Geometry & Topology

Volume 28 (2024)

Pseudo-Anosovs are exponentially generic in mapping class groups

INHYEOK CHOI

msp

Geometry & Topology **28**:4 (2024) 1923–1955
DOI: 10.2140/gt.2024.28.1923
Published: 18 July 2024

Pseudo-Anosovs are exponentially generic in mapping class groups

INHYEOK CHOI

Given a finite generating set S, let us endow the mapping class group of a closed hyperbolic surface with the word metric for S. We discuss the following question: does the proportion of non-pseudo-Anosov mapping classes in the ball of radius R converge to 0 as R tends to infinity? We show that any finite subset S' of the mapping class group is contained in a finite generating set S such that this proportion decays exponentially. Our strategy applies to weakly hyperbolic groups and does not refer to the automatic structure of the group.

20F67, 30F60, 57K20, 57M60, 60G50

1 Introduction

Let Σ be a closed hyperbolic surface. We denote by $\mathrm{Mod}(\Sigma)$, $\mathcal{T}(\Sigma)$ and $\mathcal{C}(\Sigma)$ the mapping class group, the Teichmüller space and the curve complex of Σ, respectively. When X is a Gromov hyperbolic space or $\mathcal{T}(\Sigma)$ and $g \in \mathrm{Isom}(X)$, we denote by $\tau_X(g)$ the (asymptotic) translation length of g. For a group G generated by a finite set S, we denote by $B_S(n)$ the ball of radius n with respect to the word metric for S. We also denote by $\partial B_S(n)$ the corresponding sphere of radius n. Our main result is as follows.

Theorem A (translation length grows linearly) *Let X be either a Gromov hyperbolic space or $\mathcal{T}(\Sigma)$. Let also G be a finitely generated nonelementary subgroup of $\mathrm{Isom}(X)$ and $S' \subseteq G$ be a finite subset. Then there exist $L, K > 0$ and a finite generating set $S \supseteq S'$ of G such that, for each n,*

$$\frac{\#\{g \in B_S(n) : \tau_X(g) \le Ln\}}{\#B_S(n)} \le K e^{-n/K}.$$

Non-pseudo-Anosov mapping classes have translation length zero in $\mathcal{C}(\Sigma)$. As a result, we affirmatively answer the following version of a folklore conjecture, at least for infinitely many generating sets S.

Corollary 1.1 (genericity of pAs; cf [Farb 2006, Conjecture 3.15]) *Let G be a finitely generated nonelementary subgroup of $\mathrm{Mod}(\Sigma)$. Then there exists a finite generating set $S \subseteq G$ such that the proportion of non-pseudo-Anosov mapping classes in the ball $B_S(n)$ decays exponentially as $n \to \infty$.*

Note that $\mathrm{Mod}(\Sigma)$ can act on both $\mathcal{T}(\Sigma)$ and $\mathcal{C}(\Sigma)$. Comparing the translation lengths of mapping classes on these two spaces is an interesting question. Thanks to the linear growth in Theorem A, we can deduce:

Corollary 1.2 *Let G be a finitely generated nonelementary subgroup of $\mathrm{Mod}(\Sigma)$ and $S' \subseteq G$ be a finite subset. Then there exist $L, K > 0$ and a finite generating set $S \supseteq S'$ of G such that for*

$$P := \left\{ g \in G : \frac{1}{L} \leq \frac{\tau_{\mathcal{T}(\Sigma)}(g)}{\tau_{\mathcal{C}(\Sigma)}(g)} \leq L \right\}$$

and each n,

$$\frac{\# B_S(n) \cap P^c}{\# B_S(n)} \leq K e^{-n/K}.$$

1.1 History and related problems

We remark that Theorem A may not be optimal for Gromov hyperbolic spaces. Gekhtman, Taylor and Tiozzo have proved the genericity of loxodromics in hyperbolic groups acting on separable Gromov hyperbolic spaces, in terms of the word metric for *any* finite generating set. This was generalized to relatively hyperbolic groups, RAAGs and RACGs with *particular* finite generating sets [Gekhtman et al. 2018; 2020; 2022]. Gekhtman, Taylor and Tiozzo's example [Gekhtman et al. 2020, Example 1] shows that the genericity of loxodromics is not achieved for all weakly hyperbolic groups with respect to all finite generating sets.

For relatively hyperbolic groups (and many more), Yang [2020] also established that loxodromics are exponentially generic when the action is proper and cocompact. Hence, at least for hyperbolic groups that admit a proper and cocompact action on Gromov hyperbolic spaces, Theorem A is weaker than previous results in the sense that the finite generating set cannot be arbitrary. It is however stronger in the sense that

(1) it does not require the action of G to be proper and cocompact,

(2) it deals with exponential genericity with respect to a linearly growing threshold, not a static threshold.

In fact, combining our strategy with the theory of Gekhtman, Taylor and Tiozzo yields the following.

Proposition 1.3 *Let X and G be as in Theorem A and S be a finite generating set of G. Suppose moreover that G itself is a hyperbolic group. Then there exists $\lambda > 0$ such that the following hold. Below, ν denotes the Patterson–Sullivan measure with respect to S.*

(1) *For any $x \in X$ and ν–a.e. $\eta \in \partial G$, if $(g_n)_{n \geq 0}$ is a geodesic in G converging to η, then*

$$\lim_{n \to \infty} \frac{d_X(x, g_n x)}{n} = \lambda.$$

(2) *For any $\epsilon > 0$, there exists $K > 0$ such that, for each n,*

$$\frac{\#\{g \in \partial B_S(n) : \tau_X(g) \notin [\lambda - \epsilon, \lambda + \epsilon]\}}{\# \partial B_S(n)} \leq K e^{-n/K}.$$

Note that this implies the following. Given a hyperbolic subgroup G of $\mathrm{Mod}(\Sigma)$ and *any* finite generating set S of G, let $\lambda_{\mathcal{T}}$ and $\lambda_{\mathcal{C}}$ be the escape rate of G on $\mathcal{T}(\Sigma)$ and $\mathcal{C}(\Sigma)$, respectively, in terms of the Patterson–Sullivan measure for S. Then for any $\epsilon > 0$, there exists K such that, for each n,

$$\frac{\#\{g \in B_S(n) : \lambda_{\mathcal{T}}/\lambda_{\mathcal{C}} - \epsilon \le \tau_{\mathcal{T}(\Sigma)}(g)/\tau_{\mathcal{C}(\Sigma)}(g) \le \lambda_{\mathcal{T}}/\lambda_{\mathcal{C}} + \epsilon\}}{\#B_S(n)} \le Ke^{-n/K}.$$

For the sake of completeness, we sketch the proof of Proposition 1.3 in Appendix B.

Meanwhile, Theorem A is new for the mapping class group $\mathrm{Mod}(\Sigma)$. The progress so far was that the proportion of pseudo-Anosov elements in the word metric ball stays bounded away from zero [Cumplido and Wiest 2018]. Meanwhile, in the braid group B_n for $n \ge 3$, with respect to Garside's generating set, Caruso and Wiest [2017] showed that pseudo-Anosov braids are generic in the word metric ball. See [Calvez and Wiest 2017] for a generalization to spherical Artin–Tits groups. On $\mathrm{Mod}(\Sigma)$, Yang [2020] and Erlandsson et al. [2020] discussed the genericity of pseudo-Anosovs from different viewpoints. We can further ask:

Question 1.4 *Are pseudo-Anosovs exponentially generic with respect to any finite generating set? For example, are they exponentially generic with respect to Humphries' generators? If not, are they generic at least?*

Question 1.5 *Does Proposition 1.3 hold for $G = \mathrm{Mod}(\Sigma)$ and at least one S?*

Question 1.5 is intimately related to the (geodesic) automaticity of $\mathrm{Mod}(\Sigma)$.

Let us finally mention a problem investigated by I Kapovich. Let μ be a discrete measure on a group G. We define the *nonbacktracking random walk* generated by μ as follows. The first alphabet g_1 is chosen from G with the law of μ; for each $n \ge 2$, g_n is chosen from $G \setminus \{g_{n-1}^{-1}\}$ with the law

$$\mathbb{P}(g_n = g) = \frac{1}{\mu(G \setminus \{g_{n-1}^{-1}\})}\mu(g).$$

In this setting, Gekhtman, Taylor and Tiozzo proved that $\mathbb{P}(\omega_n \text{ is loxodromic})$ tends to 1 as $n \to \infty$ [Gekhtman et al. 2020, Theorem 2.8]. With an adequate modification, our argument yields the following.

Proposition 1.6 *Let X be as in Theorem A, and μ be a nonelementary discrete measure μ on $\mathrm{Isom}(X)$. Consider the nonbacktracking random walk ω generated by μ. Then there exists $L, K > 0$ such that, for each n,*

$$\mathbb{P}\{\tau_X(\omega_n) \le Ln\} \le Ke^{-n/K}.$$

1.2 Strategy for Theorem A

One approach to the counting problem is to utilize (geodesic) automatic structures of the group. Lacking such structures, we instead consider the random walk ω on G generated by the uniform measure on S. Then the theories of Gouëzel [2022] and Baik, Choi and Kim [Baik et al. 2023] imply that the unwanted probability decays exponentially. There are at least two more theories that provide this exponential decay. One is Maher's theory [2012] for random walks with bounded support, which led to Maher and Tiozzo's more general theory [2018]. Another one is Boulanger, Mathieu, Sert and Sisto's theory [Boulanger et al. 2023] of large deviation principles for random walks with finite exponential moment.

Unavoidably, random walks cannot count lattice points in a one-to-one manner. If S is nicely populated by the self-convolution of a Schottky set S_0, however, then we have a one-to-one correspondence between some portion of lattice points and (nonbacktracking) paths of alphabets in S_0. This leads to the estimate $\#B_S(e,n) \sim \#(\text{paths from the random walk}) \cdot r^n$ with $r \sim 1$. We arrive at the desired estimate by forcing the exponential decay of probability to be much faster than the decay of r^n.

Let us bring a toy example to explain how this strategy works. Let S be a finite symmetric generating set of the free group $F_2 \simeq \langle a, b\rangle$ of rank 2. Our goal is to compare the growth rate of

$$\begin{aligned} A(n) &:= \{a_1 \cdots a_n : a_i \in S\}, \\ B(n) &:= \{a_1 \cdots a_n : a_i \in S, \tau_X(a_1 \cdots a_n) = 0\}. \end{aligned}$$

Here, S contains elements that cancel out each other. This implies that although $A(n)$ does grow exponentially, its growth rate may not equal $\#S$. Moreover, even though any nontrivial word in F_2 corresponds to a loxodromic isometry on the Cayley graph, there can be some sequences of n letters from S whose composition is trivial.

These concerns disappear if S is an alphabet for a free subsemigroup of F_2. Letters in S do not cancel out each other, and we have $A(n) \sim (\#S)^n$ and $B(n) = 0$ for $n \geq 1$. The contrast between the two growth rates persists even when a few letters of S cancel out each other. For example, let us take

$$S = \{a^2, ab, ba, a^{-2}, b^{-1}a^{-1}, a^{-1}b^{-1}\}.$$

Clearly a^2 and a^{-2}, ab and $b^{-1}a^{-1}$, and ba and $a^{-1}b^{-1}$ cancel out each other. Nonetheless, $A(n)$ still grows exponentially with the growth rate $\#S - 1$. On the other hand, the growth rate of $B(n)$ is $\#S$ times the spectral radius of the simple random walk on the homogenous tree of degree 6, which is $\frac{\sqrt{5}}{3}$. The situation gets better and better as S becomes larger and larger. For a symmetric free generating set S of a subgroup G of F_2 with $\mathrm{rk}(G) = d$, we have $A(n) \sim (2d-1)^n$ while $B(n) \sim ((2\sqrt{2d-1}/2d) \cdot 2d)^n$. As d increases, the growth rate of $A(n)$ becomes closer to $\#S$ while the growth rate of $B(n)$ stays uniformly strictly smaller than $\#S$. In summary, although cancellations may disturb the contrast between the two growth rates, such disturbance can be made negligible by taking an "almost mutually independent" set S. Such sets are called Schottky sets.

Let us now consider a choice $S = S_1 \cup S_0$, where S_1 is a (symmetric) alphabet for a free semigroup of F_2 and S_0 is an additional impurity that makes S a generating set of F_2. For simplicity let us assume the form

$$S_1 = \{c_1 \cdots c_M \cdot a^{100M} \cdot d_1 \cdots d_M : c_i, d_i \in \{a, b\}\}$$

for $M \geq 10$. Since S_1 is large enough, we can guarantee the gap between the growth rates of $A(n)$ and $B(n)$ with respect to S_1. We now want the stability of this gap with respect to the perturbation S_0. To be explicit, we hope

$$A(n) \gtrsim ((1-\rho)\,\#S)^n, \quad B(n) \lesssim (\rho'\,\#S)^n$$

for some ρ proportional to $(\#S_0)/(\#S)$ and a constant $\rho' < 1$ that works for small enough $(\#S_0)/(\#S)$. Note that we do not require any condition on the elements of S_0; we only restrict the cardinality of S_0.

The stability for $A(n)$ is straightforward, but the one for $B(n)$ is considerably harder. In contrast to the case of simple random walks, we have no information on how individual elements of S_0 interact with the elements of S_1. Perhaps a bad element of S_0 cancels out a concatenation of 10 letters in S_1, or that of 100 letters. This opens the possibility that the RV $d(o, \omega_{n+1} o) - d(o, \omega_n o)$ conditioned on ω_n has negative expectation for some ω_n. Hence, we cannot pretend as if we are summing up i.i.d. RVs with positive expectation at each single step.

Nonetheless, we may focus only on the steps chosen from S_1 and try to construct i.i.d. RVs that reflects the progresses made there. Let us construct a set $P_n = \{i(1) < \cdots < i(m)\}$ such that

(1) $g_{i(1)}, \ldots, g_{i(m)}$ are drawn from S_1, and

(2) $[o, \omega_n o]$ contains the middle 99% of $[\omega_{i(1)} o, \omega_{i(1)} g_{i(1)} o], \ldots, [\omega_{i(m)} o, \omega_{i(m)} g_{i(m)} o]$.

Note that P_n is a subset of

$$\Theta = \{1 \leq i \leq n : g_i \in S_1\}$$

whose size is sufficiently large if $(\#S_0)/(\#S)$ is small enough. Fixing the slots Θ for elements in S_1 and all the other choices $\{g_i : i \notin \Theta\}$, we are now asked to control $w_m = w_0 s_1 w_1 \cdots s_m w_m$ where w_i are fixed words in F_2 and s_i are independently drawn from S_1. Since elements of S_1 are deviating from each other early, $[o, so]$ and $[o, w_0^{-1} o]$ fellow travel for very few $s \in S_1$. Similarly, $[o, s^{-1} o]$ and $[o, w_1 o]$ are deviating early for a large probability. Due to these sorts of reasons, there is a (uniformly) high chance that the middle 99% of $[w_{k-1} o, w_{k-1} s_k o]$ is visible in $[o, w_{k-1} s_k w_k o]$. Consequently, the progress made by s_k is along $[o, w_0 s_1 \cdots s_k w_k o]$. In such a case, those progresses made by s_i for $i < k$ along $[o, w_0 s_1 \cdots w_{k-1} o]$ are still intact in $[o, w_0 s_1 \cdots s_k w_k o]$.

Still, we are worried about the situation that an unfortunate choice of s_k makes a progress that is not visible in $[o, w_0 \cdots s_k w_k o]$, or even worse, the previous Schottky progresses along $[o, w_0 \cdots s_{k-1} w_{k-1} o]$ are all lost in $[o, w_0 \cdots s_k w_k o]$. Let $s_{j(1)}, \ldots, s_{j(m')}$ be the choices from S_1 before s_k that made progresses along $[o, w_0 \cdots s_{k-1} w_{k-1} o]$. We observe that there are plenty of other choices for $s_{j(1)}, \ldots, s_{j(m')}$ that make progresses along $[o, w_0 \cdots s_{k-1} w_{k-1} o]$. This modification does not affect the positions $j(1), \ldots, j(m')$, and we call it *pivoting*. We now freeze the choice $w_{j(m')+1} s_{j(m')+1} \cdots s_k w_k$ and perform the pivoting. The progress $[o, s_{j(m')}]$ made by the $(m')^{\text{th}}$ choice $s_{j(m')}$ is aligned along $[o, s_{j(m')} w_{j(m')} s_{j(m')+1} \cdots s_k w_k o]$ with high chance, and moreover, $[o, s_{j(m'-1)}]$ is aligned along $[o, s_{j(m'-1)} w_{j'(m'-1)} \cdots w_k o]$ with high chance *regardless of the pivotal choice* $s_{j(m')}$. Continuing this, we observe that the progress made by the $(m'-l)^{\text{th}}$ choice $s_{j(m'-l)}$ — and all progresses before it — is visible in $[o, w_0 \cdots w_k o]$, outside a set of probability that decays exponentially in l. Using this, we can bound $\#P_n$ from below by the sum of i.i.d. RVs that heavily favors 1 and has an exponentially decaying tail. In particular,

$$\mathbb{P}\left(\#P_n \geq \frac{n}{K}\right) \leq Ke^{-n/K}$$

for some $K > 0$. A fuller description can be found in [Gouëzel 2022, Section 2].

