Geometry & Topology

Volume 28 (2024)

Categorical wall-crossing formula for Donaldson–Thomas theory on the resolved conifold

Yukinobu Toda

msp

Geometry & Topology **28**:3 (2024) 1341–1407
DOI: 10.2140/gt.2024.28.1341
Published: 10 May 2024

Categorical wall-crossing formula for Donaldson–Thomas theory on the resolved conifold

YUKINOBU TODA

We prove a wall-crossing formula for categorical Donaldson–Thomas invariants on the resolved conifold, which categorifies the Nagao–Nakajima wall-crossing formula for numerical DT invariants on it. The categorified Hall products are used to describe the wall-crossing formula as semiorthogonal decompositions. A successive application of the categorical wall-crossing formula yields semiorthogonal decompositions of categorical Pandharipande–Thomas stable pair invariants on the resolved conifold, which categorify the product expansion formula of the generating series of numerical PT invariants on it.

14N35; 18G80

1 Introduction

1.1 Background and summary of the paper

In this paper, we establish a wall-crossing formula for categorical Donaldson–Thomas invariants on the resolved conifold, and apply it to give a complete description of categorical Pandharipande–Thomas (PT) stable pair invariants on it.

The PT invariants count stable pairs on CY 3–folds, and were introduced by Pandharipande and Thomas [2009] in order to give a better formulation of the GW/DT correspondence conjecture of Maulik, Nekrasov, Okounkov and Pandharipande [Maulik et al. 2006]. They are special cases of Donaldson–Thomas (DT) type invariants counting stable objects in the derived category, and are now understood as fundamental enumerative invariants of curves on CY 3–folds as well as Gromov–Witten invariants and Gopakumar–Vafa invariants. Now by efforts from derived algebraic geometry due to Pantev, Toën, Vaquié and Vezzosi [Pantev et al. 2013] and Brav, Bussi and Joyce [Brav et al. 2019], the moduli spaces which define DT (in particular PT) invariants are known to be locally written as critical loci. In [Toda 2019], we proposed a study of categorical DT theory by gluing locally defined dg-categories of matrix factorizations on these moduli spaces. A definition of categorical DT invariants is introduced in the case of local surfaces in [Toda 2019] via Koszul duality and singular support quotients. We also proposed several conjectures on wall-crossing of categorical DT invariants on local surfaces, motivated by a d–critical analogue of the D/K equivalence conjecture of Bondal and Orlov [1995] and Kawamata [2002], and also categorifications of wall-crossing formulas of numerical DT invariants [Joyce and Song 2012; Kontsevich and Soibelman 2008]. In [Toda 2019], we also derived a wall-crossing formula of categorical PT invariants on local surfaces in the setting of simple wall-crossing (ie there are at most two Jordan–Hölder factors at the wall).

The purpose of this paper is to prove wall-crossing formula for categorical DT invariants on the resolved conifold, which categorifies the wall-crossing formula of Nagao and Nakajima [2011] for numerical DT invariants on it. In this case the relevant moduli spaces are global critical loci, so there is no issue with gluing dg-categories of matrix factorizations. However, the wall-crossing is not necessarily a simple wall-crossing, and the analysis of categorical wall-crossings is much harder. Our strategy is to use categorified Hall products for quivers with superpotentials introduced by Pădurariu [2019; 2023]. A key observation is that, up to Knörrer periodicity, a wall-crossing diagram for the resolved conifold locally looks like a Grassmannian flip together with some superpotential (d-critical Grassmannian flip in the sense of d-critical birational geometry [Toda 2022]). We refine the result of Ballard, Chidambaram, Favero, McFaddin and Vandermolen [Ballard et al. 2021] on derived categories of Grassmannian flips via categorified Hall products, and compare them with more global categorified Hall products under the Knörrer periodicity. The above approach via categorified Hall products yields a desired categorical wall-crossing formula. A successive iteration of wall-crossing gives a semiorthogonal decomposition of categorical PT invariants on the resolved conifold, whose semiorthogonal summands are the simplest categories of matrix factorizations over a point. We emphasize that the result of this paper is a first instance where a categorical wall-crossing formula is obtained for nonsimple wall-crossing in the context of categorical DT theory.

1.2 Categorical PT stable pair theory on the resolved conifold

The *resolved conifold* X is defined by

$$X := \mathrm{Tot}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}),$$

which is also obtained as a crepant small resolution of the conifold singularity $\{xy + zw = 0\} \subset \mathbb{C}^4$. The resolved conifold is a noncompact CY 3–fold, and an important toy model for enumerative geometry on CY 3–folds such as PT invariants.

For each $(\beta, n) \in \mathbb{Z}^2$, we denote by

$$P_n(X, \beta)$$

the moduli space of PT stable pairs (F, s) on X, ie F is a pure one-dimensional coherent sheaf on X and $s\colon \mathcal{O}_X \to F$ is surjective in dimension one, satisfying $[F] = \beta[C]$ and $\chi(F) = n$. Here $C \subset X$ is the zero section of the projection $X \to \mathbb{P}^1$, and $[F]$ is the fundamental one-cycle of F. The PT invariant $P_{n,\beta} \in \mathbb{Z}$ is defined by either taking the integration over the zero-dimensional virtual fundamental class on $P_n(X,\beta)$, or weighted Euler characteristic of the Behrend constructible function [Behrend 2009] on it. It is well-known that the generating series of PT invariants on X is given by the formula

$$\sum_{n,\beta} P_{n,\beta} q^n t^\beta = \prod_{m \geq 1} (1-(-q)^m t)^m. \tag{1-1}$$

The above formula is available in [Nagao and Nakajima 2011, Theorem 3.15], which is also obtained from the DT calculation in [Behrend and Bryan 2007] together with the DT/PT correspondence [Bridgeland 2011; Toda 2010; Stoppa and Thomas 2011].

The purpose of this paper is to give a categorification of the formula (1-1). In the case of the resolved conifold, the moduli space $P_n(X,\beta)$ is written as a global critical locus, ie there is a pair (M,w) where M is a smooth quasiprojective scheme and $w\colon M \to \mathbb{A}^1$ is a regular function such that $P_n(X,\beta)$ is isomorphic to the critical locus of w. A choice of (M,w) is not unique, and we take it using the noncommutative crepant resolution of X due to Van den Bergh [2004]; see Section 5.10. We define the *categorical PT invariant* on X to be the triangulated category of matrix factorizations of w,

$$\mathcal{DT}(P_n(X,\beta)) := \mathrm{MF}(M,w).$$

See Definition 5.23. The above triangulated category (or more precisely its dg-enhancement) recovers $P_{n,\beta}$ by taking the Euler characteristic of its periodic cyclic homology; see equation (5-59). The following is a consequence of the main result in this paper:

Theorem 1.1 (Corollary 5.24) *There exists a semiorthogonal decomposition of the form*

$$\mathcal{DT}(P_n(X,\beta)) = \langle a_{n,\beta} \text{ copies of } \mathrm{MF}(\operatorname{Spec}\mathbb{C},0)\rangle. \tag{1-2}$$

Here $a_{n,\beta}$ is defined by

$$a_{n,\beta} := \sum_{\substack{l\colon \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 0} \\ \sum_{m\geq 1} l(m)\cdot(m,1)=(n,\beta)}} \prod_{m\geq 1} \binom{m}{l(m)}. \tag{1-3}$$

Here $\mathrm{MF}(\operatorname{Spec}\mathbb{C},0)$ is the category of matrix factorizations of the zero superpotential over the point, which is equivalent to the $\mathbb{Z}/2$–periodic derived category of finite-dimensional $\mathbb{C}$–vector spaces. As the formula (1-1) is equivalent to $P_{n,\beta} = (-1)^{n+\beta} a_{n,\beta}$, by taking the periodic cyclic homologies of both sides and Euler characteristics, the result of Theorem 1.1 recovers the formula (1-1); see Remark 5.25.

1.3 Categorical wall-crossing formula

Nagao and Nakajima [2011, Theorem 3.15] derived the formula (1-1) by proving wall-crossing formula for stable perverse coherent systems on X. Under a derived equivalence of X with a noncommutative crepant

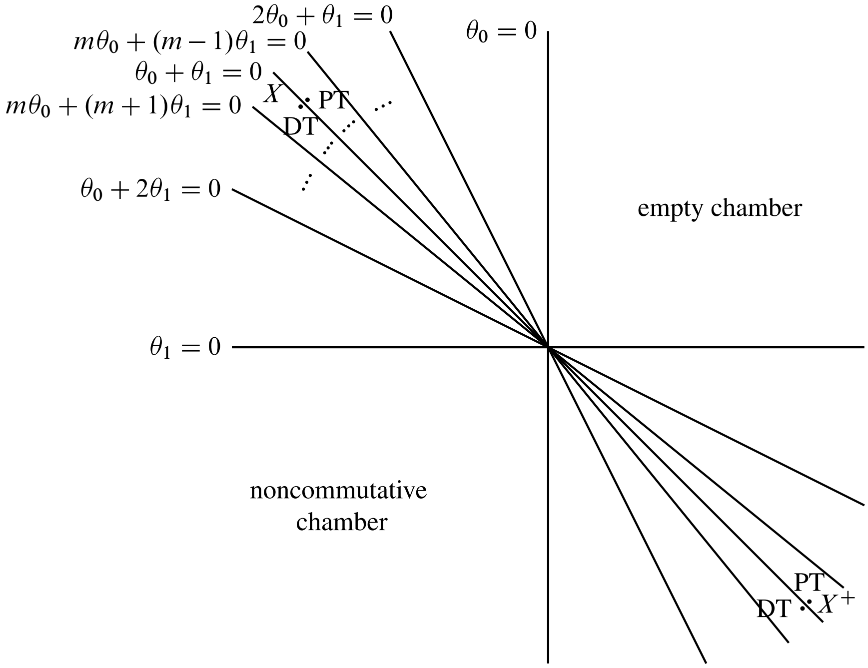


Figure 1: Wall-chamber structures.

resolution of the conifold due to Van den Bergh [2004], the category of perverse coherent systems on X is equivalent to the category of representations of the quiver with superpotential $(Q^\dagger, W)$, where

$$Q^\dagger = \begin{tikzcd} \bullet_\infty \arrow[d] & \\ \bullet_0 \arrow[r, bend left, "a_1"] \arrow[r, bend left=15, "a_2"] & \bullet_1 \arrow[l, bend left=15, "b_1"] \arrow[l, bend left, "b_2"] \end{tikzcd} \qquad \text{and} \qquad W = a_1b_1a_2b_2 - a_1b_2a_2b_1.$$

For $v=(v_0,v_1)\in\mathbb{Z}^2_{\geq 0}$, we denote by $\mathcal{M}^\dagger_Q(v)$ the $\mathbb{C}^*$–rigidified moduli stack of $Q^\dagger$–representations with dimension vector $(1,v_0,v_1)$, where 1 is the dimension vector at ∞. It is equipped with a superpotential

$$w=\mathrm{Tr}(W)\colon \mathcal{M}^\dagger_Q(v)\to\mathbb{A}^1,$$

whose critical locus is isomorphic to the moduli stack of $(Q^\dagger, W)$–representations with dimension vector $(1,v_0,v_1)$. There is also a stability parameter $\theta=(\theta_0,\theta_1)\in\mathbb{R}^2$ of $(Q^\dagger,W)$–representations, whose wall-chamber structure is pictured in Figure 1, taken from [Nagao and Nakajima 2011, Figure 1].

For $m\in\mathbb{Z}_{\geq 1}$, there is a wall in the second quadrant in Figure 1,

$$W_m := \mathbb{R}_{>0}(1-m,m)\subset\mathbb{R}^2.$$

We take a stability condition on the wall $\theta\in W_m$ and $\theta_\pm=\theta\pm(-\varepsilon,\varepsilon)$ for $\varepsilon>0$ which lie on its adjacent chambers. Let $\mathrm{DT}^{\theta_\pm}(v_0,v_1)\in\mathbb{Z}$ be the DT invariant counting $\theta_\pm$–stable $(Q^\dagger,W)$–representations with dimension vector $(1, v_0, v_1)$. We have the wall-crossing formula

$$\sum_{(v_0,v_1)\in\mathbb{Z}^2_{\geq 0}} \mathrm{DT}^{\theta_+}(v_0, v_1) q_0^{v_0} q_1^{v_1} = \Bigg(\sum_{(v_0,v_1)\in\mathbb{Z}^2_{\geq 0}} \mathrm{DT}^{\theta_-}(v_0, v_1) q_0^{v_0} q_1^{v_1} \Bigg) \cdot (1 + q_0^m(-q_1)^{m-1})^m \tag{1-4}$$

proved by Nagao and Nakajima [2011, Theorem 3.12]. The formula (1-1) is obtained from the above wall-crossing formula by applying it from $m = 1$ to $m \gg 0$ and noting that the PT invariants correspond to a chamber which is sufficiently close to (and above) the wall $\mathbb{R}_{>0}(-1, 1)$.

We prove Theorem 1.1 by giving a categorification of the formula (1-4). For $\theta \in \mathbb{R}^2$, we denote by

$$\mathcal{M}_Q^{\dagger,\theta\text{-ss}}(v) \subset \mathcal{M}_Q^{\dagger}(v)$$

the open substack of θ–semistable $Q^\dagger$–representations. The following is the main result of this paper, which gives a categorification of the formula (1-4):

Theorem 1.2 (Corollary 5.18) *For $\theta \in W_m$, by setting $s_m = (m, m-1)$, there exists a semiorthogonal decomposition*

$$\mathrm{MF}(\mathcal{M}_Q^{\dagger,\theta_+\text{-ss}}(v), w) = \Big\langle \binom{m}{l} \text{ copies of } \mathrm{MF}(\mathcal{M}_Q^{\dagger,\theta_-\text{-ss}}(v - l s_m), w) : l \geq 0 \Big\rangle. \tag{1-5}$$

There is also a precisely defined order among semiorthogonal summands in (1-2); see Corollary 5.18 for the precise statement. Again by taking the periodic cyclic homologies and the Euler characteristics, the result of Theorem 1.2 recovers the Nagao–Nakajima formula (1-4); see Remark 5.19. The result of Theorem 1.1 follows by applying Theorem 1.2 from $m = 1$ to $m \gg 0$.

We also remark that the similar categorical wall-crossing formula holds at other walls except walls at $\{\theta_0 + \theta_1 = 0\}$, ie DT/PT wall on X or on its flop; see Remark 5.21. Note that the numerical DT/PT wall-crossing formula was not directly obtained in [Nagao and Nakajima 2011], but was proved in [Bridgeland 2011; Toda 2010; Stoppa and Thomas 2011] using the full machinery of motivic Hall algebras in [Joyce and Song 2012; Kontsevich and Soibelman 2008].

1.4 Outline of the proof of Theorem 1.2

The strategy of the proof of Theorem 1.2 is to use the following ingredients:

(i) The window subcategories for GIT quotient stacks developed by Halpern-Leistner [2015] and Ballard, Favero and Katzarkov [Ballard et al. 2019].

(ii) The categorified Hall products for quivers with superpotentials introduced and studied by Pădurariu [2023; 2024; 2019].

(iii) The descriptions of derived categories under Grassmannian flips by Ballard, Chidambaram, Favero, McFaddin and Vandermolen [Ballard et al. 2021], which itself relies on earlier work by Donovan and Segal [2014] for Grassmannian flops.

For $\theta \in W_m$, let $\mathcal{M}_Q^{\dagger,\theta\text{-ss}}(v) \to M_Q^{\dagger,\theta\text{-ss}}(v)$ be the good moduli space [Alper 2013]. We have the wall-crossing diagram

$$\begin{tikzcd} M_Q^{\dagger,\theta_+\text{-ss}}(v) \arrow[rr, dashed] \arrow[dr] & & M_Q^{\dagger,\theta_-\text{-ss}}(v) \arrow[dl] \\ & M_Q^{\dagger,\theta\text{-ss}}(v) & \end{tikzcd} \tag{1-6}$$

which is shown to be a flip of smooth quasiprojective varieties. The D/K principle by Bondal and Orlov [1995] and Kawamata [2002] predicts the existence of a fully faithful functor of their derived categories or categories of matrix factorizations.

The window subcategories have been used to investigate the D/K conjecture under variations of GIT quotients. In the above setting, there exist subcategories $\mathbb{W}_{\mathrm{glob}}^{\theta_\pm}(v) \subset \mathrm{MF}(\mathcal{M}_Q^{\dagger,\theta\text{-ss}}(v), w)$, called *window subcategories*, such that the compositions

$$\mathbb{W}_{\mathrm{glob}}^{\theta_\pm}(v) \hookrightarrow \mathrm{MF}(\mathcal{M}_Q^{\dagger,\theta\text{-ss}}(v), w) \twoheadrightarrow \mathrm{MF}(\mathcal{M}_Q^{\dagger,\theta_\pm\text{-ss}}(v), w)$$

are equivalences. If we can show that $\mathbb{W}_{\mathrm{glob}}^{\theta_-}(v) \subset \mathbb{W}_{\mathrm{glob}}^{\theta_+}(v)$ for some choice of window subcategories, then we have a desired fully faithful functor

$$\mathrm{MF}(\mathcal{M}_Q^{\dagger,\theta_-\text{-ss}}(v), w) \hookrightarrow \mathrm{MF}(\mathcal{M}_Q^{\dagger,\theta_+\text{-ss}}(v), w). \tag{1-7}$$

In fact, the above argument is used in [Toda 2019, Theorem 4.3.5] to show the existence of a fully faithful functor (1-7).

We are interested in the semiorthogonal complement of the fully faithful functor (1-7). If the wall-crossing is enough simple, eg satisfying the DHT condition in [Ballard et al. 2019, Definition 4.1.4], then the above window subcategory argument also describes the semiorthogonal complement; see [Ballard et al. 2019, Theorem 4.2.1]. However our wall-crossing (1-6) does not necessary satisfy the DHT condition, and we cannot directly apply it. Instead we use categorified Hall products to describe the semiorthogonal complement of (1-7).

The categorified Hall product for quivers with superpotentials was introduced by Pădurariu [2023; 2024; 2019] in order to give a K–theoretic version of critical COHA, which was introduced in [Kontsevich and Soibelman 2011] and developed in [Davison 2017]. For $v = v_1 + v_2$ with $\theta(v_1) = 0$, it is a functor

$$*\colon \mathrm{MF}(\mathcal{M}_Q^{\theta\text{-ss}}(v_1), w) \boxtimes \mathrm{MF}(\mathcal{M}_Q^{\dagger,\theta\text{-ss}}(v_2), w) \to \mathrm{MF}(\mathcal{M}_Q^{\dagger,\theta\text{-ss}}(v), w)$$

which is defined by the pullback/pushforward with respect to the stack of short exact sequences of $Q^\dagger$–representations. We will show that, for $l \geq 0$ and a sequence of integers $0 \leq j_1 \leq \cdots \leq j_l \leq m - l$, the categorified Hall product gives a fully faithful functor

$$\boxtimes_{i=1}^{l} \mathrm{MF}(\mathcal{M}_Q^{\theta\text{-ss}}(s_m), w)_{j_i+(2i-1)(m^2-m)} \boxtimes (\mathbb{W}_{\mathrm{glob}}^{\theta_-}(v - l s_m) \otimes \chi_0^{j_l+2l(m^2-m)}) \to \mathbb{W}_{\mathrm{glob}}^{\theta_+}(v), \tag{1-8}$$

whose essential images form a semiorthogonal decomposition. Here the subscript $j_i + (2i-1)(m^2-m)$ indicates the fixed $\mathbb{C}^*$–weight part, and χ_0 is some character regarded as a line bundle on $\mathcal{M}_Q^{\dagger}(v)$; see Theorem 5.17 for details. It follows that the categorified Hall products describe the semiorthogonal complement of (1-7), which lead to a proof of Theorem 1.2.

In order to show that the functor (1-8) is fully faithful and they form a semiorthogonal decomposition, we prove these statements formally locally on the good moduli space $M_Q^{\dagger,\theta\text{-ss}}(v)$ at any point p corresponding to a θ–polystable $(Q^{\dagger}, W)$–representation R. By the étale slice theorem, one can describe the formal fibers of the diagram (1-6) at p in terms of a wall-crossing diagram of the Ext quiver $Q_p^{\dagger}$ associated with R, which is much simpler than $Q^{\dagger}$. After removing a quadratic part of the superpotential, one observes that the wall-crossing diagram for $Q_p^{\dagger}$–representations is the product of a Grassmannian flip with some trivial part. Here a *Grassmannian flip* is a birational map

$$G_{a,b}^{+}(d) \dashrightarrow G_{a,b}^{-}(d)$$

given by two GIT stable loci of the quotient stack

$$\mathcal{G}_{a,b}(d) = [(\mathrm{Hom}(A,V) \oplus \mathrm{Hom}(V,B))/\mathrm{GL}(V)],$$

where $d = \dim V$, $a = \dim A$ and $b = \dim B$ with $a \geq b$.

Donovan and Segal [2014] proved a derived equivalence $D^b(G_{a,b}^{-}(d)) \simeq D^b(G_{a,b}^{+}(d))$ in the case of $a = b$ (ie Grassmannian flop) using window subcategories, and the same argument also applies to construct a fully faithful functor $D^b(G_{a,b}^{-}(d)) \hookrightarrow D^b(G_{a,b}^{+}(d))$. However it is in a rather recent work of Ballard, Chidambaram, Favero, McFaddin and Vandermolen [Ballard et al. 2021] where the semiorthogonal complement of the above fully faithful functor is considered. We will interpret the description of semiorthogonal complement in [Ballard et al. 2021] in terms of categorified Hall products, and refine it as a semiorthogonal decomposition

$$D^b(G_{a,b}^{+}(d)) = \left\langle \binom{a-b}{l} \text{ copies of } D^b(G_{a,b}^{-}(d-l)) : 0 \leq l \leq d \right\rangle. \tag{1-9}$$

See Corollary 4.19. The above semiorthogonal decomposition unifies Kapranov's exceptional collections of derived categories of Grassmannians, and also semiorthogonal decompositions of standard toric flips, so it may be of independent interest; see Remarks 4.20 and 4.21.

A semiorthogonal decomposition similar to (1-9) also holds for categories of factorizations of a superpotential of $\mathcal{G}_{a,b}(d)$. Under the Knörrer periodicity, we compare global categorified Hall products (1-8) with local categorified Hall products giving the semiorthogonal decomposition (1-9). By combining these arguments, we see that the functor (1-8) is fully faithful and they form a semiorthogonal decomposition formally locally on $M_Q^{\dagger,\theta\text{-ss}}(v)$, hence they also hold globally.

1.5 Related works

The wall-crossing formula (1-4) was proved by Nagao and Nakajima [2011] in order to give an understanding of the product expansion formula of noncommutative DT invariants of the conifold studied by Szendrői [2008]. The wall-crossing formula (1-4) was later extended to the case of a global flopping contraction by Toda [2013] and Calabrese [2016], to the motivic DT invariants by Morrison, Mozgovoy, Nagao and Szendrői [Morrison et al. 2012], and to the DT4 invariants by Cao and Toda [2023]. Recently Tasuki Kinjo [$\geq$ 2024] has studied cohomological DT theory on the resolved conifold and proves a cohomological version of DT/PT correspondence in this case. It would be interesting to extend the argument in this paper and categorify his cohomological DT/PT correspondence.

As we already mentioned, the study of wall-crossing of categorical PT invariants was posed in [Toda 2019]. In the case of local surfaces, a categorical wall-crossing formula is conjectured in [Toda 2019, Conjecture 6.2.6] in the case of simple wall-crossing, and proved in some cases in [Toda 2019, Theorem 6.3.19] using Porta–Sala categorified Hall products for surfaces [Porta and Sala 2023]. The wall-crossing we consider in this paper is not necessary simple, so it is beyond the cases we considered in [Toda 2019, Conjecture 6.2.6]. A similar wall-crossing at $(-1,-1)$–curve is also considered in [Toda 2021, Section 7], but we only proved the existence of fully faithful functors and their semiorthogonal complements are not considered.

The categorified (K–theoretic) Hall algebras for quivers with superpotentials were introduced and studied by Tudor Pădurariu [2019; 2023]. He also proved the PBW theorem for K–theoretic Hall algebras [Pădurariu 2019;2024] via much more sophisticated combinatorial arguments (based on earlier works of Špenko and Van den Bergh [2017] and Halpern-Leistner and Sam [2020]). We expect that his arguments proving the K–theoretic PBW theorem can be applied to prove categorical (or K–theoretic) wall-crossing formula in a broader setting, including DT/PT wall-crossing in this paper.

Recently Qingyuan Jiang [2021] has studied derived categories of Quot schemes of locally free quotients, and proposed conjectural semiorthogonal decompositions of them; see [Jiang 2021, Conjecture A.5]. He proved the above conjecture in the case of rank-two quotients. His conjectural semiorthogonal decompositions resemble the one in Theorem 1.2. It would be interesting to see whether the technique in this paper can be applied to his conjecture.

Koseki Naoki [2021] has studied derived categories of moduli spaces of stable perverse coherent sheaves for a blow-up of a surface (studied by Nakajima and Yoshioka [2011]), and proved the existence of fully faithful functors under wall-crossing. As the situation is similar to the one in this paper, a similar argument to that in this paper may be applied to describe the semiorthogonal complements of his fully faithful functors.

1.6 Notation and conventions

In this paper, all the schemes or stacks are defined over $\mathbb{C}$. For an Artin stack $\mathcal{Y}$, we denote by $D^b(\mathcal{Y})$ the bounded derived category of coherent sheaves on $\mathcal{Y}$. For an algebraic group G and its representation V, we regard it as a vector bundle on BG. For a variety Y on which G acts, we denote by $V \otimes \mathcal{O}_{[Y/G]}$ the vector bundle given by the pullback of V by $[Y/G] \to BG$. For a morphism $\mathfrak{M} \to M$ from a stack $\mathfrak{M}$ to a scheme M and a closed point $y \in M$, the *formal fiber* at y is defined by

$$\widehat{\mathfrak{M}}_y := \mathfrak{M} \times_M \operatorname{Spec} \widehat{\mathcal{O}}_{M,y} \to \widehat{M}_y := \operatorname{Spec} \widehat{\mathcal{O}}_{M,y}.$$

For a triangulated category $\mathcal{D}$, its triangulated subcategory $\mathcal{D}' \subset \mathcal{D}$ is called *dense* if any object in $\mathcal{D}$ is a direct summand of an object in $\mathcal{D}'$.

Acknowledgements The author is grateful to Yalong Cao, Qingyuan Jiang, Naoki Koseki, Tudor Pădurariu and Gufang Zhao for comments on the first version of this paper. The author also thanks the referee for careful reading of the paper and several valuable comments. The author is supported by World Premier International Research Center Initiative (WPI initiative), MEXT, Japan, and Grant in Aid for Scientific Research grant 19H01779 from MEXT, Japan.

2 Preliminaries

In this section, we review triangulated categories of factorizations, the window theorem for categories of factorizations over GIT quotient stacks, and the Knörrer periodicity.

2.1 The category of factorizations

Let $\mathcal{Y}$ be a noetherian algebraic stack over $\mathbb{C}$ and take $w \in \Gamma(\mathcal{O}_{\mathcal{Y}})$. A (coherent) factorization of w consists of

$$\mathcal{P}_0 \underset{\alpha_1}{\overset{\alpha_0}{\rightleftarrows}} \mathcal{P}_1, \quad \text{where } \alpha_0 \circ \alpha_1 = \cdot w \text{ and } \alpha_1 \circ \alpha_0 = \cdot w,$$

where each $\mathcal{P}_i$ is a coherent sheaf on $\mathcal{Y}$, and the α_i are morphisms of coherent sheaves. The category of coherent factorizations naturally forms a dg-category, whose homotopy category is denoted by $\operatorname{HMF}(\mathcal{Y}, w)$. The subcategory of absolutely acyclic objects

$$\operatorname{Acy}^{\mathrm{abs}} \subset \operatorname{HMF}(\mathcal{Y}, w)$$

is defined to be the minimum thick triangulated subcategory which contains totalizations of short exact sequences of coherent factorizations of w. The triangulated category of factorizations of w is defined by

$$\operatorname{MF}(\mathcal{Y}, w) := \operatorname{HMF}(\mathcal{Y}, w)/\operatorname{Acy}^{\mathrm{abs}}.$$

(See [Orlov 2012; Efimov and Positselski 2015; Polishchuk and Vaintrob 2011].)

If $\mathcal{Y}$ is an affine scheme, then $\operatorname{MF}(\mathcal{Y}, w)$ is equivalent to Orlov's triangulated category [2009] of matrix factorizations of w. For two pairs $(\mathcal{Y}_i, w_i)$ for $i = 1, 2$, we use the notation

$$\operatorname{MF}(\mathcal{Y}_1, w_1) \boxtimes \operatorname{MF}(\mathcal{Y}_2, w_2) := \operatorname{MF}(\mathcal{Y}_1 \times \mathcal{Y}_2, w_1 + w_2).$$

For triangulated subcategories $\mathcal{C}_i \subset \operatorname{MF}(\mathcal{Y}_i, w_i)$, we denote by $\mathcal{C}_1 \boxtimes \mathcal{C}_2$ the smallest thick triangulated subcategory of $\operatorname{MF}(\mathcal{Y}_1, w_1) \boxtimes \operatorname{MF}(\mathcal{Y}_2, w_2)$ which contains $C_1 \boxtimes C_2$ for $C_i \in \mathcal{C}_i$.

It is well known that $\mathrm{MF}(\mathcal{Y}, w)$ only depends on an open neighborhood of $\mathrm{Crit}(w) \subset \mathcal{Y}$. Namely let $\mathcal{Y}' \subset \mathcal{Y}$ be an open substack such that $\mathrm{Crit}(w) \subset \mathcal{Y}'$. Then the restriction functor gives an equivalence

$$\mathrm{MF}(\mathcal{Y}, w) \xrightarrow{\sim} \mathrm{MF}(\mathcal{Y}', w|_{\mathcal{Y}'}). \tag{2-1}$$

(See [Polishchuk and Vaintrob 2011, Corollary 5.3] and [Halpern-Leistner and Sam 2020, Lemma 5.5].) Suppose that $\mathcal{Y} = [Y/G]$, where G is an algebraic group which acts on a scheme Y. Assume that $\mathbb{C}^* \subset G$ lies in the center of G, which acts on Y trivially. Then $\mathrm{MF}(\mathcal{Y}, w)$ decomposes into the direct sum

$$\mathrm{MF}(\mathcal{Y}, w) = \bigoplus_{j \in \mathbb{Z}} \mathrm{MF}(\mathcal{Y}, w)_j, \tag{2-2}$$

where each summand corresponds to the $\mathbb{C}^*$–weight j part.

2.2 Attracting loci

Let G be a reductive algebraic group with maximal torus T, which acts on a smooth affine scheme Y. We denote by M the character lattice of T and N the cocharacter lattice of T. There is a perfect pairing

$$\langle -, - \rangle \colon N \times M \to \mathbb{Z}.$$

For a one-parameter subgroup $\lambda \colon \mathbb{C}^* \to G$, let $Y^{\lambda \geq 0}$ and $Y^{\lambda = 0}$ be defined by

$$Y^{\lambda \geq 0} := \bigl\{ y \in Y : \lim_{t \to 0} \lambda(t)(y) \text{ exists} \bigr\},$$
$$Y^{\lambda = 0} := \{ y \in Y : \lambda(t)(y) = y \text{ for all } t \in \mathbb{C}^* \}.$$

The Levi subgroup and the parabolic subgroup

$$G^{\lambda = 0} \subset G^{\lambda \geq 0} \subset G$$

are also similarly defined by the conjugate G–action on G, ie $g \cdot (-) = g(-)g^{-1}$. The G–action on Y restricts to the $G^{\lambda \geq 0}$–action on $Y^{\lambda \geq 0}$, and the $G^{\lambda = 0}$–action on $Y^{\lambda = 0}$. We note that λ factors through $\lambda \colon \mathbb{C}^* \to G^{\lambda = 0}$, and it acts on $Y^{\lambda = 0}$ trivially. So we have the decomposition into λ–weight spaces

$$D^b([Y^{\lambda = 0}/G^{\lambda = 0}]) = \bigoplus_{j \in \mathbb{Z}} D^b([Y^{\lambda = 0}/G^{\lambda = 0}])_{\lambda\text{-wt} = j}.$$

We have the diagram of attracting loci

$$\begin{tikzcd}
{[Y^{\lambda \geq 0}/G^{\lambda \geq 0}]} \arrow[r, "p_\lambda"] \arrow[d, "q_\lambda"] & {[Y/G]} \\
{[Y^{\lambda = 0}/G^{\lambda = 0}]} \arrow[u, bend left, "\sigma_\lambda"]
\end{tikzcd} \tag{2-3}$$

Here p_λ is induced by the inclusion $Y^{\lambda \geq 0} \subset Y$, and q_λ is given by taking the $t \to 0$ limit of the action of $\lambda(t)$ for $t \in \mathbb{C}^*$. The morphism σ_λ is a section of q_λ induced by inclusions $Y^{\lambda = 0} \subset Y^{\lambda \geq 0}$ and $G^{\lambda = 0} \subset G^{\lambda \geq 0}$. We will use the following lemma.

Lemma 2.1 [Halpern-Leistner 2015, Corollary 3.17, Amplification 3.18]

(i) *For* $\mathscr{E}_i \in D^b([Y^{\lambda\geq 0}/G^{\lambda\geq 0}])$ *with* $i=1,2$, *suppose that*

$$\sigma_\lambda^*\mathscr{E}_1 \in D^b([Y^{\lambda=0}/G^{\lambda=0}])_{\lambda\text{-wt}\geq j} \quad \text{and} \quad \sigma_\lambda^*\mathscr{E}_2 \in D^b([Y^{\lambda=0}/G^{\lambda=0}])_{\lambda\text{-wt}<j}$$

for some j. *Then* $\mathrm{Hom}(\mathscr{E}_1,\mathscr{E}_2)=0$.

(ii) *For* $j\in\mathbb{Z}$, *the functor*

$$q_\lambda^*\colon D^b([Y^{\lambda=0}/G^{\lambda=0}])_{\lambda\text{-wt}=j} \to D^b([Y^{\lambda\geq 0}/G^{\lambda\geq 0}])$$

is fully faithful.

