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Let (L, he^{-u}) be a pseudoeffective line bundle on an n–dimensional compact Kähler manifold X. Let $h^0(X, L^k \otimes \mathcal{I}(ku))$ be the dimension of the space of sections s of L^k such that $h^k(s,s)e^{-ku}$ is integrable. We show that the limit of $k^{-n}h^0(X, L^k \otimes \mathcal{I}(ku))$ exists, and equals the nonpluripolar volume of $P[u]_{\mathcal{I}}$, the $\mathcal{I}$–model potential associated to u. We give applications of this result to Kähler quantization: fixing a Bernstein–Markov measure ν, we show that the partial Bergman measures of u converge weakly to the nonpluripolar Monge–Ampère measure of $P[u]_{\mathcal{I}}$, the partial equilibrium.

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1 Introduction

A major theme in Kähler geometry has been the approximation/quantization of natural objects in the theory, going back to a problem of Yau [1987] and early work of Tian [1988]. Initial focus was on the quantization of smooth Kähler metrics, with asymptotic expansion results due to Tian [1990], Bouche [1990], Catlin [1999], Zelditch [1998], Lu [2000] and others. Later, Donaldson [2001] proposed to not just quantize Kähler metrics, but their infinite-dimensional geometry as well. This led to a flurry of activity helping to better understand notions of stability in Kähler geometry; see the work by Berndtsson [2018], Chen and Sun [2012], Phong and Sturm [2006], Song and Zelditch [2010], Darvas, Lu and Rubinstein [Darvas et al. 2020], Zhang [2023] to only mention a few works in a fast expanding literature. We refer to the excellent textbook by Ma and Marinescu [2007] for a detailed discussion and history of many classical results in this direction.

Our work fits into this broad context; however, we consider perhaps the most singular objects one can work with: positively curved metrics on a pseudoeffective line bundle. Despite the fact that potentials of these positively curved metrics are only integrable in general, we will be able to recover their volumes and partial equilibrium measure using quantization, significantly extending the scope of previous results in the literature.

The volume of a pseudoeffective line bundle We now describe our results. Let L be a holomorphic line bundle on a compact connected Kähler manifold (X,ω) of dimension n. Let h be a smooth metric on L, and let $\theta := c_1(L,h)$ denote the Chern form of h. Let (T,h_T) be an arbitrary Hermitian holomorphic vector bundle on X of rank r, which will be used to *twist* powers of L.

By $\mathrm{PSH}(X,\theta)$ we will denote the space of quasi-plurisubharmonic (quasi-psh) functions v on X such that $\theta + \mathrm{dd^c} v = \theta + (\mathrm{i}/2\pi)\partial\bar\partial v \geq 0$ in the sense of currents. Here $\mathrm{d} = \partial + \bar\partial$ and $\mathrm{d}^c = (\mathrm{i}/4\pi)(-\partial + \bar\partial)$.

A priori $\mathrm{PSH}(X,\theta)$ may be empty, but if there exists $u \in \mathrm{PSH}(X,\theta)$, then following terminology of Demailly, we say that the pair (L, he^{-u}) is a pseudoeffective (psef) Hermitian line bundle. Moreover, to such u one can associate a nonpluripolar complex Monge–Ampère measure θ_u^n, as introduced in [Boucksom et al. 2010; Guedj and Zeriahi 2007], following ideas by Cegrell [1998] and Bedford and Taylor [1976] in the local case; see Section 2.1 for more details.

We can associate to u the so-called *$\mathcal{I}$–model potential/envelope* $P[u]_{\mathcal{I}} \in \mathrm{PSH}(X,\theta)$, defined by

$$P[u]_{\mathcal{I}} := \sup\{w \in \mathrm{PSH}(X,\theta) : w \leq 0,\ \mathcal{I}(tw) \subseteq \mathcal{I}(tu) \text{ for } t \geq 0\}. \tag{1}$$

Here $\mathcal{I}(tu)$ is a multiplier ideal sheaf, locally generated by holomorphic functions f such that $|f|^2 e^{-tu}$ is integrable. To our knowledge $P[u]_{\mathcal{I}}$ was first considered in [Kim and Seo 2020], and we studied it in detail in [Darvas and Xia 2022, Section 2.4]; see also [Trusiani 2022].

Let $H^0(X, T \otimes L^k \otimes \mathcal{I}(ku))$ be the space of global holomorphic sections s of $T \otimes L^k$ satisfying $\int_X h_T \otimes h^k(s,s)e^{-ku}\omega^n < \infty$. We also introduce the notation

$$h^0(X, T \otimes L^k \otimes \mathcal{I}(ku)) := \dim_{\mathbb{C}} H^0(X, T \otimes L^k \otimes \mathcal{I}(ku)).$$

It was conjectured by Cao [2014, page 7] and Tsuji [2007, Section 4.4] that

$$\lim_{k\to\infty} \frac{1}{k^n} h^0(X, T \otimes L^k \otimes \mathcal{I}(ku))$$

always exists. We show that this is indeed the case, and we give a precise formula for the limit in terms of the nonpluripolar volume of $P[u]_{\mathcal{I}}$:

Theorem 1.1 *Let (L, he^{-u}) be a pseudoeffective Hermitian line bundle on X, and let T be a holomorphic vector bundle of rank r on X. Then*

$$\lim_{k\to\infty} \frac{1}{k^n} h^0(X, T \otimes L^k \otimes \mathcal{I}(ku)) = \frac{r}{n!}\int_X \theta^n_{P[u]_{\mathcal{I}}}. \tag{2}$$

When L is ample and T is a line bundle, Theorem 1.2 was obtained using non-Archimedean methods in [Darvas and Xia 2022, Theorem 1.4]. As these techniques do not extend to the pseudoeffective case, we take a more elementary approach in this work. In addition, in Section 4.2 we show that the analogue of Theorem 1.1 holds for pseudoeffective $\mathbb{R}$–line bundles as well.

In the case that u has analytic singularity type with smooth remainder (see Section 2.2 for the definition), formula (2) is a well-known consequence of the Riemann–Roch theorem of Bonavero [1998, Théorème 1.1, Corollaire 1.2]; see [Darvas and Xia 2022, Theorem 2.26]. In this case, it is possible to apply a resolution of singularities to simplify/principalize the singularity locus of $\mathcal{I}(u)$, allowing for a precise asymptotic analysis. In addition, in this case one also has $\int_X \theta^n_{P[u]_{\mathcal{I}}} = \int_X \theta^n_u$ [Darvas and Xia 2022, Proposition 2.20], simplifying the right-hand side of (2). However, for general $u \in \mathrm{PSH}(X,\theta)$, one is forced to use the measures $\theta^n_{P[u]_{\mathcal{I}}}$, and this is one of the novelties of our work. Indeed, since $u - \sup_X u \le P[u]_{\mathcal{I}}$, [Witt Nyström 2019, Theorem 1.1] gives that $\int_X \theta^n_u \le \int_X \theta^n_{P[u]_{\mathcal{I}}}$, and strict inequality is possible, as pointed out in [Darvas and Xia 2022, Example 2.19].

Formula (2) is also known for $u := V_\theta := \sup\{\varphi \in \mathrm{PSH}(X,\theta) : \varphi \le 0\}$, the potential with minimal singularity type in $\mathrm{PSH}(X,\theta)$; see [Boucksom et al. 2010, Proposition 1.18]. In this case we again have $\int_X \theta^n_{P[V_\theta]_{\mathcal{I}}} = \int_X \theta^n_{V_\theta}$, recovering Boucksom's formula [Boucksom 2002b; Boucksom et al. 2010]:

$$\lim_{k\to\infty} \frac{1}{k^n} h^0(X, L^k) = \frac{1}{n!}\int_X \theta^n_{V_\theta}.$$

The above expression is called the volume of the line bundle L in the literature [Boucksom 2002a; Demailly 2012], justifying our terminology calling $(1/n!)\int_X \theta^n_{P[u]_{\mathcal{I}}}$ the *volume* of the pair (L, he^{-u}).

As T is allowed to be an arbitrary vector bundle in Theorem 1.1, one can hypothesize a version of this result with T being a coherent sheaf on X. This was pointed out to us by László Lempert.

At the slight expense of precision, we briefly describe the strategy behind the proof of Theorem 1.1. By [Witt Nyström 2019, Theorem 1.2], both the left and right sides of (2) only depend on the singularity type of the potential u. As a result, we can use the metric topology of singularity types introduced in [Darvas et al. 2021], and further developed in [Darvas and Xia 2022]. Let us very briefly recall the terminology. For $v, w \in \mathrm{PSH}(X,\theta)$ we say that

- v is more singular than w, and we write $v \preceq w$, if there exists $C \in \mathbb{R}$ such that $v \le w + C$;
- v has the same singularity type as w, and we write $v \simeq w$, if $v \preceq w$ and $w \preceq v$.

The classes $[v] \in \mathcal{S} := \mathrm{PSH}(X,\theta)/\simeq$ of this latter equivalence relation are called *singularity types*. As pointed out in [Darvas et al. 2021], and recalled in Section 2.2, $\mathcal{S}$ admits a natural pseudometric $d_{\mathcal{S}}$, making $(\mathcal{S}, d_{\mathcal{S}})$ complete (in the presence of positive mass).

By [Darvas and Xia 2022, Proposition 2.20], we have

$$H^0(X, T\otimes L^k \otimes \mathcal{I}(ku)) = H^0(X, T\otimes L^k\otimes \mathcal{I}(kP[u]_{\mathcal{I}})) \quad \text{and} \quad P[u]_{\mathcal{I}} = P[P[u]_{\mathcal{I}}]_{\mathcal{I}},$$

ie $u \to P[u]_{\mathcal{I}}$ is a projection. Hence, it is enough to prove (2) for potentials of the form $P[u]_{\mathcal{I}}$. In Section 3 we show that the singularity types $[P[u]_{\mathcal{I}}] \in \mathcal{S}$ can be $d_{\mathcal{S}}$–approximated by analytic singularity types $[u_j] \in \mathcal{S}$. It is crucial to work with potentials of the form $P[u]_{\mathcal{I}}$, as the same property does not hold for general potentials u.

The proof is then completed by an approximation argument. We take a decreasing sequence $u_j \in \mathrm{PSH}(X,\theta)$ composed of potentials with analytic singularity types such that $d_{\mathcal{S}}([u_j],[u]) \to 0$. By Bonavero's theorem we know that (2) holds for each u_j. It is known that $d_{\mathcal{S}}([u_j],[u]) \to 0$ implies $\int_X \theta^n_{P[u_j]_{\mathcal{I}}} \to \int_X \theta^n_{P[u]_{\mathcal{I}}}$, and we will prove a similar convergence result for the left-hand side of (2) as well, to finish the argument.

Let us mention applications of Theorem 1.1 that are treated elsewhere. By [Lazarsfeld and Mustaţă 2009; Kaveh and Khovanskii 2012] we can naturally assign a family of convex Okounkov bodies $\Delta(L)$ to a given big line bundle L, depending only on the numerical class of L. Moreover, $\operatorname{vol} L = \operatorname{vol} \Delta(L)$. In [Xia 2021], based on Theorem 1.1, the second author extended this construction to Hermitian pseudoeffective line bundles: it is possible to define a natural family of convex bodies $\Delta(L,\phi)$ associated with a given Hermitian pseudoeffective line bundle (L,ϕ) such that $\operatorname{vol} \Delta(L,\phi) = \operatorname{vol}(L,\phi)$.

Another application concerns automorphic forms. Consider an automorphic line bundle L on a Shimura variety or mixed Shimura variety X. The global sections of L^k correspond to certain automorphic forms. It is a natural and important question in number theory to understand the asymptotic dimensions of these automorphic forms. In general, X is not compact, but it admits natural smooth compactifications [Ash et al. 2010]. Usually the smooth equivariant metrics on L only extends to singular metrics on a compactification. In this case, Theorem 1.1 can be naturally applied. In the special case of Siegel–Jacobi modular forms, this idea has been carried out concretely in the recent preprints [Botero et al. 2022a; 2022b]. Using a particular case of Theorem 1.1, they managed to prove that the ring of Siegel–Jacobi modular forms is not finitely generated, disproving a well-known claim by Runge [1995].

Convergence of partial Bergman measures As another application of Theorem 1.1, we give a very general convergence result for partial Bergman measures to the partial equilibrium, extending the scope of numerous results in the literature.

First we recall terminology introduced in [Berman and Boucksom 2010]. A *weighted subset* of X is a pair (K,v) consisting of a closed nonpluripolar subset $K \subseteq X$ and a function $v \in C^0(K)$. Next, given $u \in \mathrm{PSH}(X,\theta)$, we tailor the definition of $\mathcal{I}$–model envelope from (1) to the pair (K,v):

$$P[u]_{\mathcal{I}}(v) := \operatorname{usc}\big(\sup\{w \in \mathrm{PSH}(X,\theta) : w|_K \le v \text{ and } \mathcal{I}(tw) \subseteq \mathcal{I}(tu),\ t \ge 0\}\big). \tag{3}$$

Here $\operatorname{usc}(\cdot)$ denotes the least upper semicontinuous envelope. In case that $K = X$, $\operatorname{usc}(\cdot)$ is unnecessary, moreover we have $P[u]_{\mathcal{I}}(0) = P[u]_{\mathcal{I}}$.

As a consequence of Corollary 5.7 below, $\theta^n_{P[u]_{\mathcal{I}}(v)}$ does not put mass on the set $(X \setminus K) \cup \{P[u]_{\mathcal{I}}(v) < v\}$. What is more, when $K = X$ and $v \in C^2(X)$, the main result of Di Nezza and Trapani [2021] implies that

$$\theta^n_{P[u]_{\mathcal{I}}(v)} = \mathbb{1}_{\{P[u]_{\mathcal{I}}(v)=v\}}\theta^n_v.$$

Analogous properties of equilibrium type measures in different contexts were obtained in [Shiffman and Zelditch 2003; Berman 2009; Ross and Witt Nyström 2017]. With this in mind, we will call the measure $\theta^n_{P[u]_{\mathcal{I}}(v)}$ the *partial equilibrium* (*measure*) associated to u and (K, v). Theorem 1.2 will further justify this choice of terminology.

Let (T, h_T) be a Hermitian line bundle. Let ν be a Borel probability measure on K. We consider the norms on $H^0(X, L^k \otimes T)$ given by

$$N^k_{v,\nu}(s) := \left(\int_K h^k \otimes h_T(s,s)e^{-kv}\,\mathrm{d}\nu\right)^{1/2} \quad \text{and} \quad N^k_{v,K}(s) := \sup_K (h^k \otimes h_T(s,s)e^{-kv})^{1/2}.$$

Note that we always have $N^k_{v,\nu}(s) \le N^k_{v,K}(s)$. The measure ν is a *Bernstein–Markov measure* with respect to (K, v) if for each $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$ such that

$$N^k_{v,K}(s) \le C_\varepsilon e^{\varepsilon k} N^k_{v,\nu}(s) \tag{4}$$

for any $s \in H^0(X, L^k \otimes T)$. A broad class of Bernstein–Markov measures are probability volume forms with respect to (X, v), where $v \in C^\infty(X)$. For more complicated examples we refer to [Berman et al. 2011, Section 1.2].

We introduce the associated *partial Bergman kernels*: for any $k \in \mathbb{N}$, $x \in K$,

$$B^k_{v,u,\nu}(x) := \sup\{h^k \otimes h_T(s,s)e^{-kv}(x) : N^k_{v,\nu}(s,s) \le 1,\ s \in H^0(X, L^k \otimes T \otimes \mathcal{I}(ku))\}.$$

The associated partial Bergman measures on X are identically zero on $X \setminus K$ and on K are defined as

$$\beta^k_{v,u,\nu} := \frac{n!}{k^n} B^k_{v,u,\nu}\,\mathrm{d}\nu. \tag{5}$$

Our next result states that the partial Bergman measures $\beta^k_{v,u,\nu}$ quantize the nonpluripolar measure $\theta^n_{P[u]_{\mathcal{I}}(v)}$, the partial equilibrium of this setting:

Theorem 1.2 *Let (L, he^{-u}) be a pseudoeffective Hermitian holomorphic line bundle on X, and let (T, h_T) be a Hermitian line bundle. Suppose that ν is a Bernstein–Markov measure with respect to a weighted subset (K, v). Then $\beta^k_{u,v,\nu} \rightharpoonup \theta^n_{P[u]_{\mathcal{I}}(v)}$ weakly as $k \to \infty$.*

To our knowledge, this result is new even in the case when L is assumed to be ample. An important particular case is when T is trivial, $K = X$, $v \equiv 0$ and $\mu = \omega^n / \int_X \omega^n$. In this case we simply put $\beta^k_u := \beta^k_{u,0,\omega^n}$ and recall that $P[u]_{\mathcal{I}} = P[u]_{\mathcal{I}}(0)$. We have the following corollary:

Corollary 1.3 *For $u \in \mathrm{PSH}(X, \theta)$ we have that $\beta^k_u \rightharpoonup \theta^n_{P[u]_{\mathcal{I}}}$ weakly as $k \to \infty$.*

When T is the trivial line bundle and u has minimal singularity, Theorem 1.2 recovers [Berman et al. 2011, Theorem B]. As part of our argument, in Sections 5 and 6 we also extend [Berman and Boucksom 2010, Theorems A and B] to our partial setting. We suspect that using our results one can now prove equidistribution theorems for (partial) Fekete point configurations, extending [Berman et al. 2011, Theorem A] to our context. However, to stay brief we omit this discussion here.

