



# *Geometry & Topology*

Volume 28 (2024)

**Riemannian manifolds with entire Grauert tube are rationally elliptic**

XIAOYANG CHEN

# Riemannian manifolds with entire Grauert tube are rationally elliptic

XIAOYANG CHEN

It was conjectured by Bott, Grove and Halperin that a compact simply connected Riemannian manifold  $M$  with nonnegative sectional curvature is rationally elliptic. We confirm this conjecture under the stronger assumption that  $M$  has entire Grauert tube, ie  $M$  is a real-analytic Riemannian manifold that has a unique adapted complex structure defined on the whole tangent bundle  $TM$ . Our result also provides a strong topological obstruction to the existence of an entire Grauert tube.

53C20

## 1 Introduction

The following conjecture is a central problem in the study of Riemannian manifolds with nonnegative sectional curvature; see Berger and Bott [2] and Grove and Halperin [7].

**Conjecture** (Bott–Grove–Halperin) A compact simply connected Riemannian manifold  $M$  with nonnegative sectional curvature is rationally elliptic.

Here  $M$  is said to be rationally elliptic if and only if it has finite-dimensional rational homotopy groups, ie all but finitely many homotopy groups of  $M$  are finite; otherwise  $M$  is said to be rationally hyperbolic. It is a well-known simple consequence of Sullivan’s minimal model theory [15] that  $M$  being rationally elliptic is equivalent to polynomial growth of the sequence of Betti numbers of its based loop space  $\Omega M$  relative to rational coefficient. If  $M$  is rationally elliptic, then there are severe topological restrictions of  $M$ . For example,  $M$  has nonnegative Euler characteristic number and  $\dim H_*(M, \mathbb{Q}) \leq 2^n$ ; see Félix, Halperin and Thomas [5] and Grove and Halperin [7].

It is known that compact simply connected homogeneous spaces and cohomogeneity one-manifolds are rationally elliptic; see Grove and Halperin [8]. Grove, Wilking and Yeager [9] confirmed the Bott–Grove–Halperin conjecture under the additional assumption that  $M$  supports an isometric action with principal orbits of codimension two.

In this paper we confirm the Bott–Grove–Halperin conjecture under the stronger assumption that  $M$  has entire Grauert tube:

**Theorem 1.1** *Let  $(M, g)$  be an  $n$ -dimensional compact simply connected real-analytic Riemannian manifold that has entire Grauert tube, then  $M$  is rationally elliptic.*

**Remark 1.2** In fact, our proof shows that  $M$  is topologically elliptic, ie the Betti numbers of its loop space relative to any field of coefficients grow at most polynomially.

Here  $(M, g)$  is said to be real-analytic if  $M$  is a real-analytic manifold with a real-analytic Riemannian metric  $g$ . Then there is a unique adapted complex structure defined on  $T^R M = \{v \in TM \mid g(v, v) < R^2\}$  for some  $R > 0$ ; see Guillemin and Stenzel [10], Lempert and Szőke [12] and Szőke [16]. When  $R = \infty$ , then  $M$  is said to have entire Grauert tube. It was shown in [12] that a Riemannian manifold with entire Grauert tube has nonnegative sectional curvature. Hence Theorem 1.1 gives a partial answer to the Bott–Grove–Halperin conjecture. On the other hand, it also provides a strong topological obstruction to the existence of an entire Grauert tube.

Aguilar [1] showed that the quotient of a Riemannian manifold with entire Grauert tube by a group of isometries acting freely also has entire Grauert tube. All known manifolds with entire Grauert tube are obtained by Aguilar’s construction: starting with a compact Lie group with a bi-invariant metric, or the product of such a group with Euclidean space, one takes the quotient by some group of isometries acting freely. Such quotient manifolds include almost all closed manifolds which are known to have Riemannian metrics with nonnegative sectional curvature.

It was conjectured by Hopf that the Euler characteristic number of a compact Riemannian manifold with nonnegative sectional curvature is nonnegative. The following corollary settles this conjecture under the stronger assumption that  $M$  has entire Grauert tube.

**Corollary 1.3** *Let  $M$  be a compact Riemannian manifold with entire Grauert tube. Then  $M$  has nonnegative Euler characteristic number.*

**Proof** If  $M$  has finite fundamental group, then its universal cover  $\tilde{M}$  with the induced Riemannian metric also has entire Grauert tube. By Theorem 1.1, the Euler characteristic number of  $\tilde{M}$  is nonnegative. Hence  $M$  has nonnegative Euler characteristic number. If  $M$  has infinite fundamental group, as  $M$  has nonnegative sectional curvature, then the Euler characteristic number of  $M$  is zero; see Cheeger and Gromoll [4].  $\square$

A related conjecture proposed by Totaro [17] predicts that a compact Riemannian manifold  $M$  with nonnegative sectional curvature has a good complexification, ie  $M$  is diffeomorphic to a smooth affine algebraic variety  $U$  over the real numbers such that the inclusion  $U(\mathbb{R}) \rightarrow U(\mathbb{C})$  is a homotopy equivalence. The Euler characteristic number of a compact manifold which has a good complexification is also nonnegative. Also, a conjecture by Burns [3] predicts that for every compact Riemannian manifold  $M$  with entire Grauert tube, the complex manifold  $TM$  is an affine algebraic variety in a natural way. If this is correct, the complex manifold  $TM$  would be a good complexification of  $M$  in the above sense. Both conjectures of Totaro and Burns are still open.

The proof of [Theorem 1.1](#) is based on the counting function introduced in Berger and Bott [\[2\]](#), Gromov [\[6\]](#) and Paternain [\[14\]](#). For  $x \in M$  and each  $T > 0$ , let

$$D_T := \{v \in T_x M \mid g(v, v) \leq T^2\}$$

be the disk of radius  $T$  in  $T_x M$ . Define the counting function  $n_T(x, y)$  by

$$n_T(x, y) := \#((\exp_x)^{-1}(y) \cap D_T).$$

In other words,  $n_T(x, y)$  counts the number of geodesic arcs joining  $x$  to  $y$  with length  $\leq T$ . When  $M$  is simply connected, then we have the crucial inequality

$$(1-1) \quad \sum_{j=0}^{k-1} \dim H_j(\Omega M, F) \leq \frac{1}{\text{Vol}_g(M)} \int_M n_{Ck}(x, y) dy,$$

where  $C$  is a positive constant independent of  $k$  and  $F$  is any field of coefficients; see [\[6; 14\]](#).

For any  $x \in M$ , Berger and Bott [\[2\]](#) proved that  $\int_M n_T(x, y) dy$  can be computed by Jacobi fields on  $M$ ; see also Paternain [\[14\]](#). Precisely, they showed that

$$(1-2) \quad \int_M n_T(x, y) dy = \int_0^T d\sigma \int_{\mathbb{S}} \sqrt{\det(g(J_j(\sigma), J_k(\sigma)))_{j,k=1,2,\dots,n-1}} d\theta,$$

where  $\mathbb{S}$  is the unit sphere of  $T_x M$ . Moreover, the  $J_j$  for  $j = 1, 2, \dots, n - 1$  are Jacobi fields along the unique geodesic  $\gamma$  determined by  $\theta \in \mathbb{S}$  (ie  $\gamma(0) = x, \gamma'(0) = \theta$ ) with initial conditions

$$J_j(0) = 0, \quad J'_j(0) = v_j,$$

where the  $v_j$  for  $j = 1, 2, \dots, n - 1$  form an orthonormal basis of  $T_\theta \mathbb{S}$ .

