompatible with (5-8) for the case $h_1 \neq h_2$ — and this will imply that $c_\pm(\chi)$ is positive.

5.2.2 The case $h_1 = h_2 = h_\star$ In that case, by writing $c_\pm(\chi) = \operatorname{tr}_s^\flat(\chi \Pi_{\pm,\partial}^{(1)}\chi \Pi_{\pm,\partial}^{(2)})$, we have

$$\operatorname{tr}_s^\flat\big((\chi \widetilde{\mathcal{S}}_\pm(s)\chi)^{2n}\big) = \frac{2c_\pm(\chi)^n}{(s-h_\star)^{2n}} + \mathcal{O}((s-h_\star)^{-2n+1}) \quad \text{as } s \to h_\star.$$

Again, provided that $c_\pm(\chi) \neq 0$, we may proceed exactly as in Section 5.1 to obtain

$$N_\pm(2n,\chi,t) \sim 2\frac{(c_\pm(\chi)t^2)^n}{(2n)!}\frac{e^{h_\star t}}{h_\star t}.$$

Remark 5.4 (continuation of Remark 5.3) If $h_1 = h_2$ and $c_\pm(\chi) = 0$, then the function $s \mapsto \operatorname{tr}_s^\flat\big((\chi \widetilde{\mathcal{S}}_\pm(s)\chi)^2\big)$ might have a pole at $s = h_\star$, of order at most 1. Therefore, reproducing the arguments of Section 5.1, we obtain

$$N_\pm(2,\chi,t) = \mathcal{O}(\exp(h_\star t)) \quad \text{as } t \to \infty. \tag{5-9}$$

Note that here, assuming $c_\pm(\chi) = 0$ only wins us a factor of t for the bound on $N_\pm(2,\chi,t)$ (with respect to the asymptotics of Theorem 2), whereas in Remarks 5.2 and 5.3 we could win a bit more. This is why we need a lower bound on $N(2,L)$ which is sharp up to a sublinear loss for the case where $h_1 = h_2$ (see Proposition 4.5 and the comments below). Indeed, we will see that $N(2,t)$ can be controlled by $N_\pm(2,\chi,t)$ whenever χ has enough support; hence, Proposition 4.5 will contradict (5-9), yielding again $c_\pm(\chi) > 0$ (see Section 6.2).

6 Proof of Theorems 1 and 2

In this section we prove Theorems 1 and 2. We will apply the asymptotic growth we obtained in the last section to some appropriate sequence of functions in $C_c^\infty(\partial \setminus \partial_0)$. Let $F \in C^\infty(\mathbb{R}, [0,1])$ be an even function such that $F \equiv 0$ on $[-1,1]$ and $F \equiv 1$ on $]-\infty,-2] \cup [2,+\infty[$. For any small $\eta > 0$, set

$$F_\eta(t) = \sum_{k\in\mathbb{Z}} F\Big(\frac{t-k\pi}{\eta}\Big).$$

Then F_η is 2π–periodic and it induces a function $F_\eta : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}_{\geq 0}$. In the coordinates from Lemma 2.3, we define

$$\chi_\eta(z) = F_\eta(\theta) \quad \text{for } z = (\tau, 0, \theta) \in \partial.$$

Then $\chi_\eta \in C_c^\infty(\partial \setminus \partial_0)$ for any $\eta > 0$ small; the function χ_η is introduced in order to forget about trajectories passing at distance not greater than η from the "glancing set" $S\gamma_\star$.

6.1 The case $\gamma_\star$ is not separating

Recall from Section 4 that we have the a priori bounds

$$C^{-1}\frac{e^{h_\star L}}{h_\star L} \leqslant N(1,L) \leqslant Ce^{h_\star L} \tag{6-1}$$

for L large enough. This estimate implies the following fact:[12]

$$\forall \varepsilon > 0 \ \exists L_0 > 0 \ \forall L_1 > 0 \ \exists L > L_1 \quad N(1, L-L_0) \leqslant \varepsilon N(1,L).$$

In particular, we see with the first part of (4-25) in Lemma 4.11 that, for any $\eta > 0$ small enough,

$$\liminf_{L\to+\infty} \frac{N(1,\eta,L)}{N(1,L)} \leqslant \frac{1}{2}, \tag{6-2}$$

where $N(1,\eta,L)$ is as defined in Section 4.3.

For $\eta > 0$ small and $L > 0$, neither $c_\pm(\chi_\eta)$ nor $N_\pm(n,\chi_\eta,L)$ (see Section 5.1) depend on $\pm$, since F is an even function. We denote them simply by $c(\eta)$ and $N(n,\chi_\eta,L)$, respectively. Then we claim that $c(\eta) > 0$ if $\eta > 0$ is small enough. Indeed, if $c(\eta) = 0$, then Remark 5.2 implies

$$N(1,\chi_\eta,L) \ll \frac{\exp(h_\star L)}{h_\star L} \quad \text{as } L \to +\infty. \tag{6-3}$$

On the other hand, $N(1,L) = N(1,\chi_\eta,L) + R(\eta,L)$ with

$$R(\eta,L) = N(1,L) - N(1,\chi_\eta,L) \leqslant N(1,2\eta,L),$$

[12] If it does not hold, then there is an $\varepsilon > 0$ such that, for any $L_0 > 0$, there is an L_1 such that, for any $n \geqslant 0$, it holds that $\varepsilon < N(1, L_1 + nL_0)/N(1, L_1 + (n+1)L_0)$, which gives $N(1, L_1 + (n+1)L_0)\varepsilon^n < N(1, L_1)$ for each n. Now, if L_0 is large enough, we see that (6-1) cannot hold, by making $n \to \infty$.

and thus, if η is small enough, (6-2) gives

$$\limsup_{L\to+\infty} \frac{N(1,\chi_\eta,L)}{N(1,L)} \geqslant \frac{1}{2}.$$

Since $C^{-1}\exp(h_\star L)/L \leqslant N(1,L)$ for large L, (6-3) cannot hold, and thus $c(\eta) > 0$.

