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Let $R > 1$ and let B be the Euclidean 4–ball of radius R with a closed subset E removed. Suppose that B embeds symplectically into the unit cylinder $\mathbb{D}^2 \times \mathbb{R}^2$. By Gromov’s nonsqueezing theorem, E must be nonempty. We prove that the Minkowski dimension of E is at least 2, and we exhibit an explicit example showing that this result is optimal at least for $R \leq \sqrt{2}$. In the appendix by Joé Brendel, it is shown that the lower bound is optimal for $R < \sqrt{3}$. We also discuss the minimum volume of E in the case that the symplectic embedding extends, with bounded Lipschitz constant, to the entire ball.

53D05, 53D35

1 Introduction

Consider \mathbb{R}^{2n} with its standard symplectic structure $\omega = \sum dx_i \wedge dy_i$. A prototypical question in symplectic geometry is to ask whether one domain of \mathbb{R}^{2n} symplectically embeds into another (ie via an embedding Φ with $\Phi^* \omega = \omega$). At the very least, a symplectic embedding preserves the standard volume form $(1/n!)\omega^n$. However, there is more rigidity in symplectic geometry than just volume. We recall the most famous result certifying this bold claim.

Let $B^{2n}(\pi R^2) \subset \mathbb{R}^{2n}$ be the open ball of radius R , and let $Z^{2n}(\pi r^2) = B^2(\pi r^2) \times \mathbb{R}^{2n-2} \subset \mathbb{R}^{2n}$ be the open cylinder of radius r .

Theorem 1.1 (Gromov’s nonsqueezing theorem [11]) *A symplectic embedding of $B^{2n}(\pi R^2)$ into $Z^{2n}(\pi r^2)$ exists if and only if $R \leq r$.*

Our goal in this paper is to try to quantify the failure of $B^{2n}(\pi R^2)$ to symplectically embed into $Z^{2n}(\pi r^2)$, when $R > r$, via the following motivating question:

Motivating question How much do we need to remove from $B^{2n}(\pi R^2)$ so that it embeds symplectically into $Z^{2n}(\pi r^2)$?

As a first attempt, one may try to use volume to answer this question. Over all possible symplectic embeddings $B^{2n}(\pi R^2) \hookrightarrow \mathbb{R}^{2n}$, what is the minimal possible volume excluded from $Z^{2n}(\pi r^2)$? This question has a straightforward answer, as the following result of Katok reveals.

Theorem 1.2 (Katok [16]) *Given a compact set X in $(\mathbb{R}^{2n}, \omega_{\text{std}})$, for every $\epsilon > 0$, there exists a Hamiltonian diffeomorphism $\phi: (\mathbb{R}^{2n}, \omega_{\text{std}}) \rightarrow (\mathbb{R}^{2n}, \omega_{\text{std}})$ such that*

$$\text{Vol}(\phi(X) \setminus Z(\pi)) \leq \epsilon.$$

We therefore modify our attempt to produce a more meaningful version of our question. We discuss two specific instances, though we focus predominantly on the first:

- (1) What is the smallest Minkowski dimension of a subset $E \subset B^{2n}(\pi R^2)$ with the property that there is a symplectic embedding $B^{2n}(\pi R^2) \setminus E \rightarrow Z^{2n}(\pi r^2)$?
- (2) Over all possible symplectic embeddings $B^{2n}(\pi R^2) \hookrightarrow \mathbb{R}^{2n}$ with Lipschitz constant at most $L > 0$, what is the minimal possible volume excluded from $Z^{2n}(\pi r^2)$?

Here Minkowski dimension stands for the lower Minkowski dimension, which is defined for any subset of $B^{2n}(\pi R^2)$. Heuristically, E having Minkowski dimension $d \in \mathbb{R}$ means that as $\epsilon \rightarrow 0$, the volume of the ϵ -neighborhood of E behaves as $c\epsilon^{2n-d}$, for some constant $c > 0$.

For each of these two questions, there is some quantity we are trying to minimize: either the Minkowski dimension of E , or the volume excluded from the cylinder under an L -Lipschitz symplectic embedding. In either case, there are two key aspects to discuss.

- **Constructive** We find an explicit symplectic embedding which provides an upper bound on the quantity that we are trying to minimize.
- **Obstructive** For a purported symplectic embedding, we find a lower bound on the quantity that we are trying to minimize.

A full answer to these two questions would require that the obstructive and constructive bounds match. Although these questions are interesting in general dimensions, in this paper we will restrict our attention only to the case of dimension $2n = 4$, save for a few open questions posed in [Section 6](#). There are also a plethora of further questions that will be posed in the final section of the paper.

We now discuss our results for each of these two questions.

1.1 The Minkowski dimension problem

Recall that here we are asking for the smallest lower Minkowski dimension of a subset $E \subset B(\pi R^2)$ with the property that there is a symplectic embedding $B(\pi R^2) \setminus E \rightarrow Z(\pi r^2)$, assuming $R > r$. (Here and henceforth we drop the superscripts from $B^4(\cdot)$ and $Z^4(\cdot)$ since the dimension is always 4.) By a scaling argument, it suffices to consider the case $r = 1$.

Let us start by discussing the constructive side. Observe that if we remove a union of codimension-one affine hyperplanes along a sufficiently fine grid, we end up with many connected components, each of which embeds into $Z(\pi)$ by translations. Interestingly, at least in a certain range of R , one can do better and find a two-dimensional submanifold E whose complement embeds into $Z(\pi)$.

To explain this result we need to introduce some notation. Let us consider the Lagrangian disk

$$L := B(2\pi) \cap \{y_1 = y_2 = 0\}.$$

Let us also define $\mathcal{E}(\pi, 4\pi) \subset \mathbb{R}^4$ to be the open ellipsoid

$$\{(x_1, y_1, x_2, y_2) \mid x_1^2 + y_1^2 + \frac{1}{4}(x_2^2 + y_2^2) < 1\},$$

and let

$$\mathcal{C} := \mathcal{E}(\pi, 4\pi) \cap \{x_2 = y_2 = 0\}.$$

Theorem 1.3 $B(2\pi) \setminus L$ is symplectomorphic to $\mathcal{E}(\pi, 4\pi) \setminus \mathcal{C}$. Consequently, $B(2\pi) \setminus L$ symplectically embeds into $Z(\pi)$.

In particular, removing a Lagrangian plane from $B(2\pi)$ halves its Gromov capacity. Our proof of [Theorem 1.3](#) has a great deal in common with Section 3 of Oakley and Usher's beautiful paper [25], where the same geometries are used for a different purpose. In fact, we show in [Section 4](#) how the projective space $\mathbb{C}\mathbb{P}^2$ is symplectomorphic to the boundary reduction of the unit cotangent bundle $D^*\mathbb{R}\mathbb{P}^2$ by using the explicit map of [25, Lemma 3.1]. [Theorem 1.3](#) can also be derived from the proof of Biran's general decomposition theorem [2, Theorem 1.A; Example 3.1.2], and Opshtein [26, Lemma 3.1]. We use the latter in our argument as well.

On the obstructive side, we show that removing a two-dimensional subset as in [Theorem 1.3](#) is the best one can do in general:

Theorem 1.4 Let E be a closed subset of \mathbb{R}^4 and let $R > 1$. Suppose that $B(\pi R^2) \setminus E$ symplectically embeds into the cylinder $Z(\pi)$. Then the lower Minkowski dimension of E is at least 2.

For the proof of [Theorem 1.4](#), we build on Gromov's original nonsqueezing argument by adding a key new ingredient: the waist inequality, which was also introduced by Gromov [12]; see also Memarian [23]. Crucially, we require the sharp version due to Akopyan and Karasev [1], as well as the Heintze–Karcher [15] bound on the volumes of tubes around minimal surfaces, in place of the monotonicity inequality for minimal surfaces.

Remark 1.5 Let $R_{\text{sup}} \in (1, \infty]$ be the supremum of the radii R such that there is a codimension-two subset of $B(\pi R^2)$ whose complement can symplectically embed into $Z(\pi)$. In the first version of this article we had conjectured that R_{sup} should be equal to $\sqrt{2}$. However shortly after its appearance, Joé Brendel informed us that using a construction inspired by Hacking and Prokhorov [13], he can prove $R_{\text{sup}} \geq \sqrt{3}$. As a consequence, we changed our conjecture to a question; see [Section 6.3](#). His construction appears in the [appendix](#).

We further remark that in [Theorem 1.3](#), we remove a Lagrangian plane. In the construction of Joé Brendel in the [appendix](#) realizing $R_{\text{sup}} \geq \sqrt{3}$, he removes a union of Lagrangians together with a symplectic divisor. In higher dimensions (see eg [Section 6.1](#)), this distinction could be interesting.

1.2 The Lipschitz problem

Recall that for fixed $L > 1$, we are asking for the smallest volume of the region

$$E(\Phi) := B(\pi R^2) \setminus \Phi^{-1}(Z(\pi)) = \Phi^{-1}(\mathbb{R}^4 \setminus Z(\pi))$$

over all symplectic embeddings $\Phi: B(\pi R^2) \hookrightarrow \mathbb{R}^4$ of Lipschitz constant bounded above by L . (We note that although we use the letter L for both the Lipschitz constant as well as for the Lagrangian disk of [Theorem 1.3](#), there will be no confusion given the context.)

On the obstructive side, we obtain the following as a corollary of the proof of the obstructive bound for the Minkowski question ([Theorem 1.4](#)):

Theorem 1.6 *Let $R > 1$. Then there exists a constant $c = c(R) > 0$ such that for all constants L and all symplectic embeddings $\Phi: B(\pi R^2) \hookrightarrow \mathbb{R}^4$ with Lipschitz constant at most L , we have*

$$\text{Vol}_4(E(\Phi)) \geq \frac{c}{L^2}.$$

It is worth noting that one may use the standard nonsqueezing theorem alone to find a weaker quantitative obstructive bound of c/L^3 as follows. Suppose we had an L -Lipschitz symplectic embedding $\phi: B(\pi R^2) \hookrightarrow \mathbb{R}^4$ for $R > 2$. Then by Gromov's nonsqueezing theorem and the Lipschitz condition, one can check that there is a ball of radius of order $1/L$ embedded inside $E(\phi)$. Hence,

$$\text{Vol}(E(\phi)) \gtrsim \frac{1}{L^4}.$$

With a little more effort, one may find order L many disjoint such balls inside $E(\phi)$, yielding the obstructive bound c/L^3 . However, jumping from c/L^3 to our obstructive bound of c/L^2 appears to require a new tool, which in our case is Gromov's waist inequality.

On the constructive side, we adapt Katok's ideas in [\[16\]](#) to prove the following:

Theorem 1.7 *Let $R > 1$. Then there exists a constant $C = C(R) > 0$ such that for all constants L , there exists a symplectic embedding $\Phi: B^4(R) \hookrightarrow \mathbb{R}^4$ with Lipschitz constant at most L such that*

$$\text{Vol}_4(E(\Phi)) \leq \frac{C}{L}.$$

Remark 1.8 As was pointed out to us by Felix Schlenk, our construction is a simplified version of multiple symplectic folding; see Schlenk [\[28, Sections 3 and 4\]](#).

One would obviously like to push the obstructive and constructive bounds together.

Organization of the paper We start by recalling some definitions and known theorems in [Section 2](#), which are then applied in [Section 3](#) to prove an obstructive bound implying [Theorem 1.4](#). In [Section 4](#) we construct the symplectomorphism of [Theorem 1.3](#). The Lipschitz problem, including [Theorems 1.6](#) and [1.7](#), are discussed in [Section 5](#). We also list several related questions in the final [Section 6](#). In the [appendix](#), written by Joé Brendel, the construction mentioned in [Remark 1.5](#) appears.

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2 Preliminaries

We use the usual asymptotic notation $f \in O(g)$ to mean $|f| \leq Cg$ for some constant C , and $f \in o(g)$ to mean that $\lim_{t \rightarrow 0} f(t)/g(t) = 0$. We write $f \in \Theta(g)$ if $f \in O(g)$ and $g \in O(f)$.

Let $S \subset \mathbb{R}^n$ be any bounded subset. Let $N_t(S)$ denote the open t -neighborhood of S with respect to the standard metric. If Σ is a compact submanifold (possibly with boundary), let $V_t(\Sigma)$ be the exponential t -tube of Σ , ie the image by the normal exponential map of the open t -neighborhood of the zero-section in the normal bundle of Σ (which is endowed with the natural metric).

We denote by Vol_n the Euclidean n -volume of a set and set

$$\alpha_l = \frac{\pi^{l/2}}{\Gamma(\frac{1}{2}l + 1)}.$$

Note that when n is a natural number, $\alpha_n = \text{Vol}_n(B^n)$ is precisely the Euclidean volume of the unit n -dimensional ball $B^n \subset \mathbb{R}^n$. For $s \geq 0$, the s -dimensional lower Minkowski content of S is defined as

$$\underline{\mathcal{M}}_s(S) := \liminf_{t \rightarrow 0^+} \frac{\text{Vol}_n(N_t(S))}{\alpha_{n-s} t^{n-s}}.$$

Note that the normalization is chosen so that if $\Sigma^k \subset \mathbb{R}^n$ is a closed k -dimensional submanifold, then $\underline{\mathcal{M}}_k(\Sigma) = \text{Vol}_k(\Sigma)$ coincides with the Euclidean k -volume of Σ .

The lower Minkowski dimension of S is defined as

$$\underline{\dim}_{\mathcal{M}}(S) := \inf_{s>0} \{\mathcal{M}_s(S) = 0\} = \sup_{s \geq 0} \{\mathcal{M}_s(S) > 0\}.$$

There are similar notions of upper Minkowski dimension and upper Minkowski content, which we will not need in this paper since a lower bound of the lower Minkowski content implies by definition the same lower bound for the upper Minkowski content. There are also equivalent definitions using ball packings. Replacing S by its closure \bar{S} does not change the Minkowski upper/lower dimensions.

2.1 Waist inequalities

The waist inequality for round spheres proved by Gromov [12] and with more details by Memarian [23] was extended to the case of maps from Euclidean balls by Akopyan and Karasev [1, Theorem 1]. The proof of the latter immediately implies the following:

Theorem 2.1 (waist inequality) *For any positive integers n and k , there exists a continuous function $h_{n,k}: (0, \infty) \rightarrow \mathbb{R}$ such that $h_{n,k} \in o(t^k)$ and the following holds: for any continuous map $f: B^n \rightarrow \mathbb{R}^k$, there exists $y \in \mathbb{R}^k$ such that*

$$(2-1) \quad \text{Vol}_n(N_t(f^{-1}(y)) \cap B^n) \geq \alpha_{n-k} \alpha_k t^k - h_{n,k}(t).$$

It will be useful for our application that the above estimate is uniform in f . This uniform estimate is indeed implied by the proof of [1, Theorem 1] as they compare the t -neighborhood of a fiber to the t -neighborhood of an equatorial unit sphere $S^{n+1-k} \subset S^{n+1}$ (cf the second-last equation of their proof, with correct normalization). The latter is independent of f , and by explicit calculation one may verify that actually $h_{n,k} \in O(t^{k+2})$.

We remark that the waist inequality for spheres [12; 23] describes a stronger property than the above statement, since it gives optimal bounds on all (not just small) neighborhoods of the big fiber.

2.2 Tubes around minimal submanifolds

The Heintze–Karcher inequality [15] estimates the volume of tubes around compact submanifolds. We need the case of minimal submanifolds in Euclidean space (covered by [15, Theorem 2.3] with $\delta = 0$ and Remark 2 on page 453 in [15]), which may be stated as follows:

Theorem 2.2 (Heintze–Karcher inequality) *For any positive integers n and k , and any smooth compact k -dimensional minimal submanifold $\Sigma^k \subset \mathbb{R}^n$ with boundary, for $t > 0$ we have*

$$(2-2) \quad \text{Vol}_n(V_t(\Sigma)) \leq \text{Vol}_k(\Sigma) \alpha_{n-k} t^{n-k}.$$

Theorem 2.2 is again a uniform estimate on the exponential t -neighborhood; one may compare it to the statement that the (upper) Minkowski k -content of a closed submanifold $\Sigma^k \subset \mathbb{R}^n$ is $\text{Vol}_k(\Sigma)$. The point of **Theorem 2.2** is that the constant in this estimate does not depend on the minimal k -submanifold Σ or t . The main ingredient for its proof is an estimate for the Jacobian determinant of the normal exponential, which in Euclidean space is 1 to lowest order. The following term is controlled by the mean curvature H , which one may expect from the interpretation of mean curvature as the first variation of $(k-)$ area. Consequently, for a general compact submanifold Σ , (2-2) will have an error term of order $c(\max_{\Sigma} |H|)t^{n-k+1}$.

2.3 Gromov foliation and maps into the cylinder

For $r < R$, recall that $B(\pi R^2)$ and $Z(\pi r^2)$ denote the open ball and open cylinder of radius R and r , respectively, in \mathbb{R}^4 . In this subsection, we give a slight modification of the holomorphic foliation argument of Gromov [11] in dimension $n = 4$.