If  $(M, g)$  has entire Grauert tube, the right-hand side in [\(1-2\)](#) can be further described by a matrix valued holomorphic function on the upper half plane. Applying Fatou’s representation theorem to this function, we will show that  $\int_M n_T(x, y) dy$  is a polynomial function of  $T$ . When  $M$  is simply connected, it follows that  $\sum_{j=0}^{k-1} \dim H_j(\Omega M, F)$  has polynomial growth for any field of coefficients. Hence  $M$  is topologically elliptic.

We finally mention that based on an iterated use of the Rauch comparison theorem for Jacobi fields, an estimate for the Betti numbers of  $\Omega M$  for manifolds with  $0 < \delta \leq \sec M \leq 1$  was derived in [\[2\]](#). Although the estimate is given in terms of the pinching constant  $\delta$ , its growth rate is exponential.

**Acknowledgements** The author is partially supported by NSFC 12171364 and 23JC1403600 (project title *On the topology of almost nonnegatively curved manifolds*). He would like to thank the referee for helpful suggestions.

## 2 Vertical and horizontal subbundles

In this section we recall some basic facts on the geometry of the tangent bundle  $TM$ . For more details, see [14]. Let  $\pi: TM \rightarrow M$  be the canonical projection, ie if  $\theta = (x, v) \in TM$ , then  $\pi(\theta) = x$ . There exists a canonical subbundle of  $TTM$ , called the vertical subbundle, whose fiber at  $\theta$  is given by the tangent vectors of curves  $\sigma: (-\epsilon, \epsilon) \rightarrow TM$  of the form:  $\sigma(t) = (x, v + t\omega)$ , where  $\omega \in T_x M$ . In other words,

$$V(\theta) = \ker((\pi_*)_{\theta}).$$

Suppose that  $M$  is endowed with a Riemannian metric  $g$ . We shall define the connection map

$$K: TTM \rightarrow TM$$

as follows: let  $\xi \in T_{\theta}TM$  and  $z: (-\epsilon, \epsilon) \rightarrow TM$  be an adapted curve to  $\xi$ , that is, with initial conditions

$$z(0) = \theta, \quad z'(0) = \xi.$$

Such a curve gives rise to a curve  $\alpha: (-\epsilon, \epsilon) \rightarrow M$  given by  $\alpha := \pi \circ z$ , and a vector field  $Z$  along  $\alpha$ ; equivalently,  $z(t) = (\alpha(t), Z(t))$ . Define

$$K_{\theta}(\xi) := (\nabla_{\alpha} Z)(0) = \lim_{t \rightarrow 0} \frac{(P_t)^{-1} Z(t) - Z(0)}{t},$$

where  $P_t: T_x M \rightarrow T_{\alpha(t)} M$  is the linear isomorphism defined by the parallel transport along  $\alpha$ . The horizontal subbundle is the subbundle of  $TTM$  whose fiber at  $\theta$  is given by

$$H(\theta) = \ker K_{\theta}.$$

Another equivalent way of constructing the horizontal subbundle is by means of the horizontal lift

$$L_{\theta}: T_x M \rightarrow T_{\theta}TM,$$

which is defined as follows. Let  $\theta = (x, v)$ . Given  $\omega \in T_x M$  and  $\alpha: (-\epsilon, \epsilon) \rightarrow M$  an adapted curve of  $\omega$ , ie  $\alpha(0) = x$ ,  $\alpha'(0) = \omega$ , let  $Z(t)$  be the parallel transport of  $v$  along  $\alpha$  and  $\sigma: (-\epsilon, \epsilon) \rightarrow TM$  be the curve  $\sigma(t) = (\alpha(t), Z(t))$ . Then

$$L_{\theta}(w) = \sigma'(0) \in T_{\theta}TM.$$

**Proposition 2.1**  $K_{\theta}$  and  $L_{\theta}$  have the following properties:

$$(\pi_*)_{\theta} \circ L_{\theta} = \text{Id} \quad \text{and} \quad K_{\theta} \circ i_* = \text{Id},$$

where  $i: T_x M \rightarrow TM$  is the inclusion map. Moreover,

$$T_{\theta}TM = H(\theta) \oplus V(\theta)$$

and the map  $j_{\theta}: T_{\theta}TM \rightarrow T_x M \times T_x M$  given by

$$j_{\theta}(\xi) = ((\pi_*)_{\theta}(\xi), K_{\theta}(\xi))$$

is a linear isomorphism.

For each  $\theta \in TM$ , there is a unique geodesic  $\gamma_\theta$  in  $M$  with initial condition  $\theta$ . Let  $\xi \in T_\theta TM$  and  $z: (-\epsilon, \epsilon) \rightarrow TM$  be an adapted curve to  $\xi$ , that is, with initial conditions

$$z(0) = \theta, \quad z'(0) = \xi.$$

Then the map  $(s, t) \mapsto \pi \circ \phi_t(z(s))$  gives rise to a variation of  $\gamma_\theta$ . Here  $\pi: TM \rightarrow M$  is the projection map and  $\phi_t$  is the geodesic flow of  $TM$ . The curves  $t \mapsto \pi \circ \phi_t(z(s))$  are geodesics and therefore the corresponding variational vector fields  $J_\xi := (\partial/\partial s)|_{s=0} \pi \circ \phi_t(z(s))$  is a Jacobi field with initial conditions

$$J_\xi(0) = (\pi_*)_\theta(\xi), \quad J'_\xi(0) = K_\theta(\xi).$$

### 3 Adapted complex structure on the tangent bundle

In this section we describe the adapted complex structure on the tangent bundle. Let  $(M, g)$  be a compact smooth Riemannian manifold. Then  $TM \setminus M$  carries a natural foliation by Riemannian surfaces defined as follows. For  $\tau \in \mathbb{R}$  denote by  $N_\tau: TM \rightarrow TM$  the smooth mapping defined by multiplication by  $\tau$  in the fibers. If  $\gamma: \mathbb{R} \rightarrow M$  is a geodesic, define an immersion  $\phi_\gamma: \mathbb{C} \rightarrow TM$  by

$$\phi_\gamma(\sigma + i\tau) = N_\tau \gamma'(\sigma).$$

If for two geodesics  $\gamma$  and  $\delta$ , it holds that  $\phi_\gamma(\mathbb{C} \setminus \mathbb{R})$  and  $\phi_\delta(\mathbb{C} \setminus \mathbb{R})$  intersect each other, then  $\gamma$  and  $\delta$  are the same geodesic traversed with different velocities, hence  $\phi_\gamma(\mathbb{C}) = \phi_\delta(\mathbb{C})$ . Therefore the images of  $\mathbb{C} \setminus \mathbb{R}$  under the mapping  $\phi_\gamma$  defines a smooth foliation of  $TM \setminus M$  by surfaces. Moreover, each leaf has complex structure that it inherits from  $\mathbb{C}$  via  $\phi_\gamma$ . The leaves along with their complex structure extend across  $M$ , but of course, on  $M$  the foliation  $\mathcal{F}$  becomes singular.