We have not discussed the linearly growing threshold of the translation length yet. For example, the word $a^{-1}b^{-1}a^3ba^2b^{-2}a^{-2}b^{-1}a^{-3}ba$ has large displacement, 18, but short translation length, 2. If we pivot the Schottky choices, e.g. modify the first or the second b^{-1} into b, its translation length will increase to 16 or 8, respectively. This illustrates that pivoting can secure large translation lengths in probability, which is the strategy of [Baik et al. 2023].

The above example is an over-simplification of weakly hyperbolic groups and mapping class groups. Gromov hyperbolic spaces and Teichmüller space are not trees, but we can still copy the above argument by using the Gromov inequality or the (partial) hyperbolicity in Teichmüller space due to Rafi.

We remark that the notions and statements in Section 2, 3 and 4 are mostly established in [Baik et al. 2023; Choi 2023; Gouëzel 2022]. Nonetheless, to make the exposition self-contained, we present all details except the proofs of Facts 2.5, 2.6, 2.8, 2.9 and A.1. See [Choi 2023] for their proofs.

Acknowledgements

The author truly thanks Hyungryul Baik, Dongryul M Kim, Jing Tao, Samuel Taylor and Wenyuan Yang for helpful discussions. The author is also grateful to Bert Wiest for drawing the author's attention to valuable references and explaining relevant backgrounds. The author also appreciates the referees' comments that led to a significant improvement on the manuscript.

The author was partially supported by the Samsung Science & Technology Foundation (SSTF-BA1702-01, SSTF-BA1301-51) and by a KIAS Individual Grant (SG091901) via the June E Huh Center for Mathematical Challenges at the Korea Institute for Advanced Study.

2 Witnessing and Schottky sets

This section is devoted to the facts about Gromov hyperbolic spaces and Teichmüller spaces. All facts in this section are proved in [Choi 2023].

Given a metric space (X, d) and a triple $x, y, z \in X$, we define the *Gromov product* of y and z with respect to x by

$$(y, z)_x = \tfrac{1}{2}[d(x, y) + d(x, z) - d(y, z)].$$

Throughout the article, we will frequently use the following property: for $x, y, z \in X$ and $g \in \mathrm{Isom}(X)$,

$$(gy, gz)_{gx} = (y, z)_x.$$

X is said to be *Gromov hyperbolic* if there exists a constant $\delta > 0$ such that every quadruple $x, y, z, w \in X$ satisfies the following inequality, called the *Gromov inequality*:

$$(x, y)_w \geq \min\{(x, z)_w, (y, z)_w\} - \delta. \tag{2-1}$$

Here, X need not be geodesic nor intrinsic; all arguments regarding Gromov hyperbolic spaces rely solely on the Gromov inequality.

From now on, we permanently fix $\delta > 0$.

In $\mathcal{T}(\Sigma)$, we say that a surface $x \in \mathcal{T}(\Sigma)$ is ϵ–thin if there exists a simple closed curve on x whose extremal length is less than ϵ; if not, we say that it is ϵ–thick. For $x, y \in \mathcal{T}(\Sigma)$, $[x, y]$ denotes the Teichmüller geodesic from x to y; it is said to be ϵ–thick if it is composed of ϵ–thick points.

In δ–hyperbolic spaces, we regard every point as ϵ–thick for any $\epsilon > 0$. Here $[x, y]$ denotes the pair of points (x, y), which is considered ϵ–thick for any $\epsilon > 0$. In either space, $[x, y]$ are called *segments* and their lengths are defined by $d(x, y)$.

Definition 2.1 (witnessing in δ–hyperbolic spaces) Let x, y, $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ be points in a δ–hyperbolic space X, and $D > 0$. We say that $[x, y]$ is *D–witnessed by* $([x_1, y_1], \dots, [x_n, y_n])$ if

(1) $(x_{i-1}, x_{i+1})_{x_i} < D$ for $i = 1, \dots, n$, where $x = x_0$ and $y = x_{n+1}$,
(2) $(y_{i-1}, y_{i+1})_{y_i} < D$ for $i = 1, \dots, n$, where $x = y_0$ and $y = y_{n+1}$, and
(3) $(y_{i-1}, y_i)_{x_i}, (x_i, x_{i+1})_{y_i} < D$ for $i = 1, \dots, n$.

Definition 2.1 seems complicated, but it is a version of Definition 2.2 in the absence of the geodesicity of the ambient space. Indeed, in a geodesic Gromov hyperbolic space, these two notions of witnessing are equivalent up to the modification of the parameter D.

Definition 2.2 (witnessing in $\mathcal{T}(\Sigma)$) Let x, y, $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ be points in $X = \mathcal{T}(\Sigma)$, and $D > 0$. We say that $[x, y]$ is *D–witnessed by* $([x_1, y_1], \dots, [x_n, y_n])$ if the geodesic $[x, y]$ contains subsegments $[x_i', y_i']$ such that

(1) x_{i+1}' is not closer to x than y_i' for $i = 1, \dots, n-1$, and
(2) $[x_i, y_i]$ and $[x_i', y_i']$ D–fellow travel.

From now on, we permanently fix X, *a* δ*–hyperbolic space or* $\mathcal{T}(\Sigma)$.

Definition 2.3 Let x, y and $\{x_i, y_i, z_i\}_{i=1}^N$ be points in X. We say that $[x, y]$ is *D–marked with* $([x_i, y_i])_{i=1}^N$, $([z_i, x_i])_{i=1}^N$ if

(1) $(y_i, z_i)_{x_i} < D$ for each $i = 1, \dots, N$, and
(2) $[x_{i-1}, x_i]$ is D–witnessed by $([x_{i-1}, y_{i-1}], [z_i, x_i])$ for each $i = 1, \dots, N+1$, where we set $x_0 = y_0 = x$ and $x_{N+1} = z_{N+1} = y$.

In this case, we also say that $[x_1, x_n]$ is fully D–marked with $([x_i, y_i])_{i=1}^{N-1}$, $([z_i, x_i])_{i=2}^N$.

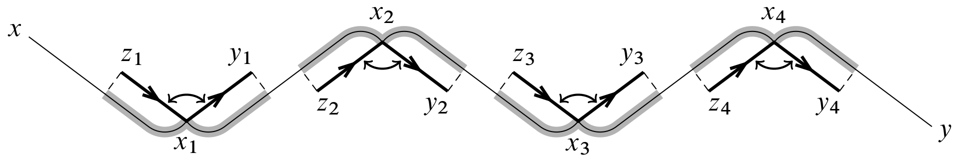

Figure 1: Schematics for $[x, y]$ being D–marked with $([x_i, y_i])_{i=1}^4$, $([z_i, x_i])_{i=1}^4$.

The following observation is immediate.

Lemma 2.4 *Let $\{x_i\}_{i=0}^N$ and $\{y_{i-1}, z_i\}_{i=1}^N$ be points in X. Suppose that for each $i = 1, \dots, N$, $[x_{i-1}, x_i]$ is fully D–marked with sequences of segments $(\gamma_{i,j})_{j=1}^{n_i-1}$, $(\eta_{i,j})_{j=2}^{n_i}$, where $\gamma_{i,1} = [x_{i-1}, y_{i-1}]$ and $\eta_{i,n_i} = [z_i, x_i]$. Suppose also that $(y_i, z_i)_{x_i} \le D$ for $i = 1, \dots, N-1$. Then $[x_0, x_N]$ is fully D–marked with*

$$(\gamma_{1,1}, \dots, \gamma_{1,n_1-1}, \gamma_{2,1}, \dots, \gamma_{2,n_2-1}, \dots, \gamma_{N,n_N-1}),$$
$$(\eta_{1,2}, \dots, \eta_{1,n_1}, \eta_{2,2}, \dots, \eta_{2,n_2}, \dots, \eta_{N,n_N}).$$

Our aim is to prove that if $[x, y]$ is D–marked with sufficiently thick and long segments, then $[x, y]$ is witnessed by those segments. In order to prove this, we need the following consequences of Rafi's fellow-traveling and thin triangle theorems [2014].

Fact 2.5 [Choi 2023, Lemma 3.9] *For each $D, \epsilon > 0$, there exist $E, L > D$ that satisfy the following. Let x_1 be an ϵ–thick point and $\{x_i\}_{i=2}^3$ and $\{y_i\}_{i=1}^5$ be points in X such that*

(1) *$[y_1, y_2]$, $[y_3, y_4]$ and $[y_4, y_5]$ are ϵ–thick and longer than L,*

(2) *$(y_3, y_5)_{y_4} \le D$,*

(3) *$[x_1, x_2]$ is D–witnessed by $([y_1, y_2], [y_3, y_4])$, and*

(4) *$[x_2, x_3]$ is E–witnessed by $[y_4, y_5]$.*

Then $[x_1, x_3]$ is E–witnessed by $[y_1, y_2]$.

Fact 2.6 [Choi 2023, Lemma 3.10] *For each $E, \epsilon > 0$, there exists $F, L > E$ that satisfies the following. Let $\{x_i\}_{i=1}^3$ and $\{y_i\}_{i=1}^3$ be points in X such that*

(1) *$[y_1, y_2]$ and $[y_2, y_3]$ are ϵ–thick and longer than L,*

(2) *$(y_1, y_3)_{y_2} \le E$,*

(3) *$[x_1, x_2]$ is E–witnessed by $[y_1, y_2]$, and*

(4) *$[x_2, x_3]$ is E–witnessed by $[y_2, y_3]$.*

Then $[x_1, x_3]$ is F–witnessed by $[y_1, y_2]$, and also by $[y_2, y_3]$. In particular, $|(x_1, x_3)_{x_2} - d(x_2, y_2)| < F$.

Combining Facts 2.5 and 2.6 yields the following.

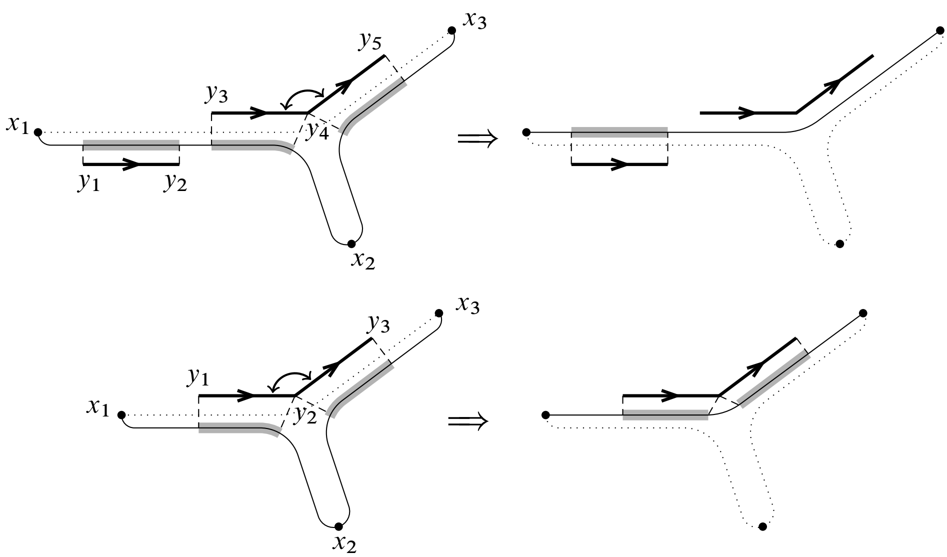

Figure 2: Schematics for Facts 2.5 and 2.6.

Corollary 2.7 *Let $D, \epsilon > 0$ and*

- $E = E(\epsilon, D)$ *and* $L_1 = L(\epsilon, D)$ *as in Fact 2.5, and*
- $F = F(\epsilon, E)$ *and* $L_2 = L(\epsilon, E)$ *as in Fact 2.6.*

Suppose that x, y and $\{x_i, y_i, z_i\}_{i=1}^N$ are points in X such that

(1) $[x, y]$ *is D–marked with* $([x_i, y_i])_{i=1}^N$, $([z_i, x_i])_{i=1}^N$, *and*

(2) $[x_i, y_i]$ *and* $[z_i, x_i]$ *are ϵ–thick and longer than* $\max(L_1, L_2)$ *for* $i = 1, \dots, N$.

Then

(1) $[x, y]$ *is F–witnessed by* $[x_i, y_i]$ *for each* $i = 1, \dots, N$,

(2) $[x, y]$ *is F–witnessed by* $[z_i, y_i]$ *for each* $i = 1, \dots, N$, *and*

(3) $(x, y)_{x_i}, (x, y)_{y_i}, (x, y)_{z_i}$ *are smaller than F for each* $i = 1, \dots, N$.

Proof It is assumed that $[x_N, y]$ is E–witnessed by $[x_N, y_N]$. Moreover, by assumption, $[x_{k-1}, x_k]$ is D–witnessed by $([x_{k-1}, y_{k-1}], [z_k, x_k])$ where $[x_{k-1}, y_{k-1}]$ and $[z_k, x_k]$ are ϵ–thick and longer than L_1. Note also that $(y_k, z_k)_{x_k} \le D$. Hence, if $[x_k, y]$ is E–witnessed by $[x_k, y_k]$, then $[x_{k-1}, y]$ is E–witnessed by $[x_{k-1}, y_{k-1}]$ by Fact 2.5. Thus, inductively, we deduce that $[x_i, y]$ is E–witnessed by $[x_i, y_i]$ for each i. Similarly, $[x, x_i]$ is E–witnessed by $[z_i, x_i]$. Now Fact 2.6 asserts that $[x, y]$ is F–witnessed by $[x_i, y_i]$ and $[z_i, y_i]$, which also implies the second item. □

We finally need two facts that guarantee witnessing by a pair of segments.

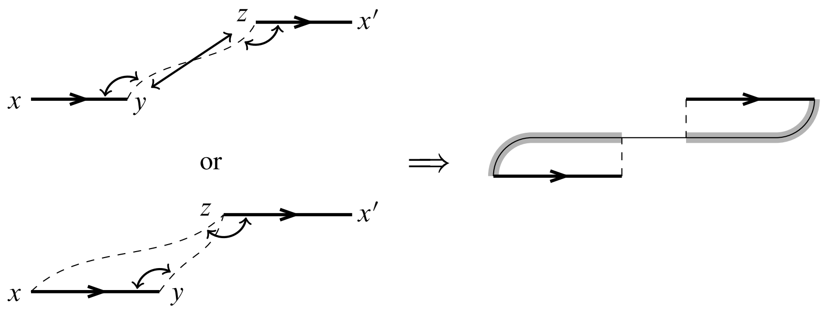

Figure 3: Schematics for Facts 2.8 and 2.9.

Fact 2.8 [Choi 2023, Lemma 3.6] *For each $C, \epsilon > 0$, there exists $D > C$ such that if $x, y, z, x' \in X$ satisfy that*

(1) $[x, y]$ *and* $[z, x']$ *are* ϵ*–thick*,

(2) $(x, z)_y, (y, x')_z < C$, *and*

(3) $d(y, z) \geq \max\{d(x, y), d(z, x'), 3D\}$,

then $[x, x']$ *is* D*–witnessed by* $([x, y], [z, x'])$.

Fact 2.9 [Choi 2023, Lemma 3.7] *For each $C, \epsilon > 0$, there exists $D > C$ such that if $x, y, x', z \in X$ satisfy that*

(1) $[x, y]$ *and* $[x', z]$ *are* ϵ*–thick*, *and*

(2) $(x, z)_y, (x, x')_z < C$,

then $[x, x']$ *is* D*–witnessed by* $([x, y], [z, x'])$.

From now on, we permanently fix a base point $o \in X$ and a finitely generated subgroup G of $\mathrm{Isom}(X)$ *that contains independent loxodromics a and b.* For $S \subset G$ and $i \in \mathbb{Z}$, we employ the notation $S^{(i)} := \{s^i : s \in S\}$.

Let us introduce the notion of Schottky sets that originated from [Gouëzel 2022].

Definition 2.10 (Schottky set) Let $K, K', \epsilon > 0$. A finite set S of isometries of X is said to be (K, K', ϵ)*–Schottky* if

(1) for all $x, y \in X$, $|\{s \in S : (x, s^i y)_o \geq K \text{ for some } i > 0\}| \leq 2$,

(2) for all $x, y \in X$, $|\{s \in S : (x, s^i y)_o \geq K \text{ for some } i < 0\}| \leq 2$,

(3) for all $s \in S$ and $i \neq 0$, $0.9995 i K' < d(o, s^i o) \leq i K'$,

(4) for all $s \in S$ and $i \in \mathbb{Z}$, the geodesic $[o, s^i o]$ is ϵ–thick,

(5) for all $x \in X$, $|\{s \in S : (x, s^i o)_o \geq K \text{ for some } i > 0\}| \leq 1$,

(6) for all $x \in X$, $|\{s \in S : (x, s^i o)_o \geq K \text{ for some } i < 0\}| \leq 1$, and

(7) for all $s_1, s_2 \in S$ and $i, j > 0$, $(s_1^i o, s_2^{-j} o)_o < K$.