2.3 Kempf–Ness stratification

Here we review Kempf–Ness stratifications associated with GIT quotients of reductive algebraic groups, and the corresponding window theorem following the convention of [Halpern-Leistner 2015, Section 2.1]. Let Y and G be as in the previous subsection. For an element $l\in\mathrm{Pic}([Y/G])_{\mathbb{R}}$, we have the open subset of l–semistable points

$$Y^{l\text{-ss}} \subset Y$$

characterized by the set of points $y\in Y$ such that for any one-parameter subgroup $\lambda\colon\mathbb{C}^*\to G$ such that the limit $z=\lim_{t\to 0}\lambda(t)(y)$ exists in Y, we have $\mathrm{wt}(l|_z)\geq 0$. Let $|*|$ be the Weyl-invariant norm on $N_{\mathbb{R}}$. The above subset of l–semistable points fits into the *Kempf–Ness* (*KN*) *stratification*

$$Y = S_1 \sqcup S_2 \sqcup \cdots \sqcup S_N \sqcup Y^{l\text{-ss}}. \tag{2-4}$$

Here for each $1\leq i\leq N$ there exists a one-parameter subgroup $\lambda_i\colon\mathbb{C}^*\to T\subset G$, an open and closed subset Z_i of $\bigl(Y\setminus\bigcup_{i'<i}S_{i'}\bigr)^{\lambda_i=0}$ (called the *center* of S_i) such that

$$S_i = G\cdot Y_i \quad \text{and} \quad Y_i := \bigl\{y\in Y^{\lambda_i\geq 0} : \lim_{t\to 0}\lambda_i(t)(y)\in Z_i\bigr\}.$$

Moreover, by setting the slope to be

$$\mu_i := -\frac{\mathrm{wt}(l|_{Z_i})}{|\lambda_i|}\in\mathbb{R},$$

we have the inequalities $\mu_1>\mu_2>\cdots>0$. We have (see [Halpern-Leistner 2015, Definition 2.2]) the diagram

$$\begin{tikzcd}
{[Y_i/G^{\lambda_i\geq 0}]} \arrow[r, "\cong"] \arrow[d] & {[S_i/G]} \arrow[r, hook, "q_i"] \arrow[dl, "p_i"'] & {\bigl[\bigl(Y\setminus\bigcup_{i'<i}S_{i'}\bigr)/G\bigr]} \\
{[Z_i/G^{\lambda_i=0}]} \arrow[urr, hook, "\tau_i"'] & &
\end{tikzcd} \tag{2-5}$$

Here the left vertical arrow is given by taking the $t\to 0$ limit of the action of $\lambda_i(t)$ for $t\in\mathbb{C}^*$, and τ_i and q_i are induced by the embeddings $Z_i\hookrightarrow Y$ and $S_i\hookrightarrow Y$, respectively.

Let $\eta_i \in \mathbb{Z}$ be defined by

$$\eta_i := \mathrm{wt}_{\lambda_i}(\det(N^{\vee}_{S_i/Y}|_{Z_i})). \tag{2-6}$$

In the case that Y is a G–representation, it is also written as

$$\eta_i = \langle \lambda_i, (Y^{\vee})^{\lambda_i>0} - (\mathfrak{g}^{\vee})^{\lambda_i>0} \rangle.$$

Here for a G–representation W and a one-parameter subgroup $\lambda\colon \mathbb{C}^* \to T$, we denote by $W^{\lambda>0} \in K(BT)$ the subspace of W spanned by weights which pair positively with λ. We will use the following version of the window theorem.

Theorem 2.2 [Halpern-Leistner 2015; Ballard et al. 2019] *For each i, take $m_i \in \mathbb{R}$. For $N' \le N$, let*

$$\mathbb{W}^l_{m_\bullet}\Big(\Big[\Big(Y \setminus \bigcup_{1\le i\le N'} S_i\Big)/G\Big]\Big) \subset D^b\Big(\Big[\Big(Y \setminus \bigcup_{1\le i\le N'} S_i\Big)/G\Big]\Big) \tag{2-7}$$

be the subcategory of objects $\mathcal{P}$ satisfying the condition

$$\tau_i^*(\mathcal{P}) \in \bigoplus_{j\in[m_i, m_i+\eta_i)} D^b([Z_i/G^{\lambda_i=0}])_{\lambda_i\text{-wt}=j} \tag{2-8}$$

for all $N' < i \le N$. Then the composition functor

$$\mathbb{W}^l_{m_\bullet}\Big(\Big[\Big(Y \setminus \bigcup_{1\le i\le N'} S_i\Big)/G\Big]\Big) \hookrightarrow D^b\Big(\Big[\Big(Y \setminus \bigcup_{1\le i\le N'} S_i\Big)/G\Big]\Big) \twoheadrightarrow D^b([Y^{l\text{-ss}}/G])$$

is an equivalence.

Let $w\colon Y \to \mathbb{A}^1$ be a G–invariant function. We will apply Theorem 2.2 for a KN stratification of $\mathrm{Crit}(w)$

$$\mathrm{Crit}(w) = S'_1 \sqcup S'_2 \sqcup \cdots \sqcup S'_N \sqcup \mathrm{Crit}(w)^{l\text{-ss}}$$

in the following way. After discarding KN strata $S_i \subset Y$ with $\mathrm{Crit}(w) \cap S_i = \varnothing$, the above stratification is obtained by restricting a KN stratification (2-4) for Y to $\mathrm{Crit}(w)$. Let $\lambda_i\colon \mathbb{C}^* \to G$ be a one-parameter subgroup for S'_i with center $Z'_i \subset S'_i$. We define $\overline{Z}_i \subset Y$ to be the union of connected components of the λ_i–fixed part of Y which contains Z'_i, and $\overline{Y}_i \subset Y$ is the set of points $y \in Y$ with $\lim_{t\to 0}\lambda_i(t)y \in \overline{Z}_i$. Similarly to (2-5), we have the diagram

$$\begin{tikzcd}
{[\overline{Y}_i/G^{\lambda_i\ge 0}]} \arrow[r, hook] \arrow[d, "\bar{p}_i"'] \arrow[rr, bend left, "\bar{q}_i"] & {[Y^{\lambda_i\ge 0}/G^{\lambda_i\ge 0}]} \arrow[r] \arrow[d] & {[Y/G]} \\
{[\overline{Z}_i/G^{\lambda_i=0}]} \arrow[r, hook] & {[Y^{\lambda_i=0}/G^{\lambda_i=0}]} \arrow[ur] &
\end{tikzcd}$$

Here the left horizontal arrows are open and closed immersions. Using the equivalence (2-1), we also have the following version of window theorem for factorization categories; see [Toda 2021, Section 2.4].

Theorem 2.3 *For each i, we take $m_i \in \mathbb{R}$. For $N' \le N$, let*

$$\mathbb{W}^l_{m_\bullet}\Big(\Big[\Big(Y \setminus \bigcup_{1\le i\le N'} S'_i\Big)/G\Big], w\Big) \subset \mathrm{MF}\Big(\Big[\Big(Y \setminus \bigcup_{1\le i\le N'} S'_i\Big)/G\Big], w\Big)$$

be the subcategory consisting of factorizations $(\mathcal{P}, d_{\mathcal{P}})$ such that

$$(\mathcal{P}, d_{\mathcal{P}})|_{[(\overline{Z}_i \setminus \bigcup_{i'<i} S'_{i'})/G^{\lambda_i=0}]} \in \bigoplus_{j\in[m_i, m_i+\overline{\eta}_i)} \mathrm{MF}\Big(\Big[\Big(\overline{Z}_i \setminus \bigcup_{i'<i} S'_{i'}\Big)/G^{\lambda_i=0}\Big], w|_{\overline{Z}_i}\Big)_{\lambda_i\text{-wt}=j} \tag{2-9}$$

for all $N' < i \le N$. Here $\overline{\eta}_i = \mathrm{wt}_{\lambda_i} \det(\mathbb{L}_{\overline{q}_i})^\vee|_{\overline{Z}_i}$. Then the composition functor

$$\mathbb{W}^l_{m_\bullet}\Big(\Big[\Big(Y \setminus \bigcup_{1\le i\le N'} S'_i\Big)/G\Big], w\Big) \hookrightarrow \mathrm{MF}\Big(\Big[\Big(Y \setminus \bigcup_{1\le i\le N'} S'_i\Big)/G\Big], w\Big) \twoheadrightarrow \mathrm{MF}([Y^{l\text{-ss}}/G], w)$$

is an equivalence.

2.4 Knörrer periodicity

Let Y be a smooth affine scheme and G be an affine algebraic group which acts on Y. Let W be a G–representation, which determines a vector bundle $\mathcal{W} \to \mathcal{Y} := [Y/G]$. Given a function $w\colon \mathcal{Y} \to \mathbb{A}^1$, we have another function on the total space of $\mathcal{W} \oplus \mathcal{W}^\vee$,

$$w + q\colon \mathcal{W} \oplus \mathcal{W}^\vee \to \mathbb{A}^1, \quad \text{where } q(x, x') = \langle x, x'\rangle.$$

We have the diagram

$$\begin{tikzcd} \mathcal{W}^\vee \arrow[r, hook, "i"] \arrow[d, "\mathrm{pr}"'] \arrow[dr, "w"] & \mathcal{W} \oplus \mathcal{W}^\vee \arrow[d, "w+q"] \\ \mathcal{Y} \arrow[r, "w"'] & \mathbb{A}^1 \end{tikzcd}$$

Here $i(x) = (0, x)$. The following is a version of Knörrer periodicity; cf [Hirano 2017a, Theorem 4.2].

Theorem 2.4 *The composition functor*

$$\Phi := i_* \,\mathrm{pr}^*\colon \mathrm{MF}(\mathcal{Y}, w) \xrightarrow{\mathrm{pr}^*} \mathrm{MF}(\mathcal{W}^\vee, w) \xrightarrow{i_*} \mathrm{MF}(\mathcal{W} \oplus \mathcal{W}^\vee, w + q) \tag{2-10}$$

is an equivalence.

Remark 2.5 Here in applying [Hirano 2017a, Theorem 4.2], we view $\mathcal{Y}$ as a closed substack $\mathcal{Y} \hookrightarrow \mathcal{W}$ cut out by the tautological section of the vector bundle $\mathcal{W} \oplus \mathcal{W} \to \mathcal{W}$ given by the second projection, and view $\mathcal{W} \oplus \mathcal{W}^\vee$ as the dual vector bundle of $\mathcal{W} \oplus \mathcal{W} \to \mathcal{W}$.

The equivalence (2-10) is given by taking the tensor product over $\mathcal{O}_{\mathcal{Y}}$ with the following factorization of q on $\mathcal{W} \oplus \mathcal{W}^\vee$

$$i_*\mathcal{O}_{\mathcal{W}^\vee} \rightleftarrows 0.$$

The above factorization is isomorphic to the Koszul factorization of q on $\mathcal{W} \oplus \mathcal{W}^\vee$, which is of the form

$$\Big(\bigwedge\nolimits^{\mathrm{even}} \mathcal{W}^\vee\Big) \otimes_{\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{W}\oplus\mathcal{W}^\vee} \rightleftarrows \Big(\bigwedge\nolimits^{\mathrm{odd}} \mathcal{W}^\vee\Big) \otimes_{\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{W}\oplus\mathcal{W}^\vee}. \tag{2-11}$$

(See [Ballard et al. 2014, Proposition 3.20].) Here each differential is given by $\lrcorner s + \wedge t$, where

$$s\colon \mathcal{W}^\vee \otimes \mathcal{O}_{\mathcal{W}\oplus\mathcal{W}^\vee} \to \mathcal{O}_{\mathcal{W}\oplus\mathcal{W}^\vee}$$

corresponds to the tautological section of $\mathcal{W} \oplus \mathcal{W} \to \mathcal{W}$ pulled back to $\mathcal{W} \oplus \mathcal{W}^\vee$, and

$$t \in \mathcal{W}^\vee \otimes_{\mathcal{O}_\mathcal{Y}} \mathcal{W} \subset \mathcal{W}^\vee \otimes_{\mathcal{O}_\mathcal{Y}} \mathcal{O}_{\mathcal{W}\oplus\mathcal{W}^\vee}$$

corresponds to $\mathrm{id} \in \mathrm{Hom}_\mathcal{Y}(\mathcal{W}, \mathcal{W})$.

Let $\lambda\colon \mathbb{C}^* \to G$ be a one-parameter subgroup. We have the diagrams

$$\begin{tikzcd} \mathcal{Y}^{\lambda\geq 0} \arrow[r, "p_\lambda"] \arrow[d, "q_\lambda"'] \arrow[dr, "w^{\lambda\geq 0}"] & \mathcal{Y}, \arrow[d, "w"] \\ \mathcal{Y}^{\lambda=0} \arrow[r, "w^{\lambda=0}"'] & \mathbb{A}^1 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} (\mathcal{W}\oplus\mathcal{W}^\vee)^{\lambda\geq 0} \arrow[r, "p'_\lambda"] \arrow[d, "q'_\lambda"'] \arrow[dr, "(w+q)^{\lambda\geq 0}"] & \mathcal{W}\oplus\mathcal{W}^\vee \arrow[d, "w+q"] \\ (\mathcal{W}\oplus\mathcal{W}^\vee)^{\lambda=0} \arrow[r, "w^{\lambda=0}+q^{\lambda=0}"'] & \mathbb{A}^1 \end{tikzcd}$$

of attracting loci. Note that we have equivalences

$$\Phi^{\lambda=0}\colon \mathrm{MF}(\mathcal{Y}^{\lambda=0}, w^{\lambda=0}) \xrightarrow{\sim} \mathrm{MF}((\mathcal{W}\oplus\mathcal{W}^\vee)^{\lambda=0}, w^{\lambda=0}+q^{\lambda=0}),$$
$$\Phi^{\lambda\geq 0}\colon \mathrm{MF}(\mathcal{Y}^{\lambda\geq 0}, w^{\lambda\geq 0}) \xrightarrow{\sim} \mathrm{MF}((\mathcal{W}^{\lambda\geq 0}\oplus(\mathcal{W}^{\lambda\geq 0})^\vee, w^{\lambda\geq 0}+q^{\lambda\geq 0}),$$

by applying Theorem 2.4 for $\mathcal{W}^{\lambda=0} \to \mathcal{Y}^{\lambda=0}$ and $\mathcal{W}^{\lambda\geq 0} \to \mathcal{Y}^{\lambda\geq 0}$, respectively.

Proposition 2.6 *The following diagram commutes*:

$$\begin{tikzcd} \mathrm{MF}(\mathcal{Y}^{\lambda=0}, w^{\lambda=0}) \arrow[r, "p_{\lambda*}q_\lambda^*"] \arrow[d, "\Phi^{\lambda=0}\circ\otimes(\det \mathcal{W}^{\lambda>0})^\vee[\dim W^{\lambda>0}]"'] & \mathrm{MF}(\mathcal{Y}, w) \arrow[d, "\Phi"] \\ \mathrm{MF}((\mathcal{W}\oplus\mathcal{W}^\vee)^{\lambda=0}, w^{\lambda=0}+q^{\lambda=0}) \arrow[r, "p'_{\lambda*}q'^*_\lambda"] & \mathrm{MF}(\mathcal{W}\oplus\mathcal{W}^\vee, w+q) \end{tikzcd}$$

Proof We have the diagram

$$\begin{tikzcd} \mathcal{W}^{\lambda=0} \arrow[d] & \mathcal{W}^{\lambda\geq 0} \arrow[l] \arrow[r] \arrow[d] & \mathcal{W} \arrow[d] \\ \mathcal{Y}^{\lambda=0} & \mathcal{Y}^{\lambda\geq 0} \arrow[l, "q_\lambda"'] \arrow[r, "p_\lambda"] & \mathcal{Y} \end{tikzcd} \tag{2-12}$$

Here all horizontal diagrams are diagrams of attracting loci, and vertical arrows are projections. From the above diagram, we construct the diagram

$$\begin{tikzcd} (\mathcal{W}\oplus\mathcal{W}^\vee)^{\lambda\geq 0} \arrow[r, "r_1"'] \arrow[d, "r_2"'] \arrow[dd, bend right, "q'_\lambda"'] \arrow[rr, bend left, "p'_\lambda"] \arrow[dr, phantom, "\square"] & \mathcal{W}^{\lambda\geq 0}\oplus p_\lambda^*\mathcal{W}^\vee \arrow[r, "f_2"'] \arrow[d, "g_2"] & \mathcal{W}\oplus\mathcal{W}^\vee \arrow[dd, "w+q"] \\ \mathcal{W}^{\lambda\geq 0}\oplus q_\lambda^*(\mathcal{W}^{\lambda=0})^\vee \arrow[r, "g_1"'] \arrow[d, "f_1"'] & \mathcal{W}^{\lambda\geq 0}\oplus(\mathcal{W}^{\lambda\geq 0})^\vee \arrow[dr, "w^{\lambda\geq 0}+q^{\lambda\geq 0}"] & \\ \mathcal{W}^{\lambda=0}\oplus(\mathcal{W}^{\lambda=0})^\vee \arrow[rr, "w^{\lambda=0}+q^{\lambda=0}"] & & \mathbb{A}^1 \end{tikzcd} \tag{2-13}$$

Here (f_1, f_2) is induced by the top horizontal diagram in (2-12), (g_1, g_2) is induced by the duals of the morphisms of vector bundles

$$q_\lambda^* \mathcal{W}^{\lambda=0} \leftarrow \mathcal{W}^{\lambda\geq 0} \to p_\lambda^* \mathcal{W}$$

on $\mathcal{Y}^{\lambda\geq 0}$, and (r_1, r_2) is induced by the diagram of attracting loci $(\mathcal{W}^\vee)^{\lambda=0} \leftarrow (\mathcal{W}^\vee)^{\lambda\geq 0} \to \mathcal{W}^\vee$ for $\mathcal{W}^\vee$.

By applying Lemma 6.1 for the right square of (2-12) (and also noting that p_λ and f_2 are proper), we have the commutative diagram

$$\begin{tikzcd} \mathrm{MF}(\mathcal{Y}^{\lambda\geq 0}, w^{\lambda\geq 0}) \arrow[r, "p_{\lambda *}"] \arrow[d, "\Phi^{\lambda\geq 0}"'] & \mathrm{MF}(\mathcal{Y}, w) \arrow[d, "\Phi"'] \\ \mathrm{MF}(\mathcal{W}^{\lambda\geq 0} \oplus (\mathcal{W}^{\lambda\geq 0})^\vee, w^{\lambda\geq 0} + q^{\lambda\geq 0}) \arrow[r, "f_{2*}g_2^*"'] & \mathrm{MF}(\mathcal{W} \oplus \mathcal{W}^\vee, w+q). \end{tikzcd}$$

Similarly by applying Lemma 6.2 for the left square of (2-12), we have the commutative diagram

$$\begin{tikzcd} \mathrm{MF}(\mathcal{Y}^{\lambda=0}, w^{\lambda=0}) \arrow[r, "q_\lambda^*"] \arrow[d, "\Phi^{\lambda=0}"'] & \mathrm{MF}(\mathcal{Y}^{\lambda\geq 0}, w^{\lambda\geq 0}) \arrow[d, "\Phi^{\lambda\geq 0}"'] \\ \mathrm{MF}((\mathcal{W} \oplus \mathcal{W}^\vee)^{\lambda=0}, w^{\lambda=0} + q^{\lambda=0}) \arrow[r, "g_{1!}f_1^*"'] & \mathrm{MF}(\mathcal{W}^{\lambda\geq 0} \oplus (\mathcal{W}^{\lambda\geq 0})^\vee, w^{\lambda\geq 0} + q^{\lambda\geq 0}) \end{tikzcd}$$

Note that we have

$$g_{1!}(-) = g_{1*}\big(- \otimes f_1^* \operatorname{pr}^* \det(\mathcal{W}^{\lambda>0})^\vee[\dim W^{\lambda>0}]\big).$$

Here $\operatorname{pr}\colon (\mathcal{W} \oplus \mathcal{W}^\vee)^{\lambda=0} \to \mathcal{Y}^{\lambda=0}$ is the projection. By the diagram (2-13) and the base change, we have the isomorphism of functors

$$f_{2*}g_2^* g_{1*} f_1^* \cong p'_{\lambda *} q_\lambda^{\prime *}\colon \mathrm{MF}(\mathcal{W}^{\lambda\geq 0} \oplus (\mathcal{W}^{\lambda\geq 0})^\vee, w^{\lambda\geq 0} + q^{\lambda\geq 0}) \to \mathrm{MF}(\mathcal{W} \oplus \mathcal{W}^\vee, w+q).$$

Therefore the proposition holds. □

3 Categorified Hall products for quivers with superpotentials

In this section, we review categorified Hall products for quivers with superpotentials introduced in [Pădurariu 2019; 2023].

3.1 Moduli stacks of representations of quivers

A *quiver* consists of data $Q = (Q_0, Q_1, s, t)$, where Q_0, Q_1 are finite sets and $s, t\colon Q_1 \to Q_0$ are maps. The set Q_0 is the set of vertices, Q_1 is the set of edges, and s, t are maps which assign source and target of each edge. A *Q–representation* consists of data

$$\mathbb{V} = \{(V_i, u_e) : i \in Q_0,\, u_e \in \operatorname{Hom}(V_{s(e)}, V_{t(e)})\},$$

where each V_i is a finite-dimensional vector space. The dimension vector $v(\mathbb{V})$ of $\mathbb{V}$ is $(\dim V_i)_{i\in Q_0}$.

For $v=(v_i)_{i\in Q_0}\in\mathbb{Z}_{\geq 0}^{Q_0}$, let $R_Q(v)$ be the vector space

$$R_Q(v)=\bigoplus_{e\in Q_1}\mathrm{Hom}(V_{s(e)},V_{t(e)}),$$

where $\dim V_i=v_i$. The algebraic group $G(v):=\prod_{i\in Q_0}\mathrm{GL}(V_i)$ acts on $R_Q(v)$ by conjugation. The stack of Q–representations of dimension vector v is given by the quotient stack

$$\mathcal{M}_Q(v):=[R_Q(v)/G(v)].$$

We discuss King's θ–stability condition [1994] on $Q^\dagger$–representations. We take

$$\theta=(\theta_i)_{i\in Q_0}\in\mathbb{R}^{Q_0}.$$

For a dimension vector $v\in\mathbb{Z}_{\geq 0}^{Q_0}$, we set $\theta(v)=\sum_{i\in Q_0}\theta_i v_i$. For a Q–representation $\mathbb{V}$, we set $\theta(\mathbb{V}):=\theta(v(\mathbb{V}))$.

Definition 3.1 A Q–representation $\mathbb{V}$ is called θ–(semi)stable if $\theta(\mathbb{V})=0$ and for any nonzero subobject $\mathbb{V}'\subsetneq\mathbb{V}$ we have $\theta(\mathbb{V}')<(\leq)\,0$.

There is an open substack

$$\mathcal{M}_Q^{\theta\text{-ss}}(v)\subset\mathcal{M}_Q(v)$$

corresponding to θ–semistable representations. By [King 1994, Proposition 3.1], if each θ_i is an integer, the above open substack corresponds to the GIT semistable locus with respect to the character

$$\chi_\theta\colon G(v)\to\mathbb{C}^*,\quad (g_i)_{i\in Q_0}\mapsto\prod_{i\in Q_0}\det g_i^{-\theta_i}.$$

By taking the GIT quotient, it admits a good moduli space [Alper 2013]

$$\pi_M\colon\mathcal{M}_Q^{\theta\text{-ss}}(v)\to M_Q^{\theta\text{-ss}}(v) \tag{3-1}$$

such that each closed point of $M_Q^{\theta\text{-ss}}(v)$ corresponds to a θ–polystable Q–representation.

Let $(a_i,b_i)\in\mathbb{Z}_{\geq 0}^2$ be a pair of nonnegative integers for each vertex $i\in Q_0$, and take $c\in\mathbb{Z}_{\geq 0}$. We define the extended quiver $Q^\dagger$ so that its vertex set is $\{\infty\}\cup Q_0$, with edges consist of edges in Q and

$$\sharp(\infty\to i)=a_i,\quad \sharp(i\to\infty)=b_i,\quad \sharp(\infty\to\infty)=c.$$

The $\mathbb{C}^*$–rigidified moduli stack of $Q^\dagger$–representations of dimension vector $(1,d)$ is given by

$$\mathcal{M}_Q^\dagger(v):=[R_{Q^\dagger}(1,v)/G(v)],$$

where 1 is the dimension vector at ∞. Note that there is a natural morphism $\mathcal{M}_{Q^\dagger}(1,v)\to\mathcal{M}_Q^\dagger(v)$ which is a trivial $\mathbb{C}^*$–gerbe. For $\theta=(\theta_\infty,\theta_i)_{i\in Q_0}$ with $\theta(1,v)=0$, the open substack of θ–semistable representations

$$\mathcal{M}_Q^{\dagger,\theta\text{-ss}}(v)\subset\mathcal{M}_Q^\dagger(v)$$

is defined in a similar way. The condition $\theta(1,v)=0$ determines θ_∞ by $\theta_\infty=-\sum_{i\in Q_0}\theta_i v_i$, so we just write $\theta=(\theta_i)_{i\in Q_0}$.

Remark 3.2 A reason for considering the extended quiver $Q^\dagger$ is to rigidify automorphisms of representations of quivers so that the resulting moduli spaces become schemes rather than stacks. Namely $(1, v)$ is the primitive dimension vector of $Q^\dagger$–representations so that $\mathcal{M}_Q^{\dagger,\theta\text{-ss}}(v)$ is a scheme (indeed smooth quasiprojective variety) for a generic choice of θ. Adding an extended vertex $\{\infty\}$ corresponds to giving a framing in PT stable pair theory.

3.2 Categorified Hall products

For a dimension vector $v \in \mathbb{Z}_{\geq 0}^{Q_0}$, let us take a decomposition

$$v = v^{(1)} + \cdots + v^{(l)}, \quad \text{where } v^{(j)} \in \mathbb{Z}_{\geq 0}^{Q_0}.$$

Let $V_i = \bigoplus_{j=1}^{l} V_i^{(j)}$ be a direct sum decomposition such that $\{V_i^{(j)}\}_{i \in Q_0}$ has dimension vector $v^{(j)}$. We take integers $\lambda^{(1)} > \cdots > \lambda^{(l)}$, and a one-parameter subgroup $\lambda\colon \mathbb{C}^* \to G(v)$ which acts on $V_i^{(j)}$ by weight $\lambda^{(j)}$. We have the stack of attracting loci

$$\mathcal{M}_Q(v^\bullet) := [R_Q(v)^{\lambda \geq 0}/G(v)^{\lambda \geq 0}].$$

The above stack is isomorphic to the stack of filtrations of Q–representations

$$0 = \mathbb{V}^{(0)} \subset \mathbb{V}^{(1)} \subset \cdots \subset \mathbb{V}^{(v)} = \mathbb{V} \tag{3-2}$$

such that each $\mathbb{V}^{(j)}/\mathbb{V}^{(j-1)}$ has dimension vector $v^{(j)}$. Moreover, we have

$$\prod_{j=1}^{l} \mathcal{M}_Q(v^{(j)}) = [R_Q(v)^{\lambda=0}/G(v)^{\lambda=0}],$$

and we have the diagram

$$\begin{tikzcd} \mathcal{M}_Q(v^\bullet) \arrow[r, "p_\lambda"] \arrow[d, "q_\lambda"'] & \mathcal{M}_Q(v) \\ \prod_{j=1}^{l} \mathcal{M}_Q(v^{(j)}) & \end{tikzcd} \tag{3-3}$$

Here p_λ sends a filtration (3-2) to $\mathbb{V}$, and q_λ sends a filtration (3-2) to its associated graded Q–representation. Since p_λ is proper, we have the functor (called the *categorified Hall product*)

$$p_{\lambda *} q_\lambda^* \colon \boxtimes_{j=1}^{l} D^b(\mathcal{M}_Q(v^{(j)})) \to D^b(\mathcal{M}_Q(v)). \tag{3-4}$$

For $\mathscr{E}^{(j)} \in D^b(\mathcal{M}_Q(v^{(j)}))$, we set

$$\mathscr{E}^{(1)} * \cdots * \mathscr{E}^{(l)} := p_{\lambda *} q_\lambda^* (\mathscr{E}^{(1)} \boxtimes \cdots \boxtimes \mathscr{E}^{(l)}).$$

The above $*$–product is associative, ie

$$(\mathscr{E}^{(1)} * \mathscr{E}^{(2)}) * \mathscr{E}^{(3)} \cong \mathscr{E}^{(1)} * (\mathscr{E}^{(2)} * \mathscr{E}^{(3)}) \cong \mathscr{E}^{(1)} * \mathscr{E}^{(2)} * \mathscr{E}^{(3)}.$$

We take $\theta = (\theta_i)_{i \in Q_0}$ such that $\theta(v^{(j)}) = 0$ for all j. Then the diagram (3-3) restricts to the diagram

$$\begin{tikzcd} \mathcal{M}_Q^{\theta\text{–ss}}(v^{\bullet}) \arrow[r, "p_\lambda"] \arrow[d, "q_\lambda"'] & \mathcal{M}_Q^{\theta\text{–ss}}(v) \\ \prod_{j=1}^{l} \mathcal{M}_Q^{\theta\text{–ss}}(v^{(j)}) & \end{tikzcd} \tag{3-5}$$

Similarly we have the functor

$$p_{\lambda *} q_{\lambda}^{*} \colon \boxtimes_{j=1}^{l} D^b(\mathcal{M}_Q^{\theta\text{–ss}}(v^{(j)})) \to D^b(\mathcal{M}_Q^{\theta\text{–ss}}(v)),$$

which coincides with (3-4) when $\theta = 0$.

Similarly, applying the above construction for the extended quiver $Q^{\dagger}$, for a decomposition

$$v = v^{(1)} + \cdots + v^{(l)} + v^{(\infty)}, \quad \text{where } v^{(j)} \in \mathbb{Z}^{Q_0}, \tag{3-6}$$

such that $\theta(v^{(j)}) = 0$ for $1 \le j \le l$, we have the functor

$$\boxtimes_{j=1}^{l} D^b(\mathcal{M}_Q^{\theta\text{–ss}}(v^{(j)})) \boxtimes D^b(\mathcal{M}_Q^{\dagger,\theta\text{–ss}}(v^{(\infty)})) \to D^b(\mathcal{M}_Q^{\dagger,\theta\text{–ss}}(v)). \tag{3-7}$$

By setting $l = 1$, it gives a left action of $\bigoplus_{\theta(v)=0} D^b(\mathcal{M}_Q^{\theta\text{–ss}}(v))$ on $\bigoplus_v D^b(\mathcal{M}_Q^{\dagger,\theta\text{–ss}}(v))$.

3.3 Categorified Hall products for quivers with superpotentials

Let W be a superpotential of a quiver Q, ie $W \in \mathbb{C}[Q]/[\mathbb{C}[Q], \mathbb{C}[Q]]$, where $\mathbb{C}[Q]$ is the path algebra of Q. Then there is a function

$$w := \operatorname{Tr}(W) \colon \mathcal{M}_Q(v) \to \mathbb{A}^1 \tag{3-8}$$

whose critical locus is identified with the moduli stack of (Q, W)–representations $\mathcal{M}_{(Q,W)}(v)$, ie Q–representations satisfying the relation ∂W.

The diagram (3-5) is extended to the diagram

$$\begin{tikzcd} \mathcal{M}_Q^{\theta\text{–ss}}(v^{\bullet}) \arrow[r, "p_\lambda"] \arrow[d, "q_\lambda"'] & \mathcal{M}_Q^{\theta\text{–ss}}(v) \arrow[d, "w"] \\ \prod_{j=1}^{l} \mathcal{M}_Q^{\theta\text{–ss}}(v^{(j)}) \arrow[r, "\sum_{j=1}^{l} w^{(j)}"] & \mathbb{A}^1 \end{tikzcd} \tag{3-9}$$

Here $w^{(j)}$ is the function (3-8) on $\mathcal{M}_Q(v^{(j)})$. Similarly to (3-4), we have the functor between triangulated categories of factorizations

$$p_{\lambda *} q_{\lambda}^{*} \colon \boxtimes_{j=1}^{l} \operatorname{MF}(\mathcal{M}_Q^{\theta\text{–ss}}(v^{(j)}), w^{(j)}) \to \operatorname{MF}(\mathcal{M}_Q^{\theta\text{–ss}}(v), w),$$

called the *categorified Hall products* for representations of quivers with superpotentials.

The superpotential naturally defines the superpotential of the extended quiver $Q^{\dagger}$, so we have the regular function $w\colon \mathcal{M}_Q^{\dagger}(v) \to \mathbb{A}^1$ as in (3-8). Similarly to (3-7), for a decomposition (3-6) we have the left action

$$\boxtimes_{j=1}^{l} \mathrm{MF}(\mathcal{M}_Q^{\theta\text{-ss}}(v^{(j)}), w^{(j)}) \boxtimes \mathrm{MF}(\mathcal{M}_Q^{\dagger,\theta\text{-ss}}(v^{(\infty)}), w^{(\infty)}) \to \mathrm{MF}(\mathcal{M}_Q^{\dagger,\theta\text{-ss}}(v), w). \tag{3-10}$$

Note that we have the decomposition (2-2) with respect to the diagonal torus $\mathbb{C}^* \subset G(v)$

$$\mathrm{MF}(\mathcal{M}_Q^{\theta\text{-ss}}(v), w) = \bigoplus_{j\in\mathbb{Z}} \mathrm{MF}(\mathcal{M}_Q^{\theta\text{-ss}}(v), w)_j.$$

We will often restrict the functor (3-10) to the fixed weight spaces

$$\boxtimes_{j=1}^{l} \mathrm{MF}(\mathcal{M}_Q^{\theta\text{-ss}}(v^{(j)}), w^{(j)})_{i_j} \boxtimes \mathrm{MF}(\mathcal{M}_Q^{\dagger,\theta\text{-ss}}(v^{(\infty)}), w^{(\infty)}) \to \mathrm{MF}(\mathcal{M}_Q^{\dagger,\theta\text{-ss}}(v), w).$$

3.4 Base change to formal fibers

Later we will take a base change of the categorified Hall product to a formal neighborhood of a point in the good moduli space (3-1). The diagram (3-9) extends to the commutative diagram

$$\begin{tikzcd}
\mathcal{M}_Q^{\theta\text{-ss}}(v^{\bullet}) \arrow[r, "p_\lambda"] \arrow[d, "q_\lambda"'] & \mathcal{M}_Q^{\theta\text{-ss}}(v) \arrow[dd, "\pi_M"] \arrow[ddr, "w"] & \\
\prod_{j=1}^{l} \mathcal{M}_Q^{\theta\text{-ss}}(v^{(j)}) \arrow[d] & & \\
\prod_{j=1}^{l} M_Q^{\theta\text{-ss}}(v^{(j)}) \arrow[r, "\oplus"] & M_Q^{\theta\text{-ss}}(v) \arrow[r] & \mathbb{A}^1
\end{tikzcd} \tag{3-11}$$

Here the bottom arrow is the morphism taking the direct sum of θ–polystable representations, which is a finite morphism (see [Meinhardt and Reineke 2019, Lemma 2.1]), and the left bottom vertical arrow is the good moduli space morphism. For a closed point $p \in M_Q^{\theta\text{-ss}}(v)$, we consider the following formal fiber

$$\widehat{\mathcal{M}}_Q^{\theta\text{-ss}}(v)_p := \mathcal{M}_Q^{\theta\text{-ss}}(v) \times_{M_Q^{\theta\text{-ss}}(v)} \widehat{M}_Q^{\theta\text{-ss}}(v)_p \to \widehat{M}_Q^{\theta\text{-ss}}(v)_p := \operatorname{Spec} \widehat{\mathcal{O}}_{M_Q^{\theta\text{-ss}}(v),p}.$$

Let $(p^{(1)}, \dots, p^{(l)}) \in \prod_{j=1}^{l} M_Q^{\theta\text{-ss}}(v^{(j)})$ be a point such that $\oplus(p^{(1)}, \dots, p^{(l)}) = p$. By taking the fiber product of the diagram (3-11) by $\widehat{M}_Q^{\theta\text{-ss}}(v)_p \to M_Q^{\theta\text{-ss}}(v)$, we obtain the diagram

$$\begin{tikzcd}
\widehat{\mathcal{M}}_Q^{\theta\text{-ss}}(v^{\bullet})_p \arrow[r, "\widehat{p}_\lambda"] \arrow[d, "\widehat{q}_\lambda"'] & \widehat{\mathcal{M}}_Q^{\theta\text{-ss}}(v)_p \\
\coprod_{p^{(\bullet)} \in \oplus^{-1}(p)} \prod_{j=1}^{l} \widehat{\mathcal{M}}_Q^{\theta\text{-ss}}(v^{(j)})_{p^{(j)}} &
\end{tikzcd} \tag{3-12}$$

The above diagram is a diagram of attracting loci for $\widehat{\mathcal{M}}_Q^{\theta\text{-ss}}(v)_p$; see [Toda 2021, Lemma 4.11].