Given  $R > 0$ , put

$$T^R M = \{v \in TM \mid g(v, v) < R^2\}.$$

A smooth complex structure on  $T^R M$  will be called adapted if the leaves of the foliation  $\mathcal{F}$  with the complex structure inherited from  $\mathbb{C}$  are complex submanifolds of  $T^R M$ .

**Theorem 3.1** [10; 12; 16] *Let  $M$  be a compact real-analytic manifold equipped with a real-analytic metric  $g$ . Then there exists some  $R > 0$  such that  $T^R M$  carries a unique adapted complex structure.*

When the adapted complex structure is defined on the whole tangent bundle, ie  $R = \infty$ , then  $M$  is said to have entire Grauert tube. It was shown in [12] that a Riemannian manifold with entire Grauert tube has nonnegative sectional curvature. The adapted complex structure on  $T^R M$  can be described as follows. For this purpose let  $\theta \in T^R M \setminus M$  and  $x = \pi(\theta)$ , where  $\pi: TM \rightarrow M$  is the projection map. Let  $\gamma$  be a geodesic determined by  $\theta$ . Choose tangent vectors  $v_1, v_2, \dots, v_{n-1}$  such that  $v_1, v_2, \dots, v_{n-1}, v_n := \gamma'(0)/|\gamma'(0)|$  form an orthonormal basis of  $T_x M$ .

Denote by  $L_\theta$  the leaf of the foliation  $\mathcal{F}$  passing through  $\theta$ . A vector  $\bar{\xi} \in T_\theta TM$  determines a vector field  $\xi$  (we call it the parallel vector field) along  $L_\theta$  by defining it to be invariant under two semigroup actions. Namely,  $\xi$  is invariant under  $N_\tau$  and the geodesic flow. For this parallel field  $\xi$ , we get that  $\xi|_{\mathbb{R}}$  is a Jacobi field along  $\gamma$ .

Now choose a set of vectors  $\bar{\xi}_1, \bar{\xi}_2, \dots, \bar{\xi}_n, \bar{\eta}_1, \bar{\eta}_2, \dots, \bar{\eta}_n \in T_\theta TM$  satisfying

$$\begin{aligned} (\pi_*)_\theta(\bar{\xi}_j) &= v_j, & K_\theta(\bar{\xi}_j) &= 0, \\ (\pi_*)_\theta(\bar{\eta}_j) &= 0, & K_\theta(\bar{\eta}_j) &= v_j. \end{aligned}$$

Here  $K: TTM \rightarrow TM$  is the connection map described in Section 2. Extend  $\bar{\xi}_j$  and  $\bar{\eta}_j$  to get parallel vector fields  $\xi_1, \xi_2, \dots, \xi_n, \eta_1, \eta_2, \dots, \eta_n$  along  $L_\theta$ . Then the Jacobi fields  $\xi_1|_{\mathbb{R}}, \xi_2|_{\mathbb{R}}, \dots, \xi_n|_{\mathbb{R}}$  are linearly independent except on a discrete subset  $S_1$  of  $\mathbb{R}$ . Hence there are smooth real-valued functions  $\phi_{jk}$  defined on  $\mathbb{R} \setminus S_1$  such that

$$\eta_k|_{\mathbb{R}} = \sum_{j=1}^n \phi_{jk} \xi_j|_{\mathbb{R}}.$$

From the presence of the adapted complex structure it follows that the functions  $\phi_{jk}$  have meromorphic extension  $f_{jk}$  over the domain

$$D = \left\{ \sigma + i\tau \in \mathbb{C} \mid |\tau| < \frac{R}{\sqrt{g(\theta, \theta)}} \right\}$$

such that for each pair  $j, k$ , the poles of  $f_{jk}$  lie on  $\mathbb{R}$  and the matrix  $\text{Im}(f_{jk})|_{D \setminus \mathbb{R}}$  is invertible. Let  $(e_{jk}) = (\text{Im } f_{jk}(i))^{-1}$ . Then the complex structure  $J$  satisfies

$$J\bar{\xi}_h = \sum_{k=1}^n e_{kh} \times \left[ \bar{\eta}_k - \sum_{j=1}^n \text{Re } f_{jk}(i) \bar{\xi}_j \right].$$

**Remark 1** Because  $\xi_1|_{\mathbb{R}}, \xi_2|_{\mathbb{R}}, \dots, \xi_{n-1}|_{\mathbb{R}}, \eta_1|_{\mathbb{R}}, \eta_2|_{\mathbb{R}}, \dots, \eta_{n-1}|_{\mathbb{R}}$  are normal Jacobi fields, while  $\xi_n|_{\mathbb{R}}$  and  $\eta_n|_{\mathbb{R}}$  are tangential Jacobi fields, for  $1 \leq j, k \leq n-1$  we have

$$\phi_{nk} = \phi_{jn} \equiv 0, \quad f_{nk} = f_{jn} \equiv 0, \quad e_{nk} = e_{jn} \equiv 0.$$

Consider the  $n$ -tuples

$$\Xi = (\xi_1, \xi_2, \dots, \xi_n) \quad \text{and} \quad H = (\eta_1, \eta_2, \dots, \eta_n),$$

and holomorphic  $n$ -tuples

$$\Xi^{1,0} = (\xi_1^{1,0}, \xi_2^{1,0}, \dots, \xi_n^{1,0}) \quad \text{and} \quad H^{1,0} = (\eta_1^{1,0}, \eta_2^{1,0}, \dots, \eta_n^{1,0}),$$

where  $\xi_j^{1,0} = \frac{1}{2}(\xi_j - iJ\xi_j)$  and  $J$  is the adapted complex structure. Then we have

$$H(\sigma) = \Xi(\sigma)f(\sigma) \quad \text{and} \quad H^{1,0}(\sigma + i\tau) = \Xi^{1,0}(\sigma + i\tau)f(\sigma + i\tau)$$

where

$$f(\sigma + i\tau) = (f_{jk}(\sigma + i\tau)) \quad \text{for } \sigma \in \mathbb{R} \setminus S_1 \text{ and } |\tau| < \frac{R}{\sqrt{g(\theta, \theta)}}.$$

The following facts are proved in [12; 16].

**Proposition 3.2** (1) The vectors  $\xi_1^{1,0}, \xi_2^{1,0}, \dots, \xi_n^{1,0}$  are linearly independent over  $\mathbb{C}$  on  $D \setminus \mathbb{R}$ . The same is true for the vectors  $\eta_1^{1,0}, \eta_2^{1,0}, \dots, \eta_n^{1,0}$ .

