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**On the subvarieties with nonsingular real loci of a real algebraic variety**

OLIVIER BENOIST

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Let  $X$  be a smooth projective real algebraic variety. We give new positive and negative results on the problem of approximating a submanifold of the real locus of  $X$  by real loci of subvarieties of  $X$ , as well as on the problem of determining the subgroups of the Chow groups of  $X$  generated by subvarieties with nonsingular real loci, or with empty real loci.

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## Introduction

We study the subvarieties with nonsingular real loci of a smooth projective real algebraic variety  $X$ . We consider their classes in the Chow groups of  $X$  and whether their real loci can approximate a fixed  $C^\infty$  submanifold of  $X(\mathbb{R})$ .

Let  $c$  and  $d$  denote the codimension and the dimension of these subvarieties. The guiding principle of our results is that, for each of the three problems that we will consider in Sections 0.1–0.3 subvarieties with nonsingular real loci are abundant when  $d < c$  (see Theorems 0.1, 0.4 and 0.6), but may be scarce for  $d \geq c$  (see Theorems 0.2, 0.5 and 0.7). The geometric rationale behind this principle, in the spirit of Whitney’s theorem in differential geometry [61], is that a  $d$ -dimensional variety mapped generically to  $X$  is expected not to self-intersect, and hence to have nonsingular image in  $X$  only if  $d < c$ .

### 0.1 Chow groups

It is an old question, going back to Borel and Haefliger [18, Section 5.17], to decide when the Chow group  $\mathrm{CH}_d(X)$  of a smooth projective variety  $X$  of dimension  $c + d$  over a field is generated by classes of smooth subvarieties of  $X$ . This is not true in general (a first counterexample is due to Hartshorne, Rees and Thomas [30, Theorem 1] for  $c = 2$  and  $d = 7$ ). The main positive result, due to Hironaka [31, Theorem, page 50], gives an affirmative answer if  $d < c$  and  $d \leq 3$  (his arguments now work over any infinite perfect field, thanks to Cossart and Piltant [22]). One may wonder whether Hironaka’s theorem holds as soon as  $d < c$ .

In real algebraic geometry, it is natural to consider, more generally, subvarieties that are smooth along their real loci. Our first theorem is a variant of Hironaka’s result in this setting, valid for all values of  $(c, d)$  such that  $d < c$ .

**Theorem 0.1** (Theorem 2.3) *Let  $X$  be a smooth projective variety of dimension  $c + d$  over  $\mathbb{R}$ . If  $d < c$ , then the group  $\mathrm{CH}_d(X)$  is generated by classes of closed subvarieties of  $X$  that are smooth along their real loci.*

Our proof is based on the smoothing technique developed by Hironaka in [31]. We need to refine it for two reasons: to control real loci, and to deal with the singularities that inevitably appear in the course of our proof if  $d > 3$ . To do so, we rely on the theory of linkage, as developed by Peskine and Szpiro [51] and Huneke and Ulrich [33]. Our argument works over an arbitrary real closed field.

Our second theorem shows that Theorem 0.1 is optimal for infinitely many values of  $c$ . We let  $\alpha(m)$  denote the number of ones in the dyadic expansion of  $m$ .

**Theorem 0.2** (Theorem 4.14) *If  $d \geq c$  are such that  $\alpha(c + 1) \geq 3$ , there exists an abelian variety  $X$  of dimension  $c + d$  over  $\mathbb{R}$  such that  $\mathrm{CH}_d(X)$  is not generated by classes of closed subvarieties of  $X$  that are smooth along their real loci.*

Theorem 0.2 is entirely new. The hypothesis that  $\alpha(c + 1) \geq 3$  cannot be weakened to  $\alpha(c + 1) \geq 2$ . Indeed, Kleiman has showed that the Chow group of codimension 2 cycles on a smooth projective fourfold or fivefold over an infinite field is generated by classes of smooth subvarieties; see [37, Theorem 5.8], where the hypothesis that the base field is algebraically closed may be discarded as a theory of Chow groups and Chern classes is now available in the required generality, due to Fulton [26].

Let us briefly explain the principle of the proof of Theorem 0.2 in the key case where  $c = d$ . Assume, to simplify, that  $\beta \in \mathrm{CH}_d(X)$  is the class of a closed subvariety  $Y \subset X$  which is smooth along  $Y(\mathbb{R})$ . Let  $g: W \rightarrow X$  be a morphism obtained by resolving the singularities of  $Y$ . The double locus of  $g$ , which is well-defined as a 0-cycle on  $W$ , has degree divisible by 4. Indeed, double points come two-by-two, and each such pair has a distinct complex conjugate. On the other hand, a double point formula due to Fulton [26, Section 9.3] computes the degree of this double locus in terms of the Chern classes of  $X$  and  $W$ , and of the self-intersection of  $Y$  in  $X$ . Divisibility results for Chern numbers due to Rees and Thomas [53, Theorem 3] now give restrictions on  $\beta$ , which sometimes lead to a contradiction.

This strategy applies as well over  $\mathbb{C}$ , and yields new examples of smooth projective complex varieties whose Chow groups are not generated by smooth subvarieties.

**Theorem 0.3** (Theorem 4.17) *If  $d \geq c$  are such that  $\alpha(c + 1) \geq 3$ , there exists a smooth projective variety  $X$  of dimension  $c + d$  over  $\mathbb{C}$  such that  $\mathrm{CH}_d(X)$  is not generated by classes of smooth closed subvarieties of  $X$ .*

This complements the counterexamples of Hartshorne, Rees and Thomas [30, Theorem 1], Debarre [24, Théorème 6] and Benoist and Debarre [12, Theorem 1.2] to the question of Borel and Haefliger. Theorem 0.3 is closely related to work of Rees and Thomas [54, Proposition 1], where the easier problem of showing that a cycle is not rationally equivalent to the class of a smooth subvariety (as opposed to a linear combination of classes of smooth subvarieties) is considered.

## 0.2 The kernel of the Borel–Haefliger map

If  $X$  is a smooth projective variety of dimension  $c + d$  over  $\mathbb{R}$ , a related problem is to determine the subgroup of  $\mathrm{CH}_d(X)$  generated by classes of subvarieties with no real points. This subgroup is included in the kernel  $\mathrm{CH}_d(X)_{\mathbb{R}\text{-hom}}$  of the Borel–Haefliger cycle class map  $\mathrm{cl}_{\mathbb{R}} : \mathrm{CH}_d(X) \rightarrow H_d(X(\mathbb{R}), \mathbb{Z}/2)$  [18] (or see [13, Section 1.6.2]) which associates with the class of an integral closed subvariety  $Y \subset X$  the homology class of its real locus. One may wonder when these two subgroups coincide. This question was known to have a positive answer for  $c = 1$  by Bröcker [19] (see also Scheiderer [56, Section 4]), for  $d = 0$  by Colliot-Thélène and Ischebeck [20, Proposition 3.2(ii)], and for  $d = 1$  and  $c = 2$  by Kucharz [41, Theorem 1.2].

Combining our improvements of Hironaka’s smoothing technique and a theorem of Ischebeck and Schülting, according to which  $\mathrm{CH}_d(X)_{\mathbb{R}\text{-hom}}$  is generated by classes of integral closed subvarieties of  $X$  whose real locus is not Zariski-dense [36, Main Theorem 4.3], we obtain an affirmative answer for all values of  $(c, d)$  such that  $d < c$ .

**Theorem 0.4** (Theorem 2.4) *Let  $X$  be a smooth projective variety of dimension  $c + d$  over  $\mathbb{R}$ . If  $d < c$ , then  $\mathrm{CH}_d(X)_{\mathbb{R}\text{-hom}}$  is generated by classes of closed integral subvarieties of  $X$  with empty real loci.*

Theorem 0.4 fails in general over nonarchimedean real closed fields (see Remark 2.5). Kucharz has shown in [41, Theorem 1.1] that the hypothesis  $d < c$  of Theorem 0.4 cannot be improved, for all even values of  $c \geq 2$ . We extend this result to all the values of  $c$  not of the form  $2^k - 1$ .

**Theorem 0.5** (Theorem 4.16) *If  $d \geq c$  are such that  $\alpha(c + 1) \geq 2$ , there exists an abelian variety  $X$  of dimension  $c + d$  over  $\mathbb{R}$  such that  $\mathrm{CH}_d(X)_{\mathbb{R}\text{-hom}}$  is not generated by classes of closed subvarieties of  $X$  with empty real loci.*

The proof of Theorem 0.5 follows the same path as that of Theorem 0.2, using additionally a new result on congruences of Chern numbers (Theorem 3.6). The hypothesis that  $\alpha(c + 1) \geq 2$  in Theorem 0.5 cannot be removed in general, in view of Bröcker’s abovementioned theorem [19] when  $c = 1$ .

## 0.3 Algebraic approximation

Let  $X$  be a smooth projective variety of dimension  $c + d$  over  $\mathbb{R}$ , and fix a closed  $d$ -dimensional  $\mathcal{C}^\infty$  submanifold  $j : M \hookrightarrow X(\mathbb{R})$ . We now focus on the classical question of whether  $M$  can be approximated by real loci of algebraic subvarieties of  $X$  (see Bochnak, Coste and Roy [16, Definition 12.4.10] or Akbulut and King [6, Section 2.8]):

**Property A** For all neighborhoods  $\mathcal{U} \subset \mathcal{C}^\infty(M, X(\mathbb{R}))$  of the inclusion, there exist  $\phi \in \mathcal{U}$  and a closed subvariety  $Y \subset X$  which is smooth along  $Y(\mathbb{R})$ , such that  $\phi(M) = Y(\mathbb{R})$ .

One must take into account the topological obstructions to the validity of Property A. The finest ones are based on cobordism theory and originate from Benedetti and Tognoli [11] or Akbulut and King [3].

If  $T$  is a topological space, recall that two continuous maps  $f_1: N_1 \rightarrow T$  and  $f_2: N_2 \rightarrow T$ , where the  $N_i$  are  $d$ -dimensional compact  $C^\infty$  manifolds, are said to be *cobordant* if there exist a compact  $C^\infty$  manifold with boundary  $C$ , a diffeomorphism  $\partial C \simeq N_1 \cup N_2$ , and a continuous map  $f: C \rightarrow T$  such that  $f|_{N_i} = f_i$  for  $i \in \{1, 2\}$ . The group (for the disjoint union) of cobordism equivalence classes of such maps is the  $d^{\text{th}}$  *unoriented cobordism group*  $\text{MO}_d(T)$  of  $T$ .

Let  $\text{MO}_d^{\text{alg}}(X(\mathbb{R})) \subset \text{MO}_d(X(\mathbb{R}))$  be the subgroup generated by cobordism classes of continuous maps of the form  $g(\mathbb{R}): W(\mathbb{R}) \rightarrow X(\mathbb{R})$ , where  $g: W \rightarrow X$  is a morphism of smooth projective varieties over  $\mathbb{R}$  and  $W$  has dimension  $d$ . The following property is a necessary condition for the validity of **Property A**:

**Property B** One has  $[j: M \hookrightarrow X(\mathbb{R})] \in \text{MO}_d^{\text{alg}}(X(\mathbb{R}))$ .

We show that it is the only obstruction to the validity of **Property A** for low values of  $d$ :

**Theorem 0.6** (**Theorem 2.7**) *Properties A and B are equivalent if  $d < c$ .*

**Theorem 0.6** was already known when  $d = 1$ , thanks to Bochnak and Kucharz [17, Theorem 1.1] for  $c = 2$ , and to Wittenberg and the author [14, Theorem 6.8] for any  $c \geq 2$  (improving earlier results by Akbulut and King [5]). **Theorem 0.6** is new for  $d \geq 2$ .

Our proof is based on a relative Nash–Tognoli theorem (see Benedetti and Tognoli [11, Proposition 4.1] or Akbulut and King [3, Proposition 0.2]), which solves the approximation problem up to unwanted singular points. To remove these singular points, we use the refinements of Hironaka’s smoothing method already mentioned in Sections 0.1–0.2. Hironaka’s smoothing technique, as developed in [31], had already been applied in the context of real algebraic approximation in the proof of [14, Theorem 6.8].

We also prove that **Theorem 0.6** is sharp: it may fail as soon as  $d \geq c$ , for infinitely many values of  $c$ . Recall that  $\alpha(m)$  is the number of ones in the dyadic expansion of  $m \geq 0$ .

**Theorem 0.7** (**Theorem 4.19**) *If  $d \geq c$  are such that  $\alpha(c + 1) = 2$ , there exist  $X$  and  $M$  such that **Property A** fails but **Property B** holds.*

To the best of our knowledge, **Theorem 0.7** features the first examples demonstrating that **Properties A** and **B** are not equivalent in general. The hypothesis that  $\alpha(c + 1) = 2$  cannot be entirely dispensed with, as **Properties A** and **B** are equivalent when  $c = 1$  (see **Proposition 5.5**). The values of  $c$  for which **Theorem 0.7** applies are  $c \in \{2, 4, 5, 8, 9, 11, 16, \dots\}$ . We have not been able to disprove the equivalence of **Properties A** and **B** for other values of  $c$ , for instance  $c = 3$ .

The proof of **Theorem 0.7** uses techniques similar to those of **Theorems 0.2** and **0.5**. Although both **Properties A** and **B** only involve real loci, the proof of **Theorem 0.7** makes use, in an essential way, of global topological properties of sets of complex points, through their classes in the complex cobordism ring  $\text{MU}_*$ .

In [43, page 269], Kucharz and van Hamel ask whether **Property A** always holds when  $X = \mathbb{P}_{\mathbb{R}}^n$ . The obstructions used in the proof of **Theorem 0.7** show that this question would have a negative answer if one replaced  $\mathbb{P}_{\mathbb{R}}^n$  with other very similar varieties, such as some products of projective spaces (**Property B** always holds for these varieties by [11, Remark 3, page 103]). This demonstrates the very particular role played by projective spaces in the question of Kucharz and van Hamel.

**Theorem 0.8** (**Theorem 4.23**) *For all  $k \geq 1$ , **Property A** fails in general for  $c = d = 2^k$  and  $X = \mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^{2^{k+1}-1}$ .*

## 0.4 Notation and conventions

A variety over a field  $k$  is a separated scheme of finite type over  $k$ . Smooth varieties over  $k$  are understood to be equidimensional. If  $f: X \rightarrow Y$  is a morphism of varieties over  $k$  and  $k'$  is a field extension of  $k$ , we let  $f(k'): X(k') \rightarrow Y(k')$  be the map induced at the level of  $k'$ -points. We denote by  $\mathbb{R}$  and  $\mathbb{C}$  the fields of real and complex numbers.

All  $C^\infty$  manifolds are assumed to be Hausdorff and second countable. We endow the set  $C^\infty(M, N)$  of  $C^\infty$  maps between two  $C^\infty$  manifolds with the weak  $C^\infty$  topology; see Hirsch [32, page 36].

For  $m \geq 0$ , we let  $\alpha(m)$  be the number of ones in the dyadic expansion of  $m$ .

## Structure

We study linkage to expand the scope of Hironaka's smoothing technique in **Section 1**, and use it in **Section 2** to prove **Theorems 0.1, 0.4 and 0.6**. Generalities about complex cobordism and an application to the divisibility of the top Segre class may be found in **Section 3**. This result and a double point formula are combined in **Section 4** to prove **Theorems 0.2, 0.3, 0.5, 0.7 and 0.8**. Finally, variants of **Properties A and B** and their interactions are considered in **Section 5**.

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## 1 Linkage

In the whole of **Section 1**, we fix an infinite field  $k$ , a smooth projective variety  $V$  over  $k$ , a very ample line bundle  $\mathcal{O}_V(1)$  on  $V$ , and a (possibly empty) Cohen–Macaulay closed subscheme  $W \subset V$  of pure codimension  $r$  in  $V$ .

We study the subvarieties of  $V$  that are linked to  $W$  by complete intersections defined by sections of multiples of  $\mathcal{O}_V(1)$  in **Section 1.1**, and their behavior in families in **Sections 1.2–1.3**, focusing in particular on their images by a morphism and on their real loci when  $k$  is a real closed field in **Sections 1.4–1.5**.

If  $\underline{l} = (l_1, \dots, l_r)$  is an  $r$ -tuple of integers and if  $\mathcal{F}$  is a coherent sheaf on  $V$ , we set  $\mathcal{F}(\underline{l}) := \bigoplus_{i=1}^r \mathcal{F}(l_i)$ . In particular,  $H^0(V, \mathcal{F}(\underline{l})) = \bigoplus_{i=1}^r H^0(V, \mathcal{F}(l_i))$ . A statement depending on an  $r$ -tuple of integers  $\underline{l} = (l_1, \dots, l_r)$  is said to hold for  $\underline{l} \gg 0$  if it holds for  $l_r \gg \dots \gg l_2 \gg l_1 \gg 0$ , i.e. if  $l_1$  is big enough, if  $l_2$  is big enough (depending on  $l_1$ ), and so forth.

### 1.1 Linked subvarieties

Let  $\mathcal{I}_W \subset \mathcal{O}_V$  be the ideal sheaf of  $W$  in  $V$ . Choose an  $r$ -tuple of integers  $\underline{l} = (l_1, \dots, l_r)$  and a section  $\underline{F} \in H^0(V, \mathcal{I}_W(\underline{l}))$  such that  $\underline{F} = (F_1, \dots, F_r)$  is a regular sequence (such  $\underline{F}$  always exists if  $\mathcal{I}_W(\underline{l})$  is generated by its global sections, for instance for  $\underline{l} \gg 0$ ).