By the derived base change, we have the commutative diagram

$$\begin{tikzcd} \boxtimes_{j=1}^{l} D^b(\mathcal{M}_Q^{\theta\text{-ss}}(v^{(j)})) \arrow[r, "p_{\lambda*}q_\lambda^*"] \arrow[d] & D^b(\mathcal{M}_Q^{\theta\text{-ss}}(v)) \arrow[d] \\ \bigoplus_{p^{(\bullet)}\in\oplus^{-1}(p)} \boxtimes_{j=1}^{l} D^b(\widehat{\mathcal{M}}_Q^{\theta\text{-ss}}(v^{(j)})_{p^{(j)}}) \arrow[r, "\widehat{p}_{\lambda*}\widehat{q}_\lambda^*"] & D^b(\widehat{\mathcal{M}}_Q^{\theta\text{-ss}}(v)_p) \end{tikzcd} \tag{3-13}$$

Here the vertical arrows are pullbacks to formal fibers.

We denote by $\widehat{w}_p : \widehat{\mathcal{M}}_Q^{\theta\text{-ss}}(v) \to \mathbb{A}^1$ the pullback of the function (3-8) to the formal fiber. Similarly to (3-13), we have the commutative diagram

$$\begin{tikzcd} \boxtimes_{j=1}^{l} \mathrm{MF}(\mathcal{M}_Q^{\theta\text{-ss}}(v^{(j)}), w) \arrow[r, "p_{\lambda*}q_\lambda^*"] \arrow[d] & \mathrm{MF}(\mathcal{M}_Q^{\theta\text{-ss}}(v)) \arrow[d] \\ \bigoplus_{p^{(\bullet)}\in\oplus^{-1}(p)} \boxtimes_{j=1}^{l} \mathrm{MF}(\widehat{\mathcal{M}}_Q^{\theta\text{-ss}}(v^{(j)})_{p^{(j)}}, \widehat{w}^{(j)}_{p^{(j)}}) \arrow[r, "\widehat{p}_{\lambda*}\widehat{q}_\lambda^*"] & \mathrm{MF}(\widehat{\mathcal{M}}_Q^{\theta\text{-ss}}(v)_p, \widehat{w}_p) \end{tikzcd} \tag{3-14}$$

4 Derived categories of Grassmannian flips

In this section, we use categorified Hall products to refine the result of [Ballard et al. 2021, Theorem 5.4.4] on variation of derived categories under Grassmannian flips.

4.1 Grassmannian flips

Let V be a vector space with dimension d, and let A and B be other vector spaces such that

$$a := \dim A \quad \text{and} \quad b := \dim B, \quad \text{with } a \geq b.$$

We form the quotient stack

$$\mathcal{G}_{a,b}(d) := [(\mathrm{Hom}(A, V) \oplus \mathrm{Hom}(V, B))/\mathrm{GL}(V)]. \tag{4-1}$$

Remark 4.1 The stack $\mathcal{G}_{a,b}(d)$ is the $\mathbb{C}^*$–rigidified moduli stack of representations of the quiver $Q_{a,b}$ of dimension vector $(1, d)$, where the vertex set is $\{\infty, 1\}$, the number of arrows from ∞ to 1 is a, that from 1 to ∞ is b, and there are no self-loops; see Section 3.1. For instance the quiver $Q_{3,2}$ is described by

$$Q_{3,2} = \begin{tikzcd} \bullet_\infty \arrow[r, bend left=60] \arrow[r, bend left=30] \arrow[r, bend left=10] & \bullet_1 \arrow[l, bend left=10] \arrow[l, bend left=40] \end{tikzcd} \tag{4-2}$$

Below we fix a basis of V, and take the maximal torus $T \subset \mathrm{GL}(V)$ to be consisting of diagonal matrices. For a one-parameter subgroup $\lambda : \mathbb{C}^* \to T$, we use the following notation for the diagram of attracting loci as in (2-3):

$$\begin{tikzcd} \mathcal{G}_{a,b}(d)^{\lambda\geq 0} \arrow[r, "p_\lambda"] \arrow[d, "q_\lambda"'] & \mathcal{G}_{a,b}(d) \\ \mathcal{G}_{a,b}(d)^{\lambda=0} & \end{tikzcd} \tag{4-3}$$

We use the determinant character

$$\chi_0\colon \mathrm{GL}(V) \to \mathbb{C}^*, \quad g \mapsto \det(g), \tag{4-4}$$

and often regard it as a line bundle on $\mathcal{G}_{a,b}(d)$. There exist two GIT quotients with respect to $\chi_0^{\pm 1}$ given by open substacks

$$G^{\pm}_{a,b}(d) \subset \mathcal{G}_{a,b}(d).$$

Here χ_0–semistable locus $G^{+}_{a,b}(d)$ consists of $(\alpha, \beta) \in \mathrm{Hom}(A, V) \oplus \mathrm{Hom}(V, B)$ such that $\alpha\colon A \to V$ is surjective, and χ_0^{-1}–semistable locus $G^{-}_{a,b}(d)$ consists of (α, β) such that $\beta\colon V \to B$ is injective. We have the diagram

$$\begin{tikzcd} G^{+}_{a,b}(d) \arrow[r, hook] \arrow[dr] & \mathcal{G}_{a,b}(d) \arrow[d] & G^{-}_{a,b}(d) \arrow[l, hook'] \arrow[dl] \\ & G^{0}_{a,b}(d) & \end{tikzcd} \tag{4-5}$$

Here the middle vertical arrow is the good moduli space for $\mathcal{G}_{a,b}(d)$.

Remark 4.2 When $a \geq d$ and $b = 0$, then $G^{-}_{a,0}(d) = \varnothing$ and $G^{+}_{a,0}(d)$ is the Grassmannian parametrizing surjections $A \twoheadrightarrow V$. If $a \geq b \geq d$, then $G^{\pm}_{a,b}(d) \to G^{0}_{a,b}(d)$ are birational and $G^{+}_{a,b}(d) \dashrightarrow G^{-}_{a,b}(d)$ is a flip $(a > b)$, flop $(a = b)$.

We have the KN stratifications with respect to $\chi_0^{\pm 1}$,

$$\mathcal{G}_{a,b}(d) = \mathcal{S}^{\pm}_0 \sqcup \mathcal{S}^{\pm}_1 \sqcup \cdots \sqcup \mathcal{S}^{\pm}_{d-1} \sqcup G^{\pm}_{a,b}(d),$$

where $\mathcal{S}^{+}_i$ consists of (α, β) such that the image of $\alpha\colon A \to V$ has dimension i, and $\mathcal{S}^{-}_i$ consists of (α, β) such that the kernel of $\beta\colon V \to B$ has dimension $d - i$. The associated one-parameter subgroups $\lambda^{\pm}_i\colon \mathbb{C}^* \to T$ are taken as

$$\lambda^{+}_i(t) = (\overbrace{1, \dots, 1}^{i}, \overbrace{t^{-1}, \dots, t^{-1}}^{d-i}),\ \lambda^{-}_i(t) = (\overbrace{t, \dots, t}^{d-i}, \overbrace{1, \dots, 1}^{i}). \tag{4-6}$$

(See [Halpern-Leistner 2015, Example 4.12].)

4.2 Window subcategories for Grassmannian flips

We fix a Borel subgroup $B \subset \mathrm{GL}(V)$ to be consisting of upper-triangular matrices, and set roots of B to be negative roots. Let $M = \mathbb{Z}^d$ be the character lattice for $\mathrm{GL}(V)$, and $M^{+}_{\mathbb{R}} \subset M_{\mathbb{R}}$ the dominant chamber. By the above choice of negative roots, we have

$$M^{+}_{\mathbb{R}} = \{(x_1, x_2, \dots, x_d) \in \mathbb{R}^d : x_1 \leq x_2 \leq \cdots \leq x_d\}.$$

We set $M^{+} := M^{+}_{\mathbb{R}} \cap M$. For $c \in \mathbb{Z}$, we set

$$\mathbb{B}_c(d) := \{(x_1, x_2, \dots, x_d) \in M^{+} : 0 \leq x_i \leq c - d\}. \tag{4-7}$$

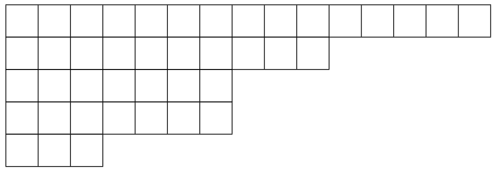


Figure 2: $(3,7,7,10,15) \in \mathbb{B}_c(d)$ for $d=5$ and $c \geq 20$.

Here $\mathbb{B}_c(d)=\varnothing$ if $c<d$. For $\chi \in \mathbb{B}_c(d)$, we assign the Young diagram whose number of boxes at the i^{th} row is x_{d-i+1}. The above assignment identifies $\mathbb{B}_c(d)$ with the set of Young diagrams with height less than or equal to d, width less than or equal to $c-d$. For example, Figure 2 illustrates the case of $(3,7,7,10,15) \in \mathbb{B}_c(d)$ for $d=5$ and $c \geq 20$.

We define the triangulated subcategory

$$\mathbb{W}_c(d) \subset D^b(\mathcal{G}_{a,b}(d)) \tag{4-8}$$

to be the smallest thick triangulated subcategory which contains $V(\chi) \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d)}$ for $\chi \in \mathbb{B}_c(d)$. Here $V(\chi)$ is the irreducible $\mathrm{GL}(V)$ representation with highest weight χ, ie it is a Schur power of V associated with the Young diagram corresponding to χ. The following proposition is well-known (see [Donovan and Segal 2014, Proposition 3.6]), which gives window subcategories for Grassmannian flips. We reprove it here using Theorem 2.2.

Proposition 4.3 *The following composition functors are equivalences*:

$$\begin{aligned}&\mathbb{W}_b(d) \subset D^b(\mathcal{G}_{a,b}(d)) \twoheadrightarrow D^b(G^-_{a,b}(d)),\\&\mathbb{W}_a(d) \subset D^b(\mathcal{G}_{a,b}(d)) \twoheadrightarrow D^b(G^+_{a,b}(d)).\end{aligned} \tag{4-9}$$

Proof We only prove the statement for $+$. Let λ_i^+ be the one-parameter subgroup in (4-6). Then η_i^+ given in (2-6) is

$$\eta_i^+ = \langle \lambda_i^+, (\mathrm{Hom}(A,V)^\vee \oplus \mathrm{Hom}(V,B)^\vee)^{\lambda_i^+>0} - \mathrm{End}(V)^{\lambda_i^+>0} \rangle = (a-i)(d-i).$$

Let $\chi'=(x_1',\dots,x_d')$ be a T–weight of $V(\chi)$ for $\chi \in \mathbb{B}_a(d)$. Then $0 \leq x_j' \leq a-d$ for $1 \leq j \leq d$, so

$$-\eta_i^+ < -(a-d)(d-i) \leq \langle \chi', \lambda_i^+ \rangle = -\sum_{j=i+1}^{d} x_j' \leq 0.$$

Therefore by setting $m_i=-\eta_i^+ + \varepsilon$ for $0<\varepsilon \ll 1$ and $l=\chi_0$ in (2-7), we have

$$\mathbb{W}_a(d) \subset \mathbb{W}_{m_\bullet}^{\chi_0}(\mathcal{G}_{a,b}(d)) \subset D^b(\mathcal{G}_{a,b}(d)).$$

It follows that the second composition functor in (4-9) is fully faithful.

In order to show that it is essentially surjective, note that the projection to $\mathrm{Hom}(A,V)$ defines a morphism

$$G^+_{a,b}(d) \to \mathrm{Gr}(a,d), \tag{4-10}$$

where $\mathrm{Gr}(a,d)$ is the Grassmannian which parametrizes d–dimensional quotients of A. By the above morphism, $G^+_{a,b}(d)$ is the total space of a vector bundle over $\mathrm{Gr}(a,d)$. The objects $V(\chi)\otimes \mathcal{O}_{\mathcal{G}_{a,b}(d)}$ for $\chi \in \mathbb{B}_a(d)$ restricted to the zero section of (4-10) forms Kapranov's exceptional collection [1984]. Since $D^b(G^+_{a,b}(d))$ is generated by pullbacks of objects from $D^b(\mathrm{Gr}(a,d))$, the essential surjectivity of (4-9) holds. □

4.3 Resolutions of categorified Hall products

Let $d = d^{(1)} + \cdots + d^{(l)} + d^{(\infty)}$ be a decomposition of d. Note that from Section 3.1, we have the categorified Hall product

$$\boxtimes_{j=1}^{l} D^b(B\mathrm{GL}(d^{(j)})) \boxtimes D^b(\mathcal{G}_{a,b}(d^{(\infty)})) \to D^b(\mathcal{G}_{a,b}(d)).$$

In particular by setting $d^{(1)}=1$ and $d^{(\infty)} = d-1$, we have the functor

$$*\colon D^b(B\mathbb{C}^*)\boxtimes D^b(\mathcal{G}_{a,b}(d-1)) \to D^b(\mathcal{G}_{a,b}(d)). \tag{4-11}$$

It is explicitly given as follows. Let $\lambda\colon \mathbb{C}^* \to T$ be given by

$$\lambda(t) = (t,1,\ldots,1). \tag{4-12}$$

Then we have the decomposition $V = V^{\lambda>0}\oplus V^{\lambda=0}$, where $V^{\lambda>0}$ is one-dimensional. Then

$$\begin{aligned}\mathcal{G}_{a,b}(d)^{\lambda=0} &= [B\mathrm{GL}(V^{\lambda>0})]\times \big[\big(\mathrm{Hom}(A,V^{\lambda=0})\oplus \mathrm{Hom}(V^{\lambda=0},B)\big)/\mathrm{GL}(V^{\lambda=0})\big]\\ &= B\mathbb{C}^*\times \mathcal{G}_{a,b}(d-1).\end{aligned}$$

The functor (4-11) is given by $p_{\lambda *}q_\lambda^*(-)$ in the diagram (4-3). The stack $\mathcal{G}_{a,b}(d)^{\lambda\geq 0}$ is the moduli stack of exact sequences of $Q_{a,b}$–representations

$$0\to \mathbb{V}^{\lambda>0}\to \mathbb{V}\to \mathbb{V}^{\lambda=0}\to 0 \tag{4-13}$$

such that $\mathbb{V}^{\lambda>0}$ has dimension vector $(0,1)$. We will often use the following lemmas:

Lemma 4.4 *For $\mathcal{E}_1\in D^b(B\mathbb{C}^*)$ and $\mathcal{E}_2 \in D^b(\mathcal{G}_{a,b}(d-1))$, we have*

$$(\mathcal{E}_1 * \mathcal{E}_2)\otimes \chi_0^j = (\mathcal{E}_1\otimes \mathcal{O}_{B\mathbb{C}^*}(j)) * (\mathcal{E}_2\otimes \chi_0^j).$$

Here we have used the same symbol χ_0 for the determinant character of $\mathrm{GL}(V^{\lambda=0})$.

Proof The lemma follows using $p_\lambda^*\chi_0 = \mathcal{O}_{B\mathbb{C}^*}(1)\boxtimes \chi_0$ and the definition of the functor (4-11). □

Lemma 4.5 *For $\mathcal{E}\in \mathbb{W}_c(d)$ and $j\geq 0$, we have $\mathcal{E}\otimes \chi_0^j \in \mathbb{W}_{c+j}(d)$.*

Proof The lemma follows since $V(\chi)\otimes \chi_0^j = V(\chi')$, where $\chi' = \chi + (j,j,\ldots,j)$. □

The following proposition is essentially proved in [Donovan and Segal 2014; Ballard et al. 2021], which we interpret in terms of categorified Hall products.

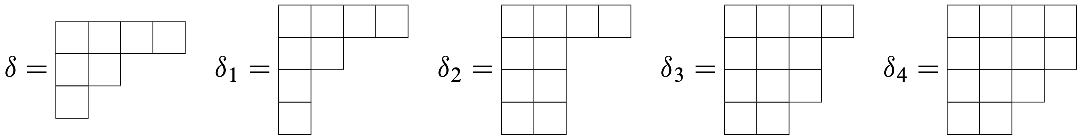


Figure 3: Algorithm for $\chi = (4, 2, 1)$, $d = 4$, $c = b = 7$.

Proposition 4.6 [Donovan and Segal 2014, Theorem A.7; Ballard et al. 2021, Proposition 5.4.6] *For $\chi \in \mathbb{B}_c(d-1)$ with $c \geq b$, let δ be the corresponding Young diagram. Then the object*

$$\mathcal{O}_{B\mathbb{C}^*} * (V(\chi) \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d-1)})$$

is a sheaf which fits into an exact sequence

$$0 \to V(\chi_K) \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d)}^{\oplus m_K} \to \cdots \to V(\chi_1) \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d)}^{\oplus m_1} \to V(\chi) \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d)} \to \mathcal{O}_{B\mathbb{C}^*} * (V(\chi) \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d-1)}) \to 0. \tag{4-14}$$

Here $\chi \in \mathbb{B}_c(d-1)$ is regarded as an element of $\mathbb{B}_{c+1}(d)$ by $(x_2, \dots, x_d) \mapsto (0, x_2, \dots, x_d)$, and each $\chi_i \in \mathbb{B}_{c+1}(d)$ in (4-14) *corresponds to a Young diagram δ_i obtained from δ by the following algorithm*:

- *The Young diagram δ_1 is obtained from δ by adding boxes to the first column until it reaches to height d.*
- *The diagram δ_i is obtained from δ_{i-1} by adding boxes to the i^{th} column until its height is one more than the height of the $(i-1)^{\text{th}}$ column of δ.*

See Figure 3. Moreover, $m_i = \dim \bigwedge^{s_i} B$ for $s_i = |\delta_i| - |\delta|$, and the sequence (4-14) *terminates when we reach a positive integer K such that $s_{K+1} > b$.*

Proof Let λ be the one-parameter subgroup (4-12). Then we have

$$\begin{aligned}(\mathrm{Hom}(A, V) \oplus \mathrm{Hom}(V, B))^{\lambda \geq 0} &= \mathrm{Hom}(A, V) \oplus \mathrm{Hom}(V^{\lambda=0}, B) \\ &\cong \mathrm{Hom}(V^\vee, A^\vee) \oplus \mathrm{Hom}(B^\vee, (V^{\lambda=0})^\vee).\end{aligned}$$

The parabolic subgroup $\mathrm{GL}(V)^{\lambda \geq 0}$ is the subgroup of $\mathrm{GL}(V)$ which preserves $V^{\lambda>0} \subset V$. Therefore there is an isomorphism of quotient stacks

$$\begin{aligned}&[(\mathrm{Hom}(A, V) \oplus \mathrm{Hom}(V, B))^{\lambda \geq 0}/\mathrm{GL}(V)^{\lambda \geq 0}] \\ &\qquad \xrightarrow{\cong} [\mathrm{Hom}(V^\vee, A^\vee) \oplus \mathrm{Hom}(B^\vee, (V^{\lambda=0})^\vee) \oplus \mathrm{Hom}^{\mathrm{inj}}((V^{\lambda=0})^\vee, V^\vee)/\mathrm{GL}(V) \times \mathrm{GL}(V^{\lambda=0})].\end{aligned}$$

Here $\mathrm{Hom}^{\mathrm{inj}}((V^{\lambda=0})^\vee, V^\vee) \subset \mathrm{Hom}((V^{\lambda=0})^\vee, V^\vee)$ is the subset consisting of injective homomorphisms. The above isomorphism is induced by the embedding into the direct summand $(V^{\lambda=0})^\vee \hookrightarrow V^\vee$ together with the natural inclusion $\mathrm{GL}(V)^{\lambda \geq 0} \hookrightarrow \mathrm{GL}(V) \times \mathrm{GL}(V^{\lambda=0})$. Under the above isomorphism, the morphism

$$p_\lambda \colon [(\mathrm{Hom}(A, V) \oplus \mathrm{Hom}(V, B))^{\lambda \geq 0}/\mathrm{GL}(V)^{\lambda \geq 0}] \to \mathcal{G}_{a,b}(d)$$

from the diagram (3-3) is identified with the morphism

$$[\mathrm{Hom}(V^\vee, A^\vee)\oplus \mathrm{Hom}(B^\vee,(V^{\lambda=0})^\vee)\oplus \mathrm{Hom}^{\mathrm{inj}}((V^{\lambda=0})^\vee, V^\vee)/\mathrm{GL}(V)\times \mathrm{GL}(V^{\lambda=0})] \xrightarrow{p_\lambda} [\mathrm{Hom}(V^\vee, A^\vee)\oplus \mathrm{Hom}(B^\vee, V^\vee)/\mathrm{GL}(V)]$$

induced by the composition of maps. The above morphism is nothing but the one considered in [Donovan and Segal 2014, Theorem A.7; Ballard et al. 2021, Proposition 5.4.6]. We then directly apply the computation of $p_{\lambda *}(-)$ for vector bundles given by Schur powers in [Donovan and Segal 2014, Theorem A.7; Ballard et al. 2021, Proposition 5.4.6] to obtain the resolution (4-14).

We also check that each χ_i in (4-14) is an element of $\mathbb{B}_{c+1}(d)$, ie δ_i has at most height d and width $c-d+1$. It is obvious that δ_i has at most height d. Let μ_j be the number of boxes of δ at the j^{th} column. Then from the algorithm defining δ_i, we have

$$s_i=(d-\mu_1)+(\mu_1+1-\mu_2)+\cdots+(\mu_{i-1}+1-\mu_i)=d+i-1-\mu_i.$$

Since $\chi\in\mathbb{B}_c(d-1)$, we have $\mu_{c-d+2}=0$, so $s_{c-d+2}=c+1>b$. Therefore we have $K\le c-d+1$. Since δ has width at most $c-d+1$, it follows that δ_i also has width at most $c-d+1$. □

Using the above proposition, we have the following lemma:

Lemma 4.7 *For $0\le j\le c-1$, we have*

$$\mathcal{O}_{B\mathbb{C}^*}(j) * (\mathbb{W}_{c-1-j}(d-1)\otimes \chi_0^j)\subset \mathbb{W}_c(d). \tag{4-15}$$

Proof We have

$$\mathcal{O}_{B\mathbb{C}^*}(j) * (\mathbb{W}_{c-1-j}(d-1)\otimes \chi_0^j)=(\mathcal{O}_{B\mathbb{C}^*} * \mathbb{W}_{c-1-j}(d-1))\otimes\chi_0^j\subset \mathbb{W}_{c-j}(d)\otimes\chi_0^j\subset\mathbb{W}_c(d).$$

Here we have used Lemma 4.4 for the first identity, Proposition 4.6 for the first inclusion and Lemma 4.5 for the last inclusion. □

4.4 Generation of window subcategories

We show that for $c\ge b$ the category $\mathbb{W}_c(d)$ is generated by its subcategory $\mathbb{W}_b(d)$ and subcategories (4-15) for $0\le j\le c-b-1$. We first prove the case of $c=b+1$, which is a variant of [Ballard et al. 2021, Lemma 5.4.9].

Lemma 4.8 *The subcategory $\mathbb{W}_{b+1}(d)\subset D^b(\mathcal{G}_{a,b}(d))$ is generated by $\mathbb{W}_b(d)$ and $\mathcal{O}_{B\mathbb{C}^*} * \mathbb{W}_b(d-1)$.*

Proof For $\chi\in\mathbb{B}_{b+1}(d)$, it is enough to show that $V(\chi)\otimes\mathcal{O}_{\mathcal{G}_{a,b}(d)}$ is generated by

$$\mathbb{W}_b(d) \quad\text{and}\quad \mathcal{O}_{B\mathbb{C}^*} * \mathbb{W}_b(d-1).$$

Let δ be the Young diagram corresponding to χ, and we denote by μ_j the number of boxes of δ at the j^{th} column. We may assume that the width of δ is exactly $b-d+1$, ie $\mu_j\ge 1$ for $1\le j\le b-d+1$ and $\mu_{b-d+2}=0$.

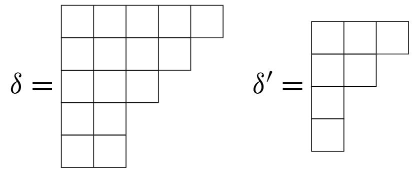


Figure 4: The diagrams δ and δ' for $\chi = (2,2,3,4,5) \in \mathbb{B}_{10}(5)$.

Suppose that the height of δ is exactly d, ie $\mu_1 = d$. We define another Young diagram δ' whose number of boxes at the j^{th} column is $\mu_{j+1} - 1$. Then the height of δ' is at most $d-1$, and the width of δ' is at most $b-d$; see Figure 4. Let $\chi' \in \mathbb{B}_{b-1}(d-1)$ be the character corresponding to δ'. As $\mathbb{B}_{b-1}(d-1) \subset \mathbb{B}_b(d-1)$, we apply Proposition 4.6 to obtain a resolution

$$0 \to V(\chi'_K) \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d)}^{\oplus m_K} \to \cdots \to V(\chi'_1) \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d)}^{\oplus m_1} \\ \to V(\chi') \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d)} \to \mathcal{O}_{B\mathbb{C}^*} * (V(\chi') \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d-1)}) \to 0. \tag{4-16}$$

for $\chi'_i \in \mathbb{B}_{b+1}(d)$. Note that we have

$$|\delta| - |\delta'| = (d - \mu_2 + 1) + (\mu_2 - \mu_3 + 1) + \cdots + (\mu_{b-d} - \mu_{b-d+1} + 1) + \mu_{b-d+1} = b.$$

From the construction of δ', the Young diagram δ is reconstructed from δ' by the algorithm given in Proposition 4.6 at the $(b-d+1)^{\text{th}}$ step. Therefore, from the above identity, it follows that there are exactly $b-d+1$ terms of the resolution (4-16), ie $K = b-d+1$, and also $m_K = 1$, $\chi'_K = \chi$. Moreover, since the width of δ' is as most $b-d$, we also have $\chi'_i \in \mathbb{B}_b(d)$ for $0 \le i < b-d+1$. Therefore $V(\chi) \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d)}$ is generated by objects in $\mathbb{W}_b(d)$ and $\mathcal{O}_{B\mathbb{C}^*} * (V(\chi') \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d-1)}) \in \mathcal{O}_{B\mathbb{C}^*} * \mathbb{W}_b(d-1)$.

Suppose that the height of δ is less than d. Then we have $\chi \in \mathbb{B}_b(d-1)$. By applying Proposition 4.6, we see that $V(\chi) \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d)}$ is generated by $\mathcal{O}_{B\mathbb{C}^*} * (V(\chi) \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d-1)}) \in \mathcal{O}_{B\mathbb{C}^*} * \mathbb{W}_b(d-1)$ and $V(\chi_i) \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d)}$ for $\chi_i \in \mathbb{B}_{b+1}(d)$. By the algorithm in Proposition 4.6, the Young diagram corresponding to χ_i has a full column, ie the height of χ_i is exactly d. Therefore by the above argument, each $V(\chi_i) \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d)}$ is generated by $\mathbb{W}_b(d)$ and $\mathcal{O}_{B\mathbb{C}^*} * \mathbb{W}_b(d-1)$. □

We then show the generation for $\mathbb{W}_c(d)$:

Lemma 4.9 *For $c \ge b$, the subcategory $\mathbb{W}_c(d) \subset D^b(\mathcal{G}_{a,b}(d))$ is generated by $\mathbb{W}_b(d)$ and*

$$\mathcal{O}_{B\mathbb{C}^*}(j) * (\mathbb{W}_{c-1-j}(d-1) \otimes \chi_0^j) \quad \textit{for } 0 \le j \le c-b-1.$$

Proof The case of $c = b+1$ is proved in Lemma 4.8. We prove the lemma for $c > b+1$ by induction on c. For $\chi \in \mathbb{B}_c(d)$, suppose that the corresponding Young diagram δ has a full column. Let δ'' be the Young diagram obtained by removing the first column, and χ'' the corresponding character. Then $\chi'' \in \mathbb{B}_{c-1}(d)$, so by the induction hypothesis $V(\chi'') \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d)}$ is generated by $\mathbb{W}_b(d)$ and $\mathcal{O}_{B\mathbb{C}^*}(j) * (\mathbb{W}_{c-2-j}(d-1) \otimes \chi_0^j)$ for $0 \le j \le c-b-2$. By taking the tensor product with χ_0 and setting $j' = j+1$, we see that

$V(\chi)\otimes\mathcal{O}_{\mathcal{G}_{a,b}(d)}$ is generated by $\mathbb{W}_b(d)\otimes\chi_0$ and $\mathcal{O}_{B\mathbb{C}^*}(j')*(\mathbb{W}_{c-1-j'}(d-1)\otimes\chi_0^{j'})$ for $1\le j'\le c-b-1$. Since $\mathbb{W}_b(d)\otimes\chi_0\subset\mathbb{W}_{b+1}(d)$, and the latter is generated by $\mathbb{W}_b(d)$ and $\mathcal{O}_{B\mathbb{C}^*}*\mathbb{W}_b(d-1)\subset\mathcal{O}_{B\mathbb{C}^*}*\mathbb{W}_{c-1}(d-1)$ by Lemma 4.8, we have the desired generation for $V(\chi)\otimes\mathcal{O}_{\mathcal{G}_{a,b}(d)}$ when δ has a full column.

If δ does not have a full column, then $\chi\in\mathbb{B}_{c-1}(d-1)$. By applying Proposition 4.6, we see that $V(\chi)\otimes\mathcal{O}_{\mathcal{G}_{a,b}(d)}$ is generated by $\mathcal{O}_{B\mathbb{C}^*}*(V(\chi)\otimes\mathcal{O}_{\mathcal{G}_{a,b}(d-1)})$ and $V(\chi_i)\otimes\mathcal{O}_{\mathcal{G}_{a,b}(d)}$ for $\chi_i\in\mathbb{B}_c(d)$. Since each Young diagram corresponding to χ_i has a full column, the desired generation of $V(\chi)\otimes\mathcal{O}_{\mathcal{G}_{a,b}(d)}$ is reduced to the case of the existence of a full column, which is proved above. □

Remark 4.10 The results of Lemmas 4.8 and 4.9 are essentially proved using [Ballard et al. 2021, Lemma 5.4.9] combined with the definition of $O_{d,s}$ in [Ballard et al. 2021, Definition 5.4.2]. Also several cohomology vanishing calculations, which will be given in Section 4.5, are also given in loc. cit. We have re-proved them in order to make them compatible with our notation, and to state them in terms of categorical Hall products.

The above generation result is stated in terms of iterated Hall products as follows:

Proposition 4.11 *For $c\ge b$, the subcategory $\mathbb{W}_c(d)\subset D^b(\mathcal{G}_{a,b}(d))$ is generated by the subcategories*

$$\mathcal{C}_{j_\bullet}:=\mathcal{O}_{B\mathbb{C}^*}(j_1)*\cdots*\mathcal{O}_{B\mathbb{C}^*}(j_l)*(\mathbb{W}_b(d-l)\otimes\chi_0^{j_l})\subset D^b(\mathcal{G}_{a,b}(d)) \tag{4-17}$$

for $0\le l\le d$ and $0\le j_1\le\cdots\le j_l\le c-b-l$. Here when $l=0$, the above subcategory is set to be $\mathbb{W}_b(d)$.

Proof We first show that (4-17) are subcategories of $\mathbb{W}_c(d)$ by the induction on c. By Lemma 4.4, the subcategory (4-17) is written as

$$\mathcal{O}_{B\mathbb{C}^*}(j_1)*\big(\big(\mathcal{O}_{B\mathbb{C}^*}(j_2-j_1)*\cdots*\mathcal{O}_{B\mathbb{C}^*}(j_l-j_1)*\big(\mathbb{W}_b((d-1)-(l-1))\otimes\chi_0^{j_l-j_1}\big)\big)\otimes\chi_0^{j_1}\big).$$

Since $j_l-j_1\le(c-1-j_1)-b-(l-1)$, by the induction hypothesis we have

$$\mathcal{O}_{B\mathbb{C}^*}(j_2-j_1)*\cdots*\mathcal{O}_{B\mathbb{C}^*}(j_l-j_1)*\big(\mathbb{W}_b((d-1)-(l-1))\otimes\chi_0^{j_l-j_1}\big)\in\mathbb{W}_{c-1-j_1}(d-1).$$

Therefore (4-17) is a subcategory of $\mathbb{W}_c(d)$ by Lemma 4.7.

We next show that $\mathbb{W}_c(d)$ is generated by subcategories (4-17) by the induction on c. By Lemma 4.9, the subcategory $\mathbb{W}_c(d)$ is generated by $\mathbb{W}_b(d)$ and $\mathcal{O}_{B\mathbb{C}^*}(j)*(\mathbb{W}_{c-1-j}(d-1)\otimes\chi_0^j)$ for $0\le j\le c-b-1$. By the induction hypothesis and Lemma 4.4, $\mathcal{O}_{B\mathbb{C}^*}(j)*(\mathbb{W}_{c-1-j}(d-1)\otimes\chi_0^j)$ is generated by

$$\begin{aligned}&\mathcal{O}_{B\mathbb{C}^*}(j)*\big(\big(\mathcal{O}_{B\mathbb{C}^*}(j_1)*\cdots*\mathcal{O}_{B\mathbb{C}^*}(j_{l'})*(\mathbb{W}_b(d-1-l')\otimes\chi_0^{j_{l'}})\big)\otimes\chi_0^j\big)\\&\qquad=\mathcal{O}_{B\mathbb{C}^*}(j)*\mathcal{O}_{B\mathbb{C}^*}(j+j_1)*\cdots*\mathcal{O}_{B\mathbb{C}^*}(j+j_{l'})*(\mathbb{W}_b(d-1-l')\otimes\chi_0^{j+j_{l'}})\end{aligned}$$

for $0\le l'\le d-1$ and $0\le j_1\le\cdots\le j_{l'}\le(c-1-j)-b-l'$. Since $j+j_{l'}\le c-b-(l'+1)$, the above subcategory is of the form (4-17) for $l=l'+1$. Therefore we obtain the desired generation. □

Remark 4.12 Let $F_j(-) := (\mathcal{O}_{B\mathbb{C}^*} * (-)) \otimes \chi_0^j = \mathcal{O}_{B\mathbb{C}^*}(j) * ((-) \otimes \chi_0^j)$. Then the repeated use of Lemma 4.4 implies that

$$\mathcal{C}_{j_\bullet} = F_{j_1} \circ F_{j_2 - j_1} \circ \cdots \circ F_{j_l - j_{l-1}}(\mathbb{W}_b(d-l)).$$

Similarly, for an intermediate step, we have

$$\mathcal{O}_{B\mathbb{C}^*}(j_i) * \cdots * \mathcal{O}_{B\mathbb{C}^*}(j_l) * (\mathbb{W}_b(d-l) \otimes \chi_0^{j_l}) = F_{j_i} \circ F_{j_{i+1} - j_i} \circ \cdots \circ F_{j_l - j_{l-1}}(\mathbb{W}_b(d-l)).$$

By the repeated use of Lemma 4.7, the above category is a subcategory of $\mathbb{W}_{b+l-i+1+j_l}(d-i+1)$.