When T is the trivial line bundle, $K = X$, $v \in C^2(X)$, $\mu = \omega^n / \int_X \omega^n$ and u has minimal or exponentially continuous singularity type, we are essentially in the setting of [Berman 2009, Theorem 1.4] and [Ross and Witt Nyström 2017, Theorem 1.4]. Our Theorem 1.2 extends these results, to the extent that our singular setting allows. Indeed, as $[u]$ is of $\mathcal{I}$–model type in these cases, we automatically get that $P[u](v) = P[u]_{\mathcal{I}}(v)$, where

$$P[u](v) := \operatorname{usc}\sup\{h \in \operatorname{PSH}(X,\theta) : h \le v,\ [h] \preceq [u]\}.$$

See Section 3 for more details. Hence, in this case the (partial) Bergman measures converge weakly to $\theta^n_{P[u](v)}$. Berman [2009] and Ross and Witt Nyström [2017] actually argue pointwise convergence of the density functions as well, on the locus where $P[u](v) = v$ and $\theta_v > 0$. As our v in Theorem 1.2 is only continuous, it is not clear how to interpret the condition $\theta_v > 0$ in our context.

Observe that $\int_X \beta^k_{v,u,\nu} = n!k^{-n}h^0(X, L^k \otimes T \otimes \mathcal{I}(ku))$. In particular, Theorem 1.2 recovers Theorem 1.1 after an integration. In fact, this plays a crucial role in the argument of Theorem 1.2. As all the measures $\beta^k_{u,v,\mu}$ have uniformly bounded masses, they form a weakly compact family. The difficulty is to prove that each subsequential limit measure is dominated by $\theta^n_{P[u]_{\mathcal{I}}(v)}$. Then the argument is concluded by simply comparing total masses of the limit measures.

The literature on partial Bergman kernels/measures has been fast expanding in many directions. One particular line of study concerns partial Bergman kernels arising from sections vanishing along a smooth divisor V, with the vanishing order increasing in the large limit. As pointed out in numerous works mentioned below, this setup is closely related to ours, when one considers L^2 integrable sections with respect to a weight that has logarithmic singularity along V. It would be interesting to study this connection in the future. One of the first works on this topic was that of Berman [2009], who proved L^1 convergence of the volume densities of the partial Bergman measures. Ross and Singer [2017] and Zelditch and Zhou [2019b] considered this problem in the presence of an S^1–symmetry near the vanishing locus, identified the forbidden region in terms of the Hamiltonian action, and gave detailed asymptotic expansions. When symmetries are not present, Coman and Marinescu [2017] proved that the partial Bergman kernel has exponential decay near the vanishing locus. For recent extensions to smooth and singular subvarieties V, see [Coman et al. 2019; Sun 2020].

Applications of partial Bergman kernels related to test configurations and geodesic rays were explored in [Ross and Witt Nyström 2014; Darvas and Xia 2022].

In another line of study, Zelditch and Zhou [2019a] initiated the study of partial Bergman kernels that arise from spectral subspaces of the Toeplitz quantization of a smooth Hamiltonian. They showed that their partial density of states also converges to an equilibrium type measure, suggesting possible connections with our Theorem 1.2. Specifically, given the Hamiltonian data (H, E) of [Zelditch and Zhou 2019a], we wonder if there exists $v \in C^\infty(X)$ and $u \in \mathrm{PSH}(X,\theta)$ such that $\{H(z) < E\} = \{P[u]_{\mathcal{I}}(v) = v\}$. If the answer to this question is affirmative, then using the terminology of [Zelditch and Zhou 2019a, Main Theorem] we would obtain that $\prod_{k,\mathcal{G}_k} \omega^n \rightharpoonup \theta^n_{P[u]_{\mathcal{I}}(v)}$.

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Organization In Section 2 we recall the relevant notions of envelopes, and adapt results in the literature about the metric topology of singularity types to our context. In Section 3 we characterize the closure of analytic singularity types in a big cohomology class. In Section 4 we prove Theorem 1.1. In Sections 5 and 6 we extend the related results of [Berman and Boucksom 2010; Berman et al. 2011] to our partial context, and prove Theorem 1.2.

2 Preliminaries

2.1 Nonpluripolar products and singularity types

Let X be a compact Kähler manifold. Let θ be a smooth real $(1,1)$–form on X. Let $\mathrm{PSH}(X,\theta)$ be the set of θ–plurisubharmonic (θ–psh) functions on X. Assume that the cohomology class of θ is pseudoeffective, ie that $\mathrm{PSH}(X,\theta)$ is nonempty.

Let $V_\theta := \sup\{v \in \mathrm{PSH}(X,\theta) : v \le 0\}$ be the potential with minimal singularity in $\mathrm{PSH}(X,\theta)$. We recall the construction of nonpluripolar product associated to $u_1, \dots, u_n \in \mathrm{PSH}(X,\theta)$ from [Boucksom et al. 2010].

Let $k \in \mathbb{N}$. Using Bedford–Taylor theory [1976], one can consider the following sequence of measures on X:

$$\mathbb{1}_{\bigcap_j \{\varphi_j > V_\theta - k\}} (\theta + dd^c \max(\varphi_1, V_\theta - k)) \wedge \cdots \wedge (\theta + dd^c \max(\varphi_n, V_\theta - k)).$$

It has been shown in [Boucksom et al. 2010, Section 1] that these measures converge weakly to the so-called *nonpluripolar product* $\theta_{\varphi_1} \wedge \cdots \wedge \theta_{\varphi_n}$ as $k \to \infty$. All complex Monge–Ampère measures will be interpreted in this sense in our work.

The resulting positive measure $\theta_{\varphi_1} \wedge \cdots \wedge \theta_{\varphi_n}$ does not charge pluripolar sets. The particular case when $u := u_1 = \cdots = u_n$ will yield θ_u^n, the nonpluripolar complex Monge–Ampère measure of u.

For any $u \in \mathrm{PSH}(X,\theta)$, let $\mathcal{I}(u)$ denote Nadel's multiplier ideal sheaf of u, namely, the coherent ideal sheaf of holomorphic functions f, such that $|f|^2 e^{-u}$ is integrable. These objects allow us to introduce an algebraic refinement of the notion of singularity type from the introduction. For $u, v \in \mathrm{PSH}(X,\theta)$ we have the following relations:

- $u \preceq_{\mathcal{I}} v$ (also written as $[u] \preceq_{\mathcal{I}} [v]$) if $\mathcal{I}(tu) \subseteq \mathcal{I}(tv)$ for all $t > 0$.
- $u \simeq_{\mathcal{I}} v$ (also written as $[u] \simeq_{\mathcal{I}} [v]$) if $u \preceq_{\mathcal{I}} v$ and $v \preceq_{\mathcal{I}} u$.

The relation $\simeq_{\mathcal{I}}$ induces equivalence classes called *$\mathcal{I}$–singularity types* $[u]_{\mathcal{I}}$, for any $u \in \mathrm{PSH}(X,\theta)$. As pointed out in [Darvas and Xia 2022], $[u]=[v]$ implies $[u]_{\mathcal{I}}=[v]_{\mathcal{I}}$, but not vice versa.

The different equivalence relations ($\simeq$ and $\simeq_{\mathcal{I}}$) admit two different envelope notions, as already alluded to in the introduction. Let us revisit them in a very general setup, that will be needed later. Let $K \subseteq X$ compact and nonpluripolar, and let $v\colon K \to [-\infty,\infty]$ measurable. To such v and $u \in \mathrm{PSH}(X,\theta)$ we associate the following notion of envelope:

$$P^\theta_K[u](v) := (\sup\{w \in \mathrm{PSH}(X,\theta) : [w] \preceq [u],\, w|_K \le v\}),$$
$$P^\theta_K[u]_{\mathcal{I}}(v) := \mathrm{usc}(\sup\{\, w \in \mathrm{PSH}(X,\theta) : [w] \preceq_{\mathcal{I}} [u],\, w|_K \le v\}).$$

Here and later $\mathrm{usc}(\cdot)$ denotes the upper semicontinuous regularization. We omit θ and X from our notation when there is no risk of confusion. In addition, we will use the following shorthand notation, ubiquitous in the literature:

$$P[u] := P^\theta_X[u](0), \qquad P[u]_{\mathcal{I}} := P^\theta_X[u]_{\mathcal{I}}(0).$$

A potential $u \in \mathrm{PSH}(X,\theta)$ is *model* if $u = P[u]$, and it is *$\mathcal{I}$–model* if $u = P[u]_{\mathcal{I}}$.

For any usc function $f\colon X \to [-\infty,\infty)$ we define

$$P^\theta(f) := \mathrm{usc}\sup\{\varphi \in \mathrm{PSH}(X,\theta) : \varphi \le f\}. \tag{6}$$

Building on the above, for usc functions $f_1,\dots,f_N$ we define a notion of rooftop envelope:

$$P^\theta(f_1,\dots,f_N) := P^\theta(\min\{f_1,\dots,f_N\}).$$

The following lemma was essentially proved in [Darvas et al. 2021]. We recall the short proof as a courtesy to the reader:

Lemma 2.1 *Let $u, v \in \mathrm{PSH}(X,\theta)$ such that $P^\theta(u,v) \in \mathrm{PSH}(X,\theta)$. If u, v are model (resp. $\mathcal{I}$–model), then $P^\theta(u,v)$ is also model (resp. $\mathcal{I}$–model).*

Proof Since $P(u,v) \le \min(u,v)$, we get that $P[P(u,v)] \le P[u] = u$ and $P[P(u,v)] \le P[v] = v$, hence $P[P(u,v)] \le P(u,v)$. This implies $P[P(u,v)] = P(u,v)$, as desired. The statement about $\mathcal{I}$–model potentials is proved in the same way. □

For any $x \in X$ and $u \in \mathrm{PSH}(X,\theta)$, we denote by $\nu(u,x)$ the Lelong number of φ at x. We recall the following result from [Boucksom et al. 2008], adapted to our context in [Darvas and Xia 2022, Corollary 2.16]:

Proposition 2.2 *Let $u, v \in \mathrm{PSH}(X, \theta)$. Then*

(i) $[u] \preceq_{\mathcal{I}} [v]$ *if and only if* $\nu(\pi^* u, y) \geq \nu(\pi^* v, y)$ *for any smooth modification* $\pi : Y \to X$ *and any* $y \in Y$.

(ii) $[u] \simeq_{\mathcal{I}} [v]$ *if and only if* $\nu(\pi^* u, y) = \nu(\pi^* v, y)$ *for any smooth modification* $\pi : Y \to X$ *and any* $y \in Y$.

Corollary 2.3 *Let $u_0, u_1, v_0, v_1 \in \mathrm{PSH}(X, \theta)$ with $[u_0] \preceq_{\mathcal{I}} [v_0]$ and $[u_1] \preceq_{\mathcal{I}} [v_1]$. For any $t \in [0, 1]$ we have $[(1-t)u_0 + tu_1] \preceq_{\mathcal{I}} [(1-t)v_0 + tv_1]$.*

Proof This follows from Proposition 2.2(i) and the additivity of Lelong numbers [Boucksom 2017, Corollary 2.10]. □

Lastly, we show concavity properties for the envelopes defined above:

Proposition 2.4 *Let $v \in C^0(K)$ and $u_0, u_1 \in \mathrm{PSH}(X, \theta)$. The following hold:*

(i) *For any $t \in [0, 1]$, let $u_t = tu_1 + (1-t)u_0$. Then*

$$tP_K[u_1]_{\mathcal{I}}(v) + (1-t)P_K[u_0]_{\mathcal{I}}(v) \leq P_K[u_t]_{\mathcal{I}}(v), \quad tP_K[u_1](v) + (1-t)P_K[u_0](v) \leq P_K[u_t](v). \tag{7}$$

(ii) *If $[u_0] \preceq_{\mathcal{I}} [u_1]$ (resp. $[u_0] \preceq [u_1]$), then $P_K[u_0]_{\mathcal{I}}(v) \leq P_K[u_1]_{\mathcal{I}}(v)$ (resp. $P_K[u_0](v) \leq P_K[u_1](v)$).*

Proof The proof of (ii) follows from the definitions. To prove (i), let $h_0, h_1 \in \mathrm{PSH}(X, \theta)$ be such that $[h_i] \preceq_{\mathcal{I}} [u_i]$ and $h_i|_K \leq v$. Then by Corollary 2.3, $[th_1 + (1-t)h_0] \preceq_{\mathcal{I}} [u_t]$. It is clear that $th_1|_K + (1-t)h_0|_K \leq v$. Hence, $th_1 + (1-t)h_0 \leq P_K[u_t]_{\mathcal{I}}(v)$.

As h_1 and h_0 are arbitrary candidates, we conclude the first inequality in (7). The proof of the second inequality is similar. □

2.2 The metric topology of singularity types

Let $\mathcal{S}(X, \theta)$ be the set of singularity types of θ–psh functions: $\mathcal{S}(X, \theta) := \mathrm{PSH}(X, \theta)/\simeq$. Let $\mathcal{A}(X, \theta) \subseteq \mathcal{S}(X, \theta)$ be the set of *analytic singularity types*, namely, all singularity types $[u]$ represented by an element $u \in \mathrm{PSH}(X, \theta)$ such that u is locally of the form

$$u = c \log \sum_{i=1}^{N} |f_i|^2 + g, \tag{8}$$

where $c \in \mathbb{Q}^+$, $f_1, \ldots, f_N$ are holomorphic functions and g is a bounded function. When g can be taken to be smooth, then following [Demailly 2018] we say that $[u]$ is a *neat* analytic singularity type.

Darvas et al. [2021] constructed a pseudometric $d_{\mathcal{S}}$ on $\mathcal{S}(X, \theta)$. As we will use the $d_{\mathcal{S}}$ topology extensively in this work, we recall here a few basic facts, and refer to [Darvas et al. 2021] for a more complete picture.

The definition of $d_{\mathcal{S}}$ involves embedding $\mathcal{S}(X,\theta)$ into the space of L^1 geodesic rays [Darvas et al. 2021, Section 3]. We do not recall the exact definition, but simply recall that there is a constant $C>0$, depending only on n, such that for any $[u],[v]\in\mathcal{S}(X,\theta)$ we have

$$(9)\quad d_{\mathcal{S}}([u],[v]) \le \sum_{j=0}^{n}\left(2\int_X \theta^j_{V_\theta}\wedge\theta^{n-j}_{\max\{u,v\}} - \int_X \theta^j_{V_\theta}\wedge\theta^{n-j}_u - \int_X \theta^j_{V_\theta}\wedge\theta^{n-j}_v\right) \le C d_{\mathcal{S}}([u],[v]).$$

Note that the term in the middle is independent of the choices of representatives u and v, as a consequence of [Darvas et al. 2018, Theorem 1.1].

Theorem 2.5 [Darvas et al. 2021, Theorem 1.1] *For any $\delta>0$, the space*

$$\mathcal{S}_\delta(X,\theta) := \left\{[u]\in\mathcal{S}(X,\theta) \;\middle|\; \int_X \theta_u^n \ge \delta\right\}$$

is $d_{\mathcal{S}}$–complete.

We paraphrase another result, to make it easily adaptable to our context:

Lemma 2.6 [Darvas et al. 2021, Lemma 4.3] *Let $u,v\in\mathrm{PSH}(X,\theta)$ be such that $[u]\preceq[v]$ and $\int_X\theta_u^n>0$. For any*

$$b\in\left(1,\left(\frac{\int_X\theta_v^n}{\int_X\theta_v^n-\int_X\theta_u^n}\right)^{1/n}\right),$$

there exists $h\in\mathrm{PSH}(X,\theta)$ such that $h+(b-1)v\le bu$. This allows to introduce:

$$(10)\qquad P(bu+(1-b)v) := \operatorname{usc}\sup\{h\in\mathrm{PSH}(X,\theta) : h+(b-1)v\le bu\}\in\mathrm{PSH}(X,\theta).$$

To clarify, when $\int_X\theta_v^n=\int_X\theta_u^n$ the condition on b in the above result is $b\in(1,\infty)$. In addition, by (10), we have that $P(bu+(1-b)v)+(b-1)v\le u$ a.e. on X, hence this inequality holds globally, since both the left- and right-hand side are quasi-psh functions.

Next we prove continuity results for the envelopes defined above.

Proposition 2.7 *Let $K\subseteq X$ be a compact and nonpluripolar subset. Let $v\in C^0(K)$. Suppose that $u_j,u\in\mathrm{PSH}(X,\theta)$ are such that $d_{\mathcal{S}}([u_j],[u])\to0$ and $\int_X\theta_u^n>0$. Then the following hold*:

- (i) *If $u_j\searrow u$ then $P_K[u_j]_{\mathcal{I}}(v)\searrow P_K[u]_{\mathcal{I}}(v)$ and $P_K[u_j](v)\searrow P_K[u](v)$.*
- (ii) *If $u_j\nearrow u$ then $P_K[u_j]_{\mathcal{I}}(v)\nearrow P_K[u]_{\mathcal{I}}(v)$ a.e. and $P_K[u_j](v)\nearrow P_K[u](v)$ a.e.*

The argument is very similar to that of [Darvas and Xia 2022, Lemma 2.21].