Let  $Z := \{F_1 = \dots = F_r = 0\} \subset V$  be the complete intersection it defines, and let  $\mathcal{I}_Z = \langle \underline{F} \rangle \subset \mathcal{O}_V$  be its ideal sheaf. Let  $W' \subset V$  be the subvariety with ideal sheaf  $\mathcal{I}_{W'} := (\mathcal{I}_Z : \mathcal{I}_W) \subset \mathcal{O}_V$ , where a local section  $s \in \mathcal{O}_V$  belongs to  $(\mathcal{I}_Z : \mathcal{I}_W)$  if multiplication by  $s$  induces a morphism  $\mathcal{I}_W \xrightarrow{s} \mathcal{I}_Z$ .

One has  $Z = W \cup W'$  set-theoretically. It is a theorem of Peskine and Szpiro [51, Proposition 1.3] that  $W' \subset V$  is also Cohen–Macaulay of pure codimension  $r$ , and that  $\mathcal{I}_W = (\mathcal{I}_Z : \mathcal{I}_{W'}) \subset \mathcal{O}_V$ . In view of the symmetry of the relation between the subschemes  $W$  and  $W'$  of  $V$ , they are said to be *linked* by the regular sequence  $\underline{F}$ . We write  $W \sim W'$ , or  $W \sim_{\underline{l}} W'$  if we want to emphasize that the regular sequence is a section of  $\mathcal{O}_V(\underline{l})$ . We also say that  $W \sim W'$  is the *link* defined by  $\underline{F}$ .

**Remarks 1.1** (i) In the whole of Section 1, we could have only considered links with respect to complete intersections of multidegree  $(l, \dots, l)$ , with  $l \gg 0$  when needed. The reason why we allow multidegrees  $\underline{l} = (l_1, \dots, l_r)$ , requiring  $\underline{l} \gg 0$  when needed, is to be able to apply directly the proof of [31, Lemma 5.1.1] in the proof of Proposition 1.6

(ii) In Sections 1.1–1.3, we could allow  $V$  to be any Gorenstein projective variety; see especially [51, Proposition 1.3].

**Lemma 1.2** *Let  $x \in V$ , and let  $g_1, \dots, g_r \in \mathcal{I}_{W,x} \subset \mathcal{O}_{V,x}$  be a regular sequence. Then, for an  $r$ -tuple of integers  $\underline{l} \gg 0$ , there exists a regular sequence  $\underline{F} \in H^0(V, \mathcal{I}_W(\underline{l}))$  such that the ideals  $\langle \underline{F} \rangle$  and  $\langle g_1, \dots, g_r \rangle$  of  $\mathcal{O}_{V,x}$  coincide.*

**Proof** Let  $Y \subset V$  and  $Y_i \subset V$  be the schematic closures of  $\{g_1 = \dots = g_r = 0\}$  and  $\{g_i = 0\}$  in  $V$ . Let  $\mathcal{I}_W, \mathcal{I}_Y, \mathcal{I}_{Y_i} \subset \mathcal{O}_V$  be the ideal sheaves of  $W, Y$  and  $Y_i$  in  $V$  and define  $\mathcal{I} := \mathcal{I}_W \cap \mathcal{I}_Y$  and  $\mathcal{I}_i := \mathcal{I}_W \cap \mathcal{I}_{Y_i}$ . The subscheme of  $V$  defined by  $\mathcal{I}$  has support  $Y \cup W$ , and hence has pure codimension  $r$  in  $V$ , and coincides with  $Y$  in a neighborhood of  $x$ .

Choose  $\underline{l} \gg 0$  so that the sheaves  $\mathcal{I}(\underline{l})$  and  $\mathcal{I}_i(l_i)$  are generated by their global sections, and choose a general element  $\underline{F} \in H^0(V, \mathcal{I}(\underline{l}))$ . Since  $\mathcal{I}_i(l_i)$  is globally generated, there exist  $G_i \in H^0(V, \mathcal{I}_i(l_i)) \subset H^0(V, \mathcal{I}(l_i))$  with  $\langle G_i \rangle = \langle g_i \rangle \subset \mathcal{O}_{V,x}$ , and hence with  $\langle G_1, \dots, G_r \rangle = \langle g_1, \dots, g_r \rangle \subset \mathcal{O}_{V,x}$ . Since  $\underline{F}$  has been chosen to be general, we deduce that  $\langle \underline{F} \rangle = \langle g_1, \dots, g_r \rangle \subset \mathcal{O}_{V,x}$ . Since  $\mathcal{I}(\underline{l})$  is globally generated, we also see that  $\underline{F}$  forms a regular sequence.  $\square$

**Lemma 1.3** *Let  $W = W_0 \sim_{l_1} W_1 \sim_{l_2} \cdots \sim_{l_j} W_j$  be links of subschemes of  $V$ . Assume that  $W$  is a local complete intersection at  $x \in W$ . For all  $r$ -tuples of integers  $l_{2j+1} \gg \cdots \gg l_{j+1} \gg 0$ , there exists a chain  $W_j \sim_{l_{j+1}} W_{j+1} \sim_{l_{j+2}} \cdots \sim_{l_{2j+1}} W_{2j+1}$  of links of subschemes of  $V$  such that  $x \notin W_{2j+1}$ .*

**Proof** For  $1 \leq i \leq j$ , let  $\underline{F}_i$  be the regular sequence yielding the link  $W_{i-1} \sim_{l_i} W_i$ . Thanks to Lemma 1.2, one may choose inductively, for  $j + 1 \leq i \leq 2j$ , an  $r$ -tuple  $l_i \gg 0$  and a regular sequence  $\underline{F}_i \in H^0(V, \mathcal{I}_{W_{i-1}}(l_i))$  such that the ideals  $\langle \underline{F}_i \rangle$  and  $\langle \underline{F}_{2j+1-i} \rangle$  of  $\mathcal{O}_{V,x}$  coincide. This gives rise to a link  $W_{i-1} \sim_{l_i} W_i$  with the property that  $W_i$  and  $W_{2j-i}$  coincide in a neighborhood of  $x$ , by the symmetry of the link construction.

The subschemes  $W_{2j}$  and  $W_0 = W$  then coincide in a neighborhood of  $x$ , and hence  $W_{2j}$  is a local complete intersection at  $x$ , defined by a regular sequence  $g_1, \dots, g_r \in \mathcal{I}_{W_{2j},x}$ . A final application of Lemma 1.2 provides us with an  $r$ -tuple  $l_{2j+1} \gg 0$ , and with a link  $W_{2j} \sim_{l_{2j+1}} W_{2j+1}$  associated with a regular sequence  $\underline{F}_{2j+1}$  such that  $\langle \underline{F}_{2j+1} \rangle = \langle g_1, \dots, g_r \rangle \subset \mathcal{O}_{V,x}$ . It follows that  $x \notin W_{2j+1}$ .  $\square$

### 1.2 Linkage in families

Let  $B$  be a smooth variety over  $k$ . Let  $\mathfrak{W} \subset V \times B$  be a closed subscheme of pure codimension  $r$  with ideal sheaf  $\mathcal{I}_{\mathfrak{W}} \subset \mathcal{O}_{V \times B}$ , such that the second projection  $f : \mathfrak{W} \rightarrow B$  is flat with Cohen–Macaulay fibers. Let  $\underline{l}$  be an  $r$ -tuple such that, letting  $p : V \times B \rightarrow B$  denote the second projection, the adjunction morphism  $p^* p_*(\mathcal{I}_{\mathfrak{W}}(\underline{l})) \rightarrow \mathcal{I}_{\mathfrak{W}}(\underline{l})$  is surjective and  $R^i p_*(\mathcal{I}_{\mathfrak{W}}(\underline{l})) = 0$  for  $i > 0$ . Note that, under these conditions, the pushforward sheaf  $E := p_*(\mathcal{I}_{\mathfrak{W}}(\underline{l}))$  is a vector bundle such that the natural morphism  $E|_b \rightarrow H^0(V, \mathcal{I}_{\mathfrak{W}_b}(\underline{l}))$  is an isomorphism for all  $b \in B$  [29, III, Theorem 12.11].

View  $E \rightarrow B$  as a geometric vector bundle over  $B$ . A point of  $E$  over  $b \in B$  corresponds to a section  $\underline{F} \in H^0(V, \mathcal{I}_{\mathfrak{W}_b}(\underline{l}))$ . Let  $B' \subset E$  be the open subset of those points such that  $\underline{F}$  forms a regular sequence, and hence defines a complete intersection in  $V$ . Let  $\mathfrak{Z} \subset V \times B'$  be the universal family of these complete intersections, and let  $\mathfrak{W}_{B'} \subset V \times B'$  be the base change of  $\mathfrak{W}$ , with ideal sheaves  $\mathcal{I}_{\mathfrak{Z}}, \mathcal{I}_{\mathfrak{W}_{B'}} \subset \mathcal{O}_{V \times B'}$ . We consider the subscheme  $\mathfrak{W}' \subset V \times B'$  with ideal sheaf  $\mathcal{I}_{\mathfrak{W}'} := (\mathcal{I}_{\mathfrak{Z}} : \mathcal{I}_{\mathfrak{W}_{B'}})$  and we let  $f' : \mathfrak{W}' \rightarrow B'$  denote the second projection.

By Proposition 1.4, this extends the construction of Section 1.1 in the relative setting.

**Proposition 1.4** *The morphism  $f' : \mathfrak{W}' \rightarrow B'$  is flat with Cohen–Macaulay fibers.*

For  $b \in B'$ , one has  $\mathcal{I}_{\mathfrak{W}'_b} = (\mathcal{I}_{\mathfrak{Z}_b} : \mathcal{I}_{\mathfrak{W}_b})$ .

**Proof** The scheme  $\mathfrak{W}$  is Cohen–Macaulay by [27, Corollaire 6.3.5(ii)], and therefore so is  $\mathfrak{W}'$  by [51, Proposition 1.3]. Since  $f'$  is equidimensional with regular base and Cohen–Macaulay total space, it is flat by [27, Proposition 6.1.5].

Choose a regular system of parameters  $x_1, \dots, x_N$  of the regular local ring  $\mathcal{O}_{B',b}$ . To show the equality of ideals  $\mathcal{I}_{\mathfrak{W}'_b} = (\mathcal{I}_{\mathfrak{Z}_b} : \mathcal{I}_{\mathfrak{W}_b})$  at a point  $v \in V_b$ , one may apply [33, Lemma 2.12]  $N$  times successively in



the local ring  $\mathcal{O}_{V_b, v}$  (this is essentially what is done in [33, Proposition 2.13]). That the fibers  $\mathfrak{W}'_b$  of  $f'$  are Cohen–Macaulay now follows from [51, Proposition 1.3].  $\square$

### 1.3 Moduli of links

We now iterate the construction of Section 1.2, thus adapting to our global setting a local construction due to Huneke and Ulrich [34; 35].

Recall that  $W \subset V$  is a Cohen–Macaulay closed subscheme of pure codimension  $r$ . We set  $L_0(W) := \text{Spec}(k)$ ,  $\mathfrak{W}_0 := W$  and  $f_0: \mathfrak{W}_0 \rightarrow L_0(W)$  to be the structural morphism. We inductively construct  $f_i: \mathfrak{W}_i \rightarrow L_i(W)$  for  $i \geq 1$  by choosing an  $r$ -tuple  $\underline{l}_i \gg 0$ , by applying the construction of Section 1.2 to  $f_{i-1}: \mathfrak{W}_{i-1} \rightarrow L_{i-1}(W)$ , and by setting  $(f_i: \mathfrak{W}_i \rightarrow L_i(W)) = (f'_{i-1}: \mathfrak{W}'_{i-1} \rightarrow L_{i-1}(W)')$ . The varieties  $L_i(W)$  are irreducible, smooth over  $k$  and  $k$ -rational, and the morphisms  $f_i$  are flat with Cohen–Macaulay fibers, as an induction argument based on Proposition 1.4 shows.

We call  $f_j: \mathfrak{W}_j \rightarrow L_j(W)$  the  $j^{\text{th}}$  moduli of links of  $W$  (with respect to the degrees  $\underline{l}_1, \dots, \underline{l}_j$ ). Its points parametrize sequences  $(\underline{F}_i)_{1 \leq i \leq j}$  of regular sequences that give rise to chains of linked subvarieties  $W = W_0 \sim_{\underline{l}_1} W_1 \sim_{\underline{l}_2} \dots \sim_{\underline{l}_j} W_j$  of  $V$ .

**Remark 1.5** The construction of the  $j^{\text{th}}$  moduli of links  $f_j: \mathfrak{W}_j \rightarrow L_j(W)$  goes through with no modifications if  $k$  is finite, but beware that  $L_j(W)(k)$  might be empty in this case.

### 1.4 Images by a morphism

In Sections 1.4–1.5, we fix a smooth morphism  $\pi: V \rightarrow X$  of smooth projective varieties over  $k$  and we let  $d$  and  $n$  be the dimensions of  $W$  and  $X$ . In Propositions 1.6, 1.7, 1.9 and 1.14, we study the images by  $\pi$  of subvarieties of  $X$  linked to  $W$ , under the assumption that  $n > 2d$ . Proposition 1.6 is due to Hironaka [31]. Proposition 1.7 is a simple Bertini theorem. Proposition 1.9, which is the main result of Section 1.4, is more delicate since one must deal with singularities of varieties linked to  $W$ . Proposition 1.14, which is the main result of Section 1.5, is specific to the case where  $k$  is a real closed field.

**Proposition 1.6** (Hironaka) *Assume that  $n > 2d$ , that  $d \leq 3$  and that  $W$  is smooth. Then, for  $j \geq 4$  and  $r$ -tuples of integers  $\underline{l}_j \gg \dots \gg \underline{l}_1 \gg 0$ , there exists a chain of linked smooth subvarieties  $W = W_0 \sim_{\underline{l}_1} W_1 \sim_{\underline{l}_2} \dots \sim_{\underline{l}_j} W_j$  of  $V$  such that  $\pi|_{W_j}: W_j \rightarrow X$  is a closed embedding.*

**Proof** Choose general links  $W = W_0 \sim_{\underline{l}_1} W_1 \sim_{\underline{l}_2} \dots \sim_{\underline{l}_j} W_j$ . Since  $d \leq 3$  and  $W$  is smooth, the  $W_i$  are smooth by [31, Corollary 3.9.1].

That  $\pi|_{W_j}: W_j \rightarrow X$  is a closed embedding may be checked over an algebraic closure of  $k$ , where it follows from the proof of [31, Lemma 5.1.1]. More precisely, define  $A(W_i) \subset W_i$  to be the closed subset of those  $x \in W_i$  such that  $\pi|_{W_i}: W_i \rightarrow X$  is not a closed embedding above a neighborhood of  $\pi(x)$ . One has  $\dim(A(W_0)) \leq \dim(W) \leq 3$ . Moreover,  $\dim(A(W_{i+1})) \leq \dim(A(W_i))$ , and the inequality is strict if  $A(W_i) \neq \emptyset$  by the proof of [31, Lemma 5.1.1]. It follows that  $A(W_i) = \emptyset$  for  $i \geq 4$ , and hence that  $\pi|_{W_j}: W_j \rightarrow X$  is a closed embedding.  $\square$

**Proposition 1.7** Assume that  $n > 2d$ , and let  $\underline{l} = (l_1, \dots, l_r)$  be an  $r$ -tuple of integers such that the linear systems  $H^0(V, \mathcal{I}_W(l_i))$  embed  $V \setminus W$  in projective spaces. Then, for  $\underline{F} \in H^0(V, \mathcal{I}_W(\underline{l}))$  general, the link  $W \sim_{\underline{l}} W'$  associated with  $\underline{F}$  satisfies:

- (i) The variety  $S := W' \setminus (W \cap W')$  is smooth.
- (ii) The morphism  $\pi|_S : S \rightarrow X$  is an embedding.
- (iii) The subsets  $\pi(S)$  and  $\pi(W)$  of  $X$  are disjoint.

**Proof** Apply Lemma 1.8 with  $Y = V \setminus W$ ,  $f = \pi|_{V \setminus W}$ , and  $F = \pi(W)$ . □

**Lemma 1.8** Let  $f : Y \rightarrow X$  be a smooth morphism of smooth varieties over  $k$ , let  $F \subset X$  be a closed subset and let  $V_1, \dots, V_r$  be linear systems on  $Y$  inducing embeddings of  $Y$  in projective spaces. If  $\dim(X) > 2(\dim(Y) - r)$  and  $\dim(X) > \dim(F) + \dim(Y) - r$ , then, for general  $\sigma_i \in V_i$ , the variety  $S := \{\sigma_i = 0\}$  is smooth, the morphism  $f|_S : S \rightarrow X$  is an embedding and  $f(S) \cap F = \emptyset$ .

**Proof** The smoothness of  $S$  follows from the Bertini theorem. For general  $\sigma_i$ , the complete intersection  $S \cap f^{-1}(F)$  in  $f^{-1}(F)$  is empty, and hence  $f(S) \cap F = \emptyset$ .