Remark 2.11 Gouëzel's original definition [2022] of Schottky set and Choi's definition [2023] do not require (5), (6) and (7); however, as remarked there, Schottky sets constructed in [Choi 2023] automatically satisfy (5), (6) and (7).

We now claim the existence of certain Schottky sets.

Proposition 2.12 (cf [Choi 2023, Proposition 4.2]) *There exists $\epsilon > 0$ such that the following holds. For any $n, n' > 0$, there exist $K(n) > 0$ and $K'(n, n') > n'$ such that G has a (K, K', ϵ)–Schottky subset S with cardinality at least n.*

Proof The proof of [Choi 2023, Proposition 4.2] implies the following:

Claim 2.13 *There exists $\epsilon, F, N_0 > 0$ such that, for all $N > N_0$,*

(a) *for any $0 \le m \le n$ and $\phi_i \in \{a, b, a^{-1}, b^{-1}\}$ such that $\phi_i \notin \phi_{i+1}^{-1}$, $[o, \phi_1^{2N} \cdots \phi_n^{2N} o]$ is ϵ–thick and $(o, \phi_1^{2N} \cdots \phi_n^{2N} o)_{\phi_1^{2N} \cdots \phi_m^{2N} o} \le F$, and*

(b) *for all n and k, the set*
$$S_{n,k,N} := \{(\phi_1^{2N} \cdots \phi_n^{2N})^{2k} : \phi_i \in \{a, b\}\}$$
satisfies properties (1), (2), (4), (5) *and* (6) *of Schottky sets for*
$$K(n, N) := \max\{d(o, \phi_1^{2N} \cdots \phi_n^{2N} o) : \phi_i \in \{a, b\}\}.$$

The proof of this claim is given in Appendix A. Assuming this, we now take large enough N such that $d(o, a^{2N} o), d(o, b^{2N} o) > 10000F$ and fix $K(n) := K(n, N)$. Let $\mathscr{S}$ be the collection of concatenations of n copies of a^{2N} and n copies of b^{2N}. For $s \in \mathscr{S}$ and k, property (a) above implies
$$0 \le [2nkd(o, a^{2N} o) + 2nkd(o, b^{2N} o)] - d(o, s^k o) \le 8nkF \le \tfrac{1}{2500} \cdot 2nk[d(o, a^{2N} o), d(o, b^{2N} o)].$$
We finally fix k such that $K'(n, n') := 2nk[d(o, a^{2N} o) + d(o, b^{2N} o)]$ is larger than n'. Then $\mathscr{S}^{(k)}$ is (K, K', ϵ)–Schottky and has cardinality $\binom{2n}{n} \ge n$. □

From now on, we permanently fix the choice $\epsilon > 0$ from Proposition 2.12. Now for each $C > 0$, we fix

- $D = D(C, \epsilon)$ that works in Facts 2.8 and 2.9,
- $E = E(D, \epsilon)$ and $L_1 = L(D, \epsilon)$ as in Fact 2.5,
- $F = F(E, \epsilon)$ and $L_2 = L(D, \epsilon)$ as in Fact 2.6,

and $L = \max(L_1, L_2)$. Note that D, F and L ultimately depend on the values of C and ϵ; we will write them as $D(C, \epsilon)$, $F(C, \epsilon)$ and $L(C, \epsilon)$.

Lemma 2.14 *Let $K > 0$, $K' > 2L(K, \epsilon) + 5000F(K, \epsilon)$ and S_1 be a (K, K', ϵ)–Schottky set. Then S_1 and $S_1^{(-1)}$ are disjoint.*

Moreover, if a nonempty sequence $(s_i)_{i=1}^N$ of elements of $S_1 \cup S_1^{(-1)}$ satisfy $s_i \neq s_{i+1}^{-1}$ for $i = 1, \dots, N-1$, then $s_1 \cdots s_N \neq \mathrm{id}$.

Proof For each $s \in S_1^{(\pm 1)}$, we claim that

$$\{s' \in S_1 \cup S_1^{(-1)} : (so, s'o)_o > K\} = \{s\}.$$

Indeed, $(so, so)_o = d(o, so) > K$, and properties (5), (6) and (7) of Schottky sets imply

$$\{s' \in S_1 \cup S_1^{(-1)} : (so, s'o)_o > K\} = \{s' \in S_1^{(\pm 1)} : (so, s'o)_o > K\} = \{s\}.$$

In particular, $s' \in S_1^{(\mp 1)}$ cannot belong to this set; this settles the first claim.

We now let $x_i = s_1 \cdots s_i o$ for $i = 0, \dots, N$. By the above claim, we realize that

$$(x_i, x_{i+2})_{x_i} = (s_{i+1}^{-1} o, s_{i+2} o)_o < K < D(K, \epsilon)$$

for $i = 0, \dots, N-2$. Then Fact 2.9 implies that $[x_i, x_{i+2}]$ is $D(K,\epsilon)$–witnessed by $([x_i, x_{i+1}], [x_{i+1}, x_2])$. Note that $[x_i, x_{i+1}]$ is trivially $D(K,\epsilon)$–witnessed by $([x_i, x_{i+1}], [x_{i+1}, x_{i+1}])$ and by $([x_i, x_i], [x_i, x_{i+1}])$. Combining these observations, we deduce that $[x_0, x_N]$ is $D(K,\epsilon)$–marked with $([x_{2i-1}, x_{2i}])_{i=1}^{\lfloor N/2 \rfloor}$, $([x_{2i-2}, x_{2i-1}])_{i=1}^{\lfloor N/2 \rfloor}$. Since $[x_i, x_{i+1}]$ are ϵ–thick and longer than $0.999K' > L(K,\epsilon)$, Corollary 2.7 implies that $(x_0, x_{i+1})_{x_i} < F(K,\epsilon)$ for each i and

$$d(x_0, x_N) \geq \sum_{i=1}^{N} d(x_{i-1}, x_i) - 2(N-1)F(K,\epsilon) \geq (0.999K' - F(K,\epsilon))N \geq 0.9K'N.$$

Hence, $s_1 \cdots s_N \neq \mathrm{id}$. □

Corollary 2.15 *Let S_1 be as in Lemma 2.14. Then the correspondence $a \mapsto a^2$ from $S_1 \cup S_1^{(-1)}$ to $S_1^{(2)} \cup S_1^{(-2)}$ is one-to-one.*

Definition 2.16 We say that a finite set S is *nicely populated* by S_0 if $S_0 \subseteq S$ and $\#S_0 \geq 0.99 \cdot \#S + 400$.

Lemma 2.17 *Given a finite set $S' \subseteq G$, there exist a (K, K', ϵ)–Schottky subset S_1 of G such that $K' > 2L(K,\epsilon) + 5000F(K,\epsilon)$, and a finite symmetric generating set S of G such that $S' \cup \{e\} \subseteq S$ and S is nicely populated by $S_1^{(2)} \cup S_1^{(-2)}$.*

Proof We first enlarge S' into a finite symmetric generating set S'' containing e. Let $n = 100\,\#S'' + 40000$ and take $K = K(n) > 0$ from Proposition 2.12. We then take $F = F(K,\epsilon)$ and $L = L(K,\epsilon)$ as described before. Using Proposition 2.12, we take $K' > n' = 2L + 5000F$ and a (K, K', ϵ)–Schottky subset S_1 of G with cardinality at least n. Thanks to Corollary 2.15, we also have $\#(S_1^{(2)} \cup S_1^{(-2)}) = 2 \cdot \#S_1 \geq n$. Hence, the union of S'', $S_1^{(2)}$ and $S_1^{(-2)}$ satisfies the desired property. □

From now on, we fix constants $K > 0$ and $K' > 2L(K,\epsilon) + 5000F(K,\epsilon)$, a (K, K', ϵ)–Schottky set S_1 with $K' > 2L(K,\epsilon) + 5000F(K,\epsilon)$, and a finite symmetric generating set $S \ni e$ of G that is nicely populated by $S_1^{(2)} \cup S_1^{(-2)}$. For $g \in G$ and $s \in S_1 \cup S_1^{(2)}$, we call $[go, gso]$ a *Schottky segment*. One should keep in mind that Schottky segments are ϵ–thick and longer than $0.999K'$. Finally, we will denote $S_1 \cup S_1^{(-1)}$ by S_0.

3 Pivoting in a random walk

We will make use of the random walk on G generated by the uniform measure μ_S on S that is constructed as follows. We consider the *step space* $(G^{\mathbb{Z}}, \mu_S^{\mathbb{Z}})$, the product space of G equipped with the product measure of μ_S. Each step path (g_n) induces a sample path (ω_n) by

$$\omega_n = \begin{cases} g_1 \cdots g_n & \text{if } n > 0, \\ e & \text{if } n = 0, \\ g_0^{-1} \cdots g_{n+1}^{-1} & \text{if } n < 0, \end{cases}$$

which constitutes a random walk with transition probability μ_S.

Given $\boldsymbol{g} = (g_1, \dots, g_n) \in G^n$, we define

$$\Theta(\boldsymbol{g}) = \{\vartheta(1) < \cdots < \vartheta(N)\} := \left\{1 \le i \le \tfrac{1}{2}n : g_{2i-1}, g_{2i} \in S_1^{(2)} \cup S_1^{(-2)}\right\}.$$

In other words, $\Theta(\boldsymbol{g})$ is the collection of steps that are chosen from $S_1^{(2)} \cup S_1^{(-2)}$. This set can well be empty, although such a situation happens with small probability. We now pick pivotal times from $\Theta(\boldsymbol{g})$.

For each $1 \le i \le N$, let a_i and b_i be the elements of $S_0 = S_1 \cup S_1^{-1}$ such that

$$a_i^2 = g_{2\vartheta(i)-1}, \quad b_i^2 = g_{2\vartheta(i)}.$$

Such a_i and b_i are uniquely determined thanks to Corollary 2.15.

We also define $w_i := g_{2\vartheta(i)+1} \cdots g_{2\vartheta(i+1)-2}$ for $1 \le i \le N-1$, with $w_0 := g_1 \cdots g_{2\vartheta(1)-2}$ and $w_N := g_{2\vartheta(N)+1} \cdots g_n$. It is clear that

$$\omega_n = g_1 g_2 \cdots g_n = w_0 a_1^2 b_1^2 w_1 \cdots a_N^2 b_N^2 w_N.$$

Remark 3.1 It will be convenient to allow the expression $\omega_{2\vartheta(N+1)-2}$ and interpret it as ω_n, even though $\vartheta(N+1)$ does not exist (there is no reason to not define $\vartheta(N+1) := (n+2)/2$ if one hopes). This way, we can say

$$\begin{aligned} \omega_{2\vartheta(k)-2} &:= w_0 a_1^2 b_1^2 \cdots w_{k-1} && (k = 1, \dots, N+1), \\ \omega_{2\vartheta(k)-1} &:= w_0 a_1^2 b_1^2 \cdots w_{k-1} a_k^2 && (k = 1, \dots, N), \\ \omega_{2\vartheta(k)} &:= w_0 a_1^2 b_1^2 \cdots w_{k-1} a_k^2 b_k^2 && (k = 1, \dots, N). \end{aligned}$$

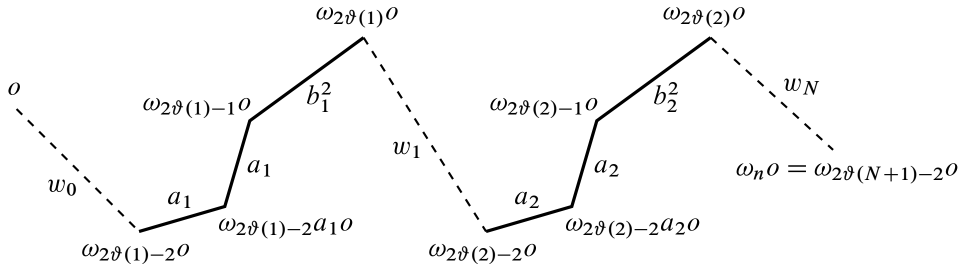

Figure 4: Words w_j, a_j and b_j that arise from a trajectory.

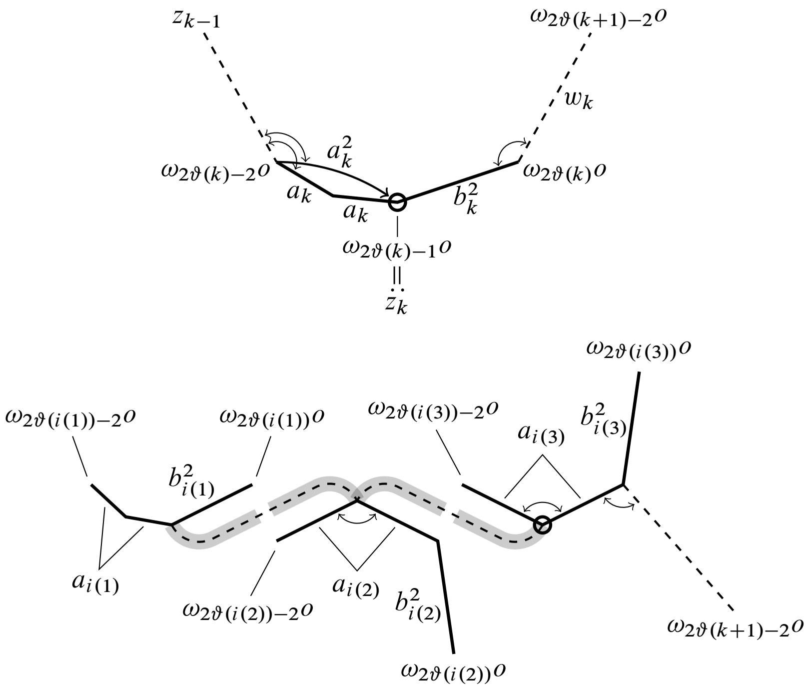

Figure 5: Schematics for criteria (A) and (B) for the construction of P_k. The upper configuration describes the situation when k is added in P_k. In the lower configuration, $\{i(1) < i(2) < i(3)\}$ satisfies items (i) and (ii) of criterion (B). Here, the shaded subsegments of the dashed lines fellow travel γ_1, η_2, γ_2 and η_3, from left to right, respectively. The newly chosen z_k is highlighted by a circle.

We now inductively define sets $P_k(\boldsymbol{g}) \subseteq \{1, \dots, k\}$ and a moving point z_k for $k = 0, \dots, N$. First take $P_0 := \varnothing$ and $z_0 := o$. Now given (P_{k-1}, z_{k-1}), (P_k, z_k) is determined as follows (see Figure 5):

(A) If $a_k \neq b_k^{-1}$, and

$$(z_{k-1}, \omega_{2\vartheta(k)-2} a_k^t o)_{\omega_{2\vartheta(k)-2}o} < K \quad \text{for } t \in \{1, 2\}, \qquad \text{(3-1)}$$
$$(\omega_{2\vartheta(k)-1} o, \omega_{2\vartheta(k+1)-2} o)_{\omega_{2\vartheta(k)}o} < K$$

hold, then we set $P_k := P_{k-1} \cup \{k\}$ and $z_k := \omega_{2\vartheta(k)-1} o$. Note that (3-1) is equivalent to

$$(\omega_{2\vartheta(k)-2}^{-1} z_{k-1}, a_k^t o)_o < K \quad \text{for } t \in \{1, 2\}, \qquad \text{(3-2)}$$
$$(b_k^{-2} o, w_k o)_o < K.$$

(B) If not, we seek sequences $\{i(1) < \cdots < i(M)\} \subseteq P_{k-1}$ with cardinality $M \geq 2$ such that

(i) $[\omega_{2\vartheta(i(1))-1} o, \omega_{2\vartheta(i(M))-2} a_{i(M)} o]$ is fully $D(K, \epsilon)$–marked with $(\gamma_j)_{j=1}^{M-1}$, $(\eta_j)_{j=2}^{M}$, where

$$\gamma_1 = [\omega_{2\vartheta(i(1))-1} o, \omega_{2\vartheta(i(1))} o],$$
$$\gamma_j = [\omega_{2\vartheta(i(j))-2} a_{i(j)} o, \omega_{2\vartheta(i(j))-1} o] \quad (2 \leq j \leq M-1),$$
$$\eta_j = [\omega_{2\vartheta(i(j))-2} o, \omega_{2\vartheta(i(j))-2} a_{i(j)} o] \quad (2 \leq j \leq M);$$

(ii) $(\omega_{2\vartheta(i(M))-2} a_{i(M)} o, \omega_{2\vartheta(k+1)-2} o)_{\omega_{2\vartheta(i(M))-1}o} < K$.

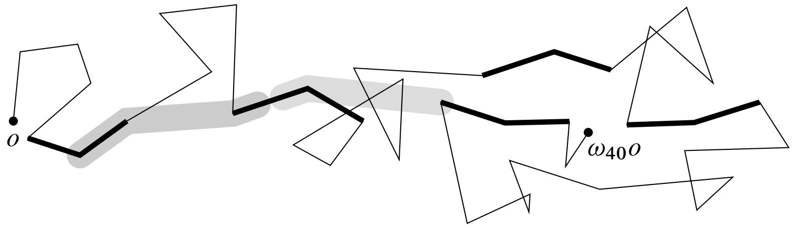

Figure 6: An example of a sample path $\boldsymbol{g}$ with length 40. The vertices represent $\omega_i o$ for $i = 0, \dots, 40$; the thick segments represent Schottky progresses. The shaded region highlights the required witnessing by Schottky segments in criterion (B) for $P_4(\boldsymbol{g})$.