Proposition 2.3 *Let $R, r > 0$. Let E be a compact subset of \mathbb{R}^4 and let $\phi: B(\pi R^2) \setminus E \rightarrow \mathbb{R}^4$ be a smooth symplectic embedding into the cylinder $Z(\pi r^2)$. Let U be the closure of an open neighborhood of $\partial B(\pi R^2) \cup E$ in \mathbb{R}^4 . Then there exists a smooth map $f: B(\pi R^2) \setminus U \rightarrow \mathbb{R}^2$ such that*

- f has no critical points on $B(\pi R^2) \setminus U$, and
- for all $y \in \mathbb{R}^2$, if $f^{-1}(y) \cap B(\pi R^2) \setminus U$ is nonempty, then it is a two-dimensional complex submanifold of Euclidean area less than πr^2 .

Proof The following argument is standard in the symplectic community. As mentioned before the statement of the proposition, the ideas are due to Gromov [11], though more thorough analytic details may be found elsewhere; see eg [22].

We define $A := \pi r^2$ for brevity. Since $B(\pi R^2) \setminus U$ has compact closure in $B(\pi R^2) \setminus E$, the image of $B(\pi R^2) \setminus U$ under ϕ lands in $B^2(\pi r_0^2) \times [-K, K]^2$ for some large constant K and $0 < r_0 < r$ (possibly depending upon U). Let $S^2(A)$ denote the 2-sphere with standard symplectic form scaled to have total area A , and let $T_K^2 = (\mathbb{R}/4K\mathbb{Z})^2$ be the 2-torus with symplectic form induced by the standard form on \mathbb{R}^2 . Then we have a symplectic embedding $B^2(\pi r_0^2) \times [-K, K]^2 \subset S^2(A) \times T_K^2$, and upon composing with ϕ , we arrive at a symplectic embedding, also denoted by ϕ (by abuse of notation),

$$\phi: B(\pi R^2) \setminus U \hookrightarrow S^2(A) \times T_K^2.$$

Let J_0 denote the standard complex structure on $B(\pi R^2)$, and let J_1 denote the standard (split) complex structure on $S^2(A) \times T_K^2$. We pick a special almost complex structure J_ϕ on $S^2(A) \times T_K^2$ which incorporates ϕ by requiring that it satisfies the following three properties:

- On the image of ϕ , $J_\phi = \phi_*(J_0)$.
- $J_\phi = J_1$ in a neighborhood of $\{\infty\} \times T_K^2$.
- Everywhere, J_ϕ is compatible with the symplectic form.

By [22, Proposition 9.4.4], which encompasses standard 4–dimensional techniques, the evaluation map

$$\text{ev}: \mathcal{M}_{0,1}(\beta, J_\phi) \rightarrow S^2(A) \times T_K^2$$

is a diffeomorphism, where $\mathcal{M}_{0,1}(\beta, J_\phi)$ is the moduli space of J_ϕ –holomorphic spheres with one marked point and in the class β . Meanwhile, the map which forgets the marked point $\tau: \mathcal{M}_{0,1}(\beta, J_\phi) \rightarrow \mathcal{M}_{0,0}(\beta, J_\phi)$ is a smooth fibration, with fibers diffeomorphic to S^2 . By positivity of intersections (see [22, Theorem 2.6.3]), each fiber of τ , which is a J_ϕ –holomorphic sphere, intersects $\{\infty\} \times T_K^2$ once and transversely, so by the implicit function theorem we have a canonical diffeomorphism g of T_K^2 with $\mathcal{M}_{0,0}(\beta, J_\phi)$. We therefore obtain a map $h: S^2(A) \times T_K^2 \rightarrow T_K^2$ by setting $h(p) = x$ when the unique sphere through p passes through (∞, x) . It is clear by construction that the diagram

$$\begin{array}{ccc} \mathcal{M}_{0,1}(\beta, J_\phi) & \xrightarrow{\text{ev}} & S^2(A) \times T_K^2 \\ \downarrow \text{for} & & \downarrow h \\ \mathcal{M}_{0,0}(\beta, J_\phi) & \xrightarrow{g} & T_K^2 \end{array}$$

commutes. In particular since the forgetful map is a smooth S^2 fiber bundle, h is smooth and has no critical points.

Notice that since the image of $B(\pi R^2) \setminus U$ under ϕ is contained in a contractible subset of $S^2(A) \times T_K^2$, we have that the composition $h \circ \phi: B(\pi R^2) \setminus U \rightarrow T_K^2$ lifts to a map $f: B(\pi R^2) \setminus U \rightarrow \mathbb{R}^2$. This is the function f we desired in the statement of the proposition, and we must now check it satisfies both of the desired properties.

The fact that $f^{-1}(y)$ is a complex submanifold is simply because it is by definition a subset of a J_ϕ –holomorphic sphere, and J_ϕ is chosen to equal $\phi_*(J_0)$ on the image of ϕ . Finally, the area bound comes from the fact that the area of $f^{-1}(y)$ is at most the symplectic area of the corresponding sphere (the one passing through $(\infty, [y])$), which is just A since symplectic area is purely homological. □

3 A quantitative obstruction to partial symplectic embeddings

As usual, for $r < R$, $B(\pi R^2)$ and $Z(\pi r^2)$ refer to the open ball and open cylinder of radius R and r , respectively, in \mathbb{R}^4 . The main estimate of this section is the following obstructive bound:

Theorem 3.1 *Let E be a compact subset of \mathbb{R}^4 and suppose that $B(\pi R^2) \setminus E$ symplectically embeds into the cylinder $Z(\pi r^2) \subset \mathbb{R}^4$. Then there is a function $k_R \in o(t^2)$ such that for any $t > 0$,*

$$\text{Vol}_4(N_t(E)) \geq \pi^2(R^2 - r^2)t^2 - k_R(t).$$

Proof Let $\Phi: B(\pi R^2) \setminus E \rightarrow Z(\pi r^2)$ be the symplectic embedding of the statement. Consider $0 < t < \frac{1}{2}(R - r)$ and take $0 < \delta < t$. Later, we will send $\delta \rightarrow 0$.

Let U_δ be the closure of $N_\delta(\partial B(\pi R^2) \cup E)$ in \mathbb{R}^4 and let $\tilde{f}_\delta: B(\pi R^2) \setminus U_\delta \rightarrow \mathbb{R}^2$ be the map given by [Proposition 2.3](#). Note that $B(\pi R^2) \setminus U_\delta = B(\pi(R-\delta)^2) \setminus \overline{N_\delta(E)}$. We then take $f_\delta: B(\pi(R-\delta)^2) \rightarrow \mathbb{R}^2$ to be any continuous extension of \tilde{f}_δ . Since f_δ agrees with \tilde{f}_δ on $B(\pi(R-\delta)^2) \setminus U_\delta$, by the conclusions of [Proposition 2.3](#) we have for any $y \in \mathbb{R}^2$ that

$$\Sigma_\delta := f_\delta^{-1}(y) \cap (B(\pi(R-\delta)^2) \setminus U_\delta)$$

is a minimal submanifold with area less than $A := \pi r^2$.

The main idea of our proof is that the waist inequality guarantees a fiber with large-volume neighborhoods, but by the area bound (and the structure of tubes) this can only happen if the fiber accumulates near the exceptional set. Accordingly, a key component is the following covering claim.

Claim 1 *Let $f_{\delta,t}$ be the restriction of f_δ to the ball $B(\pi(R-2t)^2)$. Then*

$$(3-1) \quad N_t(f_{\delta,t}^{-1}(y)) \cap B(\pi(R-2t)^2) \subset V_t(\Sigma_\delta) \cup N_{\delta+t}(E).$$

Indeed, by definition of Σ_δ and the supposition $\delta < t$ we have that

$$f_{\delta,t}^{-1}(y) \setminus \overline{N_\delta(E)} = \Sigma_\delta \cap B(\pi(R-2t)^2) \subset \Sigma_\delta.$$

Now given any submanifold Σ , its t -neighborhood $N_t(\Sigma)$ is always contained in the union of the tube $V_t(\Sigma)$ and the t -neighborhood $N_t(\partial\Sigma)$ of its boundary. So since $\partial\Sigma_\delta \subset \partial U_\delta$, we have that

$$N_t(f_{\delta,t}^{-1}(y) \setminus \overline{N_\delta(E)}) \subset N_t(\Sigma_\delta) \subset V_t(\Sigma_\delta) \cup N_t(\partial U_\delta).$$

By the triangle inequality it follows that

$$(3-2) \quad N_t(f_{\delta,t}^{-1}(y)) \subset V_t(\Sigma_\delta) \cup N_t(\partial U_\delta) \cup N_{\delta+t}(E).$$

But by definition of U_δ , we have $N_t(\partial U_\delta) \subset N_t(\partial B(\pi(R-\delta)^2)) \cup N_{\delta+t}(E)$, and since $\delta < t$, we note that $N_t(\partial B(\pi(R-\delta)^2)) \cap B(\pi(R-2t)^2) = \emptyset$. Taking the intersection of [\(3-2\)](#) with $B(\pi(R-2t)^2)$ then yields the claim.

Having established the claim, we now estimate the volume of each set in [\(3-1\)](#). First, let $h_{4,2} \in o(t^2)$ be as in [Theorem 2.1](#). By rescaling to the ball $B(\pi(R-2t)^2)$, the waist inequality [Theorem 2.1](#) applied to $f_{\delta,t}: B(\pi(R-2t)^2) \rightarrow \mathbb{R}^2$ gives that there is some $y \in \mathbb{R}^2$ for which

$$(3-3) \quad \text{Vol}_4(N_t(f_{\delta,t}^{-1}(y)) \cap B(\pi(R-2t)^2)) \geq \pi^2 t^2 (R-2t)^2 - (R-2t)^4 h_{4,2} \left(\frac{t}{R-2t} \right).$$

On the other hand, since Σ_δ is minimal with area at most A , the Heintze–Karcher inequality [Theorem 2.2](#) yields

$$(3-4) \quad \text{Vol}_4(V_t(\Sigma_\delta)) \leq \text{Vol}_2(\Sigma_\delta) \pi t^2 \leq A \pi t^2.$$

Combining the covering (3-1) with the estimates (3-3) and (3-4) yields

$$\text{Vol}_4(N_{\delta+t}(E)) \geq (\pi^2(R-2t)^2 - A\pi)t^2 - (R-2t)^4 h_{4,2} \left(\frac{t}{R-2t} \right).$$

The volume of $N_t(E)$ is nondecreasing with respect to t , and is continuous almost everywhere. Therefore, sending $\delta \rightarrow 0$ and recalling that $A = \pi r^2$ in the inequality above, we obtain for any $t > 0$ that

$$\text{Vol}_4(N_t(E)) \geq \pi^2((R-2t)^2 - r^2)t^2 - (R-2t)^4 h_{4,2} \left(\frac{t}{R-2t} \right).$$

Thus, taking

$$k_R(t) = 4R\pi^2 t^3 + R^4 h_{4,2} \left(\frac{t}{R-2t} \right),$$

for instance, concludes the proof. □

An immediate corollary is a lower bound for the lower Minkowski dimension. We will see that this bound is sharp in the next section, at least for radii R which are not too large.

Corollary 3.2 (Minkowski dimension) *Suppose that $B(\pi R^2) \setminus E$ symplectically embeds into the cylinder $Z(\pi r^2) \subset \mathbb{R}^4$. Then the two-dimensional lower Minkowski content of E satisfies*

$$\underline{\mathcal{M}}_2(E) \geq \pi(R^2 - r^2).$$

In particular, the lower Minkowski dimension $\underline{\dim}_{\mathcal{M}}(E)$ is at least 2.

4 Squeezing the complement of a Lagrangian plane

In this section it will be more convenient to use complex coordinates for the standard symplectic \mathbb{R}^4 . Therefore we consider \mathbb{C}^2 with its standard Kähler structure, ie if x and y are the complex coordinates, then the symplectic form is

$$\frac{i}{2}(dx \wedge d\bar{x} + dy \wedge d\bar{y}).$$

Let us recall the main objects in the statement of [Theorem 1.3](#) in complex notation for convenience. Let $B(2\pi) \subset \mathbb{C}^2$ be the open ball of radius $\sqrt{2}$ centered at the origin. Let $\mathbb{R}^2 \subset \mathbb{C}^2$ be the real part and define

$$L := B(2\pi) \cap \mathbb{R}^2.$$

We also define $\mathcal{E}(\pi, 4\pi) \subset \mathbb{C}^2$ to be the open ellipsoid

$$\{(x, y) \mid |x|^2 + \frac{1}{4}|y|^2 < 1\},$$

and let

$$\mathcal{C} := \mathcal{E}(\pi, 4\pi) \cap \{y = 0\}.$$

Let us introduce the main actors in the proof. Let $\mathbb{C}\mathbb{P}^2(2\pi)$ be the symplectic manifold obtained from coisotropic reduction of the sphere S^5 of radius $\sqrt{2}$ in \mathbb{C}^3 . Denoting the complex coordinates on \mathbb{C}^3 by z_1, z_2 and z_3 , we are using the symplectic structure $\frac{i}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + dz_3 \wedge d\bar{z}_3)$ on \mathbb{C}^3 . There is a canonical identification of $\mathbb{C}\mathbb{P}^2(2\pi)$ with the complex manifold $\mathbb{C}\mathbb{P}^2 := \text{Gr}_{\mathbb{C}}(1, 3)$, whose homogeneous coordinates we will denote by $[z_1 : z_2 : z_3]$. Of course, $\mathbb{C}\mathbb{P}^2(2\pi)$ is nothing but $\mathbb{C}\mathbb{P}^2$ equipped with the Fubini–Study symplectic form scaled so that a complex line (eg $\{z_1 = 0\} \subset \mathbb{C}\mathbb{P}^2$) has area 2π .

Let us also specify some submanifolds of $\mathbb{C}\mathbb{P}^2(2\pi)$ using its canonical identification with $\mathbb{C}\mathbb{P}^2$.

- $L_{\mathbb{R}\mathbb{P}}$ is the real part

$$\{[z_1 : z_2 : z_3] \mid \text{Im}(z_1) = \text{Im}(z_2) = \text{Im}(z_3) = 0\}.$$

- For $t = [t_1 : t_2 : t_3] \in \mathbb{R}\mathbb{P} := \text{Gr}_{\mathbb{R}}(1, 3)$, we define the complex lines

$$S_t := \{[z_1 : z_2 : z_3] \mid t_1 z_1 + t_2 z_2 + t_3 z_3 = 0\}.$$

- FQ is the Fermat quadric

$$\{[z_1 : z_2 : z_3] \mid z_1^2 + z_2^2 + z_3^2 = 0\}.$$

It is well-known that $\mathbb{C}\mathbb{P}^2(2\pi) \setminus (S_{[0:0:1]} \cup L_{\mathbb{R}\mathbb{P}})$ is symplectomorphic to $B(2\pi) \setminus L$; see Exercise 9.4.11 in [22].

Consider $\mathbb{R}\mathbb{P}$ as a smooth manifold in the standard way. Let λ be the tautological one-form on $T^*\mathbb{R}\mathbb{P}$ and V be the Liouville vector field, which is a vertical vector field equal to the Euler vector field in each fiber (which is defined on any vector space independently of a basis). We have

$$\omega(V, \cdot) = \lambda,$$

where $\omega = d\lambda$. We denote the zero-section submanifold on $T^*\mathbb{R}\mathbb{P}$ by $Z_{\mathbb{R}\mathbb{P}}$.

The Riemannian metric on S^2 obtained from its embedding as the round sphere of radius 1 in \mathbb{R}^3 induces a metric on its quotient by the antipodal map, which is canonically diffeomorphic to $\mathbb{R}\mathbb{P}$. We call this the round metric on $\mathbb{R}\mathbb{P}$ and denote it by $g_{\mathbb{R}\mathbb{P}}$. We have the diffeomorphism $g_{\mathbb{R}\mathbb{P}}^{\#} : T\mathbb{R}\mathbb{P} \rightarrow T^*\mathbb{R}\mathbb{P}$, which is in particular linear on the fibers. We can transport the function $K : T\mathbb{R}\mathbb{P} \rightarrow \mathbb{R}$, given by lengths of tangent vectors to

$$(4-1) \quad K^{\#} : T^*\mathbb{R}\mathbb{P} \rightarrow \mathbb{R}.$$

On $T\mathbb{R}\mathbb{P}$ we have the geodesic flow; under the identification by $g_{\mathbb{R}\mathbb{P}}^{\#}$ this becomes the Hamiltonian flow of the function $\frac{1}{2}(K^{\#})^2$. The normalized geodesic flow on $T\mathbb{R}\mathbb{P} \setminus \mathbb{R}\mathbb{P}$ becomes the Hamiltonian flow of $K^{\#}$. Let us call these the geodesic flow and normalized geodesic flow on $T^*\mathbb{R}\mathbb{P}$. Note that the normalized geodesic flow on $T^*\mathbb{R}\mathbb{P} \setminus Z_{\mathbb{R}\mathbb{P}}$ is a π -periodic action of \mathbb{R} .