Let  $H$  be the Hilbert scheme parametrizing zero-dimensional subschemes  $Z \subset Y$  of length 2 in the fibers of  $f$ . Let  $B \subset H \times V_1 \times \dots \times V_r$  be the closed subset parametrizing tuples  $([Z], \sigma_1, \dots, \sigma_r)$  such that the  $\sigma_i$  vanish on  $Z$ . Computing

$$\dim(B) = \dim(H) + \sum_i \dim(V_i) - 2r = 2 \dim(Y) - \dim(X) + \sum_i \dim(V_i) - 2r$$

shows that  $\dim(B) < \sum_i \dim(V_i)$ . To ensure that  $f|_S$  is an embedding, it suffices to choose  $(\sigma_1, \dots, \sigma_r)$  outside of the image of the projection  $B \rightarrow V_1 \times \dots \times V_r$ . □

**Proposition 1.9** Assume  $n > 2d$  and that  $W$  is a local complete intersection. For  $j \gg 0$  and  $r$ -tuples of integers  $\underline{l}_j \gg \dots \gg \underline{l}_1 \gg 0$ , there exists a chain of linked subvarieties  $W = W_0 \sim_{\underline{l}_1} W_1 \sim_{\underline{l}_2} \dots \sim_{\underline{l}_j} W_j$  of  $V$  with the property that  $\pi|_{W_j} : W_j \rightarrow X$  is geometrically injective.

**Proof** Let  $W = W_0 \sim_{\underline{l}_1} W_1 \sim_{\underline{l}_2} \dots \sim_{\underline{l}_j} W_j$  be a general chain of links. Define  $C(W_i) \subset W_i$  to be the constructible subset of those  $x \in W_i$  such that  $(\pi|_{W_i})^{-1}(\pi(x))$  has more than one geometric point. Of course, one has  $\dim(C(W_0)) \leq \dim(W) = d$ . Proposition 1.7 implies that  $C(W_{i+1}) \subset C(W_i)$  for all  $i \geq 0$ . We also claim that  $\dim(C(W_{2i+1})) < \dim(C(W_i))$  if  $2i + 1 \leq j$  and  $C(W_i) \neq \emptyset$ . These facts imply that  $C(W_j) = \emptyset$  if  $j \geq 2^{d+1} - 1$ , which concludes the proof.

It remains to prove the claim. Assume that  $2i + 1 \leq j$  and that  $C(W_i) \neq \emptyset$ . Choose finitely many points  $x_1, \dots, x_N \in C(W_i)$ , including at least one in each irreducible component of the Zariski closure of  $C(W_i)$ . By Lemma 1.3, there exists a chain of links  $W_i \sim_{\underline{l}_i} W_{i+1}^{(s)} \sim_{\underline{l}_{i+1}} \dots \sim_{\underline{l}_{2i+1}} W_{2i+1}^{(s)}$  such that  $x_s \notin W_{2i+1}^{(s)}$  for all  $1 \leq s \leq N$ . Since  $W_i \sim_{\underline{l}_i} W_{i+1} \sim_{\underline{l}_{i+1}} \dots \sim_{\underline{l}_{2i+1}} W_{2i+1}$  corresponds to a general point of the  $(i+1)^{\text{th}}$  moduli of links of  $W_i$  in the sense of Section 1.3, we deduce that  $x_s \notin W_{2i+1}$ , and hence that  $x_s \notin C(W_{2i+1})$  for  $1 \leq s \leq N$ . The chain of inclusions  $C(W_{2i+1}) \subset C(W_{2i}) \subset \dots \subset C(W_i)$  implies that  $\dim(C(W_{2i+1})) < \dim(C(W_i))$ , which proves the claim. □

**Remark 1.10** The proof of Proposition 1.9 requires a huge number of links (exponential in  $d$ ). We do not know if this is necessary, or a quirk of the proof.

### 1.5 Real loci

In Section 1.5, we keep the notation of Section 1.4 and assume moreover that  $k = R$  is a real closed field, for instance the field  $\mathbb{R}$  of real numbers.

**Lemma 1.11** Fix  $r$ -tuples of integers  $\underline{l}_1 = (l_{1,1}, \dots, l_{1,r})$  and  $\underline{l}_2 = (l_{2,1}, \dots, l_{2,r})$  with  $l_{2,i} - l_{1,i}$  nonnegative and even for  $1 \leq i \leq r$ . Let  $W \sim_{\underline{l}_1} W_1$  be a link. Then there exists a link  $W_1 \sim_{\underline{l}_2} W_2$  such that  $W_2 = W$  in a neighborhood of  $V(R)$ .

**Proof** Let  $(u_1, \dots, u_N)$  be a basis of  $H^0(V, \mathcal{O}_V(1))$ . The section  $u := \sum_{m=1}^N u_m^2$  does not vanish on  $V(R)$ . Let  $v_1, \dots, v_r \in H^0(V, \mathcal{O}_V(2))$  be general small deformations of  $u$ . They are general elements of  $H^0(V, \mathcal{O}_V(2))$  that do not vanish on  $V(R)$ .

Let  $\underline{F} = (F_1, \dots, F_r) \in H^0(V, \mathcal{I}_W(\underline{l}_1))$  be a regular sequence defining  $W \sim_{\underline{l}_1} W_1$ . There exist integers  $a_i \geq 0$  such that  $\underline{G} := (v_1^{a_1} F_1, \dots, v_r^{a_r} F_r) \in H^0(V, \mathcal{I}_{W_1}(\underline{l}_2))$ . Since the  $v_i$  are general and since  $\underline{F}$  is a regular sequence,  $\underline{G}$  is also regular sequence. Let  $W_1 \sim_{\underline{l}_2} W_2$  be the link it defines.

The ideal sheaves  $\langle \underline{F} \rangle$  and  $\langle \underline{G} \rangle$  of  $\mathcal{O}_V$  coincide in a neighborhood of  $V(R)$  since the  $v_i$  do not vanish on  $V(R)$ . It thus follows from the symmetry of linkage (see Section 1.1) that  $W_2 = W$  in a neighborhood of  $V(R)$ . □

**Lemma 1.12** Set  $W_0 := W$ . Suppose that  $W_0$  is smooth along  $W_0(R)$  and that  $n > 2d$ . Fix  $r$ -tuples of even integers  $\underline{l}_2 \gg \underline{l}_1 \gg 0$ . If  $R = \mathbb{R}$ , fix a neighborhood  $\mathcal{U} \subset C^\infty(W_0(\mathbb{R}), V(\mathbb{R}))$  of the inclusion. Then there exist links  $W_0 \sim_{\underline{l}_1} W_1 \sim_{\underline{l}_2} W_2$  with the following properties:

- (i) The variety  $W_2$  is smooth along  $W_2(R)$ .
- (ii) Let  $D(W_i) \subset W_i$  be the subset of those  $x \in W_i$  such that  $\pi|_{W_i}$  is not immersive at  $x$ . Set  $d_i := \sup_{x \in D(W_i)(R)} \dim_x D(W_i)$ . If  $D(W_0)(R) \neq \emptyset$ , then  $d_2 < d_0$ .
- (iii) If  $R = \mathbb{R}$ , there exists a diffeomorphism  $\phi: W_0(\mathbb{R}) \xrightarrow{\sim} W_2(\mathbb{R})$  such that, letting  $\iota: W_2(\mathbb{R}) \rightarrow V(\mathbb{R})$  denote the inclusion, one has  $\iota \circ \phi \in \mathcal{U}$ .

**Proof** Choose finitely many points  $x_1, \dots, x_N \in W_0(R)$ , including at least one in each irreducible component of  $D(W_0)$  that has real points. By Lemma 1.3, a link  $W_0 \sim_{\underline{l}_1} W_1$  corresponding to a general point of the first moduli of links of  $W_0$  (in the sense of Section 1.3) has the property that  $x_s \notin W_1$  for  $1 \leq s \leq N$ . Lemma 1.11 shows the existence of a link  $W_1 \sim_{\underline{l}_2} \widetilde{W}$  such that  $\widetilde{W} = W_0$  in a neighborhood of  $V(R)$ . We deduce that  $\widetilde{W}$  is smooth along  $\widetilde{W}(R) = W_0(R)$ .

Let  $\tilde{f}: \widetilde{\mathfrak{M}} \rightarrow L_1(W_1)$  be the first moduli of links of  $W_1$  with respect to the degree  $\underline{l}_2$  (as in Section 1.3) and let  $b \in L_1(W_1)(R)$  be the point associated with  $W_1 \sim_{\underline{l}_2} \widetilde{W}$ . Proposition 1.4 shows that  $\widetilde{\mathfrak{M}}_b = \widetilde{W}$  and

that  $\tilde{f}$  is flat, and hence smooth in a neighborhood of  $\tilde{\mathfrak{W}}_b(R)$ . As  $\tilde{f}$  is proper, the map  $\tilde{f}(R)$  is closed by [25, Theorem 9.6]. We deduce that there exists a Euclidean neighborhood  $\Omega$  of  $b$  in  $L_1(W_1)(R)$  such that the morphism  $\tilde{f}$  is smooth along  $\tilde{f}(R)^{-1}(\Omega)$ . Choose such an  $\Omega$  small enough. Since  $L_1(W_1)$  is smooth and irreducible (see Section 1.3), the subset  $\Omega \subset L_1(W_1)$  is Zariski-dense (apply [16, Proposition 2.8.14]). Consequently, one may choose  $a \in \Omega$  general. Let  $W_1 \sim_{l_2} W_2 := \tilde{\mathfrak{W}}_a$  be the associated link. Assertion (i) holds by our choice of  $\Omega$ .

Let  $D \subset \tilde{\mathfrak{W}}$  be the closed subset of those  $x \in \tilde{\mathfrak{W}}$  such that the morphism  $\pi|_{\tilde{f}^{-1}(\tilde{f}(x))}: \tilde{\mathfrak{W}}_{\tilde{f}(x)} \rightarrow X$  is not immersive at  $x$ . Set  $E := D \cap (W_1 \times L_1(W_1)) \subset \tilde{\mathfrak{W}}$ . By Proposition 1.7(ii), there is a proper closed subset  $F \subset L_1(W_1)$  such that  $D \subset \tilde{f}^{-1}(F) \cup (W_1 \times L_1(W_1))$  as subsets of  $V \times L_1(W_1)$ . Since  $a$  has been chosen to be general, it lies outside of  $F$ . We deduce the inclusion  $D(W_2) \subset E_a$  of subsets of  $\tilde{\mathfrak{W}}_a$ . The function  $x \mapsto \dim_x(E_{\tilde{f}(x)})$  is upper semicontinuous for the Zariski topology on  $E$  by [28, Théorème 13.1.3], and hence upper semicontinuous for the Euclidean topology on  $E(R)$ . Since  $\tilde{f}|_E: E \rightarrow L_1(W_1)$  is proper, the map  $\tilde{f}|_E(R)$  is closed by [25, Theorem 9.6]. If  $\Omega$  has been chosen small enough, we deduce at once the inequality

$$(1.13) \quad \sup_{x \in E_b(R)} \dim_x E_b \geq \sup_{x \in E_a(R)} \dim_x E_a.$$

As  $\tilde{\mathfrak{W}}_b = W_0$  in a neighborhood of  $V(\mathbb{R})$ , one has  $E_b \subset D(W_0)$  in a neighborhood of  $V(\mathbb{R})$ . Since none of the  $x_s$  belong to  $W_1$ , no irreducible component of  $D(W_0)$  that has real points is included in  $E_b$ . If  $D(W_0)(R) \neq \emptyset$ , the left-hand side of (1.13) is  $< d_0$ . On the other hand, the right-hand side of (1.13) is  $\geq d_2$  since  $D(W_2) \subset E_a$ . The inequality  $d_2 < d_0$  follows, proving (ii).

If  $R = \mathbb{R}$ , one may assume  $\Omega$  to be connected. Ehresmann’s theorem applied to the proper submersion  $\tilde{f}(\mathbb{R})|_{\tilde{f}(\mathbb{R})^{-1}(\Omega)}: \tilde{f}(\mathbb{R})^{-1}(\Omega) \rightarrow \Omega$  yields a diffeomorphism  $\psi: \Omega \times \tilde{\mathfrak{W}}_b(\mathbb{R}) \xrightarrow{\sim} \tilde{f}(\mathbb{R})^{-1}(\Omega)$  compatible with the projections to  $\Omega$ . If  $\Omega$  has been chosen small enough, the composition

$$W_0(\mathbb{R}) = \tilde{\mathfrak{W}}_b(\mathbb{R}) \xrightarrow{\psi_a} \tilde{\mathfrak{W}}_a(\mathbb{R}) = W_2(\mathbb{R}) \xrightarrow{l} V(\mathbb{R})$$

belongs to  $\mathcal{U}$  for all  $a \in \Omega$ . Assertion (iii) is proven. □

**Proposition 1.14** *Suppose that  $W$  is smooth along  $W(R)$  and that  $n > 2d$ . Let  $j \gg 0$  be even and let  $l_j \gg \dots \gg l_1 \gg 0$  be  $r$ -tuples of even integers. Define  $f_j: \mathfrak{W}_j \rightarrow L_j(W)$  as in Section 1.3. Then there exists a nonempty subset  $\Omega \subset L_j(W)(R)$  which is open for the Euclidean topology such that the following holds for all  $b \in \Omega$ :*

- (i) *The variety  $\mathfrak{W}_{j,b}$  is smooth along  $\mathfrak{W}_{j,b}(R)$ .*
- (ii) *The morphism  $\pi|_{\mathfrak{W}_{j,b}}$  is immersive along  $\mathfrak{W}_{j,b}(R)$ .*

*If moreover  $R = \mathbb{R}$  and  $\mathcal{U} \subset C^\infty(W(\mathbb{R}), V(\mathbb{R}))$  is a neighborhood of the inclusion, one may ensure that the following holds:*

- (iii) *There exists a diffeomorphism  $\phi_b: W(\mathbb{R}) \xrightarrow{\sim} \mathfrak{W}_{j,b}(\mathbb{R})$  such that, letting  $\iota_b: \mathfrak{W}_{j,b}(\mathbb{R}) \rightarrow V(\mathbb{R})$  denote the inclusion, one has  $\iota_b \circ \phi_b \in \mathcal{U}$ .*

**Proof** Choose  $j \geq 2d + 2$  even. Applying [Lemma 1.12](#)  $\frac{j}{2}$  times shows the existence of a chain of linked subvarieties  $W = W_0 \sim_{l_1} W_1 \sim_{l_2} \cdots \sim_{l_j} W_j$  of  $V$  such that  $W_j$  is smooth along  $W_j(R)$  and such that  $\pi|_{W_j}$  is immersive along  $W_j(R)$ . Moreover, if  $R = \mathbb{R}$ , [Lemma 1.12](#) ensures the existence of a diffeomorphism  $\phi: W(\mathbb{R}) \xrightarrow{\sim} W_j(\mathbb{R})$  such that  $\iota \circ \phi \in \mathcal{U}$ , where  $\iota: W_j(\mathbb{R}) \rightarrow V(\mathbb{R})$  denotes the inclusion.

Let  $a \in L_j(W)(R)$  be the point associated with  $W = W_0 \sim_{l_1} W_1 \sim_{l_2} \cdots \sim_{l_j} W_j$ . [Proposition 1.4](#) shows that  $\mathfrak{W}_{j,a} = W_j$  and that  $f_j$  is flat, and hence smooth in a neighborhood of  $\mathfrak{W}_{j,a}(R)$ . As  $f_j$  is proper, the map  $f_j(R)$  is closed by [[25](#), Theorem 9.6]. We deduce the existence of a neighborhood  $\Omega$  of  $a$  in  $L_j(W)(R)$  such that  $f_j$  is smooth along  $f_j(R)^{-1}(\Omega)$ . Assertion (i) follows. So does (ii) after maybe shrinking  $\Omega$ . If  $R = \mathbb{R}$ , that (iii) holds after shrinking  $\Omega$  further follows from Ehresmann's theorem applied to the proper submersion  $f_j(\mathbb{R})|_{f_j(\mathbb{R})^{-1}(\Omega)}: f_j(\mathbb{R})^{-1}(\Omega) \rightarrow \Omega$ .  $\square$

## 2 Smoothing by linkage

Let us apply linkage theory as developed in [Section 1](#) to prove [Theorems 0.1, 0.4 and 0.6](#).

### 2.1 Main statement

Here is the technical result from which the main theorems of [Section 2](#) will follow:

**Proposition 2.1** *Let  $g: W \rightarrow X$  be a morphism of smooth projective varieties over a real closed field  $R$ . Let  $d$  and  $n$  be the dimensions of  $W$  and  $X$  and assume that  $n > 2d$ . If  $R = \mathbb{R}$ , fix a neighborhood  $\mathcal{V} \subset \mathcal{C}^\infty(W(\mathbb{R}), X(\mathbb{R}))$  of  $g(\mathbb{R})$ . Then there exists a closed subvariety  $i: Y \hookrightarrow X$  of dimension  $d$  with the following properties:*

- (i) *The variety  $Y$  is smooth along  $Y(R)$ .*
- (ii) *If  $d \leq 3$ , then  $Y$  is smooth.*
- (iii) *The class  $g_*[W] - [Y] \in \text{CH}_d(X)$  is a linear combination of classes of smooth closed subvarieties of  $X$  with empty real loci.*
- (iv) *If  $R = \mathbb{R}$ , there is a diffeomorphism  $\psi: W(\mathbb{R}) \xrightarrow{\sim} Y(\mathbb{R})$  such that  $i(\mathbb{R}) \circ \psi \in \mathcal{V}$ .*

**Proof** Define  $V := X \times W$ , let  $\pi: V \rightarrow X$  be the first projection and let  $\mathcal{O}_V(1)$  be a very ample line bundle on  $V$ . Let  $r$  be the codimension of the closed embedding  $(g, \text{Id}): W \hookrightarrow V$ . If  $R = \mathbb{R}$ , define  $\mathcal{U} := \{\xi \in \mathcal{C}^\infty(W(\mathbb{R}), V(\mathbb{R})) \mid \pi(\mathbb{R}) \circ \xi \in \mathcal{V}\}$ , which is a neighborhood of the inclusion  $W(\mathbb{R}) \hookrightarrow V(\mathbb{R})$  by [[46](#), Theorem 11.4].