4.5 Semiorthogonal decompositions under Grassmannian flips

We show that the subcategories in Proposition 4.11 form a semiorthogonal decomposition. We prepare some lemmas.

Lemma 4.13 *For any $\chi \in \mathbb{B}_b(d)$ and $\chi' \in \mathbb{B}_c(d-1)$ for some $c \geq 0$, we have the vanishing*

$$\operatorname{Hom}_{\mathcal{G}_{a,b}(d)}\big(\mathcal{O}_{B\mathbb{C}^*}(j) * (V(\chi') \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d-1)}), V(\chi) \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d)}\big) = 0 \quad \text{for } j \geq 0. \tag{4-18}$$

Proof Let $\lambda\colon \mathbb{C}^* \to T$ be the one-parameter subgroup given by (4-12). Using the notation of the diagram (4-3), the left-hand side of (4-18) is

$$\begin{aligned}\operatorname{Hom}\big(p_{\lambda *} q_\lambda^*\big(\mathcal{O}_{B\mathbb{C}^*}(j) \boxtimes (V(\chi') \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d-1)})\big), V(\chi) \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d)}\big) \\ = \operatorname{Hom}\big(q_\lambda^*\big(\mathcal{O}_{B\mathbb{C}^*}(j) \boxtimes (V(\chi') \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d-1)})\big), p_\lambda^!\big(V(\chi) \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d)}\big)\big).\end{aligned} \tag{4-19}$$

We have the formula

$$p_\lambda^!(-) = (-) \otimes (\det V^{\lambda>0})^{d-b-1} \otimes (\det V^{\lambda=0})^{-1}[d-b-1]$$

(cf [Donovan and Segal 2014, Section A.1; Ballard et al. 2021, (5.8)]). Since $\chi \in \mathbb{B}_b(d)$ and it is a highest weight of $V(\chi)$, any T–weight $\chi'' = (x_1'', \dots, x_d'')$ of $V(\chi)$ satisfies $x_i'' \leq b - d$. Therefore any T–weight of $V(\chi) \otimes (\det V^{\lambda>0})^{d-b-1} \otimes (\det V^{\lambda=0})^{-1}$ pairs negatively with λ. On the other hand, a pairing of λ with any T–weight of the $\mathrm{GL}(V)^{\lambda=0}$–representation $(\det V^{\lambda>0})^j \boxtimes V(\chi')$ is $j \geq 0$. Therefore we have the vanishing of (4-19) by Lemma 2.1. □

Lemma 4.14 *For $\chi, \chi' \in \mathbb{B}_c(d-1)$ for some $c \geq 0$, we have the vanishing*

$$\operatorname{Hom}_{\mathcal{G}_{a,b}(d)}(\mathcal{O}_{B\mathbb{C}^*}(j) * (V(\chi) \otimes \chi_0^j \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d-1)}), \mathcal{O}_{B\mathbb{C}^*}(j') * (V(\chi) \otimes \chi_0^{j'} \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d-1)})) = 0 \tag{4-20}$$

for $j > j'$.

Proof By Lemma 4.4, we may assume that $j' = 0$. We use the notation in the proof of Lemma 4.13. Using Lemma 4.4 and the adjunction, the left-hand side of (4-20) is

$$\begin{aligned}&\operatorname{Hom}\big(\big(p_{\lambda *} q_\lambda^*\big(\mathcal{O}_{B\mathbb{C}^*} \boxtimes (V(\chi) \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d-1)})\big)\big) \otimes \chi_0^j, p_{\lambda *} q_\lambda^*\big(\mathcal{O}_{B\mathbb{C}^*} \boxtimes (V(\chi') \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d-1)})\big)\big) \\ &\qquad \cong \operatorname{Hom}\big(p_\lambda^*\big(\big(p_{\lambda *} q_\lambda^*(\mathcal{O}_{B\mathbb{C}^*} \boxtimes (V(\chi) \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d-1)}))\big) \otimes \chi_0^j\big), q_\lambda^*\big(\mathcal{O}_{B\mathbb{C}^*} \boxtimes (V(\chi') \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d-1)})\big)\big).\end{aligned}$$

By Proposition 4.6, the object

$$p_{\lambda *}q_{\lambda}^{*}(\mathcal{O}_{B\mathbb{C}^*} \boxtimes (V(\chi) \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d-1)})) \in D^b(\mathcal{G}_{a,b}(d))$$

is resolved by vector bundles of the form $V(\chi'') \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d)}$, where χ'' is either $\chi'' = \chi$, or $\chi'' \in \mathbb{B}_{c+1}(d)$ whose corresponding Young diagram has a full column. In the latter case, any T–weight of $V(\chi'')$ pairs positively with λ. Therefore in both cases, any T–weight of $V(\chi'') \otimes \chi_0^j$ for $j > 0$ pairs positively with λ. On the other hand any λ–weight of $\mathcal{O}_{B\mathbb{C}^*} \boxtimes (V(\chi') \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d-1)})$ is zero, so the desired vanishing (4-20) follows from Lemma 2.1. □

Lemma 4.15 *In the situation of Lemma 4.14, for $j \in \mathbb{Z}$ we have the isomorphism*

$$\begin{aligned}(4\text{-}21)\quad &\mathrm{Hom}_{\mathcal{G}_{a,b}(d-1)}(V(\chi) \otimes \chi_0^j \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d-1)}, V(\chi') \otimes \chi_0^j \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d-1)}) \\ &\xrightarrow{\cong} \mathrm{Hom}_{\mathcal{G}_{a,b}(d)}\big(\mathcal{O}_{B\mathbb{C}^*}(j) * (V(\chi) \otimes \chi_0^j \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d-1)}), \mathcal{O}_{B\mathbb{C}^*}(j) * (V(\chi') \otimes \chi_0^j \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d-1)})\big).\end{aligned}$$

Proof By Lemma 4.4, we may assume that $j = 0$. Let χ'' be a weight which appeared in the proof of Lemma 4.14. Note that we observed that any T–weight of $V(\chi'')$ pairs positively with λ except $\chi'' = \chi$. Therefore by Lemma 2.1(i), the right-hand side of (4-21) is isomorphic to

$$(4\text{-}22)\qquad \mathrm{Hom}\big(p_{\lambda}^{*}(V(\chi) \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d)}), q_{\lambda}^{*}\big(\mathcal{O}_{B\mathbb{C}^*} \boxtimes (V(\chi') \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d-1)})\big)\big).$$

Since $\mathcal{G}_{a,b}(d)^{\lambda \geq 0}$ parametrizes exact sequences (4-13), the object $p_{\lambda}^{*}(V(\chi) \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d)})$ admits a filtration whose associated graded is of the form $q_{\lambda}^{*}(\mathcal{O}_{B\mathbb{C}^*}(j) \boxtimes (V(\chi''') \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d-1)}))$ for $j \geq 0$ and $\chi''' \in \mathbb{B}_c(d-1)$, and $j = 0$ if and only if $\chi''' = \chi$. Therefore by Lemma 2.1(i)–(ii), the above (4-22) is isomorphic to

$$\begin{aligned}(4\text{-}23)\quad &\mathrm{Hom}\big(q_{\lambda}^{*}\big(\mathcal{O}_{B\mathbb{C}^*} \boxtimes (V(\chi) \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d-1)})\big), q_{\lambda}^{*}\big(\mathcal{O}_{B\mathbb{C}^*} \boxtimes (V(\chi') \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d-1)})\big)\big) \\ &\cong \mathrm{Hom}_{\mathcal{G}_{a,b}(d-1)}(V(\chi) \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d-1)}, V(\chi') \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d-1)}). \quad \square\end{aligned}$$

In order to state the order of semiorthogonal decompositions, we take a lexicographical order on $\mathbb{Z}^d$, ie for $m_\bullet = (m_1, \dots, m_d) \in \mathbb{Z}^d$ and $m'_\bullet = (m'_1, \dots, m'_d) \in \mathbb{Z}^d$, we write $m_\bullet \succ m'_\bullet$ if $m_i = m'_i$ for $1 \leq i \leq k$ for some $k \geq 0$ and $m_{k+1} > m'_{k+1}$.

Definition 4.16 For $j_\bullet = (j_1, j_2, \dots, j_l)$ and $j'_\bullet = (j'_1, j'_2 \dots, j'_{l'})$ with $l, l' \leq d$, we define $j_\bullet \succ j'_\bullet$ if we have $\widetilde{j}_\bullet \succ \widetilde{j}'_\bullet$, where $\widetilde{j}_\bullet$ is defined by

$$(4\text{-}24)\qquad \widetilde{j}_\bullet = (j_1, j_2, \dots, j_l, -1, \dots, -1) \in \mathbb{Z}^d.$$

The next proposition shows the semiorthogonality of subcategories (4-17) with respect to the above order.

Proposition 4.17 *For*

$$\begin{aligned} j_\bullet &= (j_1, j_2, \dots, j_l) \quad \text{with } 0 \leq l \leq d \text{ and } 0 \leq j_1 \leq j_2 \leq \dots \leq j_l, \\ j'_\bullet &= (j'_1, j'_2, \dots, j'_{l'}) \quad \text{with } 0 \leq l' \leq d \text{ and } 0 \leq j'_1 \leq j'_2 \leq \dots \leq j'_{l'}, \end{aligned}$$

suppose that $j_\bullet \succ j'_\bullet$. Then we have $\mathrm{Hom}(\mathcal{C}_{j_\bullet}, \mathcal{C}_{j'_\bullet}) = 0$. Here $\mathcal{C}_\bullet$ is defined in (4-17).

Proof Let us take $P \in \mathbb{W}_b(d-l)$ and $P' \in \mathbb{W}_b(d-l')$. We need to show the vanishing of

$$\mathrm{Hom}\big(\mathcal{O}_{B\mathbb{C}^*}(j_1) * \cdots * \mathcal{O}_{B\mathbb{C}^*}(j_l) * (P \otimes \chi_0^{j_l}), \mathcal{O}_{B\mathbb{C}^*}(j_1) * \cdots * \mathcal{O}_{B\mathbb{C}^*}(j_{l'}) * (P' \otimes \chi_0^{j'_{l}})\big). \tag{4-25}$$

We note that, by Remark 4.12, for each $i \le l, l'$ the objects

$$\mathcal{O}_{B\mathbb{C}^*}(j_{i+1}) * \cdots * \mathcal{O}_{B\mathbb{C}^*}(j_l) * (P \otimes \chi_0^{j_l}) \quad \text{and} \quad \mathcal{O}_{B\mathbb{C}^*}(j_{i+1}) * \cdots * \mathcal{O}_{B\mathbb{C}^*}(j_{l'}) * (P' \otimes \chi_0^{j'_{l}}) \tag{4-26}$$

are objects in $\mathbb{W}_{c'}(d-i)$ for some $c' \ge 0$.

From $j_\bullet \succ j'_\bullet$, we have two cases:

(i) $l > l'$ and $j_i = j'_i$ for $1 \le i \le l'$.

(ii) There is $1 \le m < l, l'$ such that $j_i = j'_i$ for $1 \le i \le m$ and $j_{m+1} > j'_{m+1}$.

In the first case, we have

$$\begin{aligned}(4\text{-}25) = \mathrm{Hom}\big(&\mathcal{O}_{B\mathbb{C}^*}(j_1) * \cdots * \mathcal{O}_{B\mathbb{C}^*}(j_{l'}) * (\mathcal{O}_{B\mathbb{C}^*}(j_{l'+1}) * \cdots * \mathcal{O}_{B\mathbb{C}^*}(j_l) * (P \otimes \chi_0^{j_l})),\\ &\mathcal{O}_{B\mathbb{C}^*}(j_1) * \cdots * \mathcal{O}_{B\mathbb{C}^*}(j_{l'}) * (P' \otimes \chi_0^{j_{l'}})\big)\\ \cong \mathrm{Hom}(&\mathcal{O}_{B\mathbb{C}^*}(j_{l'+1}) * \cdots * \mathcal{O}_{B\mathbb{C}^*}(j_l) * (P \otimes \chi_0^{j_l}), P' \otimes \chi_0^{j_{l'}}) \cong 0.\end{aligned}$$

Here the first isomorphism follows from the repeated use of Lemma 4.15, noting that (4-26) are objects in $\mathbb{W}_{c'}(d-i)$; and the second isomorphism follows from Lemma 4.13. In the second case, a similar argument as above shows that

$$\begin{aligned}(4\text{-}25) = \mathrm{Hom}\big(&\mathcal{O}_{B\mathbb{C}^*}(j_1) * \cdots * \mathcal{O}_{B\mathbb{C}^*}(j_m) * (\mathcal{O}_{B\mathbb{C}^*}(j_{m+1}) * \cdots * \mathcal{O}_{B\mathbb{C}^*}(j_l) * (P \otimes \chi_0^{j_l})),\\ &\mathcal{O}_{B\mathbb{C}^*}(j_1) * \cdots * \mathcal{O}_{B\mathbb{C}^*}(j_m) * (\mathcal{O}_{B\mathbb{C}^*}(j'_{m+1}) * \cdots * \mathcal{O}_{B\mathbb{C}^*}(j'_{l'}) * (P' \otimes \chi_0^{j'_{l'}}))\big)\\ \cong \mathrm{Hom}\big(&\mathcal{O}_{B\mathbb{C}^*}(j_{m+1}) * \cdots * \mathcal{O}_{B\mathbb{C}^*}(j_l) * (P \otimes \chi_0^{j_l}), \mathcal{O}_{B\mathbb{C}^*}(j'_{m+1}) * \cdots * \mathcal{O}_{B\mathbb{C}^*}(j'_{l'}) * (P' \otimes \chi_0^{j'_{l'}})\big) \cong 0.\end{aligned}$$

Here the first isomorphism follows from the repeated use of Lemma 4.15, and the second isomorphism follows from Lemma 4.14. □

The following is the main result in this section, which gives a refinement of a semiorthogonal decomposition in [Ballard et al. 2021, Theorem 5.4.4]:

Theorem 4.18 *For $c \ge b$, there exists a semiorthogonal decomposition*

$$\mathbb{W}_c(d) = \langle \mathcal{C}_{j_\bullet} : 0 \le j \le d, \ j_\bullet = (0 \le j_1 \le \cdots \le j_l \le c-b-l) \rangle,$$

where $\mathrm{Hom}(\mathcal{C}_{j_\bullet}, \mathcal{C}_{j'_\bullet}) = 0$ for $j_\bullet \succ j'_\bullet$, and for each $j_\bullet$ we have an equivalence $\mathbb{W}_b(d-l) \xrightarrow{\sim} \mathcal{C}_{j_\bullet}$.

Proof The generation of $\mathbb{W}_c(d)$ by $\mathcal{C}_{j_\bullet}$ is proved in Proposition 4.11, and the semiorthogonality is proved in Proposition 4.17. The equivalence $\mathbb{W}_b(d-l) \xrightarrow{\sim} \mathcal{C}_{j_\bullet}$ follows from repeated use of Lemma 4.15. □

By applying the above theorem to $c = a$ and using equation (4-9), we obtain the following corollary, which relates derived categories under Grassmannian flips.

Corollary 4.19 *There exists a semiorthogonal decomposition*

$$D^b(G^+_{a,b}(d)) = \langle D^b(G^-_{a,b}(d-l))_{j_1,\dots,j_l} : 0 \le l \le d, 0 \le j_1 \le \dots \le j_l \le a-b-l \rangle.$$

Here, $D^b(G^-_{a,b}(d-l))_{j_1,\dots,j_l}$ *is a copy of* $D^b(G^-_{a,b}(d-l))$*.*

Remark 4.20 When $b = 0$, from Remark 4.2 the semiorthogonal decomposition in Corollary 4.19 is

$$D^b(G^+_{a,0}(d)) = \langle D^b(\operatorname{Spec}\mathbb{C})_{j_1,\dots,j_d} : 0 \le j_1 \le \dots \le j_d \le a-d \rangle.$$

Each factor $D^b(\operatorname{Spec}\mathbb{C})_{j_1,\dots,j_d}$ is generated by a vector bundle, which forms Kapranov's exceptional collection [Kapranov 1984] of the Grassmannian $G^+_{a,0}(d)$.

Remark 4.21 When $d = 1$, the birational map $G^+_{a,b}(1) \dashrightarrow G^-_{a,b}(1)$ is a standard toric flip. In this case, the semiorthogonal decomposition in Corollary 4.19 is

$$D^b(G^+_{a,b}(1)) = \langle D^b(G^-_{a,b}(1)), D^b(\mathrm{pt})_{(0)}, \dots, D^b(\mathrm{pt})_{(a-b-1)} \rangle.$$

The above semiorthogonal decomposition is a (mutation of) a well-known semiorthogonal decomposition for a standard flip; see [Kawamata 2018, Example 8.8(2)].

Remark 4.22 For a fixed (a,b,l), the set of sequences of integers $(j_1,\dots,j_l)$ satisfying

$$0 \le j_1 \le \dots \le j_l \le a-b-l$$

consists of $\binom{a-b}{l}$ elements. Therefore Corollary 4.19 implies (1-9). The same applies to Corollary 5.18 and Corollary 5.24 below, so that they imply (1-5) and (1-2), respectively.

4.6 Applications to categories of factorizations

We will use the following variant of Corollary 4.19. Let Z be a smooth scheme with a closed point $z \in Z$. Let us take the formal completion of $G^0_{a,b}(d) \times Z$, where $G^0_{a,b}(d)$ is the good moduli space for $\mathcal{G}_{a,b}(d)$,

$$\widehat{G}^0_{a,b}(d)_Z := \operatorname{Spec}\widehat{\mathcal{O}}_{G^0_{a,b}(d)\times Z,(0,z)}.$$

We also take a regular function w on it,

$$w\colon \widehat{G}^0_{a,b}(d)_Z \to \mathbb{A}^1, \quad w(0,z) = 0.$$

By taking the product of the diagram (4-5) with Z and pulling it back via $\widehat{G}^0_{a,b}(d)_Z \to G^0_{a,b}(d) \times Z$, we obtain the diagram

$$\begin{tikzcd}
\widehat{G}^+_{a,b}(d)_Z \arrow[r, hook] \arrow[dr] \arrow[ddr, bend right, "w"'] & \widehat{\mathcal{G}}_{a,b}(d)_Z \arrow[d] & \widehat{G}^-_{a,b}(d)_Z \arrow[l, hook'] \arrow[dl] \arrow[ddl, bend left, "w"] \\
& \widehat{G}^0_{a,b}(d)_Z \arrow[d, "w"] & \\
& \mathbb{A}^1 &
\end{tikzcd} \tag{4-27}$$

Similarly to (4-11), we have the categorified Hall product for formal fibers (see Section 3.4)

$$*\colon \mathrm{MF}(B\mathbb{C}^*, 0) \boxtimes \mathrm{MF}(\widehat{\mathcal{G}}_{a,b}(d-1)_Z, w) \to \mathrm{MF}(\widehat{\mathcal{G}}_{a,b}(d)_Z, w).$$

The subcategory

$$\widehat{\mathbb{W}}_c(d) \subset \mathrm{MF}(\widehat{\mathcal{G}}_{a,b}(d)_Z, w)$$

is also defined, similarly to (4-8), to be the smallest thick triangulated subcategory which contains factorizations with entries $V(\chi) \otimes \mathcal{O}$ for $\chi \in \mathbb{B}_c(d)$. Note that we have the decomposition (2-2)

$$\mathrm{MF}(B\mathbb{C}^*, 0) = \bigoplus_{j \in \mathbb{Z}} \mathrm{MF}(\mathrm{Spec}\,\mathbb{C}, 0)_j$$

such that $\mathrm{MF}(\mathrm{Spec}\,\mathbb{C}, 0)_j$ is equivalent to $\mathrm{MF}(\mathrm{Spec}\,\mathbb{C}, 0)$. We then define

$$\widehat{\mathcal{C}}_{j_\bullet} := \mathrm{MF}(\mathrm{Spec}\,\mathbb{C}, 0)_{j_1} * \cdots * \mathrm{MF}(\mathrm{Spec}\,\mathbb{C}, 0)_{j_l} * (\widehat{\mathbb{W}}_b(d-l) \otimes \chi_0^{j_l}) \subset \mathrm{MF}(\widehat{\mathcal{G}}_{a,b}(d)_Z, w)) \tag{4-28}$$

for $0 \le l \le d$ and $0 \le j_1 \le \cdots \le j_l \le c-b-l$. We have the following variant of Theorem 4.18.

Corollary 4.23 *For $c \ge b$, there exists a semiorthogonal decomposition*

$$\widehat{\mathbb{W}}_c(d) = \langle \widehat{\mathcal{C}}_{j_\bullet} : 0 \le l \le d,\ j_\bullet = (0 \le j_1 \le \cdots \le j_l \le c-b-l) \rangle,$$

where $\mathrm{Hom}(\widehat{\mathcal{C}}_{j_\bullet}, \widehat{\mathcal{C}}_{j'_\bullet}) = 0$ for $j_\bullet \succ j'_\bullet$, and for each $j_\bullet$ we have an equivalence $\widehat{\mathbb{W}}_b(d-l) \xrightarrow{\sim} \widehat{\mathcal{C}}_{j_\bullet}$.

Proof The argument of the proof of Theorem 4.18 implies an analogous semiorthogonal decomposition for $D^b(\widehat{\mathcal{G}}_{a,b}(d)_Z)$. Then it is well-known that the above semiorthogonal decomposition induces the one for categories of factorizations; cf [Halpern-Leistner and Pomerleano 2020, Lemmas 1.17 and 1.18; Orlov 2006, Proposition 1.10; Pădurariu 2019, Proposition 2.7; 2023, Proposition 2.1]. □

5 Categorical Donaldson–Thomas theory for the resolved conifold

In this section, we use the result in the previous section to prove Theorem 1.2.

5.1 Geometry and algebras for the resolved conifold

Let X be the resolved conifold

$$X := \mathrm{Tot}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}}(-1)^{\oplus 2}).$$

Here we recall some well-known geometry and algebras for the resolved conifold; see [Van den Bergh 2004; Nagao and Nakajima 2011] for details. There is a birational contraction

$$f\colon X \to Y := \{xy + zw = 0\} \subset \mathbb{C}^4,$$

which contracts the zero section $C = \mathbb{P}^1 \subset X$ to the conifold singularity $0 \in Y$. Let $\mathcal{E} := \mathcal{O}_X \oplus \mathcal{O}_X(1)$, and $A := \mathrm{End}(\mathcal{E})$. Then there is an equivalence by Van den Bergh [2004],

$$\Phi := \mathbf{R}\mathrm{Hom}(\mathcal{E}, -)\colon D^b(X) \xrightarrow{\sim} D^b(\mathrm{mod}\,A). \tag{5-1}$$

Here mod A is the abelian category finitely generated right A–modules. The noncommutative algebra A is isomorphic to the path algebra associated with a quiver with superpotential (Q, W) given by

$$Q = \begin{tikzcd} \bullet_0 \arrow[rr, bend left=50, "a_1"] \arrow[rr, bend left=20, "a_2"] & & \bullet_1 \arrow[ll, bend left=20, "b_1"] \arrow[ll, bend left=50, "b_2"] \end{tikzcd} \quad \text{and} \quad W = a_1 b_1 a_2 b_2 - a_1 b_2 a_2 b_1.$$

The equivalence (5-1) restricts to the equivalences of abelian subcategories

$$\Phi\colon \mathrm{Per}(X/Y) \xrightarrow{\sim} \mathrm{mod}\, A \quad \text{and} \quad \Phi\colon \mathrm{Per}_c(X/Y) \xrightarrow{\sim} \mathrm{mod}_{\mathrm{fd}}(A).$$

Here $\mathrm{Per}(X/Y)$ is the abelian category of Bridgeland's perverse coherent sheaves [2002], given by

$$\mathrm{Per}(X/Y) = \\ \bigl\{E \in D^b(X) : \mathcal{H}^i(E) = 0 \text{ for } i \neq -1, 0,\ R^1 f_* \mathcal{H}^0(E) = f_* \mathcal{H}^{-1}(E) = 0, \mathrm{Hom}(\mathcal{H}^0(E), \mathcal{O}_C(-1)) = 0\bigr\}.$$

The subcategory $\mathrm{Per}_c(X/Y) \subset \mathrm{Per}(X/Y)$ consists of compactly supported objects, and $\mathrm{mod}_{\mathrm{fd}}(A) \subset \mathrm{mod}(A)$ consists of finite-dimensional A–modules. The simple (Q, W)–representations corresponding to the vertex $\{0, 1\}$ are given by

$$\{\mathcal{O}_C, \mathcal{O}_C(-1)[1]\} \subset \mathrm{Per}_c(X/Y).$$

An object $F \in \mathrm{Per}_c(X/Y)$ is supported on C or a zero-dimensional subscheme in X. For $F \in \mathrm{Per}_c(X/Y)$, we set

$$\mathrm{cl}(F) := (\beta, n) \in \mathbb{Z}^{\oplus 2}, \quad \text{with } [F] = \beta[C],\ \chi(F) = n,$$

where $[F]$ is the fundamental one-cycle of F. Under the equivalence Φ, an object $F \in \mathrm{Per}_c(X/Y)$ with $\mathrm{cl}(F) = (\beta, n)$ corresponds to a (Q, W)–representation with dimension vector $(n, n - \beta)$.

Following [Nagao and Nakajima 2011, Section 1], a *perverse coherent system* is defined to be a pair

$$(F, s), \quad \text{with } F \in \mathrm{Per}_c(X/Y),\ s\colon \mathcal{O}_X \to F. \tag{5-2}$$

Let $(Q^\dagger, W)$ be a quiver with superpotential, given by

$$Q^\dagger = \begin{tikzcd} \bullet_\infty \arrow[d] & & \\ \bullet_0 \arrow[rr, bend left=50, "a_1"] \arrow[rr, bend left=20, "a_2"] & & \bullet_1 \arrow[ll, bend left=20, "b_1"] \arrow[ll, bend left=50, "b_2"] \end{tikzcd} \quad \text{and} \quad W = a_1 b_1 a_2 b_2 - a_1 b_2 a_2 b_1.$$

Note that $Q^\dagger$ is an extended quiver obtained from Q as in Section 3.1. By the equivalence (5-1), giving a perverse coherent system with $\mathrm{cl}(F) = (\beta, n)$ is equivalent to giving a representation of $(Q^\dagger, W)$ with dimension vector $(v_\infty, v_0, v_1) = (1, n, n - \beta)$.

5.2 Categorical DT invariants for the resolved conifold

For a dimension vector $v=(v_0,v_1)$ of Q, let V_0 and V_1 be vector spaces with dimensions v_0 and v_1, respectively. The $\mathbb{C}^*$–rigidified moduli stack of $Q^\dagger$–representations of dimension vector $(1,v)$ in Section 3.1 is explicitly written as

$$\mathcal{M}_Q^\dagger(v)=[R_{Q^\dagger}(v)/G(v)]=[V_0\oplus \mathrm{Hom}(V_0,V_1)^{\oplus 2}\oplus \mathrm{Hom}(V_1,V_0)^{\oplus 2}/\mathrm{GL}(V_0)\times \mathrm{GL}(V_1)].$$

Let w be the function

$$w=\mathrm{Tr}(W)\colon \mathcal{M}_Q^\dagger(v)\to \mathbb{A}^1,\quad w(v,A_1,A_2,B_1,B_2)=\mathrm{Tr}(A_1B_1A_2B_2-A_1B_2A_2B_1). \tag{5-3}$$

Then its critical locus

$$\mathcal{M}_{(Q,W)}^\dagger(v):=\mathrm{Crit}(w)\subset w^{-1}(0)\subset \mathcal{M}_Q^\dagger(v) \tag{5-4}$$

is the $\mathbb{C}^*$–rigidified moduli stack of $(Q^\dagger,W)$–representations of dimension vector $(1,v)$. Here the first inclusion follows from the fact that w is a homogeneous function on $R_{Q^\dagger}(v)$ of degree four. By the equivalence (5-1), $\mathcal{M}_{(Q,W)}^\dagger(v)$ is isomorphic to the moduli stack of perverse coherent systems (5-2) satisfying $\mathrm{cl}(F)=(v_0-v_1,v_0)$.

For $\theta=(\theta_0,\theta_1)\in\mathbb{R}^2$, we denote by

$$\mathcal{M}_Q^{\dagger,\theta\text{–ss}}(v)=[R_{Q^\dagger}^{\theta\text{–ss}}(v)/G(v)]\subset \mathcal{M}_Q^\dagger(v)$$

the open substack of θ–semistable $Q^\dagger$–representations. We also have the open substack

$$\mathcal{M}_{(Q,W)}^{\dagger,\theta\text{–ss}}(v):=\mathcal{M}_Q^{\dagger,\theta\text{–ss}}(v)\cap \mathcal{M}_{(Q,W)}^\dagger(v)\subset \mathcal{M}_{(Q,W)}^\dagger(v)$$

corresponding to θ–semistable $(Q^\dagger,W)$–representations. If $\theta_i\in\mathbb{Z}$, as mentioned in Section 3.1 these open substacks are GIT semistable locus with respect to the character

$$\chi_\theta\colon G(v)=\mathrm{GL}(V_0)\times\mathrm{GL}(V_1)\to\mathbb{C}^*,\quad (g_0,g_1)\mapsto \det(g_0)^{-\theta_0}\det(g_1)^{-\theta_1}. \tag{5-5}$$

We have the good moduli spaces by taking GIT quotients

$$\pi_Q^\dagger\colon \mathcal{M}_Q^{\dagger,\theta\text{–ss}}(v)\to M_Q^{\dagger,\theta\text{–ss}}(v)\quad\text{and}\quad \pi_{(Q,W)}^\dagger\colon \mathcal{M}_{(Q,W)}^{\dagger,\theta\text{–ss}}(v)\to M_{(Q,W)}^{\dagger,\theta\text{–ss}}(v). \tag{5-6}$$

We will consider the triangulated category

$$\mathrm{MF}(\mathcal{M}_Q^{\dagger,\theta\text{–ss}}(v),w)$$

and call it the *categorical DT invariant* for the conifold quiver $(Q^\dagger,W)$. The above triangulated category (or more precisely its dg-enhancement) recovers the numerical DT invariant considered in [Nagao and Nakajima 2011]:

Lemma 5.1 *For a generic $\theta \in \mathbb{R}^2$, there is an equality*

$$e_{\mathbb{C}(\!(u)\!)}(\mathrm{HP}_*(\mathrm{MF}(\mathcal{M}_Q^{\dagger,\theta\text{–ss}}(v), w)) = (-1)^{v_1}\,\mathrm{DT}^{\theta}(v).$$

Here $\mathrm{HP}_*(-)$ is the periodic cyclic homology which is a $\mathbb{Z}/2$–graded $\mathbb{C}(\!(u)\!)$–vector space [Keller 1999], *$e_{\mathbb{C}(\!(u)\!)}(-)$ is the Euler characteristic of $\mathbb{Z}/2$–graded $\mathbb{C}(\!(u)\!)$–vector space, and $\mathrm{DT}^{\theta}(v) \in \mathbb{Z}$ is the numerical DT invariant counting $(Q^{\dagger}, w)$–representations with dimension vector $(1, v)$.*

Proof Since θ is generic and the dimension vector $(1, v)$ of $Q^{\dagger}$ is primitive, the stack $M = \mathcal{M}_Q^{\dagger,\theta\text{–ss}}(v)$ consists of only θ–stable objects and it is a smooth quasiprojective scheme. By [Efimov 2018, Theorem 5.4], there is an isomorphism of $\mathbb{Z}/2$–graded vector spaces over $\mathbb{C}(\!(u)\!)$,

$$\mathrm{HP}_*(\mathrm{MF}(M, w)) \cong H^*(M, \phi_w(\mathbb{Q}_M)) \otimes_{\mathbb{Q}} \mathbb{C}(\!(u)\!) \cong H^{*+\dim M}(M, \phi_w(\mathrm{IC}_M)) \otimes_{\mathbb{Q}} \mathbb{C}(\!(u)\!).$$

Here $\phi_w(-)$ is the vanishing cycle functor and u has degree two, and $\mathrm{IC}_M = \mathbb{Q}_M[\dim M]$. We take the Euler characteristics of both sides as $\mathbb{Z}/2$–graded vector spaces over $\mathbb{C}(\!(u)\!)$. Since we have

$$e(H^*(M, \phi_w(\mathrm{IC}_M))) = \int_M \chi_B \, de =: \mathrm{DT}^{\theta}(v),$$

where χ_B is the Behrend function [2009] on M, it is enough to show that $(-1)^{v_1} = (-1)^{\dim M}$. Let E be a $Q^{\dagger}$–representation with dimension vector $(1, v_0, v_1)$. Then we have

$$\dim M = 1 + \dim \mathrm{Ext}^1_{Q^{\dagger}}(E, E) - \dim \mathrm{Hom}_{Q^{\dagger}}(E, E) = v_0 - v_0^2 - v_1^2 + 4v_0v_1.$$

Here we have used Lemma 5.3 below for the second identity. Therefore $(-1)^{v_1} = (-1)^{\dim M}$ holds. □

Remark 5.2 If θ lies in a DT chamber in Figure 1, the invariant $\mathrm{DT}^{\theta}(v)$ reduces to the DT invariant counting ideal sheaves of compactly supported closed subschemes $Z \hookrightarrow X$ satisfying $\mathrm{cl}(\mathcal{O}_Z) = (v_0 - v_1, v_0)$, considered in [Maulik et al. 2006].

The following lemma follows immediately from the Euler pairing computations of quiver representations [Brion 2012, Corollary 1.4.3].

Lemma 5.3 *For $Q^{\dagger}$–representations E, E' with dimension vector (v_{∞}, v_0, v_1), $(v'_{\infty}, v'_0, v'_1)$, we have*

$$\dim \mathrm{Hom}_{Q^{\dagger}}(E, E') - \dim \mathrm{Ext}^1_{Q^{\dagger}}(E, E') = v_{\infty}v'_{\infty} - v_{\infty}v'_0 + v_0v'_0 - 2v_0v'_1 - 2v_1v'_0 + v_1v'_1.$$

We have the unstable locus

$$\mathcal{M}^{\dagger,\theta\text{-us}}_{(Q,W)}(v) := \mathcal{M}^{\dagger}_{(Q,W)}(v) \setminus \mathcal{M}^{\dagger,\theta\text{–ss}}_{(Q,W)}(v).$$

Then we have the open immersion

$$\mathcal{M}_Q^{\dagger,\theta\text{–ss}}(v) \subset \mathcal{M}_Q^{\dagger}(v) \setminus \mathcal{M}^{\dagger,\theta\text{-us}}_{(Q,W)}(v).$$

The next lemma shows that the categorical DT invariant can be also defined on a bigger ambient space.

Lemma 5.4 *The restriction functor*

$$\mathrm{MF}(\mathcal{M}_Q^{\dagger}(v) \setminus \mathcal{M}_{(Q,W)}^{\dagger,\theta\text{-us}}(v), w) \xrightarrow{\sim} \mathrm{MF}(\mathcal{M}_Q^{\dagger,\theta\text{-ss}}(v), w)$$

is an equivalence.