In particular, we can apply Lemma 5.1 to get $\lim_L N(n,\chi_\eta,L)(n!/(c(\eta)L)^n)(h_\star L/e^{h_\star L}) = 1$. As $N(n,L) \geqslant N(n,\chi_\eta,L)$, for L large enough,

$$C^{-1}\frac{L^n}{n!}\frac{e^{h_\star L}}{h_\star L} \leqslant N(n,L) \leqslant C\frac{L^n}{n!}\frac{e^{h_\star L}}{h_\star L}$$

(the upper bound comes from Section 4.1.2). Let $\varepsilon > 0$. Then the above estimate combined with the second part of (4-25) in Lemma 4.11 implies that, for $\eta > 0$ small enough,

$$\limsup_L R(n,\eta,L)\frac{n!}{L^n}\frac{h_\star L}{e^{h_\star L}} < \varepsilon,$$

where $R(n,\eta,L) = N(n,L) - N(n,\chi_\eta,L)$. Writing $N(n,\chi_\eta,L) \leqslant N(n,L) \leqslant N(n,\chi_\eta,L) + R(n,\eta,L)$, we obtain

$$c(\eta)^n \leqslant \liminf_L N(n,L)\frac{n!}{L^n}\frac{h_\star L}{e^{h_\star L}} \leqslant \limsup_L N(n,L)\frac{n!}{L^n}\frac{h_\star L}{e^{h_\star L}} \leqslant c(\eta)^n + \varepsilon$$

for any η small enough (depending on ε!). As $\varepsilon > 0$ is arbitrary, we finally get

$$N(n,L) \sim \frac{(c_\star L)^n}{n!}\frac{e^{h_\star L}}{h_\star L} \quad \text{as } L \to +\infty,$$

where $c_\star = \lim_{\eta\to 0} c(\eta) < +\infty$ (the limit exists as $\eta \mapsto c(\eta)$ is nonincreasing and bounded by above by (6-1)).

6.2 The case $\gamma_\star$ is separating

6.2.1 The case $h_1 \neq h_2$ In this case, recall from Section 4 that we have the bound

$$\frac{C^{-1}e^{h_\star L}}{\log(L)^2} \leqslant N(2,L) \leqslant Ce^{h_\star L}$$

for L large enough. In particular, using (4-26) in Lemma 4.11 and Remark 5.3, we may proceed exactly as in Section 6.1 to obtain

$$N(2n,L) \sim \frac{(c_\star L)^n}{n!}\frac{e^{h_\star L}}{h_\star L} \quad \text{as } L \to +\infty,$$

where $c_\star = \lim_{\eta\to 0} c_\pm(\chi_\eta)$.

6.2.2 The case $h_1 = h_2 = h_\star$ In this case, recall from Section 4 that we have the bound

$$\frac{C^{-1}Le^{h_\star L}}{\log(L)^4} \leqslant N(2,L) \leqslant CLe^{h_\star L}$$

for L large enough. In particular, using Lemma 4.11 and Remark 5.4, we may proceed exactly as in Section 6.1 to obtain

$$N(2n, L) \sim 2\frac{(c_\star L)^n}{(2n)!}\frac{e^{h_\star L}}{h_\star L} \quad \text{as } L \to +\infty,$$

where $c_\star = \lim_{\eta\to 0} c_\pm(\chi_\eta)$.

7 A Bowen–Margulis type measure

7.1 Description of the constant $c_\star$

In this subsection we describe the constant $c_\star$ in terms of Pollicott–Ruelle resonant states of the open system (M_δ, φ_t), assuming for simplicity that $\gamma_\star$ is not separating. By Section 2.6, since $\Pi_{\pm,\delta}(h_\star)$ is of rank one (see Section 5.1), we may write

$$\Pi_{\pm,\delta}(h_\star)|_{\Omega^1(M_\delta)} = u_\pm \otimes (\alpha \wedge s_\mp) \quad \text{for } u_\pm \in \mathcal{D}'^1_{E^*_{\pm,\delta}}(M_\delta) \text{ and } s_\mp \in \mathcal{D}'^1_{E^*_{\mp,\delta}}(M_\delta),$$

with $\operatorname{supp}(u_\pm, s_\pm) \subset \Gamma_{\pm,\delta}$ and $u_\pm, s_\mp \in \ker(\iota_X)$. Using the Guillemin trace formula [19] and the Ruelle zeta function $\zeta_{\Sigma_\star}$, we see that the Bowen–Margulis measure μ_0 (see [9]) of the open system (M_δ, φ_t), which is given by Bowen's formula

$$\mu_0(f) = \lim_{L\to+\infty} \sum_{\substack{\gamma\in\mathcal{P}_\delta \\ \ell(\gamma)\leqslant L}} \frac{1}{\ell(\gamma)} \int_0^{\ell(\gamma)} f(\gamma(\tau), \dot\gamma(\tau))\,\mathrm{d}\tau \quad \text{for } f \in C_c^\infty(M_\delta),$$

coincides with the distribution $f \mapsto \operatorname{tr}_s^\flat(f\,\Pi_{\pm,\delta}(h)) = \int_{M_\delta} f u_\pm \wedge \alpha \wedge s_\mp$. Note that $\operatorname{supp}(u_\pm \wedge \alpha \wedge s_\mp) \subset K_\star$, where $K_\star \subset S\Sigma_\star$ is the trapped set. On the other hand, by definition of $\Pi_{\pm,\partial}$,

$$c_\star = \lim_{\eta\to 0} \operatorname{tr}_s^\flat(\chi_\eta \Pi_{\pm,\partial}) = -\lim_{\eta\to 0} \int_\partial \chi_\eta \psi^* \iota^* u_\pm \wedge \iota^* s_\mp.$$

7.2 A Bowen–Margulis type measure

In what follows we set $S_{\gamma_\star}\Sigma = \{(x, v) \in S\Sigma : x \in \gamma_\star\}$ and, for any primitive geodesic $\gamma : \mathbb{R}/\ell(\gamma)\mathbb{Z} \to \Sigma$,

$$I_\star(\gamma) = \{z \in S_{\gamma_\star}\Sigma : z = (\gamma(\tau), \dot\gamma(\tau)) \text{ for some } \tau\}.$$

For any $n \geqslant 1$, we define the set $\Gamma_n \subset S_{\gamma_\star}\Sigma$ by

$$\complement\Gamma_n = \{z \in S_{\gamma_\star}\Sigma : (\widetilde{S}_\pm)^k(z) \text{ is well defined for } k = 1, \dots, n\}.$$

Also, we set $\ell_n(z) = \max(\ell_{+,n}(z), \ell_{-,n}(z))$, where

$$\ell_{\pm,n}(z) = \ell_\pm(z) + \ell_\pm(\widetilde{S}_\pm(z)) + \cdots + \ell_\pm(\widetilde{S}_\pm^{n-1}(z)) \quad \text{for } z \in \complement\Gamma_n,$$

and $\ell_\pm(z) = \inf\{t > 0 : \varphi_{\pm t}(z) \in S_{\gamma_\star}\Sigma\}$.