Choose an even integer  $j \gg 0$  and  $r$ -tuples of even integers  $l_j \gg \cdots \gg l_1 \gg 0$ . Let  $f_j: \mathfrak{W}_j \rightarrow L_j(W)$  be the  $j^{\text{th}}$  moduli of links of  $W$  as in [Section 1.3](#), and choose  $\Omega \subset L_j(W)(R)$  as in [Proposition 1.14](#). Since  $L_j(W)$  is smooth and irreducible (see [Section 1.3](#)), the subset  $\Omega \subset L_j(W)$  is Zariski-dense (apply [[16](#), Proposition 2.8.14]). Consequently, one may choose a general point  $b \in \Omega$ . Define  $Y := \pi(\mathfrak{W}_{j,b})$

with inclusion  $i : Y \hookrightarrow X$ . By [Proposition 1.9](#), the morphism  $\pi|_{\mathfrak{W}_{j,b}} : \mathfrak{W}_{j,b} \rightarrow Y$  is geometrically injective. By [Proposition 1.14\(i\)–\(ii\)](#), in a neighborhood of  $\mathfrak{W}_{j,b}(R)$ , the variety  $\mathfrak{W}_{j,b}$  is smooth and the morphism  $\pi|_{\mathfrak{W}_{j,b}}$  is immersive. These facts show that  $\theta := \pi(R)|_{\mathfrak{W}_{j,b}(R)} : \mathfrak{W}_{j,b}(R) \rightarrow Y(R)$  is bijective and that  $Y$  is smooth along the image of  $\theta$ . This proves (i). The argument also shows that  $\theta$  is a diffeomorphism. Consequently, if  $R = \mathbb{R}$ , one may take  $\psi := \theta \circ \phi_b$ , where  $\phi_b$  is as in [Proposition 1.14\(iii\)](#), which proves (iv). If  $d \leq 3$ , [Proposition 1.6](#) shows that  $\mathfrak{W}_{j,b}$  is smooth and that  $\pi|_{\mathfrak{W}_{j,b}}$  is a closed immersion, proving (ii).

It remains to prove (iii). The chain of links relating  $W$  and  $\mathfrak{W}_{j,b}$  shows that  $[W] - [\mathfrak{W}_{j,b}] \in \text{CH}_d(V)$  is a multiple of  $(2\lambda)^r \in \text{CH}^r(V) = \text{CH}_d(V)$ , where  $\lambda := c_1(\mathcal{O}_V(1)) \in \text{CH}^1(V)$ . Consequently,  $\pi_*[W] - \pi_*[\mathfrak{W}_{j,b}] = g_*[W] - [Y]$  is a multiple of  $\pi_*((2\lambda)^r) \in \text{CH}_d(X)$ . Let  $(u_1, \dots, u_N)$  be a basis of  $H^0(V, \mathcal{O}_V(1))$ . [Proposition 1.7](#) applied with  $W = \emptyset$ , with  $l_i = 2$ , and with the  $F_i$  chosen to be general small deformations of  $\sum_{m=1}^N u_m^2$ , shows that  $\pi_*((2\lambda)^r)$  is the class of a smooth closed subvariety of  $X$  with empty real locus. □

**Remark 2.2** Over a general real closed field, one could replace [Proposition 2.1\(iv\)](#) by the assertion that there exists a Nash diffeomorphism  $\psi : W(R) \xrightarrow{\sim} Y(R)$  such that  $g(R) : W(R) \rightarrow X(R)$  and  $i(R) \circ \psi : Y(R) \rightarrow X(R)$  are Nash-homotopic. The proof is identical, replacing the use of Ehresmann’s theorem in the proofs of [Lemma 1.12\(iii\)](#) and [Proposition 1.14](#) by its Nash analogue, proven by Coste and Shiota [[23](#), Theorem 2.4(iii)'].

## 2.2 Low-dimensional cycles

We first give applications to the Chow groups of smooth projective varieties over real closed fields.

**Theorem 2.3** *Let  $X$  be a smooth projective variety of dimension  $n$  over a real closed field  $R$ . For  $n > 2d$ , the Chow group  $\text{CH}_d(X)$  is generated by classes of closed integral subvarieties of  $X$  which are smooth along their real loci.*

**Proof** Let  $Z \subset X$  be a closed integral subvariety of  $X$ , let  $W \rightarrow Z$  be a resolution of singularities and let  $g : W \rightarrow X$  be the induced morphism. [Proposition 2.1](#) furnishes a closed subvariety  $Y \subset X$  which is smooth along  $Y(R)$  and such that  $g_*[W] - [Y] \in \text{CH}_d(X)$  is a linear combination of classes of smooth closed subvarieties of  $X$ . □

**Theorem 2.4** *Let  $X$  be a smooth projective variety of dimension  $n$  over  $\mathbb{R}$ . For  $n > 2d$ , the group  $\text{Ker}[\text{cl}_{\mathbb{R}} : \text{CH}_d(X) \rightarrow H_d(X(\mathbb{R}), \mathbb{Z}/2)]$  is generated by classes of closed integral subvarieties of  $X$  with empty real loci.*

**Proof** Ischebeck and Schülting [[36](#), Main Theorem 4.3] have shown that the group

$$\text{Ker}[\text{cl}_{\mathbb{R}} : \text{CH}_d(X) \rightarrow H_d(X(\mathbb{R}), \mathbb{Z}/2)]$$

is generated by classes of closed integral subvarieties  $Z \subset X$  with the property that  $Z(\mathbb{R})$  is not Zariski-dense in  $Z$ . Let  $W \rightarrow Z$  be a resolution of singularities of such a subvariety, and let  $g: W \rightarrow X$  be the induced morphism. Since the real locus of a smooth irreducible variety over  $\mathbb{R}$  is empty or Zariski-dense,  $W(\mathbb{R}) = \emptyset$ .

By Proposition 2.1, there is a closed subvariety  $Y \subset X$  with  $Y(\mathbb{R}) \simeq W(\mathbb{R}) = \emptyset$  and such that  $g_*[W] - [Y] \in \text{CH}_d(X)$  is a linear combination of classes of closed subvarieties of  $X$  with empty real loci.  $\square$

**Remark 2.5** The proof of Theorem 2.4 does not extend to general real closed fields because of its use of [36, Main Theorem 4.3]. As a matter of fact, the statement of Theorem 2.4 does not hold over the real closed field  $R := \bigcup_n \mathbb{R}((t^{1/n}))$  for any  $d > 0$ , as we show in the next proposition.

**Proposition 2.6** For all  $c, d \geq 1$  there exist a smooth projective variety  $X_{c,d}$  of dimension  $c + d$  over  $R := \bigcup_n \mathbb{R}((t^{1/n}))$ , and a class  $\beta_{c,d} \in \text{CH}_d(X_{c,d})$  such that:

- (i) One has  $\text{cl}_R(\beta_{c,d}) = 0 \in H_d(X_{c,d}(R), \mathbb{Z}/2)$ .
- (ii) For all identities  $\beta_{c,d} = \sum_{i \in I} n_i [Z_i] \in \text{CH}_d(X_{c,d})$  with  $n_i \in \mathbb{Z}$  and  $Z_i \subset X$  integral, there exists  $i \in I$  such that  $n_i$  is odd and  $Z_i(\mathbb{R})$  is Zariski-dense in  $Z_i$ .

**Proof** Such  $X_{1,1}$  and  $\beta_{1,1}$  have been constructed in [14, Propositions 9.17 and 9.19(ii)]; they form a counterexample to Bröcker’s EPT theorem over the field  $R$ . Let  $x \in \mathbb{P}^{c-1}(R)$  and  $y \in \mathbb{P}^{d-1}(R)$  be general points. One may then define  $X_{c,d} := X_{1,1} \times \mathbb{P}_R^{c-1} \times \mathbb{P}_R^{d-1}$  and  $\beta_{c,d} := \text{pr}_1^* \beta_{1,1} \cdot \text{pr}_2^*[x]$ . The required property of  $(X_{c,d}, \beta_{c,d})$  follows from that of  $(X_{1,1}, \beta_{1,1})$ , and from the equation  $\beta_{1,1} = (\text{pr}_1)_*(\beta_{c,d} \cdot \text{pr}_3^*[y])$ .  $\square$

### 2.3 Approximation of submanifolds

We now give an application to the existence of algebraic approximations for submanifolds of the real locus of a smooth projective variety  $X$  over  $\mathbb{R}$ . We refer to Section 0.3 for the definition of the algebraic cobordism group  $\text{MO}_d^{\text{alg}}(X(\mathbb{R}))$ .

**Theorem 2.7** Let  $X$  be a smooth projective variety of dimension  $n$  over  $\mathbb{R}$ , and let  $j: M \hookrightarrow X(\mathbb{R})$  be a closed  $C^\infty$  submanifold of dimension  $d$ . If  $n > 2d$ , the following properties are equivalent:

- (i) One has  $[j] \in \text{MO}_d^{\text{alg}}(X(\mathbb{R}))$ .
- (ii) For all neighborhoods  $\mathcal{U} \subset C^\infty(M, X(\mathbb{R}))$  of  $j$ , there exist a closed  $d$ -dimensional subvariety  $i: Y \hookrightarrow X$  smooth along  $Y(\mathbb{R})$  and a diffeomorphism  $\phi: M \xrightarrow{\sim} Y(\mathbb{R})$  such that  $i(\mathbb{R}) \circ \phi \in \mathcal{U}$ .

If moreover  $d \leq 3$ , one may choose  $Y$  to be smooth in (ii).

**Proof** Assume that (i) holds and let  $\mathcal{U}$  be as in (ii). A relative variant of the Nash–Tognoli theorem (see [11, Proposition 4.1] or [3, Proposition 0.2]) shows the existence of a morphism  $g: W \rightarrow X$  of smooth projective varieties over  $\mathbb{R}$  and of a diffeomorphism  $\chi: M \xrightarrow{\sim} W(\mathbb{R})$  such that  $g(\mathbb{R}) \circ \chi \in \mathcal{U}$ .

Assertion (ii) and the last statement of [Theorem 2.7](#) now follow by applying [Proposition 2.1](#) to the subset  $\mathcal{V} := \{\xi \circ \chi^{-1}, \xi \in \mathcal{U}\}$  of  $C^\infty(W(\mathbb{R}), X(\mathbb{R}))$  and by defining  $\phi := \psi \circ \chi$ .

Suppose conversely that (ii) holds. Applying it to a small enough neighborhood  $\mathcal{U} \subset C^\infty(M, X(\mathbb{R}))$  of  $j$  shows that  $j$  is homotopic, and hence cobordant, to a  $C^\infty$  map of the form  $i(\mathbb{R})$ , by [\[60, Proposition 4.4.4\]](#). To get (i), take  $W \rightarrow Y$  to be a resolution of singularities which is an isomorphism over  $Y(\mathbb{R})$ , define  $g: W \rightarrow X$  be the induced morphism and note that  $j$  is cobordant to  $g(\mathbb{R})$ .  $\square$

### 3 Complex cobordism and Chern numbers

After a short review of cobordism theory (in [Section 3.1](#)) and of its relation with characteristic classes (in [Section 3.2](#)), we study the top Segre class in [Section 3.3](#), our goal being [Theorem 3.6](#).

#### 3.1 The cobordism rings

Two compact  $C^\infty$  manifolds  $M_1$  and  $M_2$  of dimension  $n$  are said to be *cobordant* if there exists a compact  $C^\infty$  manifold with boundary  $C$  and a diffeomorphism  $\partial C \simeq M_1 \cup M_2$ . Let  $\text{MO}_n$  be the set of cobordism classes of such manifolds, and define  $\text{MO}_* := \bigoplus_{n \geq 0} \text{MO}_n$ .

Let  $M$  be a  $C^\infty$  manifold. A *stably almost complex structure* on  $M$  is a complex structure  $J$  on the real vector bundle  $T_M \oplus \mathbb{R}^k$  for some  $k \geq 0$ , modulo the equivalence relation generated by  $(T_M \oplus \mathbb{R}^k, J) \simeq (T_M \oplus \mathbb{R}^{k+2} = T_M \oplus \mathbb{R}^k \oplus \mathbb{C}, (J, i))$ . Two  $n$ -dimensional stably almost complex compact  $C^\infty$  manifolds  $M_1$  and  $M_2$  are said to be *complex cobordant* if there exists a stably almost complex compact  $C^\infty$  manifold with boundary  $C$  and a diffeomorphism  $\partial C \simeq M_1 \cup M_2$  compatible with the stably almost complex structures. Let  $\text{MU}_n$  be the set of complex cobordism classes of such manifolds, and define  $\text{MU}_* := \bigoplus_{n \geq 0} \text{MU}_n$ .

Disjoint union and cartesian product endow  $\text{MO}_*$  and  $\text{MU}_*$  with graded ring structures: they are the *unoriented cobordism ring* and the *complex cobordism ring*. Thom [\[58, Théorème IV.12\]](#) and Milnor [\[47\]](#) (see also Quillen [\[52, Theorem 6.5\]](#)) have computed that  $\text{MO}_* \simeq \mathbb{Z}/2[x_d]_{d \neq 2^k - 1}$  and  $\text{MU}_* \simeq \mathbb{Z}[t_{2d}]_{d \geq 1}$ , where  $x_d \in \text{MO}_d$  and  $t_{2d} \in \text{MU}_{2d}$ . A comprehensive treatment of these results may be found in [\[38\]](#). As was noted by Milnor [\[48, Lemma 1\]](#), a striking consequence of Thom's computation is that any element of  $\text{MO}_*$  may be represented by the real locus of a smooth projective variety over  $\mathbb{R}$  (a disjoint union of products of projective spaces and Milnor hypersurfaces).

Let  $\phi: \text{MU}_* \rightarrow \text{MO}_*$  be the graded ring homomorphism forgetting the stably almost complex structures. Milnor [\[48, Theorem 1\]](#) has shown that the image of  $\phi$  consists exactly of the squares in  $\text{MO}_*$ , and hence that there is a surjective ring homomorphism  $\psi: \text{MU}_* \rightarrow \text{MO}_*$  such that  $\phi(x) = \psi(x)^2$  for all  $x \in \text{MU}_*$ .

**Lemma 3.1** *The ideal  $\ker(\psi) \subset \text{MU}_*$  is generated by 2 and  $(\ker(\psi))_{2^{k+1}-2}$   $k \geq 1$ .*

**Proof** We use the isomorphisms  $\text{MO}_* \simeq \mathbb{Z}/2[x_d]_{d \neq 2^k - 1}$  and  $\text{MU}_* \simeq \mathbb{Z}[t_{2d}]_{d \geq 1}$ . Since  $\psi: \text{MU}_* \rightarrow \text{MO}_*$  is surjective,  $\psi(t_{2d}) - x_d \in \text{MO}_d$  is decomposable for  $d \neq 2^k - 1$ , and  $\psi(t_{2d})$  is of course decomposable



for  $d = 2^k - 1$ . We deduce from the surjectivity of  $\psi$  the existence of  $t'_{2d} \in \text{MU}_{2d}$  such that  $t_{2d} - t'_{2d}$  is decomposable (so that  $\text{MU}_* = \mathbb{Z}[t'_{2^d}]_{d \geq 1}$ ) and such that  $\psi(t'_{2d}) = x_d$  if  $d \neq 2^k - 1$  and  $\psi(t'_{2d}) = 0$  otherwise. It is now clear that  $\ker(\psi)$  is generated by 2 and by the  $(t'_{2^{k+1-2}})_{k \geq 1}$ .  $\square$

### 3.2 Stiefel–Whitney and Chern numbers

Let  $M$  be a compact  $C^\infty$  manifold of dimension  $n$ , and let  $w_r(M) \in H^r(M, \mathbb{Z}/2)$  be the  $r^{\text{th}}$  Stiefel–Whitney class of its tangent bundle. For a sequence of nonnegative integers  $I = (i_1, i_2, \dots)$  with  $|I| := \sum_r r i_r = n$ , we define  $w_I(M) := \langle \prod_r w_r(M)^{i_r}, [M] \rangle \in \mathbb{Z}/2$ , where  $[M]$  is the fundamental class of  $M$ . The  $w_I(M)$  are the *Stiefel–Whitney numbers* of  $M$ . Thom has shown that they only depend on the cobordism class of  $M$ , and that they determine this cobordism class [58, Théorèmes IV.3 and IV.11].