Any unparametrized oriented geodesic circle γ in $\mathbb{R}\mathbb{P}$ with its round metric defines a symplectic submanifold C_γ in $T^*\mathbb{R}\mathbb{P}$ with boundary on $Z_{\mathbb{R}\mathbb{P}}$ by taking points (q, p) such that $q \in \gamma$ and $p = g_q(v, \cdot)$, where v is nonnegatively tangent to γ . Let us denote by $-\gamma$ the geodesic circle with opposite orientation. Clearly, C_γ and $C_{-\gamma}$ intersect along $\gamma \subset Z_{\mathbb{R}\mathbb{P}}$ and form a symplectic submanifold of $T^*\mathbb{R}\mathbb{P}$, which is diffeomorphic to a cylinder.

Let $D^*\mathbb{R}\mathbb{P} \subset T^*\mathbb{R}\mathbb{P}$ be the closed unit-disk bundle, which is given by the subset $K^\# \leq 1$. Also let $U^*\mathbb{R}\mathbb{P} = (K^\#)^{-1}(1)$ be the unit-sphere bundle, ie the boundary of $D^*\mathbb{R}\mathbb{P}$, with its induced contact structure $\theta := \iota^*\lambda$.

The symplectic reduction $U^*\mathbb{R}\mathbb{P} / S^1$ is a two-sphere equipped with a canonical symplectic form. Let us call this symplectic manifold (Q, ω_Q) . The points of Q are canonically identified with unparametrized oriented geodesic circles in round $\mathbb{R}\mathbb{P}$.

Let us denote the boundary reduction symplectic manifold of $D^*\mathbb{R}\mathbb{P}$ by $\overline{D^*\mathbb{R}\mathbb{P}^2}$ (see Definition 3.9 of [30]; also note the interpretation as one half of a symplectic cut [18]).

Note that Q and $Z_{\mathbb{R}\mathbb{P}}$ sit naturally inside $\overline{D^*\mathbb{R}\mathbb{P}^2}$. The Poincaré dual of the homology class of Q is $1/\pi$ times the symplectic class. The cylinders $(C_\gamma \cup C_{-\gamma}) \cap D^*\mathbb{R}\mathbb{P}$ become symplectic 2-spheres in $\overline{D^*\mathbb{R}\mathbb{P}^2}$. They intersect Q positively in two points and $Z_{\mathbb{R}\mathbb{P}}$ along the circle γ . Let us call these spheres S_γ , now indexed by unoriented unparametrized geodesic circles on $\mathbb{R}\mathbb{P}$. Each S_γ has self-intersection number 1.

Let γ_0 be the oriented unparametrized geodesic on $\mathbb{R}\mathbb{P}$ which corresponds to the quotient of the horizontal great circle in $S^2 \subset \mathbb{R}^3$ oriented as the boundary of the lower hemisphere. We define $S := S_{\gamma_0}$.

Proposition 4.1 *There is a symplectomorphism $\overline{D^*\mathbb{R}\mathbb{P}^2} \rightarrow \mathbb{C}\mathbb{P}^2(2\pi)$ with the following properties:*

- $Z_{\mathbb{R}\mathbb{P}}$ is sent to $L_{\mathbb{R}\mathbb{P}}$.
- S is sent to $S_{[0:0:1]}$.
- Q is sent to FQ .

The proof of this proposition is postponed to Section 4.1. Let us continue with an immediate corollary.

Corollary 4.2 $\overline{D^*\mathbb{R}\mathbb{P}^2} \setminus (S \cup Z_{\mathbb{R}\mathbb{P}})$ is symplectomorphic to $\mathbb{C}\mathbb{P}^2(2\pi) \setminus (S_{[0:0:1]} \cup L_{\mathbb{R}\mathbb{P}})$, and in turn to $B(2\pi) \setminus L$.

Note that $V \cdot K^\# = K^\#$ on $T^*\mathbb{R}\mathbb{P} \setminus Z_{\mathbb{R}\mathbb{P}}$, which means that $K^\#$ is an exponentiated Liouville coordinate for V on $T^*\mathbb{R}\mathbb{P} \setminus Z_{\mathbb{R}\mathbb{P}}$. Hence, we obtain a Liouville isomorphism

$$T^*\mathbb{R}\mathbb{P} \setminus Z_{\mathbb{R}\mathbb{P}} \simeq (U^*\mathbb{R}\mathbb{P} \times (0, \infty)_r, d(r\theta)),$$

where K^\sharp is matched with the function r . The Hamiltonian vector field X_r gives the r -translation invariant Reeb vector field (using $r = 1$) on the contact levels. Finally, observe that $C_\gamma \cap (T^*\mathbb{R}\mathbb{P} \setminus Z_{\mathbb{R}\mathbb{P}})$ ’s are obtained as the traces of the Reeb orbits on $U^*\mathbb{R}\mathbb{P}$ under the Liouville flow.

In particular, we have a foliation of $\overline{D^*\mathbb{R}\mathbb{P}^2} \setminus Z_{\mathbb{R}\mathbb{P}}$ by open disks which are the reductions of the $C_\gamma \cap (D^*\mathbb{R}\mathbb{P} \setminus Z_{\mathbb{R}\mathbb{P}})$ ’s. Let us denote this Q -family of submanifolds by \mathbb{D}_γ , where $\gamma \in Q$.

Proposition 4.3 $\overline{D^*\mathbb{R}\mathbb{P}^2} \setminus Z_{\mathbb{R}\mathbb{P}}$ is symplectomorphic to an area- π standard symplectic disk bundle of (Q, ω_Q) in the sense of Biran [2, Section 2.1] in such a way that \mathbb{D}_γ is sent to the fiber over γ for every $\gamma \in Q$.

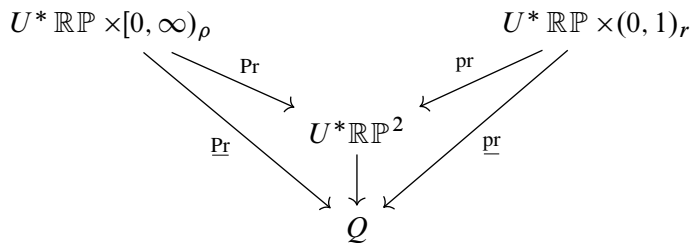
Remark 4.4 The cohomology class of ω_Q/π is integral and it admits a unique lift to $H_2(Q, \mathbb{Z})$. What we mean by an area- π standard symplectic disk bundle of (Q, ω_Q) is an area-1 standard symplectic disk bundle of $(Q, \omega_Q/\pi)$ with its symplectic form multiplied by π .

Proof The symplectomorphism $D^*\mathbb{R}\mathbb{P} \setminus Z_{\mathbb{R}\mathbb{P}} \simeq U^*\mathbb{R}\mathbb{P} \times (0, 1]_r$ induces a symplectomorphism of the boundary reductions of both sides. We note that

$$(4-2) \quad d(r\tilde{\theta}) = \underline{\text{pr}}^*\omega_Q - d((1-r)\tilde{\theta})$$

on $U^*\mathbb{R}\mathbb{P} \times (0, 1)$, where we define $\tilde{\theta} = \text{pr}^*\theta$ for clarity.

Here we use the maps in the following commutative diagram, where Pr and pr are the obvious projections, and $U^*\mathbb{R}\mathbb{P}^2 \rightarrow Q$ is the symplectic reduction map:



Notice that the map $U^*\mathbb{R}\mathbb{P} \rightarrow Q$ has the structure of a principal $U(1) = \mathbb{R}/\mathbb{Z}$ bundle structure using the Reeb flow of the contact form θ/π , and the associated complex line bundle \mathcal{L} is precisely the fiberwise blow-down of $U^*\mathbb{R}\mathbb{P} \times [0, \infty)_\rho$ with respect to its canonical projection $\underline{\text{Pr}}$ to Q . The integral Chern class of this complex line bundle is ω_Q/π (using that $H_*(Q, \mathbb{Z})$ has no torsion) and a transgression 1-form is given by the pull-back of $-\theta/\pi$ by Pr . By definition, the open unit disk bundle $\rho < 1$ inside \mathcal{L} , endowed with the symplectic form

$$(4-3) \quad \underline{\text{Pr}}^*\left(\frac{\omega_Q}{\pi}\right) + d\left(\rho^2 \text{Pr}^*\left(-\frac{\theta}{\pi}\right)\right),$$

is an area-1 standard symplectic disk bundle of $(Q, \omega_Q/\pi)$. Therefore, if we multiply this form by π , we obtain an area- π standard symplectic disk bundle of (Q, ω_Q) .

Now consider the following commutative diagram:

$$\begin{array}{ccc}
 U^* \mathbb{R}P \times [0, 1)_\rho & \xrightarrow{r=1-\rho^2} & U^* \mathbb{R}P \times (0, 1]_r \\
 \downarrow & & \downarrow \\
 \{\rho < 1\} \subset \mathcal{L} & \xrightarrow{F} & \overline{U^* \mathbb{R}P \times (0, 1]}
 \end{array}$$

Here the left map is the restriction of the fiberwise blowdown map and the right map is the boundary reduction map. The upper map sends $(x, \rho) \in U^* \mathbb{R}P \times [0, 1)_\rho$ to $(x, 1 - \rho^2)$, and it is a homeomorphism overall as well as a diffeomorphism of the interiors. By construction of symplectic boundary reduction we deduce that there is a canonical diffeomorphism F making this diagram commutative.

It automatically follows from comparing equations (4-2) and (4-3) that F is a symplectomorphism. Composing F with the symplectomorphism from the very beginning of this proof yields the desired symplectomorphism. □

Combined with Corollary 4.2, the following finishes the proof of Theorem 1.3.

Proposition 4.5 $\overline{D^* \mathbb{R}P^2} \setminus (S \cup Z_{\mathbb{R}P})$ is symplectomorphic to $\mathcal{E}(\pi, 4\pi) \setminus \mathcal{C}$.

Proof We use our Proposition 4.3 and [26, Lemma 2.1] to find an explicit symplectomorphism from the complement of \mathbb{D}_{γ_0} in $D^* \mathbb{R}P^2 \setminus Z_{\mathbb{R}P}$ to $\mathcal{E}(\pi, 4\pi)$. Here we use a symplectomorphism between $Q \setminus \{\gamma_0\}$ and the two-dimensional open ellipsoid of area 4π which sends $-\gamma_0$ to the origin, so that $\mathbb{D}_{-\gamma_0}$ is sent to \mathcal{C} by this symplectomorphism. □

4.1 Proof of Proposition 4.1

We will freely use the canonical identification of $\mathbb{C}P^n(2\pi)$ with $(\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$. Note that the homogenous coordinates $[z_1 : \dots : z_{n+1}]$ denote the class $[z_1 e_1 + \dots + z_{n+1} e_{n+1}]$, where e_i is the standard basis of \mathbb{R}^{n+1} and its complexification \mathbb{C}^{n+1} . We also realize $T^* \mathbb{R}P^n$ as

$$\{(q, p) \in S^n \times \mathbb{R}^{n+1} \mid \langle q, p \rangle = 0\} / \{\pm 1\},$$

where $S^n \subset \mathbb{R}^{n+1}$ is the unit sphere. It is a straightforward computation that the standard symplectic form on $T^* \mathbb{R}P^n$ descends from the restriction of $\sum_{i=1}^{n+1} dp_i \wedge dq_i$ on \mathbb{R}^{2n+2} under this identification. Note also that $K^\sharp([q, p]) = |p|$ away from the zero-section.

In [25, Lemma 3.1], Oakley and Usher considered the map

$$\Phi: D^* \mathbb{R}P^n \rightarrow \mathbb{C}P^n(2\pi)$$

defined by

$$(4-4) \quad \Phi([q, p]) := [\sqrt{f(|p|)}p + \frac{i}{\sqrt{f(|p|)}}q],$$

where $f(x) = (1 - \sqrt{1 - x^2})/x^2$ on $(0, 1]$ and $f(0) = \frac{1}{2}$. They proved that $\Phi|_{\text{int}(D^*\mathbb{R}P^n)}$ is a symplectomorphism onto its image $\mathbb{C}P^n(2\pi) \setminus FQ_n$, where

$$FQ_n = \left\{ [z_0 : \cdots : z_n] \in \mathbb{C}P^n(2\pi) \mid \sum z_k^2 = 0 \right\}$$

is the Fermat quadric. As before, we will denote by $\overline{D^*\mathbb{R}P^n}$ the boundary reduction of $D^*\mathbb{R}P^n$, and by $Z_{\mathbb{R}P^n}$ the zero-section. We have the following, which implies Proposition 4.1:

Proposition 4.6 *The Oakley–Usher map $\Phi: D^*\mathbb{R}P^n \rightarrow \mathbb{C}P^n(2\pi)$ descends to a symplectomorphism*

$$\bar{\Phi}: \overline{D^*\mathbb{R}P^n} \rightarrow \mathbb{C}P^n(2\pi).$$

Proof Note that $\overline{D^*\mathbb{R}P^n}$ is canonically homeomorphic to the quotient $D^*\mathbb{R}P^n / \sim$, where $x \sim y$ if $x, y \in U^*\mathbb{R}P^n$ and they are in the same orbit of the geodesic flow. Therefore, it is easy to see from the computations of Oakley–Usher that Φ descends to a bijective continuous map $\bar{\Phi}: \overline{D^*\mathbb{R}P^n} \rightarrow \mathbb{C}P^n(2\pi)$. We will show that this map is a symplectomorphism.

To do so, it suffices to show that $\bar{\Phi}$ is smooth. Indeed, if $\bar{\Phi}$ is smooth, then by continuity it follows that $\bar{\Phi}$ preserves the symplectic form. This in particular shows that $\bar{\Phi}$ is an immersion, and hence a diffeomorphism that preserves the symplectic forms.

The following point is crucial. The canonical (linear) action of the group $G = \text{SO}(n + 1)$ on \mathbb{R}^{n+1} induces an action on $D^*\mathbb{R}P^n$ (and in turn on $\overline{D^*\mathbb{R}P^n}$) and \mathbb{C}^{n+1} (and in turn on $\mathbb{C}P^n(2\pi)$). It is clear from the definition (4-4) that $\bar{\Phi}$ is G -equivariant with respect to these actions.

We first prove smoothness in the case $n = 1$. Equip $\mathbb{C}P^1(2\pi)$ with the induced Riemannian metric (the so called Fubini–Study metric), which makes it isometric to a round S^2 . We will use the fact that the image of a linear Lagrangian subspace in $\mathbb{C}^2 \setminus \{0\}$ (with standard Kähler structure) under the canonical projection to $\mathbb{C}P^1(2\pi)$ is a geodesic circle. We will denote $\mathbb{C}P^1(2\pi)$ by $\mathbb{C}P^1$ for brevity.

Note that $Z_{\mathbb{R}P^1}$ is sent to the real part $L_{\mathbb{R}P^1} \subset \mathbb{C}P^1$ under $\bar{\Phi}$. Moreover, $\overline{D^*\mathbb{R}P^1} \setminus \text{int}(D^*\mathbb{R}P^1)$ consists of two points which map to $[1 : \pm i]$. Finally, notice that the images of cotangent fibers (that is, line segments of constant q) are sent to geodesic segments connecting $[1 : i]$ and $[1 : -i]$. It is easy to see that these geodesics are orthogonal to the geodesic circle $L_{\mathbb{R}P^1}$. Also recall that $\bar{\Phi}$ is $\text{SO}(2)$ -equivariant as explained above.

Let $(D^*\mathbb{R}P^1)^+ := \{[q, p] \mid p_1q_2 - p_2q_1 \leq 0\} \subset D^*\mathbb{R}P^1$. Note that $K^\sharp(q, p) = |p|$ is a smooth function when restricted to $(D^*\mathbb{R}P^1)^+$. On $(D^*\mathbb{R}P^1)^+$ the symplectic form is easily computed to be $dK^\sharp \wedge d\theta$, where $\theta: (D^*\mathbb{R}P^1)^+ \rightarrow \mathbb{R}/\pi\mathbb{Z}$ is defined by the relation $[q, p] = [\cos(\theta), \sin(\theta), -\sin(\theta)|p|, \cos(\theta)|p|]$.

Let $\mathbb{D} \subset \mathbb{R}^2$ be the unit disk $|x|^2 + |y|^2 \leq 1$ with the symplectic form $dx \wedge dy$. It is well-known that there is a symplectic embedding $\mathbb{D} \hookrightarrow \mathbb{C}P^1$ which sends

- the origin to $[1 : i]$,
- the unit circle to $L_{\mathbb{R}P^1}$,

- radial rays to the geodesic segments that connect $L_{\mathbb{R}P^1}$ and $[1 : i]$, in a way that preserves angles at the intersections, and
- finally, the disks centered at the origin to balls around $[1 : i]$ (in the Fubini–Study metric).