Proof We first prove (i). Since $\int_X\theta^n_{u_j}\searrow\int_X\theta_u^n>0$ [Darvas et al. 2021, Proposition 4.8], by Lemma 2.6, there exists $\alpha_j\searrow0$ and $h_j:=P\big((1/\alpha_j)u+(1-(1/\alpha_j))u_j\big)\in\mathrm{PSH}(X,\theta)$ satisfying $(1-\alpha_j)u_j+\alpha_jh_j\le u$. By Proposition 2.4,

$$(1-\alpha_j)P_K[u_j]_{\mathcal{I}}(v)+\alpha_jP_K[h_j]_{\mathcal{I}}(v)\le P_K[(1-\alpha_j)u_j+\alpha_jh_j]_{\mathcal{I}}(v)\le P_K[u]_{\mathcal{I}}(v).$$

Since $\{u_j\}_j$ is decreasing, so is $\{P_K[u_j]_{\mathcal{I}}(v)\}_j$, hence $w := \lim_j P_K[u_j]_{\mathcal{I}}(v) \geq P[u]_{\mathcal{I}}(v)$ exists. Since $\alpha_j \to 0$ and $\sup_X P_K[h_j]_{\mathcal{I}}(v)$ is bounded, we can let $j \to \infty$ in the above estimate to conclude that $w = P_K[u]_{\mathcal{I}}(v)$. The same ideas yield that $P_K[u_j](v) \searrow P_K[u](v)$.

Proving (ii) is similar. Since $\int_X \omega^n_{u_j} \nearrow \int_X \omega^n_u > 0$ [Darvas et al. 2018, Theorem 2.3], by [Darvas et al. 2021, Lemma 4.3] there exists $\alpha_j \searrow 0$ and $h_j := P\big((1/\alpha_j)u_j + (1-(1/\alpha_j))u\big) \in \mathrm{PSH}(X,\theta)$ satisfying $(1-\alpha_j)u + \alpha_j h_j \leq u_j$. By Proposition 2.4,

$$(1-\alpha_j)P_K[u]_{\mathcal{I}}(v) + \alpha_j P_K[h_j]_{\mathcal{I}}(v) \leq P_K[(1-\alpha_j)u + \alpha_j h_j]_{\mathcal{I}}(v) \leq P_K[u_j]_{\mathcal{I}}(v).$$

Since $\{u_j\}_j$ is increasing, so is $\{P_K[u_j]_{\mathcal{I}}(v)\}_j$, hence $w := \operatorname{usc}\lim_j P_K[u_j]_{\mathcal{I}}(v) \leq P_K[u]_{\mathcal{I}}(v)$ exists. Since $\alpha_j \to 0$ and $\sup_X P_K[h_j]_{\mathcal{I}}(v)$ is bounded, we can let $j \to \infty$ in the above estimate to conclude that $w = P_K[u]_{\mathcal{I}}(v)$. The same proof yields that $P_K[u_j](v) \nearrow P_K[u](v)$ a.e. □

2.3 An approximation result of Demailly

Let X be a compact Kähler manifold of dimension n. Let θ be a smooth representative of a pseudoeffective $(1,1)$–class on X. Let ω be a Kähler form on X.

Following the terminology of Cao [2014, Defintion 2.3], we recall the existence of quasi-equisingular approximation for potentials in $\mathrm{PSH}(X,\theta)$. As elaborated below, this result is implicit in the proof of [Demailly et al. 2001, Theorem 2.2.1; Demailly and Paun 2004, Theorem 3.2; Demailly 2015, Theorem 1.6].

Theorem 2.8 *Let $u \in \mathrm{PSH}(X,\theta)$. Then there exists $u^D_k \in \mathrm{PSH}(X,\theta+\varepsilon_k\omega)$ with $\varepsilon_k \searrow 0$ such that*

(i) $u^D_k \searrow u$,

(ii) $[u^D_k] \in \mathcal{A}(X,\theta+\varepsilon_k\omega)$,

(iii) $\mathcal{I}\big((s2^k/(2^k-s))u^D_k\big) \subseteq \mathcal{I}(su) \subseteq \mathcal{I}(su^D_k)$ *for all* $s>0$.

Proof Parts (i) and (ii) follow from [Demailly 2015, Theorem 1.6]. The second inclusion of (iii) follows from $u \leq u^D_k$, whereas the first inclusion of (iii) follows from [Demailly 2015, Corollary 1.12]. □

As shown in [Demailly 2012, page 135, formula (13.14)] (or [Demailly and Paun 2004, Theorem 3.2(iv)]), for each u^D_k in the above theorem, there exists a holomorphic modification $\pi_k : Y_k \to X$, a smooth closed $(1,1)$–form β_k, and a $\mathbb{Q}$–divisor D_k with snc singularities on Y such that

$$\theta_{u^D_k} = [D_k] + \beta_k. \tag{11}$$

In particular, $u^D_k \circ \pi_k$ has neat analytic singularity type; recall (8).

When the pseudoeffective class is induced by a line bundle, we have a related approximation result:

Remark 2.9 *In the case that $(L,h)\to X$ is a Hermitian line bundle with $c_1(L,h)=\{\theta\}$, $(T,h_T)\to X$ is an arbitrary Hermitian line bundle, and θ_u is a Kähler current with $[u]\in\mathcal{A}(X,\theta)$, it is possible to work with the following alternative approximating sequence*:

$$\tilde{u}_k^D=\frac{1}{k}\log\sup_{\substack{s\in H^0(X,L^k\otimes T)\\ \int_X h^k\otimes h_T(s,s)e^{-ku}\omega^n\le 1}} h^k\otimes h_T(s,s). \tag{12}$$

For k big enough, this sequence will satisfy $\tilde{u}_k^D+C\log k/k\ge u$, by the Ohsawa–Takegoshi theorem. However, it is not monotone in general. On the other hand, a stronger form of condition (iii) *will hold in this case, namely $[u]\preceq[\tilde{u}_k^D]\preceq[\alpha_k u]$ for some $\alpha_k\nearrow 1$.*

Proof This is a known consequence of the Briancon–Skoda theorem [Demailly 2012], but as a courtesy to the reader we give a brief argument for the estimate $[\tilde{u}_k^D]\le[\alpha_k u]$, the only part that needs to be proved. As we point out now, this actually follows from the arguments of [Demailly 2012, Remark 5.9].

Let $\mathcal{J}$ be the coherent sheaf of holomorphic functions g satisfying $|g|\le De^{u/2c}$ with $c\in\mathbb{Q}^+$, as in (8), and $D>0$ some positive constant. As pointed out in [Demailly 2012, Remark 5.9], we may assume that the f_j in (8) are local generators of $\mathcal{J}$.

Let $\pi\colon Y\to X$ be a smooth modification such that $\pi^{-1}\mathcal{J}\cdot\mathcal{O}_Y=\mathcal{O}(-D)$, where $D=\sum_j\lambda_j D_j$ is a normal crossing divisor on Y. The existence of such π follows from Hironaka desingularization.

Now suppose that $s\in H^0(X,L^k\otimes T)$ satisfies $\int_X h^k\otimes h_T(s,s)e^{-ku}\omega^n\le 1$. By pulling back we obtain

$$\int_Y h^k\otimes h_T(s\circ\pi,s\circ\pi)e^{-ku\circ\pi}(\pi^*\omega)^n\le 1.$$

As $u\circ\pi\simeq c\sum_j\lambda_j\log g_j$ for some local generators g_j of $\mathcal{O}(-D_j)$, by Fubini's theorem $h^k\otimes h_T(s\circ\pi,s\circ\pi)$ vanishes to order at least $\lfloor kc\lambda_j\rfloor+d$ along D_j, where d is an absolute constant, only dependent on π. In particular, one can find $\alpha_k\nearrow 1$ such that $h^k\otimes h_T(s\circ\pi,s\circ\pi)$ vanishes to order at least $c\alpha_k k\lambda_j$ along D_j. Since $u\circ\pi\simeq c\sum_j\lambda_j\log g_j$, we obtain that $[(1/k)\log h^k\otimes h_T(s\circ\pi,s\circ\pi)]\preceq[\alpha_k u\circ\pi]$, which in turn gives $[\tilde{u}_k^D\circ\pi]\preceq[\alpha_k u\circ\pi]$, since $H^0(X,L^k\otimes T\otimes\mathcal{J}(ku))$ is finite-dimensional. By pushing forward, we obtain that $[\tilde{u}_k^D]\preceq[\alpha_k u]$, as desired. □

3 The closure of analytic singularity types in $\mathcal{S}(X,\theta)$

In this section we only assume that θ is a smooth representative of a big $(1,1)$–cohomology class on X. Our goal is to prove that the $d_{\mathcal{S}}$–closure of $\mathcal{A}(X,\theta)$ is the space of $\mathcal{J}$–model singularity types, in the presence of positive mass. We start with an elementary lemma:

Lemma 3.1 *Let $\pi\colon X'\to X$ be a smooth modification and $u\in\mathrm{PSH}(X,\theta)$. Then we have*

$$\pi^*P^\theta[u]_{\mathcal{J}}=P^{\pi^*\theta}[\pi^*u]_{\mathcal{J}}.$$

Proof Recall that

$$P^\theta[u]_{\mathcal{I}} = \sup\{v \in \mathrm{PSH}(X,\theta) : v \le 0, [v] \preceq_{\mathcal{I}} [u]\}. \tag{13}$$

Let $v \in \mathrm{PSH}(X,\theta)$ be a candidate of the sup in (13). Then by Proposition 2.2, for any smooth modification $p\colon Y \to X$ and any $y \in Y$, $\nu(p^*v, y) \ge \nu(p^*u, y)$. In particular, for any smooth modification $q\colon Z \to X'$ and any $z \in Z$, we have $\nu(q^*\pi^*v, z) \ge \nu(q^*\pi^*u, z)$. By Proposition 2.2 again, $[\pi^*v] \preceq_{\mathcal{I}} [\pi^*u]$. In particular, $\pi^*v \le P^{\pi^*\theta}[\pi^*u]_{\mathcal{I}}$. We arrive at the inequality

$$\pi^*(P^\theta[u]_{\mathcal{I}}) \le P^{\pi^*\theta}[\pi^*u]_{\mathcal{I}}.$$

It remains to prove the reverse inequality. There is a unique $h \in \mathrm{PSH}(X,\theta)$ such that $\pi^*h = P^\theta[\pi^*u]_{\mathcal{I}}$. We need to prove that $P^\theta[u]_{\mathcal{I}} \ge h$. It suffices to prove the following claim: for any $k > 0$, $\mathcal{I}(ku) \supseteq \mathcal{I}(kh)$. But we already know that $\mathcal{I}(k\pi^*u) = \mathcal{I}(k\pi^*h)$, while by [Demailly 2012, Proposition 5.8],

$$\mathcal{I}(ku) = \pi_*(K_{X'/X} \otimes \mathcal{I}(k\pi^*u)) \quad \text{and} \quad \mathcal{I}(kh) = \pi_*(K_{X'/X} \otimes \mathcal{I}(k\pi^*h)).$$

Hence, we conclude that $\mathcal{I}(ku) = \mathcal{I}(kh)$. □

Lemma 3.2 *If $u \in \mathrm{PSH}(X,\theta)$ satisfies $[u] \in \mathcal{A}(X,\theta)$, then $[u] = [P[u]] = [P[u]_{\mathcal{I}}]$.*

Proof Since $u \sim_{\mathcal{I}} P[u]_{\mathcal{I}}$, we get $[u] = [P[u]_{\mathcal{I}}]$ from [Kim 2015, Theorem 4.3]. Since $[u] \preceq [P[u]] \preceq [P[u]_{\mathcal{I}}]$, it also follows that $[u] = [P[u]]$. □

Proposition 3.3 *Let $u \in \mathrm{PSH}(X,\theta)$. Then $P^{\theta+\varepsilon_j\omega}[u_j^D]_{\mathcal{I}} \searrow P^\theta[u]_{\mathcal{I}}$ as $j \to \infty$, where the sequence $u_j^D \in \mathrm{PSH}(X, \theta + \varepsilon_j\omega)$ is the approximating sequence of Theorem 2.8. Moreover, if θ_u is a Kähler current, then $P^\theta[u_j^D]_{\mathcal{I}} \searrow P^\theta[u]_{\mathcal{I}}$ as $j \to \infty$.*

Proof We can suppose that $u \le 0$. Since $[u_j^D] \succeq [u]$ we have that $P^{\theta+\varepsilon_j\omega}[u_j^D]_{\mathcal{I}} \ge P^{\theta+\varepsilon_j\omega}[u]_{\mathcal{I}} \ge P^\theta[u]_{\mathcal{I}}$. Since $\{u_j^D\}_j$ is decreasing, we have that $v := \lim_j P^{\theta+\varepsilon_j\omega}[u_j^D]_{\mathcal{I}} \in \mathrm{PSH}(X,\theta)$ exists and $u \le v$.

Observe that $P^\theta[v]_{\mathcal{I}} = v$, since any candidate $h \in \mathrm{PSH}(X,\theta)$ for $P^\theta[v]_{\mathcal{I}}$ is also a candidate for each $P^{\theta+\varepsilon_j\omega}[u_j^D]_{\mathcal{I}}$. Hence, to finish the argument, it is enough to show that $\mathcal{I}(tu) = \mathcal{I}(tv)$ for all $t > 0$. By Theorem 2.8, for any $\delta > 1$ and $t > 0$ there exists $k_0(\delta, t) > 0$ such that for all $k \ge k_0$ we have $\mathcal{I}(t\delta v) \subseteq \mathcal{I}(t\delta u_k^D) \subseteq \mathcal{I}(tu)$. Letting $\delta \searrow 1$, the strong openness theorem of Guan and Zhou [2015] implies that $\mathcal{I}(tv) \subseteq \mathcal{I}(tu)$. Since the reverse inclusion is trivial, the proof of the first assertion is finished.

To prove the second assertion, assume that θ_u is a Kähler current. Hence, for j large enough, it holds that $u_j^D \in \mathrm{PSH}(X,\theta)$. On the other hand, observe that $P^{\theta+\varepsilon_j\omega}[u_j^D]_{\mathcal{I}} \ge P^\theta[u_j^D]_{\mathcal{I}} \ge P^\theta[u]_{\mathcal{I}}$, hence $P^\theta[u_j^D]_{\mathcal{I}} \searrow P^\theta[u]_{\mathcal{I}}$ as $j \to \infty$. □

We note the following important corollary of this result, which will be used numerous times in this work:

Corollary 3.4 *Let* $u \in \mathrm{PSH}(X,\theta)$. *Then*

$$\int_X (\theta+\varepsilon_j\omega)^n_{u_j^D} = \int_X (\theta+\varepsilon_j\omega)^n_{P^{\theta+\varepsilon_j\omega}[u_j^D]_{\mathcal{I}}} \searrow \int_X \theta^n_{P^\theta[u]_{\mathcal{I}}} \quad \text{as } j\to\infty,$$

where $u_j^D \in \mathrm{PSH}(X,\theta+\varepsilon_j\omega)$ *is the approximating sequence of Theorem 2.8.*

Proof The equality follows from Lemma 3.2 and [Witt Nyström 2019, Theorem 1.1].

Since $P^{\theta+\varepsilon_j\omega}[u_j^D]_{\mathcal{I}} \geq P^\theta[u]_{\mathcal{I}}$, we can start with the following inequality:

$$\varliminf_{j\to\infty} \int_X (\theta+\varepsilon_j\omega)^n_{P^{\theta+\varepsilon_j\omega}[u_j^D]_{\mathcal{I}}} \geq \varliminf_{j\to\infty} \int_X (\theta+\varepsilon_j\omega)^n_{P^\theta[u]_{\mathcal{I}}} = \int_X \theta^n_{P^\theta[u]_{\mathcal{I}}}.$$

To finish the proof, we will argue that $\varlimsup_j \int_X (\theta+\varepsilon_j\omega)^n_{P^{\theta+\varepsilon_j\omega}[u_j^D]_{\mathcal{I}}} \leq \int_X \theta^n_{P^\theta[u]_{\mathcal{I}}}$. Indeed, fixing $j_0\in\mathbb{N}$,

$$\begin{aligned}\varlimsup_{j\to\infty} \int_X (\theta+\varepsilon_j\omega)^n_{P^{\theta+\varepsilon_j\omega}[u_j^D]_{\mathcal{I}}} &= \varlimsup_{j\to\infty} \int_{\{P^{\theta+\varepsilon_j\omega}[u_j^D]_{\mathcal{I}}=0\}} (\theta+\varepsilon_j\omega)^n_{P^{\theta+\varepsilon_j\omega}[u_j^D]_{\mathcal{I}}}\\ &\leq \varlimsup_{j\to\infty} \int_{\{P^{\theta+\varepsilon_j\omega}[u_j^D]_{\mathcal{I}}=0\}} (\theta+\varepsilon_{j_0}\omega)^n_{P^{\theta+\varepsilon_j\omega}[u_j^D]_{\mathcal{I}}}\\ &\leq \int_{\{P^\theta[u]_{\mathcal{I}}=0\}} (\theta+\varepsilon_{j_0}\omega)^n_{P^\theta[u]_{\mathcal{I}}},\end{aligned}$$

where in the first line we have used that $P^{\theta+\varepsilon_j\omega}[u_j^D]_{\mathcal{I}} = P^{\theta+\varepsilon_j\omega}[u_j^D]$ from [Darvas and Xia 2022, Proposition 2.20] together with [Darvas et al. 2018, Theorem 3.8], and in the last line we have used Proposition 3.3 and [Darvas et al. 2021, Proposition 4.6]. Letting $j_0\to\infty$, we arrive at the desired conclusion:

$$\varlimsup_{j\to\infty} \int_X (\theta+\varepsilon_j\omega)^n_{u_j^D} \leq \lim_{j_0\to\infty} \int_{\{P^\theta[u]_{\mathcal{I}}=0\}} (\theta+\varepsilon_{j_0}\omega)^n_{P^\theta[u]_{\mathcal{I}}} = \int_{\{P^\theta[u]_{\mathcal{I}}=0\}} \theta^n_{P^\theta[u]_{\mathcal{I}}} \leq \int_X \theta^n_{P^\theta[u]_{\mathcal{I}}}. \qquad \square$$

Corollary 3.5 *Let* $\psi\in\mathrm{PSH}(X,\theta)$. *The following hold*:

(i) $\int_X (\theta+\varepsilon\omega+dd^c P^{\theta+\varepsilon\omega}[\psi]_{\mathcal{I}})^n \searrow \int_X \theta^n_{P^\theta[\psi]_{\mathcal{I}}}$ *as* $\varepsilon\searrow 0$.