(2) The  $2n$  vectors  $\xi_j, \eta_k$  are linearly independent at points  $\sigma + i\tau \in D \setminus \mathbb{R}$ .

**Theorem 3.3** The matrix-valued meromorphic function  $f(\sigma + i\tau)$  is symmetric (as a matrix) and satisfies

$$f(0) = 0, \quad f'(0) = \text{Id}.$$

Moreover, if  $\sigma + i\tau \in D$  with  $\tau > 0$ , then  $\text{Im } f(\sigma + i\tau)$  is a symmetric, positive definite matrix.

### 4 Growth rate of counting functions

In this section we prove Theorem 1.1.

Let  $M$  be an  $n$ -dimensional compact manifold endowed with a Riemannian metric  $g$ . For  $x \in M$  and each  $T > 0$ , let

$$D_T := \{v \in T_x M \mid g(v, v) \leq T^2\}$$

be the disk of radius  $T$  in  $T_x M$ . Define the counting function  $n_T(x, y)$  by

$$n_T(x, y) := \sharp((\exp_x)^{-1}(y) \cap D_T).$$

In other words,  $n_T(x, y)$  counts the number of geodesic arcs joining  $x$  to  $y$  with length  $\leq T$ .

The following theorems proved in [2; 6; 14] will be crucial for us.

**Theorem 4.1** We have

$$(4-1) \quad \int_M n_T(x, y) dy = \int_0^T d\sigma \int_{\mathbb{S}} \sqrt{\det(g(J_j(\sigma), J_k(\sigma)))_{j,k=1,2,\dots,n-1}} d\theta,$$

where  $\mathbb{S}$  is the unit sphere of  $T_x M$ . Moreover,  $J_j$  for  $j = 1, 2, \dots, n - 1$  are Jacobi fields along the unique geodesic  $\gamma$  determined by  $\theta \in \mathbb{S}$  (ie  $\gamma(0) = x$  and  $\gamma'(0) = \theta$ ) with initial conditions

$$J_j(0) = 0, \quad J'_j(0) = v_j,$$

where the  $v_j$  with  $j = 1, 2, \dots, n - 1$  form an orthonormal basis of  $T_\theta \mathbb{S}$ .

**Theorem 4.2** Suppose that  $M$  is an  $n$ -dimensional compact simply connected manifold endowed with a Riemannian metric  $g$ . Then

$$(4-2) \quad \sum_{j=0}^{k-1} \dim H_j(\Omega M, F) \leq \frac{1}{\text{Vol}_g(M)} \int_M n_{Ck}(x, y) dy,$$

where  $C$  is a positive constant independent of  $k$  and  $F$  is any field of coefficients.

**Remark 4.3** The assumption that  $M$  is simply connected in Theorem 4.2 is essential.



When  $M$  has entire Grauert tube, we will see that the right-hand side in [Theorem 4.1](#) can be further described by a matrix-valued holomorphic function on the upper half-plane. Applying Fatou’s representation theorem to this function, we will derive that  $\int_M n_T(x, y) dy$  has polynomial growth and hence  $M$  is topologically elliptic.

Now we give the details of the proof. Let  $\mathbb{S}$  be the unit sphere of  $T_x M$  and  $\gamma$  the unique geodesic determined by  $\theta \in \mathbb{S}$ , ie  $\gamma(0) = x, \gamma'(0) = \theta$ . Let  $v_1, v_2, \dots, v_n := \gamma'(0)$  be an orthonormal basis of  $T_x M$ .

As in [Section 3](#), choose a set of vectors  $\bar{\xi}_1, \bar{\xi}_2, \dots, \bar{\xi}_n, \bar{\eta}_1, \bar{\eta}_2, \dots, \bar{\eta}_n \in T_\theta T M$  satisfying

$$\begin{aligned} \pi_*(\bar{\xi}_j) &= v_j, & K\bar{\xi}_j &= 0, \\ \pi_*(\bar{\eta}_j) &= 0, & K\bar{\eta}_j &= v_j. \end{aligned}$$

Here  $K: T T M \rightarrow T M$  is the connection map described in [Section 2](#). Extend  $\bar{\xi}_j$  and  $\bar{\eta}_j$  to get parallel vector fields  $\xi_1, \xi_2, \dots, \xi_n, \eta_1, \eta_2, \dots, \eta_n$ . Then  $J_j := \eta_j|_{\mathbb{R}}$  for  $j = 1, 2, \dots, n - 1$  are normal Jacobi fields along  $\gamma$  with initial conditions

$$J_j(0) = 0, \quad J'_j(0) = v_j.$$

Moreover,  $\xi_1|_{\mathbb{R}}, \xi_2|_{\mathbb{R}}, \dots, \xi_n|_{\mathbb{R}}$  are linearly independent except on a discrete subset  $S_1$  of  $\mathbb{R}$ . Hence there are smooth real-valued functions  $\phi_{jk}$  defined on  $\mathbb{R} \setminus S_1$  such that

$$\eta_k|_{\mathbb{R}} = \sum_{j=1}^n \phi_{jk} \xi_j|_{\mathbb{R}}.$$

As  $M$  has entire Grauert tube, it follows that the functions  $\phi_{jk}$  have a meromorphic extension  $f_{jk}$  over the whole complex plane such that for each pair  $j, k$ , the poles of  $f_{jk}$  lie on  $\mathbb{R}$  and the matrix  $\text{Im}(f_{jk})|_{\mathbb{C} \setminus \mathbb{R}}$  is invertible.

Consider the  $n$ -tuples

$$\Xi = (\xi_1, \xi_2, \dots, \xi_n) \quad \text{and} \quad H = (\eta_1, \eta_2, \dots, \eta_n),$$

and holomorphic  $n$ -tuples

$$\Xi^{1,0} = (\xi_1^{1,0}, \xi_2^{1,0}, \dots, \xi_n^{1,0}) \quad \text{and} \quad H^{1,0} = (\eta_1^{1,0}, \eta_2^{1,0}, \dots, \eta_n^{1,0}),$$

where  $\xi_j^{1,0} = \frac{1}{2}(\xi_j - iJ\xi_j)$  and  $J$  is the adapted complex structure.