We now consider the induced continuous map $\Pi : (D^*\mathbb{R}P^1)^+ \rightarrow \mathbb{D}$ defined by the commutative diagram

$$\begin{array}{ccc}
 (D^*\mathbb{R}P^1)^+ & \xrightarrow{\Phi} & \mathbb{C}P^1 \\
 & \searrow \Pi & \uparrow \\
 & & \mathbb{D}
 \end{array}$$

Let (ρ, ϕ) denote polar coordinates on $\mathbb{D} \setminus \{0\}$. We deduce, from our discussion thus far, the following facts:

- There exists a constant c such that $\Pi^*\phi = -2\theta + c$.
- There exists a function $h : [0, 1) \rightarrow (0, 1]$ such that

$$\Pi^*\rho = h \circ K^\sharp.$$

- π restricts to a symplectomorphism $(D^*\mathbb{R}P^1)^+ \setminus U^*\mathbb{R}P^1 \rightarrow \mathbb{D} \setminus \{0\}$.

These facts imply that h satisfies the differential equation $h'(x)h(x) = -\frac{1}{2}$, which, with the initial condition $h(0) = 1$, has the unique solution $h(x) = \sqrt{1-x}$.¹ We thus observe that the map $\Pi : (D^*\mathbb{R}P^1)^+ \rightarrow \mathbb{D}$ is a model for the boundary reduction of $(D^*\mathbb{R}P^1)^+$ at $U^*\mathbb{R}P^1$; see [30, equation (3.1)]. This proves that the map $\bar{\Phi}$ is smooth at both points of $D^*\mathbb{R}P^1 \setminus \text{int}(D^*\mathbb{R}P^1)$ as we can repeat the same argument on the other half of $D^*\mathbb{R}P^1$.

We now move on to the case $n > 1$. We start with a preliminary lemma.

Lemma 4.7 *Let G be a Lie group acting smoothly on M and N . Suppose that S is a smooth submanifold of M and that the multiplication map $G \times S \rightarrow M$ is a surjective submersion. If $\phi : M \rightarrow N$ is a G -equivariant map and $\phi|_S$ is smooth, then ϕ is also smooth.*

Proof We have the following commutative diagram in which each map is known to be smooth except the bottom map:

$$\begin{array}{ccc}
 G \times S & \xrightarrow{\text{id} \times \phi} & G \times N \\
 \downarrow \mu_M & & \downarrow \mu_N \\
 M & \xrightarrow{\phi} & N
 \end{array}$$

Since the left map is smooth surjective submersion, it has local smooth sections $M \supset U \rightarrow G \times S$. The commutativity then implies that ϕ is smooth. □

¹This form of h can also be recovered from the explicit forms of Φ and the embedding $\mathbb{D} \hookrightarrow \mathbb{C}P^1$.

Note that the orbits of $G = \text{SO}(n + 1)$ on $\overline{D^*\mathbb{R}P^n}$ are the submanifolds of constant $|p|$. Fix any unoriented geodesic circle γ in $\mathbb{R}P^n$ with its round metric g . We obtain an embedding of $T^*\mathbb{R}P^1$ in $T^*\mathbb{R}P^n$ by taking points (q, p) such that $q \in \gamma$ and $p = g_q(v, \cdot)$, where v is tangent to γ . This restricts to a smooth embedding of $D^*\mathbb{R}P^1$ into $D^*\mathbb{R}P^n$, and of $\overline{D^*\mathbb{R}P^1}$ into $\overline{D^*\mathbb{R}P^n}$ (the last point is particularly clear in the description of the boundary reductions at hand as in the proof of Proposition 4.3, which we keep in mind for the next point as well.)

It is easy to see that the multiplication map $G \times \overline{D^*\mathbb{R}P^1} \rightarrow \overline{D^*\mathbb{R}P^n}$ is indeed a surjective submersion using that $G \times T^*\mathbb{R}P^1 \rightarrow T^*\mathbb{R}P^n$ is one. Applying the lemma with $S = \overline{D^*\mathbb{R}P^1}$, smoothness of $\bar{\Phi}$ follows from the smoothness of the $n = 1$ case $\bar{\Phi}_1: D^*\mathbb{R}P^1 \rightarrow \mathbb{C}P^1$. □

Remark 4.8 In [29, Chapter V], Seade gives a description of $\mathbb{C}P^n$ as a double mapping cylinder via the natural $\text{SO}(n + 1)$ action. One may follow this discussion to obtain the corresponding description of $\overline{D^*\mathbb{R}P^n}$, and that the map Φ factors as the normal exponential map of $\mathbb{R}P^n \hookrightarrow \mathbb{C}P^n$ (with respect to the Fubini–Study metric) composed with the map $D^*\mathbb{R}P^n \rightarrow T^*\mathbb{R}P^n \simeq N\mathbb{R}P^n$ induced by $|p| \mapsto \frac{1}{2} \sin^{-1} |p|$; note that $xf(x) = \tan(\frac{1}{2} \sin^{-1} x)$ and that the focal set of $\mathbb{R}P^n$ is precisely FQ_n . This yields an alternative construction of Φ as well as its extension $\bar{\Phi}$.

5 Lipschitz symplectic embeddings of balls

In this section, we aim to prove Theorems 1.6 and 1.7 from the introduction. To begin, we introduce some slightly more general notation, in which we also vary the radius of the cylinder. Suppose that $\Phi: B(\pi R^2) \rightarrow \mathbb{R}^4$ is a symplectic embedding with Lipschitz constant $L > 0$. Then we may set

$$E(\Phi, r) := \Phi^{-1}(\mathbb{R}^4 \setminus Z(\pi r^2)).$$

Recall now that Theorem 1.6 is the statement that $\text{Vol}_4(E(\Phi, 1)) \geq c/L^2$ for some constant $c = c(R) > 0$.

Proof of Theorem 1.6 Let δ be any number strictly between 0 and $R - 1$. Observe that by the Lipschitz bound, we have

$$N_{\delta/L}(E(\Phi, 1 + \delta)) \subset E(\Phi, 1).$$

Applying Theorem 3.1 with

$$E = E(\Phi, 1 + \delta)$$

to the symplectic embedding

$$\Phi|_{B(\pi R^2) \setminus E(\Phi, 1 + \delta)}: B(\pi R^2) \setminus E(\Phi, 1 + \delta) \rightarrow Z(1 + \delta),$$

we obtain

$$\text{Vol}_4(E(\Phi, 1)) \geq \pi^2(R^2 - (1 + \delta)^2) \frac{\delta^2}{L^2} - o\left(\frac{\delta^2}{L^2}\right),$$

which implies the desired bound after fixing some value for δ , for instance $\delta = \frac{1}{2}(R - 1)$. □

Remark 5.1 We may more generally ask about the volume of the region $E(\Phi, r)$ for $R > r$. By a scaling argument, we find that

$$\text{Vol}_4(E(\Phi, r)) \geq \frac{r^4 c(R/r)}{L^2}.$$

Hence, we lose no information by restricting to the case $r = 1$.

Remark 5.2 The proof demonstrates that we may take $c(R) \sim R^4$ as R grows large.

On the other hand, we wish to prove the constructive bound, in which we must find an embedding $\Phi: B^4(R) \rightarrow \mathbb{R}^4$ of Lipschitz constant bounded by L such that

$$\text{Vol}_4(E(\Phi)) \leq \frac{C}{L}$$

for some $C = C(R) > 0$. (Recall here that $E(\Phi) = E(\Phi, 1)$.)

Proof of Theorem 1.7 The fact that $\text{Vol}_4(E(\Phi))$ can be made arbitrarily small if there is no restriction on the Lipschitz constant is exemplified by ideas of Katok [16]. Our proof is a quantitative refinement obtained using constructions which appear in symplectic folding. The basic idea is to break the ball $B(\pi R^2)$ into a number of cubes and Hamiltonian isotope each of these cubes into $Z(\pi)$, where the cubes are separated by walls of width $1/L$. To begin, we make three simplifications. First, we replace the domain $B(\pi R^2)$ with the cube $K(R) = [-R, R]^4$, as the volume defect can only increase in size. Second, we replace $K(R)$ with the rectangular prism

$$K'(R) = ([0, 4R^2] \times [0, 1]) \times ([0, 4R^2] \times [0, 1]),$$

where the parentheses indicate a symplectic splitting. Explicitly, the factors refer to the coordinates x_1, y_1, x_2, y_2 , with symplectic form $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. Notice that the natural symplectomorphism between $K(R)$ and $K'(R)$ has Lipschitz constant $2R$, and in particular, the effect of replacing $K(R)$ with $K'(R)$ only affects our proposition by a factor of R which gets absorbed into the constant C . And finally, we allow the Lipschitz constant to be $O_R(L)$, by which we mean it is bounded by AL , where A is a constant which again depends upon R ; we arrive at the proposition as stated by absorbing this constant in C .

Consider now each symplectic factor $[0, 4R^2] \times [0, 1] \subset \mathbb{R}^2$ of $K'(R)$. For $i \in \mathbb{Z}$ with $0 \leq i \leq 4R^2$, let X_i be the region in this rectangle with

$$i \leq x \leq i + 1 - \frac{1}{L}.$$

Our goal is to fit each $X_i \times X_j \subset K'(R)$ into $Z(\pi)$ —indeed, the complement of the union of these regions has volume $\Theta(R^4/L)$; the R^4 factor gets absorbed by the constant C .

To begin, there is an area-preserving map of the rectangle $[0, 4R^2] \times [0, 1]$ into \mathbb{R}^2 of Lipschitz constant at most $O(L)$ which translates X_i in the x -direction by i . That is, X_0 stays fixed, X_1 gets shifted to the right

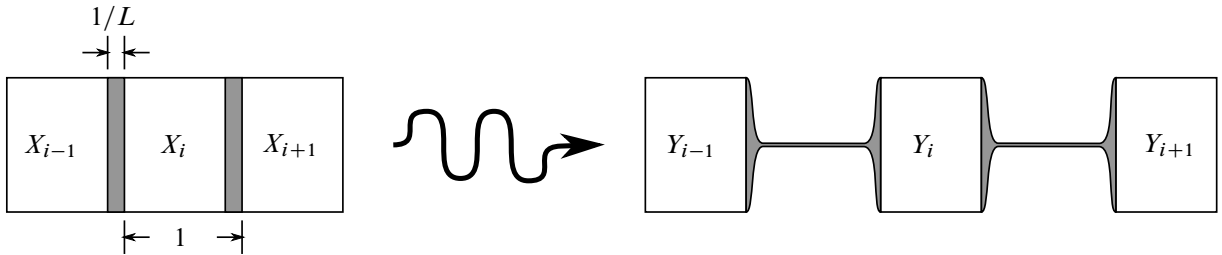


Figure 1: We stretch the region between X_i and X_{i+1} in an area-preserving way so that the images Y_i and Y_{i+1} are separated by distance just over 1.

by 1, and the region between them gets stretched like taffy in an area-preserving way to accommodate for this shift. See Figure 1. Let Y_i be the image of X_i under this map. Then Y_i and Y_{i+1} are separated by distance $1 + 1/L$, with

$$Y_i = \left[2i, 2i + 1 - \frac{1}{L} \right] \times [0, 1].$$

Explicitly, a model for the taffy-stretching map is given as $\phi: [0, 1/L] \times [0, 1] \rightarrow [0, 1 + 1/L] \times [0, 1]$ of the form

$$\phi(x, y) = \left(f(x), \frac{1}{2} + \frac{y - \frac{1}{2}}{f'(x)} \right)$$

(which is automatically area-preserving; see $D\phi$ computed below) with $f: [0, 1/L] \rightarrow [0, 1 + 1/L]$ a family of functions, depending upon L , such that

- $f'(x) = 1$ for x in an open neighborhood of the endpoints 0 and $1/L$,
- $f(0) = 0$ and $f(1/L) = 1 + 1/L$,
- $1 \leq f'(x) \in O(L)$,
- $|f''(x)/(f'(x))^2| \in O(L)$.

The first two conditions imply that the constructed stretching map ϕ glues to the rigid translations of the X_i in a C^∞ manner. The latter two conditions imply the desired Lipschitz constant bound of $O(L)$, since

$$D\phi = \begin{pmatrix} f'(x) & 0 \\ -(y - \frac{1}{2}) \frac{f''(x)}{(f'(x))^2} & \frac{1}{f'(x)} \end{pmatrix},$$

with each entry in this matrix of order $O(L)$.

An example of such a desired function f may be given in the form²

$$f(x) = \int_0^x \frac{1}{1 - Cg(y)} dy,$$

²There are of course many such choices for function f , and we do not claim to make an optimal choice. In an earlier draft, we claimed that the Lipschitz constant of the stretching map could be made less than $2L$. We only need that the Lipschitz constant is $O(L)$ for our purposes, and it is unclear whether $2L$ can be achieved. We thank the reviewer for pointing out this lack of clarity.

where C is a constant dependent upon L and g (as we will explain soon), and where $g : [0, 1/L] \rightarrow [0, 1/4L]$ is a smooth function which is C^0 -close to the continuous function

$$g_0(x) = \begin{cases} x & \text{if } 0 \leq x < 1/4L, \\ 1/4L & \text{if } 1/4L \leq x < 3/4L, \\ 1/L - x & \text{if } 3/4L \leq x \leq 1/L. \end{cases}$$

We may take g so that $g(x) \equiv 0$ identically near the endpoints $x = 0$ and $x = 1/L$, and such that $|g'(x)| \leq 1 + \epsilon$ for any chosen $\epsilon > 0$. The constant C is chosen so that $f(1/L) = 1 + 1/L$, ie such that

$$I(C) := \int_0^{1/L} \frac{1}{1 - Cg(y)} dy = 1 + \frac{1}{L}.$$

We claim that such a constant C exists. Notice that the value of the integral $I(C)$, as a function of C , is continuous and monotonically nondecreasing on the interval $[0, 1/\sup\{g(x)\})$, with

$$I(0) = \frac{1}{L} < 1 + \frac{1}{L} < \infty = \lim_{C \rightarrow 1/\sup\{g(x)\}} I(C),$$

where the limit on the right follows because g attains its maximum value on the interior of the interval $[0, 1/L]$, and because g is smooth, we have $g' = 0$ at this maximum value. The existence of C now follows from the intermediate value theorem, and monotonicity implies uniqueness. We notice that because $\sup\{g(x)\} \approx 1/4L$ and $C < 1/\sup\{g(x)\}$, we have that $C \in O(L)$.

With these choices, all of the conditions on f are now met, so long as we take a close enough approximation g of g_0 (where the closeness depends upon L). The first bullet point follows because $g(x) \equiv 0$ near the endpoints. The second follows because we chose C accordingly. The fourth is guaranteed since

$$\frac{f''(x)}{(f'(x))^2} = Cg'(x),$$

and we have constructed g so that $|g'(x)| \leq 1 + \epsilon$ and $C \in O(L)$.

That leaves the third bullet point, which we now verify. The fact that $f'(x) \geq 1$ is clear, so it suffices to check $f'(x) \in O(L)$. Solving for C_0 using g_0 , we find

$$1 + \frac{1}{L} = \int_0^{1/L} \frac{1}{1 - C_0g_0(x)} dx > \int_{1/4L}^{3/4L} \frac{1}{1 - C_0/4L} dx = \frac{1}{2L} \cdot \frac{1}{1 - C_0/4L},$$

so that $C_0 < 4L(1 - 1/(2(L + 1)))$. Notice that the value of C depends continuously on the function g (in the C^0 -topology), so for any $\epsilon > 0$ there is a choice of approximation g so that

$$\sup\{f'(x)\} = \sup\left\{\frac{1}{1 - Cg(x)}\right\} < \sup\left\{\frac{1}{1 - C_0g_0(x)}\right\} + \epsilon = \frac{1}{1 - C_0/4L} + \epsilon < 2(L + 1) + \epsilon.$$

Hence, $f'(x) \in O(L)$, as required.

Applying the stretching map to each symplectic factor, each region $X_i \times X_j$ is sent to $Y_i \times Y_j$. It suffices now to find a symplectomorphism of \mathbb{R}^4 of Lipschitz constant $O_R(1)$ so that each $Y_i \times Y_j$ has image in

the cylinder $Z(\pi)$, since in such a case, we compose with our stretching map, and each $X_i \times X_j$ under this composition lands in $Z(\pi)$, where the composition map has total Lipschitz constant $O_R(L)$. We construct this by sliding each $Y_i \times Y_j$ in two steps. We begin by separating the y_2 -coordinates of the various blocks based on their x_1 -coordinates. That is, we translate $Y_i \times Y_j$ in the y_2 -direction by $2i$ units, in other words so that it gets translated to

$$\left[2i, 2i + 1 - \frac{1}{L}\right] \times [0, 1] \times \left[2j, 2j + 1 - \frac{1}{L}\right] \times [2i, 2i + 1].$$

Explicitly, we use a Hamiltonian of the form $H = -\rho(x_1)x_2$, where ρ is a step function with the properties

- $\rho(x_1) = 2i$ when $2i \leq x_1 \leq 2i + 1$ and $0 \leq i \leq 4R^2$,
- $\|\rho'\|_\infty \in O(1)$,
- $\|\rho''\|_\infty \in O(1)$.