If such a sequence exists, let $\{i(1) < \cdots < i(M)\}$ be such a sequence with maximal $i(1)$; we set $P_k := P_{k-1} \cap \{1, \dots, i(1)\}$ and $z_k := \omega_{2\vartheta(i(M))-2} a_{i(M)} o$. If such a sequence does not exist, then we set $P_k := \varnothing$ and $z_k := o$.[1]

Figure 6 illustrates how P_k evolves as k increases. The path $\boldsymbol{g}$ under consideration has

$$\Theta(\boldsymbol{g}) = \{3, 6, 10, 13, 19\}.$$

Note that

$$\omega_0 o = o, \quad \omega_{2\vartheta(1)-2} o = \omega_4 o, \quad \omega_{2\vartheta(1)-1} o = \omega_5 o, \quad \omega_{2\vartheta(1)} = \omega_6 o, \quad \omega_{2\vartheta(2)-2} o = \omega_{10} o$$

are arranged as required in criterion (A), which implies $P_1(\boldsymbol{g}) = \{1\}$. Since $\omega_5 o$, $[\omega_{10} o, \omega_{11} o]$, $[\omega_{11} o, \omega_{12} o]$ and $\omega_{18} o$ are arranged as desired, $P_2(\boldsymbol{g}) = \{1, 2\}$ (even though $(\omega_{11} o, \omega_i o)_{\omega_{12} o}$ is not always small for all $i > 12$). By similar reasoning, $P_3(\boldsymbol{g}) = \{1, 2, 3\}$.

Since $\omega_{36} o$ is on the left of $[\omega_{25} o, \omega_{26} o]$, $P_4(\boldsymbol{g})$ is not $\{1, 2, 3, 4\}$. If $\omega_{36} o$ were on the right of $[\omega_{24} o, \omega_{25} o]$, then $P_4(\boldsymbol{g}) = \{1, 2, 3\}$ might have held; but it is not the case. Since $\omega_{36} o$ is not on the right of $[\omega_{18} o, \omega_{19} o]$, $P_4(\boldsymbol{g}) = \{1, 2\}$ cannot hold either; we only have $P_4(\boldsymbol{g}) = \{1\}$. $P_5(\boldsymbol{g})$ then becomes $\{1, 5\}$.

Note the following facts:

(1) $P_k(\boldsymbol{g})$ is measurable with respect to the choice of g_i.

(2) $i \in P_m$ only if $P_i = P_{i-1} \cup \{i\}$ and i survives during stage $i+1, \dots, m$.

(3) If $i \in P_m$ and $i \in P_n$, then $\{1, \dots, i\} \cap P_m = \{1, \dots, i\} \cap P_n$.

Lemma 3.2 *The following holds for any $0 \le k \le n$ and $\boldsymbol{g} \in G^n$. Let $l < m$ be consecutive elements in $P_k = P_k(\boldsymbol{g})$, i.e. $l, m \in P_k$ and $l = \max(P_k \cap \{1, \dots, m-1\})$. Let also $t \in \{1, 2\}$. Then there exists a sequence*

$$\{l = i(1) < \cdots < i(M') = m\} \subseteq P_k$$

[1] When there are several sequences that realize maximal $i(1)$, we choose the maximum in the lexicographic order on the length of sequences and $i(2), i(3), \dots$.

with cardinality $M' \ge 2$ *such that* $[\omega_{2\vartheta(l)-1}o, \omega_{2\vartheta(m)-2}a_m^t o]$ *is fully* $D(K,\epsilon)$*–marked with* $(\gamma_j)_{j=1}^{M'-1}$, $(\eta_j)_{j=2}^{M'}$, *where*

$$\begin{aligned} \gamma_1 &= [\omega_{2\vartheta(i(1))-1}o, \omega_{2\vartheta(i(1))}o], \\ \gamma_j &= [\omega_{2\vartheta(i(j))-2}a_{i(j)}o, \omega_{2\vartheta(i(j))-1}o] \qquad (2 \le j \le M'-1), \\ \eta_j &= [\omega_{2\vartheta(i(j))-2}o, \omega_{2\vartheta(i(j))-2}a_{i(j)}o] \qquad (2 \le j \le M'-1), \\ \eta_{M'} &= [\omega_{2\vartheta(i(M))-2}o, \omega_{2\vartheta(i(M'))-2}a_m^t o]. \end{aligned} \tag{3-3}$$

Proof If $l, m \in P_k$ then $l \in P_l$ and $l, m \in P_m$. In particular, l (resp. m) is newly chosen at stage l (resp. m) by fulfilling criterion (A). Hence, $(\omega_{2\vartheta(l)-1}o, \omega_{2\vartheta(l+1)-2}o)_{\omega_{2\vartheta(l)}o} < K$ and $z_l = \omega_{2\vartheta(l)-1}o$. Moreover, $P_m = P_{m-1} \cup \{m\}$ and $l = \max P_{m-1}$.

If $l = m-1$, then m is newly chosen at stage $m = l+1$. In this case we have

$$(\omega_{2\vartheta(l)-1}o, \omega_{2\vartheta(m)-2}a_m^t o)_{\omega_{2\vartheta(m)-2}o} = (z_l, \omega_{2\vartheta(m)-2}a_m^t o)_{\omega_{2\vartheta(m)-2}o} < K$$

from criterion (A) for m. Then Fact 2.9 implies that $[\omega_{2\vartheta(l)-1}o, \omega_{2\vartheta(m)-2}a_m^t o]$ is fully $D(K,\epsilon)$–marked with

$$[\omega_{2\vartheta(l)-1}o, \omega_{2\vartheta(l)}o], \quad [\omega_{2\vartheta(m)-2}o, \omega_{2\vartheta(m)-2}a_m^t o].$$

Hence, $\{l = i(1) < i(2) = m\}$ works.

If $l < m-1$, then $l = \max P_{m-1}$ has survived at stage $m-1$ by fulfilling criterion (B). This means that there exist $l = i(1) < \cdots < i(M)$ in P_{m-2} with $M \ge 2$ such that $[\omega_{2\vartheta(l)-1}o, \omega_{2\vartheta(i(M))-2}a_{i(M)}o]$ is fully $D(K,\epsilon)$–marked with $(\gamma_j)_{j=1}^{M-1}$, $(\eta_j)_{j=2}^{M}$. Here, the γ_j and η_j are as in (3-3). Moreover,

$$(\omega_{2\vartheta(i(M))-2}a_{i(M)}o, \omega_{2\vartheta(m)-2}o)_{\omega_{2\vartheta(i(M))-1}o} < K \tag{3-4}$$

and $z_{m-1} = \omega_{2\vartheta(i(M))-2}a_{i(M)}o$.

We now claim that $i(1) < \cdots < i(M)$ and $i(M+1) := m$ together serve as the desired sequence (hence $M' = M+1 \ge 3$). First, since m is newly chosen based on criterion (A),

$$(\omega_{2\vartheta(i(M))-2}a_{i(M)}o, \omega_{2\vartheta(m)-2}a_m^t)_{\omega_{2\vartheta(m)-2}o} = (z_{m-1}, \omega_{2\vartheta(m)-2}a_m^t)_{\omega_{2\vartheta(m)-2}o} < K. \tag{3-5}$$

Now combining inequalities (3-4) and (3-5), Fact 2.9 implies that $[\omega_{2\vartheta(i(M))-2}a_{i(M)}o, \omega_{2\vartheta(m)-2}a_m^t o]$ is $D(K,\epsilon)$–witnessed by

$$([\omega_{2\vartheta(i(M))-2}a_{i(M)}o, \omega_{2\vartheta(i(M))-1}o], [\omega_{2\vartheta(m)-2}o, \omega_{2\vartheta(m)-2}a_m^t o]) = (\gamma_M, \eta_M).$$

Finally,

$$(\omega_{2\vartheta(i(M))-1}o, \omega_{2\vartheta(i(M))-2}o)_{\omega_{2\vartheta(i(M))-2}a_{i(M)}o} = (a_{i(M)}o, a_{i(M)}^{-1}o)_o < K < D(K,\epsilon)$$

thanks to property (7) of the (K, K', ϵ)–Schottky set S_1. This implies that $[\omega_{2\vartheta(l)-1}o, \omega_{2\vartheta(m)-2}a_m^t o]$ is fully $D(K,\epsilon)$–marked with $(\gamma_j)_{j=1}^{M}$, $(\eta_j)_{j=2}^{M+1}$ that are as in (3-3). □

Lemma 3.3 *The following holds for any $n > 0$ and $\boldsymbol{g} \in G^n$. Let*

$$\Theta(\boldsymbol{g}) = \{\vartheta(1) < \cdots < \vartheta(N)\} \quad (N = \#\Theta(\boldsymbol{g})),$$
$$P_N(\boldsymbol{g}) = \{\iota(1) < \cdots < \iota(m)\} \qquad (m = \#P_N(\boldsymbol{g})).$$

Then there exist $M \geq m$, Schottky segments $(\gamma_l)_{l=1}^M$ and $(\eta_l)_{l=1}^M$, and $1 \leq l(1) < \cdots < l(m) \leq M$ such that

1. *$[o, \omega_n o]$ is $D(K,\epsilon)$–marked with $(\gamma_l)_l$, $(\eta_l)_l$, and*
2. *$\gamma_{l(t)} = [\omega_{2\vartheta(\iota(t))-1} o, \omega_{2\vartheta(\iota(t))} o]$ and $\eta_{l(t)} = [\omega_{2\vartheta(\iota(t))-2} o, \omega_{2\vartheta(\iota(t))-1} o]$.*

Proof We will apply Lemma 2.4. First recall Lemma 3.2: for each $t = 2, \ldots, m$,

$$[\omega_{2\vartheta(\iota(t-1))-1} o, \omega_{2\vartheta(\iota(t))-1} o]$$

is fully $D(K,\epsilon)$–marked with some Schottky sequences $(\gamma_{l;t})_l$, $(\eta_{l;t})_l$, whose forms are given by (3-3). Here, note that the length of these sequences need not be 1; this leads to the possibility that $l(t)-l(t-1) > 1$.

Given the above result, it suffices to prove that

1. $[o, \omega_{2\vartheta(\iota(1))-1} o]$ is fully $D(K,\epsilon)$–marked with $[o,o]$, $[\omega_{2\vartheta(\iota(1))-2} o, \omega_{2\vartheta(\iota(1))-1} o]$,
2. $[\omega_{2\vartheta(\iota(m))-1} o, \omega_n o]$ is fully $D(K,\epsilon)$–marked with some sequences $(\gamma'_j)_{j=1}^{M'}$, $(\eta'_j)_{j=2}^{M'+1}$ of Schottky segments, where $\gamma'_1 = [\omega_{2\vartheta(\iota(m))-1} o, \omega_{2\vartheta(\iota(m))} o]$, and
3. $(\omega_{2\vartheta(\iota(t))-2} o, \omega_{2\vartheta(\iota(t))} o)_{\omega_{2\vartheta(\iota(t))-1} o} < D(K,\epsilon)$ for each $t = 1, \ldots, m$.

First, $\iota(1) = \min P_N$ implies that $P_{\iota(1)-1} = \varnothing$, $z_{\iota(1)-1} = o$ and that $\iota(1)$ is newly chosen at stage $\iota(1)$. Hence,

$$(o, \omega_{2\vartheta(\iota(1))-1} o)_{\omega_{2\vartheta(\iota(1))-2} o} = (z_{\iota(1)-1}, \omega_{2\vartheta(\iota(1))-1} o)_{\omega_{2\vartheta(\iota(1))-2} o} < K$$

and Fact 2.9 implies the first item.

Next, we observe how $\iota(m)$ survived in P_N. If $\iota(m) = N$, then it was newly chosen at stage N; $(\omega_{2\vartheta(\iota(m))-1} o, \omega_n o)_{\omega_{2\vartheta(\iota(m))} o} < K$ holds and Fact 2.9 implies that $[\omega_{2\vartheta(\iota(m))-1} o, \omega_n o]$ is fully $D(K,\epsilon)$–marked with

$$[w_{2\vartheta(\iota(m))-1} o, w_{2\vartheta(\iota(m))} o], \quad [\omega_n o, \omega_n o].$$

If $\iota(m) \neq N$, then it has survived at stage N by fulfilling criterion (B). Thus, there exist $i \in P_{N-1}$ such that $(\omega_{2\vartheta(i)-2} a_i o, \omega_n o)_{\omega_{2\vartheta(i)-1} o} < K$ and Schottky segments $(\gamma'_j)_{j=1}^{M'-1}$, $(\eta'_j)_{j=2}^{M'}$ such that

$$[\omega_{2\vartheta(\iota(m))-1} o, \omega_{2\vartheta(i)-2} a_i o]$$

is fully $D(K,\epsilon)$–marked with (γ'_j), (η'_j), where

$$\gamma'_1 = [\omega_{2\vartheta(\iota(m))-1} o, \omega_{2\vartheta(\iota(m))} o], \quad \eta'_{M'} = [\omega_{2\vartheta(i)-2} o, \omega_{2\vartheta(i)-2} a_i o].$$

Furthermore, the second item of criterion (B) and Fact 2.9 imply that $[\omega_{2\vartheta(i)-2} a_i o, \omega_n o]$ is $D(K,\epsilon)$–witnessed by

$$([\omega_{2\vartheta(i)-2} a_i o, \omega_{2\vartheta(i)-1} o], [\omega_n o, \omega_n o]).$$

Finally, recall that

$$(\omega_{2\vartheta(i)-2}o, \omega_{2\vartheta(i)-1}o)_{\omega_{2\vartheta(i)-2}a_i o} = (a_i^{-1}o, a_i o)_o < K < D(K,\epsilon)$$

by property (7) of the (K, K', ϵ)–Schottky set S_1. Combining these, we conclude that $[\omega_{2\vartheta(\iota(m))-1}o, \omega_n o]$ is fully $D(K,\epsilon)$–marked with

$$(\gamma'_1, \dots, \gamma'_{M'-1}, [\omega_{2\vartheta(i)-2}a_i o, \omega_{2\vartheta(i)-1}o]), \quad (\eta'_2, \dots, \eta'_{M'}, [\omega_n o, \omega_n o]).$$

This settles the second item.

For the third item let $t \in \{1, \dots, m\}$. Since $\iota(t) \in P_N(\boldsymbol{g})$, $\iota(t)$ was newly chosen at stage $\iota(t)$ by fulfilling criterion (A); hence, $a_{\iota(t)}^{-1} \neq b_{\iota(t)}$ and we deduce that

$$(\omega_{2\vartheta(\iota(t))-2}o, \omega_{2\vartheta(\iota(t))}o)_{\omega_{2\vartheta(\iota(t))-1}o} = (a_{\iota(t)}^{-2}o, b_{\iota(t)}^2 o)_o < K$$

from properties (5), (6), (7) of Schottky sets. □

The same proof also yields the following lemma:

Lemma 3.4 *Let $k < k'$ be elements of $P_N(\boldsymbol{g})$ and $t \in \{1,2\}$. Then there exist some sequences $(\gamma_l)_{l=1}^{M-1}$, $(\eta_l)_{l=2}^{M}$ of Schottky segments such that*

1. *$[\omega_{2\vartheta(k)-1}o, \omega_{2\vartheta(k')-2}a_{k'}^t o]$ is fully $D(K,\epsilon)$–marked with (γ_l), (η_l), and*
2. *$\gamma_1 = [\omega_{2\vartheta(k)-1}o, \omega_{2\vartheta(k)}o]$ and $\eta_M = [\omega_{2\vartheta(k')-2}o, \omega_{2\vartheta(k')-2}a_{k'}^t o]$.*

Having established the properties of $P_N(\boldsymbol{g})$, our next goal is to estimate the size of $P_N(\boldsymbol{g})$. We first fix N, $\Theta = \{\vartheta(1) < \cdots < \vartheta(N)\}$ and the choices $(g_{2j-1}, g_{2j}) \notin (S_1^{(2)} \cup S_1^{(-2)})^2$ for $j \notin \Theta$. Conditioned on these choices, we draw $(g_{2\vartheta(1)-1}, g_{2\vartheta(1)}, \dots, g_{2\vartheta(N)-1}, g_{2\vartheta(N)})$ from $(S_1^{(2)} \cup S_1^{(-2)})^{2N}$ with the product measure of the uniform measure on $S_1 \cup S_1^{-1}$. In particular, we regard $\boldsymbol{g}$, the ω_j and P_k as RVs of $s = (a_1, b_1, \dots, a_N, b_N)$; here, a_i and b_i are independently drawn from S_0 with the uniform measure.

We will often modify the given choice s; the modified choices will be denoted by $\tilde{s} = (\tilde{a}_1, \dots, \tilde{b}_N)$ or $\bar{s} = (\bar{a}_1, \dots, \bar{b}_N)$. We will then denote by $\tilde{\omega}_j$ or $\bar{\omega}_j$ the sample path arising from the modified choices, respectively.