Then we have

$$H(\sigma) = \Xi(\sigma)f(\sigma) \quad \text{and} \quad H^{1,0}(\sigma + i\tau) = \Xi^{1,0}(\sigma + i\tau)f(\sigma + i\tau),$$

where

$$f(\sigma + i\tau) = (f_{jk}(\sigma + i\tau)) \quad \text{for } \sigma \in \mathbb{R} \setminus S_1.$$

**Lemma 4.4** *If  $\sigma + i\tau \in \mathbb{C} \setminus \mathbb{R}$ , then  $\text{Im } f^{-1}(\sigma + i\tau)$  is invertible.*

**Proof** The proof is almost identical to the proof of Proposition 6.8 in [12]. Suppose there is a nonzero column vector  $v = (v_j) \in \mathbb{R}^n$  such that  $\text{Im } f^{-1}(\sigma + i\tau)v = 0$ ,  $\tau \neq 0$ , ie  $\omega = (\omega_k) = f^{-1}(\sigma + i\tau)v \in \mathbb{R}^n$ . By Proposition 3.2,  $f^{-1}$  exists on  $\mathbb{C} \setminus \mathbb{R}$ . Then we have

$$\Xi^{1,0} = H^{1,0} f^{-1}$$

at the point  $\sigma + i\tau$ . Hence

$$\sum \xi_j^{1,0} v_j = \Xi^{1,0} v = H^{1,0} f^{-1} v = H^{1,0} \omega = \sum \eta_k^{1,0} \omega_k.$$

Taking real parts, we get

$$\sum \xi_j v_j = \sum \eta_k \omega_k,$$

in contradiction with Proposition 3.2. □

**Lemma 4.5**  $G(\zeta) := -f^{-1}(\zeta)$  is a matrix-valued meromorphic function on  $\mathbb{C}$  whose pole lies in a discrete subset of  $\mathbb{R}$  and  $\text{Im } G(\zeta)$  is positive definite for  $\zeta = \sigma + i\tau \in \mathbb{C}^+$ , where  $\mathbb{C}^+$  is the upper half-plane.

**Proof** Since  $H^{1,0}$  and  $\Xi^{1,0}$  are invertible on  $\mathbb{C}$  except on a discrete subset, combined with  $H^{1,0} = \Xi^{1,0} f$  we get that  $G(\zeta)$  is a matrix-valued meromorphic function on  $\mathbb{C}$  whose pole lies in a discrete subset of  $\mathbb{R}$ . By Theorem 3.3, we have

$$f(0) = 0, \quad f'(0) = \text{Id}.$$

Then for small positive  $\tau$ , we get

$$\begin{aligned} \text{Im } G(i\tau) &= \text{Im}(-f^{-1}(i\tau)) = \text{Im}(-(f(0) + i\tau f'(0) + O(\tau^2))^{-1}) \\ &= \text{Im}(-i\tau \text{Id} + O(\tau^2))^{-1} = \text{Im}\left(\frac{i}{\tau}(\text{Id} + O(\tau))^{-1}\right). \end{aligned}$$

Hence  $\text{Im } G(i\tau)$  is positive definite for small positive  $\tau$ . As  $\text{Im } G(\zeta)$  is nondegenerate on  $\mathbb{C}^+$  by Lemma 4.4,  $\text{Im } G(\zeta)$  is positive definite for  $\zeta = \sigma + i\tau \in \mathbb{C}^+$ . □

Let  $f_1 = (f_{jk})$  with  $j, k = 1, 2, \dots, n-1$ . Then we have:

**Lemma 4.6** There exists a discrete subset  $S_2 \subset \mathbb{R}$  such that for  $\sigma \in \mathbb{R} \setminus S_2$ , we have

$$(4-3) \quad \det(g(J_j(\sigma), J_k(\sigma))_{j,k=1,2,\dots,n-1}) = \frac{1}{\det((-f_1^{-1})'(\sigma))},$$

where  $J_j$  for  $j = 1, 2, \dots, n$  are normal Jacobi fields along  $\gamma$  with initial conditions

$$J_j(0) = 0, \quad J'_j(0) = v_j,$$

and  $v_1, v_2, \dots, v_n := \gamma'(0)$  is an orthonormal basis of  $T_x M$ .

**Proof** We can view  $\Xi(\sigma)$  and  $H(\sigma)$  as linear mappings  $\mathbb{R}^n \rightarrow T_{\gamma(\sigma)}M$ , given by

$$(\omega_j) = \omega \mapsto \Xi(\sigma)\omega = \sum_{j=1}^n \omega_j \xi_j(\sigma)$$

and similarly for  $H(\sigma)$ . Denote by  $\Xi^*(\sigma)$  and  $H^*(\sigma)$  the adjoints of  $\Xi(\sigma)$  and  $H(\sigma)$ , respectively (adjoint defined using the Euclidean scalar product on  $\mathbb{R}^n$  and the Riemannian metric on  $T_{\gamma(\sigma)}M$ ). Let  $\{e_j\}$  be the standard orthonormal basis of  $\mathbb{R}^n$ . Then we have

$$g(J_j(\sigma), J_k(\sigma)) = g(H(\sigma)e_j, H(\sigma)e_k) = \langle H^*(\sigma)H(\sigma)e_j, e_k \rangle.$$

By the proof of Proposition 6.11 in [12], we get

$$\Xi^*(\sigma)\Xi(\sigma)f'(\sigma) = \text{Id} \quad \text{for } \sigma \in (0, c)$$

for some small positive constant  $c$ . On the other hand, we have

$$\Xi(\sigma)e_j = \xi_j(\sigma) \quad \text{and} \quad \Xi^*(\sigma)\Xi(\sigma)e_j = g(\xi_j(\sigma), \xi_k(\sigma))e_k.$$

Hence  $\Xi^*(\sigma)\Xi(\sigma)$  is real-analytic over  $\mathbb{R}$  and so it has a holomorphic extension to a small open set in  $\mathbb{C}$  containing  $\mathbb{R}$ . As  $M$  has entire Grauert tube, it follows that  $f(\sigma)$  has a meromorphic extension over the whole complex plane such that its poles lie on a discrete subset  $S_1 \subset \mathbb{R}$ . Then we have

$$\Xi^*(\sigma)\Xi(\sigma)f'(\sigma) = \text{Id} \quad \text{for } \sigma \in \mathbb{R} \setminus S_1.$$

On the other hand,  $f^{-1}(\sigma)$  exists on  $\sigma \in \mathbb{R} \setminus S'_1$  for some discrete subset  $S'_1$ . Moreover,  $f(\sigma)$  is symmetric, by Proposition 6.11 in [12] and analytic continuation. Let  $S_2 = S_1 \cup S'_1$ . For  $\sigma \in \mathbb{R} \setminus S_2$ , we have

$$\begin{aligned} H^*(\sigma)H(\sigma) &= (\Xi(\sigma)f(\sigma))^* \Xi(\sigma)f(\sigma) = f(\sigma)\Xi^*(\sigma)\Xi(\sigma)f(\sigma) \\ &= f(\sigma)(f'(\sigma))^{-1}f(\sigma) = ((-f^{-1})'(\sigma))^{-1}. \end{aligned}$$

Since  $f_{jn} = f_{nk} = 0$  for  $j, k = 1, 2, \dots, n-1$ , we see that

$$\det(g(J_j(\sigma), J_k(\sigma)))_{j,k=1,2,\dots,n-1} = \frac{1}{\det((-f_1^{-1})'(\sigma))} \quad \text{for } \sigma \in \mathbb{R} \setminus S_2. \quad \square$$

The following version of Fatou’s representation theorem will be crucial for us.