(ii) *If* θ_ψ *is a Kähler current, then* $\int_X (\theta-\varepsilon\omega+dd^c P^{\theta-\varepsilon\omega}[\psi]_{\mathcal{I}})^n \nearrow \int_X \theta^n_{P^\theta[\psi]_{\mathcal{I}}}$ *as* $\varepsilon\searrow 0$.

Proof We approximate ψ with $\psi_j^D\in\mathrm{PSH}(X,\theta+\varepsilon_j\omega)$ from Theorem 2.8. For $\varepsilon>0$, applying Corollary 3.4 for $\psi\in\mathrm{PSH}(X,\theta+\varepsilon\omega)$ (for the same approximating sequence $\psi_j^D\in\mathrm{PSH}(X,\theta+(\varepsilon+\varepsilon_j)\omega)$ independent of ε) we get that

$$\int_X (\theta+\varepsilon\omega+dd^c P^{\theta+\varepsilon\omega}[\psi]_{\mathcal{I}})^n = \lim_{j\to\infty}\int_X (\theta+(\varepsilon+\varepsilon_j)\omega+dd^c\psi_j^D)^n,$$

$$\int_X (\theta+dd^c P^{\theta}[\psi]_{\mathcal{I}})^n = \lim_{j\to\infty}\int_X (\theta+\varepsilon_j\omega+dd^c\psi_j^D)^n.$$

Using the multilinearity of nonpluripolar products, (i) follows. The proof of (ii) follows the same pattern and is left to the reader. $\square$

Proposition 3.6 *Let $u \in \mathrm{PSH}(X,\theta)$ be such that $\int_X \theta_u^n > 0$. Then there exists $v \in \mathrm{PSH}(X,\theta)$ such that $u \geq v$ and $\theta_v \geq \delta\omega$ for some $\delta > 0$.*

Proof We may assume that $u \leq 0$. Since $u \leq V_\theta$ and $\int_X \theta_{V_\theta}^n \geq \int_X \theta_u^n > 0$, by Lemma 2.6 there exists $b > 0$ such that $h := P((1+b)u - bV_\theta) \in \mathrm{PSH}(X,\theta)$ and

$$\frac{b}{b+1}V_\theta + \frac{1}{b+1}h \leq u.$$

By [Boucksom 2002b], there exists $w \in \mathrm{PSH}(X,\theta)$ such that $w \leq 0$ and $\theta_w \geq \delta'\omega$ for some $\delta' > 0$. Since $w \leq V_\theta$, we obtain that

$$v := \frac{b}{b+1}w + \frac{1}{b+1}h \leq u$$

and $\theta_v \geq (b\delta'/(b+1))\omega$. □

Next we extend [Darvas and Xia 2022, Theorem 2.24] to big cohomology classes.

Lemma 3.7 *Let $u \in \mathrm{PSH}(X,\theta)$. Assume that θ_u is a Kähler current. Let u_k^D be the approximation sequence in Theorem 2.8. Then*

$$d_{\mathcal{S}}([u_k^D], P^\theta[u]_{\mathcal{I}}) \to 0 \quad \text{as } k \to \infty. \tag{14}$$

In particular,

$$\lim_{k\to\infty} \int_X \theta_{u_k^D}^n = \int_X \theta_{P^\theta[u]_{\mathcal{I}}}^n. \tag{15}$$

Proof First observe that $u_k^D \in \mathrm{PSH}(X,\theta)$ when k is large enough, so (14) indeed makes sense. The second assertion follows from the first and [Darvas et al. 2021, Lemma 3.7], so it suffices to prove the first. By Proposition 3.3, $P^\theta[u_k^D]_{\mathcal{I}}$ decreases to $P^\theta[u]_{\mathcal{I}}$ as $k \to \infty$.

Since the potentials $P^\theta[u_k^D]_{\mathcal{I}}$ are model [Darvas and Xia 2022, Proposition 2.18(i)], by [Darvas et al. 2021, Lemma 3.6, Proposition 4.8] we obtain that $d_{\mathcal{S}}([P^\theta[u]_{\mathcal{I}}],[P^\theta[u_k^D]]_{\mathcal{I}}) \to 0$ as $k \to \infty$. By Lemma 3.2 we conclude (14) . □

Theorem 3.8 *Suppose that $u \in \mathrm{PSH}(X,\theta)$ is such that $\int_X \theta_u^n > 0$. Then $[u] \in \overline{\mathcal{A}(X,\theta)}^{d_{\mathcal{S}}}$ if and only if $[P[u]] = [P[u]_{\mathcal{I}}]$. Additionally, if $[P[u]] = [P[u]_{\mathcal{I}}]$ and θ_u is a Kähler current, then the regularization sequence $\{[u_k^D]\}_k$ of Theorem 2.8 $d_{\mathcal{S}}$–converges to $[u]$.*

Here the notation $\overline{\mathcal{A}(X,\theta)}^{d_{\mathcal{S}}}$ means the closure of $\mathcal{A}(X,\theta)$ in $\mathcal{S}(X,\theta)$ with respect to the $d_{\mathcal{S}}$–metric.

Proof To begin, let $v \in \mathrm{PSH}(X,\theta)$ be such that $v \leq u$ and $\theta_v \geq \delta\omega$ for some $\delta > 0$. Such v exists by Proposition 3.6. Let $v_t := (1-t)v + tu$, with $t \in [0,1]$. Then θ_{v_t} is a Kähler current for $t \in [0,1)$ and $v_t \nearrow u$ a.e. as $t \nearrow 1$.

Assume first that $[P[u]_{\mathcal{I}}] = [P[u]]$. By replacing u with $P[u]_{\mathcal{I}}$, we can additionally assume that $u = P[u]_{\mathcal{I}}$. By [Darvas and Xia 2022, Lemma 2.21(iii)] we obtain that $P[v_t]_{\mathcal{I}} \nearrow P[u]_{\mathcal{I}} = u$ a.e. as $t \to 1$. In particular, by [Darvas et al. 2021, Lemma 4.1] we obtain that $d_{\mathcal{S}}(P[v_t]_{\mathcal{I}}, [u]) \to 0$ as $t \to 1$.

Let us fix $t \in [0,1)$. By the above, it is enough to argue that $[P[v_t]_{\mathcal{I}}] \in \overline{\mathcal{A}}^{d_{\mathcal{S}}}$. For this we apply the regularization method of Theorem 2.8 to v_t, obtaining $v^D_{t,k} \in \mathrm{PSH}(X,\theta)$ such that $[v^D_{t,k}] \in \mathcal{A}(X,\theta)$ (we used here that θ_{v_t} is a Kähler current). By Lemma 3.7, $d_{\mathcal{S}}([v^D_{t,k}],[v_t]) \to 0$ as $k \to \infty$. So $[P[v_t]_{\mathcal{I}}] \in \overline{\mathcal{A}}^{d_{\mathcal{S}}}$, and we conclude.

In the reverse direction, suppose there exists $[v_j] \in \mathcal{A}(X,\theta)$ such that $d_{\mathcal{S}}([v_j],[u]) \to 0$. By Lemma 3.2, we can assume that $v_j = P[v_j]_{\mathcal{I}} = P[v_j]$. In addition, we can assume that $u = P[u]$, since $d_{\mathcal{S}}(u, P[u]) = 0$ [Darvas et al. 2021, Theorem 3.3]. Since $\int_X \theta_u^n > 0$, after possibly restricting to a subsequence of v_j, we can use [Darvas et al. 2021, Theorem 5.6] to conclude existence of an increasing sequence of model potentials $\{w_j\} \in \mathrm{PSH}(X,\theta)$ such that $w_j \le v_j$ and $d_{\mathcal{S}}([w_j],[u]) \to 0$. As pointed out after the statement of [Darvas et al. 2021, Theorem 5.6], after possibly taking a subsequence of the v_j, we can take

$$w_j := \lim_{k\to\infty} P(v_j, v_{j+1}, \dots, v_{j+k}).$$

Since all the v_j are $\mathcal{I}$–model, an iterated application of Lemma 2.1 implies that so is

$$h_{j,k} := P(v_j, v_{j+1}, \dots, v_{j+k}).$$

Moreover, since w_j is the decreasing limit of the $h_{j,k}$, then w_j is $\mathcal{I}$–model too [Darvas and Xia 2022, Lemma 2.21(i)]. Lastly, since u is the increasing limit of the w_j, then u is $\mathcal{I}$–model as well [Darvas and Xia 2022, Lemma 2.21(iii)]. □

4 Proof of Theorem 1.1

Let X be a connected compact Kähler manifold of dimension n. For this section, let T be an arbitrary holomorphic vector bundle on X, with rank r.

4.1 The case of integral line bundles

Let L be a pseudoeffective line bundle on X. Let h be a smooth Hermitian metric on L such that $\theta := c_1(L,h)$. We fix a Kähler form ω on X such that $\omega - \theta$ is a Kähler form.

Proposition 4.1 *Suppose that $u \in \mathrm{PSH}(X,\theta)$. Then*

$$\varlimsup_{k\to\infty} \frac{1}{k^n} h^0(X, T \otimes L^k \otimes \mathcal{I}(ku)) \le \frac{r}{n!} \int_X \theta^n_{P[u]_{\mathcal{I}}}.$$

Proof Since $P[P[u]_{\mathcal{I}}]_{\mathcal{I}} = P[u]_{\mathcal{I}}$ and $\mathcal{I}(sP[u]_{\mathcal{I}}) = \mathcal{I}(su)$ for all $s > 0$ [Darvas and Xia 2022, Proposition 2.18(ii)], we can replace u with $P[u]_{\mathcal{I}}$ to assume that u is $\mathcal{I}$–model.

Next we apply the regularization method of Theorem 2.8 to u, obtaining $u^D_j \in \mathrm{PSH}(X, \theta + \varepsilon_j\omega)$ such that $[u^D_j] \in \mathcal{A}(X, \theta + \varepsilon_j\omega)$ and $u^D_j \searrow u$. Let $\pi_k : Y_k \to X$ be the smooth resolution of singularities of (11).

By [Demailly 2012, Proposition 5.8] and [Bonavero 1998, Théorème 2.1] applied to $q=0$ on Y_k (see also [Ma and Marinescu 2007, Theorem 2.3.18]), we obtain that

$$\begin{aligned}\varlimsup_{k\to\infty}\frac{1}{k^n}h^0(X,T\otimes L^k\otimes \mathcal{J}(ku)) &\le \varlimsup_{k\to\infty}\frac{1}{k^n}h^0(X,T\otimes L^k\otimes\mathcal{J}(ku_j^D))\\ &= \varlimsup_{k\to\infty}\frac{1}{k^n}h^0(Y,\pi_k^*T\otimes(\pi_k^*L)^k\otimes K_{Y/X}\otimes\mathcal{J}(ku_j^D\circ\pi_k))\\ &\le \frac{r}{n!}\int_{Y_k(0)}\pi_k^*\theta^n_{u_j^D}=\frac{r}{n!}\int_{\pi_k(Y_k(0))}\theta^n_{u_j^D}\\ &\le \frac{r}{n!}\int_{\pi_k(Y_k(0))}(\theta+\varepsilon_j\omega)^n_{u_j^D}\le\frac{r}{n!}\int_X(\theta+\varepsilon_j\omega)^n_{u_j^D},\end{aligned}$$

where $Y_k(0)\subseteq Y_k$ is the set contained in the smooth locus of the $(1,1)$–current $\pi_k^*\theta_{u_j^D}$ where the eigenvalues of $\pi_k^*\theta_{u_j^D}$ are all positive. By Corollary 3.4, $\lim_{j\to\infty}\int_X(\theta+\varepsilon_j\omega)^n_{u_j^D}=\int_X\theta_u^n$, finishing the argument. □

Lemma 4.2 *Let $u\in\mathrm{PSH}(X,\theta)$ such that θ_u is a Kähler current. Let $\beta\in(0,1)$. Then there exists $k_0:=k_0(u,\beta)$ such that for all $k\ge k_0$ there exists $v_{\beta,k}\in\mathrm{PSH}(X,\theta)$ satisfying*

(i) $P[u]_{\mathcal{J}}\ge(1-\beta)u_k^D+\beta v_{\beta,k}$,

(ii) $\int_X\theta^n_{v_{\beta,k}}>0$.

Proof Due to Lemma 3.7, we have that $\int_X\theta^n_{u_k^D}\searrow\int_X\theta^n_{P[u]_{\mathcal{J}}}$. In particular, there exists $k_0>0$ such that

$$\frac{1}{\beta^n}<\frac{\int_X\theta^n_{u_k^D}}{\int_X\theta^n_{u_k^D}-\int_X\theta^n_{P[u]_{\mathcal{J}}}}\quad\text{for all } k\ge k_0.$$

By Lemma 2.6 we obtain that

$$v_{k,\beta}:=P\Big(\frac{1}{\beta}P[u]_{\mathcal{J}}-\frac{1-\beta}{\beta}u_k^D\Big)\in\mathrm{PSH}(X,\theta)\quad\text{and}\quad P[u]_{\mathcal{J}}\ge(1-\beta)u_k^D+\beta v_{\beta,k}.$$

Now we show that $v_{\beta,k}$ has positive mass. Pick $\beta'\in(0,\beta)$ such that

$$\frac{1}{\beta'^n}<\frac{\int_X\theta^n_{u_k^D}}{\int_X\theta^n_{u_k^D}-\int_X\theta^n_{P[u]_{\mathcal{J}}}}\quad\text{for all } k\ge k_0.$$

Then

$$h:=P\bigg(\frac{1}{\beta'}P[u]_{\mathcal{J}}-\frac{1-\beta'}{\beta'}u_k^D\bigg)\in\mathrm{PSH}(X,\theta)$$

is defined as well, and $v_{k,\beta}\ge(\beta'/\beta)h+((\beta-\beta')/\beta)u_k^D\in\mathrm{PSH}(X,\theta)$, implying that

$$\int_X\theta^n_{v_{k,\beta}}\ge\frac{(\beta-\beta')^n}{\beta^n}\int_X\theta^n_{u_k^D}\ge\frac{(\beta-\beta')^n}{\beta^n}\int_X\theta_u^n>0,$$

where we applied [Witt Nyström 2019, Theorem 1.1] twice. □

Proposition 4.3 *Suppose that $u \in \mathrm{PSH}(X,\theta)$ with $\theta_u > \delta\omega$ for some $\delta > 0$. Then*

$$\varliminf_{j\to\infty} \frac{1}{j^n} h^0(X, T\otimes L^j \otimes \mathcal{I}(ju)) \geq \frac{r}{n!}\int_X \theta^n_{P[u]_{\mathcal{I}}}.$$

Proof To start, we fix a number $\beta = p/q \in (0,\min(\delta,1))\cap\mathbb{Q}$. It suffices to show that there is a constant $C>0$, only dependent on r, n and θ, such that

$$\varliminf_{j\to\infty} \frac{1}{j^n} h^0(X, T\otimes L^j \otimes \mathcal{I}(ju)) \geq \frac{r}{n!}\int_X \theta^n_{P[u]_{\mathcal{I}}} - C\beta.$$

Writing $j = aq+b$ for some $b = 0,\dots,q-1$, observe that

$$h^0(X, T\otimes L^j \otimes \mathcal{I}(ju)) \geq h^0\big(X, T\otimes L^{b-q}\otimes L^{(a+1)q}\otimes \mathcal{I}((a+1)qu)\big).$$

Absorbing L^{b-q} into T, and noticing that $b-q$ can only take a finite number of values, we find that it suffices to prove that

$$\varliminf_{j\to\infty} \frac{1}{j^n q^n} h^0(X, T\otimes L^{jq} \otimes \mathcal{I}(jqu)) \geq \frac{r}{n!}\int_X \theta^n_{P[u]_{\mathcal{I}}} - C\beta \tag{16}$$

for an arbitrary twisting bundle T.