We will now prove Theorem 3, which says that, for any $f \in C^\infty(S_{\gamma_\star}\Sigma)$, the limit

$$\mu_n(f) = \lim_{L\to+\infty} \frac{1}{N(n, L)} \sum_{\gamma\in\mathcal{P}_n} \frac{1}{n} \sum_{z\in I_\star(\gamma)} f(z) \tag{7-1}$$

exists and defines a probability measure μ_n on $S_{\gamma_\star}\Sigma$ supported in Γ_n. We will also prove that, in the nonseparating case,

$$\mu_n(f) = c_\star^{-n} \lim_{\eta\to 0} \operatorname{tr}_{\mathrm{s}}^\flat(f(\chi_\eta \Pi_{\pm,\partial}\chi_\eta)^n), \tag{7-2}$$

where $c_\star > 0$ is the constant appearing in Theorem 1. Note that here we identify f with its lift $p_\star^* f$ (which is a function on ∂), so that the above formula makes sense (recall that $p_\star\colon S\Sigma_\star \to S\Sigma$ is the natural projection which identifies both components of $\partial S\Sigma_\star = \partial$). Of course, a similar formula holds in the nonseparating case, but we omit it here.

Proof of Theorem 3 Let $f \in C^\infty(S_{\gamma_\star}\Sigma)$ be a nonnegative function. Then, reproducing the arguments in the proof of Proposition 3.7, for $\operatorname{Re}(s)$ big enough,

$$\operatorname{tr}_{\mathrm{s}}^\flat\big(f(\chi_\eta \widetilde{\mathcal{S}}_\pm(s)\chi_\eta)^n\big) = \sum_{i(\gamma,\gamma_\star)=n} \bigg(\sum_{z\in I_\star(\gamma)} f(z)\bigg) e^{-s\ell(\gamma)} I_\star(\gamma,\chi_\eta),$$

where χ_η is as defined in Section 6 and $I_\star(\gamma,\chi_\eta) = I_{\star,\pm}(\gamma,\chi_\eta)$ (see Section 5; this does not depend on $\pm$, as the function F used to construct χ_η is even). Now, as f is nonnegative, we may proceed exactly as in Section 5, replacing $g_{n,\chi}(t)$ by

$$g_{n,\chi_\eta,f}(t) = \sum_{\substack{\gamma\in\mathcal{P}\\ i(\gamma,\gamma_\star)=n}} \bigg(\sum_{z\in I_\star(\gamma)} f(z)\bigg) \sum_{\substack{k\geqslant 1\\ k\ell(\gamma)\leqslant t}} I_\star(\gamma,\chi_\eta) \quad \text{for } t \geqslant 0,$$

to obtain that

$$\lim_{L\to\infty} \frac{n!}{L^n}\frac{h_\star L}{e^{h_\star L}} \sum_{\substack{\gamma\in\mathcal{P}\\ i(\gamma_\star,\gamma)=n\\ \ell(\gamma)\leqslant L}} \bigg(\sum_{z\in I_\star(\gamma)} f(z)\bigg) I_\star(\gamma,\chi_\eta) = \operatorname{Res}_{s=h_\star} \operatorname{tr}_{\mathrm{s}}^\flat\big(f(\chi_\eta \widetilde{\mathcal{S}}_\pm(s)\chi_\eta)^n\big). \tag{7-3}$$

We denote by $\nu_{n,\eta}(f)$ the left-hand side of (7-3). Then $\eta \mapsto \nu_{n,\eta}(f)$ is a nonnegative and nonincreasing function which is bounded by above by $nc_\star^n\|f\|_\infty$ by Theorem 1. In particular, the formula

$$\mu_n(f) = \lim_{\eta\to 0} \frac{1}{nc_\star^n}\nu_{n,\eta}(f) \quad \text{for } f \in C^\infty(S_{\gamma_\star}\Sigma, \mathbb{R}_{\geqslant 0})$$

defines a measure μ_n on $S_{\gamma_\star}\Sigma$ whose total mass is not greater than 1. In fact, its total mass is equal to 1, since, by definition of $c_\star$,

$$\mu_n(1) = \lim_{\eta\to 0} \frac{nc_\pm(\chi_\eta)^n}{nc_\star^n} = 1.$$

Let $\varepsilon > 0$. Then, for each $f \in C^\infty(S_{\gamma_\star}\Sigma, \mathbb{R}_{\geqslant 0})$, one has, by Lemma 4.11,

$$\sum_{\substack{\gamma\in\mathcal{P}\\ i(\gamma_\star,\gamma)=n\\ \ell(\gamma)\leqslant L}} \bigg(\sum_{z\in I_\star(\gamma)} f(z)\bigg)(1 - I_\star(\gamma,\chi_\eta)) \leqslant nN(n,\eta,L)\|f\|_\infty \leqslant \varepsilon nN(n,L)\|f\|_\infty$$

for large L whenever η is small enough. In particular, setting

$$\mu_n^+(f) = \limsup_L \frac{A_f(n,L)}{nN(n,L)} \quad \text{and} \quad \mu_n^-(f) = \liminf_L \frac{A_f(n,L)}{nN(n,L)},$$

where

$$A_f(n,L) = \sum_{\substack{\gamma\in\mathcal{P} \\ i(\gamma_\star,\gamma)=n \\ \ell(\gamma)\leqslant L}} \biggl(\sum_{z\in I_\star(\gamma)} f(z)\biggr),$$