Similarly, if  $M$  is a stably almost complex compact  $C^\infty$  manifold of dimension  $n$ , we let  $c_r(M) \in H^{2r}(M, \mathbb{Z})$  denote the  $r^{\text{th}}$  Chern class of its stable tangent bundle, and we define the *Chern numbers*  $c_I(M) := \langle \prod_r c_r(M)^{i_r}, [M] \rangle \in \mathbb{Z}$  of  $M$  for  $|I| = \frac{n}{2}$ . These numbers only depend on the complex cobordism class of  $M$ , and determine it; see [52, Theorem 6.5].

**Lemma 3.2** *For all  $y \in \text{MU}_{2d}$  and all  $I = (i_1, i_2, \dots)$  with  $|I| = d$ , the reduction modulo 2 of  $c_I(y)$  is equal to  $w_I(\psi(y))$ .*

**Proof** Represent  $y$  by a stably almost complex compact  $C^\infty$  manifold  $M$  and  $\psi(y)$  by a compact  $C^\infty$  manifold  $N$ . For  $I' = (0, i_1, 0, i_2, \dots)$ ,

$$c_I(M) = w_{I'}(M) = w_{I'}(N \times N) = w_I(N) \in \mathbb{Z}/2,$$

where the first equality holds by [49, Problem 14-B], the second since  $M$  is cobordant to  $N \times N$ , and the third by [48, Lemma 2].  $\square$

The relation  $(\sum_r c_r(M))(\sum_r s_r(M)) = 1$  defines the *Segre classes* or *normal Chern classes*  $s_r(M) \in H^{2r}(M, \mathbb{Z})$  of  $M$ . Pairing the top Segre class with  $[M]$  yields a morphism  $s_d: \text{MU}_{2d} \rightarrow \mathbb{Z}$  which is a linear combination of Chern numbers. This characteristic number is multiplicative in the following sense:

**Lemma 3.3** *For  $y \in \text{MU}_{2d}$  and  $y' \in \text{MU}_{2d'}$ , one has  $s_{d+d'}(yy') = s_d(y)s_{d'}(y')$ .*

**Proof** Represent  $y$  and  $y'$  by stably almost complex compact  $C^\infty$  manifolds and apply the Whitney sum formula [49, (14.7)].  $\square$

### 3.3 Divisibility of the top Segre class

In [53], Rees and Thomas study divisibility properties of some Chern numbers. In [Theorem 3.4](#), we recall one of their results, which we complement in [Theorem 3.6](#). Recall from [Section 0.4](#) that we let  $\alpha(m)$  be the number of ones in the dyadic expansion of  $m$ .

**Theorem 3.4** (Rees and Thomas) *For  $d \geq 0$  and  $e \geq 1$ , the function  $s_d: \text{MU}_{2d} \rightarrow \mathbb{Z}$  is divisible by  $2^e$  if and only if  $\alpha(d + e - 1) > 2(e - 1)$ .*

**Proof** Rees and Thomas [53, Theorem 3] show that  $s_d: \text{MU}_{2d} \rightarrow \mathbb{Z}$  is divisible by  $2^e$  if and only if  $\alpha(d + f) > 2f$  for all  $0 \leq f \leq e - 1$ , and it is easily verified that  $\alpha(d + f) > 2f$  implies that  $\alpha(d + f - 1) > 2(f - 1)$  for all  $f \geq 1$ .  $\square$

We point out for later use an easy corollary of Lemma 3.3 and Theorem 3.4:

**Corollary 3.5** For  $d \geq 1$ , the function  $s_d: \text{MU}_{2d} \rightarrow \mathbb{Z}$  takes even values, and takes values divisible by 4 on decomposable elements.

Here is the main result of Section 3:

**Theorem 3.6** Let  $d \geq 0$  and  $e \geq 1$  be such that  $\alpha(d + e - 1) > 2(e - 1)$ . Then  $s_d/2^e: \text{MU}_{2d} \rightarrow \mathbb{Z}$  coincides modulo 2 with an integral linear combination of Chern numbers if and only if  $\alpha(d + e) \geq 2e$ .

**Proof** Assume first that  $\alpha(d + e) < 2e$ , and let  $d + e = 2^{a_1} + \dots + 2^{a_f}$  be the dyadic expansion of  $d + e$ , with  $f \leq 2e - 1$ . Define  $d_1 := 2^{a_1} + \dots + 2^{a_{f-1}} - (e - 1)$  and  $d_2 := 2^{a_f} - 1$ . One has  $\alpha(d_1 + e - 1) = f - 1 \leq 2e - 2$ . It thus follows from Theorem 3.4 that there exists  $y_1 \in \text{MU}_{2d_1}$  such that  $s_{d_1}(y_1)$  is not divisible by  $2^e$ . Theorem 3.4 also shows the existence of  $y_2 \in \text{MU}_{2d_2}$  such that  $s_{d_2}(y_2)$  is not divisible by 4. We deduce from Lemma 3.3 that  $s_d(y_1 y_2)/2^e \in \mathbb{Z}$  is odd.

Since the map  $\psi: \text{MU}_* \rightarrow \text{MO}_*$  is surjective and  $\text{MO}_{2^{a_f}-1}$  contains no indecomposable element (see Section 3.1), there exists a decomposable element  $z_2 \in \text{MU}_{2d_2}$  with  $\psi(z_2) = \psi(y_2)$ . By Corollary 3.5, one may replace  $y_2$  with  $y_2 - z_2$  and thus assume that  $\psi(y_2) = 0$ . But then  $\psi(y_1 y_2) = \psi(y_1)\psi(y_2) = 0$ , which shows, in view of Lemma 3.2, that all the Chern numbers of  $y_1 y_2$  are even. Consequently,  $s_d(y_1 y_2)/2^e \pmod{2}$  cannot be a linear combination with  $\mathbb{Z}/2$  coefficients of Chern numbers of  $y_1 y_2$ . We have thus proven the direct implication of the theorem.

Assume that  $\alpha(d + e) \geq 2e$ . Let  $k$  be such that  $1 \leq 2^k - 1 \leq d$ . We claim that  $\text{MU}_{2^{k+1}-2} \cdot \text{MU}_{2d-2^{k+1}+2}$  is included in the kernel of the morphism  $\chi: \text{MU}_{2d} \rightarrow \mathbb{Z}/2$  obtained by reducing  $s_d/2^e: \text{MU}_{2d} \rightarrow \mathbb{Z}$  modulo 2. To see this, choose  $y \in \text{MU}_{2^{k+1}-2}$  and  $z \in \text{MU}_{2d-2^{k+1}+2}$ . We now compute

$$\alpha(d - 2^k + 1 + (e - 1)) = \alpha(d - 2^k + e) \geq \alpha(d + e) - 1 > 2(e - 1),$$

and Theorem 3.4 shows that  $s_{d-2^k+1}(z)$  is divisible by  $2^e$ . Since  $s_{2^k-1}(y)$  is even by Corollary 3.5, Lemma 3.3 shows that  $s_d(yz)$  is divisible by  $2^{e+1}$ , as wanted.

We deduce from Lemma 3.1 that the kernel of  $\psi: \text{MU}_{2d} \rightarrow \text{MO}_d$  is included in the kernel of  $\chi$ , and hence that  $\chi = \mu \circ \psi$  for some morphism  $\mu: \text{MO}_d \rightarrow \mathbb{Z}/2$ . Since a class in  $\text{MO}_d$  is determined by its Stiefel–Whitney numbers (see Section 3.2), the morphism  $\mu$  is a linear combination of Stiefel–Whitney numbers. Lemma 3.2 now implies that  $\chi$  is the reduction modulo 2 of an integral linear combination of Chern numbers, which concludes the proof.  $\square$

**Example 3.7** The first interesting case of Theorem 3.6 is  $d = 2$  and  $e = 1$ . Since  $\alpha(d + e) = \alpha(3) = 2 = 2e$ , it predicts the existence of an integral linear combination of Chern numbers which coincides modulo 2 with  $\frac{1}{2}s_2: \text{MU}_4 \rightarrow \mathbb{Z}$ .

We claim that this linear combination may be chosen to be  $c_2$ . Indeed,  $MU_4$  is generated by classes of projective complex surfaces (see [2, II, Corollary 10.8]), and for such a surface  $S$ , our claim follows from Noether's formula

$$s_2(S) = (c_1^2 - c_2)(S) = 12\chi(S, \mathcal{O}_S) - 2c_2(S).$$

## 4 The double point class

We use the results of Section 3 in combination with a double point formula. We give applications to Chow groups in Section 4.3, proving Theorems 0.2, 0.3 and 0.5. We also construct new examples of submanifolds of real loci of smooth projective varieties over  $\mathbb{R}$  without algebraic approximations in Section 4.5, proving Theorems 0.7 and 0.8.

### 4.1 A consequence of Fulton's double point formula

Formulas for the rational equivalence class of the double point locus of a morphism go back to Todd [59, (7.01)] and Laksov [44, Theorem 26] under strong assumptions on the morphism. The following proposition is an application of a refined double point formula of Fulton [26, Theorem 9.3], which is valid for an arbitrary morphism.

**Proposition 4.1** *Let  $g: W \rightarrow X$  be a morphism of smooth projective varieties over  $\mathbb{R}$ . Let  $d$  be the dimension of  $W$  and assume that  $X$  has dimension  $2d$ . Let  $N_{W/X} := [g^*T_X] - [T_W]$  be the virtual normal bundle of  $g$ .*

(i) *If  $g$  is an embedding, then*

$$(4.2) \quad \deg((g_*[W])^2) = \deg(c_d(N_{W/X})).$$

(ii) *If  $g$  is an embedding in a neighborhood of  $g(\mathbb{C})^{-1}(X(\mathbb{R}))$ , then*

$$(4.3) \quad \deg((g_*[W])^2) \equiv \deg(c_d(N_{W/X})) \pmod{4}.$$

**Proof** Let  $D(g) \subset W$  be the closed subset consisting of those  $x \in W$  such that  $g$  is not an embedding above a neighborhood of  $g(x)$ . Let  $\mathbb{D}(g) \in \text{CH}_0(D(g))$  be the double point class of  $g$  defined in [26, Section 9.3]. By [26, Theorem 9.3],

$$\mathbb{D}(g) = g^*g_*[W] - c_d(N_{W/X}) \in \text{CH}_0(W).$$

Since  $g_*g^*g_*[W] = (g_*[W])^2$  by the projection formula, we deduce that

$$(4.4) \quad \deg(g_*\mathbb{D}(g)) = \deg((g_*[W])^2) - \deg(c_d(N_{W/X})) \in \mathbb{Z}.$$

In (i), one has  $D(g) = \emptyset$ , and hence  $\mathbb{D}(g) = 0$ , and (4.4) implies (4.2).

Define  $\bar{D}(g) := g(D(g))$ . By [26, Example 9.3.14] there exists a 0-cycle  $\bar{\mathbb{D}}(g) \in \text{CH}_0(\bar{D}(g))$  such that  $g_*\mathbb{D}(g) = 2\bar{\mathbb{D}}(g) \in \text{CH}_0(\bar{D}(g))$ . The hypothesis of (ii) implies that  $\bar{D}(g) \subset X$  has no real points, and so  $\deg(g_*\mathbb{D}(g)) = 2\deg(\bar{\mathbb{D}}(g))$  is divisible by 4. The desired congruence (4.3) now follows from (4.4).  $\square$

**Remark 4.5** Proposition 4.1(i) is of course much easier than Proposition 4.1(ii). It follows for instance from [26, Corollary 6.3].

### 4.2 Weil restrictions of scalars and quotients of abelian varieties

We gather here the geometric constructions that will be used in Section 4.3.

We let  $(A, \lambda)$  denote a very general principally polarized abelian variety of dimension  $d \geq 1$  over  $\mathbb{C}$ . Let  $e_1, \dots, e_d, f_1, \dots, f_d \in H^1(A(\mathbb{C}), \mathbb{Z})$  be a basis such that the principal polarization  $\lambda \in H^2(A(\mathbb{C}), \mathbb{Z}) = \wedge^2 H^1(A(\mathbb{C}), \mathbb{Z})$  of  $A$  is equal to  $\sum_i e_i \wedge f_i$ . Denote by  $(A', \lambda' = \sum_i e'_i \wedge f'_i)$  another copy of  $(A, \lambda)$ , which we identify with the dual of  $A$  by means of the principal polarization. Let  $\mu \in H^2(A(\mathbb{C}) \times A'(\mathbb{C}), \mathbb{Z})$  be the class of the Poincaré bundle. As in [15, Lemma 14.1.10] (whose notation is different from ours), one computes that  $\mu = \sum_i (e_i \wedge f'_i + e'_i \wedge f_i)$ .

**Lemma 4.6** *The subring  $\text{Hdg}^*(A(\mathbb{C}) \times A'(\mathbb{C}), \mathbb{Q}) \subset H^*(A(\mathbb{C}) \times A'(\mathbb{C}), \mathbb{Q})$  of Hodge classes is generated by  $\lambda, \lambda'$  and  $\mu$ .*

**Proof** Let  $V := H_1(A(\mathbb{C}), \mathbb{Q})$ . In [50], two subgroups  $\text{Hod}(A) \subset L(A)$  of the symplectic group  $\text{Sp}(V, \lambda)$  are considered. As  $A$  is very general,  $\text{Hod}(A)$  is equal to  $\text{Sp}(V, \lambda)$ ; see [15, Proposition 17.4.2]. It follows that  $\text{Hod}(A) = L(A) = \text{Sp}(V, \lambda)$  and that  $\text{End}(A)_{\mathbb{Q}} = \mathbb{Q}$ ; see e.g. [15, Section 17.6(3)]. This implies that  $A$  has no factors of type III (in the sense of [50, page 199]). We can now apply [50, Theorem 3.1] to show that  $\text{Hdg}^*(A(\mathbb{C}) \times A'(\mathbb{C}), \mathbb{Q})$  is generated as a ring by  $\text{Hdg}^2(A(\mathbb{C}) \times A'(\mathbb{C}), \mathbb{Q})$ .

The Künneth formula induces an isomorphism of Hodge structures

$$H^2(A(\mathbb{C}) \times A'(\mathbb{C}), \mathbb{Q}) \simeq H^2(A(\mathbb{C}), \mathbb{Q}) \oplus H^2(A'(\mathbb{C}), \mathbb{Q}) \oplus \text{End}(H^1(A(\mathbb{C}), \mathbb{Q})).$$

The  $\mathbb{Q}$ -vector spaces  $\text{Hdg}^2(A(\mathbb{C}), \mathbb{Q})$  and  $\text{Hdg}^2(A'(\mathbb{C}), \mathbb{Q})$  are respectively generated by  $\lambda$  and  $\lambda'$ , by Mattuck's theorem [15, Theorem 17.4.1]. As  $\text{End}(A)_{\mathbb{Q}} = \mathbb{Q}$ , the  $\mathbb{Q}$ -vector space of Hodge classes in  $\text{End}(H^1(A(\mathbb{C}), \mathbb{Q}))$  is one-dimensional, generated by  $\mu$ . □

In the next lemmas, we let  $\mathbb{Z}/2$  act on  $A \times A'$  by exchanging the two factors. We define  $D := \text{Hdg}^{2d}(A(\mathbb{C}) \times A'(\mathbb{C}), \mathbb{Z})$  and  $E := D^{\mathbb{Z}/2}$ , and we consider the subgroup  $F \subset H^1(A(\mathbb{C}) \times A'(\mathbb{C}), \mathbb{Z})$  generated by  $e_2, \dots, e_d, f_1, \dots, f_d, e'_2, \dots, e'_d, f'_1, \dots, f'_d, 2e_1, e_1 + e'_1, 2e'_1$ . We also define  $\varepsilon_{k,l,m} := (\lambda^k/k!)((\lambda')^l/l!)(\mu^m/m!)$ , and we set  $\zeta_{k,l,m} := \varepsilon_{k,l,m} + \varepsilon_{l,k,m}$  if  $k > l$  and  $\zeta_{k,k,m} := \varepsilon_{k,k,m}$ .

**Lemma 4.7** (i) *The classes  $\varepsilon_{k,l,m}$ , where  $(k, l, m)$  ranges over the triples of nonnegative integers such that  $k + l + m = d$ , form a  $\mathbb{Z}$ -basis of  $D$ .*

(ii) *The classes  $\zeta_{k,l,m}$ , where  $(k, l, m)$  ranges over the triples of nonnegative integers such that  $k \geq l$  and  $k + l + m = d$ , form a  $\mathbb{Z}$ -basis of  $E$ .*

(iii) *The subgroups  $E \cap \wedge^{2d} F$  and  $2E$  of  $H^{2d}(A(\mathbb{C}) \times A'(\mathbb{C}), \mathbb{Z})$  are equal.*

**Proof** By Lemma 4.6, the integral classes listed in (i) span  $\text{Hdg}^{2d}(A(\mathbb{C}) \times A'(\mathbb{C}), \mathbb{Q})$  as a  $\mathbb{Q}$ -vector space. To prove (i), it remains to show that these classes are linearly independent, and that they span a

primitive sublattice of  $\text{Hdg}^{2d}(A(\mathbb{C}) \times A'(\mathbb{C}), \mathbb{Z})$ . Both assertions follow from the fact that, when one decomposes them in a basis of  $H^{2d}(A(\mathbb{C}) \times A'(\mathbb{C}), \mathbb{Z}) = \bigwedge^{2d}(\mathbb{Z}e_1 \oplus \dots \oplus \mathbb{Z}f'_d)$  consisting of wedges of elements of  $\{e_1, \dots, f'_d\}$ , the coefficient of the basis element which equals

$$(4.8) \quad \left( \bigwedge_{i=1}^k e_i \wedge f_i \right) \wedge \left( \bigwedge_{i=k+1}^{k+l} e'_i \wedge f'_i \right) \wedge \left( \bigwedge_{i=k+l+1}^d e_i \wedge f'_i \right)$$

up to a sign is nonzero only in the decomposition of  $\varepsilon_{k,l,m}$ .