**Proposition 4.7** *Suppose that  $F$  is an  $n \times n$  matrix-valued holomorphic function on the upper half-plane  $\mathbb{C}^+ = \{\zeta \in \mathbb{C} \mid \text{Im } \zeta > 0\} \cup (\mathbb{R} \setminus P)$ , where  $P$  is a discrete subset of  $\mathbb{R}$  consisting of poles of  $F$ . Suppose that for every  $\zeta \in \mathbb{C}^+$ ,  $\text{Im } F(\zeta)$  is a symmetric, positive definite matrix, whereas for  $\zeta \in \mathbb{R} \setminus P$ ,  $\text{Im } F(\zeta) = 0$ . Then there is an  $n \times n$  symmetric matrix  $\mu = (\mu_{jk})$  whose entries are real-valued, signed Borel measures on  $\mathbb{R}$  such that:*

(1°)  $\mu_{jk}$  does not have mass on any interval which does not contain a pole of  $F$ .

(2°)  $\int_{-\infty}^{+\infty} \frac{|d\mu_{jk}(t)|}{1+t^2} < \infty.$

(3°)  $\mu$  is positive semidefinite in the sense that for any  $(\omega_j) \in \mathbb{R}^n$ , the measure  $\sum \omega_j \omega_k \mu_{jk}$  is non-negative.

(4°) For  $\zeta \in \mathbb{C}^+$ ,

$$F'(\zeta) = A + \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{d\mu(t)}{(\zeta - t)^2},$$

where  $A$  is a symmetric, positive semidefinite constant matrix. In fact, we have

$$A = \lim_{\tau \rightarrow +\infty} \frac{\text{Im } F(i\tau)}{\tau},$$

and  $d\mu(\sigma)$  is the weak limit of  $\text{Im } F(\sigma + i\tau)$  as  $\tau \rightarrow 0^+$ .

**Proof** See [11] and Proposition 7.4 in [12]. The only difference is that we require  $F$  has a holomorphic extension to  $\mathbb{R} \setminus P$ , hence we get that  $\mu_{jk}$  does not have mass on any interval which does not contain a pole of  $F$ . □

Now we are going to finish the proof of Theorem 1.1. Applying Proposition 4.7 to the matrix-valued holomorphic function  $(-f_1^{-1})$  on the upper half-plane, we get

$$(4-4) \quad (-f_1^{-1})'(\zeta) = A + \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{d\mu(t)}{(\zeta - t)^2} \quad \text{for } \zeta \in \mathbb{C}^+,$$

where  $A = (a_{jk})$  is a symmetric, positive semidefinite constant matrix and  $\mu$  is an  $n \times n$  positive semidefinite symmetric matrix whose entries are real-valued, signed Borel measures on  $\mathbb{R}$ . By analytic continuation, equation (4-4) also holds on  $\mathbb{R}$  except a discrete subset. Moreover,  $\mu$  does not have mass on any interval which does not contain a pole of  $-f_1^{-1}$ . This yields that

$$(-f_1^{-1})'(\sigma) = A + \frac{1}{\pi} \sum_j \frac{\mu(t_j)}{(\sigma - t_j)^2} \quad \text{for } \sigma \in \mathbb{R} \setminus \{t_1, t_2, \dots\},$$

where  $\{t_1, t_2, \dots\}$  are poles of  $-f_1^{-1}$ . As  $f(0) = 0$ , we see that 0 is pole of  $-f_1^{-1}$ .

**Lemma 4.8**  $\mu(0) = \pi \text{ Id.}$

**Proof** By Proposition 4.7, we get

$$\begin{aligned} \mu(0) &= \lim_{\delta \rightarrow 0^+} \mu(-\delta, \delta) = \lim_{\delta \rightarrow 0^+} \lim_{\tau \rightarrow 0^+} \int_{-\delta}^{\delta} \text{Im}(-f_1^{-1}(\sigma + i\tau)) d\sigma \\ &= \lim_{\delta \rightarrow 0^+} \lim_{\tau \rightarrow 0^+} \int_{-\delta}^{\delta} \text{Im}\left(-\left(f_{jk}(0) + f'_{jk}(0)(\sigma + i\tau) + O(\sigma + i\tau)^2\right)_{1 \leq j, k \leq n-1}^{-1}\right) d\sigma \\ &= \lim_{\delta \rightarrow 0^+} \lim_{\tau \rightarrow 0^+} \int_{-\delta}^{\delta} \text{Im}\left(-\left((\sigma + i\tau) \text{ Id} + O(\sigma + i\tau)^2\right)^{-1}\right) d\sigma \\ &= \lim_{\delta \rightarrow 0^+} \lim_{\tau \rightarrow 0^+} \int_{-\delta}^{\delta} \text{Im}\left(-\frac{1}{\sigma + i\tau} (\text{Id} + O(\sigma + i\tau))^{-1}\right) d\sigma \\ &= \lim_{\delta \rightarrow 0^+} \lim_{\tau \rightarrow 0^+} \int_{-\delta}^{\delta} \text{Im}\left(-\frac{1}{\sigma + i\tau} \text{ Id} + O(1)\right) d\sigma = \lim_{\delta \rightarrow 0^+} \lim_{\tau \rightarrow 0^+} \int_{-\delta}^{\delta} \frac{\tau}{\sigma^2 + \tau^2} d\sigma \text{ Id} = \pi \text{ Id.} \quad \square \end{aligned}$$

Given Lemma 4.8, then we have

$$(-f_1^{-1})'(\sigma) = \frac{1}{\sigma^2} \text{Id} + B,$$

where

$$B = A + \frac{1}{\pi} \sum_{t_j \neq 0} \frac{\mu(t_j)}{(\sigma - t_j)^2}$$

is positive semidefinite.

**Lemma 4.9** Let  $A_1$  and  $A_2$  be two  $k \times k$  Hermitian positive semidefinite complex matrix, then

$$\det(A_1 + A_2) \geq \det A_1 + \det A_2.$$

**Proof** It follows from the Minkowski determinant theorem [13, page 115] that

$$(\det(A_1 + A_2))^{1/k} \geq (\det A_1)^{1/k} + (\det A_2)^{1/k}. \quad \square$$

By Theorem 3.3, we get that  $f(\sigma + i\tau)$  is a symmetric matrix, so is  $-f_1^{-1}(\sigma + i\tau)$ . By Proposition 4.7, we see that  $A$  and  $\mu(t_j)$  are real-valued symmetric positive semidefinite matrix. By Lemma 4.9, we get

$$\frac{1}{\det((-f_1^{-1})'(\sigma))} \leq \sigma^{2n-2}.$$

By Theorem 4.1 and Lemma 4.6, we see

$$\int_M n_T(x, y) dy \leq p(T),$$

where  $p(T)$  is a polynomial of degree at most  $n$ . By Theorem 4.2,  $\sum_{j=0}^{k-1} \dim H_j(\Omega M, F)$  has polynomial growth for any field of coefficients. It follows that  $M$  is topologically elliptic.