we see that, for each $\varepsilon > 0$ and η small depending on ε,

$$|\mu_n^\pm(f) - \nu_{n,\eta}(f)| \leqslant \varepsilon \|f\|_\infty.$$

Indeed, setting

$$A_f(n,\eta,L) = \sum_{\substack{\gamma\in\mathcal{P} \\ i(\gamma_\star,\gamma)=n \\ \ell(\gamma)\leqslant L}} \biggl(\sum_{z\in I_\star(\gamma)} f(z)\biggr) I_\star(\gamma,\chi_\eta),$$

we have

$$\limsup_L \left| \left(\frac{1}{nN(n,L)} - \frac{n!L^n}{nc_\star^n e^{h_\star L}} \right) A_f(n,\eta,L) \right| = 0$$

by Theorem 1, since $A_f(n,\eta,L) \leqslant nN(n,L)$. Now we may let $\eta\to 0$ to get $|\mu_n^\pm(f)-\mu_n(f)| \leqslant \varepsilon\|f\|_\infty$; since ε is arbitrary, this yields $\mu_n^\pm(f) = \mu_n(f)$. This implies that the limit (7-1) exists, and moreover (7-2) holds by (7-3) (provided that $\gamma_\star$ is not separating).

Next, take a general $f \in C^\infty(S_{\gamma_\star}\Sigma)$, which we no longer assume to be nonnegative. Choose some smooth functions $f_{\delta,\pm}$, $\delta \in\,]0,1[$ with the property that $\|f - (f_{\delta,+} + f_{\delta,-})\|_\infty \leqslant \delta$ and $\pm f_{\delta,\pm} \geqslant 0$, and write $f_\delta = f_{\delta+} + f_{\delta-}$. By nonnegativeness of $\pm f_{\delta,\pm}$, the arguments above imply that $A_{f_\delta}(n,L)/(nN(n,L)) \to \mu_n(f_\delta)$ as $L\to\infty$. On the other hand, $|A_f(n,L) - A_{f_\delta}(n,L)| \leqslant A_{|f-f_\delta|}(n,L) \leqslant \delta n N(n,L)$. Letting $L\to\infty$, this yields

$$\mu_n(f_\delta) - \delta \leqslant \liminf_L \frac{A_f(n,L)}{nN(n,L)} \leqslant \limsup_L \frac{A_f(n,L)}{nN(n,L)} \leqslant \mu_n(f_\delta) + \delta.$$

Since $\mu_n(f_\delta) \to \mu_n(f)$ as $\delta\to 0$, (7-1) and (7-2) are valid for f.

Finally, if $f \in C_c^\infty(S_{\gamma_\star}\Sigma \setminus \Gamma_n)$ then there is $L > 0$ such that

$$\ell_n(z) \leqslant L \quad \text{for } z \in \operatorname{supp}(f).$$

In particular, for any $\gamma\in\mathcal{P}$ such that $i(\gamma,\gamma_\star) = n$ and $\ell(\gamma) > L$, we have $f(z) = 0$ for any $z \in I_\star(\gamma)$. This shows that $\mu_n(f) = 0$, and the support condition for μ_n follows. □

8 A large deviation result

The goal of this section, which is independent of the rest of the paper, is to prove the following result, which is a consequence of a classical large deviation result by Kifer [25]:

Proposition 8.1 *There exists $I_\star > 0$ such that the following holds. For any $\varepsilon > 0$, there are $C, \delta > 0$ such that, for large L,*

$$\frac{1}{N(L)} \#\left\{ \gamma \in \mathcal{P} : \ell(\gamma) \leqslant L \text{ and } \left| \frac{i(\gamma, \gamma_\star)}{\ell(\gamma)} - I_\star \right| \geqslant \varepsilon \right\} \leqslant C \exp(-\delta L). \tag{8-1}$$

In fact, $I_\star = 4i(\bar{m}, \delta_{\gamma_\star})$, where i is Bonahon's intersection form [6], $\delta_{\gamma_\star}$ is the Dirac measure on $\gamma_\star$ and $\bar{m}$ is the renormalized Bowen–Margulis measure on M (here we see the intersection form as a function on the space of φ–invariant measures on $S\Sigma$, as described below). Lalley [28] showed a similar result for self-intersection numbers; see also [41] for self-intersection numbers with prescribed angles.

8.1 Bonahon's intersection form

Let $\mathcal{M}_\varphi(S\Sigma)$ be the set of finite positive measures on $S\Sigma$ invariant by the geodesic flow, endowed with the vague topology. For any closed geodesic γ, we denote by $\delta_\gamma \in \mathcal{M}_\varphi(S\Sigma)$ the Lebesgue measure of γ parametrized by arc length (thus of total mass $\ell(\gamma)$). Let $\mu \in \mathcal{M}_\varphi(S\Sigma)$ be the Liouville measure, that is, the measure associated to the volume form $\frac{1}{2}\alpha \wedge \mathrm{d}\alpha$.

Proposition 8.2 (Bonahon [7]; see also Otal [34]) *There exists a continuous function*

$$i : \mathcal{M}_\varphi(S\Sigma) \times \mathcal{M}_\varphi(S\Sigma) \to \mathbb{R}_+$$

which is additive and positively homogeneous with respect to each variable and such that $i(\mu, \mu) = 2\pi \operatorname{vol}(\Sigma)$ and

$$i(\delta_\gamma, \delta_{\gamma'}) = i(\gamma, \gamma') \quad \text{and} \quad i(\mu, \delta_\gamma) = 2\ell(\gamma),$$

for any closed geodesics γ and γ'.

Remark 8.3 (i) Actually, Bonahon's intersection form is a pairing on the space of *geodesic currents*. This space is naturally identified with the space of φ–invariant measures on $S\Sigma$ which are also invariant by the flip $R\colon (x, v) \mapsto (x, -v)$. By $i(\nu, \nu')$ for general $\nu, \nu' \in \mathcal{M}_\varphi(S\Sigma)$ we simply mean $i(\Phi(\nu), \Phi(\nu'))$ where $\Phi\colon \nu \mapsto \nu + R^*\nu$ (note that $\varphi_t R = R\varphi_{-t}$ for $t \in \mathbb{R}$).