Proof The lemma follows since the category of factorizations only depends on an open neighborhood of the critical locus; see the equivalence (2-1). □

5.3 Wall-chamber structure

There is a wall-chamber structure for the θ–stability as in Figure 1 in the introduction, taken from [Nagao and Nakajima 2011, Figure 1].

In Figure 1, if θ lies in the first quadrant then $\mathcal{M}_{(Q,W)}^{\dagger,\theta\text{-ss}}(v) = \varnothing$ unless $v = 0$, so it is called an *empty chamber*. In this case, the categorical DT invariants are given in the following lemma.

Lemma 5.5 *Let $\theta_{\mathrm{en}} \in \mathbb{R}^2$ lie in an empty chamber. Then*

$$\mathrm{MF}(\mathcal{M}_Q^{\dagger,\theta_{\mathrm{en}}\text{-ss}}(v), w) = \begin{cases} \mathrm{MF}(\operatorname{Spec}\mathbb{C}, 0) & \text{if } v = 0, \\ 0 & \text{if } v \neq 0. \end{cases}$$

Proof If $v \neq 0$, then $\mathrm{MF}(\mathcal{M}_Q^{\dagger,\theta_{\mathrm{en}}\text{-ss}}(v), w) = 0$ by the equivalence (2-1), since the critical locus of w is empty. If $v = 0$, then $\mathcal{M}_Q^{\dagger,\theta_{\mathrm{en}}\text{-ss}}(v) = \operatorname{Spec}\mathbb{C}$ and $w = 0$. □

We focus on the walls in the second quadrant, classified by $m \in \mathbb{Z}_{\geq 1}$:

$$W_m := \mathbb{R}_{>0} \cdot (1-m, m) \subset \mathbb{R}^2.$$

If θ lies between W_m and W_{m+1}, then the moduli stack $\mathcal{M}_{(Q,W)}^{\dagger,\theta\text{-ss}}(v)$ is constant, consisting of θ–stable objects. So $\mathcal{M}_{(Q,W)}^{\dagger,\theta\text{-ss}}(v)$ is a quasiprojective scheme, and the good moduli space morphism $\pi_{(Q,W)}^{\dagger}$ in (5-6) is an isomorphism. If θ is also sufficiently close to the wall W_m, then $\mathcal{M}_Q^{\dagger,\theta\text{-ss}}(v)$ also consists of θ–stable objects and the morphism $\pi_Q^{\dagger}$ in (5-6) is an isomorphism. The categorical DT invariant is also constant when θ deforms inside a chamber:

Lemma 5.6 *The triangulated category $\mathrm{MF}(\mathcal{M}_Q^{\dagger,\theta\text{-ss}}(v), w)$ is constant (up to equivalence) when θ deforms inside a chamber in Figure 1.*

Proof Suppose that θ lies in a chamber in Figure 1. Although θ does not lie in a wall for $(Q^{\dagger}, W)$–representations, it may lie on a wall for $Q^{\dagger}$–representations. However, the destabilizing locus in $\mathcal{M}_Q^{\dagger,\theta\text{-ss}}(v)$ is disjoint from $\mathrm{Crit}(w) = \mathcal{M}_{(Q,W)}^{\dagger,\theta\text{-ss}}(v)$, so by (2-1) the triangulated categories $\mathrm{MF}(\mathcal{M}_Q^{\dagger,\theta\text{-ss}}(v), w)$ are equivalent under wall-crossing inside a chamber of Figure 1. □

For $\theta \in W_m$, there is a unique (up to isomorphism) θ–stable (Q, W)–representation S_m — that is, $(Q^{\dagger}, W)$–representation whose dimension vector at ∞ is zero — given by

$$S_m := \left(\begin{tikzcd} \mathbb{C}^m \arrow[r, bend left=40, "0"] \arrow[r, shift left, "0"] & \mathbb{C}^{m-1} \arrow[l, shift left, "B_1^0"] \arrow[l, bend left=40, "B_2^0"] \end{tikzcd} \right), \quad B_1^0(f_i) = e_i, \quad B_2^0(f_i) = e_{i+1}. \tag{5-7}$$

See [Nagao and Nakajima 2011, Theorem 3.5]. Here $\{e_1, \dots, e_m\}$, $\{f_1, \dots, f_{m-1}\}$ are bases of $\mathbb{C}^m$ and $\mathbb{C}^{m-1}$, respectively. Note that S_m has dimension vector $s_m = (m, m-1)$ so that $\theta(S_m) = 0$ when $\theta \in W_m$. Under the equivalence Φ in (5-1), we have the relation

$$\Phi(\mathcal{O}_C(m-1)) = S_m. \tag{5-8}$$

See [Nagao and Nakajima 2011, Remark 3.6]. Since $s_m = (m, m-1)$ is primitive, the moduli stack $\mathcal{M}_Q^{\theta\text{–ss}}(s_m)$ consists of θ–stable Q–representations, and the good moduli space morphism

$$\mathcal{M}_Q^{\theta\text{–ss}}(s_m) \to M_Q^{\theta\text{–ss}}(s_m) \tag{5-9}$$

is a $\mathbb{C}^*$–gerbe. There is a function defined similarly to (5-3),

$$w = \mathrm{Tr}(W) \colon \mathcal{M}_Q^{\theta\text{–ss}}(s_m) \to \mathbb{A}^1,$$

whose critical locus $\mathcal{M}_{(Q,W)}^{\theta\text{–ss}}(s_m)$ is the moduli stack of θ–stable (Q, W)–representation. Note that $\mathcal{M}_{(Q,W)}^{\theta\text{–ss}}(s_m)$ consists of one point, corresponding to the unique θ–stable (Q, W)–representation S_m.

Lemma 5.7 *For any $j \in \mathbb{Z}$, there is an equivalence*

$$\mathrm{MF}(\mathcal{M}_Q^{\theta\text{–ss}}(s_m), w)_j \simeq \mathrm{MF}(\operatorname{Spec} \mathbb{C}, 0).$$

Proof Let $V_0 = \mathbb{C}^m$, $V_1 = \mathbb{C}^{m-1}$ and $B_1^0, B_2^0 \colon V_1 \to V_0$ be maps as in (5-7). Note that we have

$$\mathcal{M}_Q^{\theta\text{–ss}}(s_m) = [(\mathrm{Hom}(V_0, V_1)^{\oplus 2} \oplus \mathrm{Hom}(V_1, V_0)^{\oplus 2})^{\theta\text{–ss}} / \mathrm{GL}(V_0) \times \mathrm{GL}(V_1)].$$

It admits a projection

$$\mathcal{M}_Q^{\theta\text{–ss}}(s_m) \to [\mathrm{Hom}(V_1, V_0)^{\oplus 2} / \mathrm{GL}(V_0) \times \mathrm{GL}(V_1)]. \tag{5-10}$$

The target of the above morphism is identified with the moduli stack of representations of the Kronecker quiver Q_K (ie two vertices $\{0, 1\}$ with two arrows from 1 to 0). We have the Beilinson equivalence

$$\mathbf{R}\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1), -) \colon D^b(\mathbb{P}^1) \xrightarrow{\sim} D^b(\mathrm{Rep}(Q_K)).$$

Under the above equivalence, $\mathcal{O}_{\mathbb{P}^1}(m-1)$ corresponds to (B_1^0, B_2^0), which is a θ–stable Q_K–representation. Since $\mathcal{O}_{\mathbb{P}^1}(m-1)$ is rigid in $\mathbb{P}^1$ with automorphism $\mathbb{C}^*$, there is a $\mathrm{GL}(V_0) \times \mathrm{GL}(V_1)$–invariant open neighborhood

$$(B_1^0, B_2^0) \in \mathcal{U} \subset (\mathrm{Hom}(V_1, V_0)^{\oplus 2})^{\theta\text{–ss}}$$

such that $[\mathcal{U}/\mathrm{GL}(V_0)\times\mathrm{GL}(V_1)]$ is isomorphic to $B\mathbb{C}^*$, where $\mathbb{C}^*$ is the diagonal torus in $\mathrm{GL}(V_0)\times\mathrm{GL}(V_1)$, ie $t\mapsto(t\cdot\mathrm{id}_{V_0}, t\cdot\mathrm{id}_{V_1})$. By pulling it back by the projection (5-10), we see that there is an open immersion

$$\mathrm{Hom}(V_0,V_1)^{\oplus 2}\times B\mathbb{C}^*\subset\mathcal{M}_Q^{\theta\text{–ss}}(s_m),\quad (A_1,A_2)\mapsto(A_1,A_2,B_1^0,B_2^0), \tag{5-11}$$

such that the image of $0\in\mathrm{Hom}(V_0,V_1)^{\oplus 2}$ is $\{S_m\}=\mathrm{Crit}(w)$. By the equivalence (2-1), the restriction functor gives an equivalence

$$\mathrm{MF}(\mathcal{M}_Q^{\theta\text{–ss}}(s_m),w)\xrightarrow{\sim}\mathrm{MF}(\mathrm{Hom}(V_0,V_1)^{\oplus 2}\times B\mathbb{C}^*,w). \tag{5-12}$$

The function w restricted (5-11) is a quadratic function by the definition of w, which must be nondegenerate as its critical locus is one point. Since $\mathrm{Hom}(V_0,V_1)^{\oplus 2}$ is even-dimensional, for a suitable choice of basis of $\mathrm{Hom}(V_0.V_1)^{\oplus 2}$ the function w is written as $y_1z_1+\cdots+y_nz_n$, where n is the dimension of $\mathrm{Hom}(V_0,V_1)$. Therefore the right-hand side of (5-12) is equivalent to $\mathrm{MF}(B\mathbb{C}^*,0)$ by the Knörrer periodicity in Theorem 2.4. □

5.4 Descriptions of formal fibers

By the above classification of θ–stable (Q,W)–representations, a θ–polystable $(Q^\dagger,W)$–representation of dimension vector $(1,v_0,v_1)$ at the wall $\theta\in W_m$ is of the form

$$R=R_\infty\oplus(V\otimes S_m), \tag{5-13}$$

where V is a finite-dimensional vector space and R_∞ is a θ–stable $(Q^\dagger,W)$–representation. By setting $d:=\dim V$, the dimension vector of R_∞ is $(1,v_0-dm,v_1-d(m-1))$. By regarding R as a θ–polystable $Q^\dagger$–representation, it determines a point $p\in M_Q^{\dagger,\theta\text{–ss}}(v)$.

Remark 5.8 The vector space V in (5-13) will play the same role of the vector space V in Section 4. Below we fix a basis of V and use the same convention of the dominant chamber in Section 4.2.

Below, we fix a θ–polystable object (5-13), and $p\in M_Q^{\dagger,\theta\text{–ss}}(v)$ is the corresponding point as above. We will give a description of the formal fiber of the good moduli space morphism $\pi_Q^\dagger:\mathcal{M}_Q^{\dagger,\theta\text{–ss}}(v)\to M_Q^{\dagger,\theta\text{–ss}}(v)$ at p. We set

$$G_p:=\mathrm{Aut}(R)=\mathrm{GL}(V).$$

It acts on $\mathrm{Ext}^1_{Q^\dagger}(R,R)$ by the conjugation, and we have the good moduli space morphism

$$[\mathrm{Ext}^1_{Q^\dagger}(R,R)/G_p]\to\mathrm{Ext}^1_{Q^\dagger}(R,R)/\!\!/G_p. \tag{5-14}$$

Let $q\in R_{Q^\dagger}(v)$ be a point corresponding to the polystable object (5-13). Note that $\mathrm{Ext}^1_{Q^\dagger}(R,R)$ is the tangent space of the stack $\mathcal{M}_Q^{\dagger,\theta\text{–ss}}(v)$ at q. By Luna's étale slice theorem, there exists a G_p–invariant locally closed subset $q \in W_p \subset R_{Q^\dagger}(v)$ and a commutative diagram

$$\begin{tikzcd}
\big([\mathrm{Ext}^1_{Q^\dagger}(R,R)/G_p],0\big) \arrow[d] \arrow[dr, phantom, "\square"] & ([W_p/G_p],q) \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & (\mathcal{M}^{\dagger,\theta\text{-ss}}_Q(v),q) \arrow[d] \\
(\mathrm{Ext}^1_{Q^\dagger}(R,R)/\!\!/G_p,0) & (W_p/\!\!/G_p,q) \arrow[l] \arrow[r] & (M^{\dagger,\theta\text{-ss}}_Q(v),p)
\end{tikzcd} \tag{5-15}$$

such that each horizontal arrows are étale.

We have the following decomposition of $\mathrm{Ext}^1_{Q^\dagger}(R,R)$ as G_p–representations:

$$\begin{aligned}
\mathrm{Ext}^1_{Q^\dagger}(R,R) &= \mathrm{Ext}^1_{Q^\dagger}(R_\infty,R_\infty) \oplus (V \otimes \mathrm{Ext}^1_{Q^\dagger}(R_\infty,S_m)) \\
&\qquad \oplus (V^\vee \otimes \mathrm{Ext}_{Q^\dagger}(S_m,R_\infty)) \oplus (\mathrm{End}(V) \otimes \mathrm{Ext}^1_Q(S_m,S_m)) \\
&= (\mathrm{Ext}^1_{Q^\dagger}(R_\infty,R_\infty) \oplus \mathrm{Ext}^1_Q(S_m,S_m)) \oplus (V \otimes \mathrm{Ext}^1_{Q^\dagger}(R_\infty,S_m)) \\
&\qquad \oplus (V^\vee \otimes \mathrm{Ext}^1_{Q^\dagger}(S_m,R_\infty)) \oplus (\mathrm{End}_0(V) \otimes \mathrm{Ext}^1_Q(S_m,S_m)).
\end{aligned} \tag{5-16}$$

Here $\mathrm{End}_0(V)$ is the kernel of the trace map $\mathrm{Tr}\colon \mathrm{End}(V) \to \mathbb{C}$, which is an irreducible G_p–representation. The last identity gives a direct-sum decomposition of $\mathrm{Ext}^1_{Q^\dagger}(R,R)$ into its irreducible G_p–representations whose irreducible factors are $\mathbb{C}$ (the trivial representation), V, $V^\vee$ and $\mathrm{End}_0(V)$. The number of summands is calculated as follows:

Lemma 5.9 *We have the identities*

$$\begin{aligned}
a_{v,m,d} &:= \mathrm{ext}^1_{Q^\dagger}(R_\infty,S_m) = C_{v,m} + m + d(-2m^2+2m+1), \\
b_{v,m,d} &:= \mathrm{ext}^1_{Q^\dagger}(S_m,R_\infty) = C_{v,m} + d(-2m^2+2m+1), \\
C_{v,m} &:= (m-2)v_0 + (m+1)v_1, \\
c_m &:= \mathrm{ext}^1_{Q^\dagger}(S_m,S_m) = 2m^2-2m.
\end{aligned} \tag{5-17}$$

Proof The lemma easily follows from Lemma 5.3, noting that

$$\mathrm{Hom}(R_\infty,S_m) = \mathrm{Hom}(S_m,R_\infty) = 0 \quad \text{and} \quad \mathrm{Hom}(T_m,S_m) = \mathbb{C}.$$

For example, since the dimension vectors of R_∞ and S_m are $(1, v_0-md, v_1-(m-1)d)$ and $(0,m,m-1)$, respectively, we have

$$\begin{aligned}
-a_{v,m,d} &= \hom(R_\infty,S_m) - \mathrm{ext}^1(R_\infty,S_m) \\
&= -m + (v_0-md)m - 2(v_0-md)(m-1) - 2(v_1-(m-1)d)m + (m-1)(v_1-(m-1)d) \\
&= -(m-2)v_0 - (m+1)v_1 - m - d(-2m^2+2m+1).
\end{aligned}$$

□

The left vertical arrow in (5-15) is also identified with a moduli stack of some quiver representations and its good moduli space. We define Q_p to be the Ext quiver for $\{S_m\}$ and $Q_p^\dagger$ to be the Ext quiver for $\{R_\infty, S_m\}$. Namely Q_p is the quiver with one vertex $\{1\}$ and the number of loops at 1 is c_m. The quiver $Q_p^\dagger$ consists of two vertices $\{\infty, 1\}$, the number of arrows from ∞ to 1 is $a_{v,m,d}$, from 1 to ∞ is $b_{v,m,d}$, and the number of loops at ∞ (resp. 1) is $\mathrm{ext}^1_{Q^\dagger}(R_\infty, R_\infty)$ (resp. c_m). From (5-16), we have the identification

$$\begin{tikzcd} \mathcal{M}^\dagger_{Q_p}(d) \arrow[r, equal] \arrow[d] & {[\mathrm{Ext}^1_{Q^\dagger}(R,R)/G_p]} \arrow[d] \\ M^\dagger_{Q_p}(d) \arrow[r, equal] & \mathrm{Ext}^1_{Q^\dagger}(R,R)/\!\!/G_p \end{tikzcd} \tag{5-18}$$

By combining the diagrams (5-15), (5-18) and taking the formal fibers, we have a commutative diagram

$$\begin{tikzcd} \mathcal{M}^\dagger_{Q_p}(d) \arrow[d] \arrow[dr, phantom, "\square"] & \widehat{\mathcal{M}}^\dagger_{Q_p}(d) \arrow[l] \arrow[r, equal] \arrow[d] \arrow[rr, bend left, "\eta_p"] & \left[\widehat{\mathrm{Ext}}^1_{Q^\dagger}(R,R)/G_p\right] \arrow[r, "\cong"] \arrow[d] & \widehat{\mathcal{M}}^{\dagger,\theta\text{-ss}}_Q(v)_p \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{M}^{\dagger,\theta\text{-ss}}_Q(v) \arrow[d] \\ M^\dagger_{Q_p}(d) & \widehat{M}^\dagger_{Q_p}(d) \arrow[l] \arrow[r, equal] & \widehat{\mathrm{Ext}}^1_{Q^\dagger}(R,R)/\!\!/G_p \arrow[r, "\cong"] & \widehat{M}^{\dagger,\theta\text{-ss}}_Q(v)_p \arrow[r] & M^{\dagger,\theta\text{-ss}}_Q(v) \end{tikzcd} \tag{5-19}$$

Here each vertical arrow is a good moduli space morphism, the vertical arrow second from the right (resp. left) is the formal fiber of the right (resp. left) one at p (resp. origin), the middle vertical arrow is the formal fiber of the morphism (5-14) at the origin. The square second from the right is obtained by the formal completions of good moduli spaces in the diagram (5-15), where the horizontal arrows are isomorphisms since the horizontal arrows in the diagram (5-15) are étale.

We then compare the semistable loci under the isomorphism η_p in the diagram (5-19). We take $\theta = (\theta_0, \theta_1) \in W_m$ and $\theta_\pm$ of the form

$$\theta_\pm = (\theta_0 \mp \varepsilon, \theta_1 \pm \varepsilon), \quad \text{with } \varepsilon > 0. \tag{5-20}$$

We take (θ_0, θ_1) and ε to be integers, and $\theta_\pm$ to lie on chambers adjacent to W_m which are sufficiently close to W_m, eg take $\varepsilon = 1$ and $(\theta_0, \theta_1) = N \cdot (1-m, m)$ for a sufficiently large integer N. We have the open substacks

$$\mathcal{M}^{\dagger,\theta_\pm\text{-ss}}_Q(v) \subset \mathcal{M}^{\dagger,\theta\text{-ss}}_Q(v) \quad \text{and} \quad \widehat{\mathcal{M}}^{\dagger,\theta_\pm\text{-ss}}_Q(v)_p \subset \widehat{\mathcal{M}}^{\dagger,\theta\text{-ss}}_Q(v)_p$$

corresponding to $\theta_\pm$–semistable representations.

On the other hand, as in (4-4) we set $\chi_0\colon \mathrm{GL}(V) \to \mathbb{C}^*$ to be the determinant character $g \mapsto \det(g)$. We have the open substacks

$$\mathcal{M}^{\dagger,\chi_0^{\pm 1}\text{-ss}}_{Q_p}(d) \subset \mathcal{M}^\dagger_{Q_p}(d) \quad \text{and} \quad \widehat{\mathcal{M}}^{\dagger,\chi_0^{\pm 1}\text{-ss}}_{Q_p}(d) \subset \widehat{\mathcal{M}}^\dagger_{Q_p}(d)$$

corresponding to $\chi_0^{\pm 1}$–semistable $Q_p^\dagger$–representations. We have the following lemma.

Lemma 5.10 *The isomorphism η_p in* (5-19) *restricts to the isomorphisms*

$$\eta_p\colon \widehat{\mathcal{M}}^{\dagger,\chi_0^{\pm 1}\text{-ss}}_{Q_p}(d) \xrightarrow{\cong} \widehat{\mathcal{M}}^{\dagger,\theta_\pm\text{-ss}}_{Q}(v)_p. \tag{5-21}$$

Proof Let us consider the composition

$$G_p = \mathrm{GL}(V) \hookrightarrow \mathrm{GL}(V_0) \times \mathrm{GL}(V_1) \xrightarrow{\chi_{\theta_\pm}} \mathbb{C}^*. \tag{5-22}$$

We see that the above composition is given by $g \mapsto \det(g)^{\pm\varepsilon}$, where $\chi_{\theta_\pm}$ is the character (5-5) applied to $\theta_\pm$. Indeed, we have

$$V_0 = (V \otimes \mathbb{C}^m) \oplus \mathbb{C}^{v_0 - dm} \quad \text{and} \quad V_1 = (V \otimes \mathbb{C}^{m-1}) \oplus \mathbb{C}^{v_1 - d(m-1)}.$$

The embedding $\mathrm{GL}(V) \hookrightarrow \mathrm{GL}(V_0) \times \mathrm{GL}(V_1)$ is given by

$$g \mapsto ((g \otimes 1_{\mathbb{C}^m}) \oplus 1_{\mathbb{C}^{v_0-dm}}, (g \otimes 1_{\mathbb{C}^{m-1}}) \oplus 1_{\mathbb{C}^{v_1-d(m-1)}}).$$

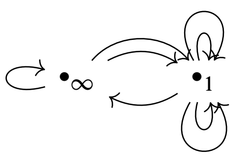
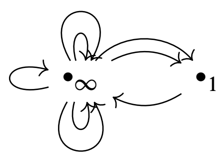
By composing it with $\chi_{\theta_\pm}$, we see that the composition (5-22) is given by $g \mapsto \det(g)^{\pm\varepsilon}$. Therefore under the isomorphism η_p in (5-19), the line bundle on $\widehat{\mathcal{M}}^{\dagger,\theta\text{-ss}}_{Q}(v)$ determined by $\chi_{\theta_\pm}$ corresponds to that on $\widehat{\mathcal{M}}^{\dagger}_{Q_p}(d)$ determined by $\chi_0^{\pm\varepsilon}$. Therefore the lemma holds. □

5.5 Reduced Ext quiver

We define the *reduced Ext quiver* $Q_p^{\mathrm{red},\dagger}$ to be the quiver obtained from $Q_p^\dagger$ by removing all the loops at the vertex $\{1\}$, and adding c_m loops at the vertex $\{\infty\}$, where c_m is as given in (5-17). It contains the full subquiver

$$Q_p^{\mathrm{red}} \subset Q_p^{\mathrm{red},\dagger} \tag{5-23}$$

consisting of the vertex $\{1\}$ and no loops. See the diagrams

$Q_p^\dagger =$  and $Q_p^{\mathrm{red},\dagger} =$ 

Let $\mathcal{M}^\dagger_{Q_p^{\mathrm{red}}}(d)$ be the $\mathbb{C}^*$–rigidified moduli stack of $Q_p^{\mathrm{red},\dagger}$–representations with dimension vector $(1,d)$. It is described as

$$\begin{aligned}\mathcal{M}^\dagger_{Q_p^{\mathrm{red}}}(d) &= \bigl[\bigl(\mathbb{C}^{\mathrm{ext}^1_{Q^\dagger}(R_\infty,R_\infty)+c_m} \oplus V^{\oplus a_{v,m,d}} \oplus (V^\vee)^{\oplus b_{v,m,d}}\bigr)/\mathrm{GL}(V)\bigr], \\ &= \mathbb{C}^{\mathrm{ext}^1_{Q^\dagger}(R_\infty,R_\infty)+c_m} \times \mathcal{G}_{a_{v,m,d},b_{v,m,d}}(d),\end{aligned} \tag{5-24}$$

ie it is obtained from (5-16) by removing the last factor $\mathrm{End}_0(V) \otimes \mathrm{Ext}^1_Q(S_m,S_m)$, and taking the quotient by $\mathrm{GL}(V)$. Here $\mathcal{G}_{a,b}(d)$ is the quotient stack (4-1) studied in Section 4. We also denote by $\widehat{\mathcal{M}}^\dagger_{Q_p^{\mathrm{red}}}(d)$ the formal fiber for the good moduli space morphism

$$\mathcal{M}^\dagger_{Q_p^{\mathrm{red}}}(d) \to M^\dagger_{Q_p^{\mathrm{red}}}(d)$$

at the origin.

By restricting the function (5-3) to the formal fiber of the good moduli space morphism (5-6) and pulling it back by the isomorphism η_p in (5-19), we have the function

$$w_p\colon \widehat{\mathscr{M}}^{\dagger}_{Q_p}(d) = [\widehat{\mathrm{Ext}}^1_{Q^\dagger}(R,R)/G_p] \to \mathbb{A}^1.$$

We see that the above function is a sum of a function from $\widehat{\mathscr{M}}^{\dagger}_{Q_p^{\mathrm{red}}}(d)$ and some nondegenerate quadratic form. Let us take a (noncanonical) isomorphism of $\mathbb{C}$–vector spaces

$$\mathrm{Ext}^1_Q(S_m,S_m) \cong H \oplus H^\vee, \tag{5-25}$$

where the dimension of H is m^2-m. Since there is also an isomorphism $\mathrm{End}_0(V) \cong \mathrm{End}_0(V)^\vee$ of G_p–representations, we have an isomorphism of G_p–representations

$$\mathrm{End}_0(V) \otimes \mathrm{Ext}^1_Q(S_m,S_m) \cong W \oplus W^\vee,$$

where $W = \mathrm{End}_0(V) \otimes H$. In particular, we have the nondegenerate symmetric quadratic form

$$q = \langle -,- \rangle \colon \mathrm{End}_0(V) \otimes \mathrm{Ext}^1_Q(S_m,S_m) \to \mathbb{A}^1 \tag{5-26}$$

defined to be the natural pairing on W and $W^\vee$. Note that (5-25) is a summand of $\mathrm{Ext}^1_{Q^\dagger}(R,R)$ by the decomposition (5-16). We will use the following proposition, whose proof will be given in Section 6.4.

Proposition 5.11 *By replacing the isomorphisms in* (5-19) *and* (5-25) *if necessary, the function w_p is written as*

$$w_p = w_p^{\mathrm{red}} + q. \tag{5-27}$$

Here w_p^{red} is nonzero and does not contain variables from $\mathrm{End}_0(V) \otimes \mathrm{Ext}^1_Q(S_m,S_m)$ under the decomposition (5-16).

The $\mathrm{GL}(V)$–representation W determines the vector bundle $\mathscr{W}$ on $\widehat{\mathscr{M}}^{\dagger}_{Q_p^{\mathrm{red}}}(d)$. By Proposition 5.11, we have the commutative diagram

$$\begin{tikzcd}
\mathscr{W}^\vee \arrow[r, hook, "i"] \arrow[d, "\mathrm{pr}"] & \mathscr{W} \oplus \mathscr{W}^\vee \arrow[dr, "w_p^{\mathrm{red}}+q"'] & \widehat{\mathscr{M}}^{\dagger}_{Q_p}(d) \arrow[l, "\iota"'] \arrow[r, "\eta_p", "\cong"'] \arrow[d, "w_p"] & \widehat{\mathscr{M}}^{\dagger,\theta\text{-ss}}_{Q}(v)_p \arrow[dl, "w_p"] \\
\widehat{\mathscr{M}}^{\dagger}_{Q_p^{\mathrm{red}}}(d) \arrow[rr, "w_p^{\mathrm{red}}"'] & & \mathbb{A}^1 &
\end{tikzcd} \tag{5-28}$$

Here pr is the projection, i is given by $i(x) = (0,x)$, ι is the natural morphism by the formal completion (see Lemma 6.4) and η_p is the isomorphism in (5-19).

Proposition 5.12 *There is an equivalence*

$$\Phi_p := \iota^* i_* \mathrm{pr}^* \colon \mathrm{MF}(\widehat{\mathscr{M}}^{\dagger}_{Q_p^{\mathrm{red}}}(d), w_p^{\mathrm{red}}) \xrightarrow{\sim} \mathrm{MF}(\widehat{\mathscr{M}}^{\dagger}_{Q_p}(d), w_p). \tag{5-29}$$

Proof The composition functor

$$i_* \operatorname{pr}^* \colon \operatorname{MF}(\widehat{\mathcal{M}}^{\dagger}_{Q_p^{\mathrm{red}}}(d), w_p^{\mathrm{red}}) \xrightarrow{\operatorname{pr}^*} \operatorname{MF}(\mathcal{W}^{\vee}, w_p^{\mathrm{red}}) \xrightarrow{i_*} \operatorname{MF}(\mathcal{W} \oplus \mathcal{W}^{\vee}, w_p^{\mathrm{red}} + q) \tag{5-30}$$

is an equivalence by Theorem 2.4. By Lemma 6.4, the functor

$$\iota^* \colon \operatorname{MF}(\mathcal{W} \oplus \mathcal{W}^{\vee}, w_p^{\mathrm{red}} + q) \to \operatorname{MF}(\widehat{\mathcal{M}}^{\dagger}_{Q_p}(d), w_p) \tag{5-31}$$

is fully faithful with dense image. By Lemma 6.3 and the equivalence (5-30), the left-hand side of (5-31) is idempotent complete, so the functor (5-31) is an equivalence. Therefore we obtain the proposition. □

5.6 Window subcategories

In this subsection, we define several window subcategories for moduli stacks of representations of quivers and their formal fibers discussed in the previous subsection. The notation is summarized in Table 1.

Global window subcategory $\mathbb{W}^{\theta_\pm}_{\mathrm{glob}}(v)$ We take $\theta \in W_m$ and $\theta_\pm$ as in (5-20) which are sufficiently close to the wall W_m. Then the KN stratification of $\mathcal{M}^{\dagger}_{Q}(v)$ for $\chi_{\theta_\pm}$ is finer than those for χ_θ. So we have KN stratifications for $\mathcal{M}^{\dagger,\theta\text{-ss}}_{Q}(v)$ with respect to $\chi_{\theta_\pm}$,

$$\mathcal{M}^{\dagger,\theta\text{-ss}}_{Q}(v) = \mathcal{S}^{\pm}_1 \sqcup \cdots \sqcup \mathcal{S}^{\pm}_{N^\pm} \sqcup \mathcal{M}^{\dagger,\theta_\pm\text{-ss}}_{Q}(v), \tag{5-32}$$

with associated one-parameter subgroups $\lambda^{\pm}_i \colon \mathbb{C}^* \to \operatorname{GL}(V_0) \times \operatorname{GL}(V_1)$ and the associated number $\eta^{\pm}_i \in \mathbb{Z}$ as in (2-6). By Theorem 2.3 (and also noting Lemma 5.4), for each choice of real numbers $m^{\pm}_{\bullet} = \{(m^{\pm}_i)\}_{1 \le i \le N^\pm}$ we have the subcategories

$$\mathbb{W}^{\theta_\pm}_{m^\pm_\bullet}(v) \subset \operatorname{MF}(\mathcal{M}^{\dagger,\theta\text{-ss}}_{Q}(v), w) \tag{5-33}$$

such that the compositions

$$\mathbb{W}^{\theta_\pm}_{m^\pm_\bullet}(v) \hookrightarrow \operatorname{MF}(\mathcal{M}^{\dagger,\theta\text{-ss}}_{Q}(v), w) \twoheadrightarrow \operatorname{MF}(\mathcal{M}^{\dagger,\theta_\pm\text{-ss}}_{Q}(v), w) \tag{5-34}$$

are equivalences. The subcategory (5-33) consists of objects whose $\lambda^{\pm}_i$–weights at each center of $\mathcal{S}^{\pm}_i$ are contained in $[m^{\pm}_i, m^{\pm}_i + \eta^{\pm}_i)$.

We define the character

$$\chi_0 \colon \operatorname{GL}(V_0) \times \operatorname{GL}(V_1) \to \mathbb{C}^*, \quad (g_0, g_1) \mapsto \det(g_0) \cdot \det(g_1)^{-1},$$

ie $\chi_0 = \chi_{(-1,1)}$ in equation (5-5).

	moduli stack	formal fiber	windows
conifold quiver $Q^{\dagger}$	$\mathcal{M}^{\dagger,\theta\text{-ss}}_{Q}(v)$	$\widehat{\mathcal{M}}^{\dagger,\theta\text{-ss}}_{Q}(v)_p$	$\mathbb{W}^{\theta_\pm}_{\mathrm{glob}}(v)$, $\mathbb{W}^{\theta_\pm}_{\mathrm{loc}}(v)_p$
Ext quiver $Q^{\dagger}_p$	$\mathcal{M}^{\dagger}_{Q_p}(d)$	$\widehat{\mathcal{M}}^{\dagger}_{Q_p}(d)$	$\mathbb{W}^{\pm}(d)_p$
reduced Ext quiver $Q^{\mathrm{red},\dagger}_p$	$\mathcal{M}^{\dagger}_{Q_p^{\mathrm{red}}}(d)$	$\widehat{\mathcal{M}}^{\dagger}_{Q_p^{\mathrm{red}}}(d)$	$\mathbb{W}_c(d)_p$

Table 1: Notation of moduli spaces and windows.

As we discussed in (5-22), the composition

$$G_p = \mathrm{GL}(V) \hookrightarrow \mathrm{GL}(V_0)\times \mathrm{GL}(V_1) \xrightarrow{\chi_0} \mathbb{C}^*$$

coincides with the determinant character $\chi_0\colon \mathrm{GL}(V)\to\mathbb{C}^*$. For $m_\bullet^{\pm}$ we use the special choices

$$\begin{aligned} m_i^+ &= -\tfrac{1}{2}\eta_i^+ + \left(\tfrac{1}{2}C_{v,m}+\tfrac{1}{2}m\right)\langle\lambda_i^+,\chi_0\rangle, \\ m_i^- &= -\tfrac{1}{2}\eta_i^- + \tfrac{1}{2}C_{v,m}\langle\lambda_i^-,\chi_0\rangle. \end{aligned} \tag{5-35}$$

Here $C_{v,m}$ is given in (5-17). We then define

$$\mathbb{W}^{\theta_\pm}_{\mathrm{glob}}(v) \subset \mathrm{MF}(\mathcal{M}_Q^{\dagger,\theta\text{-ss}}(v), w) \tag{5-36}$$

to be the window subcategories (5-33) for the choices of $m_\bullet^{\pm}$ as (5-35).