Lemma 3.5 *For $0 < k \le N$ and partial choices $s \in S_0^{2(k-1)}$,*

$$\mathbb{P}\big(\#P_k(s, a_k, b_k) = \#P_{k-1}(s) + 1\big) \ge \tfrac{9}{10}.$$

Proof Recall criterion (A) for $\#P_k = \#P_{k-1} + 1$. Note that the condition

$$(\omega_{2\vartheta(k)-1}o, \omega_{2\vartheta(k+1)-2}o)_{\omega_{2\vartheta(k)}o} = (b_k^{-2}o, w_k o)_o < K \tag{3-6}$$

depends only on b_k and not on other a_i's or b_i's. This holds for at least $(\#S_1 - 1)$ choices of b_k in S_1 and $(\#S_1 - 1)$ choices in $S_1^{(-1)}$.

Let us now fix a choice $b_k \in S_1^{(\pm)}$ satisfying condition (3-6) and $a_1, b_1, \dots, a_{k-1}, b_{k-1}$ that determine $\omega_{2\vartheta(k)-2}$ and z_{k-1}. Then the remaining conditions

$$(3\text{-}7)\qquad \begin{gathered} (z_{k-1}, \omega_{2\vartheta(k)-2}a_k o)_{\omega_{2\vartheta(k)-2}o} = (\omega_{2\vartheta(k)-2}^{-1} z_{k-1}, a_k o)_o < K, \\ (z_{k-1}, \omega_{2\vartheta(k)-2}a_k^2 o)_{\omega_{2\vartheta(k)-2}o} = (\omega_{2\vartheta(k)-2}^{-1} z_{k-1}, a_k^2 o)_o < K, \\ a_k \neq b_k^{-1} \end{gathered}$$

hold for at least $(\#S_1 - 1)$ choices of a_k in $S_1^{(\pm 1)}$ and $(\#S_1 - 2)$ choices in $S_1^{(\mp 1)}$, due to properties (5), (6) and (7) of Schottky sets. Since conditions (3-6) and (3-7) together constitute criterion (A), we obtain

$$\mathbb{P}(\#P_k = \#P_{k-1} + 1) \geq \frac{2\#S_1 - 2}{2\#S_1} \cdot \frac{2\#S_1 - 3}{2\#S_1} \geq 0.9. \qquad \square$$

Given $a_1, b_1, \dots, a_{k-1}, b_{k-1}$ and b_k, we define the set $\widetilde{S}_k'$ of elements a_k in S_0 that satisfy condition (3-7). In the proof above, we have observed that $\#[S_0 \setminus \widetilde{S}_k'] \leq 3$.

Lemma 3.6 *Let $i \in P_k(s)$ for a choice $s = (a_1, b_1, \dots, a_N, b_N)$, and $\bar{s}$ be obtained from s by replacing a_i with $\bar{a}_i \in \widetilde{S}_i'(a_1, b_1, \dots, a_{i-1}, b_{i-1}, b_i)$. Then $P_l(s) = P_l(\bar{s})$ and $\widetilde{S}_l'(s) = \widetilde{S}_l'(\bar{s})$ for any $1 \leq l \leq k$.*

Proof Since $a_1, b_1, \dots, a_{i-1}, b_{i-1}$ are intact, $P_l(s) = P_l(\bar{s})$ and $\widetilde{S}_l'(s) = \widetilde{S}_l'(\bar{s})$ hold for $l = 0, \dots, i-1$. At stage i, b_i satisfies condition (3-6) (since $i \in P_k(s)$) and $\bar{a}_i$ satisfies condition (3-7); hence, $i \in P_k(\bar{s})$ and $P_i(s) = P_i(\bar{s})$. We also have $\widetilde{S}_i'(s) = \widetilde{S}_i'(\bar{s})$. At this stage, however,

$$\bar{z}_i = \bar{\omega}_{2\vartheta(i)-1} o = g\omega_{2\vartheta(i)-1} o = g z_i,$$

where

$$g := \omega_{2\vartheta(i)-1} \bar{a}_i^2 (\omega_{2\vartheta(i)-1} a_i^2)^{-1}.$$

More generally,

$$(3\text{-}8)\qquad \bar{\omega}_j = g\omega_j \quad (j \geq 2\vartheta(i) - 1),$$

or in other words,

$$(3\text{-}9)\qquad \begin{aligned} \bar{\omega}_{2\vartheta(j)-1} &= g\omega_{2\vartheta(j)-1} \quad (j \geq i), \\ \bar{\omega}_{2\vartheta(j)-2} &= g\omega_{2\vartheta(j)-2} \quad (j > i). \end{aligned}$$

Recall again that the intermediate words w_j in between Schottky steps are unchanged.

We now claim the following for $i < l \leq k$:

1. If s fulfills criterion (A) at stage l, then so does $\bar{s}$.
2. If not and $\{i(1) < \cdots < i(M)\} \subseteq P_{l-1}(s)$ is the maximal sequence for s in criterion (B) at stage l, then it is also the maximal one for $\bar{s}$ at stage l.
3. In both cases, we have $P_l(s) = P_l(\bar{s})$ and $\bar{z}_l = g z_l$.

Assuming the third item for $l-1$: $P_{l-1}(s) = P_{l-1}(\bar{s})$ and $\bar{z}_{l-1} = gz_{l-1}$ let us test inequality (3-2) for $\bar{\omega}$ in criterion (A). If s fulfills criterion (A) at stage l, then

$$(\bar{\omega}_{2\vartheta(l)-2}^{-1}\bar{z}_{l-1}, a_l^t o)_o = (\omega_{2\vartheta(l)-2}^{-1} g^{-1} \cdot gz_{l-1}, a_l^t o)_o < K$$

for $t = 1, 2$ and $(b_l^{-2} o, w_l o)_o < K$. Hence we obtain the first item. In this case we also deduce that

$$P_l(s) = P_{l-1}(s) \cup \{l\} = P_{l-1}(\bar{s}) \cup \{l\} = P_l(\bar{s}), \quad \bar{z}_l = \bar{\omega}_{2\vartheta(l)-1} o = g\omega_{2\vartheta(l)-1} o = gz_l,$$

which constitute the third item for l.

Let us now check the second item. Due to equality (3-9), a sequence $\{i(1) < \cdots < i(M)\}$ in

$$P_{l-1}(s) \cap \{i, \dots, l-1\} = P_{l-1}(\bar{s}) \cap \{i, \dots, l-1\}$$

works for s in criterion (B) if and only if it works for $\bar{s}$. Furthermore, i belongs to $P_l(s)$ since $i \in P_k(s)$ and $i < l \le k$; hence, such sequences exist and the maximal sequence is chosen among them. Therefore, the maximal sequence $\{i(1) < \cdots < i(M)\}$ (with cardinality $M \ge 2$) for s is also maximal for $\bar{s}$. We then deduce that

$$P_l(s) = P_{l-1}(s) \cap \{1, \dots, i(1)\} = P_{l-1}(\bar{s}) \cap \{1, \dots, i(1)\} = P_l(\bar{s}),$$
$$\bar{z}_l = \bar{\omega}_{2\vartheta(i(M))-2} a_{i(M)} o = g\omega_{2\vartheta(i(M))-2} a_{i(M)} o = gz_l,$$

which constitute the third item for l. Here we used the condition $M \ge 2$ and $i(M) > i$; beware that $\bar{\omega}_{2\vartheta(i)-2}\bar{a}_i o$ and $g\omega_{2\vartheta(i)-2} a_i o$ may differ.

Since we have the base case $\bar{z}_i = gz_i$, an induction shows that $P_l(s) = P_l(\bar{s})$ for each $i < l \le k$. Moreover, equality (3-9) and $\bar{z}_{l-1} = gz_{l-1}$ imply that $\widetilde{S}'_l(s) = \widetilde{S}'_l(\bar{s})$. □

Given $1 \le k \le N$ and a partial choice $s = (a_1, b_1, \dots, a_k, b_k)$, we say that $\bar{s} = (\bar{a}_1, \bar{b}_1, \dots, \bar{a}_k, \bar{b}_k)$ is *pivoted from s* if

(1) $b_j = \bar{b}_j$ for all $1 \le j \le k$,

(2) $\bar{a}_i \in \widetilde{S}'_i(s)$ for $i \in P_k(s)$, and

(3) $a_j = \bar{a}_j$ for all other $j \notin P_k(s)$.

Lemma 3.6 then asserts that being pivoted from each other is an equivalence relation. For each $s \in S_0^{2k}$, let $\mathcal{E}_k(s)$ be the equivalence class of s.

Lemma 3.7 *For $0 \le k < N$, $j \ge 0$ and $s \in S_0^{2k}$,*

$$\mathbb{P}\big(\#P_{k+1}(\tilde{s}, a_{k+1}, b_{k+1}) < \#P_k(s) - j \mid \tilde{s} \in \mathcal{E}_k(s), (a_{k+1}, b_{k+1}) \in S_0^2\big) \le 1/10^{j+1}.$$

Proof Let us fix $s = (a_1, b_1, \dots, a_k, b_k) \in S_0^{2k}$ and

$$\mathcal{A} := \big\{(a_{k+1}, b_{k+1}) \in S_0^2 : \#P_{k+1}(s, a_{k+1}, b_{k+1}) = \#P_k(s) + 1\big\}.$$

Then Lemma 3.5 implies that $\mathbb{P}(\mathscr{A} \mid S_0^2) \geq 0.9$. Moreover, for $(a_{k+1}, b_{k+1}) \in \mathscr{A}$ and $\tilde{s} \in \mathscr{E}_k(s)$, $(\tilde{s}, a_{k+1}, b_{k+1})$ is pivoted from (s, a_{k+1}, b_{k+1}) since $(\tilde{s}, a_{k+1}, b_{k+1})$ and (s, a_{k+1}, b_{k+1}) differ at slots in $P_k(s) \subseteq P_{k+1}(s, a_{k+1}, b_{k+1})$. Lemma 3.6 then implies that

$$P_{k+1}(\tilde{s}) = P_{k+1}(s) = P_k(s) \cup \{k+1\} = P_k(\tilde{s}) \cup \{k+1\}.$$

In other words,

$$\mathbb{P}\big(\#P_{k+1}(\tilde{s}, a_{k+1}, b_{k+1}) < \#P_k(\tilde{s}) \mid (a_{k+1}, b_{k+1}) \in S_0^2\big) \leq 1 - \mathbb{P}(\mathscr{A}) \leq \tfrac{1}{10}$$

for each $\tilde{s} \in \mathscr{E}_k(s)$. Gathering all the cases, we deduce that

$$\mathbb{P}\big(\#P_{k+1}(\tilde{s}, a_{k+1}, b_{k+1}) < \#P_k(\tilde{s}) \mid \tilde{s} \in \mathscr{E}_k(s), (a_{k+1}, b_{k+1}) \in S_0^2\big) \leq \tfrac{1}{10}.$$

This settles the case $j = 0$.

Now let $j = 1$. The event under discussion becomes void when $\#P_k(s) \leq 2$. Excluding such cases, let $l < m$ be the last 2 elements of $P_k(s)$. For each choice $\tilde{s}$ in $\mathscr{E}_k(s)$ and each subset A of $\widetilde{S}'_m(s)$, we define

$$E(\tilde{s}, A) := \big\{\bar{s} = (\bar{a}_i, \bar{b}_i)_{i=1}^k : \bar{b}_i = \tilde{b}_i \text{ for all } i, \bar{a}_i = \tilde{a}_i \text{ for } i \neq m, \bar{a}_m \in A\big\}.$$

In plain words, $E(\tilde{s}, A)$ is a set of choices that are pivoted from $\tilde{s}$ only at stage m, such that the pivotal choice belongs to A. Then $\{E(\tilde{s}, \widetilde{S}'_m(s)) : \tilde{s} \in \mathscr{E}_k(s)\}$ partitions $\mathscr{E}_k(s)$ by Lemma 3.6.

We now fix $(a_{k+1}, b_{k+1}) \in S_0^2$ and $\tilde{s} = (\tilde{a}_1, \dots, \tilde{b}_k) \in \mathscr{E}_k(s)$. Let $A' \subseteq \widetilde{S}'_m(s)$ be the collection of elements $\bar{a}_m$ that satisfies

$$\text{(3-10)} \quad \big(\bar{a}_m^{-1} o, (\tilde{\omega}_{2\vartheta(m)-1})^{-1} \tilde{\omega}_{2\vartheta(k)-2} a_{k+1}^2 b_{k+1}^2 w_{k+1} o\big)_o = \big(\bar{a}_m^{-1} o, \tilde{b}_m^2 w_m \cdots \tilde{a}_k^2 \tilde{b}_k^2 w_k a_{k+1}^2 b_{k+1}^2 w_{k+1} o\big)_o < K.$$

Note that A' depends on $\tilde{s}$, a_{k+1} and b_{k+1}. By properties (5) and (6) of Schottky sets, $\#[\widetilde{S}'_m(s) \setminus A'] \leq 2$.

We now claim that $\#P_{k+1}(\bar{s}, a_{k+1}, b_{k+1}) \geq \#P_k(s) - 1$ for $\bar{s} \in E(\tilde{s}, A')$. First, since $l < m$ are consecutive elements in $P_k(\bar{s})$, Lemma 3.2 gives a sequence $\{l = i(1) < \cdots < i(M) = m\} \subseteq P_k$ with $M \geq 2$ such that $[\bar{\omega}_{2\vartheta(l)-1} o, \bar{\omega}_{2\vartheta(m)-2} \bar{a}_m o]$ is fully $D(K, \epsilon)$–marked with $(\gamma_j)_{j=1}^{M-1}$, $(\eta_j)_{j=2}^{M}$, where

$$\begin{aligned}
\gamma_1 &= [\bar{\omega}_{2\vartheta(i(1))-1} o, \bar{\omega}_{2\vartheta(i(1))} o],\\
\gamma_j &= [\bar{\omega}_{2\vartheta(i(j))-2} a_{i(j)} o, \bar{\omega}_{2\vartheta(i(j))-1} o] \quad (2 \leq j \leq M-1),\\
\eta_j &= [\bar{\omega}_{2\vartheta(i(j))-2} o, \bar{\omega}_{2\vartheta(i(j))-2} a_{i(j)} o] \quad (2 \leq j \leq M).
\end{aligned}$$

Moreover, condition (3-10) implies that

$$(\bar{\omega}_{2\vartheta(i(M))-2} a_{i(M)} o, \bar{\omega}_{2\vartheta(k+1)-2} o)_{\bar{\omega}_{2\vartheta(i(M))-1} o} < K.$$

In summary, $\{l = i(1) < \cdots < i(M)\} \subseteq P_k(\bar{s})$ works for $\bar{s}$ in criterion (B) at stage $k+1$, which implies $P_{k+1}(\bar{s}) \supseteq P_k(\bar{s}) \cap \{1, \dots, l\}$, hence the claim.

As a result, we deduce that

$$\mathbb{P}\big(\#P_{k+1}(\bar{s}, a_{k+1}, b_{k+1}) < \#P_k(s) - 1 \mid \bar{s} \in E(\tilde{s}, \widetilde{S}'_m)\big) \le \frac{\#[\widetilde{S}'_m(s) \setminus A']}{\#\widetilde{S}'_m(s)} \le \frac{2}{\#S_0 - 3} \le 0.1$$

for any $\tilde{s} \in \mathscr{E}_k(s)$ and $(a_{k+1}, b_{k+1}) \in S_0^2$. Since $E(\tilde{s}, \widetilde{S}'_m)$'s for $\tilde{s} \in \mathscr{E}_k(s)$ partition $\mathscr{E}_k(s)$, we deduce that

$$\mathbb{P}\big(\#P_{k+1}(\tilde{s}, a_{k+1}, b_{k+1}) < \#P_k(s) - 1 \mid \tilde{s} \in \mathscr{E}_k(s)\big) \le 0.1$$

for any $(a_{k+1}, b_{k+1}) \in S_0^2$. Moreover, the above probability vanishes when $(a_{k+1}, b_{k+1}) \in \mathscr{A}$. Since $\mathbb{P}(\mathscr{A} \mid S_0^2) \ge 0.9$, we deduce that

$$\mathbb{P}\big(\#P_{k+1}(\tilde{s}, a_{k+1}, b_{k+1}) < \#P_k(s) - 1 \mid \tilde{s} \in \mathscr{E}_k(s), (a_{k+1}, b_{k+1}) \in S_0^2\big) \le 0.01. \tag{3-11}$$

This settles the case $j = 1$.