The corresponding Hamiltonian vector field X_H is of the form $2i \partial_{y_2}$ when $2i \leq x_1 \leq 2i + 1$, and hence translates $Y_i \times Y_j$ as desired. Explicitly, the time-1 Hamiltonian flow is

$$\phi_H^1(x_1, y_1; x_2, y_2) = (x_1, y_1 + \rho'(x_1)x_2; x_2, y_2 + \rho(x_1)),$$

with derivative

$$D\phi_H^1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \rho''(x_1)x_2 & 1 & \rho'(x_1) & 0 \\ 0 & 0 & 1 & 0 \\ \rho'(x_1) & 0 & 0 & 1 \end{pmatrix}.$$

All terms are $O(1)$ except for $\rho''(x_1)x_2$, because x_2 can grow large. But on the image of our cube after the taffy-stretching step, x_2 is at most $4R^2$, and so the relevant Lipschitz constant of this sliding step is $O(R^2)$.

A similar construction, using a Hamiltonian of the form $-\rho(y_2)y_1$ for the same function ρ , allows us to then take each of these new blocks and translate them in the x_1 -direction. After we complete both of these steps, the image of $Y_i \times Y_j$ is

$$\left[0, 1 - \frac{1}{L}\right] \times [0, 1] \times \left[2j, 2j + 1 - \frac{1}{L}\right] \times [2i, 2i + 1].$$

A final translation simultaneously in the x_1 and y_1 coordinates by $-\frac{1}{2}$ lands each of these blocks in the cylinder $Z(\pi)$, concluding the proof. □

Remark 5.3 As in Remark 5.1, we may also vary the radius of the cylinder r , but where the new constant is $r^4 C(R/r)$.

Remark 5.4 The construction as presented has $C(R) \sim R^9$. Indeed, our volume defect came with a factor of R^4 , but the Lipschitz constant has an extra factor of R^5 . The first factor of R appearing in the Lipschitz constant came from replacing $K(R)$ with $K'(R)$. An extra two factors of R^2 came from our

slide moves. One can optimize a little by performing a single diagonal slide move instead of two separate orthogonal slide moves. Hence, in the end, we may take $C(R) \sim R^7$. We suspect this is far from optimal, though decreasing the exponent appears to require a new idea.

6 Further questions

6.1 Minkowski dimension problem in higher dimensions

In this section we pose the simplest Minkowski dimension question that one could ask in dimensions higher than four, and make a couple of remarks about it. Let us assume $n > 2$ throughout this section.

Question 1 What is the smallest $d \in \mathbb{R}$ such that for some $A > \pi$, there exists a closed subset $E \subset B^{2n}(A)$ of Minkowski dimension d such that $B^{2n}(A) \setminus E$ symplectically embeds into $Z^{2n}(\pi)$?

Assume that for some $A > \pi$ and a closed subset $E \subset B^{2n}(A)$ of Minkowski dimension d , $B^{2n}(A) \setminus E$ symplectically embeds into $Z^{2n}(\pi)$. We find it plausible that a version of our obstructive argument would still give $d \geq 2$, even though we do not have a proof of this.

The argument in [Proposition 2.3](#) suggests that the problem is related to the question of how the 2-width of a round ball changes after the removal of a closed subset. One can explicitly see that for E being the intersection of $B^{2n}(A)$ with a linear Lagrangian subspace L of \mathbb{R}^{2n} , an n -dimensional submanifold, there is a (holomorphic!) sweepout of $B^{2n}(A) \setminus E$ with width $\frac{1}{2}A$. Namely, we take the foliation by half-disks that are the connected components of the intersections with $B^{2n}(A) \setminus E$ of affine complex planes that intersect L nontransversely.

6.2 Capacity after removing a linear plane

Throughout this section let $B := B^4(2\pi) \subset \mathbb{C}^2$, where \mathbb{C}^2 is equipped with its standard Kähler structure. Let us denote the complex coordinates by x and y .

Let us denote by c_{Gr} the Gromov width, which is a capacity defined on any symplectic manifold Y^{2n} as the supremum of πr^2 , where $B^{2n}(\pi r^2)$ symplectically embeds into Y .

Definition 6.1 Let $V \subset \mathbb{C}^2$ be a real subspace of dimension 2. We define the symplecticity of V as $|\omega_{\text{st}}(e_1, e_2)|$, where e_1, e_2 is any orthonormal basis of T_0V .

Notice that V is a Lagrangian plane if and only if its symplecticity is 0. On the other extreme, V is a complex plane if and only if its symplecticity is 1.

The symplecticity defines a surjective continuous function

$$\text{Gr}_{\mathbb{R}}(2, 4) \rightarrow [0, 1].$$

Lemma 6.2 Two elements of $\text{Gr}_{\mathbb{R}}(2, 4)$ have the same symplecticity if and only if there is an element of $U(2)$ sending one to the other.

We can therefore define the function (symplecticity to capacity)

$$\text{stc}: [0, 1] \rightarrow (0, \infty)$$

by $s \mapsto c_{\text{Gr}}(B \setminus V_s)$, where $V_s \in \text{Gr}_{\mathbb{R}}(2, 4)$ with symplecticity s .

The following remarkable statement is due to Traynor.

Proposition 6.3 (Traynor [31, Proposition 5.2]) *B is symplectomorphic to*

$$B \setminus (\{xy = 0\} \cup \{(e^{it}x, y) \cup (x, e^{it}y) \mid \text{Im}(x) = \text{Im}(y) = 0, \text{Re}(x) \geq 0, \text{Re}(y) \geq 0, t \in [0, 2\pi]\}).$$

Hence, we have that $\text{stc}(1) = 2\pi$. On the other hand, it follows immediately from our [Theorem 1.3](#) that $\text{stc}(0) = \pi$. We finish with the obvious question.

Question 2 What is the function stc ? Is it continuous?

6.3 Minkowski dimension problem for large R

Let $R_{\text{sup}} \in (1, \infty]$ be the supremum of the radii R such that there is a closed subset E of Minkowski dimension 2 inside $B(\pi R^2)$ whose complement symplectically embeds into $Z(\pi)$. In [Section 4](#), we showed that $R_{\text{sup}} \leq \sqrt{2}$. This inequality will be improved by Joé Brendel to $R_{\text{sup}} \leq \sqrt{3}$ using a different construction; see [Remark 1.5](#). An intriguing aspect of both of the squeezing constructions is that they fail for large radii R . This motivates the following:

Question 3 Is R_{sup} a finite number?

6.4 Minkowski dimension problem for extendable embeddings

Here is a variant of our Minkowski dimension question, which is also more directly related to the Lipschitz number question. Assume $R > r$. What is the smallest Minkowski dimension of a subset $E \subset B^4(\pi R^2)$ with the property that for any neighborhood U of E , there is a symplectic embedding $B^4(\pi R^2) \rightarrow \mathbb{C}^2$ such that $B^4(\pi R^2) \setminus U$ maps inside $Z^4(\pi r^2)$?

Our obstructive [Theorem 1.4](#) still gives a bound, but our construction in [Section 4](#) does not apply. Recall $B := B^4(2\pi)$.

Proposition 6.4 *Consider the embedding $B \setminus L \hookrightarrow Z(\pi)$ that we constructed in [Theorem 1.3](#). There exists an embedded circle η in $iL \cap B$ that is disjoint from L , and which maps into $\{x_1 = y_1 = 0\}$.*

Proof Recall that our symplectomorphism first sends $B \setminus L$ to $\mathbb{C}\mathbb{P}^2(2\pi) \setminus (L_{\mathbb{R}\mathbb{P}} \cup S_{[0:0:1]})$ via the restriction of a symplectomorphism $b: B \rightarrow \mathbb{C}\mathbb{P}^2(2\pi) \setminus S_{[0:0:1]}$. The image of $L \cap B$ under b is $L_{\mathbb{R}\mathbb{P}} \setminus S_{[0:0:1]}$, whereas the image of $iL \cap B$ is $iL_{\mathbb{R}\mathbb{P}} \setminus S_{[0:0:1]}$, where we define

$$iL_{\mathbb{R}\mathbb{P}} := \{[z_1 : z_2 : z_3] \mid \text{Re}(z_1) = \text{Re}(z_2) = \text{Im}(z_3) = 0\}.$$

FQ and $iL_{\mathbb{R}P}$ intersect along the circle

$$\widehat{C} = \{[i \cos \theta : i \sin \theta : 1] \mid \theta \in [0, 2\pi]\}.$$

Note that \widehat{C} is disjoint from $L_{\mathbb{R}P} \cup S_{[0:0:1]}$.

Now recall that to complete our symplectomorphism from $B \setminus L$ to $\mathcal{E}(\pi, 4\pi) \setminus \mathcal{C}$ we use the Oakley–Usher symplectomorphism between $\mathbb{C}P^2(2\pi) \setminus (L_{\mathbb{R}P} \cup S_{[0:0:1]})$ and $D^*\mathbb{R}P^2 \setminus (Z_{\mathbb{R}P} \cup S)$, and then the Opshtein symplectomorphism.

From Opshtein’s formula we see that all the points on $Q \setminus (Z_{\mathbb{R}P} \cup S)$ and therefore on $FQ \setminus (L_{\mathbb{R}P} \cup S_{[0:0:1]})$ are sent to points in $\{x_1 = y_1 = 0\}$ in $\mathcal{E}(\pi, 4\pi)$. This means that $b^{-1}(\widehat{C})$ satisfies the condition in the statement. □

Corollary 6.5 *Let U be a neighborhood of L that is disjoint from η . Then, we cannot extend the restriction to $B \setminus U$ of our symplectic embedding $B \setminus L$ into $Z(\pi)$ to a symplectic embedding $B \rightarrow \mathbb{C}^2$.*

Proof If there were such an embedding, the action of the image of η would have to be simultaneously zero and nonzero, which is absurd. □

6.5 Bounds on Minkowski content of the defect region

We have shown in [Corollary 3.2](#) that the lower Minkowski dimension bound $\dim_{\mathcal{M}}(E) \geq 2$ is optimal in the range $R \in (1, \sqrt{2}]$. Is the estimate on the 2–content $\underline{\mathcal{M}}_2(E) \geq \pi(R^2 - 1)$ also sharp? That is, does there exist $R \in (1, \sqrt{2}]$ and E with $\underline{\mathcal{M}}_2(E) = \pi(R^2 - 1)$ such that $B(\pi R^2) \setminus E$ symplectically embeds into $Z(\pi)$?

6.6 Speculations on the Lipschitz question

Consider the volume loss function

$$VL(L, R) := \inf \{ \text{Vol}(E(\Phi)) \},$$

where the infimum is taken over all symplectic embeddings $\Phi: B^4(\pi R^2) \rightarrow \mathbb{R}^4$ with Lipschitz constant at most L , and

$$E(\Phi) = \Phi^{-1}(\mathbb{R}^4 \setminus Z^4(\pi)).$$

In [Section 5](#), we proved [Theorems 1.6](#) and [1.7](#), which may be summarized by the statement that

$$\frac{c(R)}{L^2} \leq VL(L, R) \leq \frac{C(R)}{L},$$

where we tacitly assume $R > 1$. Even more, we noted in [Remarks 5.2](#) and [5.4](#) that our methods show that we may take $c(R) \sim R^4$ and $C(R) \sim R^7$.

One natural question is whether we can bring the two bounds closer together. In terms of factors of R , we suspect that the asymptotics should indeed be $\Theta(R^4)$, ie growing like the total volume, though we could not find constructions which remain under this upper bound for large L . As for the factors of L , the jury is very much out. We nonetheless formulate a precise conjecture.

Conjecture 1 *For any $\ell, r > 0$, the limit*

$$\lim_{R \rightarrow \infty} \lim_{L \rightarrow \infty} \frac{\text{VL}(\ell L, rR)}{\text{VL}(L, R)}$$

exists and is positive.

The second question is to what extent our methods work in higher dimensions. On the constructive side, we may simply take our constructed symplectic embedding in 4 dimensions and extend it to $2n$ dimensions by acting by the identity on the symplectically complementary $2n - 4$ dimensions. This yields a constructive bound of $\Theta(R^{2n+3}/L)$. As for the obstructive bound, a modification of the techniques presented in this paper, in which we obtain a sweepout instead of a foliation if we follow Gromov's nonsqueezing argument, should probably yield a bound of $O(R^{2n}/L^{2n-2})$.

Finally, we describe a quantity which we believe could be interesting. Although we are working with balls $B^4(\pi R^2)$, we could in principle replace these with other subdomains in \mathbb{R}^4 . To be precise, suppose we fix $X \subset \mathbb{R}^4$ a bounded domain. Consider the generalized volume loss function

$$\text{VL}_X(L, R) := \inf\{\text{Vol}_4(E(\Phi))\},$$

where the infimum is taken over all symplectic embeddings $\Phi: RX \rightarrow \mathbb{R}^4$ of Lipschitz embedding at most L , and we take

$$E(\Phi) := \Phi^{-1}(\mathbb{R}^4 \setminus Z^4(\pi)).$$

In the case $X = B^4(\pi)$, we recover the usual volume loss function above.

We offer the following reasonable-looking conjecture.

Conjecture 2 *For all bounded domains X , the limit*

$$s_X := \lim_{R \rightarrow \infty} \lim_{L \rightarrow \infty} \frac{\text{VL}_X(L, R)}{\text{VL}(L, R)}$$

exists and is strictly positive.

Notice that if there exists a symplectic embedding $rX \hookrightarrow Y$ of Lipschitz constant ℓ , then

$$\text{VL}_X(\ell L, rR) \leq \text{VL}_Y(L, R).$$

Should [Conjecture 2](#) hold, one might hope to use this inequality to compare the values of s_X and s_Y for bounded domains X and Y .

Appendix Squeezing and degenerations of the complex projective plane

by Joé Brendel

A.1 Introduction and main theorem

Our goal is to show the following.

Theorem A.1 *For every $\alpha < 3$, there is a set $\Sigma \subset B^4(\alpha)$ of Minkowski dimension 2 such that $B^4(\alpha) \setminus \Sigma$ symplectically embeds into $Z^4(1) = \mathbb{R}^2 \times D^2(1) \subset \mathbb{R}^4$.*

Our notation corresponds to that of the main body of the text by setting $\alpha = \pi R^2$. The idea of the proof is to view $B^4(\alpha)$ as $\mathbb{C}\mathbb{P}^2(\alpha) \setminus \mathbb{C}\mathbb{P}^1$ and to use almost toric fibrations of $\mathbb{C}\mathbb{P}^2$. As observed in [32], for every Markov triple (a, b, c) , there is a triangle $\Delta_{a,b,c}(\alpha) \subset \mathbb{R}^2$ and an almost toric fibration on $\mathbb{C}\mathbb{P}^2(\alpha)$ with a base diagram whose underlying polytope is $\Delta_{a,b,c}(\alpha)$. Now note that the toric moment map image of $Z^4(1)$ is the half-strip $\mathcal{S} = \mathbb{R}_{\geq 0} \times [0, 1)$. We shall show that if the triangle $\Delta_{a,b,c}(\alpha)$ fits into \mathcal{S} (after applying an integral affine transformation), then there is a symplectic embedding of $\mathbb{C}\mathbb{P}^2(\alpha)$ into $Z^4(1)$ at the cost of removing a certain subset Σ' from $\mathbb{C}\mathbb{P}^2(\alpha)$. The point here is that one can get a good understanding of the subset one needs to remove. Indeed, we show that Σ' is a union of three Lagrangian pinwheels (defined as in [7]) and a symplectic torus. In particular, this set has Minkowski dimension 2. A combinatorial argument shows that for every $\alpha < 3$, there is a Markov triple (a, b, c) and an inclusion $\Delta_{a,b,c}(\alpha) \subset \mathcal{S}$; see Lemma A.5.

Remark A.2 As was pointed out to us by Leonid Polterovich, our results can be combined with Gromov's nonsqueezing to show that any symplectic ball $B^4(1 + \varepsilon) \subset \mathbb{C}\mathbb{P}^2(\alpha)$ intersects the set $\Sigma' \subset \mathbb{C}\mathbb{P}^2$ discussed above. See Corollary A.9 for more details.

Remark A.3 The same strategy may work to produce symplectic embeddings

$$D^2(\alpha) \times D^2(\alpha) \setminus \Sigma \hookrightarrow Z^4(1)$$

of the polydisk of capacity $\alpha < 2$ minus a union of some two-dimensional manifolds into the cylinder. Indeed, one can view the polydisk as the affine part of $S^2 \times S^2$ and use almost toric fibrations of the latter space to carry out the same argument.