To illustrate the idea of the above proof, we give two examples here. Let  $M$  be an  $n$ -dimensional compact manifold of constant sectional curvature  $c$ . From the proof of Theorem 2.5 in [16], we have

$$f_1(\sigma + i\tau) = \begin{cases} (\sigma + i\tau) \text{Id} & \text{if } c = 0, \\ (tg(\sigma + i\tau)) \text{Id} & \text{if } c = 1. \end{cases}$$

**Case 1** When  $c = 0$ , then  $-f_1^{-1}(\sigma + i\tau) = (-1/(\sigma + i\tau)) \text{Id}$ . Hence

$$(-f_1^{-1})'(\sigma) = \frac{1}{\sigma^2} \text{Id}.$$

Let  $F(\sigma + i\tau) := -f_1^{-1}(\sigma + i\tau)$ . In this case, the matrix  $A$  and measure  $\mu$  in Proposition 4.7 can be computed by

$$A = \lim_{\tau \rightarrow +\infty} \frac{\text{Im } F(i\tau)}{\tau} = 0,$$

$$\mu(0) = \lim_{\delta \rightarrow 0^+} \mu(-\delta, \delta) = \lim_{\delta \rightarrow 0^+} \lim_{\tau \rightarrow 0^+} \int_{-\delta}^{\delta} \text{Im } F(\sigma + i\tau) d\sigma = \pi \text{Id}.$$

Then  $\int_M n_T(x, y) dy$  has polynomial growth of degree  $n$ .

**Case 2** When  $c = 1$ , then  $-f_1^{-1}(\sigma + i\tau) = (-\cot(\sigma + i\tau)) \text{Id}$ . Hence

$$(-f_1^{-1})'(\sigma) = \frac{1}{\sin^2(\sigma)} \text{Id}.$$

Let  $F(\sigma + i\tau) := -f_1^{-1}(\sigma + i\tau)$ . In this case, the matrix  $A$  and measure  $\mu$  in [Proposition 4.7](#) can be computed by

$$A = \lim_{\tau \rightarrow +\infty} \frac{\text{Im } F(i\tau)}{\tau} = 0,$$

$$\mu(j\pi) \equiv \mu(0) = \lim_{\delta \rightarrow 0^+} \mu(-\delta, \delta) = \lim_{\delta \rightarrow 0^+} \lim_{\tau \rightarrow 0^+} \int_{-\delta}^{\delta} \text{Im } F(\sigma + i\tau) d\sigma = \pi \text{Id} \quad \text{for } j \in \mathbb{Z}.$$

Then  $\int_M n_T(x, y) dy$  has linear growth.

## References

- [1] **R M Aguilar**, *Symplectic reduction and the homogeneous complex Monge–Ampère equation*, Ann. Global Anal. Geom. 19 (2001) 327–353 [MR](#) [Zbl](#)
- [2] **M Berger, R Bott**, *Sur les variétés à courbure strictement positive*, Topology 1 (1962) 301–311 [MR](#) [Zbl](#)
- [3] **D M Burns, Jr, K K Leung**, *The complex Monge–Ampère equation, Zoll metrics and algebraization*, Math. Ann. 371 (2018) 1–40 [MR](#) [Zbl](#)
- [4] **J Cheeger, D Gromoll**, *On the structure of complete manifolds of nonnegative curvature*, Ann. of Math. 96 (1972) 413–443 [MR](#) [Zbl](#)
- [5] **Y Félix, S Halperin, J-C Thomas**, *Rational homotopy theory*, Graduate Texts in Math. 205, Springer (2001) [MR](#) [Zbl](#)
- [6] **M Gromov**, *Homotopical effects of dilatation*, J. Differential Geom. 13 (1978) 303–310 [MR](#) [Zbl](#)
- [7] **K Grove, S Halperin**, *Contributions of rational homotopy theory to global problems in geometry*, Inst. Hautes Études Sci. Publ. Math. 56 (1982) 171–177 [MR](#) [Zbl](#)
- [8] **K Grove, S Halperin**, *Dupin hypersurfaces, group actions and the double mapping cylinder*, J. Differential Geom. 26 (1987) 429–459 [MR](#) [Zbl](#)
- [9] **K Grove, B Wilking, J Yeager**, *Almost non-negative curvature and rational ellipticity in cohomogeneity two*, Ann. Inst. Fourier (Grenoble) 69 (2019) 2921–2939 [MR](#) [Zbl](#)
- [10] **V Guillemin, M Stenzel**, *Grauert tubes and the homogeneous Monge–Ampère equation*, J. Differential Geom. 34 (1991) 561–570 [MR](#) [Zbl](#)
- [11] **P Koosis**, *Introduction to  $H_p$  spaces*, Lond. Math. Soc. Lect. Note Ser. 40, Cambridge Univ. Press (1980) [MR](#) [Zbl](#)
- [12] **L Lempert, R Szöke**, *Global solutions of the homogeneous complex Monge–Ampère equation and complex structures on the tangent bundle of Riemannian manifolds*, Math. Ann. 290 (1991) 689–712 [MR](#) [Zbl](#)
- [13] **M Marcus, H Minc**, *A survey of matrix theory and matrix inequalities*, revised edition, Dover, New York (2010)
- [14] **G P Paternain**, *Geodesic flows*, Progr. Math. 180, Birkhäuser, Boston, MA (1999) [MR](#) [Zbl](#)

- [15] **D Sullivan**, *Infinitesimal computations in topology*, Inst. Hautes Études Sci. Publ. Math. 47 (1977) 269–331  
[MR](#) [Zbl](#)
- [16] **R Szőke**, *Complex structures on tangent bundles of Riemannian manifolds*, Math. Ann. 291 (1991) 409–428  
[MR](#) [Zbl](#)
- [17] **B Totaro**, *Complexifications of nonnegatively curved manifolds*, J. Eur. Math. Soc. 5 (2003) 69–94 [MR](#) [Zbl](#)

*School of Mathematical Sciences*

*Key Laboratory of Intelligent Computing and Applications (Ministry of Education)*

*Institute for Advanced Study, Tongji University*

*Shanghai, China*

[xychen100@tongji.edu.cn](mailto:xychen100@tongji.edu.cn)

Proposed: Gang Tian

Seconded: John Lott, Dmitri Burago

Received: 3 January 2021

Revised: 5 May 2022

# GEOMETRY & TOPOLOGY

[msp.org/gt](http://msp.org/gt)

## MANAGING EDITOR

András I Stipsicz    Alfréd Rényi Institute of Mathematics  
[stipsicz@renyi.hu](mailto:stipsicz@renyi.hu)