For $j = 2$, we similarly assume $\#P_k(s) \ge 3$ and let $l' < l < m$ be the last 3 elements. We define the set $\mathscr{A}_1$ of $(\bar{a}_m, a_{k+1}, b_{k+1})$ in $\widetilde{S}'_m(s) \times S_0^2$ such that

$$\#P_{k+1}(\underbrace{a_1, b_1, \dots, \bar{a}_m, b_m, \dots, a_k, b_k}_{\text{obtained from } s \text{ by replacing } a_m \text{ with } \bar{a}_m}, a_{k+1}, b_{k+1}) \ge \#P_k(s) - 1,$$

or, equivalently,

$$P_k(s) \cap \{1, \dots, l\} \subseteq P_{k+1}(a_1, b_1, \dots, \bar{a}_m, b_m, \dots, a_k, b_k, a_{k+1}, b_{k+1}).$$

Now, if $\tilde{s} = (\tilde{a}_1, \tilde{b}_1, \dots, \tilde{a}_k, \tilde{b}_k) \in \mathscr{E}_k(s)$ is such that $(\tilde{a}_m, a_{k+1}, b_{k+1}) \in \mathscr{A}_1$, then $(\tilde{s}, a_{k+1}, b_{k+1})$ is pivoted from $(a_1, b_1, \dots, \tilde{a}_m, b_m, \dots, a_{k+1}, b_{k+1})$ since they only differ at the a_i for i in

$$P_k(s) \cap \{1, \dots, l\} \subseteq P_{k+1}(a_1, b_1, \dots, \tilde{a}_m, b_m, \dots, a_{k+1}, b_{k+1}).$$

Lemma 3.6 then implies that $P_{k+1}(\tilde{s}, a_{k+1}, b_{k+1})$ also contains $P_k(s) \cap \{1, \dots, l\}$. This fact and inequality (3-11) implies that

$$\begin{aligned}\mathbb{P}(\mathscr{A}_1 \mid \widetilde{S}'_m(s) \times S_0^2) &= \mathbb{P}\big((\tilde{a}_m, a_{k+1}, b_{k+1}) \in \mathscr{A}_1 \mid \tilde{s} \in \mathscr{E}_k(s), (a_{k+1}, b_{k+1}) \in S_0^2\big) \\ &\ge \mathbb{P}\big(\#P_{k+1}(\tilde{s}, a_{k+1}, b_{k+1}) \ge \#P_k(s) - 1 \mid \tilde{s} \in \mathscr{E}_k(s), (a_{k+1}, b_{k+1}) \in S_0^2\big) \\ &\ge 0.99.\end{aligned}$$

We now define for $\tilde{s} \in \mathscr{E}_k(s)$ and each $A \subseteq \widetilde{S}'_l(s)$,

$$E_1(\tilde{s}, A) := \big\{\bar{s} = (\bar{a}_i, \bar{b}_i)_{i=1}^k : \bar{b}_i = \tilde{b}_i \text{ for all } i, \bar{a}_i = \tilde{a}_i \text{ for } i \ne l, a_l \in A\big\}.$$

Then $\{E(\tilde{s}, \widetilde{S}'_l(s)) : \tilde{s} \in \mathscr{E}_k(s)\}$ partitions $\mathscr{E}_k(s)$ by Lemma 3.6.

Now fixing $(a_{k+1}, b_{k+1}) \in S_0^2$ and $\tilde{s} \in \mathscr{E}_k(s)$, let $A'_1 \subseteq \widetilde{S}'_l(s)$ be the collection of elements $\bar{a}_l \in \widetilde{S}'_l(s)$ that satisfies

$$\begin{aligned}&\big(\bar{a}_l^{-1} o, (\tilde{\omega}_{2\vartheta(l)-1})^{-1} \tilde{\omega}_{2\vartheta(k)-2} a_{k+1}^2 b_{k+1}^2 w_{k+1} o\big)_o \\ &\qquad = \big(\bar{a}_l^{-1} o, \tilde{b}_l^2 w_l \cdots \tilde{a}_k^2 \tilde{b}_k^2 w_k a_{k+1}^2 b_{k+1}^2 w_{k+1} o\big)_o < K.\end{aligned} \tag{3-12}$$

By properties (5) and (6) of Schottky sets, $\#[\widetilde{S}'_m(s) \setminus A'_1] \le 2$.

We now claim that $\#P_{k+1}(\bar{s}, a_{k+1}, b_{k+1}) \geq \#P_k(s) - 2$ for $\bar{s} \in E_1(\tilde{s}, A'_1)$. First, since $l' < l$ are consecutive elements in $P_k(\bar{s})$, Lemma 3.2 gives a sequence $\{l' = i(1) < \cdots < i(M) = l\} \subseteq P_k$ such that $[\bar{\omega}_{2\vartheta(l')-1}o, \bar{\omega}_{2\vartheta(l)-2}\bar{a}_l o]$ is fully $D(K,\epsilon)$–marked with $(\gamma_j)_{j=1}^{M-1}$, $(\eta_j)_{j=2}^{M}$, where

$$\begin{aligned}
\gamma_1 &= [\bar{\omega}_{2\vartheta(i(1))-1}o, \bar{\omega}_{2\vartheta(i(1))}o],\\
\gamma_j &= [\bar{\omega}_{2\vartheta(i(j))-2}a_{i(j)}o, \bar{\omega}_{2\vartheta(i(j))-1}o] \quad (j = 2, \dots, M-1),\\
\eta_j &= [\bar{\omega}_{2\vartheta(i(j))-2}o, \bar{\omega}_{2\vartheta(i(j))-2}a_{i(j)}o] \quad (j = 2, \dots, M).
\end{aligned}$$

Moreover, condition (3-10) implies that

$$(\bar{\omega}_{2\vartheta(i(M))-2}a_{i(M)}o, \bar{\omega}_{2\vartheta(k+1)-2}o)_{\bar{\omega}_{2\vartheta(i(M))-1}o} < K.$$

In summary, $\{l' = i(1) < \cdots < i(M)\} \subseteq P_k(\bar{s})$ works for $\bar{s}$ in criterion (B) at stage $k+1$, which implies that $P_{k+1}(\bar{s}) \supseteq P_k(\bar{s}, a_{k+1}, b_{k+1}) \cap \{1, \dots, l'\}$, hence the claim.

As a result, we deduce that

$$\mathbb{P}\big(\#P_{k+1}(\bar{s}, a_{k+1}, b_{k+1}) < \#P_k(s) - 2 \mid \bar{s} \in E_1(\tilde{s}, \widetilde{S}'_l)\big) \leq 0.1$$

for each $\tilde{s} \in \mathscr{E}_k(s)$ and $(a_{k+1}, b_{k+1}) \in S_0^2$. Here, for $\tilde{s}$ and (a_{k+1}, b_{k+1}) such that $(\tilde{a}_m, a_{k+1}, b_{k+1}) \in \mathscr{A}_1$, the above probability vanishes. Since

$$\begin{aligned}
\mathbb{P}\Big[\bigcup\{E_1(\tilde{s}, \widetilde{S}'_l) \times (a_{k+1}, b_{k+1}) : (\tilde{a}_m, a_{k+1}, b_{k+1}) \notin \mathscr{A}_1\} \,\Big|\, \mathscr{E}_k(s) \times S_0^2\Big]&\\
= \mathbb{P}\big[(\tilde{a}_m, a_{k+1}, b_{k+1}) \notin \mathscr{A}_1 \mid \widetilde{S}'_m(s) \times S_0^2\big] &\leq 0.01,
\end{aligned}$$

we sum up the conditional probabilities to obtain

$$\mathbb{P}\big(\#P_{k+1}(\tilde{s}, a_{k+1}, b_{k+1}) < \#P_k(s) - 2 \mid \tilde{s} \in \mathscr{E}_k(s)\big) \leq 0.001. \tag{3-13}$$

We repeat this procedure to cover all $j < \#P_k(s)$. The case $j \geq \#P_k(s)$ is void. □

Corollary 3.8 *Conditioned on paths $\boldsymbol{g} \in G^n$ such that $\#\Theta(\boldsymbol{g}) = N$, $\#P_N(\boldsymbol{g})$ is greater in distribution than the sum of N i.i.d. X_i, whose distribution is given by*

$$\mathbb{P}(X_i = j) = \begin{cases} \frac{9}{10} & \text{if } j = 1,\\ 9/10^{-j+1} & \text{if } j < 0,\\ 0 & \text{otherwise.} \end{cases} \tag{3-14}$$

The RV X_i in the above corollary satisfies

$$\begin{aligned}
\mathbb{E}[X_i] &= \tfrac{9}{10} - \tfrac{9}{10}\big(\tfrac{1}{10} + \tfrac{2}{10^2} + \cdots\big) = \tfrac{9}{10} - \tfrac{1}{9} = \tfrac{71}{90},\\
\mathbb{E}[1.4^{-X_i}] &= \tfrac{5}{7} \cdot \tfrac{9}{10} + \sum_{j=1}^{\infty} \tfrac{9}{10} \cdot \big(\tfrac{7}{50}\big)^j = \tfrac{9}{14} + \tfrac{9}{10} \cdot \tfrac{7}{50} + \frac{1}{1 - \frac{7}{50}} = \tfrac{1188}{1505}.
\end{aligned}$$

Proof Lemmas 3.5 and 3.7 imply that for $0 \le k < N$ and any i,

$$\mathbb{P}\big(\#P_{k+1}(\boldsymbol{g}) \ge i + j \mid \#P_k(\boldsymbol{g}) = i\big) \ge \begin{cases} 1 - \frac{1}{10} & \text{if } j = 1, \\ 1 - 1/10^{-j+1} & \text{if } j < 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3-15}$$

Hence, there exists a nonnegative RV U_k such that $\#P_{n+1} - U_k$ and $\#P_k + X'$ have the same distribution, where X' is an i.i.d. copy of X_{k+1} that is independent from $\#P_k$.

For each $1 \le k \le N$, we claim that $\mathbb{P}(\#P_k(\boldsymbol{g}) \ge i) \ge \mathbb{P}(X_1 + \cdots + X_k \ge i)$ for each i. For $k = 1$, we have $\#P_{k-1}(\boldsymbol{g}) = 0$ always and the claim follows from inequality (3-15). Given the claim for k,

$$\begin{aligned} \mathbb{P}(\#P_{k+1} \ge i) &\ge \mathbb{P}(\#P_k + X' \ge i) \\ &= \sum_j \mathbb{P}(\#P_k \ge j)\mathbb{P}(X' = i - j) \\ &\ge \sum_j \mathbb{P}(X_1 + \cdots + X_k \ge j)\mathbb{P}(X_{k+1} = i - j) \\ &= \mathbb{P}(X_1 + \cdots + X_k + X_{k+1} \ge i). \end{aligned}$$

□

Given $\boldsymbol{g} \in G^n$ with

$$\Theta(\boldsymbol{g}) = \{\vartheta(1) < \cdots < \vartheta(N)\}, \quad P_N(\boldsymbol{g}) = \{\iota(1) < \cdots < \iota(m)\} \subseteq \{1, \dots, N\},$$

we finally define the l^{th} pivotal time of $\boldsymbol{g}$ by $2\vartheta(\iota(l)) - 1$ and the *set of pivotal times* $P_n^*(\boldsymbol{g})$ by

$$P_n^*(\boldsymbol{g}) := \{2\vartheta(i) - 1 : i \in P_N(\boldsymbol{g})\}.$$

We also define $\widetilde{S}_{2\vartheta(\iota(l))-1}(\boldsymbol{g}) := \widetilde{S}'_{\iota(l)}(s)$ for $l = 1, \dots, m$; $\bar{\boldsymbol{g}} \in G^n$ is said to be *pivoted from* $\boldsymbol{g}$ if $g_j = \bar{g}_j$ unless $j \in P_n^*(\boldsymbol{g})$, in which case we require $\bar{g}_j \in \widetilde{S}_j(\boldsymbol{g})$.

Lemma 3.3 and Corollary 2.7 imply that

$$(\omega_i o, \omega_k)_{\omega_j o} < F(K, \epsilon) < \tfrac{1}{4000} K' \tag{3-16}$$

for $i, j, k \in P_n^*(\boldsymbol{g}) \cup \{0, n\}$ such that $i \le j \le k$. Moreover, for any $i, j, k \in P_n^*(\boldsymbol{g}) \cup \{0\}$ such that $i < j \le k$,

$$1.999K' \le d(\omega_{k-1} o, \omega_k o) \le 2K', \quad (\omega_i o, \omega_k o)_{\omega_{i-1} o} < F(K, \epsilon) < \tfrac{1}{4000} K'. \tag{3-17}$$

The first inequality is due to the fact that $g_k \in S_1^{(2)} \cup S_1^{(-2)}$; the second inequality follows from Lemma 3.2 and Corollary 2.7.

4 Pivoting and translation lengths

We will now define another equivalence relation on paths with sufficiently many pivots. Let us fix $\boldsymbol{g} \in G^n$ with $P_n^*(g) = \{i(1) < \cdots < i(m)\}$, where $m := \#P_n^*(\boldsymbol{g})$ satisfies $\frac{1}{5}n \le m \le \frac{1}{2}n$. For convenience, let also $i(0)=0$. We now define quantities

$$D_f(\boldsymbol{g}) = \sum_{l=1}^{\lfloor n/12\rfloor} [d(\omega_{i(l-1)}o, \omega_{i(l)-1}o) + 2K'],$$
$$D_b(\boldsymbol{g}) = \sum_{l=m-\lfloor n/12\rfloor+1}^{m} [d(\omega_{i(l-1)}o, \omega_{i(l)-1}o) + 2K'] + d(\omega_{i(m)}o, \omega_n o),$$
$$D_t(\boldsymbol{g}) = \sum_{l=1}^{m} [d(\omega_{i(l-1)}o, \omega_{i(l)-1}o) + 2K'] + d(\omega_{i(m)}o, \omega_n o).$$

Note the inequality

$$D_f(\boldsymbol{g}) \ge \sum_{l=1}^{\lfloor n/12\rfloor} [d(\omega_{i(l-1)}o, \omega_{i(l)-1}o) + d(\omega_{i(l)-1}o, \omega_{i(l)}o)] \ge d(o, \omega_{i(k)-1}), d(o, \omega_{i(k)}) \tag{4-1}$$

for $k = 1, \dots, \lfloor \frac{1}{12}n \rfloor$. Similarly, D_t dominates $d(o, \omega_n o)$. Moreover, due to inequalities (3-16) and (3-17), we have $|d(o,\omega_n o) - D_t| < 2F(K,\epsilon)n \le \frac{1}{1000}K'n$. We also observe that at least one of D_f and D_b is smaller than $\frac{1}{2}D_t - \frac{1}{20}K'n$; indeed, $D_t - D_f - D_b$ is the sum of at least $\frac{1}{20}n$ terms of the form $d(\omega_{i(l-1)}o, \omega_{i(l)-1}o) + 2K' \ge 2K'$.

If $D_f \le D_b$, then we allow pivoting at the first $\lfloor \frac{1}{12}n \rfloor$ pivotal times. Otherwise, we allow pivoting at the last $\lfloor \frac{1}{12}n \rfloor$ pivotal times. Since D_f, D_b, D_t and the set of pivotal times are invariant under pivoting, this rule partitions $\{\boldsymbol{g} \in G^n : \#P_n^*(\boldsymbol{g}) \ge \frac{1}{5}n\}$ into equivalence classes $\mathscr{F}(\boldsymbol{g})$.

We are now ready to prove the core lemma for Theorem A.

Lemma 4.1 *Let $n > 25$ and suppose that $\boldsymbol{g} \in G^n$ satisfies*

$$\#P_n^*(\boldsymbol{g}) \ge \tfrac{1}{5}n, \quad D_f(\boldsymbol{g}) \le D_b(\boldsymbol{g}).$$

Let also $1 \le k < k' \le \lfloor \frac{1}{12}n \rfloor$ and $\tilde{g}_{i(l)} \in \widetilde{S}_{i(l)}(\boldsymbol{g})$ for $l = 1, \dots, k-1, k'+1, \dots, \lfloor \frac{1}{12}n \rfloor$.

Then there exist $A \subseteq \widetilde{S}_{i(k)}(\boldsymbol{g})$ and $A' \subseteq \widetilde{S}_{i(k')}(\boldsymbol{g})$, each of cardinality at most 2, such that for any $\bar{\boldsymbol{g}} \in \mathscr{F}(\boldsymbol{g})$ such that $\bar{g}_{i(l)} = \tilde{g}_l$ for $l = 1, \dots, k-1, k'+1, \dots, \lfloor \frac{1}{12}n \rfloor$ and $\bar{g}_{i(k)} \notin A^{(2)}$, $\bar{g}_{i(k')} \notin A'^{(2)}$, we have $\tau(\bar{\omega}_n) \ge \frac{1}{12}K'n$.

Proof By the assumption $\bar{\boldsymbol{g}} \in \mathscr{F}(\boldsymbol{g})$, $\bar{\boldsymbol{g}}$ can differ only at step $i(1), \dots, i(\lfloor \frac{1}{12}n \rfloor)$. Hence, for $\bar{\boldsymbol{g}} \in \mathscr{F}(\boldsymbol{g})$ such that $\bar{g}_{i(l)} = \tilde{g}_{i(l)}$ for $l = 1, \dots, k-1, k'+1, \dots, \lfloor \frac{1}{12}n \rfloor$, the isometry

$$v := (\bar{\omega}_{i(k')})^{-1}\bar{\omega}_n\bar{\omega}_{i(k)-1} = \bar{g}_{i(k')+1}\cdots\bar{g}_n \cdot \bar{g}_1 \cdots \bar{g}_{i(k)-1}$$

is uniform. We define

$$A := \{g \in \widetilde{S}_k(\boldsymbol{g}) : (v^{-1}o, g^2 o)_o \ge K\}, \quad A' := \{g \in \widetilde{S}_{k'}(\boldsymbol{g}) : (g^{-1}o, vo)_o \ge K\}.$$

Since $\widetilde{S}_k(\boldsymbol{g}) \subseteq S_0 = S_1 \cup S_1^{(-1)}$, properties (5) and (6) of Schottky sets imply that $\#A \le 2$. Similarly we have $\#A' \le 2$.