By Lemma 4.2, there is $k_0 > 0$ depending on β and u such that for $k \geq k_0$, there exists a potential $v_{\beta,k} \in \mathrm{PSH}(X,\theta)$ of positive mass such that

$$P[u]_{\mathcal{I}} \geq w_{\beta,k} := (1-\beta)u^D_k + \beta v_{\beta,k} \quad \text{for all } k \geq k_0.$$

For big enough k_0 we also have $\theta_{u^D_k} > \beta\omega \geq \beta\theta$ for all $k \geq k_0$. In particular, $u^D_k \in \mathrm{PSH}(X,(1-\beta)\theta)$. We have $H^0(X, T\otimes L^{jq}\otimes \mathcal{I}(jqu)) \supseteq H^0(X, T\otimes L^{jq}\otimes \mathcal{I}(jqw_{\beta,k}))$, hence

$$h^0(X, T\otimes L^{jq}\otimes \mathcal{I}(jqu)) \geq h^0(X, T\otimes L^{jq}\otimes \mathcal{I}(jqw_{\beta,k})). \tag{17}$$

For each fixed $k>0$, we can take a resolution of singularities $\pi : Y \to X$ such that $\pi^* u^D_k$ has neat analytic singularities along a normal crossing $\mathbb{Q}$–divisor, as described in (11). By [Demailly 2012, Proposition 5.8] and the projection formula,

$$h^0(X, T\otimes L^{jq}\otimes \mathcal{I}(jqw_{\beta,k})) = h^0(Y, \pi^*T\otimes K_{Y/X}\otimes (\pi^*L)^{jq}\otimes \mathcal{I}(jq\pi^* w_{\beta,k})). \tag{18}$$

Since $\int_Y (\pi^*\theta + \mathrm{dd^c}\pi^* v_{\beta,k})^n = \int_X \theta^n_{v_{\beta,k}} > 0$, there exists a nonzero section

$$s_j \in H^0(Y, \pi^* L^{\beta j q}\otimes \mathcal{I}(\beta j q \pi^* v_{\beta,k})) = H^0(Y, \pi^* L^{jp}\otimes \mathcal{I}(jp\pi^* v_{\beta,k}))$$

for all j large enough, by Lemma 4.4. Hence applying Lemma 4.5 for $T := \pi^*T\otimes K_{Y/X}$, $E_1 = \pi^*L^{q-p}$, $E_2 = \pi^* L^p$, $\chi_1 := q\pi^* u^D_k$, $\chi_2 := p\pi^* v_{\beta,k}$, $s_j := s_j$ and $\varepsilon := \beta$, we find

$$\begin{aligned} &h^0(Y, \pi^*T\otimes K_{Y/X}\otimes \pi^*L^{jq}\otimes \mathcal{I}(jq\pi^* w_{k,\beta})) \\ &\quad = h^0\big(Y, \pi^*T\otimes K_{Y/X}\otimes \pi^*L^{(q-p)j}\otimes \pi^*L^{pj}\otimes \mathcal{I}((1-\beta)jq\pi^* u^D_k + jp\pi^* v_{\beta,k})\big) \\ &\quad \geq h^0(Y, \pi^*T\otimes K_{Y/X}\otimes \pi^*L^{(q-p)j}\otimes \mathcal{I}(jq\pi^* u^D_k)) \end{aligned} \tag{19}$$

for j large enough (depending on k).

Since $\theta_{u_k^D} > \beta\omega \geq \beta\theta$, we notice that $qu_k^D \in \mathrm{PSH}(X, \theta(q-p))$. Hence, by [Bonavero 1998, Théorème 2.1, Corollaire 2.2] (see also [Darvas and Xia 2022, Theorem 2.26]), we have the estimates

$$\begin{aligned}(20)\quad \varliminf_{j\to\infty} \frac{1}{j^n q^n} h^0(Y, \pi^*T \otimes K_{Y/X} \otimes \pi^* L^{(1-\beta)qj} \otimes \mathcal{I}(jq\pi^* u_k^D))& \\ = \varliminf_{j\to\infty} \frac{1}{j^n q^n} h^0(Y, \pi^*T \otimes K_{Y/X} \otimes \pi^* L^{(q-p)j} \otimes \mathcal{I}(jq\pi^* u_k^D))& \\ = \frac{r}{q^n n!}\int_Y ((q-p)\pi^*\theta + q\,\mathrm{dd^c}\pi^* u_k^D)^n = \frac{r}{n!}\int_X ((1-\beta)\theta + \mathrm{dd^c} u_k^D)^n& \\ \geq \frac{r}{n!}\int_X \theta^n_{u_k^D} - C\beta,&\end{aligned}$$

where $C > 0$ depends only on r, n and θ. Putting together (17)–(20), we obtain

$$\varliminf_{j\to\infty} \frac{1}{j^n} h^0(X, T \otimes L^j \otimes \mathcal{I}(ju)) \geq \frac{r}{n!}\int_X \theta^n_{u_k^D} - C\beta.$$

Letting $k \to \infty$ and applying Lemma 3.7, we conclude (16). □

Lemma 4.4 *Suppose that $L \to X$ is a big line bundle, with smooth Hermitian metric h. Let $\theta = c_1(L, h)$. Let $v \in \mathrm{PSH}(X, \theta)$ with $\int_X \theta_v^n > 0$. Then for m big enough there exists an $s \in H^0(X, L^m \otimes \mathcal{I}(mv))$ which is nonvanishing.*

Proof By Proposition 3.6 there exists $w \in \mathrm{PSH}(X, \theta)$ such that $w \leq v$ and $\theta_w \geq \delta\omega$. By [Demailly 2012, Theorem 13.21], for m big enough there exists an $s \in H^0(X, L^m \otimes \mathcal{I}(mw))$ which is nonzero. Since $w \leq v$, we get that $s \in H^0(X, L^m \otimes \mathcal{I}(mv))$. □

Lemma 4.5 *Suppose that E_1, E_2 and T are vector bundles over a connected complex manifold Y, with* rank $E_2 = 1$, *and that χ_1 and χ_2 are quasi-psh functions on Y, with χ_1 having normal crossing divisorial singularity type. Suppose that there exists a nonzero section $s_j \in H^0(Y, E_2^{\otimes j} \otimes \mathcal{I}(j\chi_2))$ for all j big enough. Then for any $\varepsilon \in (0,1)$, the map $w \mapsto w \otimes s_j$ between the vector spaces*

$$H^0(Y, T \otimes E_1^{\otimes j} \otimes \mathcal{I}(j\chi_1)) \to H^0\big(Y, T \otimes E_1^{\otimes j} \otimes E_2^{\otimes j} \otimes \mathcal{I}(j(1-\varepsilon)\chi_1 + j\chi_2)\big)$$

is well defined and injective for all j big enough.

Proof Suppose that the singularity type of χ_1 is given by the effective normal crossing $\mathbb{R}$–divisor $\sum_j \alpha_j D_j$ with $\alpha_j > 0$. By [Demailly 2012, Remark 5.9] we have that

$$\mathcal{I}(j\chi_1) = \mathcal{O}_Y\Big(-\sum_m \lfloor \alpha_m j \rfloor D_j\Big).$$

We obtain that $we^{-j(1-\varepsilon)\chi_1}$ is bounded for any $w \in H^0(Y, T \otimes E_1^j \otimes \mathcal{I}(j\chi_1))$ and j big enough. Since $s_j \in H^0(Y, E_2^{\otimes j} \otimes \mathcal{I}(j\chi_2))$, we obtain that

$$w \otimes s_j \in H^0\big(Y, T \otimes E_1^{\otimes j} \otimes E_2^{\otimes j} \otimes \mathcal{I}(j(1-\varepsilon)\chi_1 + j\chi_2)\big).$$

Injectivity of $w \mapsto w \otimes s_j$ follows from the identity theorem of holomorphic functions. □

Theorem 4.6 *Suppose that $u \in \mathrm{PSH}(X,\theta)$. Then*

$$\lim_{k\to\infty} \frac{1}{k^n} h^0(X, T \otimes L^k \otimes \mathcal{I}(ku)) = \frac{r}{n!} \int_X \theta^n_{P[u]_{\mathcal{I}}}. \tag{21}$$

Proof Since both sides of (21) only depend on $P[u]_{\mathcal{I}}$, we can assume that $P[u]_{\mathcal{I}} = u$. Proposition 4.1 implies (21) for $\int_X \theta_u^n = 0$, so we can also assume that $\int_X \theta_u^n > 0$. In particular, L is a big line bundle and X is projective. By Proposition 3.6, there exists $v \le u$ such that θ_v is a Kähler current. Let $v_t := (1-t)v + tu$. Then θ_{v_t} is a Kähler current for $t \in [0,1)$, so we can apply Proposition 4.3 to obtain that

$$\varliminf_{k\to\infty} \frac{1}{k^n} h^0(X, T \otimes L^k \otimes \mathcal{I}(ku)) \ge \varliminf_{k\to\infty} \frac{1}{k^n} h^0(X, T \otimes L^k \otimes \mathcal{I}(kv_t)) \ge \frac{r}{n!} \int_X \theta^n_{P[v_t]_{\mathcal{I}}}.$$

Letting $t \to 0$ and using [Darvas and Xia 2022, Lemma 2.21(iii)], we obtain that

$$\varliminf_{k\to\infty} \frac{1}{k^n} h^0(X, T \otimes L^k \otimes \mathcal{I}(ku)) \ge \frac{r}{n!} \int_X \theta^n_{P[u]_{\mathcal{I}}}.$$

The reverse inequality follows from Proposition 4.1. □

4.2 The case of $\mathbb{R}$–line bundles

In this subsection we extend Theorem 4.6 to $\mathbb{R}$–line bundles. First we deal with the case of $\mathbb{Q}$–line bundles.

Corollary 4.7 *Let L be a pseudoeffective $\mathbb{Q}$–line bundle on X, represented by an effective $\mathbb{Q}$–divisor D. Let θ be a smooth form representative of $c_1(L)$. Let $u \in \mathrm{PSH}(X,\theta)$. Then*

$$\lim_{k\to\infty} \frac{1}{k^n} h^0(X, T \otimes \mathcal{O}_X(\lfloor kD \rfloor) \otimes \mathcal{I}(ku)) = \frac{r}{n!} \int_X \theta^n_{P[u]_{\mathcal{I}}}. \tag{22}$$

Proof We may assume that $D' := aD$ is a line bundle L' for some $a \in \mathbb{N}$. For each $k \in \mathbb{N}$, write

$$k = k_0 a + k', \quad \text{where } k_0 \in \mathbb{N},\ k' \in [0, a-1).$$

Note that the difference $\lfloor kD \rfloor - k_0 D'$ can represent only a finite number of different line bundles. Hence, in order to prove (22), it suffices to establish the following: for each fixed $k' \in [0, a-1)$,

$$\lim_{k_0\to\infty} \frac{1}{k_0^n a^n} h^0(X, T \otimes L'^{k_0} \otimes \mathcal{I}(k_0 au + k'u)) = \frac{r}{n!} \int_X \theta^n_{P[u]_{\mathcal{I}}}.$$

Observe that $\mathcal{I}(k_0 au + k'u) \subseteq \mathcal{I}(k_0 au)$, so by Theorem 4.6 we have

$$\varlimsup_{k_0\to\infty} \frac{1}{k_0^n a^n} h^0(X, T \otimes L'^{k_0} \otimes \mathcal{I}(k_0 au + k'u)) \le \frac{r}{n!} \int_X \theta^n_{P[u]_{\mathcal{I}}}.$$

On the other hand, as $\mathcal{I}(k_0 au + k'u) \supseteq \mathcal{I}((k_0+1)au)$, we get

$$\begin{aligned}\varliminf_{k_0\to\infty} \frac{1}{k_0^n a^n} h^0(X, T\otimes L'^{k_0}\otimes \mathcal{I}(k_0au+k'u)) &\geq \varliminf \frac{1}{k_0^n a^n} h^0\big(X, T\otimes L'^{k_0}\otimes \mathcal{I}((k_0+1)au)\big)\\ &= \varliminf_{k_0\to\infty} \frac{1}{((k_0+1)a)^n} h^0\big(X, T\otimes L'^{*}\otimes L'^{k_0+1}\otimes \mathcal{I}((k_0+1)au)\big)\\ &= \frac{r}{n!}\int_X \theta^n_{P[u]_{\mathcal{I}}},\end{aligned}$$

where in the last step we again used Theorem 4.6, finishing the proof. □

Corollary 4.8 *Assume that X is projective. Let D be a big $\mathbb{R}$–divisor on X. Let θ be a smooth form representing the cohomology class $[D]$. Let $u \in \mathrm{PSH}(X,\theta)$. Then*

$$\lim_{k\to\infty} \frac{1}{k^n} h^0(X, T\otimes \mathcal{O}_X(\lfloor kD\rfloor)\otimes \mathcal{I}(ku)) = \frac{r}{n!}\int_X \theta^n_{P[u]_{\mathcal{I}}}. \tag{23}$$

Proof We first deal with the $\leq$ direction in (23). Fix $\delta > 0$. Fix $\varepsilon > 0$, so that

$$\int_X (\theta+\varepsilon\omega+\mathrm{dd^c} P^{\theta+\varepsilon\omega}[u]_{\mathcal{I}})^n < \int_X \theta^n_{P[u]_{\mathcal{I}}} + \delta.$$

This is possible by Corollary 3.5(i). Take a $\mathbb{Q}$–divisor D^δ such that the cohomology class $\{D^\delta - D\}$ has a smooth positive representative $\theta^\delta \leq \varepsilon\omega$. As a result, $D^\delta - D$ is ample. This is possible as X is projective. We have $u \in \mathrm{PSH}(X, \theta+\theta^\delta)$. Then $\mathcal{O}_X(\lfloor kD^\delta\rfloor - \lfloor kD\rfloor)$ has a nonzero global section s for k big enough. As a result, the map $H^0(X, T\otimes\mathcal{O}_X(\lfloor kD\rfloor)\otimes\mathcal{I}(ku)) \to H^0(X, T\otimes\mathcal{O}_X(\lfloor kD^\delta\rfloor)\otimes\mathcal{I}(ku))$ given by $s'\mapsto s'\otimes s$ is injective, allowing us to write the estimates

$$\begin{aligned}\varlimsup_{k\to\infty}\frac{1}{k^n}h^0(X, T\otimes\mathcal{O}_X(\lfloor kD\rfloor)\otimes\mathcal{I}(ku)) &\leq \varlimsup_{k\to\infty}\frac{1}{k^n}h^0(X, T\otimes\mathcal{O}_X(\lfloor kD^\delta\rfloor)\otimes\mathcal{I}(ku))\\ &= \frac{r}{n!}\int_X (\theta+\theta^\delta+\mathrm{dd^c}P^{\theta+\theta^\delta}[u]_{\mathcal{I}})^n\\ &\leq \frac{r}{n!}\int_X (\theta+\varepsilon\omega+\mathrm{dd^c}P^{\theta+\varepsilon\omega}[u]_{\mathcal{I}})^n\\ &\leq \frac{r}{n!}\int_X \theta^n_{P[u]_{\mathcal{I}}} + \frac{r\delta}{n!},\end{aligned}$$

where in the second line we have used Theorem 4.6. Letting $\delta\to 0+$, we conclude the $\leq$ direction in (23).

For the reverse direction, we can replace u by $P[u]_{\mathcal{I}}$, as in the proof of Theorem 4.6. Hence, we can assume that u is $\mathcal{I}$–model. If $\int_X \theta_u^n = 0$, we are done by the previous arguments, so we can assume that $\int_X \theta_u^n > 0$.

We first treat the case where $\theta_u > \varepsilon_0\omega$ for some $\varepsilon_0 > 0$. Fix $\delta > 0$. Fix $\varepsilon \in (0,\varepsilon_0)$, so that

$$\int_X (\theta-\varepsilon\omega+\mathrm{dd^c}P^{\theta-\varepsilon\omega}[u]_{\mathcal{I}})^n > \int_X \theta^n_{P[u]_{\mathcal{I}}} - \delta.$$

This is possible by Corollary 3.5(ii). Take a $\mathbb{Q}$–divisor D^δ so that $\{D-D^\delta\}$ has a smooth positive representative $\theta^\delta \le \varepsilon\omega$. As a result, $D-D^\delta$ is ample. Then we have $u \in \mathrm{PSH}(X,\theta-\theta^\delta)$. As before, we have the estimates

$$\begin{aligned}\varliminf_{k\to\infty}\frac{1}{k^n}h^0(X,T\otimes\mathcal{O}_X(\lfloor kD\rfloor)\otimes\mathcal{I}(ku)) &\ge \varliminf_{k\to\infty}\frac{1}{k^n}h^0(X,T\otimes\mathcal{O}_X(\lfloor kD^\delta\rfloor)\otimes\mathcal{I}(ku))\\ &=\frac{r}{n!}\int_X(\theta-\theta^\delta+\mathrm{dd^c}P^{\theta-\theta^\delta}[u]_{\mathcal{I}})^n\\ &\ge\frac{r}{n!}\int_X(\theta-\varepsilon\omega+\mathrm{dd^c}P^{\theta-\varepsilon\omega}[u]_{\mathcal{I}})^n\\ &\ge\frac{r}{n!}\int_X\theta^n_{P[u]_{\mathcal{I}}}-\frac{r\delta}{n!},\end{aligned}$$

where in the second line we have used Theorem 4.6. Letting $\delta\to 0+$, we conclude (23) in this case.