Local window subcategories $\mathbb{W}^{\theta_\pm}_{\mathrm{loc}}(v)_p$ Let us take a θ–polystable object R as in (5-13), and the corresponding closed point $p\in M_Q^{\dagger,\theta\text{-ss}}(v)$. Then we have the diagram of formal fibers (5-19). By restricting the KN stratification (5-32) to the formal fiber, we obtain the KN stratification of $\widehat{\mathcal{M}}_Q^{\dagger,\theta\text{-ss}}(v)_p$

$$\widehat{\mathcal{M}}_Q^{\dagger,\theta\text{-ss}}(v)_p = \widehat{\mathcal{S}}^{\pm}_{1,p}\sqcup\cdots\sqcup\widehat{\mathcal{S}}^{\pm}_{N^\pm,p}\sqcup\widehat{\mathcal{M}}_Q^{\dagger,\theta_\pm\text{-ss}}(v)_p. \tag{5-37}$$

We define local window subcategories

$$\mathbb{W}^{\theta_\pm}_{\mathrm{loc}}(v)_p \subset \mathrm{MF}(\widehat{\mathcal{M}}_Q^{\dagger,\theta\text{-ss}}(v)_p, w_p)$$

similarly to (5-36) as in Theorem 2.3, with respect to the KN stratifications (5-37) and the choices of $m_\bullet^{\pm}$ in (5-35). The following lemma follows immediately from the definition of window subcategories:

Lemma 5.13 *An object $\mathcal{E}\in\mathrm{MF}(\mathcal{M}_Q^{\dagger,\theta\text{-ss}}(v),w)$ is an object in $\mathbb{W}^{\theta_\pm}_{\mathrm{glob}}(v)$ if and only if for any closed point $p\in M_Q^{\dagger,\theta\text{-ss}}(v)$ represented by an object of the form* (5-13) *we have $\mathcal{E}|_{\widehat{\mathcal{M}}_Q^{\dagger,\theta\text{-ss}}(v)_p}\in\mathbb{W}^{\theta_\pm}_{\mathrm{loc}}(v)_p$.*

Proof Since the defining conditions of window subcategories $\mathbb{W}^{\theta_\pm}_{\mathrm{glob}}(v)$ are local on the good moduli space, $\mathcal{E}$ is an object in $\mathbb{W}^{\theta_\pm}_{\mathrm{glob}}(v)$ if and only if $\mathcal{E}|_{\widehat{\mathcal{M}}_Q^{\dagger,\theta\text{-ss}}(v)_p}\in\mathbb{W}^{\theta_\pm}_{\mathrm{loc}}(v)_p$ for any $p\in M_Q^{\dagger,\theta\text{-ss}}(v)$. If p is not represented by an object of the form (5-13), then the formal fiber $\widehat{\mathcal{M}}_Q^{\dagger,\theta\text{-ss}}(v)_p$ does not intersect with the critical locus of w, so $\mathrm{MF}(\widehat{\mathcal{M}}_Q^{\dagger,\theta\text{-ss}}(v)_p,w_p)=0$. □

Window subcategories $\mathbb{W}^{\pm}(d)_p$ for the Ext quiver By pulling the KN stratification (5-37) back to $\widehat{\mathcal{M}}^{\dagger}_{Q_p}(d)$ by the isomorphism η_p in (5-19), we have the KN stratification of $\widehat{\mathcal{M}}^{\dagger}_{Q_p}(d)$ with respect to $\chi_0^{\pm 1}$

$$\widehat{\mathcal{M}}^{\dagger}_{Q_p}(d) = \widetilde{\mathcal{S}}^{\pm}_{1,p}\sqcup\cdots\sqcup\widetilde{\mathcal{S}}^{\pm}_{N^\pm,p}\sqcup\widehat{\mathcal{M}}^{\dagger,\chi_0^{\pm1}}_{Q_p}(d). \tag{5-38}$$

We define window subcategories

$$\mathbb{W}^{\pm}(d)_p\subset\mathrm{MF}(\widehat{\mathcal{M}}^{\dagger}_{Q_p}(d),w_p)$$

as in Theorem 2.3, with respect to the KN stratifications (5-38) and the choices of $m_\bullet^{\pm}$ in (5-35). By the isomorphism η_p in (5-19), we have the equivalence

$$\eta_p^*\colon \mathbb{W}^{\theta_\pm}_{\mathrm{loc}}(v)_p \xrightarrow{\sim} \mathbb{W}^{\pm}(d)_p. \tag{5-39}$$

Window subcategories $\mathbb{W}_c(d)_p$ for the reduced Ext quiver For $c \in \mathbb{Z}_{\geq 0}$, we also define

$$\mathbb{W}_c(d)_p \subset \mathrm{MF}(\widehat{\mathcal{M}}^{\dagger}_{Q_p^{\mathrm{red}}}(d), w_p^{\mathrm{red}})$$

to be the thick closure of matrix factorizations whose entries are of the form $V(\chi) \otimes \mathcal{O}$ for $\chi \in \mathbb{B}_c(d)$, where $\mathbb{B}_c(d)$ is as defined in (4-7). By the description (5-24) of $\mathcal{M}^{\dagger}_{Q_p^{\mathrm{red}}}(d)$ in terms of the stack $\mathcal{G}_{a_{v,m,d}, b_{v,m,d}}(d)$, the argument of Proposition 4.3 (also see the argument of Corollary 4.23) implies that the following composition functors are equivalences:

$$\begin{aligned} &\mathbb{W}_{a_{v,m,d}}(d)_p \hookrightarrow \mathrm{MF}(\widehat{\mathcal{M}}^{\dagger}_{Q_p^{\mathrm{red}}}(d), w_p^{\mathrm{red}}) \twoheadrightarrow \mathrm{MF}(\widehat{\mathcal{M}}^{\dagger, \chi_0\text{-ss}}_{Q_p^{\mathrm{red}}}(d), w_p^{\mathrm{red}}), \\ &\mathbb{W}_{b_{v,m,d}}(d)_p \hookrightarrow \mathrm{MF}(\widehat{\mathcal{M}}^{\dagger}_{Q_p^{\mathrm{red}}}(d), w_p^{\mathrm{red}}) \twoheadrightarrow \mathrm{MF}(\widehat{\mathcal{M}}^{\dagger, \chi_0^{-1}\text{-ss}}_{Q_p^{\mathrm{red}}}(d), w_p^{\mathrm{red}}). \end{aligned} \tag{5-40}$$

5.7 Comparison of window subcategories

We compare the window subcategories in the previous subsection under the Knörrer periodicity:

Proposition 5.14 *The equivalence* (5-29) *restricts to the equivalences*

$$\Phi_p \colon \mathbb{W}_{a_{v,m,d}}(d)_p \otimes \chi_0^{d(m^2-m)} \xrightarrow{\sim} \mathbb{W}^+(d)_p,$$
$$\Phi_p \colon \mathbb{W}_{b_{v,m,d}}(d)_p \otimes \chi_0^{d(m^2-m)} \xrightarrow{\sim} \mathbb{W}^-(d)_p.$$

Proof We only give a proof for the $+$ part. Let $\mathcal{W} \to \widehat{\mathcal{M}}^{\dagger}_{Q_p^{\mathrm{red}}}(d)$ be the vector bundle as in the diagram (5-28). The KN stratifications (5-32) are pullbacks of the KN stratifications

$$\mathcal{W} \oplus \mathcal{W}^{\vee} = \overline{\mathcal{S}}_1^{\pm} \sqcup \cdots \sqcup \overline{\mathcal{S}}_{N^{\pm}}^{\pm} \sqcup (\mathcal{W} \oplus \mathcal{W}^{\vee})^{\chi_0^{\pm 1}\text{-ss}} \tag{5-41}$$

of $\mathcal{W} \oplus \mathcal{W}^{\vee}$ with respect to $\chi_0^{\pm 1}$ by the morphism ι in (5-28). We denote by

$$\overline{\mathbb{W}}^{\pm}(d)_p \subset \mathrm{MF}(\mathcal{W} \oplus \mathcal{W}^{\vee}, w_p^{\mathrm{red}} + q)$$

the window subcategories with respect to the above stratifications (5-41) and $m_{\bullet}^{\pm} \in \mathbb{R}$ given by (5-35). By the definition of the above window subcategories, the equivalence (5-31) restricts to the equivalence

$$\iota^* \colon \overline{\mathbb{W}}^{\pm}(d)_p \xrightarrow{\sim} \mathbb{W}^{\pm}(d)_p.$$

Therefore it is enough to show that the equivalence (5-30) restricts to the equivalence

$$i_* \operatorname{pr}^* \colon \mathbb{W}_{a_{v,m,d}}(d)_p \otimes \chi_0^{d(m^2-m)} \xrightarrow{\sim} \overline{\mathbb{W}}^+(d)_p.$$

We have the commutative diagram

$$\begin{tikzcd} \mathbb{W}_{a_{v,m,d}}(d)_p \otimes \chi_0^{d(m^2-m)} \arrow[r, hook] \arrow[d, dotted] & \mathrm{MF}(\widehat{\mathcal{M}}^{\dagger}_{Q_p^{\mathrm{red}}}(d), w_p^{\mathrm{red}}) \arrow[r] \arrow[d, "i_* \operatorname{pr}^*"', "\sim"] & \mathrm{MF}(\widehat{\mathcal{M}}^{\dagger, \chi_0\text{-ss}}_{Q_p^{\mathrm{red}}}(d), w_p^{\mathrm{red}}) \arrow[d, dotted, "\sim"] \\ \overline{\mathbb{W}}^+(d)_p \arrow[r, hook] & \mathrm{MF}(\mathcal{W} \oplus \mathcal{W}^{\vee}, w_p^{\mathrm{red}} + q) \arrow[r] & \mathrm{MF}((\mathcal{W} \oplus \mathcal{W}^{\vee})^{\chi_0\text{-ss}}, w_p^{\mathrm{red}} + q) \end{tikzcd} \tag{5-42}$$

The composition of top arrows is an equivalence by the equivalence in (5-40), and that of bottom arrows is also an equivalence by Theorem 2.3. We see that the middle vertical arrow descends to an equivalence of the right vertical dotted arrow. Note that we have the isomorphism

$$\mathrm{Crit}(w_p^{\mathrm{red}}) \cap \widehat{\mathcal{M}}^{\dagger,\chi_0\text{-ss}}_{Q_p^{\mathrm{red}}} \xhookrightarrow{\cong} \mathrm{Crit}(w_p^{\mathrm{red}}+q) \cap (\mathcal{W} \oplus \mathcal{W}^\vee)^{\chi_0\text{-ss}}$$

induced by the zero section $\widehat{\mathcal{M}}^{\dagger}_{Q_p^{\mathrm{red}}}(d) \hookrightarrow \mathcal{W} \oplus \mathcal{W}^\vee$. In particular, we have the inclusion

$$\mathrm{Crit}(w_p^{\mathrm{red}}+q) \cap (\mathcal{W} \oplus \mathcal{W}^\vee)^{\chi_0\text{-ss}} \subset (\mathcal{W} \oplus \mathcal{W}^\vee) \times_{\widehat{\mathcal{M}}^{\dagger}_{Q_p^{\mathrm{red}}}(d)} \widehat{\mathcal{M}}^{\dagger,\chi_0\text{-ss}}_{Q_p^{\mathrm{red}}}(d). \tag{5-43}$$

The desired equivalence is given by the composition

$$\begin{aligned}\mathrm{MF}(\widehat{\mathcal{M}}^{\dagger,\chi_0\text{-ss}}_{Q_p^{\mathrm{red}}}(d), w_p^{\mathrm{red}}) &\xrightarrow{\sim} \mathrm{MF}\big((\mathcal{W} \oplus \mathcal{W}^\vee) \times_{\widehat{\mathcal{M}}^{\dagger}_{Q_p^{\mathrm{red}}}(d)} \widehat{\mathcal{M}}^{\dagger,\chi_0\text{-ss}}_{Q_p^{\mathrm{red}}}(d), w_p^{\mathrm{red}}+q\big) \\ &\xrightarrow{\sim} \mathrm{MF}((\mathcal{W} \oplus \mathcal{W}^\vee)^{\chi_0\text{-ss}}, w_p^{\mathrm{red}}+q).\end{aligned}$$

Here the first equivalence is Knörrer periodicity in Theorem 2.4, and the second equivalence follows from (5-43) and the equivalence (2-1).

Therefore it is enough to show that the middle vertical arrow in (5-42) restricts to the left dotted arrow, ie for $\mathcal{P} \in \mathbb{W}_{a_{v,m,d}}(d)_p \otimes \chi_0^{d(m^2-m)}$, we show that the object $i_* \,\mathrm{pr}^*(\mathcal{P})$ lies in $\overline{\overline{\mathbb{W}}}{}^+(d)_p$. Note that the critical locus of $w_p^{\mathrm{red}}+q$ lies in the zero section $\widehat{\mathcal{M}}^{\dagger}_{Q_p^{\mathrm{red}}}(d) \subset \mathcal{W} \oplus \mathcal{W}^\vee$. From Theorem 2.3, it is enough to show that $i_* \,\mathrm{pr}^*(\mathcal{P})$ satisfies the condition (2-9) for one-parameter subgroups which appear in the KN stratification of $\widehat{\mathcal{M}}^{\dagger}_{Q_p^{\mathrm{red}}}(d)$. From the description (5-24) of $\mathcal{M}^{\dagger}_{Q_p^{\mathrm{red}}}(d)$, its KN stratifications with respect to $\chi_0^{\pm 1}$ are KN stratifications of $\mathcal{G}_{a_{v,m,d},b_{v,m,d}}(d)$ discussed in Section 4.1, up to a product with a trivial factor. Therefore they are of the form

$$\widehat{\mathcal{M}}^{\dagger}_{Q_p^{\mathrm{red}}}(d) = \mathcal{S}_0^{\pm} \sqcup \cdots \sqcup \mathcal{S}_{d-1}^{\pm} \sqcup \widehat{\mathcal{M}}^{\dagger,\chi_0^{\pm 1}\text{-ss}}_{Q_p^{\mathrm{red}}}(d)$$

such that each associated one-parameter subgroup $\lambda_i^{\pm} : \mathbb{C}^* \to G_p = \mathrm{GL}(V)$ is given by (4-6), ie λ_i^+ is

$$\lambda_i^+(t) = (\overbrace{1, \ldots, 1}^{i}, \overbrace{t^{-1}, \ldots, t^{-1}}^{d-i}). \tag{5-44}$$

Therefore in order to show that the object $i_* \,\mathrm{pr}^*(\mathcal{P})$ lies in $\overline{\overline{\mathbb{W}}}{}^+(d)_p$, it is enough to check the weight conditions (2-9) for the above λ_i^+.

Since the object $i_* \,\mathrm{pr}^*(\mathcal{P})$ is given by taking the tensor product with the Koszul factorization (2-11), it is isomorphic to a direct summand of a matrix factorization whose entries are of the form

$$V(\chi) \otimes \textstyle\bigwedge^k W \otimes \chi_0^{d(m^2-m)} \otimes \mathcal{O}, \quad \text{where } \chi \in \mathbb{B}_{a_{v,m,d}}(d) \text{ and } 0 \le k \le \dim W.$$

For each one-parameter subgroup $\lambda : \mathbb{C}^* \to G_p$, we set

$$\gamma_\lambda := \langle \lambda, W^{\lambda>0} \rangle = -\langle \lambda, W^{\lambda<0} \rangle,$$

where the second identity holds as $W = \mathrm{End}_0(V) \otimes \mathbb{C}^{m^2-m}$ is a self-dual G_p–representation. Then we have the following inclusions of the set of λ_i^+–weights of $V(\chi) \otimes \bigwedge^k W \otimes \chi_0^{d(m^2-m)}$:

$$\begin{aligned}
&\mathrm{wt}_{\lambda_i^+}\big(V(\chi) \otimes \textstyle\bigwedge^k W \otimes \chi_0^{d(m^2-m)}\big) \\
&\qquad \subset \bigcup_{\chi' \in \mathrm{wt}(V(\chi))} \Big[-\sum_{j=i+1}^{d} x_j' - (d-i)\cdot d(m^2-m) - \gamma_{\lambda_i^+}, -\sum_{j=i+1}^{d} x_j' - (d-i)\cdot d(m^2-m) + \gamma_{\lambda_i^+}\Big] \\
&\qquad \subset [(d-i)(-a_{v,m,d} + d - dm^2 + dm) - \gamma_{\lambda_i^+}, (d-i)(-dm^2+dm) + \gamma_{\lambda_i^+}].
\end{aligned}$$

Here $\mathrm{wt}(V(\chi))$ is the set of T–weights of $V(\chi)$ for the maximal torus $T \subset G_p$, and we have written $\chi' \in \mathrm{wt}(V(\chi))$ as $\chi' = (x_1', \dots, x_d')$ satisfying $0 \le x_j' \le a_{v,m,d} - d$.

We show that the above set of weights is contained in $[m_i^+, m_i^+ + \eta_i^+)$. From the decomposition (5-16), the $\eta_i^+ \in \mathbb{Z}$ which appear in (5-35) for the one-parameter subgroup (5-44) are calculated as in the proof of Proposition 4.3:

$$\begin{aligned}
\eta_i^+ &= \langle \lambda_i^+, (\mathrm{Ext}^1_{Q^\dagger}(R,R)^\vee)^{\lambda_i^+>0} - (\mathfrak{g}_p^\vee)^{\lambda_i^+>0}\rangle \\
&= \langle \lambda_i^+, ((V^\vee)^{\oplus a_{v,m,d}} \oplus V^{\oplus b_{b,m,d}} \oplus W \oplus W^\vee)^{\lambda_i^+>0} - \mathrm{End}(V)^{\lambda_i^+>0}\rangle \\
&= (a_{v,m,d} - i)(d-i) + 2\gamma_{\lambda_i^+}.
\end{aligned}$$

Here $\mathfrak{g}_p = \mathrm{End}(V)$ is the Lie algebra of $G_p = \mathrm{GL}(V)$. Therefore we have

$$\begin{aligned}
&[m_i^+, m_i^+ + \eta_i^+) \\
&\quad = \big[-\tfrac{1}{2}\eta_i^+ + \big(\tfrac{1}{2}C_{v,m} + \tfrac{1}{2}m\big)\langle \lambda_i^+, \chi_0\rangle, \tfrac{1}{2}\eta_i^+ + \big(\tfrac{1}{2}C_{v,m} + \tfrac{1}{2}m\big)\langle \lambda_i^+, \chi_0\rangle\big) \\
&\quad = \big[(d-i)\big(-a_{v,m,d} + \tfrac{1}{2}i + \tfrac{1}{2}d - dm^2 + dm\big) - \gamma_{\lambda_i^+}, (d-i)\big(-dm^2 + dm + \tfrac{1}{2}d - \tfrac{1}{2}i\big) + \gamma_{\lambda_i^+}\big).
\end{aligned}$$

Since $0 \le i \le d-1$, we conclude the inclusion

$$\mathrm{wt}_{\lambda_i^+}\big(V(\chi) \otimes \textstyle\bigwedge^k W \otimes \chi_0^{d(m^2-m)}\big) \subset [m_i^+, m_i^+ + \eta_i^+).$$

Therefore the weight condition (2-9) for $i_* \mathrm{pr}^* \mathcal{P}$ with respect to λ_i^+ is satisfied. ☐

Let $s_m = (m, m-1)$ be the dimension vector of the stable Q–representation S_m, defined in (5-7). Let $q_m \in M_Q^{\theta\text{-ss}}(s_m)$ be the corresponding closed point. We consider the formal fiber of the good moduli space morphism (5-9) at q_m

$$\widehat{\mathcal{M}}_Q^{\theta\text{-ss}}(s_m) \to \widehat{M}_Q^{\theta\text{-ss}}(s_m).$$

Similarly to (5-15), the étale slice theorem implies an isomorphism

$$\widehat{\mathcal{M}}_{Q_p}(1) = [\widehat{\mathrm{Ext}}^1_Q(S_m, S_m)/\mathrm{Aut}(S_m)] \xrightarrow{\cong} \widehat{\mathcal{M}}_Q^{\theta\text{-ss}}(s_m). \tag{5-45}$$

Here $\mathrm{Aut}(S_m) = \mathbb{C}^*$ acts on $\mathrm{Ext}^1_Q(S_m, S_m)$ trivially. We will also use the following lemma, which compares window subcategories for quivers without framings.

Lemma 5.15 *For any $j \in \mathbb{Z}$, we have equivalences*

$$\mathrm{MF}(\widehat{\mathcal{M}}_{Q_p^{\mathrm{red}}}(1), w_p^{\mathrm{red}})_j \xrightarrow{\sim} \mathrm{MF}(\widehat{\mathcal{M}}_{Q_p}(1), w_p)_j \xrightarrow{\sim} \mathrm{MF}(\widehat{\mathcal{M}}_Q^{\theta\text{-ss}}(s_m), w)_j, \tag{5-46}$$

and all of them are equivalent to $\mathrm{MF}(\operatorname{Spec}\mathbb{C}, 0)$*. Here the first equivalence is given by the Knörrer periodicity in Theorem 2.4.*

Proof By the definition of Q_p^{red} in (5-23), we have $(\widehat{\mathcal{M}}_{Q_p^{\mathrm{red}}}(1), w_p^{\mathrm{red}}) = (B\mathbb{C}^*, 0)$. On the other hand, the isomorphisms (5-25), (5-45) and an argument of Proposition 5.11 imply an isomorphism

$$(\widehat{\mathcal{M}}_{Q_p}(1), w_p) \cong \big([\widehat{(H \oplus H^\vee)}/\mathbb{C}^*], q\big), \tag{5-47}$$

where $\mathbb{C}^*$ acts on $H = \mathbb{C}^{m^2-m}$ trivially and q is the natural paring on H and its dual. By the Knörrer periodicity in Theorem 2.4, we have an equivalence

$$\mathrm{MF}(\widehat{\mathcal{M}}_{Q_p^{\mathrm{red}}}(1), w_p^{\mathrm{red}})_j = \mathrm{MF}(\operatorname{Spec}\mathbb{C}, 0) \xrightarrow{\sim} \mathrm{MF}(H \oplus H^\vee, q).$$

The natural functor by the formal completion

$$\mathrm{MF}(H \oplus H^\vee, q) \to \mathrm{MF}(\widehat{H \oplus H^\vee}, q)$$

is an equivalence; see [Brown 2016, Remark 2.18]. Therefore we obtain the desired equivalences (5-46). □

5.8 Comparison of Hall products

As in the previous subsections, we take a stability condition on the wall $\theta \in W_m$ for $m \geq 1$. As in Section 3.3, we have the categorified Hall product

$$\mathrm{MF}(\mathcal{M}_Q^{\theta\text{-ss}}(s_m), w)_{j_1} \boxtimes \cdots \boxtimes \mathrm{MF}(\mathcal{M}_Q^{\theta\text{-ss}}(s_m), w)_{j_l} \boxtimes \mathrm{MF}(\mathcal{M}_Q^{\dagger,\theta\text{-ss}}(v - ls_m), w) \to \mathrm{MF}(\mathcal{M}_Q^{\dagger,\theta\text{-ss}}(v), w). \tag{5-48}$$

Here $s_m = (m, m-1)$ is the dimension vector of S_m. We take a θ–polystable representation $Q^\dagger$–representation R of the form (5-13), ie $R = R_\infty \oplus (V \otimes S_m)$ with $\dim V = d$, and the corresponding closed point $p \in M_Q^{\dagger,\theta\text{-ss}}(v)$. By taking the base change of the above categorified Hall product to the formal completion at p (see Section 3.4), we obtain the functor

$$\mathrm{MF}(\widehat{\mathcal{M}}_Q^{\theta\text{-ss}}(s_m), w)_{j_1} \boxtimes \cdots \boxtimes \mathrm{MF}(\widehat{\mathcal{M}}_Q^{\theta\text{-ss}}(s_m), w)_{j_l} \boxtimes \mathrm{MF}(\widehat{\mathcal{M}}_Q^{\dagger,\theta\text{-ss}}(v - ls_m)_{p_l}, w) \to \mathrm{MF}(\widehat{\mathcal{M}}_Q^{\dagger,\theta\text{-ss}}(v)_p, w). \tag{5-49}$$

Here $p_l \in M_Q^{\dagger,\theta\text{-ss}}(v - ls_m)$ corresponds to the θ–polystable representation $R_\infty \oplus (V' \otimes S_m)$ with $\dim V' = d - l$. We note that by the isomorphism η_p in (5-19) and the isomorphism (5-45), the above functor is identified with the functor

$$\mathrm{MF}(\widehat{\mathcal{M}}_{Q_p}(1), w_p)_{j_1} \boxtimes \cdots \boxtimes \mathrm{MF}(\widehat{\mathcal{M}}_{Q_p}(1), w_p)_{j_l} \boxtimes \mathrm{MF}(\widehat{\mathcal{M}}_{Q_p}^\dagger(d-l), w_p) \to \mathrm{MF}(\widehat{\mathcal{M}}_{Q_p}^\dagger(d), w_p) \tag{5-50}$$

obtained by the categorified Hall products for $Q_p^\dagger$–representations and the completions at the origins.

A similar construction also gives the categorified Hall product for $Q_p^{\mathrm{red},\dagger}$–representations

$$\mathrm{MF}(\widehat{\mathcal{M}}_{Q_p^{\mathrm{red}}}(1),0)_{j_1}\boxtimes\cdots\boxtimes\mathrm{MF}(\widehat{\mathcal{M}}_{Q_p^{\mathrm{red}}}(1),0)_{j_l}\boxtimes\mathrm{MF}(\widehat{\mathcal{M}}^{\dagger}_{Q_p^{\mathrm{red}}}(d-l),w_p^{\mathrm{red}})\to\mathrm{MF}(\widehat{\mathcal{M}}^{\dagger}_{Q_p^{\mathrm{red}}}(d),w_p^{\mathrm{red}}). \tag{5-51}$$

We compare the above categorified Hall products under the Knörrer periodicity:

Proposition 5.16 *The following diagram commutes*:

$$\begin{tikzcd}
\boxtimes_{i=1}^{l}\mathrm{MF}(\widehat{\mathcal{M}}_{Q_p^{\mathrm{red}}}(1),0)_{j_i}\boxtimes\mathrm{MF}(\widehat{\mathcal{M}}^{\dagger}_{Q_p^{\mathrm{red}}}(d-l),w_p^{\mathrm{red}}) \arrow[r] \arrow[d] & \mathrm{MF}(\widehat{\mathcal{M}}^{\dagger}_{Q_p^{\mathrm{red}}}(d),w_p^{\mathrm{red}}) \arrow[d] \\
\boxtimes_{i=1}^{l}\mathrm{MF}(\widehat{\mathcal{M}}_{Q_p}(1),w_p)_{j_i+(2i-d-1)(m^2-m)}\boxtimes\mathrm{MF}(\widehat{\mathcal{M}}^{\dagger}_{Q_p}(d-l),w_p) \arrow[r] & \mathrm{MF}(\widehat{\mathcal{M}}^{\dagger}_{Q_p}(d),w_p)
\end{tikzcd} \tag{5-52}$$

Here the horizontal arrows are given by categorized Hall products (5-50) *and* (5-51), *the right vertical arrow is given in Proposition 5.12, and the left vertical arrow is a composition of the functors in Proposition 5.12 and Lemma 5.15 with the equivalence*

$$\begin{aligned}\boxtimes_{i=1}^{l}\otimes\mathcal{O}_{B\mathbb{C}^*}((2i-d-1)(m^2-m))\boxtimes\otimes\chi_0^{l(m^2-m)}\big[\big(dl-\tfrac{1}{2}l-\tfrac{1}{2}l^2\big)(m^2-m)\big]:&\\ \boxtimes_{i=1}^{l}\mathrm{MF}(\widehat{\mathcal{M}}_{Q_p^{\mathrm{red}}}(1),0)_{j_i}\boxtimes\mathrm{MF}(\widehat{\mathcal{M}}^{\dagger}_{Q_p^{\mathrm{red}}}(d-l),w_p^{\mathrm{red}})&\\ \xrightarrow{\sim}\boxtimes_{i=1}^{l}\mathrm{MF}(\widehat{\mathcal{M}}_{Q_p^{\mathrm{red}}}(1),0)_{j_i+(2i-d-1)(m^2-m)}\boxtimes\mathrm{MF}(\widehat{\mathcal{M}}^{\dagger}_{Q_p^{\mathrm{red}}}(d-l),w_p^{\mathrm{red}}).&\end{aligned} \tag{5-53}$$

Proof We take $\lambda\colon\mathbb{C}^*\to G_p=\mathrm{GL}(V)$ by

$$\lambda(t)=(t^l,t^{l-1},\dots,t,\overbrace{1,\dots,1}^{d-l}).$$

Then the top arrow in the diagram (5-52) is obtained from the diagram of attracting loci for $\widehat{\mathcal{M}}^{\dagger}_{Q_p^{\mathrm{red}}}(d)$ with respect to the above λ. In the diagram (5-28), the vector bundle $\mathcal{W}\to\widehat{\mathcal{M}}^{\dagger}_{Q_p^{\mathrm{red}}}(d)$ is induced by the $\mathrm{GL}(V)$–representation $W=\mathrm{End}_0(V)\otimes H$ for $H=\mathbb{C}^{m^2-m}$ by its definition. By Proposition 2.6, the categorified Hall products in (5-52) commute with Knörrer periodicity equivalences up to equivalence:

$$\begin{aligned}\otimes\det(\mathrm{End}_0(V)^{\lambda>0}\otimes H)^{\vee}[\dim(\mathrm{End}_0(V)^{\lambda>0}\otimes H)]:&\\ \boxtimes_{i=1}^{l}\mathrm{MF}(\widehat{\mathcal{M}}_{Q_p^{\mathrm{red}}}(1),0)\boxtimes\mathrm{MF}(\widehat{\mathcal{M}}^{\dagger}_{Q_p^{\mathrm{red}}}(d-l),w_p^{\mathrm{red}})&\\ \xrightarrow{\sim}\boxtimes_{i=1}^{l}\mathrm{MF}(\widehat{\mathcal{M}}_{Q_p^{\mathrm{red}}}(1),0)\boxtimes\mathrm{MF}(\widehat{\mathcal{M}}^{\dagger}_{Q_p^{\mathrm{red}}}(d-l),w_p^{\mathrm{red}}).&\end{aligned} \tag{5-54}$$

It is enough to show that the equivalence (5-54) restricts to the equivalence (5-53). Let $V=\bigoplus_{i=0}^{l}V_i$ be the decomposition into λ–weight parts, ie V_i has λ–weight i so that $\dim V_i=1$ for $1\le i\le l$ and $\dim V_0=d-l$. We have

$$\mathrm{End}_0(V,V)^{\lambda>0}=\Big(\bigoplus_{0\le i<j\le l}V_i^{\vee}\otimes V_j\Big).$$

We compute that

$$\det(\mathrm{End}_0(V,V)^{\lambda>0})^\vee = \bigotimes_{0\le i<j\le l} \det(V_i\otimes V_j^\vee) = \bigotimes_{1\le j\le l}\det(V_0\otimes V_j^\vee)\otimes \bigotimes_{1\le i<j\le l}\det(V_i\otimes V_j^\vee)$$
$$= (\det V_0)^l \otimes \bigotimes_{i=1}^{l} (\det V_i)^{2l-2i+1-d}.$$

We note that $\otimes \det V_i = \otimes \mathcal{O}_{B\mathbb{C}^*}(1)$ on the factor $\mathrm{MF}(\widehat{\mathcal{M}}_{Q_p^{\mathrm{red}}}(1),0)_{j_{l-i+1}}$. We also have

$$\dim \mathrm{End}_0(V,V)^{\lambda>0} = dl - \tfrac{1}{2}l - \tfrac{1}{2}l^2.$$

Therefore the equivalence (5-54) restricts to the equivalence (5-53). □

5.9 Semiorthogonal decomposition of global window subcategories

The following is the main result in this section:

Theorem 5.17 *For $l\ge 0$ and $0\le j_1\le\cdots\le j_l\le m-l$, the categorified Hall product* (5-48) *restricts to the fully faithful functor*

$$\Upsilon_{j_\bullet}\colon \boxtimes_{i=1}^{l} \mathrm{MF}(\mathcal{M}_Q^{\theta\text{-ss}}(s_m),w)_{j_i+(2i-1)(m^2-m)} \boxtimes \left(\mathbb{W}_{\mathrm{glob}}^{\theta-}(v-ls_m)\otimes \chi_0^{j_l+2l(m^2-m)}\right) \to \mathbb{W}_{\mathrm{glob}}^{\theta+}(v) \tag{5-55}$$

such that, by setting $\mathcal{C}_{j_\bullet}$ to be the essential image of the above functor $\Upsilon_{j_\bullet}$, we have the semiorthogonal decomposition

$$\mathbb{W}_{\mathrm{glob}}^{\theta+}(v) = \langle \mathcal{C}_{j_\bullet} : l\ge 0, 0\le j_1\le\cdots\le j_l\le m-l\rangle, \tag{5-56}$$

where $\mathrm{Hom}(\mathcal{C}_{j_\bullet},\mathcal{C}_{j'_\bullet})=0$ *for* $j_\bullet \succ j'_\bullet$ (see Definition 4.16).

Proof We take a θ–polystable representation R of the form (5-13), ie $R=R_\infty\oplus(V\otimes S_m)$ with $\dim V=d$, the corresponding closed point $p\in M_Q^{\dagger,\theta\text{-ss}}(v)$, and consider the quivers $Q_p^\dagger$, $Q_p^{\dagger,\mathrm{red}}$ as in the previous subsections. Note that if we remove the loops at the vertex $\{\infty\}$ from $Q_p^\dagger$, then we obtain the quiver $Q_{a,b}$ for $a=a_{v,m,d}$ and $b=b_{v,m,d}$ considered in Remark 4.1. By applying Corollary 4.23 for the above $Q_{a,b}$, and then taking the tensor product with $\chi_0^{d(m^2-m)}$, we obtain the semiorthogonal decomposition

$$\mathbb{W}_{a_{v,m,d}}(d)_p\otimes\chi_0^{d(m^2-m)} = \left\langle \boxtimes_{i=1}^{l} \mathrm{MF}(\widehat{\mathcal{M}}_{Q_p^{\mathrm{red}}}(1),0)_{j_i+d(m^2-m)} \boxtimes \left(\mathbb{W}_{b_{v,m,d}}(d-l)_{p_l}\otimes \chi_0^{(d-l)(m^2-m)}\otimes\chi_0^{j_l+l(m^2-m)}\right)\right\rangle.$$

Here $l\ge 0$, $p_l\in M_Q^{\dagger,\theta\text{-ss}}(v-ls_m)$ corresponds to $R_\infty\oplus(V'\otimes S_m)$ with $\dim V'=d-l$, and

$$0\le j_1\le\cdots\le j_l\le a_{v,m,d}-b_{v,m,d}-l=m-l. \tag{5-57}$$

Applying Proposition 5.14, Lemma 5.15 and Proposition 5.16, we get the semiorthogonal decomposition

$$\mathbb{W}^{+}(d)_p = \left\langle \boxtimes_{i=1}^{l} \mathrm{MF}(\widehat{\mathcal{M}}_{Q_p}(1), w_p)_{j_i+(2i-1)(m^2-m)} \boxtimes \left(\mathbb{W}^{-}(d-l)_{p_l} \otimes \chi_0^{j_l+2l(m^2-m)}\right)\right\rangle.$$

By the identification of categorified Hall products (5-49) with (5-50) together with the equivalence (5-39), we obtain the semiorthogonal decomposition

$$\mathbb{W}^{\theta+}_{\mathrm{loc}}(v)_p = \left\langle \boxtimes_{i=1}^{l} \mathrm{MF}(\widehat{\mathcal{M}}^{\theta\text{–ss}}_{Q}(s_m), w)_{j_i+(2i-1)(m^2-m)} \boxtimes \left(\mathbb{W}^{\theta-}_{\mathrm{loc}}(v-ls_m)_{p_l} \otimes \chi_0^{j_l+2l(m^2-m)}\right)\right\rangle. \tag{5-58}$$

A key observation is that in the above semiorthogonal decomposition there is no term involving $d=\dim V$ (which depends on a choice of θ–polystable object (5-13)), so we can globalize it. Indeed we have globally defined functors (5-55) and, noting Lemma 5.13, in order to show that they are fully faithful and forms a semiorthogonal decomposition it is enough to check these properties formally locally at each closed point of $M^{\dagger,\theta\text{–ss}}_{Q}(v)$ corresponding to a θ–polystable $(Q^{\dagger}, W)$–representation; see the arguments in [Toda 2021, Proposition 6.9, Theorem 6.11], for example.