Let us now fix $h \in \widetilde{S}_k(\boldsymbol{g}) \setminus A$ and $h' \in \widetilde{S}_{k'}(\boldsymbol{g}) \setminus A'$, and consider $\bar{\boldsymbol{g}} \in \mathscr{F}(\boldsymbol{g})$ such that $\bar{g}_{\rho(l)} = \tilde{g}_l$ for $l = 1, \dots, k-1, k'+1, \dots, \lfloor \frac{1}{12} n \rfloor$ and $\bar{g}_{\rho(k)} = h^2$, $\bar{g}_{\rho(k')} = h'^2$.

Since $i(k), i(k') \in P_n^*(\boldsymbol{g}) = P_n^*(\bar{\boldsymbol{g}})$ and $h'^2 = \bar{g}_{i(k')}$, Lemma 3.4 gives Schottky segments $(\gamma_l)_{l=1}^{M-1}$ and $(\eta_l)_{l=2}^{M}$ such that $[\bar{\omega}_{i(k)} o, \bar{\omega}_{i(k')-1} h' o]$ is fully $D(K, \epsilon)$–marked with (γ_l), (η_l), where

$$\gamma_1 = [\bar{\omega}_{i(k)} o, \bar{\omega}_{i(k)+1} o], \quad \eta_M = [\bar{\omega}_{i(k')-1} o, \bar{\omega}_{i(k')-1} h' o].$$

Next, inequality (4-1) implies that

$$\begin{aligned} d(o, vo) &\geq d(o, \bar{\omega}_n o) - d(o, \bar{\omega}_{i(k')} o) - d(o, \bar{\omega}_{i(k)-1} o) \\ &\geq \bigl(D_t - \tfrac{1}{1000} K' n\bigr) - 2D_f \geq \tfrac{1}{12} K' n \geq 2K' + 3D(K, \epsilon). \end{aligned}$$

Since we also have $(h'^{-1} o, vo)_o \leq K$ and $(o, vh^2 o)_{vo} \leq K$, Fact 2.8 implies that $[h'^{-1} o, vh^2 o]$ is $D(K, \epsilon)$–witnessed by $([h'^{-1} o, o], [vo, vh^2 o])$. By applying isometry $\bar{\omega}_n^{i-1} \bar{\omega}_{i(k')}$, we deduce that

$$[\bar{\omega}_n^{i-1} \bar{\omega}_{i(k')-1} h' o, \bar{\omega}_n^i \bar{\omega}_{i(k)} o]$$

is $D(K, \epsilon)$–witnessed by Schottky segments

$$[\bar{\omega}_n^{i-1} \bar{\omega}_{i(k')-1} h' o, \bar{\omega}_n^{i-1} \bar{\omega}_{i(k')} o], \quad [\bar{\omega}_n^i \bar{\omega}_{i(k)-1} o, \bar{\omega}_n^i \bar{\omega}_{i(k)} o].$$

We now claim that $[\bar{\omega}_{i(k)} o, \bar{\omega}_n^i \bar{\omega}_{i(k)} o]$ is fully $D(K, \epsilon)$–witnessed by

$$\begin{gathered} \bigl(\gamma_1, \dots, \gamma_{M-1}, [\bar{\omega}_{i(k')-1} h' o, \bar{\omega}_{i(k')} o], \bar{\omega}_n \gamma_1, \dots, \bar{\omega}_n \gamma_{M-1}, [\bar{\omega}_n \bar{\omega}_{i(k')-1} h' o, \bar{\omega}_n \bar{\omega}_{i(k')} o], \dots\bigr), \\ \bigl(\eta_2, \dots, \eta_M, [\bar{\omega}_n \bar{\omega}_{i(k)-1} o, \bar{\omega}_n \bar{\omega}_{i(k)} o], \bar{\omega}_n \eta_2, \dots, \bar{\omega}_n \eta_M, [\bar{\omega}_n^2 \bar{\omega}_{i(k)-1} o, \bar{\omega}_n^2 \bar{\omega}_{i(k)} o], \dots\bigr). \end{gathered}$$

This claim will follow from Lemma 2.4 once we check

$$\begin{aligned} (\bar{\omega}_{i(k')-1} o, \bar{\omega}_{i(k')} o)_{\bar{\omega}_{i(k')-1} h' o} &= (h'^{-1} o, h' o)_o < D(K, \epsilon), \\ (\bar{\omega}_{i(k)-1} o, \bar{\omega}_{i(k)+1} o)_{\bar{\omega}_{i(k)} o} &= (h^{-2} o, g_{i(k)+1} o)_o < D(K, \epsilon). \end{aligned}$$

The first item follows from property (7) of Schottky sets, and the second item follows from $h \in \widetilde{S}_k(\boldsymbol{g})$; hence the claim. In particular, Corollary 2.7 implies $(\bar{\omega}_{i(k)} o, \bar{\omega}_n^i \bar{\omega}_{i(k)} o)_{\bar{\omega}_n^{i-1} \bar{\omega}_{i(k)} o} < F(K, \epsilon)$ for each $i \geq 1$ and

$$\begin{aligned} \frac{1}{i} d(\bar{\omega}_{i(k)} o, \bar{\omega}_n^i \bar{\omega}_{i(k)} o) &\geq d(\bar{\omega}_{i(k)} o, \bar{\omega}_n \bar{\omega}_{i(k)} o) - F(K, \epsilon) \\ &\geq [d(o, \omega_n o) - d(o, \bar{\omega}_{i(k)} o) - d(\bar{\omega}_n o, \bar{\omega}_n \bar{\omega}_{i(k)} o)] - \tfrac{1}{1000} K' n \\ &\geq D_t - 2D_f - \tfrac{1}{500} K' n \geq \tfrac{1}{12} K' n. \end{aligned}$$

By sending $i \to \infty$, we deduce that $\tau(\bar{\omega}_n) \geq \frac{1}{12} K' n$. □

5 Proof of Theorem A

Proof of Theorem A Let $S' \subseteq G$ be the given finite set. By using Lemma 2.17, we take a (K, K', ϵ)–Schottky subset S_1 of G such that $K' > 2L(K, \epsilon) + 5000 F(K, \epsilon)$ and a finite symmetric generating set $S \supseteq S'$ such that $e \in S$ and S is nicely populated by $S_1^{(2)} \cup S_1^{(-2)}$.

As before, we consider the random walk on G generated by the uniform measure μ_S on S. We first claim

$$\mathbb{P}\big(\#P_n^* \le \tfrac{1}{5}n\big) \le \tfrac{1}{3}n \cdot 0.9^n + 0.9886^n.$$

for large n. The first term of the right-hand side is for the event of trajectories $\boldsymbol{g}$ with $\#\Theta(\boldsymbol{g}) \le \frac{1}{3}n$, whose probability is at most

$$\sum_{i=0}^{\lfloor n/3\rfloor} \binom{\lfloor n/2\rfloor}{i} \cdot 0.99^i \cdot 0.01^{\lfloor n/2\rfloor - i} \le \sum_{i=0}^{\lfloor n/3\rfloor} \binom{\lfloor n/2\rfloor}{\lfloor n/3\rfloor} \cdot 0.99^{\lfloor n/3\rfloor} \cdot 0.01^{n/6-1} \le \tfrac{1}{3}n \cdot 0.9^n.$$

Here the final inequality is deduced from the fact that

$$\binom{3(m+1)}{2(m+1)} \cdot 0.01^{m+1} = \binom{3m}{2m} \cdot \frac{(3m+3)(3m+2)(3m+1)}{(m+1)(2m+1)(2m+2)} \cdot 0.01^{m+1} \le \binom{3m}{2m} \cdot 0.01^m \cdot 0.07$$

for sufficiently large m, and that $0.07^{1/6} < 0.9$.

The second term is an estimation for the sum of m i.i.d. RVs X_i of the distribution in (3-14). Recall that X_i is an RV with exponential tail and $\mathbb{E}[X_i] = \frac{71}{90}$. Hence for $\lambda < \frac{71}{90}$, the theory of large deviation says that $\mathbb{P}\big(\sum_{i=1}^m X_i < \lambda m\big) \le e^{-\kappa(\lambda)m}$ for some $\kappa(\lambda) > 0$. The easiest way to show this (for suitable λ) is to take an intermediate base $\lambda < \lambda_0 < \mathbb{E}[X_i]$ and apply Markov's inequality to the RV $\lambda_0^{\sum_i X_i}$. Indeed, Markov's inequality tells us that

$$\mathbb{P}\bigg(\sum_{i=1}^m X_i < \tfrac{1}{5}n\bigg) \cdot 1.4^{-n/5} \le \mathbb{E}\big[1.4^{-\sum_{i=1}^m X_i}\big] = \prod_{i=1}^m \mathbb{E}[1.4^{-X_i}] = \big(\tfrac{1188}{1505}\big)^m$$

and the desired estimate follows for $m \ge \frac{1}{3}n$.

We now consider an equivalence class $\mathcal{F}(\boldsymbol{g})$ of $\boldsymbol{g} \in G^n$ such that $\#P_n(\boldsymbol{g}) \ge \frac{1}{5}n$ and $D_f(\boldsymbol{g}) \le D_b(\boldsymbol{g})$. For $h_i \in \widetilde{S}_i(\boldsymbol{g})$ and $k = 1, \dots, \lfloor \frac{1}{24}n \rfloor$, Lemma 4.1 gives the sets

$$A_k(\boldsymbol{g}, \{h_l, h_{\lfloor n/12\rfloor - l}\}_{l=1}^{k-1}) \subseteq \widetilde{S}_{i(k)}(\boldsymbol{g}), \quad A'_k(\boldsymbol{g}, \{h_l, h_{\lfloor n/12\rfloor - l}\}_{l=1}^{k-1}) \subseteq \widetilde{S}_{i(\lfloor n/12\rfloor - k)}(\boldsymbol{g}),$$

with cardinality at most 2, such that $\bar{g} \in \mathcal{F}(\boldsymbol{g})$ satisfies $\tau(\bar{\omega}_n) < \frac{1}{12}K'n$ only if

$$\bar{g}_{\rho(k)} \in A_k(\boldsymbol{g}, \{\bar{g}_{i(l)}, \bar{g}_{i(\lfloor n/12\rfloor - l)}\}_{l=1}^{k-1}) \quad \text{or} \quad \bar{g}_{i(\lfloor n/12\rfloor - k)} \in A'_k(\boldsymbol{g}, \{\bar{g}_{i(l)}, \bar{g}_{u(\lfloor n/12\rfloor - l)}\}_{l=1}^{k-1})$$

for each $k = 1, \dots, \lfloor \frac{1}{24}n \rfloor$. This implies that

$$\begin{aligned}
\mathbb{P}\big(\tau(\bar{\omega}_n) \le \tfrac{1}{12}K'n \mid \bar{\boldsymbol{g}} \in \mathcal{F}(\boldsymbol{g})\big) &\le \prod_{k=1}^{\lfloor n/24\rfloor} \frac{(\#\widetilde{S}_{i(k)})(\#\widetilde{S}_{i(\lfloor n/12\rfloor - k)}) - (\#\widetilde{S}_{i(k)} - 2)(\#\widetilde{S}_{i(\lfloor n/12\rfloor - k)} - 2)}{(\#\widetilde{S}_{i(k)})(\#\widetilde{S}_{i(\lfloor n/20\rfloor - k)})} \\
&\le \prod_{k=1}^{\lfloor n/24\rfloor} \bigg[\frac{2}{\#\widetilde{S}_{i(k)}} + \frac{2}{\#\widetilde{S}_{i(\lfloor n/12\rfloor - k)}}\bigg] \\
&\le \prod_{k=1}^{\lfloor n/24\rfloor} \bigg[\frac{2}{0.99\#S} + \frac{2}{0.99\#S}\bigg] = (0.2475\#S)^{-\lfloor n/24\rfloor}.
\end{aligned}$$

A similar argument leads to the same conclusion for any $\mathscr{F}(\boldsymbol{g})$ with $D_f(\boldsymbol{g}) > D_b(\boldsymbol{g})$. Since these $\mathscr{F}(\boldsymbol{g})$ partition $\{\#P_n \geq \frac{1}{5}n\}$, we conclude that the number of n–step trajectories ω such that $\tau(\omega_n) \leq \frac{1}{12}K'n$ is bounded by

$$(\#S)^n \cdot [0.91^n + 0.9886^n + (0.2475\,\#S)^{-n/24}] \leq 0.999^n \cdot (0.99\,\#S)^n$$

for sufficiently large n. Since any mapping class in $B_S(n)$ is obtained from an n–step trajectory, we conclude that the number of mapping classes in $B_S(n)$ with translation length less than $\frac{1}{12}K'n$ is bounded by $0.999^n \cdot (0.99\,\#S)^n$.

Meanwhile, the set

$$S_{\text{Schottky}} = \{(a_1, \dots, a_n) : a_i \in S_1 \cup S_1^{-1}, a_i \neq a_{i+1}^{-1}\}$$

is composed of at least $(\#S_0 - 1)^n \geq (0.99\,\#S)^n$ sequences. We claim that if $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ are distinct sequences in S_{Schottky}, then $a_1^2 \cdots a_n^2$ and $b_1^2 \cdots b_2^2$ are distinct elements in $B_S(n)$. Indeed, the sequence

$$(a_n^{-1}, a_n^{-1}, \dots, a_1^{-1}, a_1^{-1}, b_1, b_1, \dots, b_n, b_n)$$

will not completely cancel out and their product will not become an identity by Lemma 2.14. Hence, we have at least $(0.99\,\#S)^n$ distinct elements in $B_S(n)$. We thus finally have, for large n,

$$\frac{\#\{g \in B_S(n) : \tau_X(g) \leq \frac{1}{12}K'n\}}{\#B_S(n)} \leq 0.999^n. \qquad \square$$

Appendix A The proof of Claim 2.13

In this section, we prove Claim 2.13 in the proof of Proposition 2.12. We first recall the following lemma:

Fact A.1 [Choi 2023, Lemma 3.12] *For each $F, \epsilon > 0$, there exists $H, L > F$ that satisfies the following. If $x, y, z, p_1, p_2 \in X$ satisfy that*

1. *$[p_1, p_2]$ is ϵ–thick and longer than L,*
2. *$[x, y]$ is F–witnessed by $[p_1, p_2]$, and*
3. *$(x, z)_y \geq d(p_1, y) - F$,*

then $[z, y]$ is H–witnessed by $[p_1, p_2]$.

Recall that we have fixed $o \in X$ and independent loxodromics $a, b \in G$. By [Choi 2023, Lemmas 4.3 and 4.4], there exists $\epsilon_0, C_0 > 0$ such that

1. $[o, a^i o]$ and $[o, b^i o]$ are ϵ_0–thick for all $i \in \mathbb{Z}$, and
2. $(\phi^i o, \psi^j o)_o < C_0$ for all $i, j > 0$ and $\phi, \psi \in \{a, b, a^{-1}, b^{-1}\}$ such that $\phi \neq \psi$.

We then define

- $D_0 = D(C = C_0, \epsilon_0)$ as in Fact 2.9;
- $E_0 = E(D = D_0, \epsilon_0)$ and $L_0 = L(D = D_0, \epsilon_0)$ as in Fact 2.5;

- $F_0 = F(E = E_0, \epsilon_0)$ and $L_1 = L(E = E_0, \epsilon_0)$ as in Fact 2.6;
- $H_0 = H(F = F_0, \epsilon_0)$ and $L_2 = L(F = F_0, \epsilon_0)$ as in Fact A.1;
- $H_1 = H(F = H_0, \epsilon_0)$ $L_3 = L(F = H_0, \epsilon_0)$ as in Fact A.1;
- $F_1 = 2F_0 + H_1 + \delta + 1$;
- $F_2 = F(E = H_0, \epsilon_0)$ and $L_4 = L(E = H_0, \epsilon_0)$ as in Fact 2.6;
- $L_5 = \max(L_0, L_1, L_2, L_3, L_4, 2F_0 + F_1 + 2F_2)$.

There exists N_0 such that $d(o, \phi^N o) > L_5$ for all $\phi \in \{a, b, a^{-1}, b^{-1}\}$ and $N > N_0$. Let us fix $N > N_0$. We now consider a sequence $\{\phi_i\}$ in $\{a, b, a^{-1}, b^{-1}\}$ such that $\phi_i \neq \phi_{i+1}^{-1}$. Then we observe that

(1) $(\phi_i^{-N} o, \phi_{i+1}^N)_o < C_0$ by the assumption $\phi_i \neq \phi_{i+1}^{-1}$;

(2) $[o, \phi_i^N \phi_{i+1}^N o]$ is D_0–witnessed by $[o, \phi_i^N o]$ and $[\phi^N o, \phi^N \phi_{i+1}^N o]$ by Fact 2.9;

(3) $[o, \phi_i^N o]$ are ϵ–thick and longer than L_5.