(ii) The formulae for $i(\mu, \mu)$ and $i(\mu, \delta_\gamma)$ differ from [7]; this is due to our convention, since here the Liouville measure μ corresponds to twice the Liouville current considered in [7].

8.2 Large deviations

For any $\nu \in \mathcal{M}_\varphi(S\Sigma)$ we denote by $h(\nu)$ the measure-theoretical entropy of φ with respect to ν. Then we have the following result:

Proposition 8.4 (Kifer [25]) *Let $F \subset \mathcal{M}^1_\varphi(S\Sigma)$ be a closed set, where $\mathcal{M}^1_\varphi(S\Sigma)$ is the set of φ–invariant probability measures on $S\Sigma$. Then*

$$\limsup_L \frac{1}{L} \log \frac{1}{N(L)} \#\left\{ \gamma \in \mathcal{P} : \ell(\gamma) \leqslant L \text{ and } \frac{\delta_\gamma}{\ell(\gamma)} \in F \right\} \leqslant \sup_{\nu \in F} h(\nu) - h,$$

where h is the entropy of the geodesic flow.

Proof of Proposition 8.1 We denote by $\bar{m} \in \mathcal{M}^1_\varphi(S\Sigma)$ the unique probability measure of maximal entropy, that is,

$$\bar{m} = \lim_{L\to+\infty} \sum_{\substack{\gamma\in\mathcal{P}\\ \ell(\gamma)\leqslant L}} \frac{\delta_\gamma}{\ell(\gamma)},$$

where the convergence holds in the weak sense. Let $\varepsilon > 0$. Define

$$F_\varepsilon = \{\nu \in \mathcal{M}^1_\varphi(S\Sigma) : |i(\nu, \delta_{\gamma_\star}) - i(\bar{m}, \delta_{\gamma_\star})| \geqslant \varepsilon\}.$$

Then F_ε is closed in $\mathcal{M}^1_\varphi(S\Sigma)$, and thus compact by the Banach–Alaoglu theorem, and $\bar{m} \in \complement F_\varepsilon$ so that $\delta = h - \sup_{\nu\in F_\varepsilon} h(\nu) > 0$. In particular, for large L,

$$\frac{1}{N(L)} \#\left\{\gamma \in \mathcal{P} : \frac{\delta_\gamma}{\ell(\gamma)} \in F_\varepsilon\right\} \leqslant C \exp(-\delta' L)$$

for some $0<\delta'<\delta$ and $C>0$. Now, by Proposition 8.2, $\delta_\gamma/\ell(\gamma) \in F_\varepsilon$ gives $|i(\gamma,\gamma_\star)/\ell(\gamma) - i(\bar{m}, \delta_{\gamma_\star})| \geqslant \varepsilon$. Let $I_\star = i(\bar{m}, \delta_{\gamma_\star})$. Then it is a well-known fact that $\bar{m}$ has full support in $S\Sigma$, which implies $I_\star > 0$ by definition of $i(\bar{m}, \delta_{\gamma_\star})$; see [34]. □

Remark 8.5 (i) It is not hard to see that Proposition 8.1 implies

$$\frac{1}{N(L)} \sum_{\ell(\gamma)\leqslant L} i(\gamma, \gamma_\star) \sim I_\star L$$

as $L \to +\infty$. Thus we recover [39, Theorem 4].

(ii) If (Σ, g) is hyperbolic, then $\bar{m}$ is the renormalized Liouville measure and, with Proposition 8.2, we find

$$I_\star = \frac{\ell(\gamma_\star)}{2\pi^2(\mathrm{g}-1)}.$$

(iii) If $\varepsilon < I_\star$ then every closed geodesic γ which does not intersect $\gamma_\star$ satisfies $\delta_\gamma/\ell(\gamma) \in F_\varepsilon$. In particular, the right-hand side of (8-1) is bounded from below by $C\exp((h_\star - h)L)$, where we used that $N(0, L) \sim \exp(h_\star L)/h_\star L$ and $N(L) \sim \exp(hL)/hL$ as $L \to \infty$.

Appendix A Closed geodesics minimize intersection numbers

In this section we prove Lemma 2.1. We proceed by contradiction and assume that $i(\gamma_1, \gamma_2) < |\gamma_1 \cap \gamma_2|$. As γ_1 and γ_2 are not powers of each other, the images of γ_1 and γ_2 intersect transversally (otherwise their images would coincide by uniqueness of the geodesic equation). Since $i(\gamma_1, \gamma_2) < |\gamma_1 \cap \gamma_2|$, we may find loops $\alpha_j : \mathbb{R}/\mathbb{Z} \to \Sigma$ for $j = 1, 2$ with $\alpha_j \sim \gamma_j$ and $|\alpha_1 \cap \alpha_2| < |\gamma_1 \cap \gamma_2|$, and we may moreover assume that α_1 and α_2 intersect transversally. Let $H_j : [0,1] \times \mathbb{R}/\mathbb{Z} \to \Sigma$ for $j = 1, 2$ be smooth homotopies between γ_j and α_j, and define $H : [0,1] \times \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \to \Sigma \times \Sigma$ by setting

$$H(s, \tau_1, \tau_2) = (H_1(s, \tau_1), H_2(s, \tau_2)) \quad \text{for } (s, \tau_1, \tau_2) \in [0,1] \times \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}.$$

Let $\Delta(\Sigma) = \{(x,x) : x \in \Sigma\}$ be the diagonal in Σ. Then $H(0,\cdot)$ and $H(1,\cdot)$ are transverse to $\Delta(\Sigma)$, in the sense that, for every $k = 0, 1$ and $(\tau_1, \tau_2) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ with $H(k, \tau_1, \tau_2) \in \Delta(\Sigma)$,

$$\mathrm{d}H(k,\tau_1,\tau_2) T_{(k,\tau_1,\tau_2)}(\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}) + T_{H(k,\tau_1,\tau_2)}\Delta(\Sigma) = T_{H(k,\tau_1,\tau_2)}(\Sigma \times \Sigma).$$