Assertion (ii) follows from (i) and from the fact that the  $\mathbb{Z}/2$ -action exchanges  $\lambda$  and  $\lambda'$  and preserves  $\mu$ .

We now prove (iii). That  $2E \subset E \cap \bigwedge^{2d} F$  is obvious, and we check the reverse inclusion. Let  $Q$  be the quotient of  $H^{2d}(A(\mathbb{C}) \times A'(\mathbb{C}), \mathbb{Z}) = \bigwedge^{2d}(\mathbb{Z}e_1 \oplus \dots \oplus \mathbb{Z}f'_d)$  by the subgroup generated by elements of the form  $(e_1 - e'_1) \wedge g_2 \wedge \dots \wedge g_{2d}$  with  $g_2, \dots, g_{2d} \in \{e_1, \dots, f'_d\}$  on the one hand, and by elements of the form  $g_1 \wedge \dots \wedge g_{2d}$  with  $g_1, \dots, g_{2d} \in \{e_1, \dots, f'_d\} \setminus \{e_1, e'_1\}$  on the other hand. Consider the  $\mathbb{Z}/2$ -basis of  $Q \otimes_{\mathbb{Z}} \mathbb{Z}/2$  consisting of the images of the elements of the form  $e_1 \wedge g_2 \wedge \dots \wedge g_{2d}$  with  $g_2, \dots, g_{2d} \in \{e_1, \dots, f'_d\} \setminus \{e_1, e'_1\}$ . Decompose in this basis the images in  $Q \otimes_{\mathbb{Z}} \mathbb{Z}/2$  of the classes  $\zeta_{k,l,m}$  considered in (ii). For  $k \geq l$ , the basis element (4.8) appears with nonzero coefficient only in the decomposition of the image of  $\zeta_{k,l,m}$ , with one exception: when  $(k, l, m) = (1, 0, d - 1)$ , the basis element (4.8) appears with nonzero coefficient only in the decomposition of the images of  $\zeta_{1,0,d-1}$  and of  $\zeta_{0,0,d}$ . This shows that the classes  $\zeta_{k,l,m}$  are  $\mathbb{Z}/2$ -linearly independent in  $Q \otimes_{\mathbb{Z}} \mathbb{Z}/2$ . As the image of  $\bigwedge^{2d} F$  in  $Q \otimes_{\mathbb{Z}} \mathbb{Z}/2$  is zero, we deduce that all the coefficients appearing in the decomposition of an element of  $E \cap \bigwedge^{2d} F$  in the  $\mathbb{Z}$ -basis of  $E$  described in (ii) must be even. This shows that  $E \cap \bigwedge^{2d} F \subset 2E$ , as wanted.  $\square$

**Lemma 4.9** *If  $\delta, \delta' \in E$ , then  $\deg(\delta\delta')$  is even.*

**Proof** It suffices to prove the lemma when  $\delta$  and  $\delta'$  belong to the  $\mathbb{Z}$ -basis of  $E$  described in Lemma 4.7(ii). We may thus assume that  $\delta = \zeta_{k,l,m}$  and  $\delta' = \zeta_{k',l',m'}$ . If  $k > l$ , then

$$\deg(\delta\delta') = \deg((\varepsilon_{k,l,m} + \varepsilon_{l,k,m})\delta') = 2 \deg(\varepsilon_{k,l,m}\delta')$$

by  $\mathbb{Z}/2$ -invariance of  $\delta'$ , and this number is even. The same argument applies if  $k' > l'$ . Assume now that  $k = l$  and  $k' = l'$ . Then one has

$$(4.10) \quad \delta\delta' = \zeta_{k,k,m}\zeta_{k',k',m'} = \binom{k+k'}{k}^2 \binom{m+m'}{m} \varepsilon_{k+k',k+k',m+m'}.$$

As a consequence of [9, Lemme 1, page 247] and of the projection formula applied to the morphism  $A \times A' \rightarrow A'$ , one computes that

$$(4.11) \quad \deg(\varepsilon_{k+k',k+k',m+m'}) = (-1)^{d-k-k'} \deg_{A'} \left( \frac{(\lambda')^{k+k'}}{(k+k')!} \frac{(\lambda')^{d-k-k'}}{(d-k-k')!} \right) = (-1)^{d-k-k'} \binom{d}{k+k'}.$$

Combining (4.10) and (4.11) yields  $\deg(\delta\delta') = (-1)^{d-k-k'} \binom{k+k'}{k}^2 \binom{m+m'}{m} \binom{d}{k+k'}$ .

Assume for contradiction that this number is odd. Let us say that two integers  $n, n' \geq 0$  are *dyadically disjoint* if their dyadic expansions do not share any nonzero digit. The formula for the 2-adic valuation

of the factorial appearing in [55, page 241] shows that  $\binom{n+n'}{n}$  is odd if and only if  $n$  and  $n'$  are dyadically disjoint. It follows that  $k$  and  $k'$  are dyadically disjoint, and that so are  $m = d - 2k$  and  $m' = d - 2k'$ , and  $k + k'$  and  $d - k - k'$ . As  $d - 2k$  and  $d - 2k'$  are dyadically disjoint, the integer  $d$  is even. Write  $d = 2e$ . Then  $e - k$  and  $e - k'$  are dyadically disjoint. As  $k$  and  $k'$ , as well as  $e - k$  and  $e - k'$  and also  $k + k'$  and  $(e - k) + (e - k')$  are dyadically disjoint, the four integers  $k, k', e - k$  and  $e - k'$  are pairwise dyadically disjoint. Since  $k + (e - k) = k' + (e - k')$ , this is impossible unless these four numbers all vanish. This contradicts the assumption that  $d \geq 1$ .  $\square$

Let  $G := \text{Gal}(\mathbb{C}/\mathbb{R})$ , and consider the  $G$ -module  $\mathbb{Z}(j) := (\sqrt{-1})^j \mathbb{Z} \subset \mathbb{C}$ . If  $X$  is a variety over  $\mathbb{R}$ , letting  $G$  act both on  $X(\mathbb{C})$  and on  $\mathbb{Z}(j)$  endows  $H^k(X(\mathbb{C}), \mathbb{Z}(j))$  with an action of  $G$ .

**Proposition 4.12** *For all  $d \geq 1$ , there exists an abelian variety  $X$  of dimension  $2d$  over  $\mathbb{R}$  and a class  $\beta \in \text{CH}^d(X)$  with the following properties:*

- (i) *If  $\gamma, \gamma' \in H^{2d}(X(\mathbb{C}), \mathbb{Z}(d))$  are Hodge and  $G$ -invariant, then  $\deg(\gamma\gamma')$  is even.*
- (ii) *One has  $\deg(\beta^2) \equiv 2 \pmod{4}$ .*
- (iii)  *$\text{cl}_{\mathbb{R}}(\beta) = 0 \in H_d(X(\mathbb{R}), \mathbb{Z}/2)$ .*

**Proof** Let  $(A, \lambda)$  be a very general principally polarized abelian variety of dimension  $d$  which is defined over  $\mathbb{R}$  (by Baire's theorem, one may choose a very general real point of the moduli space  $M$  of  $d$ -dimensional principally polarized abelian varieties with level 3 structure, since  $M$  is a smooth variety). Write  $x \mapsto \bar{x}$  for the action of complex conjugation on  $A(\mathbb{C})$ .

Let  $(A', \lambda')$  be another copy of  $(A, \lambda)$ . The real abelian variety  $A \times A'$  has set of complex points  $A(\mathbb{C}) \times A'(\mathbb{C})$  with an action of complex conjugation given by  $(x, y) \mapsto (\bar{x}, \bar{y})$ . Define  $X := \text{Res}_{\mathbb{C}/\mathbb{R}}(A)$  to be the Weil restriction of scalars of  $A_{\mathbb{C}}$ . It is the abelian variety over  $\mathbb{R}$  whose set of complex points is  $X(\mathbb{C}) = A(\mathbb{C}) \times A'(\mathbb{C})$ , with an action of complex conjugation given by  $(x, y) \mapsto (\bar{y}, \bar{x})$ . The subvariety  $A_{\mathbb{C}} \times \{0\} \cup \{0\} \times A'_{\mathbb{C}}$  of  $A_{\mathbb{C}} \times A'_{\mathbb{C}}$  descends to a subvariety  $Z \subset X$ , and we define  $\beta := [Z] \in \text{CH}^d(X)$ . Since the normalization of  $Z$  has no real points,  $\text{cl}_{\mathbb{R}}(\beta) = 0$ . Moreover,  $\deg(\beta^2) = 2$ . Assertions (ii) and (iii) are proven.

Let  $\mu \in H^2(A(\mathbb{C}) \times A'(\mathbb{C}), \mathbb{Z}(1))$  be the class of the Poincaré bundle. As the cycle class map

$$\text{CH}^1(A_{\mathbb{C}} \times A'_{\mathbb{C}}) \rightarrow H^2(A(\mathbb{C}) \times A'(\mathbb{C}), \mathbb{Z}(1))$$

is  $G$ -equivariant, the classes  $\lambda, \lambda'$  and  $\mu$  all belong to  $H^2(A(\mathbb{C}) \times A'(\mathbb{C}), \mathbb{Z}(1))^G$ . It then follows from Lemma 4.7(i) that the group  $\text{Hdg}^{2d}(A(\mathbb{C}) \times A'(\mathbb{C}), \mathbb{Z}(d))$  consists exclusively of  $G$ -invariant classes.

As the action of complex conjugation on  $X(\mathbb{C})$  is the composition of the action of complex conjugation on  $A(\mathbb{C}) \times A'(\mathbb{C})$  and the exchange of the two factors, we deduce that group of  $G$ -invariant Hodge classes in  $H^{2d}(X(\mathbb{C}), \mathbb{Z}(d))$  is exactly the group  $E$  described in Lemma 4.7(ii). Assertion (i) now follows from Lemma 4.9.  $\square$

The next proposition is a variant of [Proposition 4.12](#) which works over the complex numbers, but which is slightly more complicated.

**Proposition 4.13** *For all  $d \geq 1$ , there exists a  $2d$ -dimensional smooth projective variety  $X$  over  $\mathbb{C}$  and a class  $\beta \in \text{CH}^d(X)$  with the following properties:*

- (i) *If  $\gamma, \gamma' \in H^{2d}(X(\mathbb{C}), \mathbb{Z})$  are Hodge, then  $\deg(\gamma\gamma')$  is even.*
- (ii) *One has  $\deg(\beta^2) \equiv 2 \pmod{4}$ .*
- (iii) *All higher Chern classes of  $X$  are torsion, i.e.  $c(X) = 1 \in \text{CH}^*(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ .*

**Proof** Let  $(A, \lambda)$  be a very general principally polarized abelian variety of dimension  $d$  over  $\mathbb{C}$ . Let  $e_1, \dots, e_d, f_1, \dots, f_d \in H^1(A(\mathbb{C}), \mathbb{Z})$  be a basis such that  $\lambda = \sum_i e_i \wedge f_i$ . Let  $\tau \in A(\mathbb{C})[2] \simeq H_1(A(\mathbb{C}), \mathbb{Z}/2)$  be the 2-torsion point associated with the morphism  $H^1(A(\mathbb{C}), \mathbb{Z}) \rightarrow \mathbb{Z}/2$  sending  $e_1$  to 1 and  $e_2, \dots, e_d, f_1, \dots, f_d$  to 0. Denote by  $(A', \lambda' = \sum_i e'_i \wedge f'_i, \tau')$  another copy of  $(A, \lambda = \sum_i e_i \wedge f_i, \tau)$ . Let  $\mathbb{Z}/4$  act on  $A \times A'$  via  $(x, x') \mapsto (x' + \tau, x)$ . Let  $p: A \times A' \rightarrow X$  (resp.  $q: A \times A' \rightarrow B$ ) be the quotient of  $A \times A'$  by  $\mathbb{Z}/4$  (resp. by the subgroup  $\mathbb{Z}/2 \subset \mathbb{Z}/4$ ). Since  $\mathbb{Z}/2$  acts on  $A \times A'$  via  $(x, x') \mapsto (x + \tau, x' + \tau')$ , we see that  $B$  is an abelian variety, and that  $q^*(H^1(B(\mathbb{C}), \mathbb{Z})) \subset H^1(A(\mathbb{C}) \times A'(\mathbb{C}), \mathbb{Z})$  is the subgroup generated by  $e_2, \dots, e_d, f_1, \dots, f_d, e'_2, \dots, e'_d, f'_1, \dots, f'_d, 2e_1, e_1 + e'_1, 2e'_1$ .

Assertion (iii) follows at once from the fact that  $c(A \times A') = 1 \in \text{CH}^*(A \times A')$  since  $p: A \times A' \rightarrow X$  is finite étale.

Consider  $Z := p(A \times \{0\}) \subset A \times A'$ , and define  $\beta := [Z] \in \text{CH}^d(X)$ . As

$$p^*\beta = [A \times \{0\}] + [A \times \{\tau'\}] + [\{0\} \times A'] + [\{\tau\} \times A'],$$

one computes that  $\deg(p^*\beta^2) = 8$ . Since  $\deg(p) = 4$ , one has  $\deg(\beta^2) = 2$ . This proves (ii).

Let  $\gamma, \gamma' \in \text{Hdg}^{2d}(X(\mathbb{C}), \mathbb{Z})$  be Hodge classes. As translations act trivially on the cohomology of  $A \times A'$ , the automorphism  $(x, x') \mapsto (x' + \tau, x)$  acts in cohomology as  $(x, x') \mapsto (x', x)$ . It follows that  $\delta := p^*\gamma$  and  $\delta' := p^*\gamma'$  are invariant under the involution of  $\text{Hdg}^{2d}(A(\mathbb{C}) \times A'(\mathbb{C}), \mathbb{Z})$  exchanging the two factors. In other words, the classes  $\delta$  and  $\delta'$  belong to the group denoted by  $E$  in [Lemma 4.7](#). Since moreover  $\delta$  and  $\delta'$  belong to  $\text{Im}(q^*)$ , [Lemma 4.7\(iii\)](#) shows that  $\delta$  and  $\delta'$  are divisible by 2 in  $E$ . [Lemma 4.9](#) now implies that  $\deg(\delta\delta') \equiv 0 \pmod{8}$ . Since  $\deg(p) = 4$ , we deduce that  $\deg(\gamma\gamma') \equiv 0 \pmod{2}$ , which proves (i).  $\square$

### 4.3 High-dimensional cycles

Here are applications to Chow groups.

**Theorem 4.14** *Let  $d \geq c$  be such that  $\alpha(c+1) \geq 3$ . Then there exists an abelian variety  $X$  of dimension  $c+d$  over  $\mathbb{R}$  such that  $\text{CH}_d(X)$  is not generated by classes of closed subvarieties of  $X$  that are smooth along their real loci.*

**Proof** Suppose first that  $d = c$ , and let  $X$  and  $\beta$  be as in Proposition 4.12. Assume for contradiction that  $\beta = \sum_i n_i [Y_i] \in \text{CH}_d(X)$  where  $n_i \in \mathbb{Z}$  and the  $Y_i$  are integral closed subvarieties of  $X$  that are smooth along their real loci. One computes

$$(4.15) \quad \deg(\beta^2) = \sum_i n_i^2 \deg([Y_i]^2) + 2 \sum_{i < j} n_i n_j \deg([Y_i][Y_j]).$$

The existence of Krasnov’s cycle class map  $\text{cl}: \text{CH}_d(X) \rightarrow H_G^{2d}(X(\mathbb{C}), \mathbb{Z}(d))$  to  $G$ -equivariant Betti cohomology refining the usual complex cycle class map  $\text{cl}_{\mathbb{C}}: \text{CH}_d(X) \rightarrow H^{2d}(X(\mathbb{C}), \mathbb{Z}(d))$  to Betti cohomology [40, Theorem 0.6], the fact that the image of  $\text{cl}_{\mathbb{C}}$  consists of Hodge classes and Proposition 4.12(i) combine to show that  $2 \sum_{i < j} n_i n_j \deg([Y_i][Y_j])$  is divisible by 4. Let  $g_i: W_i \rightarrow Y_i$  be a resolution of singularities which is an isomorphism above  $Y_i(\mathbb{R})$ . Proposition 4.1(ii) and the fact that all higher Chern classes of the abelian variety  $X$  vanish show that  $\deg([Y_i]^2) \equiv \deg(s_d(W_i)) \pmod{4}$ . Since  $\alpha(d + 1) \geq 3$  by hypothesis, Theorem 3.4 implies that  $\deg(s_d(W_i)) = s_d([W_i(\mathbb{C})]) \equiv 0 \pmod{4}$ . These congruences and Proposition 4.12(ii) contradict (4.15).