Then as in Corollary 2.7, Facts 2.5 and 2.6 imply that $[o, \phi_1^{2N} \cdots \phi_n^{2N} o]$ is F_0–witnessed by ϵ_0–thick segments

$$[o, \phi_1^N o], [\phi_1^N o, \phi_1^N \phi_2^N o], \dots, [\phi_1^N \cdots \phi_{n-1}^N o, \phi_1^N \cdots \phi_n^N o].$$

Consequently, $[o, \phi_1^N \cdots \phi_n^N o]$ is ϵ–thick for $\epsilon = \epsilon_0 e^{-8F_0}$. It is also clear that

$$(o, \phi_1^N \cdots \phi_n^N o)_{\phi_1^N \cdots \phi_m^N o} < F_0 \quad (0 \le m \le n)$$

and

$$d(o, \phi_1^N \cdots \phi_n^N o) \ge d(o, \phi_1^N \cdots \phi_{n-1}^N o) + L_1 - 2F_0 \ge d(o, \phi_1^N \cdots \phi_{n-1}^N o) + F_1. \tag{A-1}$$

We now define

$$\begin{aligned} S_{n,N} &:= \{g_1, \dots, g_{2^n}\} = \{\phi_1^{2N} \cdots \phi_n^{2N} : \phi_i \in \{a, b\}\}, \\ V(g_i^{\pm}) &:= \{x \in X : (x, g_i^{\pm 2} o)_o \ge d(o, g_i^{\pm 1} o) - F_1\}, \\ V'(g_i^{\pm}) &:= \{x \in X : (x, g_i^{\pm 2} o)_o \ge d(o, g_i^{\pm 1} o)\}. \end{aligned}$$

Our first claim is that $V(g_1^+), \dots, V(g_{2^n}^+), V(g_1^-), \dots, V(g_{2^n}^-)$ are all disjoint. To show this, let $(\phi_i)_{i=1}^n$ and $(\psi_i)_{i=1}^n$ be distinct sequences in $\{a, b\}^n \cup \{a^{-1}, b^{-1}\}^n$, and $\Phi = \phi_1^{2N} \cdots \phi_n^{2N}$ and $\Psi = \psi_1^{2N} \cdots \psi_n^{2N}$. Let $t = \min\{1 \le i \le 10 : \phi_i \neq \psi_i\}$ and $w = \phi_1^{2N} \cdots \phi_{t-1}^{2N}$. Now suppose that a point $x \in X$ belongs to both $V(\Phi)$ and $V(\Psi)$. First, $x \in V(\Phi)$ implies

$$(x, \Phi^2 o)_o \ge d(o, \Phi o) - F_1 \ge d(o, \phi_1^{2N} \cdots \phi_{t-1}^{2N} \phi_t^N o) = d(o, w\phi_t^N o)$$

by inequality (A-1). Since $[o, \Phi^2 o]$ is F_0–witnessed by $[wo, w\phi_t^N o]$, Fact A.1 asserts that $[o, x]$ is H_0–witnessed by $[wo, w\phi_t^N o]$. By a similar reason, $(x, \Psi^2 o)_o \ge d(o, w\psi_t^N o)$ and $[o, x]$ is H_0–witnessed by $[wo, w\psi_t^N o]$. Since $(\phi_t o, \psi_t o)_o < C_0 < H_0$, Fact 2.6 implies that $[x, x]$ is F_2–witnessed by $[wo, w\psi_t o]$, whose length is at least $L_5 > 2F_2$; such an x does not exist.

The next claim is that if $x \notin V(g_i^-)$, then $g_i^2 x \in V'(g_i)$. Indeed, we know that $(o, g_i^2 o)_{g_i o} \le F_0 \le \frac{1}{2} F_1$ and

$$(g_i^2 x, g_i^2 o)_o = (x, o)_{g_i^{-2} o} = d(o, g_i^{-2} o) - (x, g_i^{-2} o)_o \ge d(o, g_i^2 o) - d(o, g_i o) + F_1 \ge d(o, g_i o).$$

Since $V'(g_i) \subseteq V(g_i)$ and $V(g_i) \cap V(g_i^-) = \varnothing$, we can iterate this to deduce that $g_i^{2k} x \in V'(g_i)$ for $k > 0$. Similarly, if $x \notin V(g_i)$, then $g_i^{-2k} x \in V'(g_i^-)$ for $k > 0$.

Now let $x, y \in X$. Since $\{V(g_i^+), V(g_i^-)\}$ are disjoint, $y \in V(g_i^-)$ for at most one $g_i \in S_{n,N}$ and $x \in V(g_j^+)$ for at most one $g_j \in S_{n,N}$. Suppose $s = \phi_1^{2N} \cdots \phi_n^{2N} \in S_{n,N}$ is neither of them, and let $k > 0$. We then have

$$(x, s^2 o)_o < d(o, so) - F_1, \quad (s^{2k} y, s^2 o)_o \ge d(o, so).$$

Since $[o, s^2 o]$ is F_0–witnessed by $[s\phi_n^{-N} o, so]$, Fact A.1 implies that $[o, s^{2k} y]$ is H_0–witnessed by $[s\phi_n^{-N} o, so]$. Now if we suppose that $(x, s^{2k} y)_o > d(o, so)$, then $[o, x]$ is also H_1–witnessed by $[s\phi_n^{-N} o, so]$, again by Fact A.1. Meanwhile, if X is a δ–hyperbolic space, we deduce that

$$\begin{aligned}(x, s^2 o)_o \ge \min\{(x, so)_o, (so, s^2 o)_o\} - \delta &= d(o, so) - \max\{(x, o)_{so}, (s^2 o, o)_{so}\} - \delta \\ &\ge d(o, so) - (H_1 + F_0 + \delta) \ge d(o, so) - F_1.\end{aligned}$$

Alternatively, if $X = \mathcal{T}(\Sigma)$, then we deduce that

$$(x, s^2 o)_o \ge d(o, so) - d(so, [o, x]) - d(so, [o, s^2 o]) \ge d(o, so) - (H_1 + F_0) \ge d(o, so) - F_1.$$

In either case we obtain a contradiction. Hence, $(x, s^{2k} y)_o \le d(o, so)$.

Similarly, if $s \ne g_i$ such that $y \in V(g_i^+)$ and $s \ne g_j$ such that $x \in V(g_j^-)$, then $(x, s^{-2k} y)_o \le d(o, s^{-1} o)$ for all $k > 0$. Finally, note that $y = o$ cannot belong to any of $V(g_j^{\pm})$ since $d(o, g_j^{\pm 1} o) \ge L_5 > F_1$ for any $g_j \in S_{n,N}$. This settles the desired claim.

Appendix B Sketch of the proof of Proposition 1.3

We borrow the definitions and notations in [Gekhtman et al. 2018].

In [Gekhtman et al. 2018], the authors consider the automatic structure of G, a directed graph Γ that records exactly one geodesic between e and g for each $g \in G$. Hence, the vertex set of its universal cover $\widetilde{\Gamma}$ and G are in one-to-one correspondence. Let LG be the set of vertices with large growth. For $g \in G$ and $0 < \epsilon < 1$, we denote by $\hat{g}_\epsilon$ the element along the path from e to g at distance ϵn from e. Then for any $0 < \epsilon < 1$, the ratio

$$\frac{\#\{g \in \partial B_S(n) : \hat{g}_\epsilon \notin LN\}}{\#\partial B_S(n)} \tag{B-1}$$

decays exponentially; see [Gekhtman et al. 2018, Proposition 2.5].

The authors then construct a Markov chain on Γ whose n–step distribution is denoted by $\mathbb{P}^n$. There exists $c > 1$ such that for any $A \subseteq \tilde{\Gamma}$, the proportion of $A \cap LG$ in $\partial B_S(n)$ is at least $(1/c)\mathbb{P}^n(A)$ and at most $c\mathbb{P}^n(A)$.

We now denote by $\mathcal{R}$ the set of recurrent vertices. For $v \in \mathcal{R}$, the loop semigroup Γ_v associated to v is nonelementary, i.e. there exist independent loxodromics $a_v, b_v \in \Gamma_v$ [Gekhtman et al. 2018, Corollary 6.11]. Let us now condition on the paths growing from v. Let $n(k, v, \omega)$ be the k^{th} return time to v and $T_v = \mathbb{E}_v n(1, v, \omega)$. Then for each $\epsilon > 0$,

$$\mathbb{P}_v\left\{\left|\frac{n(k,v,\omega)}{k} - T_v\right| > \epsilon\right\} \tag{B-2}$$

is exponentially decaying as $k \to \infty$; see [Gekhtman et al. 2018, Lemma 6.13]. For each n, we also define the last return time $\tilde{n}(\omega) = \max(\{n(k, v, \omega) : k \in \mathbb{N}\} \cap \{1, \dots, n\})$ to v. Then for each $\epsilon > 0$,

$$\mathbb{P}_v^n\left\{\frac{n - \tilde{n}(\omega)}{n} > \epsilon\right\} \tag{B-3}$$

decays exponentially.

We now strengthen [Gekhtman et al. 2018, Theorem 6.14]. For each $\epsilon > 0$,

$$\mathbb{P}_v\left\{\left|\frac{d(\omega_{n(k,v)}o, o)}{k} - l_v\right| > \epsilon\right\} \tag{B-4}$$

decays exponentially as $k \to \infty$, since μ_v actually has finite exponential moment. The proof for deviations from above can be found in [Boulanger et al. 2023]. For deviations from below, [Boulanger et al. 2023] and [Gouëzel 2022] deal with the case that X is Gromov hyperbolic. When X is the Teichmüller space, one can use Choi's modification of Gouëzel's construction in [Choi 2023]. Now together with the control on quantities (B-2) and (B-3), we obtain that for any $\epsilon > 0$,

$$\mathbb{P}_v^n\left\{\left|\frac{d(\omega_n o, o)}{n} - \frac{l_v}{T_v}\right| > \epsilon\right\} \tag{B-5}$$

decays exponentially. Now the proof of [Gekhtman et al. 2018, Theorem 6.14] shows that l_v/T_v is uniform for all $v \in \mathcal{R}$, which we denote by λ. Since the arrival time at $\mathcal{R}$ (beginning at e) also has finite exponential moment, we conclude that

$$\mathbb{P}_e^n\left\{\left|\frac{d(\omega_n o, o)}{n} - \lambda\right| > \epsilon\right\} \tag{B-6}$$

decays exponentially. By combining this with the decay of (B-1), we deduce that

$$\frac{\#\{g \in \partial B_S(n) : |d(o, go)/n - \lambda| > \epsilon\}}{\#\partial B_S(n)}$$

decays exponentially; cf [Gekhtman et al. 2018, Theorem 7.3].

We now need to discuss translation lengths instead of displacements. For each recurrent component C of Γ, we pick $v = v_C \in C$ and take a Schottky set as a subset of $\{w = g_1 \cdots g_n : g_i = a_v \text{ or } b_v\}$. We now consider the loop random walk generated by μ_v; recall the decomposition of the random walk into usual steps and "Schottky steps" for the pivot construction in [Choi 2023; Gouëzel 2022].

Until step $\frac{1}{4}T_v n$ in the loop random walk, we have Kn slots for Schottky steps for some $K > 0$ outside an event of exponentially decaying probability. Moreover, $\frac{1}{4}T_v n$ steps in the loop random walk occur before step $\frac{1}{2}n$ in the Markov process based at v_C, outside an event of exponentially decaying probability (quantity (B-2)). Finally, the Markov process beginning from e arrives at $\{v_C : C \text{ is recurrent}\}$ within step $\frac{1}{2}n$ outside an event of exponentially decaying probability.

In summary, giving up an event of exponentially decaying probability, a random path in the Markov process has at least Kn slots for Schottky loops based at some v_C. By pivoting the choice of Schottky loops at these slots, we can guarantee at least $K'n$ eventual pivots until step n for some $K' > 0$, outside an event of exponentially decaying probability.

Given these results, it now suffices to focus on the elements g such that

(1) $\check{g}_{1-\epsilon} \in LG$ and $d(o, \check{g}_{1-\epsilon}o) \geq (1-2\epsilon)\lambda n$,

(2) $d(o, \check{g}_\epsilon o) \leq 2\epsilon\lambda n$, and

(3) the subpath $[e, \check{g}_\epsilon]$ possesses at least $K'\epsilon n$ pivots for $[e, g]$.

We then consider the equivalence class by pivoting at the first $K'\epsilon n$ pivots. By early pivoting, one can show that only few elements inside the equivalence class satisfy $\tau_X(g) \leq (1 - 4\epsilon - MK')n$, for some suitable $M > 0$. By modulating ϵ and K', we establish the desired result.

References

[Baik et al. 2023] **H Baik**, **I Choi**, **D M Kim**, *Linear growth of translation lengths of random isometries on Gromov hyperbolic spaces and Teichmüller spaces*, J. Inst. Math. Jussieu (online publication November 2023)

[Boulanger et al. 2023] **A Boulanger**, **P Mathieu**, **Ç Sert**, **A Sisto**, *Large deviations for random walks on Gromov-hyperbolic spaces*, Ann. Sci. École Norm. Sup. 56 (2023) 885–944 MR Zbl

[Calvez and Wiest 2017] **M Calvez**, **B Wiest**, *Acylindrical hyperbolicity and Artin–Tits groups of spherical type*, Geom. Dedicata 191 (2017) 199–215 MR Zbl

[Caruso and Wiest 2017] **S Caruso**, **B Wiest**, *On the genericity of pseudo-Anosov braids, II: Conjugations to rigid braids*, Groups Geom. Dyn. 11 (2017) 549–565 MR Zbl

[Choi 2023] **I Choi**, *Central limit theorem and geodesic tracking on hyperbolic spaces and Teichmüller spaces*, Adv. Math. 431 (2023) art. id. 109236 MR Zbl

[Cumplido and Wiest 2018] **M Cumplido**, **B Wiest**, *A positive proportion of elements of mapping class groups is pseudo-Anosov*, Bull. Lond. Math. Soc. 50 (2018) 390–394 MR Zbl

[Erlandsson et al. 2020] **V Erlandsson**, **J Souto**, **J Tao**, *Genericity of pseudo-Anosov mapping classes, when seen as mapping classes*, Enseign. Math. 66 (2020) 419–439 MR Zbl

[Farb 2006] **B Farb**, *Some problems on mapping class groups and moduli space*, from "Problems on mapping class groups and related topics" (B Farb, editor), Proc. Sympos. Pure Math. 74, Amer. Math. Soc., Providence, RI (2006) 11–55 MR Zbl

[Gekhtman et al. 2018] **I Gekhtman**, **S J Taylor**, **G Tiozzo**, *Counting loxodromics for hyperbolic actions*, J. Topol. 11 (2018) 379–419 MR Zbl

[Gekhtman et al. 2020] **I Gekhtman**, **S J Taylor**, **G Tiozzo**, *Counting problems in graph products and relatively hyperbolic groups*, Israel J. Math. 237 (2020) 311–371 MR Zbl

[Gekhtman et al. 2022] **I Gekhtman**, **S J Taylor**, **G Tiozzo**, *Central limit theorems for counting measures in coarse negative curvature*, Compos. Math. 158 (2022) 1980–2013 MR Zbl

[Gouëzel 2022] **S Gouëzel**, *Exponential bounds for random walks on hyperbolic spaces without moment conditions*, Tunis. J. Math. 4 (2022) 635–671 MR Zbl

[Maher 2012] **J Maher**, *Exponential decay in the mapping class group*, J. Lond. Math. Soc. 86 (2012) 366–386 MR Zbl

[Maher and Tiozzo 2018] **J Maher**, **G Tiozzo**, *Random walks on weakly hyperbolic groups*, J. Reine Angew. Math. 742 (2018) 187–239 MR Zbl

[Rafi 2014] **K Rafi**, *Hyperbolicity in Teichmüller space*, Geom. Topol. 18 (2014) 3025–3053 MR Zbl

[Yang 2020] **W-y Yang**, *Genericity of contracting elements in groups*, Math. Ann. 376 (2020) 823–861 MR Zbl

June E Huh Center for Mathematical Challenges, KIAS
Seoul, South Korea

inhyeokchoi48@gmail.com

Proposed: Mladen Bestvina | Received: 9 November 2021
Seconded: Benson Farb, David Fisher | Revised: 18 August 2022

GEOMETRY & TOPOLOGY
msp.org/gt

See inside back cover or msp.org/gt for submission instructions.

The subscription price for 2024 is US $805/year for the electronic version, and $1135/year (+$70, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Geometry & Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Geometry & Topology (ISSN 1465-3060 printed, 1364-0380 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

GT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY
mathematical sciences publishers
nonprofit scientific publishing
http://msp.org/

Geometry & Topology

Volume 28 Issue 4 (pages 1501–1993) 2024