## BOARD OF EDITORS

Mohammed Abouzaid	Stanford University <a href="mailto:abouzaid@stanford.edu">abouzaid@stanford.edu</a>	Mark Gross	University of Cambridge <a href="mailto:mgross@dpms.cam.ac.uk">mgross@dpms.cam.ac.uk</a>
Dan Abramovich	Brown University <a href="mailto:dan_abramovich@brown.edu">dan_abramovich@brown.edu</a>	Rob Kirby	University of California, Berkeley <a href="mailto:kirby@math.berkeley.edu">kirby@math.berkeley.edu</a>
Ian Agol	University of California, Berkeley <a href="mailto:ianagol@math.berkeley.edu">ianagol@math.berkeley.edu</a>	Bruce Kleiner	NYU, Courant Institute <a href="mailto:bkleiner@cims.nyu.edu">bkleiner@cims.nyu.edu</a>
Arend Bayer	University of Edinburgh <a href="mailto:arend.bayer@ed.ac.uk">arend.bayer@ed.ac.uk</a>	Sándor Kovács	University of Washington <a href="mailto:skovacs@uw.edu">skovacs@uw.edu</a>
Mark Behrens	University of Notre Dame <a href="mailto:mbehrens1@nd.edu">mbehrens1@nd.edu</a>	Urs Lang	ETH Zürich <a href="mailto:urs.lang@math.ethz.ch">urs.lang@math.ethz.ch</a>
Mladen Bestvina	University of Utah <a href="mailto:bestvina@math.utah.edu">bestvina@math.utah.edu</a>	Marc Levine	Universität Duisburg-Essen <a href="mailto:marc.levine@uni-due.de">marc.levine@uni-due.de</a>
Martin R Bridson	University of Oxford <a href="mailto:bridson@maths.ox.ac.uk">bridson@maths.ox.ac.uk</a>	Ciprian Manolescu	University of California, Los Angeles <a href="mailto:cm@math.ucla.edu">cm@math.ucla.edu</a>
Jim Bryan	University of British Columbia <a href="mailto:jbryan@math.ubc.ca">jbryan@math.ubc.ca</a>	Haynes Miller	Massachusetts Institute of Technology <a href="mailto:hmr@math.mit.edu">hmr@math.mit.edu</a>
Dmitri Burago	Pennsylvania State University <a href="mailto:burago@math.psu.edu">burago@math.psu.edu</a>	Tomasz Mrowka	Massachusetts Institute of Technology <a href="mailto:mrowka@math.mit.edu">mrowka@math.mit.edu</a>
Tobias H Colding	Massachusetts Institute of Technology <a href="mailto:colding@math.mit.edu">colding@math.mit.edu</a>	Aaron Naber	Northwestern University <a href="mailto:anaber@math.northwestern.edu">anaber@math.northwestern.edu</a>
Simon Donaldson	Imperial College, London <a href="mailto:s.donaldson@ic.ac.uk">s.donaldson@ic.ac.uk</a>	Peter Ozsváth	Princeton University <a href="mailto:petero@math.princeton.edu">petero@math.princeton.edu</a>
Yasha Eliashberg	Stanford University <a href="mailto:eliash-gt@math.stanford.edu">eliash-gt@math.stanford.edu</a>	Leonid Polterovich	Tel Aviv University <a href="mailto:polterov@post.tau.ac.il">polterov@post.tau.ac.il</a>
Benson Farb	University of Chicago <a href="mailto:farb@math.uchicago.edu">farb@math.uchicago.edu</a>	Colin Rourke	University of Warwick <a href="mailto:gt@maths.warwick.ac.uk">gt@maths.warwick.ac.uk</a>
David M Fisher	Rice University <a href="mailto:davidfisher@rice.edu">davidfisher@rice.edu</a>	Roman Sauer	Karlsruhe Institute of Technology <a href="mailto:roman.sauer@kit.edu">roman.sauer@kit.edu</a>
Mike Freedman	Microsoft Research <a href="mailto:michaelf@microsoft.com">michaelf@microsoft.com</a>	Stefan Schwede	Universität Bonn <a href="mailto:schwede@math.uni-bonn.de">schwede@math.uni-bonn.de</a>
David Gabai	Princeton University <a href="mailto:gabai@princeton.edu">gabai@princeton.edu</a>	Natasa Sesum	Rutgers University <a href="mailto:natasas@math.rutgers.edu">natasas@math.rutgers.edu</a>
Stavros Garoufalidis	Southern U. of Sci. and Tech., China <a href="mailto:stavros@mpim-bonn.mpg.de">stavros@mpim-bonn.mpg.de</a>	Gang Tian	Massachusetts Institute of Technology <a href="mailto:tian@math.mit.edu">tian@math.mit.edu</a>
Cameron Gordon	University of Texas <a href="mailto:gordon@math.utexas.edu">gordon@math.utexas.edu</a>	Ulrike Tillmann	Oxford University <a href="mailto:tillmann@maths.ox.ac.uk">tillmann@maths.ox.ac.uk</a>
Jesper Grodal	University of Copenhagen <a href="mailto:jg@math.ku.dk">jg@math.ku.dk</a>	Nathalie Wahl	University of Copenhagen <a href="mailto:wahl@math.ku.dk">wahl@math.ku.dk</a>
Misha Gromov	IHÉS and NYU, Courant Institute <a href="mailto:gromov@ihes.fr">gromov@ihes.fr</a>	Anna Wienhard	Universität Heidelberg <a href="mailto:wienhard@mathi.uni-heidelberg.de">wienhard@mathi.uni-heidelberg.de</a>

---

See inside back cover or [msp.org/gt](http://msp.org/gt) for submission instructions.

---

The subscription price for 2024 is US \$805/year for the electronic version, and \$1135/year (+\$70, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Geometry & Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

---

Geometry & Topology (ISSN 1465-3060 printed, 1364-0380 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

---

GT peer review and production are managed by EditFLOW<sup>®</sup> from MSP.

PUBLISHED BY

 **mathematical sciences publishers**  
nonprofit scientific publishing  
<http://msp.org/>

© 2024 Mathematical Sciences Publishers



# GEOMETRY & TOPOLOGY

Volume 28 Issue 3 (pages 1005–1499) 2024

---

Homological mirror symmetry for hypertoric varieties, I: Conic equivariant sheaves	1005
MICHAEL MCBREEN and BEN WEBSTER	
Moduli spaces of Ricci positive metrics in dimension five	1065
MCFEELY JACKSON GOODMAN	
Riemannian manifolds with entire Grauert tube are rationally elliptic	1099
XIAOYANG CHEN	
On certain quantifications of Gromov’s nonsqueezing theorem	1113
KEVIN SACKEL, ANTOINE SONG, U MUT VAROLGUNES and JONATHAN J ZHU	
Zariski dense surface groups in $SL(2k + 1, \mathbb{Z})$	1153
D DARREN LONG and MORWEN B THISTLETHWAITE	
Scalar and mean curvature comparison via the Dirac operator	1167
SIMONE CECCHINI and RUDOLF ZEIDLER	
Symplectic capacities, unperturbed curves and convex toric domains	1213
DUSA MCDUFF and KYLER SIEGEL	
Quadric bundles and hyperbolic equivalence	1287
ALEXANDER KUZNETSOV	
Categorical wall-crossing formula for Donaldson–Thomas theory on the resolved conifold	1341
YUKINOBU TODA	
Nonnegative Ricci curvature, metric cones and virtual abelianness	1409
JIAYIN PAN	
The homology of the Temperley–Lieb algebras	1437
RACHAEL BOYD and RICHARD HEPWORTH	