To deal with the general case, apply the  $d = c$  case to get a smooth projective variety  $X'$  of dimension  $2c$  over  $\mathbb{R}$  and a class  $\beta' \in \text{CH}_c(X')$  that is not a linear combination of classes of subvarieties of  $X'$  that are smooth along their real loci. Define  $X := X' \times A$  where  $A$  is any abelian variety of dimension  $d - c$  over  $\mathbb{R}$ , and  $\beta := \text{pr}_1^* \beta' \in \text{CH}_d(X)$ . That  $\beta$  is not a linear combination of classes of subvarieties of  $X$  that are smooth along their real loci follows from the corresponding property of  $\beta'$  and from the Bertini theorem.  $\square$

**Theorem 4.16** *If  $d \geq c$  are such that  $\alpha(c + 1) \geq 2$ , there exists an abelian variety  $X$  of dimension  $c + d$  over  $\mathbb{R}$  such that  $\text{Ker}[\text{cl}_{\mathbb{R}}: \text{CH}_d(X) \rightarrow H_d(X(\mathbb{R}), \mathbb{Z}/2)]$  is not generated by classes of closed subvarieties of  $X$  with empty real loci.*

**Proof** The proof is almost identical to the proof of Theorem 4.14, replacing *that are smooth along their real loci* by *with empty real loci* everywhere. Only the argument used in the  $d = c$  case to show that  $s_d([W_i(\mathbb{C})]) \equiv 0 \pmod{4}$  needs to be modified, as follows. Since  $W_i(\mathbb{R}) = \emptyset$ , one has  $\psi([W_i(\mathbb{C})]) = 0 \in \text{MO}_d$  by [21, Theorem 22.4]. We deduce from Lemma 3.2 that all the Chern numbers of  $[W_i(\mathbb{C})] \in \text{MU}_{2d}$  are even. Theorem 3.6 and the hypothesis that  $\alpha(d + 1) \geq 2$  imply that  $s_d([W_i(\mathbb{C})]) \equiv 0 \pmod{4}$ , as wanted.  $\square$

**Theorem 4.17** *If  $d \geq c$  are such that  $\alpha(c + 1) \geq 3$ , there exists a smooth projective variety  $X$  of dimension  $c + d$  over  $\mathbb{C}$  such that  $\text{CH}_d(X)$  is not generated by classes of smooth closed subvarieties of  $X$ .*

**Proof** The proof is similar to that of Theorem 4.14. The argument at the end of the proof of Theorem 4.14 shows that we may assume that  $d = c$ .

Let  $X$  and  $\beta$  be as in Proposition 4.13. Assume that  $\beta = \sum_i n_i [Y_i] \in \text{CH}_d(X)$  where  $n_i \in \mathbb{Z}$  and the  $Y_i$  are smooth closed subvarieties of  $X$ , and consider (4.15). Since the Betti cohomology classes of the  $Y_i$  are Hodge, Proposition 4.13(i) shows that  $2 \sum_{i < j} n_i n_j \deg([Y_i][Y_j])$  is divisible by 4. Proposition 4.13(iii)



and [26, Corollary 6.3] together imply that  $\deg([Y_i]^2) = \deg(s_d(W_i))$ . Since  $\alpha(d + 1) \geq 3$ , Theorem 3.4 implies that  $\deg(s_d(W_i))$  is divisible by 4. Proposition 4.13(ii) now contradicts (4.15).  $\square$

### 4.4 Hypersurfaces in abelian varieties

We give here a geometric construction based on a Noether–Lefschetz argument, on which the proof of Theorem 4.19 relies.

**Proposition 4.18** *For all  $d, e \geq 1$ , there exists a  $2d$ –dimensional smooth projective variety  $X$  over  $\mathbb{R}$  with the following properties:*

- (i) *The total Chern class  $c(X) \in \text{CH}^*(X)$  of  $X$  satisfies  $c(X) \equiv 1 \pmod{2^{e+1}}$ .*
- (ii) *The subgroup of Hodge classes  $\text{Hdg}^{2d}(X(\mathbb{C}), \mathbb{Z}) \subset H^{2d}(X(\mathbb{C}), \mathbb{Z})$  is generated by a class  $\eta \in \text{Hdg}^{2d}(X(\mathbb{C}), \mathbb{Z})$  with  $\deg(\eta^2) \equiv 0 \pmod{2^{e+1}}$ .*
- (iii) *One has  $X(\mathbb{R}) \neq \emptyset$ .*

**Proof** Arguing as in the proof of Proposition 4.12, choose a very general principally polarized abelian variety  $A$  of dimension  $2d + 1$  over  $\mathbb{R}$ . The principal polarization of  $A$  is represented by an ample line bundle  $\mathcal{L}$  on  $A$  which is defined over  $\mathbb{R}$ ; see [57, Theorem 4.1]. The group  $\text{Hdg}^{2d}(A(\mathbb{C}), \mathbb{Z})$  of degree  $2d$  Hodge classes on  $A_{\mathbb{C}}$  is generated by  $(1/d!)c_1(\mathcal{L})^d$  by Mattuck’s theorem (see [15, Theorem 17.4.1]) since  $(1/d!)c_1(\mathcal{L})^d$  is a primitive integral cohomology class.

Let  $l \gg 0$  be such that  $2^{e+1} \mid l$  and  $\mathcal{L}^{\otimes l}$  is very ample. Choose a Lefschetz pencil of sections  $\mathcal{L}^{\otimes l}$ , and let  $X \subset A$  be a very general member of this pencil with  $X(\mathbb{R}) \neq \emptyset$ .

The restriction morphism  $H^{2d}(A(\mathbb{C}), \mathbb{Z}) \rightarrow H^{2d}(X(\mathbb{C}), \mathbb{Z})$  is injective with torsion free cokernel  $\Lambda$  by the weak Lefschetz theorem [8, Theorem 2]. Let  $\Xi \subset \Lambda_{\mathbb{C}}$  be the subspace of Hodge classes. Since  $X$  was chosen to be very general, any class in  $\Xi$  remains Hodge when transported horizontally using the Gauss–Manin connection of the Lefschetz pencil. It follows that  $\Xi$  is stabilized by the monodromy of the Lefschetz pencil on  $\Lambda_{\mathbb{C}}$ . Since this action is irreducible as a consequence of the hard Lefschetz theorem (see [45, Theorem 7.3.2]) and since the Hodge structure on  $\Lambda_{\mathbb{C}}$  is not trivial (as the restriction map  $H^{2d}(A, \mathcal{O}_A) \rightarrow H^{2d}(X, \mathcal{O}_X)$  is not surjective), we deduce that  $\Xi = 0$ . This shows that all Hodge classes in  $H^{2d}(X(\mathbb{C}), \mathbb{Z})$  are in the image of the injective restriction map  $H^{2d}(A(\mathbb{C}), \mathbb{Z}) \rightarrow H^{2d}(X(\mathbb{C}), \mathbb{Z})$ , and hence that  $\text{Hdg}^{2d}(X(\mathbb{C}), \mathbb{Z})$  is generated by the restriction  $\eta$  of  $(1/d!)c_1(\mathcal{L})^d$ . We now compute  $\deg(\eta^2) = \deg((1/(d!)^2)c_1(\mathcal{L})^{2d}c_1(\mathcal{L}^{\otimes l})) = (l(2d + 1)!)/(d!)^2$ , which proves (ii).

The normal exact sequence  $0 \rightarrow T_X \rightarrow (T_A)|_X \rightarrow \mathcal{L}^{\otimes l}|_X \rightarrow 0$  shows that  $c(X) = c(A)|_X c(\mathcal{L}^{\otimes l}|_X)^{-1} = (1 + lc_1(\mathcal{L})|_X)^{-1} \equiv 1 \pmod{2^{e+1}}$ , proving (i).  $\square$

### 4.5 Submanifolds with no algebraic approximations

Now come the promised applications to algebraic approximation. We refer to Section 0.3 for the definition of the cobordism group  $\text{MO}_d(X(\mathbb{R}))$  of the real locus of a smooth projective variety  $X$  over  $\mathbb{R}$ , and of its

subgroup  $\text{MO}_d^{\text{alg}}(X(\mathbb{R}))$  of algebraic cobordism classes. Recall that  $\text{MO}_d$  denotes the cobordism ring, which is the cobordism group of the point (see Section 3.1).

**Theorem 4.19** *Assume that  $c, d, e \geq 1$  are such that  $d \geq c$  and  $\alpha(c + e) = 2e$ . Then there exist a smooth projective variety  $X$  of dimension  $c + d$  over  $\mathbb{R}$  and a  $d$ -dimensional closed  $C^\infty$  submanifold  $j : M \hookrightarrow X(\mathbb{R})$  such that the following properties hold for all  $d$ -dimensional closed subvarieties  $i : Y \hookrightarrow X$ :*

- (i) *One has  $[j] \in \text{MO}_d^{\text{alg}}(X(\mathbb{R}))$ .*
- (ii) *If  $Y$  is smooth, then  $[M] \neq [Y(\mathbb{R})] \in \text{MO}_d$ .*
- (iii) *If  $e = 1$  and  $Y$  is smooth along  $Y(\mathbb{R})$ , then  $[M] \neq [Y(\mathbb{R})] \in \text{MO}_d$ .*

We first prove Theorem 4.19 in the particular case where  $c = d$ .

**Lemma 4.20** *If  $c = d$ , Theorem 4.19 holds with any  $X$  as in Proposition 4.18.*

**Proof** Recall from Section 3.2 that if  $I = (i_1, i_2, \dots)$  is a sequence of nonnegative integers, we set  $|I| := \sum_r r i_r$ . By Theorem 3.6, there exists a degree 1 homogeneous polynomial  $P \in \mathbb{Z}[x_I]_{|I|=d}$  such that  $s_d(y) \equiv 2^e P(c_I(y)) \pmod{2^{e+1}}$  for all  $y \in \text{MU}_{2d}$ .

Theorem 3.4 shows the existence of  $y_0 \in \text{MU}_{2d}$  such that  $s_d(y_0) \not\equiv 0 \pmod{2^{e+1}}$ , and hence such that  $P(c_I(y_0)) \not\equiv 0 \pmod{2}$ . Letting  $M$  be a compact  $C^\infty$  manifold representing  $\psi(y_0) \in \text{MO}_d$ , Lemma 3.2 shows that  $P(w_I(M)) \not\equiv 0 \in \mathbb{Z}/2$ .

A theorem of Whitney [62, Theorem 5] asserts that any  $d$ -dimensional compact  $C^\infty$  manifold may be embedded in  $\mathbb{R}^{2d}$ . It follows that one may choose an embedding  $j : M \hookrightarrow X(\mathbb{R})$  of  $M$  in a small ball of  $X(\mathbb{R})$ . Since this small ball is contractible,  $j$  is homotopic (and hence cobordant) to a constant map  $M \rightarrow X(\mathbb{R})$ . As  $M$  is cobordant to the real locus of a smooth projective variety over  $\mathbb{R}$  (see Section 3.1), one deduces that  $[j] \in \text{MO}_d^{\text{alg}}(X(\mathbb{R}))$ , proving (i).

Let  $W \rightarrow Y$  be a desingularization of  $Y$  which is an isomorphism above  $Y(\mathbb{R})$ , and let  $g : W \rightarrow X$  be the induced morphism. Proposition 4.1 shows, under the assumptions of either (ii) or (iii), that

$$(4.21) \quad \deg([Y]^2) = \deg((g_*[W])^2) \equiv \deg(c_d(N_{W/X})) \pmod{2^{e+1}}.$$

Since the Betti cohomology class of  $Y$  is a Hodge class,  $\deg([Y]^2) \equiv 0 \pmod{2^{e+1}}$  by Proposition 4.18(ii). The total Chern class of  $N_{W/X}$  is  $c(N_{W/X}) = c(g^*T_X)c(T_W)^{-1} = g^*c(X)s(W)$ , where  $s(W) \in \text{CH}^*(W)$  denotes the total Segre class of  $W$ . Proposition 4.18(iii) then implies  $c_d(N_{W/X}) \equiv s_d(W) \pmod{2^{e+1}}$ . Together with (4.21), these facts show that  $s_d(W) \equiv 0 \pmod{2^{e+1}}$ .

So  $s_d(W(\mathbb{C})) \equiv 0 \pmod{2^{e+1}}$ . By our choice of  $P$ , we deduce that  $P(c_I(W(\mathbb{C}))) \equiv 0 \pmod{2}$ . Conner and Floyd [21, Theorem 22.4] have proven that  $W(\mathbb{C})$  and  $W(\mathbb{R}) \times W(\mathbb{R})$  are cobordant, and it follows that  $\psi([W(\mathbb{C})]) = [W(\mathbb{R})] \in \text{MO}_d$ . Lemma 3.2 now shows that  $P(w_I(W(\mathbb{R}))) = 0 \in \mathbb{Z}/2$ , and hence that  $P(w_I(Y(\mathbb{R}))) = P(w_I(W(\mathbb{R}))) \neq P(w_I(M))$ . Since Stiefel–Whitney numbers are cobordism-invariants (see Section 3.2), we have  $[M] \neq [Y(\mathbb{R})] \in \text{MO}_d$ . □

The proof of [Theorem 4.19](#) in general easily reduces to the above lemma.

**Proof of Theorem 4.19** [Lemma 4.20](#) produces a smooth projective variety  $X'$  of dimension  $2c$  over  $\mathbb{R}$  and a  $c$ -dimensional closed  $C^\infty$  submanifold  $j': M' \hookrightarrow X'(\mathbb{R})$  satisfying properties (i)–(iii) of [Theorem 4.19](#) (with  $d$  replaced by  $c$ ). Define  $X := X' \times \mathbb{P}_{\mathbb{R}}^{d-c}$  and  $M := M' \times \mathbb{P}^{d-c}(\mathbb{R})$ , and consider the embedding  $j := (j', \text{Id}): M \hookrightarrow X(\mathbb{R})$ .

Our choice of  $X'$  and  $M'$  shows the existence of a morphism  $g': W' \rightarrow X'$  of smooth projective varieties over  $\mathbb{R}$  such that  $j'$  is cobordant to  $g'(\mathbb{R})$ . Defining  $W := W' \times \mathbb{P}_{\mathbb{R}}^{d-c}$  and  $g := (g', \text{Id})$ , we see that  $j$  is cobordant to  $g(\mathbb{R})$ , and hence that  $[j] \in \text{MO}_d^{\text{alg}}(X(\mathbb{R}))$ , which proves (i).

Suppose now that  $i: Y \hookrightarrow X$  is as in the statement of [Theorem 4.19](#) and satisfies the hypothesis of either (ii) or (iii). Let  $x \in \mathbb{P}^{d-c}(\mathbb{R})$  be a general point, and define  $Y' := Y \cap (X' \times \{x\})$  and  $i': Y' \hookrightarrow X' \times \{x\} \simeq X'$  to be the natural inclusion. Bertini's theorem ensures that  $Y'$  is smooth in case (ii) and that  $Y'$  is smooth along  $Y'(\mathbb{R})$  in case (iii). If  $M$  were cobordant to  $Y(\mathbb{R})$ , Sard's theorem would imply that  $M'$  is cobordant to  $Y'(\mathbb{R})$ . This contradicts our choice of  $M'$  and proves (ii) and (iii). □

**Remarks 4.22** (i) [Theorem 4.19](#)(i) and (iii) prove [Theorem 0.7](#).

(ii) It is striking that the obstructions to  $M$  being approximable by real loci of algebraic subvarieties of  $X$  provided by [Theorem 4.19](#)(ii)–(iii) involve cobordism theory, although  $[j] \in \text{MO}_d^{\text{alg}}(X(\mathbb{R}))$  by [Theorem 4.19](#)(i). Loosely speaking, the map  $j$  is cobordant to an algebraic map, but not to an algebraic embedding.

(iii) Complex cobordism and [Theorem 3.6](#) are not needed to prove [Theorem 4.19](#) for  $c = 2$ . One may use Noether's formula instead, as in [Example 3.7](#).

(iv) In the setting of [Theorem 4.19](#), one could moreover arrange that the inclusion  $j: M \hookrightarrow X(\mathbb{R})$  be approximable in the  $C^\infty$  topology by the inclusion of the set of smooth real points of an algebraic subvariety  $Z \subset X$  with compact set of smooth real points (which necessarily also has some singular real points if  $e = 1$ , in view of [Theorem 4.19](#)(iii)). To prove it when  $c = d$ , one can use linkage and general position arguments as in Sections 1 and 2. We do not give a detailed proof here. The general case reduces to the case  $c = d$  as in the proof of [Theorem 4.19](#).

The proof of [Theorem 4.23](#) is a variant of the proof of [Lemma 4.20](#). Since  $c(\mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^{2k+1-1}) \not\equiv 1 \pmod{4}$ , the argument is slightly more complicated.

The relation  $(\sum_r w_r(M))(\sum_r \bar{w}_r(M)) = 1$  defines the *normal Stiefel–Whitney classes*  $\bar{w}_r(M) \in H^r(M, \mathbb{Z}/2)$  of a compact  $C^\infty$  manifold  $M$ . We also recall that if  $i: Y \rightarrow X$  is a morphism of varieties over  $\mathbb{R}$ , we let  $i(\mathbb{R}): Y(\mathbb{R}) \rightarrow X(\mathbb{R})$  be the induced map between sets of real points.