Here we give some more details for how to derive the global semiorthogonal decomposition (5-56) from the formal local one (5-58). We first note that the categorified Hall product (5-48) restricts to the functor (5-55). This follows from the fact that the categorified Hall products commute with base change to the formal completion of good moduli spaces (see the diagram (3-14)), the fact (which follows from (5-58)) that formally locally over $M^{\dagger,\theta\text{–ss}}_{Q}(v)$ the categorified Hall product restricts to the functor

$$\boxtimes_{i=1}^{l} \mathrm{MF}(\widehat{\mathcal{M}}^{\theta\text{–ss}}_{Q}(s_m), w)_{j_i+(2i-1)(m^2-m)} \boxtimes \left(\mathbb{W}^{\theta-}_{\mathrm{loc}}(v-ls_m)_{p_l} \otimes \chi_0^{j_l+2l(m^2-m)}\right) \to \mathbb{W}^{\theta+}_{\mathrm{loc}}(v)_p,$$

and noting Lemma 5.13.

By Lemma 6.6 below, the functor $\Upsilon_{j_\bullet}$ admits a right adjoint $\Upsilon^{R}_{j_\bullet}$. Now in order to show that $\Upsilon_{j_\bullet}$ is fully faithful, it is enough to show that the adjunction morphism

$$(-) \to \Upsilon^{R}_{j_\bullet} \circ \Upsilon_{j_\bullet}(-)$$

is an isomorphism. Equivalently, it is enough to show that the cone of the above morphism is zero. By Lemma 6.5, this is a property formally locally over $M^{\dagger,\theta\text{–ss}}_{Q}(v)$. So from the semiorthogonal decomposition (5-58) we conclude that $\Upsilon_{j_\bullet}$ is fully faithful. A similar argument also shows that $\mathscr{C}_{j_\bullet}$ for $j_\bullet$ given in (5-57) are semiorthogonal.

In order to show that $\mathscr{C}_{j_\bullet}$ for $j_\bullet$ given in (5-57) generate $\mathbb{W}^{\theta+}_{\mathrm{glob}}(v)$, let us take $\mathscr{E} \in \mathbb{W}^{\theta+}_{\mathrm{glob}}(v)$ and $j_\bullet$ so that $j_\bullet$ is maximal in the order of Definition 4.16. We have the distinguished triangle

$$\Upsilon_{j_\bullet}\Upsilon^{R}_{j_\bullet}(\mathscr{E}) \to \mathscr{E} \to \mathscr{E}', \quad \text{where } \mathscr{E}' \in \mathscr{C}^{\perp}_{j_\bullet}.$$

By applying the above construction for $\mathscr{E}'$ and the second maximal $j_\bullet$ and repeating, we obtain the distinguished triangle

$$\mathscr{E}_1 \to \mathscr{E} \to \mathscr{E}_2, \quad \text{where } \mathscr{E}_1 \in \langle \mathscr{C}_{j_\bullet}\rangle \text{ and } \mathscr{E}_2 \in \langle \mathscr{C}_{j_\bullet}\rangle^{\perp}.$$

Here $\langle \mathcal{C}_{j_\bullet} \rangle$ is the right-hand side of (5-56). From the semiorthogonal decomposition (5-58), we have $\mathcal{E}_2|_{\widehat{\mathcal{M}}_Q^{\dagger,\theta\text{-ss}}(v)_p} = 0$ for any closed point $p \in M_Q^{\dagger,\theta\text{-ss}}(v)$, therefore $\mathcal{E}_2 = 0$ by Lemma 6.5. Therefore $\mathcal{E} \in \langle \mathcal{C}_{j_\bullet} \rangle$, and we have the desired semiorthogonal decomposition (5-56). $\square$

The following corollary, which is an immediate consequence from Theorem 5.17, categorifies wall-crossing formula of the associated DT invariants in [Nagao and Nakajima 2011].

Corollary 5.18 *There exists a semiorthogonal decomposition of the form*

$$\mathrm{MF}(\mathcal{M}_Q^{\dagger,\theta+}(v), w) = \langle \mathrm{MF}(\mathcal{M}_Q^{\dagger,\theta-}(v - l s_m), w)_{j_\bullet} : l \geq 0, 0 \leq j_1 \leq \cdots \leq j_l \leq m - l \rangle.$$

Here $\mathrm{MF}(\mathcal{M}_Q^{\dagger,\theta-}(v - l s_m), w)_{j_\bullet}$ *is a copy of* $\mathrm{MF}(\mathcal{M}_Q^{\dagger,\theta-}(v - l s_m), w)$.

Proof By the equivalences (5-34), the left-hand side of (5-56) is equivalent to $\mathrm{MF}(\mathcal{M}_Q^{\dagger,\theta+}(v), w)$. On the other hand, the subcategory $\mathcal{C}_{j_\bullet}$ in (5-56) is equivalent to $\mathrm{MF}(\mathcal{M}_Q^{\dagger,\theta-}(v - l s_m), w)$ by the equivalences (5-34) together with Lemma 5.7. $\square$

Remark 5.19 The semiorthogonal decomposition in Corollary 5.18 recovers the numerical wall-crossing formula (1-4). Indeed the periodic cyclic homologies are additive with respect to semiorthogonal decompositions [Tabuada 2005, Theorem 6.3, Section 6.1], so we have

$$\mathrm{HP}_*(\mathrm{MF}(\mathcal{M}_Q^{\dagger,\theta+\text{-ss}}(v), w)) = \bigoplus_{l \geq 0} \mathrm{HP}_*(\mathrm{MF}(\mathcal{M}_Q^{\dagger,\theta-\text{-ss}}(v - l s_m), w))^{\oplus \binom{m}{l}}.$$

By taking the Euler characteristics and using Lemma 5.1, we obtain the formula (1-4).

By applying Corollary 5.18 from the empty chamber in Figure 1 to the wall-crossing at W_m, and noting Lemma 5.6, we obtain the following.

Corollary 5.20 *For* $\theta \in W_m$, *there exists a semiorthogonal decomposition*

$$\mathrm{MF}(\mathcal{M}_Q^{\dagger,\theta+}(v), w) = \langle \mathcal{C}_{j_\bullet^{(*)}} \rangle.$$

Here each $\mathcal{C}_{j_\bullet^{(*)}}$ *is equivalent to* $\mathrm{MF}(\operatorname{Spec} \mathbb{C}, 0)$ *and* $j_\bullet^{(*)}$ *is a collection of nonpositive integers of the form*

$$j_\bullet^{(*)} = \{(0 \leq j_1^{(i)} \leq \cdots \leq j_{l_i}^{(i)} \leq i - l_i)\}_{1 \leq i \leq m}$$

for some integers $l_i \geq 0$ *satisfying*

$$(v_0, v_1) = \sum_{i=1}^{m} l_i \cdot (i, i-1).$$

We have $\mathrm{Hom}(\mathcal{C}_{j_\bullet^{(*)}}, \mathcal{C}_{j_\bullet'^{(*)}}) = 0$ *if* $j_\bullet^{(i)} = j_\bullet'^{(i)}$ *for* $k < i \leq m$ *for some* k *and* $j_\bullet^{(k)} \succ j_\bullet'^{(k)}$.

Proof Let $\theta_{\mathrm{en}} \in \mathbb{R}^2$ lie in the empty chamber in Figure 1. By Lemma 5.6, a successive application of Corollary 5.18 gives the semiorthogonal decomposition

$$\mathrm{MF}(\mathcal{M}_Q^{\dagger,\theta+}(v), w) = \langle \mathrm{MF}(\mathcal{M}_Q^{\dagger,\theta_{\mathrm{en}}}(v - l_m s_m - l_{m-1} s_{m-1} - \cdots - l_1 s_1), w)_{j_\bullet^{(m)}, j_\bullet^{(m-1)}, \ldots, j_\bullet^{(1)}} \rangle.$$

Here $l_i \geq 0$ are integers and $0 \leq j_1^{(1)} \leq \cdots \leq j_{l_i}^{(i)} \leq i - l_i$ for $1 \leq i \leq m$. By applying Lemma 5.5, we obtain the corollary. □

Remark 5.21 The arguments of Theorem 5.17 and Corollary 5.18 work for other walls except walls at $\{\theta_0 + \theta_1 = 0\}$. For example, let us consider, in Figure 1, the wall

$$W_m' := \mathbb{R}_{>0}(-m-1, m), \quad \text{where } m \in \mathbb{Z}_{\geq 0}.$$

Then for $\theta \in W_m'$, there is a unique θ–stable (Q, W)–representation S_m' of dimension vector $s_m' = (m, m+1)$, which corresponds to $\mathcal{O}_C(-m-1)[1]$ under the equivalence Φ in (5-1); see [Nagao and Nakajima 2011, Remark 3.6]. The arguments of Theorem 5.17 and Corollary 5.18 work verbatim by replacing S_m and s_m with S_m' and s_m', so that we have the semiorthogonal decomposition

$$\mathrm{MF}(\mathcal{M}_Q^{\dagger,\theta_+}(v), w) = \langle \mathrm{MF}(\mathcal{M}_Q^{\dagger,\theta_-}(v - l s_m'), w)_{j_\bullet} : l \geq 0, 0 \leq j_1 \leq \cdots \leq j_l \leq m - l \rangle.$$

Remark 5.22 On the other hand, the above arguments do not work at walls in $\{\theta_0 + \theta_1 = 0\}$. For example at the DT/PT wall $\theta \in \mathbb{R}_{>0}(-1, 1)$, there exist an infinite number of θ–stable (Q, W)–representations corresponding to closed points in X, and the associated Ext quivers are more complicated. At the DT/PT wall, we expect the categorical wall-crossing formula

$$\mathrm{MF}(\mathcal{M}_Q^{\dagger,\theta_+}(v), w) = \left\langle \boxtimes_{i=1}^{k} \mathbb{S}(d_i)_{v_i} \boxtimes \mathrm{MF}\left(\mathcal{M}_Q^{\dagger,\theta_-}\left(v - \sum_{i=1}^{k} d_i \cdot s_\infty\right), w\right)\right\rangle.$$

Here $s_\infty = (1, 1)$, $(d_i, v_i) \in \mathbb{Z}_{>0} \times \mathbb{Z}$ satisfy $-1 < v_1/d_1 < \cdots < v_k/d_k \leq 1$ and $\mathbb{S}(v)_d$ is the subcategory

$$\mathbb{S}(d)_v \subset \mathrm{MF}(\mathcal{M}_Q^{\theta\text{-ss}}(d s_\infty), w)_v$$

defined similarly to the quasi-BPS category in [Pădurariu and Toda 2022]. Some details may be pursued in a future work.

5.10 Semiorthogonal decompositions of categorical stable pair theory

By definition a *PT stable pair* [Pandharipande and Thomas 2009] on X is a pair (F, s) where F is a pure one-dimensional coherent sheaf on X and $s \colon \mathcal{O}_X \to F$ is surjective in dimension one. For $(\beta, n) \in \mathbb{Z}^2$, we denote by

$$P_n(X, \beta)$$

the moduli space of PT stable pair moduli space (F, s) on X satisfying $[F] = \beta[C]$ and $\chi(F) = n$, where $[F]$ is the fundamental one-cycle of F. Since any such a sheaf F is supported on C, the moduli space $P_n(X, \beta)$ is a projective scheme.

Nakao and Nakajima [2011, Proposition 2.11] proved that the equivalence (5-1) induces the isomorphism

$$\Phi_* \colon P_n(X, \beta) \xrightarrow{\cong} \mathcal{M}_{(Q,W)}^{\dagger,\theta_{\mathrm{PT}}}(n, n - \beta),$$

where $\theta_{\mathrm{PT}} := (-1+\varepsilon, 1+\varepsilon)$ for $0<\varepsilon \ll 1$. The right-hand side is the critical locus of the function $w\colon \mathcal{M}_Q^{\dagger,\theta_{\mathrm{PT}}}(n,n-\beta) \to \mathbb{A}^1$ defined by (5-3). Based on the above isomorphism, the categorical PT invariant is defined as follows.

Definition 5.23 We define the categorical PT invariant for the resolved conifold X to be

$$\mathcal{DT}(P_n(X,\beta)) := \mathrm{MF}(\mathcal{M}_Q^{\dagger,\theta_{\mathrm{PT}}}(n,n-\beta), w).$$

Similarly to Lemma 5.1, the categorical PT invariant recovers the numerical PT invariant by

$$P_{n,\beta} = (-1)^{n+\beta} e_{\mathbb{C}(\!(u)\!)}\big(\mathrm{HP}_*\big(\mathcal{DT}(P_n(X,\beta))\big)\big). \tag{5-59}$$

By applying Corollary 5.20 for $m \gg 0$, we obtain the following.

Corollary 5.24 *For any $(\beta,n)\in\mathbb{Z}^2$, there exists a semiorthogonal decomposition*

$$\mathcal{DT}(P_n(X,\beta)) = \langle \mathcal{C}_{j_\bullet^{(*)}} \rangle.$$

Here each $\mathcal{C}_{j_\bullet^{(*)}}$ is equivalent to $\mathrm{MF}(\operatorname{Spec}\mathbb{C},0)$ and $j_\bullet^{(*)}$ is a collection of nonpositive integers of the form

$$j_\bullet^{(*)} = \{(0\le j_1^{(i)} \le \cdots \le j_{l_i}^{(i)} \le i-l_i)\}_{i\ge 1}$$

for some integers $l_i \ge 0$ satisfying

$$(\beta,n) = \sum_{i\ge 1} l_i\cdot(1,i).$$

We have $\mathrm{Hom}(\mathcal{C}_{j_\bullet^{(*)}}, \mathcal{C}_{j_\bullet'^{(*)}})=0$ if $j_\bullet^{(i)} = j_\bullet'^{(i)}$ for $i>k$ for some k and $j_\bullet^{(k)} \succ j_\bullet'^{(k)}$.

Remark 5.25 Similarly to Remark 5.19, the semiorthogonal decomposition in Corollary 5.24 implies

$$\mathrm{HP}_*(\mathcal{DT}(P_n(X,\beta))) = \mathrm{HP}_*(\mathrm{MF}(\operatorname{Spec}\mathbb{C},0))^{\oplus a_{n,\beta}},$$

where $a_{n,\beta}$ is given by (1-3). Taking the Euler characteristics of both sides, we obtain $P_{n,\beta} = (-1)^{n+\beta}a_{n,\beta}$, which recovers the formula (1-1).

6 Some technical lemmas

In this section, we give proofs of some postponed technical lemmas.

6.1 Functoriality of Knörrer periodicity

Let $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be stacks of the form $\mathcal{Y}_i = [Y_i/G_i]$, where Y_i is a smooth affine scheme and G_i is a reductive algebraic group which acts on Y_i. Let $\mathcal{W}_i \to \mathcal{Y}_i$ be vector bundles. Then by Theorem 2.4, we have equivalences

$$\Phi_i\colon \mathrm{MF}(\mathcal{Y}_i, w_i) \xrightarrow{\sim} \mathrm{MF}(\mathcal{W}_i \oplus \mathcal{W}_i^\vee, w_i+q_i), \tag{6-1}$$

where q_i is a natural quadratic form on $\mathcal{W}_i \oplus \mathcal{W}_i^\vee$, ie $q_i(x,x') = \langle x, x' \rangle$. On the other hand, the categories of quasicoherent factorizations $\mathrm{MF}_{\mathrm{qcoh}}(\mathcal{Y}_i, w_i)$ are compactly generated by $\mathrm{MF}(\mathcal{Y}_i, w_i)$ (see [Ballard et al. 2014, Proposition 3.15]), so it is equivalent to the ind-completion of $\mathrm{MF}(\mathcal{Y}_i, w_i)$. Therefore by taking ind-completions of both sides in (6-1), the above equivalences extend to equivalences

$$\Phi_i : \mathrm{MF}_{\mathrm{qcoh}}(\mathcal{Y}_i, w_i) \xrightarrow{\sim} \mathrm{MF}_{\mathrm{qcoh}}(\mathcal{W}_i \oplus \mathcal{W}_i^\vee, w_i + q_i).$$

Suppose that we have a commutative diagram

$$\begin{tikzcd} \mathcal{W}_1 \arrow[r, "g"] \arrow[d] & \mathcal{W}_2 \arrow[d] \\ \mathcal{Y}_1 \arrow[r, "f"] & \mathcal{Y}_2 \end{tikzcd}$$

where f is a morphism of stacks, and the top arrow is induced by a morphism of vector bundles $g : \mathcal{W}_1 \to f^*\mathcal{W}_2$. We have the induced diagram

$$\mathcal{W}_1 \oplus \mathcal{W}_1^\vee \xleftarrow{h_1} \mathcal{W}_1 \oplus f^*\mathcal{W}_2^\vee \xrightarrow{h_2} \mathcal{W}_2 \oplus \mathcal{W}_2^\vee, \tag{6-2}$$

where $h_1 = (\mathrm{id}_{\mathcal{W}_1}, g^\vee)$ and $h_2 = (g, f)$. The following lemma is a variant of [Toda 2019, Lemma 2.4.4].

Lemma 6.1 *The following diagram commutes*:

$$\begin{tikzcd} \mathrm{MF}_{\mathrm{qcoh}}(\mathcal{Y}_1, w_1) \arrow[r, "f_*"] \arrow[d, "\Phi_1"', "\sim"] & \mathrm{MF}_{\mathrm{qcoh}}(\mathcal{Y}_2, w_2) \arrow[d, "\Phi_2", "\sim"'] \\ \mathrm{MF}_{\mathrm{qcoh}}(\mathcal{W}_1 \oplus \mathcal{W}_1^\vee, w_1 + q_1) \arrow[r, "h_{2*}h_1^*"] & \mathrm{MF}_{\mathrm{qcoh}}(\mathcal{W}_2 \oplus \mathcal{W}_2^\vee, w_2 + q_2) \end{tikzcd} \tag{6-3}$$

Proof We have the commutative diagram

$$\begin{tikzcd} & f^*\mathcal{W}_2^\vee \arrow[r, hook, "h_4"] \arrow[dl, "h_5"'] \arrow[d, "h_3"'] \arrow[rr, bend left, "h_6"] \arrow[dr, phantom, "\square"] & \mathcal{W}_1 \oplus f^*\mathcal{W}_2^\vee \arrow[r, "h_2"] \arrow[d, "h_1"'] & \mathcal{W}_2 \oplus \mathcal{W}_2^\vee \\ \mathcal{Y}_1 & \mathcal{W}_1^\vee \arrow[l, "\mathrm{pr}_1"'] \arrow[r, hook, "i_1"] & \mathcal{W}_1 \oplus \mathcal{W}_1^\vee & \end{tikzcd}$$

Here pr_1 is the projection and $i_1(x) = (0, x)$. By the above diagram together with derived base change, we have

$$h_{2*}h_1^*\Phi_1(-) \cong h_{2*}h_1^* i_{1*}\,\mathrm{pr}_1^*(-) \cong h_{2*}h_{4*}h_3^*\,\mathrm{pr}_1^*(-) \cong h_{6*}h_5^*(-).$$

On the other hand, we have the commutative diagram

$$\begin{tikzcd} f^*\mathcal{W}_2^\vee \arrow[r, "h_7"] \arrow[d, "h_5"'] \arrow[rr, bend left, "h_6"] \arrow[dr, phantom, "\square"] & \mathcal{W}_2^\vee \arrow[r, hook, "i_2"] \arrow[d, "\mathrm{pr}_2"] & \mathcal{W}_2 \oplus \mathcal{W}_2^\vee \\ \mathcal{Y}_1 \arrow[r, "f"] & \mathcal{Y}_2 & \end{tikzcd}$$

Here pr_2 is the projection and $i_2(x)=(0,x)$. Similarly we have

$$\Phi_2 f_*(-) \cong i_{2*}\,\mathrm{pr}_2^* f_* \cong i_{2*}h_{7*}h_5^* \cong h_{6*}h_5^*(-).$$

Therefore the diagram (6-3) commutes. □

We also have the following lemma, which is a variant of [Toda 2019, Lemma 2.4.7].

Lemma 6.2 *Suppose that $g\colon \mathcal{W}_1 \to f^*\mathcal{W}_2$ is a surjective morphism of vector bundles on $\mathcal{Y}_1$. Then we have the commutative diagram*

$$\begin{tikzcd} \mathrm{MF}(\mathcal{Y}_2, w_2) \arrow[r, "f^*"] \arrow[d, "\Phi_2"', "\sim"] & \mathrm{MF}(\mathcal{Y}_1, w_1) \arrow[d, "\Phi_1", "\sim"'] \\ \mathrm{MF}(\mathcal{W}_2 \oplus \mathcal{W}_2^\vee, w_2+q_2) \arrow[r, "h_{1!}h_2^*"] & \mathrm{MF}(\mathcal{W}_1 \oplus \mathcal{W}_1^\vee, w_1+q_1) \end{tikzcd}$$

Proof The assumption that $g\colon \mathcal{W}_1 \to f^*\mathcal{W}_2$ is surjective implies that the morphism h_1 in (6-2) is a closed immersion, hence $h_{1!}$ gives a left adjoint of h_1^*. The lemma now follows by taking left adjoints of horizontal arrows in (6-3) and restrict to coherent factorizations. □

6.2 The categories of factorizations on formal fibers

Let G be a reductive algebraic group and Y be a finite-dimensional G–representation. We denote by $\widehat{Y}$ the formal fiber of the quotient morphism $Y \to Y/\!\!/G$ at the origin; see Section 1.6 for the definition of formal fiber. Then

$$[\widehat{Y}/G] \to \widehat{Y}/\!\!/G = \mathrm{Spec}\,\widehat{\mathcal{O}}_{Y/\!\!/G,0}$$

is a good moduli space for $[\widehat{Y}/G]$, and is isomorphic to the formal fiber of the morphism $[Y/G] \to Y/\!\!/G$ at 0. We take an element $w \in \Gamma(\mathcal{O}_{[\widehat{Y}/G]}) = \widehat{\mathcal{O}}_{Y/\!\!/G,0}$ with $w(0)=0$. We have the following lemma:

Lemma 6.3 *For $w \neq 0$, the triangulated category $\mathrm{MF}([\widehat{Y}/G], w)$ is idempotent complete.*

Proof Let $\widehat{Z} \subset \widehat{Y}$ be the closed subscheme defined by the zero locus of w. We have the following version of Orlov equivalence [2009] relating the categories of factorizations and those of singularities (see [Polishchuk and Vaintrob 2011, Theorem 3.14])

$$\mathrm{MF}([\widehat{Y}/G], w) \xrightarrow{\sim} D^b([\widehat{Z}/G])/\mathrm{Perf}([\widehat{Z}/G]).$$

Let $\boldsymbol{m}_0 \subset \mathcal{O}_{\widehat{Z}}$ be the maximal ideal which defines $0 \in \widehat{Z}$, and denote by $\widehat{\mathcal{O}}_{\widehat{Z}}$ the formal completion of $\mathcal{O}_{\widehat{Z}}$ at $\boldsymbol{m}_0$. Let $Z^{(n)} := \mathrm{Spec}\,\mathcal{O}_{\widehat{Z}}/\boldsymbol{m}_0^n$ and $\overline{Z} := \mathrm{Spec}\,\widehat{\mathcal{O}}_{\widehat{Z}}$. By the coherent completeness for the stacks $[\widehat{Z}/G]$ and $[\overline{Z}/G]$ (see [Alper et al. 2019, Theorem 1.6]), we have the equivalences

$$\mathrm{Coh}([\widehat{Z}/G]) \xrightarrow{\sim} \varprojlim_n \mathrm{Coh}([Z^{(n)}/G]) \xleftarrow{\sim} \mathrm{Coh}([\overline{Z}/G]).$$

In particular, we have an equivalence

$$D^b([\widehat{Z}/G]) \xrightarrow{\sim} D^b([\overline{Z}/G]),$$

which restricts to the equivalence for subcategories of perfect objects. Therefore we obtain the equivalence

$$\mathrm{MF}([\widehat{Y}/G], w) \xrightarrow{\sim} D^b([\overline{Z}/G])/\mathrm{Perf}([\overline{Z}/G]).$$

Since $\widehat{\mathcal{O}}_{\widehat{Z}}$ is a complete local ring, the singularity category $D^b(\overline{Z})/\mathrm{Perf}(\overline{Z})$ is well-known to be idempotent complete; for example, see [Dyckerhoff 2011, Lemma 5.6; Kalck and Yang 2018, Lemma 5.5]. The argument can be easily extended to the G–equivariant setting. Indeed, following the proof of [Kalck and Yang 2018, Lemma 5.5], it is enough to show that for a G–equivariant maximal Cohen–Macaulay $\widehat{\mathcal{O}}_{\widehat{Z}}$–module M and an idempotent $e \in \underline{\mathrm{End}}^G(M)$, it is lifted to a G–invariant idempotent in $\mathrm{End}(M)$. Here $\underline{\mathrm{End}}^G(M)$ is the set of morphisms in the G–equivariant stable category of maximal Cohen–Macaulay modules over $\widehat{\mathcal{O}}_{\widehat{Z}}$. For an idempotent $e \in \underline{\mathrm{End}}^G(M)$, we lift it to $a \in \mathrm{End}(M)$, which we can assume to be G–invariant as G is reductive. Then as in the proof of [Curtis and Reiner 1981, Theorem 6.7], the limit $\widetilde{e} := \lim f_j(a)$ exists, and is an idempotent in $\mathrm{End}(M)$ which lifts e. Here $f_j(x)$ is given by

$$f_j(x) = \sum_{i=0}^{n} \binom{2n}{i} x^{2n-i}(1-x)^i.$$

By construction $\widetilde{e}$ is G–invariant, so we obtain the desired lifting property of the idempotents. $\square$

Let W be another finite-dimensional G–representation and $q \colon W \to \mathbb{A}^1$ be a G–invariant nondegenerate quadratic form. We take $w \in \widehat{\mathcal{O}}_{Y/\!\!/G,0}$ with $w(0) = 0$. We have the following lemma:

Lemma 6.4 *There is a natural morphism of stacks*

$$\iota \colon [(\widehat{Y \oplus W})/G] \to [(\widehat{Y} \times W)/G] \tag{6-4}$$

such that the induced functor

$$\iota^* \colon \mathrm{MF}\big([(\widehat{Y} \times W)/G], w+q\big) \to \mathrm{MF}\big([(\widehat{Y \oplus W})/G], w+q\big) \tag{6-5}$$

is fully faithful with dense image.

Proof Let π_Y and $\pi_{Y\oplus W}$ be the quotient morphisms

$$\pi_Y \colon Y \to Y/\!\!/G, \quad \pi_{Y\oplus W} \colon Y \oplus W \to (Y \oplus W)/\!\!/G.$$

Then we have $\pi_{Y\oplus W}^{-1}(0,0) \subset \pi_Y^{-1}(0) \times W$, therefore we have the induced natural morphism (6-4) by the definition of formal fibers.

Note that we have $\mathrm{Crit}(w+q) = \mathrm{Crit}(w) \times \{0\}$, so the morphism (6-4) induces the isomorphism of critical loci of $w+q$ on $\widehat{Y} \times W$ and $\widehat{Y \oplus W}$, and also their formal neighborhoods. Therefore the functor (6-5) is fully faithful with dense image by [Orlov 2011, Theorem 2.10] (in loc. cit. it is stated without G–action, but the same argument applies verbatim to the G–equivariant setting). $\square$

Suppose that Y is quasiprojective variety with an action of a reductive algebraic group G such that the good moduli space $\pi \colon [Y/G] \to Y/\!\!/G$ exists. For each closed point $y \in Y/\!\!/G$, we denote by $[\widehat{Y}_y/G]$ the formal fiber of π at y. For a regular function $w\colon [Y/G]\to\mathbb{A}^1$, we denote by $\widehat{w}_y$ its restriction to $[\widehat{Y}_y/G]$, and $\widehat{\pi}_y\colon[\widehat{Y}_y/G]\to\widehat{Y}_y/\!\!/G$ its good moduli space. We have the following lemma:

Lemma 6.5 *For $\mathscr{E}\in\mathrm{MF}([Y/G],w)$, suppose that $\mathscr{E}|_{[\widehat{Y}_y/G]}\in\mathrm{MF}([\widehat{Y}_y/G],\widehat{w}_y)$ is isomorphic to zero for any closed point $y\in Y/\!\!/G$. Then we have $\mathscr{E}\cong 0$.*

Proof The inner homomorphism $\mathscr{H}om^\bullet(\mathscr{E},\mathscr{E})$ is an object in $\mathrm{MF}([Y/G],0)$, which is equivalent to the $\mathbb{Z}/2$–periodic derived category of coherent sheaves on $[Y/G]$. By the derived base change, we have

$$\pi_*\mathscr{H}om^\bullet(\mathscr{E},\mathscr{E})\otimes_{\mathscr{O}_{Y/\!\!/G}}\widehat{\mathscr{O}}_{Y/\!\!/G,y}\cong\widehat{\pi}_{y*}\mathscr{H}om^\bullet(\mathscr{E}|_{[\widehat{Y}_y/G]},\mathscr{E}|_{[\widehat{Y}_y/G]})\cong 0$$

in the $\mathbb{Z}/2$–periodic derived category of quasicoherent sheaves on $\widehat{Y}_y/\!\!/G$. The object $\pi_*\mathscr{H}om^\bullet(\mathscr{E},\mathscr{E})$ is an object in the $\mathbb{Z}/2$–periodic derived category of quasicoherent sheaves on $Y/\!\!/G$ whose formal completions at any $y\in Y/\!\!/G$ is zero, so it is isomorphic to zero. Then we have $\mathrm{Hom}^\bullet(\mathscr{E},\mathscr{E})=\mathbf{R}\Gamma(\mathscr{H}om^\bullet(\mathscr{E},\mathscr{E}))=0$, so $\mathscr{E}\cong 0$. □

6.3 Right adjoint functor

Lemma 6.6 *The functor $\Upsilon_{j_\bullet}$ in (5-55) admits a right adjoint $\Upsilon^R_{j_\bullet}$.*

Proof We consider the diagram

(6-6)
$$\begin{tikzcd}
 & \mathcal{M}_Q^{\dagger,\theta\text{-ss}}(v^\bullet) \arrow[d, "q"'] \arrow[dr, "p"] & & \\
\begin{array}{c}\mathcal{M}_Q^{\theta\text{-ss}}(s_m)^{\times l}\\ \times\mathcal{M}_Q^{\dagger,\theta_-\text{-ss}}(v-ls_m)\end{array} \arrow[r, hook] \arrow[d] & \begin{array}{c}\mathcal{M}_Q^{\theta\text{-ss}}(s_m)^{\times l}\\ \times\mathcal{M}_Q^{\dagger,\theta\text{-ss}}(v-ls_m)\end{array} \arrow[d] & \mathcal{M}_Q^{\dagger,\theta\text{-ss}}(v) \arrow[d] & \mathcal{M}_Q^{\dagger,\theta_+\text{-ss}}(v) \arrow[l, hook'] \arrow[d] \\
\begin{array}{c}M_Q^{\theta\text{-ss}}(s_m)^{\times l}\\ \times M_Q^{\dagger,\theta_-\text{-ss}}(v-ls_m)\end{array} \arrow[r] & \begin{array}{c}M_Q^{\theta\text{-ss}}(s_m)^{\times l}\\ \times M_Q^{\dagger,\theta\text{-ss}}(v-ls_m)\end{array} \arrow[r, "\oplus"'] & M_Q^{\dagger,\theta\text{-ss}}(v) \arrow[d, "w"'] & M_Q^{\dagger,\theta_+\text{-ss}}(v) \arrow[l] \\
 & & \mathbb{A}^1 &
\end{tikzcd}$$

Similarly to (5-36), let

$$\widetilde{\mathbb{W}}^{\theta_\pm}_{\mathrm{glob}}(v)\subset D^b(\mathcal{M}_Q^{\dagger,\theta\text{-ss}}(v))$$

be the window subcategory (2-7) for the choice $m_\bullet^\pm$ in (5-35). We consider the composition functor

(6-7)
$$\begin{aligned}
D^b(M_Q^{\theta\text{-ss}}(s_m))^{\boxtimes l}\boxtimes D^b(M_Q^{\dagger,\theta\text{-ss}}(v-ls_m))
&\xrightarrow{\sim}\mathop{\boxtimes}_{i=1}^{l}D^b(\mathcal{M}_Q^{\theta\text{-ss}}(s_m))_{j_i+(2i-1)(m^2-m)}\boxtimes\widetilde{\mathbb{W}}^{\theta_-}_{\mathrm{glob}}(v-ls_m)\\
&\to D^b(\mathcal{M}_Q^{\dagger,\theta\text{-ss}}(v))\to D^b(M_Q^{\dagger,\theta_+\text{-ss}}(v)).
\end{aligned}$$

Here the first equivalence is due to window theorem in Theorem 2.2 together with the fact that (5-9) is a $\mathbb{C}^*$–gerbe, the second arrow is the categorified Hall product (ie p_*q^* in the diagram (6-6)), and the last arrow is the restriction to the semistable locus. The first arrow is of Fourier–Mukai type by Lemma 6.7 below, and the second and the third arrows are also of Fourier–Mukai type by their constructions. Therefore the above composition functor is of Fourier–Mukai type. So we have the kernel object

$$\mathcal{P} \in D^b\big((M_Q^{\theta\text{-ss}}(s_m)^{\times l} \times M_Q^{\dagger,\theta_-\text{-ss}}(v-ls_m)) \times M_Q^{\dagger,\theta_+\text{-ss}}(v)\big).$$

Moreover the kernel objects of the second and the third arrows in (6-7) are pushforwards from the fiber products over $M_Q^{\dagger,\theta\text{-ss}}(v)$ by their constructions. By Lemma 6.7 below, the kernel object of the first arrow in (6-7) is a pushforward from the fiber product over $\mathbb{A}^1$ and supported on the fiber product over $M_Q^{\dagger,\theta\text{-ss}}(v)$. Therefore the object $\mathcal{P}$ is a pushforward of an object

$$\mathcal{P}_w \in D^b\big((M_Q^{\theta\text{-ss}}(s_m)^{\times l} \times M_Q^{\dagger,\theta_-\text{-ss}}(v-ls_m)) \times_{\mathbb{A}^1} M_Q^{\dagger,\theta_+\text{-ss}}(v)\big) \tag{6-8}$$

supported on the fiber product over $M_Q^{\dagger,\theta\text{-ss}}(v)$. Since $M_Q^{\dagger,\theta_+\text{-ss}}(v)$ and $M_Q^{\theta\text{-ss}}(s_m)^{\times l} \times M_Q^{\dagger,\theta_-\text{-ss}}(v-ls_m)$ are proper over $M_Q^{\dagger,\theta\text{-ss}}(v)$, the functor (6-7) admits a right adjoint given by the Fourier–Mukai kernel $\mathcal{P}^R$ defined by

$$\mathcal{P}^R := \mathcal{P}^\vee \boxtimes \omega_{M_Q^{\theta\text{-ss}}(s_m)^{\times l} \times M_Q^{\dagger,\theta_-\text{-ss}}(v-ls_m)}[\dim M_Q^{\theta\text{-ss}}(s_m)^{\times l} \times M_Q^{\dagger,\theta_-\text{-ss}}(v-ls_m)].$$