In particular, by [20, Corollary page 73] we may assume that H is globally transverse to $\Delta(\Sigma)$, so that $H^{-1}(\Delta(\Sigma))$ is a smooth 1–dimensional submanifold of $[0,1] \times (\mathbb{R}/\mathbb{Z})^2$. Now

$$|\gamma_1 \cap \gamma_2| = |H^{-1}(\Delta(\Sigma)) \cap (\{0\} \times (\mathbb{R}/\mathbb{Z})^2)| \quad \text{and} \quad |\alpha_1 \cap \alpha_2| = |H^{-1}(\Delta(\Sigma)) \cap (\{1\} \times (\mathbb{R}/\mathbb{Z})^2)|.$$

Since $|\gamma_1 \cap \gamma_2| > |\alpha_1 \cap \alpha_2|$ and because $H^{-1}(\Delta(\Sigma))$ is smooth, we may find a smooth path $c\colon [0,1] \to [0,1] \times (\mathbb{R}/\mathbb{Z})^2$ such that $c(0) \neq c(1)$ and

$$\mathrm{Im}(c) \subset H^{-1}(\Delta(\Sigma)) \quad \text{and} \quad c(0), c(1) \in \{0\} \times (\mathbb{R}/\mathbb{Z})^2.$$

Write $c = (S, T_1, T_2)$ for some smooth functions $S\colon [0,1] \to [0,1]$ and $T_j\colon [0,1] \to \mathbb{R}/\mathbb{Z}$, and for $u \in [0,1]$ define the path $c_u = (uS, T_1, T_2)\colon [0,1] \to [0,1] \times (\mathbb{R}/\mathbb{Z})^2$. Let $x_k = H(c(k)) \in \Sigma$ for $k = 0, 1$. Then define the paths

$$\beta_{j,u} = \pi_j \circ H \circ c_u\colon [0,1] \to \Sigma \quad \text{for } j = 1,2 \text{ and } u \in [0,1],$$

where $\pi_1, \pi_2\colon \Sigma \times \Sigma \to \Sigma$ are the projections over the first and second factor, respectively. As $c_1 = c$ and $\mathrm{Im}(c) \subset H^{-1}(\Delta(\Sigma))$, we have $\beta_{1,1} = \beta_{2,1}$. In particular, the paths $\beta_{1,0}$ and $\beta_{2,0}$ are homotopic within the space of curves linking x_0 and x_1, since for each u, one has $\beta_{j,u}(k) = x_k$ for $j = 1,2$ and $k = 0,1$. Moreover, the paths $\beta_{1,0}$ and $\beta_{2,0}$ are subpaths of γ_1 and γ_2, respectively, and are in particular geodesic paths. Let $\widetilde{\Sigma}$ be a universal cover of Σ and take $\tilde{x}_0 \in \widetilde{\Sigma}$ a lift of x_0. For $j = 1,2$, let $\tilde{\beta}_j\colon [0,1] \to \widetilde{\Sigma}$ be the unique lift of $\beta_{j,0}$ starting at $\tilde{x}_0$. Then $\tilde{\beta}_1(1) = \tilde{\beta}_2(1)$ since the paths $\beta_{j,0}$ for $j = 1,2$ are homotopic in Σ via a homotopy preserving endpoints. In particular, we have found two distinct geodesic segments of $\widetilde{\Sigma}$ joining $\tilde{x}_0$ and $\tilde{\beta}_0(1)$ (the image of the paths $\tilde{\beta}_{j,0}$ for $j = 1,2$ cannot coincide since $c(0) \neq c(1)$ and the intersection $\gamma_1 \cap \gamma_2$ is transverse). Thus the exponential map $\exp_{\tilde{x}_0}\colon T_{\tilde{x}_0}\widetilde{\Sigma} \to \widetilde{\Sigma}$ at $\tilde{x}_0$ is not a diffeomorphism, and $\widetilde{\Sigma}$ cannot be negatively curved by virtue of the Cartan–Hadamard theorem (see for example [29, Theorem 11.5]). This completes the proof.

Appendix B An elementary fact about pullbacks of distributions

Lemma B.1 *Let $K \in \mathcal{D}'(\mathbb{R}^d \times \mathbb{R}^d)$ be a compactly supported distribution. We assume that $\mathrm{WF}(K) \subset \Gamma$, where $\Gamma \subset T^*(\mathbb{R}^d \times \mathbb{R}^d)$ is a closed conical subset such that*

$$\Gamma \cap N^*\Delta = \varnothing, \quad \text{where } N^*\Delta = \{(x,\xi,x,-\xi) : (x,\xi) \in T^*\mathbb{R}^d\}.$$

In particular, the pullback i^*K, where $i : x \mapsto (x,x)$, is well defined. Then, for $N \in \mathbb{N}_{\geqslant 1}$ large enough, the following holds. Let $u \in C_c^N(\mathbb{R}^d)$ and assume that the pullback $i^*(\pi_1^* u K)$ is well defined, where $\pi_1\colon (x,x) \mapsto x$ is the projection on the first factor. Then

$$i^*(\pi_1^* u \cdot K) = u \cdot i^* K.$$

Proof Let $K_\varepsilon \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, $\varepsilon \in]0,1]$, be a sequence of distributions supported in a fixed compact set such that $K_\varepsilon \to K$ in $\mathcal{D}'_\Gamma(\mathbb{R}^d \times \mathbb{R}^d)$. Let $\Gamma' \subset T^*(\mathbb{R}^d \times \mathbb{R}^d)$ be an open conical subset containing $N^*\Delta$. As K_ε is compactly supported, we may assume that $|t-q| > \delta_0$ for any $(t,q) \in \Gamma \times \Gamma'$ such that $|t| = |q| = 1$ for some $\delta_0 > 0$. By definition of the convergence in $\mathcal{D}'_\Gamma(\mathbb{R}^d \times \mathbb{R}^d)$ (see [23, Definition 8.2.2]), for every N there is $C_N > 0$ such that, for any $\varepsilon > 0$ small enough,

$$|\widehat{K}_\varepsilon(q)| \leqslant C_N \langle q\rangle^{-N} \quad \text{for } q \in \Gamma'. \tag{B-1}$$