**Theorem 4.23** Fix  $k \geq 1$  and define  $X := \mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^{2k+1-1}$ . There exists a  $2^k$ -dimensional closed  $C^\infty$  submanifold  $j: M \hookrightarrow X(\mathbb{R})$  such that  $[j] \neq [i(\mathbb{R})] \in \text{MO}_{2^k}(X(\mathbb{R}))$  for all  $2^k$ -dimensional closed subvarieties  $i: Y \hookrightarrow X$  that are smooth along  $Y(\mathbb{R})$ .

**Proof** Since  $\alpha(2^k + 1) = 2$ , [Theorem 3.6](#), shows that there is a degree 1 homogeneous polynomial  $P \in \mathbb{Z}[x_I]_{|I|=2^k}$  with  $s_{2^k}(y) \equiv 2P(c_I(y)) \pmod{4}$  for all  $y \in \text{MU}_{2^{k+1}}$ . By [Theorem 3.4](#), one may find  $y_0 \in \text{MU}_{2^{k+1}}$  such that  $s_{2^k}(y_0) \not\equiv 0 \pmod{4}$ , and hence such that  $P(c_I(y_0)) \not\equiv 0 \pmod{2}$ . Letting  $M$  be a compact  $C^\infty$  manifold representing  $\psi(y_0) \in \text{MO}_{2^k}$ , [Lemma 3.2](#) shows that  $P(w_I(M)) \not\equiv 0 \pmod{2}$ . By Whitney’s theorem [[62](#), [Theorem 5](#)], one may embed  $j : M \hookrightarrow X(\mathbb{R})$  in a small ball of  $X(\mathbb{R})$ .

Let  $i : Y \hookrightarrow X$  be as in the statement of [Theorem 4.23](#). Assume for contradiction that  $[j] = [i(\mathbb{R})] \in \text{MO}_{2^k}(X(\mathbb{R}))$ . Let  $W \rightarrow Y$  be a desingularization of  $Y$  which is an isomorphism above  $Y(\mathbb{R})$  and let  $g : W \rightarrow X$  be the induced morphism. [Proposition 4.1\(ii\)](#) shows that  $\deg([Y]^2) = \deg((g_*[W])^2) \equiv \deg(c_{2^k}(N_{W/X})) \pmod{4}$ , and hence that

$$(4.24) \quad \deg([Y]^2) \equiv \sum_{r=0}^{2^k} \deg(g^*c_r(X)s_{2^k-r}(W)) \pmod{4}.$$

Consider the Borel–Haefliger cycle class map  $\text{cl}_{\mathbb{R}} : \text{CH}^*(X) \rightarrow H^*(X(\mathbb{R}), \mathbb{Z}/2)$  [[18](#)]; see also [[13](#), [Section 1.6.2](#)]. Since  $[j] = [i(\mathbb{R})] \in \text{MO}_{2^k}(X(\mathbb{R}))$ , one has  $\text{cl}_{\mathbb{R}}([Y]) = [M] = 0 \in H^{2^k}(X(\mathbb{R}), \mathbb{Z}/2)$ . Set

$$H_1 := c_1(\mathcal{O}_{\mathbb{P}_{\mathbb{R}}^1}(1)) \in \text{CH}^1(\mathbb{P}_{\mathbb{R}}^1) \quad \text{and} \quad H_2 := c_1(\mathcal{O}_{\mathbb{P}_{\mathbb{R}}^{2^k+1-1}}(1)) \in \text{CH}^1(\mathbb{P}_{\mathbb{R}}^{2^k+1-1}).$$

As  $\text{CH}^{2^k}(X)$  is generated by  $(H_2)^{2^k}$  and  $H_1(H_2)^{2^k-1}$ , the kernel of  $\text{cl}_{\mathbb{R}} : \text{CH}^{2^k}(X) \rightarrow H^{2^k}(X(\mathbb{R}), \mathbb{Z}/2)$  is generated by  $2(H_2)^{2^k}$  and  $2H_1(H_2)^{2^k-1}$ . As a consequence,  $[Y] \in \text{CH}^{2^k}(X)$  is a multiple of 2, and hence  $\deg([Y]^2)$  is divisible by 4.

The Euler exact sequences

$$0 \rightarrow \mathcal{O}_{\mathbb{P}_{\mathbb{R}}^N} \rightarrow \mathcal{O}_{\mathbb{P}_{\mathbb{R}}^N}(1)^{\oplus N+1} \rightarrow T_{\mathbb{P}_{\mathbb{R}}^N} \rightarrow 0$$

and the Whitney sum formula yield  $c(X) = (1 + H_1)^2(1 + H_2)^{2^k} \in \text{CH}^*(X)$ . Since  $H_1^2 = H_2^{2^k} = 0$ , we deduce that  $c(X) \equiv 1 \pmod{2}$ . For  $r \geq 1$ , let  $\gamma_r \in \text{CH}^r(X)$  be such that  $c_r(X) = 2\gamma_r$ . Since Borel and Haefliger have shown that  $\text{cl}_{\mathbb{R}}(c(W)) = w(W(\mathbb{R}))$  [[18](#), [Section 5.18](#)], see also [[39](#), [Proposition 3.5.1](#)],  $\text{cl}_{\mathbb{R}}(s(W)) = \bar{w}(W(\mathbb{R}))$ . We deduce that, for  $r \geq 1$ ,

$$(4.25) \quad \begin{aligned} \deg(\text{cl}_{\mathbb{R}}(g^*\gamma_r s_{2^k-r}(W))) &= \deg(g(\mathbb{R})^* \text{cl}_{\mathbb{R}}(\gamma_r) \bar{w}_{2^k-r}(W(\mathbb{R}))) \\ &= \deg(j^* \text{cl}_{\mathbb{R}}(\gamma_r) \bar{w}_{2^k-r}(M)) = 0 \in \mathbb{Z}/2, \end{aligned}$$

where the first equality follows from the functorial properties of  $\text{cl}_{\mathbb{R}}$  (see [[13](#), [Section 1.6.2](#)]), the second from the equality of the Stiefel–Whitney numbers of the cobordant maps  $j$  and  $g(\mathbb{R}) = i(\mathbb{R})$  (see [[21](#), [Theorem 17.3](#)]), and the third holds since  $j^* : H^r(X(\mathbb{R}), \mathbb{Z}/2) \rightarrow H^r(M, \mathbb{Z}/2)$  vanishes for  $r \geq 1$  because the image of  $j$  is included in a small ball of  $X(\mathbb{R})$ .

Equation (4.25) demonstrates that  $\deg(g^*\gamma_r s_{2^k-r}(W)) \in \mathbb{Z}$  is even, and so  $\deg(g^*c_r(X)s_{2^k-r}(W))$  is divisible by 4, for all  $r \geq 1$ . Plugging the congruences we have obtained into (4.24) shows that  $\deg(s_{2^k}(W)) \equiv 0 \pmod{4}$ , and hence that  $s_{2^k}(W(\mathbb{C})) \equiv 0 \pmod{4}$ . Our choice of  $P$  implies that  $P(c_I(W(\mathbb{C}))) \equiv 0 \pmod{2}$ .

By [21, Theorem 22.4],  $\psi([W(\mathbb{C})]) = [W(\mathbb{R})] \in \text{MO}_{2k}$ . Lemma 3.2 shows that  $P(w_I(W(\mathbb{R}))) = 0 \in \mathbb{Z}/2$ , and hence that  $P(w_I(Y(\mathbb{R}))) \neq P(w_I(M))$ . We deduce that  $[M] \neq [Y(\mathbb{R})] \in \text{MO}_{2k}$  by cobordism-invariance of Stiefel–Whitney numbers. A fortiori,  $[j] \neq [i(\mathbb{R})] \in \text{MO}_{2k}(X(\mathbb{R}))$ , a contradiction.  $\square$

**Remarks 4.26** (i) Theorem 4.23 implies Theorem 0.8 by [60, Proposition 4.4.4].

(ii) Theorem 4.23 is false for  $k = 0$  by [16, Theorem 12.4.11]. The proof fails in this case because  $\alpha(2^k + 1) < 2$  precisely for this value of  $k$ .

(iii) The simplest particular case of Theorem 4.23 is the following. Embed  $\mathbb{P}^2(\mathbb{R})$  in  $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3$  and let  $j : \mathbb{P}^2(\mathbb{R}) \hookrightarrow \mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^3(\mathbb{R})$  be the induced embedding. Then  $j$  is not cobordant to the inclusion of the real locus of a closed subvariety of  $\mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^3$  which is smooth along its real locus. A fortiori,  $j$  may not be isotoped to such a real locus. As in Remark 4.22(iii), the use of Theorem 3.6 may be replaced by Noether’s formula in the proof of this particular case.

(iv) The conclusion of Theorem 4.23, is not explained by a difference between the groups  $\text{MO}_*^{\text{alg}}(X(\mathbb{R}))$  and  $\text{MO}_*(X(\mathbb{R}))$ , as they coincide by [11, Remark 3, page 103] or [36, Corollary 1, page 314].

## 5 Algebraic approximation and algebraic homology

In this section, fix a smooth projective variety  $X$  of dimension  $c + d$  over  $\mathbb{R}$ , and a closed  $d$ -dimensional  $\mathcal{C}^\infty$  submanifold  $j : M \hookrightarrow X(\mathbb{R})$ . Recall from Section 0.3 the definition of the approximation property (Property A) and of its necessary condition Property B based on cobordism. We now study variants of these properties, our goal being Theorem 5.6.

### 5.1 A stronger approximation property

It is natural to consider the following strengthening of Property A considered in Section 0.3:

**Property A'** For all neighborhoods  $\mathcal{U} \subset \mathcal{C}^\infty(M, X(\mathbb{R}))$  of the inclusion, there exist  $\phi \in \mathcal{U}$  and a smooth closed subvariety  $Y \subset X$  such that  $\phi(M) = Y(\mathbb{R})$ .

The following two theorems are analogues of Theorems 0.6 and 0.7 for Property A'. They are consequences of Theorems 2.7 and 4.19, respectively.

**Theorem 5.1** Properties A' and B are equivalent if  $d < c$  and  $d \leq 3$ .

**Theorem 5.2** Let  $d \geq c$  and  $e \geq 1$  be such that  $\alpha(c + e) = 2e$ . Then there exist  $X$  and  $M$  such that Property A' fails but Property B holds.

**Remarks 5.3** (i) We do not know if it is possible to get rid of the hypothesis that  $d \leq 3$  in Theorem 5.1.

(ii) Examples where Property B (and even Property A) hold but Property A' fails had already been obtained by Akbulut and King [7, Theorem 4], and refined by Kucharz [42, Theorem 1.1]. Their examples work for all  $(c, d)$  with  $c \geq 2$  and  $d \geq c + 2$ . The range of pairs  $(c, d)$  that we reach in Theorem 5.2 is different.

## 5.2 Homology and cobordism obstructions

Define  $H_d^{\text{alg}}(X(\mathbb{R}), \mathbb{Z}/2)$  to be the image of the Borel–Haefliger cycle class map

$$\text{cl}_{\mathbb{R}} : \text{CH}_d(X) \rightarrow H_d(X(\mathbb{R}), \mathbb{Z}/2).$$

Consider the following property:

**Property H** One has  $j_*[M] \in H_d^{\text{alg}}(X(\mathbb{R}), \mathbb{Z}/2)$ .

**Property B** implies **Property H** since algebraic homology classes are preserved by pushforwards; see for instance [13, Section 1.6.2]. It follows that **Property H** is an obstruction to **Properties A** and **A'** that is weaker than **Property B**. In fact, the cobordism obstruction **Property B** was first used by Bochnak and Kucharz [17, Corollary 1.3] to give examples where **Property A** fails but **Property H** holds, when  $d \geq 3$  and  $c \geq 2$ .

**Lemma 5.4** *If  $d \leq 2$ , then **Properties B** and **H** are equivalent.*

**Proof** Since  $\text{MO}_1 = 0$ , an isomorphism

$$(H_{d-2}(X(\mathbb{R}), \mathbb{Z}/2) \otimes \text{MO}_2) \oplus H_d(X(\mathbb{R}), \mathbb{Z}/2) \xrightarrow{\sim} \text{MO}_d(X(\mathbb{R}))$$

is constructed in [21, Theorem 17.2]. It restricts to an isomorphism

$$(H_{d-2}(X(\mathbb{R}), \mathbb{Z}/2) \otimes \text{MO}_2) \oplus H_d^{\text{alg}}(X(\mathbb{R}), \mathbb{Z}/2) \xrightarrow{\sim} \text{MO}_d^{\text{alg}}(X(\mathbb{R}))$$

by Ischebeck and Schülting [36, Corollary 1, page 314], since  $H_0^{\text{alg}}(X(\mathbb{R}), \mathbb{Z}/2) = H_0(X(\mathbb{R}), \mathbb{Z}/2)$ . We deduce that the two conditions  $j_*[M] \in H_d^{\text{alg}}(X(\mathbb{R}), \mathbb{Z}/2)$  and  $[j : M \hookrightarrow X(\mathbb{R})] \in \text{MO}_d^{\text{alg}}(X(\mathbb{R}))$  are equivalent.  $\square$

## 5.3 Hypersurfaces

The following proposition is a well-known improvement of [16, Theorem 12.4.11], which goes back to the work of Benedetti and Tognoli [10, Proposition 1, page 227]; see also [4, Theorem A].

**Proposition 5.5** *If  $c = 1$ , then **Properties A'** and **H** are equivalent.*

**Proof** Assume that **Property H** holds. Let  $\mathcal{U} \subset \mathcal{C}^\infty(M, X(\mathbb{R}))$  be a neighborhood of the inclusion. By [16, Theorem 12.4.11], there exist  $\psi \in \mathcal{U}$ , an open neighborhood  $U$  of  $X(\mathbb{R})$  in  $X$  and a smooth closed hypersurface  $Z \subset U$  with  $\psi(M) = Z(\mathbb{R})$ . Let  $\bar{Z} \subset X$  be the Zariski closure of  $Z$ . Since  $X$  is smooth, there exist a line bundle  $\mathcal{L}$  on  $X$  and a section  $s \in H^0(X, \mathcal{L})$  with  $\bar{Z} = \{s = 0\}$ .

Fix a very ample line bundle  $\mathcal{O}_X(1)$  on  $X$ . Let  $(u_1, \dots, u_N)$  be a basis of  $H^0(X, \mathcal{O}_X(1))$ . The section  $v := \sum_{m=1}^N u_m^2 \in H^0(X, \mathcal{O}_X(2))$  vanishes nowhere on  $X(\mathbb{R})$ . Choose  $l \gg 0$  with  $\mathcal{M} := \mathcal{L}(2l)$  very ample, let  $t \in H^0(X, \mathcal{M})$  be a general small deformation of  $sv^l$  and set  $Y := \{t = 0\}$ , which is smooth by Bertini.

That  $Y$  has the required properties follows from [1, Section 20]. More precisely, applying the proofs of [1, Lemmas 20.3 and 20.4] with  $X = X(\mathbb{R})$ ,  $Y = \mathcal{M}(\mathbb{R})$ ,  $W \subset Y$  the zero section,  $r \geq 1$  and  $A = \mathcal{C}^{r+1}(X(\mathbb{R}), \mathcal{M}(\mathbb{R}))$  shows that if  $t$  is close to  $sv^l$ , then the inclusions  $Z(\mathbb{R}) \subset X(\mathbb{R})$  and  $Y(\mathbb{R}) \subset X(\mathbb{R})$  are isotopic, by an isotopy which is  $C^\infty$  because  $t$  and  $sv^l$  are, and small in the  $C^\infty$  topology (see the use of the implicit section theorem in the proof of [1, Lemma 20.3]).  $\square$

## 5.4 A question of Bochnak and Kucharz

In [17, pages 685–686], Bochnak and Kucharz ask for which values of  $c$  and  $d$  are Properties **A** and **H** equivalent. We obtain a full answer to that question, and disprove the expectation raised in [17, page 686] that Properties **A** and **H** are not equivalent for  $d = 2$  and  $c \geq 3$ .

**Theorem 5.6** *Properties **H**, **B**, **A** and **A'** are all equivalent in the following cases: if  $c \leq 1$ , if  $d \leq 1$ , or if  $d = 2$  and  $c \geq 3$ .*

*For all other values of  $c$  and  $d$ , there exist  $X$  and  $M$  satisfying **Property H** but not **Property A**.*

**Proof** The theorem is trivial if  $c = 0$  or  $d = 0$ . It follows from [Proposition 5.5](#) if  $c = 1$ . The cases with  $d \geq 3$  and  $c \geq 2$  are covered by Bochnak and Kucharz in [17, Corollary 1.3]. If  $d \leq 2$ , then **Property H** is equivalent to **Property B** by [Lemma 5.4](#). The cases where  $d = 1$  and  $c \geq 2$ , or where  $d = 2$  and  $c \geq 3$ , now follow from [Theorem 5.1](#). Finally, when  $c = d = 2$ , one may apply [Theorem 5.2](#)  $\square$

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Département de mathématiques et applications, École normale supérieure, CNRS  
Paris, France

[olivier.benoist@ens.fr](mailto:olivier.benoist@ens.fr)

Proposed: Haynes R Miller  
Seconded: Dan Abramovich, Mark Gross

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