The relationship between Markov triples and the complex and symplectic geometry of $\mathbb{C}\mathbb{P}^2$ has generated a lot of interest in recent years. It first appeared in the work of Galkin and Usnich [8], who conjectured that for every Markov triple there is an exotic Lagrangian torus in $\mathbb{C}\mathbb{P}^2$. This conjecture was proved and generalized by Vianna [32; 33] by the use of almost toric fibrations; see also Symington [30]. On the algebrogeometric side, Hacking and Prokhorov [13] showed that a complex surface X with quotient singularities admits a \mathbb{Q} -Gorenstein smoothing to $\mathbb{C}\mathbb{P}^2$ if and only if X is a weighted projective

space $\mathbb{C}\mathbb{P}(a^2, b^2, c^2)$ and (a, b, c) forms a Markov triple. In [7], Evans and Smith studied embeddings of *Lagrangian pinwheels* into $\mathbb{C}\mathbb{P}^2$. This is directly related to [13], since Lagrangian pinwheels appear naturally as vanishing cycles of the smoothings of $\mathbb{C}\mathbb{P}(a^2, b^2, c^2)$ to $\mathbb{C}\mathbb{P}^2$. See also the recent work by Casals and Vianna [5] and the forthcoming paper joint with Mikhalkin and Schlenk [4] for other applications of almost toric fibrations to symplectic embedding problems.

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A.2 Some geometry of Markov triangles

Let us recall some facts about Markov numbers and their associated triangles.

Definition A.4 A triple of natural numbers $a, b, c \in \mathbb{N}_{>0}$ is called a *Markov triple* if it solves the *Markov equation*

$$(A-1) \quad a^2 + b^2 + c^2 = 3abc.$$

If (a, b, c) is a Markov triple, then so is $(a, b, 3ab - c)$. Starting from the solution $(1, 1, 1)$, we obtain the so-called *Markov tree* by mutations $(a, b, c) \rightarrow (a, b, 3ab - c)$. The first few Markov triples are

$$(A-2) \quad (1, 1, 1), \quad (1, 1, 2), \quad (1, 2, 5), \quad (1, 5, 13), \quad (2, 5, 29), \quad (1, 13, 34), \quad \dots$$

Given $\alpha > 0$, let $\mathbb{C}\mathbb{P}^2$ be equipped with the Fubini–Study symplectic form ω normalized so that $\int_{\mathbb{C}\mathbb{P}^1} \omega = \alpha$. For every Markov triple (a, b, c) , there is an almost toric fibration of $\mathbb{C}\mathbb{P}^2$ with almost toric base diagram a rational triangle $\Delta_{a,b,c}(\alpha) \subset \mathbb{R}^2$, which we call the *Markov triangle* associated to the Markov triple (a, b, c) . The first Markov triangle is the (honest) toric moment map image of $\mathbb{C}\mathbb{P}^2$ in our normalization,

$$(A-3) \quad \Delta_{1,1,1}(\alpha) = \{(x, y) \in \mathbb{R}_{\geq 0}^2 \mid x + y \leq \alpha\}.$$

In fact, we will slightly abuse notation and at times think of $\Delta_{a,b,c}(\alpha)$ as a triangle in \mathbb{R}^2 and at other times as the equivalence class of triangles \mathbb{R}^2 under the group of toric symmetries given by *integral affine transformations*, ie elements in $\text{Aff}(2; \mathbb{Z}) = \mathbb{R}^2 \rtimes \text{GL}(2; \mathbb{Z})$. For every mutation of Markov triples $(a, b, c) \rightarrow (a, b, 3ab - c)$, there is a corresponding mutation of triangles $\Delta_{a,b,c}(\alpha) \rightarrow \Delta_{a,b,3ab-c}(\alpha)$, defined by cutting the triangle in two halves and applying a shear map to one of the halves and gluing it back to the other half. This is called a *branch move* and we refer to Symington [30, Sections 5.3 and 6] and Vianna [32, Section 2] for details. For a concrete description of the triangles $\Delta_{a,b,c}(\alpha)$, see (A-23)

and the surrounding discussion. The area of the Markov triangles is well-defined, since it is invariant under $\text{Aff}(2; \mathbb{Z})$. Furthermore, the area is invariant under the mutation of triangles, and hence we obtain

$$(A-4) \quad \text{area}(\Delta_{a,b,c}(\alpha)) = \text{area}(\Delta_{1,1,1}(\alpha)) = \frac{1}{2}\alpha^2.$$

Recall that we are interested in *embedding Markov triangles into the half-strip* $\mathcal{S} = \mathbb{R}_{\geq 0} \times [0, 1)$. This means that, given $\alpha > 0$, we look for a Markov triple (a, b, c) such that $\Delta_{a,b,c}(\alpha) \subset \mathcal{S}$ up to applying an element in $\text{Aff}(2; \mathbb{Z})$. We will prove the following.

Lemma A.5 *For every $0 < \alpha < 3$, there is a Markov triple (a, b, c) such that the Markov triangle $\Delta_{a,b,c}(\alpha) \subset \mathbb{R}^2$ is in $\mathcal{S} = \mathbb{R}_{\geq 0} \times [0, 1)$ up to applying an element in $\text{Aff}(2; \mathbb{Z})$. This result is sharp in the sense that for $\alpha \geq 3$, there is no such Markov triple.*

Let us introduce some definitions from integral affine geometry. A vector $v \in \mathbb{Z}^2$ is called *primitive* if $\beta v \notin \mathbb{Z}^2$ for all $0 < \beta < 1$. Note that to every vector $w \in \mathbb{R}^2$ with rational slope, we can associate a unique primitive vector v such that $w = \gamma v$ for $\gamma > 0$. We call γ the *affine length* of w and denote it by $\ell_{\text{aff}}(w)$. Let $l \subset \mathbb{R}^2$ be a rational affine line (or line segment) with primitive directional vector $v \in \mathbb{Z}^2$ and $p \in \mathbb{R}^2$ be a point. Then the *affine distance* is defined as $d_{\text{aff}}(p, l) = |\det(v, u)|$, where u is any vector such that $p + u \in l$. This does not depend on any of the choices we have made and these quantities are $\text{Aff}(2; \mathbb{Z})$ -invariant. See McDuff [21] for more details.

For a given rational (not necessarily Markov) triangle Δ , we denote by E_1, E_2, E_3 its edges and by v_1, v_2, v_3 its vertices such that v_i lies opposite to the edge E_i . We call $\ell_{\text{aff}}(E_1) + \ell_{\text{aff}}(E_2) + \ell_{\text{aff}}(E_3)$ the *affine perimeter* of Δ . Note that the affine perimeter of a Markov triangle $\Delta_{a,b,c}(\alpha)$ is equal to 3α . We have the following formula for the area of Δ ,

$$(A-5) \quad \text{area}(\Delta) = \frac{1}{2} \ell_{\text{aff}}(E_i) d_{\text{aff}}(v_i, E_i).$$

This follows from the definition of affine distance and affine length.

Proof of Lemma A.5 For the first part of the proof we only need one branch of the Markov tree, namely the one where the maximal entry grows the fastest. More precisely, let Δ_n be the Markov triangle associated to the triple (m_{n+2}, m_{n+1}, m_n) , where m_k is recursively defined as

$$(A-6) \quad m_0 = m_1 = m_2 = 1, \quad m_{k+2} = 3m_{k+1}m_k - m_{k-1}.$$

The first few terms of this sequence are given by $\{m_k\}_{k \in \mathbb{N}} = \{1, 1, 1, 2, 5, 29, 433, \dots\}$. We clearly have $m_k \rightarrow \infty$. Recall from [32] that the affine side lengths of a Markov triangle $\Delta_{a,b,c}(\alpha)$ are given by $\lambda a^2, \lambda b^2, \lambda c^2$ for a proportionality constant $\lambda > 0$. Since the affine perimeter of the Markov triangle is 3α , we obtain $\lambda(a^2 + b^2 + c^2) = 3\alpha$. Together with the Markov equation (A-1) this yields that the longest edge E_n in $\Delta_n(\alpha)$ has affine length

$$(A-7) \quad \ell_{\text{aff}}(E_n) = \frac{\alpha m_{n+2}}{m_{n+1}m_n} \stackrel{(A-6)}{=} \alpha \left(3 - \frac{m_{n-1}}{m_{n+1}m_n} \right).$$

Since the second summand goes to 0 for large n , we obtain $\ell_{\text{aff}}(E_n) \rightarrow 3\alpha$. Let v_n be the vertex opposite to E_n . By (A-4) and the area formula (A-5) we obtain

$$(A-8) \quad \ell_{\text{aff}}(E_n) d_{\text{aff}}(v_n, E_n) = \alpha^2.$$

This implies that $d_{\text{aff}}(v_n, E_n) \rightarrow \frac{1}{3}\alpha$. Now note that we can assume, up to $\text{Aff}(2; \mathbb{Z})$, that the maximal edge E_n lies in $\mathbb{R}_{\geq 0} \times \{0\}$ and that $\Delta_n(\alpha)$ lies in the upper half-plane. Then the (Euclidean) height of $\Delta_n(\alpha)$ is equal to $d_{\text{aff}}(v_n, E_n)$. This implies that for all $\alpha < 3$, we have $\Delta_n(\alpha) \subset \mathcal{S}$ for large enough $n \in \mathbb{N}$.

Now let $\alpha \geq 3$ and suppose that there is a Markov triangle $\Delta_{a,b,c}(\alpha) \subset \mathcal{S}$. Let E be the longest edge of $\Delta_{a,b,c}(\alpha)$ and v the opposite vertex. We first prove that E is parallel to $e_1 = (1, 0)$. Indeed, suppose it is not. Then we can write $E = \ell_{\text{aff}}(E)v$ for a primitive vector $v = (v_1, v_2) \in \mathbb{Z}^2$ with $v_2 \geq 1$. But since the affine perimeter of $\Delta_{a,b,c}(\alpha)$ is 3α and E is the longest edge, we obtain $\ell_{\text{aff}}(E) \geq \alpha$ and thus $\Delta_{a,b,c}(\alpha)$ is not contained in \mathcal{S} . Now if E is parallel to e_1 , then the (Euclidean) height of $\Delta_{a,b,c}(\alpha)$ is $d_{\text{aff}}(v, E)$. The affine perimeter 3α is strictly larger than $\ell_{\text{aff}}(E)$ from which we deduce $d_{\text{aff}}(v, E) > \frac{1}{3}\alpha \geq 1$ by the area formula (A-5). \square

A.3 Proof of the main theorem

Definition A.6 Let p be a positive integer. The topological space obtained from the unit disk D by quotienting out the action of the group of the p^{th} roots of unity on ∂D is called a p -pinwheel.

For example, the 2-pinwheel is $\mathbb{R}\mathbb{P}^2$. The image of ∂D in the quotient is called the *core circle*. For all $p > 2$ the p -pinwheels are not smooth at points of the core circle. A *Lagrangian pinwheel* in a symplectic manifold M is a Lagrangian embedding of a p -pinwheel into M ; see [7, Definition 2.3] for the meaning of *embedding* in this context. As it turns out, for every Lagrangian p -pinwheel, there is an additional extrinsic parameter $q \in \{1, \dots, p-1\}$ measuring the twisting of the pinwheel around its core circle. We call such an object (p, q) -pinwheel and denote it by $L_{p,q}$ when this causes no confusion. For us, the following result is key.

Proposition A.7 Let (a, b, c) be a Markov triple and let $s_a, s_b, s_c \in \mathbb{C}\mathbb{P}(a^2, b^2, c^2)$ be the orbifold singular points of the corresponding weighted projective space. Then there is a surjective map

$$\tilde{\phi}: \mathbb{C}\mathbb{P}^2 \rightarrow \mathbb{C}\mathbb{P}(a^2, b^2, c^2)$$

and there are (mutually disjoint) Lagrangian pinwheels $L_{a,q_a}, L_{b,q_b}, L_{c,q_c} \subset \mathbb{C}\mathbb{P}^2$ with

$$(A-9) \quad \tilde{\phi}(L_{a,q_a}) = s_a, \quad \tilde{\phi}(L_{b,q_b}) = s_b, \quad \tilde{\phi}(L_{c,q_c}) = s_c$$

such that $\tilde{\phi}$ restricts to a symplectomorphism

$$(A-10) \quad \phi: \mathbb{C}\mathbb{P}^2 \setminus (L_{a,q_a} \sqcup L_{b,q_b} \sqcup L_{c,q_c}) \rightarrow \mathbb{C}\mathbb{P}(a^2, b^2, c^2) \setminus \{s_a, s_b, s_c\}.$$

Furthermore, the preimage of the set $\mathfrak{D} \subset \mathbb{C}\mathbb{P}(a^2, b^2, c^2)$ — see (A-11) — consists of the union of three Lagrangian pinwheels and a symplectic two-torus intersecting the pinwheels in their respective core circles.

Remark A.8 For the construction of this symplectomorphism, it seems plausible that one can use the existence of a \mathbb{Q} -Gorenstein smoothing of $\mathbb{C}\mathbb{P}(a^2, b^2, c^2)$ to $\mathbb{C}\mathbb{P}^2$ with vanishing locus consisting of a union of three pinwheels. Such smoothings were constructed in [13], by showing that there are no obstructions to piecing together local smoothings of the cyclic quotient singularities holomorphically. Our proof follows the same strategy in the symplectic set-up; see also [7, Examples 2.5–2.6] for a discussion of this, and [17, Section 3; 6, Section 1] for the local smoothings.

We now turn to the proof of [Theorem A.1](#) using [Proposition A.7](#), the proof of which we postpone to [Section A.4](#).

Proof of Theorem A.1 Step 1 Let $\alpha < 3$ and choose α' so that $\alpha < \alpha' < 3$. Pick a Markov triple (a, b, c) and an associated Markov triangle $\Delta_{a,b,c} = \Delta_{a,b,c}(\alpha')$ which lies in $\mathcal{S} = \mathbb{R}_{\geq 0} \times [0, 1)$. This is possible by [Lemma A.5](#). Note that $\Delta_{a,b,c}$ is the image of the toric orbifold moment map $\mu: \mathbb{C}\mathbb{P}(a^2, b^2, c^2) \rightarrow \Delta_{a,b,c}$, provided we normalize the orbifold symplectic form appropriately. See the discussion surrounding [\(A-26\)](#) for details on the toric structure on weighted projective space. Let

$$(A-11) \quad \mathcal{D} = \mu^{-1}(\partial\Delta_{a,b,c}) \subset \mathbb{C}\mathbb{P}(a^2, b^2, c^2)$$

be the preimage of the boundary. The set \mathcal{D} is a union of complex suborbifolds

$$\mathbb{C}\mathbb{P}(a^2, b^2) \cup \mathbb{C}\mathbb{P}(b^2, c^2) \cup \mathbb{C}\mathbb{P}(a^2, c^2)$$

such that each of these suborbifolds projects to one edge of the triangle $\Delta_{a,b,c}$. The complement of \mathcal{D} admits a symplectic embedding

$$(A-12) \quad \psi: \mathbb{C}\mathbb{P}(a^2, b^2, c^2) \setminus \mathcal{D} \hookrightarrow Z^4(1).$$

Indeed, this follows from the inclusion $\text{int}(\Delta_{a,b,c}) \subset \text{int}(\mathcal{S})$ and the fact that inclusions of toric moment map images which respect the boundary stratifications yield (equivariant) symplectic embeddings; see for example [30].

Step 2 By [Proposition A.7](#), there is a symplectomorphism ϕ from the complement of Lagrangian pinwheels $L_{a,q_a}, L_{b,q_b}, L_{c,q_c}$ to the complement of the orbifold points of $\mathbb{C}\mathbb{P}(a^2, b^2, c^2)$. By restricting ϕ , we obtain the symplectomorphism

$$(A-13) \quad \phi': \mathbb{C}\mathbb{P}^2 \setminus \Sigma' \rightarrow \mathbb{C}\mathbb{P}(a^2, b^2, c^2) \setminus \mathcal{D}.$$

Again by [Proposition A.7](#), the set Σ' consists of the union of three pinwheels and a symplectic two-torus.

Step 3 The standard embedding $B^4(\alpha) \subset \mathbb{C}\mathbb{P}^2(\alpha')$ together with ϕ' and ψ yields an embedding $B^4(\alpha) \setminus \Sigma \hookrightarrow Z^4(1)$. Here Σ denotes $\Sigma = \Sigma' \cap B^4(\alpha)$. The set Σ has Minkowski dimension two. Indeed, the embedding $B^4(\alpha) \subset \mathbb{C}\mathbb{P}^2(\alpha')$ is bilipschitz (its image being contained in a closed ball) and volume-preserving and the set Σ' consists of the union of three pinwheels and a symplectic two-torus. \square

As was pointed out to us by Leonid Polterovich, one can combine [Theorem A.1](#) with Gromov's nonsqueezing theorem to get certain rigidity results, reminiscent of [\[2, Theorem 1.B\]](#).