Let $\Gamma'' \subset \Gamma'$ be another open conical subset containing $N^*\Delta$, and let $\delta > 0$ be such that, for any $q \in \Gamma''$ and $t \in \mathbb{R}^{2d}$,

$$|t-q| < \delta|q| \implies t \in \Gamma'. \tag{B-2}$$

Then, for any $q \in \Gamma''$,

$$\begin{aligned}(2\pi)^{2d}|\widehat{K_\varepsilon\pi_1^* u}(q)| &\leqslant \int_{\mathbb{R}_t^{2d}} |\widehat{K}_\varepsilon(t)| \cdot |\widehat{\pi_1^* u}(q-t)|\,\mathrm{d}t \\ &\leqslant \int_{|t-q|<\delta|q|} |\widehat{K}_\varepsilon(t)| \cdot |\widehat{\pi_1^* u}(q-t)|\,\mathrm{d}t + \int_{|t-q|\geqslant\delta|q|} |\widehat{K}_\varepsilon(t)| \cdot |\widehat{\pi_1^* u}(q-t)|\,\mathrm{d}t.\end{aligned}$$

Let $N_1, N_2 \in \mathbb{N}_{\geqslant 1}$ and $\langle t\rangle = \sqrt{1+|t|^2}$. Then, using (B-1), (B-2) and Peetre's inequality, and assuming that $u \in C_c^{N_2}(\mathbb{R}^d)$ with $N_2 \geqslant 2d+1$,

$$\begin{aligned}\int_{|t-q|<\delta|t|} |\widehat{K}_\varepsilon(t)| \cdot |\widehat{\pi_1^* u}(q-t)|\,\mathrm{d}t &\leqslant C_{N_1,N_2} \int_{|t-q|<\delta|q|} \langle t\rangle^{-N_1} \langle q-t\rangle^{-N_2}\,\mathrm{d}t \\ &\leqslant C'_{N_1,N_2} \langle q\rangle^{-N_1+N_2} \int_{\mathbb{R}^d} \langle t\rangle^{-N_2}\,\mathrm{d}t.\end{aligned}$$

On the other hand, if k is the order of K and $N_3 \in \mathbb{N}_{\geqslant 1}$ is such that $u \in C_c^{N_3}(\mathbb{R}^d)$, then

$$\begin{aligned}\int_{|t-q|\geqslant\delta|q|} |\widehat{K}_\varepsilon(t)| \cdot |\widehat{\pi_1^* u}(q-t)|\,\mathrm{d}t &\leqslant C_{k,N_3} \int_{|t-q|\geqslant\delta|q|} \langle t\rangle^{k} \langle q-t\rangle^{-N_3} \\ &\leqslant C'_{k,N_3} \langle q\rangle^{-N_3+(k+2d+1)} \int_{\mathbb{R}^{2d}} \langle t\rangle^{-2d-1}\,\mathrm{d}t.\end{aligned}$$

Therefore, if $u \in C^N(\mathbb{R}^d)$ with $N = k+2d+1+N'$,

$$(2\pi)^{2d}|\widehat{K_\varepsilon\pi_1^* u}(q)| \leqslant C_N \langle q\rangle^{-N'} \quad \text{for } q \in \Gamma''. \tag{B-3}$$

Note that, for $\varphi \in C_c^\infty(\mathbb{R}^d)$,

$$\langle i^*(K_\varepsilon\pi_1^* u), \varphi\rangle = \int_{\mathbb{R}_x^d} \varphi(x) \int_{\mathbb{R}_\xi^d \times \mathbb{R}_\eta^d} \widehat{K_\varepsilon\pi_1^* u}(\xi,\eta) e^{ix(\xi+\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}x.$$

Indeed, (B-3) shows that the integral in (ξ,η) converges near $N^*\Delta$ if $N' \geqslant 2d+1$, and far from $N^*\Delta$ we can use the stationary phase method to get enough convergence in (ξ,η), so the above integral makes sense as an oscillatory integral and coincides with $\langle i^*(K_\varepsilon\pi_1^* u), \varphi\rangle$, since this formula is obviously true if u is smooth. Moreover, all the above estimates are uniform in ε and thus, letting $\varepsilon \to 0$, we obtain the desired result, since obviously $i^*(K_\varepsilon\pi_1^* u) = u(i^* K_\varepsilon)$ for each $\varepsilon \in]0,1]$. □

References

[1] **N Anantharaman**, *Precise counting results for closed orbits of Anosov flows*, Ann. Sci. École Norm. Sup. 33 (2000) 33–56 MR Zbl

[2] **D V Anosov**, *Geodesic flows on closed Riemann manifolds with negative curvature*, Proceedings of the Steklov Institute of Mathematics 90, Amer. Math. Soc., Providence, RI (1969) MR Zbl

[3] **M F Atiyah**, **R Bott**, *A Lefschetz fixed point formula for elliptic complexes, I*, Ann. of Math. 86 (1967) 374–407 MR Zbl

[4] **V Baladi**, **M F Demers**, *On the measure of maximal entropy for finite horizon Sinai billiard maps*, J. Amer. Math. Soc. 33 (2020) 381–449 MR Zbl

[5] **V Baladi**, **M F Demers**, **C Liverani**, *Exponential decay of correlations for finite horizon Sinai billiard flows*, Invent. Math. 211 (2018) 39–177 MR Zbl

[6] **F Bonahon**, *Bouts des variétés hyperboliques de dimension* 3, Ann. of Math. 124 (1986) 71–158 MR Zbl

[7] **F Bonahon**, *The geometry of Teichmüller space via geodesic currents*, Invent. Math. 92 (1988) 139–162 MR Zbl

[8] **Y Bonthonneau**, *Les résonances du Laplacien sur les variétés à pointes*, PhD thesis, Université Paris Sud (2015) Available at `http://www.theses.fr/2015PA112141.pdf`

[9] **R Bowen**, *The equidistribution of closed geodesics*, Amer. J. Math. 94 (1972) 413–423 MR Zbl

[10] **R Bowen**, *Symbolic dynamics for hyperbolic flows*, Amer. J. Math. 95 (1973) 429–460 MR Zbl

[11] **M R Bridson**, **A Haefliger**, *Metric spaces of non-positive curvature*, Grundl. Math. Wissen. 319, Springer (1999) MR Zbl