Finally, we treat the general case. By Proposition 3.6, there exists $v\in\mathrm{PSH}(X,\theta)$, such that $v\le u$ and θ_v is a Kähler current. Set $u_t:=(1-t)u+tv$ for $t\in[0,1]$. For $t\in(0,1]$, θ_{u_t} is still a Kähler current. By the special case treated above, we get

$$\text{(24)}\qquad \begin{aligned}\varliminf_{k\to\infty}\frac{1}{k^n}h^0(X,T\otimes\mathcal{O}_X(\lfloor kD\rfloor)\otimes\mathcal{I}(ku)) &\ge \lim_{k\to\infty}\frac{1}{k^n}h^0(X,T\otimes\mathcal{O}_X(\lfloor kD\rfloor)\otimes\mathcal{I}(ku_t))\\ &=\frac{r}{n!}\int_X\theta^n_{P[u_t]_{\mathcal{I}}}\end{aligned}$$

for $t\in(0,1]$. As $t\searrow 0$ we have $u_t\nearrow u$, hence $P[u_t]_{\mathcal{I}}\nearrow P[u]_{\mathcal{I}}$ a.e. by Proposition 2.7(ii). By [Darvas et al. 2018, Theorem 2.3], $\int_X\theta^n_{P[u_t]_{\mathcal{I}}}\nearrow\int_X\theta^n_{P[u]_{\mathcal{I}}}$. Letting $t\searrow 0$ in (24), we find the desired inequality

$$\varliminf_{k\to\infty}\frac{1}{k^n}h^0(X,T\otimes\mathcal{O}_X(\lfloor kD\rfloor)\otimes\mathcal{I}(ku))\ge\frac{r}{n!}\int_X\theta^n_{P[u]_{\mathcal{I}}}.\qquad\square$$

5 Envelopes of singularity types with respect to compact sets

Let X be a connected compact Kähler manifold of dimension n. For this whole section, let $K\subseteq X$ be a closed nonpluripolar set. Let θ be a closed real $(1,1)$–form on X representing a pseudoeffective cohomology class. Let $u\in\mathrm{PSH}(X,\theta)$.

Let $v\colon K\to[-\infty,\infty)$ be a function. We introduce the following K–relative envelopes and their regularizations, refining the definitions from Section 2.1:

$$E^\theta_K[u]_{\mathcal{I}}(v):=\sup\{h\in\mathrm{PSH}(X,\theta):h|_K\le v\text{ and }[h]\preceq_{\mathcal{I}}[u]\},\quad P^\theta_K[u]_{\mathcal{I}}(v):=\mathrm{usc}(E^\theta_K[u]_{\mathcal{I}}(v)),$$
$$E^\theta_K[u](v):=\sup\{h\in\mathrm{PSH}(X,\theta):h|_K\le v\text{ and }[h]\preceq[u]\},\qquad P^\theta_K[u](v):=\mathrm{usc}(E^\theta_K[u](v)).$$

We omit θ and K from the notation when there is no risk of confusion. When v is bounded, neither of the above candidate sets are empty: one can always take $h=u-C$ for a large enough constant C.

We note the following maximum principles, that follow from the above definitions:

Lemma 5.1 *Let $v \in C^0(K)$. Let $h \in \mathrm{PSH}(X,\theta)$. Assume that $[h] \preceq [u]$, then*

$$\sup_K (h-v) = \sup_{X\setminus\{h=-\infty\}} (h - E_K[u](v)) = \sup_{X\setminus\{E_K[u](v)=-\infty\}} (h - E_K[u](v)). \tag{25}$$

Proof We prove the first equality first. We write $S = \{h=-\infty\}$.

By definition, $E_K[u](v)|_K \le v$, so

$$(h - E_K[u](v))|_{K\setminus S} \ge h|_{K\setminus S} - v|_{K\setminus S}.$$

This implies that $\sup_K (h-v) \le \sup_{X\setminus S}(h - E_K[u](v))$.

Conversely, observe that $\sup_K(h-v) > -\infty$ as K is nonpluripolar. Let $h' := h - \sup_K(h-v)$. Then h' is a candidate in the definition of $E_K[u](v)$, hence $h' \le E_K[u](v)$, namely

$$h - \sup_K(h-v) \le E_K[u](v),$$

the latter implies that $\sup_K(h-v) \ge \sup_{X\setminus S}(h - E_K[u](v))$, finishing the proof of the first identity.

We have $\{E_K[u](v) = -\infty\} \subseteq S$, and we notice that points in $S \setminus \{E_K[u](v) = -\infty\}$ do not contribute to the supremum of $h - E_K[u](v)$ on $X \setminus \{E_K[u](v) = -\infty\}$, hence the last equality of (25) also follows. □

Next we make the following observations about the singularity types of our envelopes:

Lemma 5.2 *For any $v \in C^0(K)$ we have $[P_K[u](v)] = [P[u]]$ and $[P_K[u]_{\mathcal{I}}(v)] = [P[u]_{\mathcal{I}}]$. In particular, if $[u] \in \mathcal{A}(X,\theta)$, then $P_K[u](v) = P_K[u]_{\mathcal{I}}(v)$.*

Proof Let $C>0$ such that $-C \le v \le C$. Then $P[u] - C \le P_K[u](v)$. Since K is nonpluripolar, for $h \in \mathrm{PSH}(X,\theta)$ the condition $h|_K \le v \le C$ implies that $h \le \widetilde{C}$ on X for some $\widetilde{C} := \widetilde{C}(C,K) > 0$; see [Guedj and Zeriahi 2007, Corollary 4.3]. This implies that $P_K[u](v) \le P[u] + \widetilde{C}$, giving $[P_K[u](v)] = [P[u]]$. The exact same argument applies in case of the $P[\,\cdot\,]_{\mathcal{I}}$ envelope as well. Finally, when $[u] \in \mathcal{A}(X,\theta)$, we have that $[u] = [P[u]_{\mathcal{I}}] = [P[u]]$ (Lemma 3.2). We claim that for $h \in \mathrm{PSH}(X,\theta)$, $[h] \preceq [u]$ if and only if $[h] \preceq_{\mathcal{I}} [u]$. This claim immediately gives $P_K[u](v) = P_K[u]_{\mathcal{I}}(v)$. The forward direction of the claim is trivial, so suppose $[h] \preceq_{\mathcal{I}} [u]$. We then have $P[h]_{\mathcal{I}} \le P[u]_{\mathcal{I}}$. This implies that $[h] \preceq [P[h]_{\mathcal{I}}] \preceq [P[u]_{\mathcal{I}}] = [u]$. □

Corollary 5.3 *Let $u \in \mathrm{PSH}(X,\theta)$ and $v \in C^0(X)$. Then $P_K[u]_{\mathcal{I}}(v) = P_K[P_K[u]_{\mathcal{I}}(v)]_{\mathcal{I}}(v)$.*

Proof By definition, the right-hand side is the usc regularization of

$$\sup\{h \in \mathrm{PSH}(X,\theta) : h|_K \le v, [h] \preceq_{\mathcal{I}} P_K[u]_{\mathcal{I}}(v)\}.$$

By Lemma 5.2 and [Darvas and Xia 2022, Proposition 2.18(ii)], this expression can be rewritten as

$$\sup\{h \in \mathrm{PSH}(X,\theta) : h|_K \le v, [h] \preceq_{\mathcal{I}} [u]\}.$$

The usc regularization of the latter expression is just $P_K[u]_{\mathcal{I}}(v)$. □

Lemma 5.4 *Let $u \in \mathrm{PSH}(X,\theta)$ be a potential with positive mass. Let $v \in C^0(K)$. Let $S \subseteq X$ be a pluripolar set. Let $h \in \mathrm{PSH}(X,\theta)$ satisfy $[h] \preceq [u]$. Assume that h has positive mass and $h|_{K\setminus S} \leq v|_{K\setminus S}$. Then $h \leq P_K[u](v)$.*

Proof By the global Josefson theorem [Guedj and Zeriahi 2005, Theorem 7.2], there is $\chi \in \mathrm{PSH}(X,\theta)$ such that $S \subseteq \{\chi = -\infty\}$. We claim that we can choose χ so that $\chi \leq h$. In fact, since $\int_X \theta_h^n > 0$, fixing some χ and $\varepsilon > 0$ small enough, we have

$$\int_X \theta^n_{\varepsilon\chi+(1-\varepsilon)V_\theta} + \int_X \theta_h^n > \int_X \theta^n_{V_\theta}.$$

Thus, by [Darvas et al. 2021, Lemma 5.1], we have $P(\varepsilon\chi+(1-\varepsilon)V_\theta, h) \in \mathrm{PSH}(X,\theta)$. Since we have $P(\varepsilon\chi+(1-\varepsilon)V_\theta, h) \leq \varepsilon\chi$, the claim is proved by replacing χ with $P(\varepsilon\chi+(1-\varepsilon)V_\theta, h)$.

Fix $\chi \leq h$ as above. For any $\delta \in (0,1)$, we have

$$(1-\delta)h|_K + \delta\chi|_K \leq v \quad \text{and} \quad [(1-\delta)h+\delta\chi] \preceq [u].$$

Hence, $(1-\delta)h+\delta\chi \leq P_K[u](v)$. Letting $\delta \searrow 0$, we conclude that $h \leq P_K[u](v)$. □

Corollary 5.5 *Let $u \in \mathrm{PSH}(X,\theta)$ be a potential with positive mass. Let $v \in C^0(K)$. Then*

$$P_K[u](v) = P_X[u](P_K[V_\theta](v)).$$

Proof It is clear that $P_K[u](v) \leq P_X[u](P_K[V_\theta](v))$. For the reverse direction, it suffices to prove that any $h \in \mathrm{PSH}(X,\theta)$ such that $[h] \preceq [u]$, $h \leq P_K[V_\theta](v)$ satisfies $h \leq P_K[u](v)$. As u has positive mass, we can assume that h has positive mass as well. Let $S = \{P_K[V_\theta](v) > E_K[V_\theta](v)\}$. By [Bedford and Taylor 1982, Theorem 7.1], S is a pluripolar set. Observe that $h|_{K\setminus S} \leq v|_{K\setminus S}$, hence by Lemma 5.4, $h \leq P_K[u](v)$ and we conclude. □

The next result motivates our terminology to call the measures $\theta^n_{P_K[u](v)}$ the *partial equilibrium measures* of our context.

Lemma 5.6 *Let $v \in C^0(K)$. Let $u \in \mathrm{PSH}(X,\theta)$. Then $\theta^n_{P_K[u](v)}$ does not charge $X \setminus K$. Moreover, $P_K[u](v)|_K = v$ a.e. with respect to $\theta^n_{P_K[u](v)}$. More precisely, we have*

$$\theta^n_{P_K[u](v)} \leq \mathbb{1}_{K\cap\{P_K[u](v)=P_K[V_\theta](v)=v\}}\, \theta^n_{P_K[V_\theta](v)}. \tag{26}$$

Proof First we address the case when $u = V_\theta$.

Let $S \subseteq X$ be a closed pluripolar set such that V_θ is locally bounded on $X \setminus S$.

For the first assertion, it suffices to show that $\theta^n_{P_K[V_\theta](v)}$ does not charge any open ball $B \Subset X \setminus (S \cup K)$.

By Choquet's lemma, we can take an increasing sequence $h_j \in \mathrm{PSH}(X,\theta)$ converging to $P_K[V_{\min}](v)$ a.e. and $h_j|_K \leq v$. By [Bedford and Taylor 1982, Proposition 9.1], we can find $w_j \in \mathrm{PSH}(X,\theta)$ such that $(\theta + \mathrm{dd^c} w_j|_B)^n = 0$ and w_j agrees with h_j outside B. Note that w_j is clearly increasing and $w_j \geq h_j$, along with $w_j|_K \leq v$. It follows that w_j converges to $P_K[V_\theta](v)$ as well. By continuity of the

Monge–Ampère operator along increasing bounded sequences [Darvas et al. 2018, Theorem 2.3], we find that $\theta^n_{P_K[V_\theta](v)}$ does not charge B, as desired.

For the second assertion, let $x \in (X \setminus S) \cap K$ be a point such that $P_K[V_\theta](v)(x) < v(x) - \varepsilon$ for some $\varepsilon > 0$. Let B be a ball centered at x, small enough so that θ has a local potential on B, allowing us to identify θ–psh functions with psh functions (on B). By shrinking B, we can further guarantee

(i) $\bar{B} \subseteq X \setminus S$,

(ii) $P_K[V_\theta](v)|_{\bar{B}} < v(x) - \varepsilon$,

(iii) $v|_{\bar{B}\cap K} > v(x) - \varepsilon$.

Construct the sequences h_j and w_j as above. On B, by choosing a local potential of θ, we may identify h_j and w_j with the corresponding psh functions in a neighborhood of $\bar{B}$. By 2. we have $w_j \le v(x) - \varepsilon$ on ∂B, hence by the comparison principle, $w_j|_B \le v(x) - \varepsilon$. By (3) we have $w_j|_{B\cap K} \le v|_{B\cap K}$. Thus, we conclude that $\theta^n_{P_K[V_\theta](v)}$ does not charge B, as in the previous paragraph.

For the general case, we can assume $\int_X \theta^n_u > 0$. Indeed, due to Lemma 5.2, we have $\int_X \theta^n_{P_K[u](v)} = \int_X \theta^n_u$, hence there is nothing to prove if $\int_X \theta^n_u = 0$. By Corollary 5.5, $P_K[u](v) = P_X[u](P_K[V_\theta](v))$. Now [Darvas et al. 2018, Theorem 3.8] gives

$$\theta^n_{P_K[u](v)} \le \mathbb{1}_{\{P_K[u](v)=P_K[V_\theta](v)\}}\theta^n_{P_K[V_\theta](v)} \le \mathbb{1}_{\{P_K[u](v)=v\}}\theta^n_{P_K[V_\theta](v)},$$

where in the last inequality we have used the first part of the argument. □

Corollary 5.7 *Let $v \in C^0(K)$. Let $u \in \mathrm{PSH}(X,\theta)$. Then $\theta^n_{P_K[u](v)}$ (resp. $\theta^n_{P_K[u]_\mathcal{I}(v)}$) does not charge $(X \setminus K) \cup \{P_K[u](v) < v\}$ (resp. $(X \setminus K) \cup \{P_K[u]_\mathcal{I}(v) < v\}$).*

Proof The first part of the corollary follows from Lemma 5.6. For the second part, we can assume that $\int_X \theta^n_{P_K[u]_\mathcal{I}(v)} > 0$, otherwise there is nothing to prove. By definition, we have $P_K[u]_\mathcal{I}(v) = P_K[P[u]_\mathcal{I}]_\mathcal{I}(v)$. Next we show that $P_K[P[u]_\mathcal{I}]_\mathcal{I}(v) = P_K[P[u]_\mathcal{I}](v)$. The inequality $P_K[P[u]_\mathcal{I}]_\mathcal{I}(v) \ge P_K[P[u]_\mathcal{I}](v)$ is trivial. By Lemma 5.2 we get that $[P_K[P[u]_\mathcal{I}]_\mathcal{I}(v)] = [P[u]_\mathcal{I}]$. Due to Choquet's lemma, we get that $P_K[P[u]_\mathcal{I}]_\mathcal{I}(v) \le v$ on $K \setminus S$, where S is pluripolar. As a result, due to the nonvanishing mass assumption, Lemma 5.4 allows to conclude that $P_K[P[u]_\mathcal{I}]_\mathcal{I}(v) \le P_K[P[u]_\mathcal{I}](v)$.

Since $P_K[P[u]_\mathcal{I}]_\mathcal{I}(v) = P_K[u]_\mathcal{I}(v)$, we get that $\theta^n_{P_K[u]_\mathcal{I}(v)}$ does not charge $(X \setminus K) \cup \{P_K[u]_\mathcal{I}(v) < v\}$, using the first part of the corollary. □

Proposition 5.8 *Let $u \in \mathrm{PSH}(X,\theta)$ be a potential with positive mass. Let $v \in C^0(K)$. Then*

$$P_K[u](v) = P_K[P[u]](v). \tag{27}$$

In particular, $P_K[u](v) = P_K[P_K[u](v)](v)$.

Proof It is obvious that $P_K[u](v) \le P_K[P[u]](v)$. We to prove the reverse inequality. As $P_K[u](v)$ and $P_K[P[u]](v)$ have the same singularity types (Lemma 5.2), by the domination principle [Darvas et al. 2018, Corollary 3.10], it suffices to show that $P_K[u](v) \ge P_K[P[u]](v)$ a.e. with respect to $\theta^n_{P_K[u](v)}$.

By (26), $P_K[u](v) = P_K[V_\theta](v) = v$ a.e. with respect to $\theta^n_{P_K[u](v)}$. Hence, $P_K[P[u]](v) = v$ a.e. with respect to $\theta^n_{P_K[u](v)}$. We conclude that $P_K[u](v) = P_K[P[u]](v)$.

Finally, that $P_K[u](v) = P_K[P_K[u](v)](v)$ follows from Lemma 5.2 and (27). □

Lemma 5.9 *Fix a Kähler form ω on X. For $v \in C^0(K)$ there exists an increasing bounded sequence $\{v_j^-\}_j$ in $C^\infty(X)$ and a decreasing bounded sequence $\{v_j^+\}_j$ in $C^\infty(X)$ such that for all $u \in \mathrm{PSH}(X,\theta)$ with $\int_X \theta_u^n > 0$, and $\delta \in [0,1]$, we have*

- (i) $P_X^{\theta+\delta\omega}[u](v_j^+) \searrow P_K^{\theta+\delta\omega}[u](v)$,
- (ii) $P_X^{\theta+\delta\omega}[u](v_j^-) \nearrow P_K^{\theta+\delta\omega}[u](v)$ *a.e.,*
- (iii) $\sup_X |v_j^-| \le C(\|v\|_{C^0(K)}, K, \theta+\omega)$ *and* $\sup_X |v_j^+| \le C(\|v\|_{C^0(K)}, K, \theta+\omega)$.