Corollary A.9 *Let $\Sigma' \subset \mathbb{C}\mathbb{P}^2(\alpha)$ be one of the above sets such that $\mathbb{C}\mathbb{P}^2(\alpha) \setminus \Sigma'$ embeds into $Z^4(1)$ for some $1 < \alpha < 3$. Then every symplectic ball $B^4(1 + \varepsilon) \subset \mathbb{C}\mathbb{P}^2$ for $\varepsilon > 0$ intersects Σ' .*

Proof Assume $B^4(1 + \varepsilon) \subset \mathbb{C}\mathbb{P}^2(\alpha)$ does not intersect Σ' . The embedding $\mathbb{C}\mathbb{P}^2(\alpha) \setminus \Sigma' \hookrightarrow Z^4(1)$ yields a symplectic embedding $B^4(1 + \varepsilon) \hookrightarrow Z^4(1)$, contradicting nonsqueezing. \square

Note that for a fixed $1 < \alpha < 3$ we get infinitely many sets $\Sigma' \subset \mathbb{C}\mathbb{P}^2$ to which [Corollary A.9](#) applies and all of these consist of a union of a symplectic torus and Lagrangian pinwheels.

A.4 Proof of [Proposition A.7](#)

Following the exposition in [\[7, Example 2.5\]](#) we consider smoothings of certain orbifold quotients of \mathbb{C}^2 . This yields the local version from [Lemma A.10](#) of the symplectomorphism in [Proposition A.7](#).

Let a and q be coprime integers with $1 \leq q < a$ and take the quotient of \mathbb{C}^2 by the action of $(a^2)^{\text{th}}$ roots of unity

$$(A-14) \quad \zeta \cdot (z_1, z_2) = (\zeta z_1, \zeta^{aq-1} z_2), \quad \text{where } \zeta^{a^2} = 1.$$

We denote this quotient by $\mathbb{C}^2 / \Gamma_{a,q}$. It can be embedded as $\{w_1 w_2 = w_3^a\}$ into the quotient $\mathbb{C}^3 / \mathbb{Z}_a$ by the action

$$(A-15) \quad \eta \cdot (w_1, w_2, w_3) = (\eta w_1, \eta^{-1} w_2, \eta^q w_3), \quad \eta^a = 1.$$

The smoothing is given by

$$(A-16) \quad \mathcal{X} = \{w_1 w_2 = w_3^a + t\} \subset \mathbb{C}^3 / \mathbb{Z}_a \times \mathbb{C}_t,$$

which we view as a degeneration by projecting to the t -component, $\pi: \mathcal{X} \rightarrow \mathbb{C}_t$. We denote the fibers by $X_t = \pi^{-1}(t)$. The smooth fiber X_1 is a rational homology ball and the vanishing cycle of the degeneration is a Lagrangian pinwheel $L_{a,q}$. This follows from the description of \mathcal{X} as \mathbb{Z}_a -quotient of an A_{a-1} -Milnor fiber. Let $s \in X_0$ be the unique isolated singularity of X_0 , and $\mathcal{X}^{\text{reg}} = \mathcal{X} \setminus \{s\}$ its complement. The restriction of the standard symplectic form ω_0 on $\mathbb{C}^4 = \mathbb{C}^3 \times \mathbb{C}_t$ yields a symplectic manifold $(\mathcal{X}^{\text{reg}}, \Omega)$. Note that the smooth loci of the fibers $X_{t \neq 0}$ and $X_0 \setminus \{s\}$ are symplectic submanifolds. Let us now construct a symplectomorphism

$$(A-17) \quad \psi: X_1 \setminus L_{a,q} \rightarrow X_0 \setminus \{s\} = \mathbb{C}^2 / \Gamma_{a,q} \setminus \{0\}.$$

For this, we take the connection on \mathcal{X}^{reg} defined as the symplectic complement to the vertical distribution,

$$(A-18) \quad \xi_x = (\ker(\pi_*)_x)^\Omega = \{v \in T_x \mathcal{X}^{\text{reg}} \mid \Omega(v, w) = 0 \text{ for all } w \in T_x \pi^{-1}(\pi(x))\}.$$

This connection is symplectic in the sense that its parallel transport maps are symplectomorphisms whenever they are defined. In particular, we get a symplectomorphism between any two regular fibers X_{t_1} and X_{t_2} by picking a curve in \mathbb{C}^\times with endpoints t_1 and t_2 . Since we are interested in the singular fiber for the construction of (A-17), take the curve $\gamma(r) = 1 - r \in \mathbb{C}$. For every $r < 1$, this yields a symplectomorphism $\psi^r : X_1 \rightarrow X_{1-r}$. As it turns out, setting $\psi(x) = \lim_{r \rightarrow 1} \psi^r(x)$ for $x \in X_1$ yields a well-defined surjective map $\tilde{\psi} : X_1 \rightarrow X_0$ with vanishing cycle a Lagrangian pinwheel, $L_{a,q} = \tilde{\psi}^{-1}(s)$, and which restricts to the desired symplectomorphism (A-17). For the fact that $\tilde{\psi}$ is well-defined, see the unpublished notes by Evans [6, Lemma 1.2]. We refer to [17, Section 3.1] for more details on the specific degeneration we consider above and to [6, Lemma 1.20] for the fact that the vanishing cycle is a Lagrangian pinwheel.

Lemma A.10 (Evans [6]) *Let $\mathbb{C}^2 / \Gamma_{a,q}$ be the quotient by the action (A-14) and $S_{a,q}$ its smoothing as above. Then there is a surjective map $\tilde{\psi}_a : S_{a,q} \rightarrow \mathbb{C}^2 / \Gamma_{a,q}$ with $\tilde{\psi}_a(L_{a,q}) = \{0\}$ and which restricts to a symplectomorphism*

$$(A-19) \quad \psi_a : S_{a,q} \setminus L_{a,q} \rightarrow (\mathbb{C}^2 / \Gamma_{a,q}) \setminus \{0\}.$$

Furthermore, the preimage of the set $\mathfrak{D}_{a,q_a} = \{z_1 z_2 = 0\} / \Gamma_{a,q}$ under $\tilde{\psi}_a$ is the union of $L_{a,q}$ and a symplectic cylinder intersecting $L_{a,q}$ in its core circle.

Proof As explained above, the main statement of the lemma follows from [6]. We only need to identify the preimage of $\mathfrak{D}_{a,q_a} = \{z_1 z_2 = 0\} / \Gamma_{a,q}$, which can be done by keeping track of the parallel transport in the explicit model (A-16). Under the identification of $\mathbb{C}^2 / \Gamma_{a,q}$ with X_0 (which we will tacitly use throughout), the set $\{z_1 z_2 = 0\} / \Gamma_{a,q}$ corresponds to $\{w_3 = 0\} / \mathbb{Z}_a \cap X_0$. We claim that the set $\bigcup_t (\{w_3 = 0\} / \mathbb{Z}_a \cap X_t) \subset \mathcal{X}$ is invariant under the symplectic parallel transport induced by (A-18). This proves that $\tilde{\psi}_a^{-1}(\{w_3 = 0\} / \mathbb{Z}_a \cap X_0) = (\{w_3 = 0\} / \mathbb{Z}_a \cap X_1) \cup L_{a,q}$. Indeed, the preimage of $0 \in \mathbb{C}^2 / \Gamma_{a,q}$ is $L_{a,q}$ and on the complement of $L_{a,q}$, the map $\tilde{\psi}_a$ is a diffeomorphism. To see that $\{w_3 = 0\} / \mathbb{Z}_a \cap X_1$ is a symplectic cylinder intersecting $L_{a,q}$ in its core circle, one can consider the singular fibration structure of the map $X_s \rightarrow \mathbb{C}$ given by $(w_1, w_2, w_3) \mapsto w_3^p$; see [6, Section 1.2.3] for more details. Thus it remains to show the invariance of $\bigcup_t (\{w_3 = 0\} / \mathbb{Z}_a \cap X_t)$.

Let $x \in \{w_3 = 0\} / \mathbb{Z}_a \cap X_t$, meaning that x is a \mathbb{Z}_a -class of a point $(x_1, x_2, 0) \in \mathbb{C}^3$ with $x_1 x_2 = t$. This gives a natural inclusion $T_{(x,t)}\mathcal{X} = T_x X_t \oplus T_t \mathbb{C} \subset \mathbb{C}^3 \oplus \mathbb{C}$. Since $\Omega = \omega_{\mathbb{C}^3} \oplus \omega_{\mathbb{C}}$, the horizontal lift by the symplectic connection (A-18) of a vector $v \in T_t \mathbb{C} = \mathbb{C}$ is given by $u + v \in \xi_{(x,t)}$, where $u \in (T_x X_t)^{\omega_{\mathbb{C}^3}}$. Note that the subspace $\{u_1 = u_2 = 0\} \subset T_x \mathbb{C}^3 = \mathbb{C}^3$ is contained in $T_x X_t$ and hence its symplectic complement $\{u_3 = 0\} \subset \mathbb{C}^3$ contains the symplectic complement $(T_x X_t)^{\omega_{\mathbb{C}^3}}$. Since u is contained in $(T_x X_t)^{\omega_{\mathbb{C}^3}}$, this proves that u is tangent to the subset $\{w_3 = 0\}$ and hence this subset is preserved under parallel transport. □

We turn to the proof of Proposition A.7. The main idea is to use the fact that $\Delta_{a,b,c}(\alpha)$ is both the almost toric base polytope of $\mathbb{C}\mathbb{P}^2(\alpha)$ and the toric base of $\mathbb{C}\mathbb{P}(a^2, b^2, c^2)$ with a suitably normalized

symplectic form. This shows that there is a symplectomorphism which intertwines the (almost) toric structures on the preimages of a complement of neighborhoods of the vertices. For example, one can choose $W \subset \Delta_{a,b,c}$ as in Figure 2. We use Lemma A.10 to extend this symplectomorphism.

Let us make a few preparations. In particular, we discuss how to use the quotient $\mathbb{C}^2/\Gamma_{a,q}$ as a local toric model for $\mathbb{C}\mathbb{P}(a^2, b^2, c^2)$. This is the orbifold version of the toric ball embedding into $\mathbb{C}\mathbb{P}^2$ one obtains by the inclusion of the simplex with one edge removed into the standard simplex $\Delta_{1,1,1}$. The toric structure on $\mathbb{C}^2/\Gamma_{a,q}$ is induced by the standard toric structure $(z_1, z_2) \mapsto (\pi|z_1|^2, \pi|z_2|^2)$ on \mathbb{C}^2 . Indeed, note that the $\Gamma_{a,q}$ -action is obtained by restricting the standard $T^2 = (\mathbb{R}/\mathbb{Z})^2$ -action to a discrete subgroup \mathbb{Z}_{a^2} . This implies that we obtain an induced action by $T^2/\Gamma_{a,q}$ on $\mathbb{C}^2/\Gamma_{a,q}$. This action is Hamiltonian and its moment map image, under a suitable identification $T^2 \cong T^2/\Gamma_{a,q}$, is given by

$$(A-20) \quad \angle_{a,q} = \{x_1w_1 + x_2w_2 \mid x_1, x_2 \geq 0\}, \quad \text{where } w_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } w_2 = \begin{pmatrix} aq - 1 \\ a^2 \end{pmatrix}.$$

See for example [7, Remark 2.7] or [30, Section 9]. Note that a ball $B^4(d) = \{\pi(|z_1|^2 + |z_2|^2) < d\} \subset \mathbb{C}^2$ quotients to an orbifold ball $B^4(d)/\Gamma_{a,q} \subset \mathbb{C}^2/\Gamma_{a,q}$, which is fibered by the induced toric structure on the quotient. Furthermore, the boundary sphere $S^3(d) \subset \mathbb{C}^2$ quotients to a lens space $\Sigma_{a,q}(d) = S^3(d)/\Gamma_{a,q}$ of type $(aq - 1, a^2)$ equipped with its canonical contact structure and which fibers over a segment in $\angle_{a,q}$. We will use this fact in the proof of Proposition A.7.

Let us now show that, for a suitable choice of q , the toric system $\mathbb{C}^2/\Gamma_{a,q} \rightarrow \angle_{a,q}$ can be used as a local model around one of the orbifold points of the toric system on $\mathbb{C}\mathbb{P}(a^2, b^2, c^2)$. In order to get a concrete description of this toric system on the weighted projective space, recall that the symplectic orbifold $\mathbb{C}\mathbb{P}(a^2, b^2, c^2)$ can be defined as a symplectic quotient of \mathbb{C}^3 ,

$$(A-21) \quad \mathbb{C}\mathbb{P}(a^2, b^2, c^2) = H^{-1}(a^2b^2c^2)/S^1, \quad \text{where } H = a^2|z_1|^2 + b^2|z_2|^2 + c^2|z_3|^2.$$

This description (A-21) has the advantage that it is naturally equipped with a Hamiltonian T^2 -action inherited from the standard T^3 -action on \mathbb{C}^3 . This induced action is toric and its moment map image is given by the intersection of the plane defined by H and the positive orthant,

$$(A-22) \quad \tilde{\Delta}_{a,b,c} = \{a^2y_1 + b^2y_2 + c^2y_3 = a^2b^2c^2\} \cap \mathbb{R}_{\geq 0}^3.$$

Note that this is a polytope in \mathbb{R}^3 and not \mathbb{R}^2 . We get a Markov triangle $\Delta_{a,b,c}(abc) \subset \mathbb{R}^2$ as in Section A.2 by setting

$$(A-23) \quad \Delta_{a,b,c}(abc) = \Phi^{-1}(\tilde{\Delta}_{a,b,c})$$

for Φ an integral affine embedding (see Definition A.11) containing $\tilde{\Delta}_{a,b,c}$ in its image. Recall that this produces the same triangles (up to integral affine equivalence) as those obtained from almost toric fibrations of $\mathbb{C}\mathbb{P}^2$ as discussed in Section A.2; see [32, Section 2]. Hence it makes sense to denote them by $\Delta_{a,b,c}(abc)$. The normalization $\alpha = abc$ of the triangle comes from the choice of level at which we have reduced in (A-21).

Definition A.11 An affine map $\Phi: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ given by $x \mapsto Ax + b$ for $A \in \mathbb{Z}^{3 \times 2}$ and $b \in \mathbb{R}^3$ is called an *integral affine embedding* if it is injective and if $A(\mathbb{Z}^2) = A(\mathbb{R}^2) \cap \mathbb{Z}^3$.

By the definition of integral affine embedding, the definition (A-23) makes sense and the polytope it defines is independent of the choice of Φ up to applying an integral affine transformation. We now show that there is a natural number q and an integral affine embedding $\Phi_{a,q}$ such that the triangle $\Phi_{a,q}^{-1}(\tilde{\Delta}_{a,b,c}) \subset \mathbb{R}^2$ is obtained by intersecting $\angle_{a,q}$ with a half-plane. Indeed, set

$$(A-24) \quad \Phi_{a,q}: \mathbb{R}^2 \rightarrow \mathbb{R}^3, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} -b^2 & 1 + \frac{b}{a}(bq - 3c) \\ a^2 & 1 - aq \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b^2c^2 \\ 0 \\ 0 \end{pmatrix},$$

where q satisfies $bq = 3c \pmod{a}$; see also [7, Example 2.6]. The map $\Phi_{a,q}$ has image

$$\{a^2 y_1 + b^2 y_2 + c^2 y_3 = a^2 b^2 c^2\}$$

and it is an integral affine embedding, as can be checked by a computation. Furthermore Φ maps 0 to the vertex $(b^2c^2, 0, 0)$ (corresponding to a^2) of $\tilde{\Delta}_{a,b,c}$, and v_1 and v_2 to the outgoing edges at $(b^2c^2, 0, 0)$. This means that there is an integral vector (ξ_1, ξ_2) defining a half-plane $K = \{\xi_1 x_1 + \xi_2 x_2 \leq k\}$ such that

$$(A-25) \quad \Phi_{a,q}^{-1}(\tilde{\Delta}_{a,b,c}) = K \cap \angle_{a,q}.$$

From this we deduce the desired toric model. Let $\tilde{E}_a \subset \tilde{\Delta}_{a,b,c}$ be the edge opposite the vertex $(b^2c^2, 0, 0)$.

Proposition A.12 *The subset in $\mathbb{C}\mathbb{P}(a^2, b^2, c^2)$ fibering over $\tilde{\Delta}_{a,b,c} \setminus \tilde{E}_a$ is fibered (orbifold) symplectomorphic to the subset in $\mathbb{C}^2 / \Gamma_{a,q}$ fibering over $\text{Int } K \cap \angle_{a,q}$.*

Proof We have shown above that $\tilde{\Delta}_{a,b,c} \setminus \tilde{E}_a$ and $\text{Int } K \cap \angle_{a,q}$ are integral affine equivalent. This implies the claim by the classification of compact toric orbifolds by their moment map images; see [19]. Compactness is not a problem here, since we can compactify the subset fibering over $K \cap \angle_{a,q}$ by performing a symplectic cut at $\{\xi_1 x_1 + \xi_2 x_2 = k\}$. □

Let us now fix a moment map

$$(A-26) \quad \mu: \mathbb{C}\mathbb{P}(a^2, b^2, c^2) \rightarrow \Delta_{a,b,c} = \Delta_{a,b,c}(abc) \subset \mathbb{R}^2$$

by composing the moment map $\mathbb{C}\mathbb{P}(a^2, b^2, c^2) \rightarrow \tilde{\Delta}_{a,b,c}$ with the inverse of a suitable integral affine embedding as described above. Until the end of the proof of Proposition A.7, we simplify notation by writing $\Delta_{a,b,c} = \Delta_{a,b,c}(abc)$.