[12] **F Dal'bo**, *Remarques sur le spectre des longueurs d'une surface et comptages*, Bol. Soc. Brasil. Mat. 30 (1999) 199–221 MR Zbl

[13] **N V Dang**, **G Rivière**, *Poincaré series and linking of Legendrian knots*, preprint (2020) arXiv 2005.13235

[14] **H Delange**, *Généralisation du théorème de Ikehara*, Ann. Sci. École Norm. Sup. 71 (1954) 213–242 MR Zbl

[15] **S Dyatlov**, **C Guillarmou**, *Pollicott–Ruelle resonances for open systems*, Ann. Henri Poincaré 17 (2016) 3089–3146 MR Zbl

[16] **S Dyatlov**, **M Zworski**, *Dynamical zeta functions for Anosov flows via microlocal analysis*, Ann. Sci. Éc. Norm. Supér. 49 (2016) 543–577 MR Zbl

[17] **V Erlandsson**, **J Souto**, *Counting curves in hyperbolic surfaces*, Geom. Funct. Anal. 26 (2016) 729–777 MR Zbl

[18] **C Guillarmou**, *Lens rigidity for manifolds with hyperbolic trapped sets*, J. Amer. Math. Soc. 30 (2017) 561–599 MR Zbl

[19] **V Guillemin**, *Lectures on spectral theory of elliptic operators*, Duke Math. J. 44 (1977) 485–517 MR Zbl

[20] **V Guillemin**, **A Pollack**, *Differential topology*, Prentice-Hall, Englewood Cliffs, NJ (1974) MR Zbl

[21] **L Guillopé**, *Sur la distribution des longueurs des géodésiques fermées d'une surface compacte à bord totalement géodésique*, Duke Math. J. 53 (1986) 827–848 MR Zbl

[22] **G Higman**, **B H Neumann**, **H Neumann**, *Embedding theorems for groups*, J. London Math. Soc. 24 (1949) 247–254 MR Zbl

[23] **L Hörmander**, *The analysis of linear partial differential operators, I: Distribution theory and Fourier analysis*, 2nd edition, Grundl. Math. Wissen. 256, Springer (1990) MR Zbl

[24] **A Katsuda**, **T Sunada**, *Homology and closed geodesics in a compact Riemann surface*, Amer. J. Math. 110 (1988) 145–155 MR Zbl

[25] **Y Kifer**, *Large deviations, averaging and periodic orbits of dynamical systems*, Comm. Math. Phys. 162 (1994) 33–46 MR Zbl

[26] **S P Lalley**, *Closed geodesics in homology classes on surfaces of variable negative curvature*, Duke Math. J. 58 (1989) 795–821 MR Zbl

[27] **S P Lalley**, *Renewal theorems in symbolic dynamics, with applications to geodesic flows, non-Euclidean tessellations and their fractal limits*, Acta Math. 163 (1989) 1–55 MR Zbl

[28] **S P Lalley**, *Self-intersections of closed geodesics on a negatively curved surface: statistical regularities*, from "Convergence in ergodic theory and probability" (V Bergelson, P March, J Rosenblatt, editors), Ohio State Univ. Math. Res. Inst. Publ. 5, de Gruyter, Berlin (1996) 263–272 MR Zbl

[29] **J M Lee**, *Riemannian manifolds: an introduction to curvature*, Graduate Texts in Math. 176, Springer (1997) MR Zbl

[30] **R C Lyndon**, **P E Schupp**, *Combinatorial group theory*, Ergebnisse der Math. 89, Springer (1977) MR Zbl

[31] **G A Margulis**, *Certain applications of ergodic theory to the investigation of manifolds of negative curvature*, Funkcional. Anal. i Priložen. 3 (1969) 89–90 MR Zbl In Russian; translated in Funct. Anal. Appl. 3 (1969) 335–336

[32] **M Mirzakhani**, *Growth of the number of simple closed geodesics on hyperbolic surfaces*, Ann. of Math. 168 (2008) 97–125 MR Zbl

[33] **M Mirzakhani**, *Counting mapping class group orbits on hyperbolic surfaces*, preprint (2016) arXiv 1601.03342

[34] **J-P Otal**, *Le spectre marqué des longueurs des surfaces à courbure négative*, Ann. of Math. 131 (1990) 151–162 MR Zbl

[35] **W Parry**, **M Pollicott**, *An analogue of the prime number theorem for closed orbits of Axiom A flows*, Ann. of Math. 118 (1983) 573–591 MR Zbl

[36] **W Parry**, **M Pollicott**, *Zeta functions and the periodic orbit structure of hyperbolic dynamics*, Astérisque 187–188, Soc. Math. France, Paris (1990) MR Zbl

[37] **F Paulin**, **M Pollicott**, **B Schapira**, *Equilibrium states in negative curvature*, Astérisque 373, Soc. Math. France, Paris (2015) MR Zbl

[38] **R Phillips**, **P Sarnak**, *Geodesics in homology classes*, Duke Math. J. 55 (1987) 287–297 MR Zbl

[39] **M Pollicott**, *Asymptotic distribution of closed geodesics*, Israel J. Math. 52 (1985) 209–224 MR Zbl

[40] **M Pollicott**, *Homology and closed geodesics in a compact negatively curved surface*, Amer. J. Math. 113 (1991) 379–385 MR Zbl

[41] **M Pollicott**, **R Sharp**, *Angular self-intersections for closed geodesics on surfaces*, Proc. Amer. Math. Soc. 134 (2006) 419–426 MR Zbl

[42] **T Roblin**, *Ergodicité et équidistribution en courbure négative*, Mém. Soc. Math. Fr. 95, Soc. Math. France, Paris (2003) MR Zbl

[43] **P C Sarnak**, *Prime geodesic theorems*, PhD thesis, Stanford University (1980) MR Available at https://www.proquest.com/docview/303065936

[44] **Y G Sinai**, *Dynamical systems with elastic reflections: ergodic properties of dispersing billiards*, Uspehi Mat. Nauk 25 (1970) 141–192 MR Zbl In Russian; translated in Russian Math. Surveys 25 (1970) 137–189

[45] **I M Singer**, **J A Thorpe**, *Lecture notes on elementary topology and geometry*, Scott, Foresman and Co, Glenview, IL (1967) MR Zbl

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