By Theorem 2.3, the functor $\Upsilon_{j_\bullet}$ in (5-55) is regarded as a functor

$$\Upsilon_{j_\bullet}\colon \mathrm{MF}(M_Q^{\theta\text{-ss}}(s_m), w)^{\boxtimes l} \boxtimes \mathrm{MF}(M_Q^{\dagger,\theta_-\text{-ss}}(v-ls_m), w) \to \mathrm{MF}(M_Q^{\dagger,\theta_+\text{-ss}}(v), w). \tag{6-9}$$

The above functor is a Fourier–Mukai functor with kernel given by $\Xi(\mathcal{P}_w)$, where Ξ is the natural functor (see [Hirano 2017b, Theorem 5.5])

$$\begin{aligned}\Xi\colon D^b\big(\big(M_Q^{\theta\text{-ss}}(s_m)^{\times l} \times M_Q^{\dagger,\theta_-\text{-ss}}(v-ls_m)\big) \times_{\mathbb{A}^1} M_Q^{\dagger,\theta_+\text{-ss}}(v)\big)& \\ \to \mathrm{MF}\big(\big(M_Q^{\theta\text{-ss}}(s_m)^{\times l} \times M_Q^{\dagger,\theta_-\text{-ss}}(v-ls_m)\big) \times M_Q^{\dagger,\theta_+\text{-ss}}(v), w \boxplus (-w)\big)&\end{aligned}$$

By the Grothendieck Riemann–Roch theorem, the object $\mathcal{P}^R$ is the pushforward of an object $\mathcal{P}_w^R$ in the right-hand side of (6-8). Then the right adjoint of (6-9) is obtained by the Fourier–Mukai kernel $\Xi(\mathcal{P}_w^R)$. $\square$

Lemma 6.7 *In the setting of Theorem 2.2, let $\mathcal{Y} = [Y/G]$ and $\mathcal{Y}^{\mathrm{ss}} = [Y^{l\text{-ss}}/G]$, and assume that $\mathcal{Y}^{\mathrm{ss}}$ is a projective scheme over $Y /\!\!/ G$. Then the splitting of $D^b(\mathcal{Y}) \twoheadrightarrow D^b(\mathcal{Y}^{\mathrm{ss}})$ in Theorem 2.2 (applied to $N'=0$) is of Fourier–Mukai type, with kernel object $\mathcal{P} \in D^b(\mathcal{Y} \times \mathcal{Y}^{\mathrm{ss}})$ supported on $\mathcal{Y} \times_{Y/\!\!/G} \mathcal{Y}^{\mathrm{ss}}$. Moreover for any nonconstant $w\colon Y/\!\!/G \to \mathbb{A}^1$, we have $\mathcal{P} = i_*\mathcal{P}_w$ for some $\mathcal{P}_w \in D^b(\mathcal{Y} \times_{\mathbb{A}^1} \mathcal{Y}^{\mathrm{ss}})$. Here $\mathcal{Y} \times_{\mathbb{A}^1} \mathcal{Y}^{\mathrm{ss}}$ is given by the diagram*

$$\begin{tikzcd} \mathcal{Y} \times_{\mathbb{A}^1} \mathcal{Y}^{\mathrm{ss}} \arrow[r, "i"] \arrow[d] & \mathcal{Y} \times \mathcal{Y}^{\mathrm{ss}} \arrow[d, "w \boxplus (-w)"] \\ 0 \arrow[r] & \mathbb{A}^1 \end{tikzcd}$$

Proof The KN stratification of $\mathcal{Y}$ pulls back to the one on $\mathcal{Y}\times\mathcal{Y}^{\mathrm{ss}}$ via the first projection, thus by a choice of $m_\bullet$ in Theorem 2.2 we have the splitting Ψ of $D^b(\mathcal{Y}\times\mathcal{Y}^{\mathrm{ss}}) \twoheadrightarrow D^b(\mathcal{Y}^{\mathrm{ss}}\times\mathcal{Y}^{\mathrm{ss}})$. From its construction, Ψ is linear over $\mathrm{Perf}(Y/\!\!/G\times Y/\!\!/G)$. Therefore for any nonconstant w, by [Halpern-Leistner 2015, Proposition 5.5] there is a splitting Φ_w of $D^b(\mathcal{Y}\times_{\mathbb{A}^1}\mathcal{Y}^{\mathrm{ss}}) \twoheadrightarrow D^b(\mathcal{Y}^{\mathrm{ss}}\times_{\mathbb{A}^1}\mathcal{Y}^{\mathrm{ss}})$ such that the following diagram commutes:

$$\begin{tikzcd}
D^b(\mathcal{Y}^{\mathrm{ss}}\times_{\mathbb{A}^1}\mathcal{Y}^{\mathrm{ss}}) \arrow[r, "\Phi_w"] \arrow[d, "i_*"'] & D^b(\mathcal{Y}\times_{\mathbb{A}^1}\mathcal{Y}^{\mathrm{ss}}) \arrow[d, "i_*"'] \\
D^b(\mathcal{Y}^{\mathrm{ss}}\times\mathcal{Y}^{\mathrm{ss}}) \arrow[r, "\Phi"] & D^b(\mathcal{Y}\times\mathcal{Y}^{\mathrm{ss}})
\end{tikzcd}$$

Since $\mathcal{Y}^{\mathrm{ss}}$ is a quasiprojective scheme, we have $\mathcal{O}_\Delta\in D^b(\mathcal{Y}^{\mathrm{ss}}\times\mathcal{Y}^{\mathrm{ss}})$. We set $\mathcal{P}=\Phi(\mathcal{O}_\Delta)$ and $\mathcal{P}_w=\Phi_w(\mathcal{O}_\Delta)$. Then $\mathcal{P}=i_*\mathcal{P}_w$. Since this holds for any w, the object $\mathcal{P}$ is supported on $\mathcal{Y}\times_{Y/\!\!/G}\mathcal{Y}^{\mathrm{ss}}$. Then the object $\mathcal{P}$ induces the Fourier–Mukai functor $D^b(\mathcal{Y}^{\mathrm{ss}})\to D^b(\mathcal{Y})$ which gives the splitting in Theorem 2.2 by the argument in [Halpern-Leistner 2015, Section 2.3]. □

6.4 Proof of Proposition 5.11

Proof The assertion is trivial if $\dim V\leq 1$. Below we assume that $\dim V\geq 2$. Note that $\mathrm{ord}_0(w_p)\geq 2$, where $\mathrm{ord}_0(w_p)$ is the vanishing order of w_p at 0. This is because $w_p(0)=0$ by the first inclusion in (5-4) together with the fact that $0\in\mathrm{Crit}(w_p)\neq\varnothing$.

Let us consider the Hessian of w_p,

$$\mathrm{Hess}(w_p)\colon \mathrm{Ext}^1_{Q^\dagger}(R,R)\otimes\mathcal{O}_{\widehat{\mathcal{M}}^\dagger_{Q_p}(d)}\to\mathrm{Ext}^1_{Q^\dagger}(R,R)^\vee\otimes\mathcal{O}_{\widehat{\mathcal{M}}^\dagger_{Q_p}(d)}.$$

The kernel of the above morphism at the origin is $\mathrm{Ext}^1_{(Q^\dagger,W)}(R,R)$. By the relation (5-8), we have

$$\mathrm{Ext}^1_{(Q,W)}(S_m,S_m)=\mathrm{Ext}^1_X(\mathcal{O}_C(m-1),\mathcal{O}_C(m-1))=0.$$

It follows that

$$\mathrm{Ker}(\mathrm{Hess}(w_p)|_0)\cap(\mathrm{End}(V)\otimes\mathrm{Ext}^1_Q(S_m,S_m))=0. \tag{6-10}$$

By Lemma 6.8 below, by replacing the isomorphism η_p in (5-19) if necessary, there exist linear subspaces

$$W_1\subset\mathrm{Ext}^1_{Q^\dagger}(R_\infty,R_\infty),\quad W_2\subset\mathrm{Ext}^1_{Q^\dagger}(R_\infty,S_m),\quad W_3\subset\mathrm{Ext}^1_{Q^\dagger}(S_m,R_\infty)$$

such that $w_p=w_1+w_2$, where w_1 does not contain variables from $\mathrm{End}(V)\otimes\mathrm{Ext}^1_Q(S_m,S_m)$ with $\deg(w_1)\geq 3$, and w_2 is a nondegenerate G–invariant quadratic form on

$$\begin{aligned}W_1&\oplus(W_2\otimes V)\oplus(W_3\otimes V^\vee)\oplus(\mathrm{End}(V)\otimes\mathrm{Ext}^1_Q(S_m,S_m))\\&=(W_1\oplus\mathrm{Ext}_Q(S_m,S_m))\oplus(W_2\otimes V)\oplus(W_3\otimes V^\vee)\oplus(\mathrm{End}_0(V)\otimes\mathrm{Ext}^1_Q(S_m,S_m)).\end{aligned}$$

As we assumed that $\dim V\geq 2$, the $\mathrm{GL}(V)$–representation $\mathrm{End}_0(V)$ is a nontrivial irreducible $\mathrm{GL}(V)$–representation, and it is not isomorphic to V nor $V^\vee$. Therefore $w_2=w_3+q$, where w_3 does not contain variables from $\mathrm{End}_0(V)\otimes\mathrm{Ext}^1_Q(S_m,S_m)$ and q is a nondegenerate $\mathrm{GL}(V)$–invariant quadratic form on $\mathrm{End}_0(V) \otimes \mathrm{Ext}^1_Q(S_m, S_m)$. Moreover, w_3 is nonzero, since otherwise it contradicts with (6-10) and $\mathrm{End}_0(V) \subsetneq \mathrm{End}(V)$. By replacing the isomorphism (5-25) if necessary, we can also assume that q coincides with (5-26). Therefore we obtain a desired form (5-27). □

We have used the following lemma, whose proof is a variant of [Joyce 2015, Proposition 2.24]:

Lemma 6.8 *Let G be a reductive algebraic group and V be a finite-dimensional G–representation. Let $w\colon \widehat{V} \to \mathbb{A}^1$ be a G–invariant formal function such that $\mathrm{ord}_0(w) \geq 2$. Let V_1 be the kernel of the Hessian at the origin*

$$V_1 = \mathrm{Ker}\big(\mathrm{Hess}(w)|_0\colon V \to V^\vee\big).$$

Then there exists a direct sum decomposition $V = V_1 \oplus V_2$ of G–representations and a G–equivariant isomorphism $\phi\colon \widehat{V} \xrightarrow{\cong} \widehat{V}$ such that $\phi^* w = w_1 + w_2$, where $w_1 \in \mathcal{O}_{\widehat{V}_1}$ is G–invariant with $\mathrm{ord}_0(w_1) \geq 3$, and $w_2 \in \mathrm{Sym}^2(V_2^\vee)$ is a G–invariant nondegenerate quadratic form on V_2.

Proof As w is G–invariant, the Hessian of w at the origin $\mathrm{Hess}(w)|_0\colon V \to V^\vee$ is G–equivariant. As G is reductive, there is a splitting $V = V_1 \oplus V_2$ as G–representations and the Hessian at the origin is written as

$$\mathrm{Hess}(w)|_0 = \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix} : V_1 \oplus V_2 \to V_1^\vee \oplus V_2^\vee,$$

where q is a G–equivariant isomorphism $q\colon V_2 \xrightarrow{\cong} V_2^\vee$ with $q^\vee = q$. We identify q as an element $q \in \mathrm{Sym}^2(V_2^\vee)^G$, which is a G–invariant nondegenerate quadratic form q on V_2. For $(y_1, y_2) \in V_1 \oplus V_2$, we can write $w(y_1, y_2)$ as

$$w(y_1, y_2) = w^{\geq 3}(y_1, y_2) + q(y_2), \tag{6-11}$$

where $w^{\geq 3}(y_1, y_2)$ consists of terms with degrees bigger than or equal to three. We set $d_i = \dim V_i$ and fix bases of V_1 and V_2, writing their elements as $y_1 = \{y_1^{(i)}\}_{1 \leq i \leq d_1}$ and $y_2 = \{y_2^{(i)}\}_{1 \leq i \leq d_2}$, respectively. Here we take an orthonormal basis for V_2, so q is written as

$$q(y_2) = \frac{1}{2} \sum_{i=1}^{d_1} (y_2^{(i)})^2.$$

Then the closed subscheme

$$\left\{ \frac{\partial w}{\partial y_2^{(i)}} = 0 : 1 \leq i \leq d_2 \right\} = \left\{ y_2^{(i)} + \frac{\partial w^{\geq 3}}{\partial y_2^{(i)}} = 0 : 1 \leq i \leq d_2 \right\} \subset \widehat{V}$$

is smooth of codimension d_2. By the variable change

$$y_2^{(i)} \mapsto \frac{\partial w}{\partial y_2^{(i)}} = y_2^{(i)} + \frac{\partial w^{\geq 3}}{\partial y_2^{(i)}}, \tag{6-12}$$

we may assume that $\mathrm{Crit}(w)$ is contained in $\{y_2 = 0\} \subset \widehat{V}$. The variable change (6-12) can be described without coordinates as follows. Let dw be the morphism given by the derivation of w,

$$dw\colon V \otimes \mathcal{O}_{\widehat{V}} \to \mathcal{O}_{\widehat{V}}. \tag{6-13}$$

We have the morphisms

$$\phi\colon V^\vee = V_1^\vee \oplus V_2^\vee \xrightarrow{(\mathrm{id},q^{-1})} V_1^\vee \oplus V_2 \xrightarrow{(\mathrm{id},dw|_{V_2})} \widehat{\mathcal{O}}_V.$$

The above composition induces the isomorphism $\mathcal{O}_{\widehat{V}} \xrightarrow{\cong} \mathcal{O}_{\widehat{V}}$, which is identified with the variable change (6-12). The above construction is G–equivariant, so the variable change (6-12) is G–equivariant.

The condition that $\mathrm{Crit}(w) \subset \{y_2 = 0\}$ implies that each $y_2^{(i)}$ is written as

$$y_2^{(i)} = \sum_{j=1}^{d_1} a_{ij} \frac{\partial w}{\partial y_1^{(j)}} + \sum_{j=1}^{d_2} b_{ij} \frac{\partial w}{\partial y_2^{(j)}}$$

for some $a_{ij}, b_{ij} \in \mathcal{O}_{\widehat{V}}$. By writing $b_{ij} = b_{ij}(0) + b_{ij}^{\geq 1}$ and comparing the degree-one terms for y_2, we see that $b_{ij}(0) = \delta_{ij}$. Therefore we obtain the relation

$$-\frac{\partial w^{\geq 3}}{\partial y_2^{(i)}} = \sum_{j=1}^{d_1} a_{ij} \frac{\partial w^{\geq 3}}{\partial y_1^{(i)}} + \sum_{j=1}^{d_2} b_{ij}^{\geq 1} \left(y_2^{(j)} + \frac{\partial w^{\geq 3}}{\partial y_2^{(j)}} \right).$$

The Nakayama lemma implies the inclusion of ideals

$$\left(\frac{\partial w^{\geq 3}}{\partial y_2^{(i)}} : 1 \leq i \leq d_2 \right) \subset \left(\frac{\partial w^{\geq 3}}{\partial y_1^{(j)}}, y_2^{(i)} : 1 \leq j \leq d_1,\ 1 \leq i \leq d_2 \right) \tag{6-14}$$

in $\widehat{\mathcal{O}}_{\widehat{V}}$, the formal completion at the maximal ideal of $\mathcal{O}_{\widehat{V}}$. Since these are G–invariant ideals, by the coherent completeness of $[\widehat{V}/G]$ the inclusion (6-14) also holds in $\mathcal{O}_{\widehat{V}}$; see the proof of Lemma 6.3. In particular, there is a relation of the form

$$\frac{\partial w}{\partial y_2^{(i)}}\bigg|_{y_2=0} = \sum_{i,j} c_{ij} \frac{\partial w}{\partial y_1^{(j)}}\bigg|_{y_2=0} \tag{6-15}$$

for some $c_{ij} \in \mathcal{O}_{\widehat{V}_1}$. We apply the variable change

$$\widetilde{y}_1^{(i)} = y_1^{(i)} + \sum_j c_{ij} y_2^{(i)} \quad \text{and} \quad \widetilde{y}_2^{(i)} = y_2^{(i)}. \tag{6-16}$$

Then we have

$$\frac{\partial w}{\partial \widetilde{y}_2^{(i)}}\bigg|_{\widetilde{y}_2=0} = \left(\sum_j \frac{\partial y_1^{(j)}}{\partial \widetilde{y}_2^{(i)}} \frac{\partial w}{\partial y_1^{(j)}} + \sum_j \frac{\partial y_2^{(j)}}{\partial \widetilde{y}_2^{(i)}} \frac{\partial w}{\partial y_2^{(j)}} \right)\bigg|_{\widetilde{y}_2=0} = -\sum_j c_{ij} \frac{\partial w}{\partial y_1^{(j)}}\bigg|_{\widetilde{y}_2=0} + \frac{\partial w}{\partial y_2^{(i)}}\bigg|_{\widetilde{y}_2=0} = 0.$$

It follows that we can assume that $(\partial w/\partial y_2^{(i)})|_{y_2=0} = 0$.

We see that the variable change (6-16) can be taken to be G–equivariant. For the morphism (6-13), we can write $dw \otimes \mathcal{O}_{\widehat{V}_1}$ as

$$dw \otimes \mathcal{O}_{\widehat{V}_1} = \alpha^{(1)} \oplus \alpha^{(2)}\colon (V_1 \otimes \mathcal{O}_{\widehat{V}_1}) \oplus (V_2 \otimes \mathcal{O}_{\widehat{V}_1}) \to \mathcal{O}_{\widehat{V}_1}.$$

Then the ideals of $\mathcal{O}_{\widehat{V}_1}$

$$I_1 = \left(\frac{\partial w}{\partial y_1^{(i)}} \bigg|_{y_2=0} \right) \quad \text{and} \quad I_2 = \left(\frac{\partial w}{\partial y_2^{(i)}} \bigg|_{y_2=0} \right)$$

are generated by the images of $\alpha^{(1)}$ and $\alpha^{(2)}$, respectively, so in particular they are G–invariant. By the relation (6-15) we have $I_2 \subset I_1$. We have the G–equivariant diagram

$$\begin{tikzcd} V_2 \otimes \mathcal{O}_{\widehat{V}_1} \arrow[r, "\alpha^{(2)}"] \arrow[d, dotted, "\phi"'] & I_2 \arrow[d, hook] \\ V_1 \otimes \mathcal{O}_{\widehat{V}_1} \arrow[r, "\alpha^{(1)}"] & I_1 \end{tikzcd}$$

where each horizontal arrow is a surjection. As G is reductive, from the above diagram there is a G–equivariant dotted arrow ϕ which makes the above diagram commutative. A choice of ϕ corresponds to a choice of c_{ij} in (6-15). Then we have the G–equivariant morphism

$$V^\vee = V_1^\vee \oplus V_2^\vee \xrightarrow{(\mathrm{id}+\phi^\vee,\mathrm{id})} \mathcal{O}_{\widehat{V}}.$$

The above morphism induces the G–equivariant isomorphism $\widehat{\mathcal{O}}_V \xrightarrow{\cong} \widehat{\mathcal{O}}_V$, which corresponds to the variable change (6-16). In particular, we can choose c_{ij} so that (6-16) is G–equivariant.

Finally, we set

$$g(y_1, y_2) := w(y_1, y_2) - w(y_1, 0).$$

Then from the above arguments we have $g(y_1, 0) = 0$ and $(\partial g/\partial y_2^{(i)})|_{y_2=0} = 0$. It follows that $g(y_1, y_2)$ is written as

$$g(y_1, y_2) = \sum_{i,j} y_2^{(i)} y_2^{(j)} Q_{ij}(y_1, y_2)$$

for some $Q_{ij} \in \mathcal{O}_{\widehat{V}}$. As the quadratic term of $g(y_1, y_2)$ coincides with q by (6-11), we have $Q_{ij}(0) = \frac{1}{2}\delta_{ij}$. It follows that the critical locus of $g(y_1, y_2)$ is $\{y_2 = 0\} \subset \widehat{V}$, so the G–equivariant Morse lemma (see [Arnold et al. 1985, Section 17.3]) applied for g implies that by a G–equivariant variable change of the form $\tilde{y}_1^{(i)} = y_1^{(i)}$ and $\tilde{y}_2^{(i)} = \sum_{i,j} \alpha^{(ij)}(y_1, y_2) y_2^{(j)}$ we can make $g(\tilde{y}_1, \tilde{y}_2) = q(\tilde{y}_2)$. As $\mathrm{ord}_0(w(y_1, 0)) \geq 3$ from (6-11), the lemma is proved. □

References

[Alper 2013] **J Alper**, *Good moduli spaces for Artin stacks*, Ann. Inst. Fourier (Grenoble) 63 (2013) 2349–2402 MR Zbl

[Alper et al. 2019] **J Alper**, **J Hall**, **D Rydh**, *The étale local structure of algebraic stacks*, preprint (2019) arXiv 1912.06162

[Arnold et al. 1985] **V I Arnold**, **S M Guseĭn-Zade**, **A N Varchenko**, *Singularities of differentiable maps, I: The classification of critical points, caustics and wave fronts*, Monogr. Math. 82, Birkhäuser, Boston, MA (1985) MR Zbl

[Ballard et al. 2014] **M Ballard**, **D Favero**, **L Katzarkov**, *A category of kernels for equivariant factorizations and its implications for Hodge theory*, Publ. Math. Inst. Hautes Études Sci. 120 (2014) 1–111 MR Zbl

[Ballard et al. 2019] **M Ballard**, **D Favero**, **L Katzarkov**, *Variation of geometric invariant theory quotients and derived categories*, J. Reine Angew. Math. 746 (2019) 235–303 MR Zbl

[Ballard et al. 2021] **M R Ballard**, **N K Chidambaram**, **D Favero**, **P K McFaddin**, **R R Vandermolen**, *Kernels for Grassmann flops*, J. Math. Pures Appl. 147 (2021) 29–59 MR Zbl

[Behrend 2009] **K Behrend**, *Donaldson–Thomas type invariants via microlocal geometry*, Ann. of Math. 170 (2009) 1307–1338 MR Zbl

[Behrend and Bryan 2007] **K Behrend**, **J Bryan**, *Super-rigid Donaldson–Thomas invariants*, Math. Res. Lett. 14 (2007) 559–571 MR Zbl

[Bondal and Orlov 1995] **A Bondal**, **D Orlov**, *Semiorthogonal decomposition for algebraic varieties*, preprint (1995) arXiv alg-geom/9506012

[Brav et al. 2019] **C Brav**, **V Bussi**, **D Joyce**, *A Darboux theorem for derived schemes with shifted symplectic structure*, J. Amer. Math. Soc. 32 (2019) 399–443 MR Zbl

[Bridgeland 2002] **T Bridgeland**, *Flops and derived categories*, Invent. Math. 147 (2002) 613–632 MR Zbl

[Bridgeland 2011] **T Bridgeland**, *Hall algebras and curve-counting invariants*, J. Amer. Math. Soc. 24 (2011) 969–998 MR Zbl

[Brion 2012] **M Brion**, *Representations of quivers*, from "Geometric methods in representation theory, I" (M Brion, editor), Sémin. Congr. 24, Soc. Math. France, Paris (2012) 103–144 MR Zbl

[Brown 2016] **M K Brown**, *Knörrer periodicity and Bott periodicity*, Doc. Math. 21 (2016) 1459–1501 MR Zbl

[Calabrese 2016] **J Calabrese**, *Donaldson–Thomas invariants and flops*, J. Reine Angew. Math. 716 (2016) 103–145 MR Zbl

[Cao and Toda 2023] **Y Cao**, **Y Toda**, *Counting perverse coherent systems on Calabi–Yau* 4*–folds*, Math. Ann. 385 (2023) 1379–1429 MR Zbl

[Curtis and Reiner 1981] **C W Curtis**, **I Reiner**, *Methods of representation theory*, *I*: *With applications to finite groups and orders*, Wiley, New York (1981) MR Zbl

[Davison 2017] **B Davison**, *The critical CoHA of a quiver with potential*, Q. J. Math. 68 (2017) 635–703 MR Zbl

[Donovan and Segal 2014] **W Donovan**, **E Segal**, *Window shifts, flop equivalences and Grassmannian twists*, Compos. Math. 150 (2014) 942–978 MR Zbl

[Dyckerhoff 2011] **T Dyckerhoff**, *Compact generators in categories of matrix factorizations*, Duke Math. J. 159 (2011) 223–274 MR Zbl

[Efimov 2018] **A I Efimov**, *Cyclic homology of categories of matrix factorizations*, Int. Math. Res. Not. 2018 (2018) 3834–3869 MR Zbl

[Efimov and Positselski 2015] **A I Efimov**, **L Positselski**, *Coherent analogues of matrix factorizations and relative singularity categories*, Algebra Number Theory 9 (2015) 1159–1292 MR Zbl

[Halpern-Leistner 2015] **D Halpern-Leistner**, *The derived category of a GIT quotient*, J. Amer. Math. Soc. 28 (2015) 871–912 MR Zbl

[Halpern-Leistner and Pomerleano 2020] **D Halpern-Leistner**, **D Pomerleano**, *Equivariant Hodge theory and noncommutative geometry*, Geom. Topol. 24 (2020) 2361–2433 MR Zbl

[Halpern-Leistner and Sam 2020] **D Halpern-Leistner**, **S V Sam**, *Combinatorial constructions of derived equivalences*, J. Amer. Math. Soc. 33 (2020) 735–773 MR Zbl

[Hirano 2017a] **Y Hirano**, *Derived Knörrer periodicity and Orlov's theorem for gauged Landau–Ginzburg models*, Compos. Math. 153 (2017) 973–1007 MR Zbl

[Hirano 2017b] **Y Hirano**, *Equivalences of derived factorization categories of gauged Landau–Ginzburg models*, Adv. Math. 306 (2017) 200–278 MR Zbl

[Jiang 2021] **Q Jiang**, *Derived categories of Quot schemes of locally free quotients*, preprint (2021) arXiv 2107.09193

[Joyce 2015] **D Joyce**, *A classical model for derived critical loci*, J. Differential Geom. 101 (2015) 289–367 MR Zbl

[Joyce and Song 2012] **D Joyce**, **Y Song**, *A theory of generalized Donaldson–Thomas invariants*, Mem. Amer. Math. Soc. 1020, Amer. Math. Soc., Providence, RI (2012) MR Zbl

[Kalck and Yang 2018] **M Kalck**, **D Yang**, *Relative singularity categories*, *II*: *DG models*, preprint (2018) arXiv 1803.08192

[Kapranov 1984] **M M Kapranov**, *Derived category of coherent sheaves on Grassmann manifolds*, Izv. Akad. Nauk SSSR Ser. Mat. 48 (1984) 192–202 MR Zbl In Russian; translated in Math. USSR-Izv. 24 (1985) 183–192

[Kawamata 2002] **Y Kawamata**, *D–equivalence and K–equivalence*, J. Differential Geom. 61 (2002) 147–171 MR Zbl

[Kawamata 2018] **Y Kawamata**, *Birational geometry and derived categories*, from "Surveys in differential geometry" (H-D Cao, J Li, R M Schoen, S-T Yau, editors), Surv. Differ. Geom. 22, International, Somerville, MA (2018) 291–317 MR Zbl

[Keller 1999] **B Keller**, *On the cyclic homology of exact categories*, J. Pure Appl. Algebra 136 (1999) 1–56 MR Zbl

[King 1994] **A D King**, *Moduli of representations of finite-dimensional algebras*, Q. J. Math. 45 (1994) 515–530 MR Zbl

[Kinjo ≥ 2024] **T Kinjo**, *Cohomological DT/PT correspondence for the resolved conifold*, in preparation

[Kontsevich and Soibelman 2008] **M Kontsevich**, **Y Soibelman**, *Stability structures, motivic Donaldson–Thomas invariants and cluster transformations*, preprint (2008) arXiv 0811.2435

[Kontsevich and Soibelman 2011] **M Kontsevich**, **Y Soibelman**, *Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson–Thomas invariants*, Commun. Number Theory Phys. 5 (2011) 231–352 MR Zbl

[Koseki 2021] **N Koseki**, *Birational geometry of moduli spaces of perverse coherent sheaves on blow-ups*, Math. Z. 299 (2021) 2379–2404 MR Zbl

[Maulik et al. 2006] **D Maulik**, **N Nekrasov**, **A Okounkov**, **R Pandharipande**, *Gromov–Witten theory and Donaldson–Thomas theory, I*, Compos. Math. 142 (2006) 1263–1285 MR Zbl

[Meinhardt and Reineke 2019] **S Meinhardt**, **M Reineke**, *Donaldson–Thomas invariants versus intersection cohomology of quiver moduli*, J. Reine Angew. Math. 754 (2019) 143–178 MR Zbl

[Morrison et al. 2012] **A Morrison**, **S Mozgovoy**, **K Nagao**, **B Szendrői**, *Motivic Donaldson–Thomas invariants of the conifold and the refined topological vertex*, Adv. Math. 230 (2012) 2065–2093 MR Zbl

[Nagao and Nakajima 2011] **K Nagao**, **H Nakajima**, *Counting invariant of perverse coherent sheaves and its wall-crossing*, Int. Math. Res. Not. 2011 (2011) 3885–3938 MR Zbl

[Nakajima and Yoshioka 2011] **H Nakajima**, **K Yoshioka**, *Perverse coherent sheaves on blow-up, I: A quiver description*, from "Exploring new structures and natural constructions in mathematical physics" (K Hasegawa, T Hayashi, S Hosono, Y Yamada, editors), Adv. Stud. Pure Math. 61, Math. Soc. Japan, Tokyo (2011) 349–386 MR Zbl

[Orlov 2006] **D O Orlov**, *Triangulated categories of singularities, and equivalences between Landau–Ginzburg models*, Mat. Sb. 197 (2006) 117–132 MR Zbl In Russian; translated in Sb. Math. 197 (2006) 1827–1840

[Orlov 2009] **D Orlov**, *Derived categories of coherent sheaves and triangulated categories of singularities*, from "Algebra, arithmetic, and geometry, II" (Y Tschinkel, Y Zarhin, editors), Progr. Math. 270, Birkhäuser, Boston, MA (2009) 503–531 MR Zbl

[Orlov 2011] **D Orlov**, *Formal completions and idempotent completions of triangulated categories of singularities*, Adv. Math. 226 (2011) 206–217 MR Zbl

[Orlov 2012] **D Orlov**, *Matrix factorizations for nonaffine LG-models*, Math. Ann. 353 (2012) 95–108 MR Zbl

[Pandharipande and Thomas 2009] **R Pandharipande**, **R P Thomas**, *Curve counting via stable pairs in the derived category*, Invent. Math. 178 (2009) 407–447 MR Zbl

[Pantev et al. 2013] **T Pantev**, **B Toën**, **M Vaquié**, **G Vezzosi**, *Shifted symplectic structures*, Publ. Math. Inst. Hautes Études Sci. 117 (2013) 271–328 MR Zbl

[Polishchuk and Vaintrob 2011] **A Polishchuk**, **A Vaintrob**, *Matrix factorizations and singularity categories for stacks*, Ann. Inst. Fourier (Grenoble) 61 (2011) 2609–2642 MR Zbl

[Porta and Sala 2023] **M Porta**, **F Sala**, *Two-dimensional categorified Hall algebras*, J. Eur. Math. Soc. 25 (2023) 1113–1205 MR Zbl

[Pădurariu 2019] **T Pădurariu**, *K–theoretic Hall algebras for quivers with potential*, preprint (2019) arXiv 1911.05526

[Pădurariu 2023] **T Pădurariu**, *Categorical and K–theoretic Hall algebras for quivers with potential*, J. Inst. Math. Jussieu 22 (2023) 2717–2747 MR Zbl

[Pădurariu 2024] **T Pădurariu**, *Generators for K–theoretic Hall algebras of quivers with potential*, Selecta Math. 30 (2024) art. id. 4 MR Zbl

[Pădurariu and Toda 2022] **T Pădurariu**, **Y Toda**, *Categorical and K–theoretic Donaldson–Thomas theory of $\mathbb{C}^3$, I* (2022) arXiv 2207.01899 To appear in Duke Math. J.

[Špenko and Van den Bergh 2017] **Š Špenko**, **M Van den Bergh**, *Non-commutative resolutions of quotient singularities for reductive groups*, Invent. Math. 210 (2017) 3–67 MR Zbl

[Stoppa and Thomas 2011] **J Stoppa**, **R P Thomas**, *Hilbert schemes and stable pairs: GIT and derived category wall crossings*, Bull. Soc. Math. France 139 (2011) 297–339 MR Zbl

[Szendrői 2008] **B Szendrői**, *Non-commutative Donaldson–Thomas invariants and the conifold*, Geom. Topol. 12 (2008) 1171–1202 MR Zbl

[Tabuada 2005] **G Tabuada**, *Une structure de catégorie de modèles de Quillen sur la catégorie des dg-catégories*, C. R. Math. Acad. Sci. Paris 340 (2005) 15–19 MR Zbl

[Toda 2010] **Y Toda**, *Curve counting theories via stable objects, I: DT/PT correspondence*, J. Amer. Math. Soc. 23 (2010) 1119–1157 MR Zbl

[Toda 2013] **Y Toda**, *Curve counting theories via stable objects, II: DT/ncDT flop formula*, J. Reine Angew. Math. 675 (2013) 1–51 MR Zbl

[Toda 2019] **Y Toda**, *Categorical Donaldson–Thomas theory for local surfaces*, preprint (2019) arXiv 1907.09076

[Toda 2021] **Y Toda**, *Semiorthogonal decompositions for categorical Donaldson–Thomas theory via Θ–stratifications*, preprint (2021) arXiv 2106.05496

[Toda 2022] **Y Toda**, *Birational geometry for d-critical loci and wall-crossing in Calabi–Yau* 3*–folds*, Algebr. Geom. 9 (2022) 513–573 MR Zbl

[Van den Bergh 2004] **M Van den Bergh**, *Three-dimensional flops and noncommutative rings*, Duke Math. J. 122 (2004) 423–455 MR Zbl

Kavli Institute for the Physics and Mathematics of the Universe (WPI), University of Tokyo
Kashiwa, Japan

`yukinobu.toda@ipmu.jp`

Proposed: Jim Bryan Received: 20 January 2022
Seconded: Mark Gross, Richard P Thomas Revised: 14 August 2022

GEOMETRY & TOPOLOGY
msp.org/gt

See inside back cover or msp.org/gt for submission instructions.

The subscription price for 2024 is US $805/year for the electronic version, and $1135/year (+$70, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Geometry & Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Geometry & Topology (ISSN 1465-3060 printed, 1364-0380 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

GT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

GEOMETRY & TOPOLOGY

Volume 28 Issue 3 (pages 1005–1499) 2024