Proof We fix $\delta \in [0,1]$. First we prove the existence of $\{v_j^-\}_j$. Let

$$C_{K,v} := \sup\Bigl\{\sup_X w : w \in \mathrm{PSH}(X,\theta+\omega),\ w|_K \le v\Bigr\}.$$

Since K is nonpluripolar, we have that $C_{K,v} \in \mathbb{R}$. Now let $\tilde v : X \to \mathbb{R}$ so that $\tilde v|_K = v$ and $\tilde v|_{X\setminus K} = C_{K,v}+1$. Since $\tilde v$ is lsc, there exists an increasing and uniformly bounded sequence $\{v_j^-\}_j$ in $C^\infty(X)$ such that $v_j^- \nearrow \tilde v$.

Observe that $P_X^{\theta+\delta\omega}[u](v_j^-)$ is increasing in j, and that $P_X^{\theta+\delta\omega}[u](v_j^-) \le P_K^{\theta+\delta\omega}[u](v)$. To prove that $P_X^{\theta+\delta\omega}[u](v_j^-) \nearrow P_K^{\theta+\delta\omega}[u](v)$ a.e., let w be a candidate for $P_K^{\theta+\delta\omega}[u](v)$ such that $\sup_K(w-v) < 0$. Then, since w is usc and $w < \tilde v$, by Dini's lemma there exists j_0 such that $w < v_j^-$ for $j \ge j_0$, ie $w \le P_X^{\theta+\delta\omega}[u](v_j^-)$, proving existence of $\{v_j^-\}_j$.

Next we prove the existence of $\{v_j^+\}_j$. Since $h := \max(P_K^{\theta+\omega}[V_{\theta+\omega}](v), \inf_K v - 1)$ is usc, there exists a decreasing and uniformly bounded sequence $\{v_j^+\}_j$ in $C^\infty(X)$ such that $v_j^+ \searrow h$. Trivially, $\chi := \lim_{j\to\infty} P_X^{\theta+\delta\omega}[u](v_j^+) \ge P_K^{\theta+\delta\omega}[u](v)$. In particular, χ has positive mass, since it has the same singularity types as $P_K^{\theta+\delta\omega}[u](v)$ (Lemma 5.2). We introduce

$$S := \{E_K^{\theta+\omega}[V_{\theta+\omega}](v) < P_K^{\theta+\omega}[V_{\theta+\omega}](v)\}.$$

By [Bedford and Taylor 1982, Theorem 7.1], S is a pluripolar set. Observe that $P_X^{\theta+\delta\omega}[u](v_j^+) \le v_j^+$ for all j. Thus, $\chi \le h$. On the other hand, $h \le v$ on $K \setminus S$. So in particular, $\chi|_{K\setminus S} \le v|_{K\setminus S}$. By Lemma 5.2 we also have that $[\chi] = [P_K^{\theta+\delta\omega}[u](v)]$. Hence, by Lemma 5.4, $\chi \le P_K^{\theta+\delta\omega}[P_K^{\theta+\delta\omega}[u](v)](v) = P_K^{\theta+\delta\omega}[u](v)$, where we also used the last statement of Proposition 5.8. □

We recall the relative Monge–Ampère energy $I^\theta_{[u]} : \mathcal{E}^1(X,\theta;P[u]) \to \mathbb{R}$ from [Darvas et al. 2018]:

$$I^\theta_{[u]}(\varphi) := \frac{1}{n+1}\sum_{i=0}^n \int_X (\varphi - P[u])\,\theta^i_\varphi \wedge \theta^{n-i}_{P[u]}. \tag{28}$$

Using integration by parts (see [Xia 2019; Lu 2021, Theorem 1.2; Vu 2022, Theorem 2.6], and compare with [Boucksom et al. 2010, Theorem 1.14]), the argument of Berman and Boucksom [2010, Corollary 4.2] can be reproduced line by line to yield the following cocycle property: for $\varphi_1, \varphi_2 \in \mathcal{E}^1(X,\theta; P[u])$ such that $[\varphi_i] = [P[u]]$ for $i = 1, 2$, we have

$$I^\theta_{[u]}(\varphi_1) - I^\theta_{[u]}(\varphi_2) = \frac{1}{n+1}\sum_{i=0}^{n}\int_X (\varphi_1 - \varphi_2)\,\theta^i_{\varphi_1} \wedge \theta^{n-i}_{\varphi_2}. \tag{29}$$

Following Berman and Boucksom [2010], we define the *partial equilibrium energy* $\mathcal{I}^\theta_{[u],K}$ of $v \in C^0(K)$:

$$\mathcal{I}^\theta_{[u],K}(v) := I^\theta_{[u]}(P_K[u](v)). \tag{30}$$

Berman and Boucksom [2010] used the symbol $\mathcal{E}$ for the above quantity. Due to potential confusion with the notation for (relative) full mass classes (that also uses the symbol $\mathcal{E}$), we use the symbol $\mathcal{I}$ instead.

Next we extend [Berman and Boucksom 2010, Theorem B] using the arguments of [Darvas 2019, Proposition 4.32], itself reproducing ideas from [Lu and Nguyen 2015]:

Proposition 5.10 *Let $K \subseteq X$ be a closed nonpluripolar set, consider $v, f \in C^0(K)$ and $u \in \mathrm{PSH}(X,\theta)$ satisfying $\int_X \theta^n_u > 0$. Then $t \mapsto \mathcal{I}^\theta_{[u],K}(v + tf)$ for $t \in \mathbb{R}$ is differentiable and*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}^\theta_{[u],K}(v + tf) = \int_K f\,\theta^n_{P_K[u](v+tf)}. \tag{31}$$

In this work, we will only need this result in the case that $[u] \in \mathcal{A}(X,\theta)$.

Proof It suffices to prove (31) at $t = 0$, which is equivalent to

$$\lim_{t\to 0}\frac{I^\theta_{[u]}(P_K[u](v+tf)) - I^\theta_{[u]}(P_K[u](v))}{t} = \int_K f\,\theta^n_{P_K[u](v)}. \tag{32}$$

By switching f to $-f$, we may assume that $t > 0$ in the above limit.

By the comparison principle [Darvas et al. 2018, Proposition 3.5] and (29), we find that

$$\begin{aligned}\mathcal{I}^\theta_{[u],K}(v+tf) - \mathcal{I}^\theta_{[u],K}(v) &= \frac{1}{n+1}\sum_{i=0}^{n}\int_X \big(P_K[u](v+tf) - P_K[u](v)\big)\,\theta^i_{P_K[u](v+tf)} \wedge \theta^{n-i}_{P_K[u](v)} \\ &\le \int_X (P_K[u](v+tf) - P_K[u](v))\,\theta^n_{P_K[u](v)}.\end{aligned}$$

By Lemma 5.6,

$$\int_X (P_K[u](v+tf) - P_K[u](v))\,\theta^n_{P_K[u](v)} \le t\int_K f\,\theta^n_{P_K[u](v)}.$$

Thus, we get the inequality

$$\varlimsup_{t\to 0+}\frac{I^\theta_{[u]}(P_K[u](v+tf)) - I^\theta_{[u]}(P_K[u](v))}{t} \le \int_K f\,\theta^n_{P_K[u](v)}.$$

Similarly, we have

$$I^{\theta}_{[u]}(P_K[u](v+tf)) - I^{\theta}_{[u]}(P_K[u](v)) \geq \int_X (P_K[u](v+tf) - P_K[u](v))\, \theta^n_{P_K[u](v+tf)}$$
$$\geq t \int_K f\, \theta^n_{P_K[u](v+tf)}.$$

Together with the above, this implies (32). □

In the next lemma, we prove convergence results for the partial equilibrium energy:

Lemma 5.11 *Let $v \in C^0(K)$ and $u \in \mathrm{PSH}(X,\theta)$ with $\int_X \theta_u^n > 0$. Let v_j^- and v_j^+ be the sequences constructed in Lemma 5.9. Then*

$$\lim_{j\to\infty} \mathcal{I}^{\theta}_{[u],X}(v_j^-) = \mathcal{I}^{\theta}_{[u],K}(v) \quad \textit{and} \quad \lim_{j\to\infty} \mathcal{I}^{\theta}_{[u],X}(v_j^+) = \mathcal{I}^{\theta}_{[u],K}(v).$$

Proof This follows from Lemmas 5.2 and 5.9, and [Darvas et al. 2018, Theorem 2.3]. □

6 Quantization of partial equilibrium measures

In this section, we give a proof for Theorem 1.2. Throughout the section, $L \to X$ is a pseudoeffective line bundle and h is a Hermitian metric on L such that $\theta := c_1(L,h)$. Let $T \to X$ be a Hermitian line bundle on X with a smooth Hermitian metric h_T. We normalize the Kähler metric ω on X so that

$$\int_X \omega^n = 1.$$

6.1 Bernstein–Markov measures

Let $K \subseteq X$ be a closed nonpluripolar subset. Let v be a measurable function on K and let ν be a positive Borel probability measure on K. We introduce the following functions on $H^0(X, L^k \otimes T)$, with values possibly equaling ∞:

$$N^k_{v,\nu}(s) := \left(\int_K h^k \otimes h_T(s,s) e^{-kv}\, \mathrm{d}\nu\right)^{1/2} \quad \text{and} \quad N^k_{v,K}(s) := \sup_{K\setminus\{v=-\infty\}} (h^k \otimes h_T(s,s)e^{-kv})^{1/2}.$$

We start with the following elementary observation.

Lemma 6.1 *Let $v_1 \leq v_2$ be two measurable functions on X. Assume that $\{v_1 = -\infty\} = \{v_2 = -\infty\}$. Then for any $s \in H^0(X, L^k \otimes T)$ and any $k > 0$, we have*

$$N^k_{v_1,K}(s) \geq N^k_{v_2,K}(s).$$

If ν puts no mass on $\{v = -\infty\}$, then we always have

$$N^k_{v,\nu}(s) \leq N^k_{v,K}(s). \tag{33}$$

We recall terminology introduced in [Berman and Boucksom 2010], providing a natural context in which the converse of (33) holds, with subexponential growth. A *weighted subset* of X is a pair (K,v) consisting of a closed nonpluripolar subset $K \subseteq X$ and a function $v \in C^0(K)$.

Let (K, v) be a weighted subset of X. A positive Borel probability measure ν on K is *Bernstein–Markov* with respect to (K, v) if for each $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$ such that

$$N^k_{v,K}(s) \le C_\varepsilon e^{\varepsilon k} N^k_{v,\nu}(s) \tag{34}$$

for any $s \in H^0(X, L^k \otimes T)$ and any $k \in \mathbb{N}$. We write $\mathrm{BM}(K, v)$ for the set of Bernstein–Markov measures with respect to (K, v). As pointed out in [Berman et al. 2011], any volume form measure on X is Bernstein–Markov with respect to (X, v), with $v \in C^\infty(X)$.

Proposition 6.2 *Assume that (K, v) is a weighted subset of X. Then:*

(i) *$N^k_{v,K}$ is a norm on $H^0(X, L^k \otimes T)$.*

(ii) *For any $\nu \in \mathrm{BM}(K, v)$, $N^k_{v,\nu}$ is a norm on $H^0(X, L^k \otimes T)$.*

Proof (i) As v is bounded, $N^k_{v,K}$ is clearly finite on $H^0(X, L^k \otimes T)$. In order to show that it is a norm, it suffices to show that for any $s \in H^0(X, L^k \otimes T)$, $N^k_{v,K}(s) = 0$ implies that $s = 0$. In fact, we have $s|_K = 0$; hence $s = 0$ by the connectedness of X.

(ii) As v is bounded, clearly $N^k_{v,\nu}$ is finite and satisfies the triangle inequality. Nondegeneracy follows from the fact that $N^k_{v,K}$ is a norm and (34). □

6.2 Partial Bergman kernels

In this section, with the terminology and context of the previous section, we fix a weighted subset (K, v) of X and $\nu \in \mathrm{BM}(K, v)$. We introduce the associated partial Bergman kernels: for any $k \in \mathbb{N}$ and $x \in K$,

$$B^k_{v,u,\nu}(x) := \sup\{h^k \otimes h_T(s,s)e^{-kv}(x) : N^k_{v,\nu}(s,s) \le 1, s \in H^0(X, L^k \otimes T \otimes \mathcal{I}(ku))\}. \tag{35}$$

The associated partial Bergman measures on X are identically zero on $X \setminus K$, and on K are defined as

$$\beta^k_{v,u,\nu} := \frac{n!}{k^n} B^k_{v,u,\nu}\, \mathrm{d}\nu. \tag{36}$$

Observe that

$$\int_K \beta^k_{v,u,\nu} = \frac{n!}{k^n} h^0(X, L^k \otimes T \otimes \mathcal{I}(ku)). \tag{37}$$

Our aim is to show the following weak convergence result:

$$\beta^k_{v,u,\nu} \rightharpoonup \theta^n_{P_K[u]_{\mathcal{I}}(v)}, \quad \text{as } k \to \infty. \tag{38}$$

We focus momentarily on the case when $\mathrm{d}\nu = \omega^n$ and $K = X$. That (38) holds in this particular case follows from [Ross and Witt Nyström 2017, Theorem 1.4]. Relying on the recent paper of Di Nezza and Trapani [2021], we give here a short proof of this result, borrowing ideas from Berman [2009] as well.

Proposition 6.3 *Let $u \in \mathrm{PSH}(X, \theta)$ be such that θ_u is a Kähler current and $[u] \in \mathcal{A}(X, \theta)$. If $v \in C^\infty(X)$, then $\beta^k_{v,u,\omega^n} \rightharpoonup \theta^n_{P_X[u]_{\mathcal{I}}(v)} = \theta^n_{P_X[u](v)}$ as $k \to \infty$.*

Proof That $\theta^n_{P_X[u]_{\mathcal{I}}(v)} = \theta^n_{P_X[u](v)}$ follows from Lemma 5.2. We start with noticing that as $k \to \infty$,

$$\beta^k_{v,u,\omega^n} \le \beta^k_{v,V_\theta,\omega^n} \rightharpoonup \theta^n_{P_X[V_\theta](v)} = \mathbb{1}_{\{v = P_X[V_\theta](v)\}} \theta^n_v,$$

where the convergence follows from [Berman 2009, Theorem 1.2], and the last identity is due to [Di Nezza and Trapani 2021, Corollary 3.4]. Letting μ be the weak limit of a subsequence of β^k_{v,u,ω^n}, we obtain

$$\mu \le \lim_{k\to\infty} \beta^k_{v,V_\theta,\omega^n} = \mathbb{1}_{\{v = P_X[V_\theta](v)\}} \theta^n_v. \tag{39}$$

Let $s \in H^0(X, L^k \otimes T \otimes \mathcal{I}(ku))$ be a section such that $N^k_{v,\omega^n}(s,s) \le 1$. Then by [Berman 2009, Lemma 4.1], there exists $C > 0$ such that $h^k \otimes h_T(s,s)e^{-kv} \le B^k_{v,u,\omega^n} \le B^k_{v,V_\theta,\omega^n} \le k^n C$. This implies

$$\frac{1}{k} \log h^k \otimes h_T(s,s) \le v + \frac{\log C}{k} + n\frac{\log k}{k}.$$

However, we also have that $[(1/k)\log h^k \otimes h_T(s,s)] \le [\tilde{u}^D_k] \le \alpha_k[u]$, where $\tilde{u}^D_k$ is as defined in Remark 2.9, and $\alpha_k \in (0,1)$ is also from the notation of Remark 2.9. Let $\varepsilon > 0$. Observe that for all $k \ge k_0(\varepsilon)$, we have $(1/k)\log h^k \otimes h_T(s,s) \in \mathrm{PSH}(X, \theta + \varepsilon\omega)$. In particular,

$$\frac{1}{k} \log h^k \otimes h_T(s,s) - \frac{\log C}{k} - n\frac{\log k}{k} \le P^{\theta+\varepsilon\omega}_X[\alpha_k u](v).$$

Now, taking the supremum over all candidates s, we obtain that

$$B^k_{v,u,\omega^n} \le Ck^n e^{k(P^{\theta+\varepsilon\omega}_X[\alpha_k u](v) - v)} \quad \text{for } k \ge k_0. \tag{40}$$

We claim that μ does not put mass on $\{P^{\theta+\varepsilon\omega}_X[u](v) < v\}$ for any $\varepsilon > 0$. Since by Proposition 2.7 $P^{\theta+\varepsilon\omega}_X[\alpha_k u](v) \searrow P^{\theta+\varepsilon\omega}_X[u](v)$, we get that $\{P^{\theta+\varepsilon\omega}_X[\alpha_k u](v) < v\} \nearrow \{P^{\theta+\varepsilon\omega}_X[u](v) < v\}$. As a result, to argue the claim, it suffices to show that μ does not put mass on the set $\{P^{\theta+\varepsilon\omega}_X[\alpha_k u](v) < v\}$ for any k. Note that the latter set is open, hence (40) implies our claim.