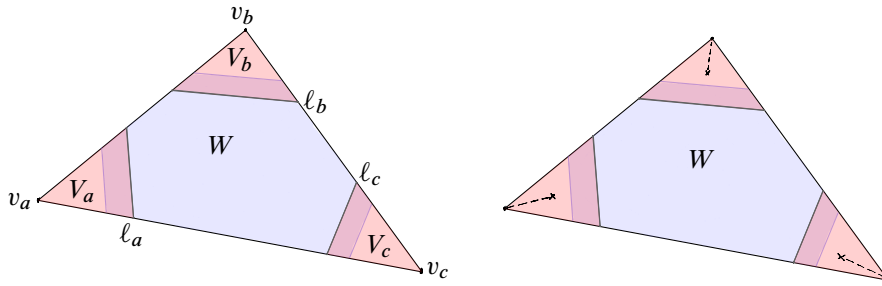


Figure 2: The triangle $\Delta_{a,b,c}$ as union $W \cup V_a \cup V_b \cup V_c$, on the left as the toric moment polytope of $\mathbb{C}\mathbb{P}(a^2, b^2, c^2)$ and on the right as almost toric base diagram of $\mathbb{C}\mathbb{P}^2$. In both cases the fibration is toric over W and lens spaces fiber over the segments ℓ_a, ℓ_b, ℓ_c .

Proof of Proposition A.7 The main part of the proof will be concerned with proving the existence of the symplectomorphism (A-10) and for readability, we postpone the proof of the existence of the global map $\tilde{\phi}$ and the computation of $\tilde{\phi}^{-1}(\mathcal{D})$ to Step 5.

Step 1 We start by setting up some notation on the side of the weighted projective space. The orbifold points $s_a, s_b, s_c \in \mathbb{C}\mathbb{P}(a^2, b^2, c^2)$ are mapped to the vertices $v_a, v_b, v_c \in \Delta_{a,b,c}$ under the moment map $\mu: \mathbb{C}\mathbb{P}(a^2, b^2, c^2) \mapsto \Delta_{a,b,c}$. Let us first focus on the orbifold point s_a . Denote the edge opposite to v_a by E_a . By Proposition A.12, there is an orbifold symplectomorphism

$$(A-27) \quad \rho_a: \mu^{-1}(\Delta_{a,b,c} \setminus E_a) \rightarrow \mu_{\mathbb{C}^2/\Gamma_{a,q}}^{-1}(\text{Int } K \cap \angle_{a,q}),$$

which intertwines the toric structures. Now let $\overline{B}^4(d)/\Gamma_{a,q} \subset \mathbb{C}^2/\Gamma_{a,q}$ be a closed orbifold ball for $\overline{B}^4(d) = \{\pi(|z_1|^2 + |z_2|^2) \leq d\}$ and $d > 0$. Its boundary $S^3(d)/\Gamma_{a,q} \subset \mathbb{C}^2/\Gamma_{a,q}$ is a lens space equipped with the standard contact structure. Note that both the orbifold ball and its boundary are fibered by the moment map $\mu_{\mathbb{C}^2/\Gamma_{a,q}}$. Since ρ_a intertwines the toric structures, the image sets

$$(A-28) \quad B_a^{\text{orb}} = \rho_a^{-1}(\overline{B}^4(d)/\Gamma_{a,q}) \quad \text{and} \quad \Sigma_a = \rho_a^{-1}(S^3(d)/\Gamma_{a,q})$$

are fibered by μ . Then the image of the pair (B_a, Σ_a) under μ is a pair (V_a, ℓ_a) consisting of a segment contained in a triangle around the vertex v_a . Note also that the lens space Σ_a is naturally equipped with its standard contact structure. We do the same procedure around the remaining vertices v_b and v_c , and denote the corresponding sets by $B_b^{\text{orb}}, B_c^{\text{orb}}, \Sigma_b, \Sigma_c \subset \mathbb{C}\mathbb{P}(a^2, b^2, c^2)$ and by $V_b, V_c, \ell_b, \ell_c \subset \Delta_{a,b,c}$. We choose the sizes so that $B_a^{\text{orb}}, B_b^{\text{orb}}$ and B_c^{orb} are mutually disjoint. Furthermore, we choose a set $W \subset \Delta_{a,b,c}$ such that $\Delta_{a,b,c} = W \cup V_a \cup V_b \cup V_c$ and such that the overlap $W \cap V_j$ is a strip around ℓ_j for all $j \in \{a, b, c\}$. Again, see Figure 2.

Step 2 Now consider the almost toric fibration of $\mathbb{C}\mathbb{P}^2$ associated to the triangle $\Delta_{a,b,c}$. In the conventions of [30; 32] the triangle $\Delta_{a,b,c}$ is decorated with three dashed line segments of prescribed slope between the vertices and the nodal points. The latter are usually marked by a cross. There is a map $\pi: \mathbb{C}\mathbb{P}^2 \rightarrow \Delta_{a,b,c}$ which is a standard toric fibration away from the dashed lines, but which is

only continuous on the preimages of the dashed lines (which encode monodromy of the integral affine structure). By applying *nodal slides* if necessary, we may assume that the dashed lines lie outside of the subset $W \subset \Delta_{a,b,c}$. Since the projection π is standard toric away from the dashed lines, this implies that there is a symplectomorphism

$$(A-29) \quad \phi_0 : \mathbb{C}\mathbb{P}^2 \supset \pi^{-1}(W) \rightarrow \mu^{-1}(W) \subset \mathbb{C}\mathbb{P}(a^2, b^2, c^2),$$

which intertwines π and μ . Define the preimages

$$(A-30) \quad B'_a = \pi^{-1}(V_a) \quad \text{and} \quad \Sigma'_a = \pi^{-1}(\ell_a).$$

By [30, Section 9], the set B'_a is a closed rational homology ball and Σ'_a is a lens space of type $(aq - 1, a^2)$ equipped with its standard contact structure. In fact, ϕ_0 maps the copy Σ'_a of the lens space to the copy Σ_a . However, contrary to B_a^{orb} , the rational homology ball B'_a is smooth. The same discussion holds for b and c .

Step 3 The key part of the proof is finding extensions

$$(A-31) \quad \phi_j : B'_j \setminus L_{j,q_j} \rightarrow B_j^{\text{orb}} \setminus \{s_j\}$$

of the map $\phi_0|_{\Sigma'_j}$, where L_{j,q_j} are Lagrangian pinwheels for $j \in \{a, b, c\}$. For this we use Lemma A.10. Again, restricting our attention to a , let ψ_a be the symplectomorphism from Lemma A.10. Note that we have already established the correspondence between B_a^{orb} and $\mathbb{C}^2/\Gamma_{a,q}$ by ρ_a and that this correspondence is compatible with the toric picture. We now establish a correspondence between the rational homology sphere B'_a and the space $S_{a,q}$ coming from the smoothing in Lemma A.10. Define yet another copy Σ''_a of the lens space by setting $\Sigma''_a = \psi_a^{-1}(\rho_a(\Sigma_{a,q}))$. This lens space is also equipped with the standard contact structure and it bounds a rational homology ball B''_a by [7, Example 2.5]. We now have two pairs (B'_a, Σ'_a) and (B''_a, Σ''_a) consisting of a rational homology ball bounded by a lens space carrying its standard contact structure. By [9, Proposition A.2], which relies on [20], this implies that (B'_a, Σ'_a) and (B''_a, Σ''_a) are equivalent up to symplectic deformation. Let $\chi_a : (B'_a, \Sigma'_a) \rightarrow (B''_a, \Sigma''_a)$ be the diffeomorphism we obtain from this. Note that we cannot directly use the symplectic deformation to conclude, since the symplectomorphism obtained from a Moser-type argument may not restrict to the desired map on the boundary. More precisely, we obtain a diagram of diffeomorphisms of lens spaces

$$(A-32) \quad \begin{array}{ccc} B'_a \supset \Sigma'_a & \xrightarrow{\phi_0|_{\Sigma'_a}} & \Sigma_a \subset B_a^{\text{orb}} \\ \downarrow \chi_a|_{\Sigma'_a} & & \downarrow \rho_a|_{\Sigma_a} \\ B''_a \supset \Sigma''_a & \xrightarrow{\psi_a|_{\Sigma''_a}} & S^3(d)/\Gamma_{a,q} \subset B^4(d)/\Gamma_{a,q} \end{array}$$

and this diagram *does not commute*. We may, however, correct the diffeomorphism χ_a so that (A-32) commutes. Recall that $\Sigma_{a,q}$ is a lens space of type $(aq - 1, a^2)$. Since $(aq - 1)^2 \not\equiv \pm 1 \pmod{a^2}$, it follows from [3, Théorème 3(a)] that the space of diffeomorphisms of Σ''_a has two components, namely the one

of the identity and the one of the involution τ induced by the involution $(z_1, z_2) \mapsto (\bar{z}_1, \bar{z}_2)$ of S^3 . The diffeomorphism τ extends to a diffeomorphism $\tilde{\tau}$ of $B''_{p,q}$. Up to postcomposing χ_a with $\tilde{\tau}$, we may thus assume that $(\psi_a^{-1} \rho_a \phi_0 \chi_a^{-1})|_{\Sigma'_a}$ is isotopic to the identity by an isotopy φ_t . Using this isotopy, we can correct the diffeomorphism χ_a such that the diagram (A-32) commutes. Indeed, the set $\pi^{-1}(V_a \cap W)$ is a collar neighborhood $\Sigma'_a \times [0, 2)$ and thus we can use the collar coordinate together with the isotopy φ_t to define a corrected diffeomorphism $\tilde{\chi}_a$, which coincides with the original diffeomorphism χ_a on $\mu^{-1}(V_a \setminus W)$ and with $(\psi_a^{-1} \rho_a \phi_0)|_{\Sigma'_a}$ on Σ'_a . Recall that two collar neighborhoods which agree on the boundary coincide up to applying a smooth isotopy; see Munkres [24, Lemma 6.1]. This means that, after applying an isotopy in (B''_a, Σ'_a) , we can assume that the corrected version of χ_a and $\psi_a^{-1} \rho_a \phi_0$ agree on a smaller collar $\Sigma_a \times [0, 1)$. Denoting the diffeomorphism we obtain in this way by $\tilde{\chi}_a$, this allows us to define a diffeomorphism

$$(A-33) \quad \phi_a = \rho_a^{-1} \circ \psi_a \circ \tilde{\chi}_a|_{B'_a \setminus L_{a,q_a}} : B'_a \setminus L_{a,q_a} \rightarrow B_a^{\text{orb}} \setminus \{s_a\},$$

which extends ϕ_0 in the sense that it agrees with ϕ_0 on a collar of Σ'_a . Since ψ_a is defined outside of a Lagrangian pinwheel $L_{a,q_a} \subset B''_a$, the diffeomorphism ϕ_a is defined outside of a pinwheel (which we again denote by L_{a,q_a}) in B'_a . We repeat this procedure for b and c to obtain diffeomorphisms ϕ_b and ϕ_c .

Step 4 By construction, the diffeomorphisms ϕ_a, ϕ_b and ϕ_c extend the initial symplectomorphism $\phi_0|_{\pi^{-1}(W \setminus (V_a \cup V_b \cup V_c))}$ and hence we obtain a diffeomorphism

$$(A-34) \quad \hat{\phi} : \mathbb{C}\mathbb{P}^2 \setminus (L_{a,q_a} \sqcup L_{b,q_b} \sqcup L_{c,q_c}) \rightarrow \mathbb{C}\mathbb{P}(a^2, b^2, c^2) \setminus \{s_a, s_b, s_c\}.$$

We now turn to the symplectic forms. On $\mathbb{C}\mathbb{P}^2$, we define a symplectic form $\hat{\omega}$ which turns $\hat{\phi}$ into a symplectomorphism as follows. On $\pi^{-1}(W \setminus (V_a \cup V_b \cup V_c))$ we define $\hat{\omega}$ to be the usual Fubini–Study form ω . On V_j we define $\hat{\omega}$ as the pullback form $\tilde{\chi}_j^* \omega_{B''_j}$, where $\tilde{\chi}_j$ is the corrected diffeomorphism constructed at the end of Step 3. This yields a well-defined symplectic form which turns $\hat{\phi}$ into a symplectomorphism. Indeed, this follows from the fact that the maps ϕ_0, ψ_j and ρ_j are symplectomorphisms and ϕ_j is defined as their composition (A-33). This also implies that the symplectic form $\hat{\omega}$ has the same total volume as the Fubini–Study form. By the Gromov–Taubes theorem [22, Remark 9.4.3(ii)], the form $\tilde{\omega}$ is symplectomorphic to the Fubini–Study form and hence postcomposing $\hat{\phi}$ from (A-34) with this symplectomorphism yields the desired symplectomorphism (A-10).

Step 5 The definition of the global map $\tilde{\phi} : \mathbb{C}\mathbb{P}^2 \rightarrow \mathbb{C}\mathbb{P}(a^2, b^2, c^2)$ is obtained by replacing ϕ_a from (A-33) by $\tilde{\phi}_a = \rho_a^{-1} \circ \tilde{\psi}_a \circ \tilde{\chi}_a$ and carrying out the rest of the construction as above. The map $\tilde{\psi}_a$ is given by Lemma A.10. Let us now identify $\tilde{\phi}^{-1}(\mathcal{D})$, where $\mathcal{D} = \mu^{-1}(\partial\Delta_{a,b,c})$. Let $\tilde{W} \subset W$ be the subset of W where $\tilde{\phi}$ coincides with ϕ_0 . Since ϕ_0 intertwines the toric structures on $\pi^{-1}(W) \subset \mathbb{C}\mathbb{P}^2$ and $\mu^{-1}(W) \subset \mathbb{C}\mathbb{P}(a^2, b^2, c^2)$, the set $\tilde{\phi}^{-1}(\mathcal{D}) \cap \pi^{-1}(\tilde{W})$ fibers over the three pieces of the boundary given by $\tilde{W} \cap \partial\Delta_{a,b,c}$ and hence consists of three disjoint symplectic cylinders.

We use Lemma A.10 to prove that the missing pieces $\tilde{\phi}^{-1}(\mathcal{D}) \cap B'_j$ for $j \in \{a, b, c\}$ are also given by symplectic cylinders and that the union of the six cylinders is given by a torus. We again discuss the case

of $j = a$ since the other two are completely analogous. Recall that $\rho_a: B_a^{\text{orb}} \rightarrow B^4(d)/\Gamma_{a,q}$ is compatible with the toric structure and thus $\rho_a(\mathcal{D} \cap B_a^{\text{orb}}) = \mathcal{D}_{a,q_a} \cap \rho_a(B_a^{\text{orb}})$, where $\mathcal{D}_{a,q_a} = \{z_1 z_2 = 0\}/\Gamma_{a,q_a}$ as in Lemma A.10. Indeed, this follows from the fact that \mathcal{D}_{a,q_a} fibers over the boundary of the moment map image \angle_{a,q_a} . By Lemma A.10, we deduce that $\tilde{\chi}_a^{-1}(\tilde{\phi}_a^{-1}(\rho_a(\mathcal{D} \cap B_a^{\text{orb}})))$ is given by the union of a pinwheel with a piece of a cylinder. Furthermore, recall that near the lens spaces at the respective boundaries, the map $\tilde{\rho}_a^{-1}\tilde{\psi}_a\chi_a$ coincides with ϕ_0 . This implies that the cylinder contained in B'_a has two boundary components at $\partial B'_a = \Sigma'_a$, which are smoothly identified in a collar neighborhood with boundary components of the set $\tilde{\phi}^{-1}(\mathcal{D}) \cap \pi^{-1}(\tilde{W})$ discussed in the previous paragraph. This proves the claim. \square

Remark A.13 We suspect that there are shorter and more natural proofs of Proposition A.7. In particular, one should be able to avoid Gromov–Taubes. One possibility we have hinted at above is working with a global degeneration and trying to analyze its vanishing cycle. This would completely avoid the use almost toric fibrations. Another possibility, in the spirit of [27; 14], is to equip the explicit local degeneration from [17] with a family of integrable systems avoiding the pinwheel and extending the given toric structure on the boundary. The symplectomorphism from Proposition A.7 then follows from the usual toric arguments and it is automatically equivariant. Although this construction is elementary, it is somewhat outside the scope of this appendix and we hope to carry out the details elsewhere. This is also reminiscent of [10, Section 7], and it is plausible one can apply results from this paper to prove the same result.

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