Since $u \in \mathcal{A}(X,\theta)$, we have that $P^{\theta+\varepsilon\omega}_X[u](v) = [u]$ for all $\varepsilon \ge 0$ by Lemma 5.2. As a result, $P^{\theta+\varepsilon\omega}_X[u](v) \searrow P^\theta_X[u](v)$. We can let $\varepsilon \searrow 0$ to conclude that μ does not put mass on $\{P_X[u](v) < v\} = \bigcup_{\varepsilon>0}\{P^{\theta+\varepsilon\omega}_X[u](v) < v\}$. Putting this together with (39), we obtain that

$$\mu \le \mathbb{1}_{\{P_X[u](v) = v\}} \theta^n_v = \theta^n_{P_X[u](v)},$$

where the last equality is due to [Di Nezza and Trapani 2021, Corollary 3.4]. Comparing total masses (via (37) and Theorem 1.1), we conclude that $\mu = \theta^n_{P_X[u](v)}$. As μ is an arbitrary limit point of β^k_{v,u,ω^n}, we conclude that β^k_{v,u,ω^n} converges weakly to $\theta^n_{P_X[u](v)}$ as $k \to \infty$. □

Let $\mathrm{Norm}\big(H^0(X, L^k \otimes T \otimes \mathcal{I}(ku))\big)$ be the space of $\mathbb{C}$–norms on the vector space $H^0(X, L^k \otimes T \otimes \mathcal{I}(ku))$ and let $\mathcal{L}_{k,u} : \mathrm{Norm}\big(H^0(X, L^k \otimes T \otimes \mathcal{I}(ku))\big) \to \mathbb{R}$ be the *partial Donaldson functional*, extending the definition from [Berman and Boucksom 2010],

$$\mathcal{L}_{k,u}(H) = \frac{n!}{k^{n+1}} \log \frac{\mathrm{vol}\{s : H(s) \le 1\}}{\mathrm{vol}\{s : N^k_{0,\omega^n}(s) \le 1\}},$$

where vol is simply the Euclidean volume.

Proposition 6.4 *Let $w, w' \in C^0(X)$. Suppose that $u \in \mathrm{PSH}(X,\theta)$ is such that θ_u is a Kähler current and $[u] \in \mathcal{A}(X,\theta)$. Then*

$$\lim_{k\to\infty}\big(\mathcal{L}_{k,u}(N^k_{w,\omega^n}) - \mathcal{L}_{k,u}(N^k_{w',\omega^n})\big) = \mathcal{I}^{\theta}_{[u],X}(w) - \mathcal{I}^{\theta}_{[u],X}(w'). \tag{41}$$

In particular,

$$\lim_{k\to\infty}\mathcal{L}_{k,u}(N^k_{w,\omega^n}) = \mathcal{I}^{\theta}_{[u],X}(w). \tag{42}$$

Proof First observe that by Proposition 6.2, for any $k > 0$, both N^k_{w,ω^n} and N^k_{w',ω^n} are norms, hence the expressions inside the limit in (41) make sense.

To start, we make the following classical observation:

$$\mathcal{L}_{k,u}(N^k_{w,\omega^n}) - \mathcal{L}_{k,u}(N^k_{w',\omega^n}) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{k,u}(N^k_{w+t(w'-w),\omega^n})\,\mathrm{d}t = \int_0^1\int_X (w'-w)\,\beta^k_{w+t(w'-w),u,\omega^n}\,\mathrm{d}t.$$

By Proposition 6.3, we have

$$\lim_{k\to\infty}\int_X (w'-w)\,\beta^k_{w+t(w'-w),u,\omega^n} = \int_X (w'-w)\,\theta^n_{P_X[u](w+t(w'-w))}.$$

By Theorem 1.1 we have $\big|\int_X (w'-w)\beta^k_{w+t(w'-w),u,\omega^n}\big| \le C\sup_X|w-w'|$. Hence, by the dominated convergence theorem we obtain that

$$\lim_{k\to\infty}\big(\mathcal{L}_{k,u}(N^k_{w,\omega^n}) - \mathcal{L}_{k,u}(N^k_{w',\omega^n})\big) = \int_0^1\int_X (w'-w)\,\theta^n_{P_X[u](w+t(w'-w))}\,\mathrm{d}t = \mathcal{I}^{\theta}_{[u],X}(w) - \mathcal{I}^{\theta}_{[u],X}(w'),$$

where in the last equality we have used Proposition 5.10.

Finally, (42) is just a special case of (41) with $w' = 0$. □

Lemma 6.5 *Let $u \in \mathrm{PSH}(X,\theta)$. Let (K, v) be a weighted subset of X. Let $\nu \in \mathrm{BM}(K, v)$. Then*

$$\lim_{k\to\infty}\big(\mathcal{L}_{k,u}(N^k_{v,K}) - \mathcal{L}_{k,u}(N^k_{v,\nu})\big) = 0. \tag{43}$$

This is a direct consequence of the definition of Bernstein–Markov measures (34).

Corollary 6.6 *Take $w \in C^0(X)$ and $u \in \mathrm{PSH}(X,\theta)$ such that θ_u is a Kähler current and $[u] \in \mathcal{A}(X,\theta)$. Then*

$$\lim_{k\to\infty}\mathcal{L}_{k,u}(N^k_{w,X}) = \mathcal{I}^{\theta}_{[u],X}(w).$$

Proof This follows from Lemma 6.5 and Proposition 6.4 and the fact that $\omega^n \in \mathrm{BM}(X, 0)$. □

Using these preliminary facts we extend Proposition 6.4 for much less regular data, again relying on ideas from [Berman and Boucksom 2010]:

Proposition 6.7 *Let $u \in \mathrm{PSH}(X,\theta)$ be such that θ_u is a Kähler current and $[u] \in \mathcal{A}(X,\theta)$. Let (K,v) and (K',v') be two weighted subsets of X. Then*

$$\lim_{k\to\infty}\big(\mathcal{L}_{k,u}(N^k_{v,K}) - \mathcal{L}_{k,u}(N^k_{v',K'})\big) = \mathcal{I}^{\theta}_{[u],K}(v) - \mathcal{I}^{\theta}_{[u],K'}(v'). \tag{44}$$

In particular,

$$\lim_{k\to\infty} \mathcal{L}_{k,u}(N^k_{v,K}) = \mathcal{I}^{\theta}_{[u],K}(v). \tag{45}$$

Proof First observe that by Proposition 6.2, for any $k>0$, both $N^k_{v,K}$ and $N^k_{v',K'}$ are norms, hence the expressions inside the limit in (44) make sense. Moreover, (45) is just a special case of (44) for $K'=X$ and $v'=0$.

To prove (44) it is enough to show that for any fixed $w \in C^\infty(X)$ we have

$$\lim_{k\to\infty}\big(\mathcal{L}_{k,u}(N^k_{v,K}) - \mathcal{L}_{k,u}(N^k_{w,\omega^n})\big) = \mathcal{I}^{\theta}_{[u],K}(v) - \mathcal{I}^{\theta}_{[u],X}(w). \tag{46}$$

For $\varepsilon \in (0,1)$ small enough we have that $\theta_{(1-\varepsilon)u}$ is still a Kähler current. Let us fix such ε, along with an arbitrary $\varepsilon' \in (0,1)$.

Let $\{v_j^-\}_j$ and $\{v_j^+\}_j$ be the sequence of smooth global functions constructed in Lemma 5.9 for the data (K,v).

By Remark 2.9 there exists $k_0(\varepsilon,\varepsilon') \in \mathbb{N}$ such that $[(1/k)\log h^k \otimes h_T(s,s)] \preceq [(1-\varepsilon)u]$, as well as $(1/k)\log h^k \otimes h_T(s,s) \in \mathrm{PSH}(X,\theta+\varepsilon'\omega)$ for any $s \in H^0(X, T\otimes L^k \otimes \mathcal{I}(ku))$ and $k \ge k_0(\varepsilon,\varepsilon')$.

In particular, Lemma 5.1 gives that

$$\begin{aligned}
N^k_{E_K^{\theta+\varepsilon'\omega}[(1-\varepsilon)u](v),X}(s) &= N^k_{v,K}(s),\\
N^k_{E_X^{\theta+\varepsilon'\omega}[(1-\varepsilon)u](v_j^-),X}(s) &= N^k_{v_j^-,X}(s),\\
N^k_{E_X^{\theta+\varepsilon'\omega}[(1-\varepsilon)u](v_j^+),X}(s) &= N^k_{v_j^+,X}(s).
\end{aligned}$$

As $E_X^{\theta+\varepsilon'\omega}[(1-\varepsilon)u](v_j^-) \le E_K^{\theta+\varepsilon'\omega}[(1-\varepsilon)u](v) \le E_X^{\theta+\varepsilon'\omega}[(1-\varepsilon)u](v_j^+)$, by Lemma 6.1 we have

$$N^k_{v_j^+,X}(s) \le N^k_{v,K}(s) \le N^k_{v_j^-,X}(s) \quad \text{for } s \in H^0(X,T\otimes L^k \otimes \mathcal{I}(ku)) \text{ and } k \ge k_0(\varepsilon,\varepsilon').$$

Composing with $\mathcal{L}_{k,u}$, we arrive at

$$\mathcal{L}_{k,u}(N^k_{v_j^-,X}) \le \mathcal{L}_{k,u}(N^k_{v,K}) \le \mathcal{L}_{k,u}(N^k_{v_j^+,X}) \quad \text{for } k \ge k_0(\varepsilon,\varepsilon').$$

For any $j>0$, by Corollary 6.6 we get

$$\begin{aligned}
\mathcal{I}^{\theta}_{[u],X}(v_j^-) - \mathcal{I}^{\theta}_{[u],X}(w) &= \lim_{k\to\infty}\big(\mathcal{L}_{k,u}(N^k_{v_j^+,X}) - \mathcal{L}_{k,u}(N^k_{w,X})\big)\\
&\le \varliminf_{k\to\infty}\big(\mathcal{L}_{k,u}(N^k_{v,K}) - \mathcal{L}_{k,u}(N^k_{w,X})\big) \le \varlimsup_{k\to\infty}\big(\mathcal{L}_{k,u}(N^k_{v,K}) - \mathcal{L}_{k,u}(N^k_{w,X})\big)\\
&\le \lim_{k\to\infty}\big(\mathcal{L}_{k,u}(N^k_{v_j^-,X}) - \mathcal{L}_{k,u}(N^k_{w,X})\big) = \mathcal{I}^{\theta}_{[u],X}(v_j^+) - \mathcal{I}^{\theta}_{[u],X}(w).
\end{aligned}$$

Using Lemma 5.11, we can let $j \to \infty$ to arrive at

$$\mathcal{I}^\theta_{[u],K}(v) - \mathcal{I}^\theta_{[u],K}(w) \le \varliminf_{k\to\infty} \big(\mathcal{L}_{k,u}(N^k_{v,K}) - \mathcal{L}_{k,u}(N^k_{w,X})\big) \le \varlimsup_{k\to\infty} \big(\mathcal{L}_{k,u}(N^k_{v,K}) - \mathcal{L}_{k,u}(N^k_{w,X})\big) \le \mathcal{I}^\theta_{[u],K}(v) - \mathcal{I}^\theta_{[u],K}(w).$$

Hence, (46) follows. □

Corollary 6.8 *Let $u \in \mathrm{PSH}(X,\theta)$ such that θ_u is a Kähler current and $[u] \in \mathcal{A}(X,\theta)$. Let (K,v) be a weighted subset of X. Assume that $\nu \in \mathrm{BM}(K,v)$. Then*

$$\lim_{k\to\infty} \mathcal{L}_{k,u}(N^k_{v,\nu}) = \mathcal{I}^\theta_{[u],K}(v).$$

Proof Our claim follows from Proposition 6.7 and Lemma 6.5. □

Proposition 6.9 *Suppose that $u \in \mathrm{PSH}(X,\theta)$ with $[u] \in \mathcal{A}(X,\theta)$, and assume that θ_u is a Kähler current. Let (K,v) be a weighted subset of X. Let $\nu \in \mathrm{BM}(K,v)$. Then $\beta^k_{v,u,\nu} \rightharpoonup \theta^n_{P_K[u]_\mathcal{I}(v)} = \theta^n_{P_K[u](v)}$ weakly as $k \to \infty$.*

The following proof is similar to that of [Berman et al. 2011, Theorem B].

Proof For $w \in C^0(X)$, let

$$f_k(t) = \mathcal{L}_{k,u}(N^k_{v+tw,\nu}) \quad \text{and} \quad g(t) := \mathcal{I}^\theta_{[u],K}(v+tw).$$

By Corollary 6.8 $\varliminf_{k\to\infty} f_k(t) = g(t)$. Note that f_k is concave by Hölder's inequality (see [Berman et al. 2011, Proposition 2.4]), so by [Berman and Boucksom 2010, Lemma 7.6], $\lim_{k\to\infty} f'_k(0) = g'(0)$, which is equivalent to $\beta^k_{v,u,\nu} \rightharpoonup \theta^n_{P_K[u](v)}$, by Proposition 5.10. □

Next we deal with the case of Kähler currents:

Proposition 6.10 *Suppose that $u \in \mathrm{PSH}(X,\theta)$ such that θ_u is a Kähler current. Let (K,v) be a weighted subset of X and $\nu \in \mathrm{BM}(K,v)$. Then $\beta^k_{u,v,\nu} \rightharpoonup \theta^n_{P_K[u]_\mathcal{I}(v)}$ as $k \to \infty$.*

Proof Let μ be the weak limit of a subsequence of $\beta^k_{v,u,\nu}$. We claim that

$$\mu \le \theta^n_{P_K[u]_\mathcal{I}(v)}. \tag{47}$$

Observe that this claim implies the conclusion. In fact, by Theorem 1.1, we have equality of the total masses, so equality holds in (47). As μ is an arbitrary subsequential limit of the weak compact sequence $\{\beta^k_{v,u,\nu}\}_k$, we get that $\beta^k_{v,u,\nu} \rightharpoonup \theta^n_{P_K[u]_\mathcal{I}(v)}$ as $k \to \infty$.

We prove the claim. Let $\{u^D_j\}_j$ be the approximation sequence of Theorem 2.8. By Lemmas 5.2 and 3.7, we know that $d_\mathcal{S}([u^D_j],[P_K[u]_\mathcal{I}]) = d_\mathcal{S}([u^D_j],[P_K[u]_\mathcal{I}(v)]) \to 0$. In particular,

$$\lim_{j\to\infty} \int_X \theta^n_{P_K[u^D_j]_\mathcal{I}(v)} = \int_X \theta^n_{P_K[u]_\mathcal{I}(v)}. \tag{48}$$

We know that $\theta_{u^D_j}$ are Kähler currents for high enough j. Since $u \le u^D_j$, we trivially obtain that $\beta^k_{v,u,\nu} \le \beta^k_{v,u^D_j,\nu}$ for any $k \ge 1$. As $\nu \in \mathrm{BM}(K,v)$, by Proposition 6.9, $\mu \le \theta^n_{P_K[u^D_j]_\mathcal{I}(v)}$ for any $j \ge 1$

fixed. By Proposition 2.7, $P_K[u_j^D]_{\mathcal{I}}(v) \searrow P_K[u]_{\mathcal{I}}(v)$ as $j \to \infty$. Hence, by (48) and [Darvas et al. 2018, Theorem 2.3], (47) follows. □

Finally, the main result:

Theorem 6.11 *Suppose that $u \in \mathrm{PSH}(X,\theta)$. Let (K,v) be a weighed subset of X, let $\nu \in \mathrm{BM}(K,v)$. Then $\beta^k_{v,u,\nu} \rightharpoonup \theta^n_{P_K[u]_{\mathcal{I}}(v)}$ as $k \to \infty$.*

Proof By Lemma 5.2 and [Darvas and Xia 2022, Proposition 2.18] we have that

$$H^0\big(X, L^k \otimes T \otimes \mathcal{I}(ku)\big) = H^0\big(X, L^k \otimes T \otimes \mathcal{I}(kP[u]_{\mathcal{I}})\big) = H^0\big(X, L^k \otimes T \otimes \mathcal{I}(kP[u]_{\mathcal{I}}(v))\big).$$

This allows us to replace u with $P_K[u]_{\mathcal{I}}(v)$. In addition, by Theorem 1.1 we can also assume that $\int_X \theta^n_u > 0$, otherwise there is nothing to prove.

By Proposition 3.6, there exists $u_j \in \mathrm{PSH}(X,\theta)$ such that $u_j \nearrow u$ a.e. and θ_{u_j} are Kähler currents. This gives $\beta^k_{v,u_j,\nu} \le \beta^k_{v,u,\nu}$. Let μ be the weak limit of a subsequence of $\beta^k_{v,u,\nu}$. Then by Proposition 6.10, $\theta^n_{P_K[u_j]_{\mathcal{I}}(v)} \le \mu$. By Proposition 2.7 and [Darvas et al. 2018, Theorem 2.3], $\theta^n_{P_K[u_j]_{\mathcal{I}}(v)} \nearrow \theta^n_{P_K[u]_{\mathcal{I}}(v)}$. Hence,

$$\theta^n_{P_K[u]_{\mathcal{I}}(v)} \le \mu. \tag{49}$$

A comparison of total masses ((37) and Theorem 1.1) gives that equality holds in (49). As μ is an arbitrary subsequential limit of the weak compact sequence $\{\beta^k_{v,u,\mu}\}_k$, we obtain that $\beta^k_{v,u,\nu} \rightharpoonup \theta^n_{P_K[u]_{\mathcal{I}}(v)}$ as $k \to \infty$. □

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