Geometry & Topology

Volume 28 (2024)

Packing Lagrangian tori

Richard Hind

Ely Kerman

msp

Geometry & Topology **28**:5 (2024) 2207–2257
DOI: 10.2140/gt.2024.28.2207
Published: 24 August 2024

Packing Lagrangian tori

RICHARD HIND
ELY KERMAN

We consider the problem of packing a symplectic manifold with integral Lagrangian tori, that is, Lagrangian tori whose area homomorphisms take only integer values. We prove that the Clifford torus in $S^2 \times S^2$ is a maximal integral packing, in the sense that any other integral Lagrangian torus must intersect it. In the other direction, we show that in any symplectic polydisk $P(a,b)$ with $a, b > 2$, there is at least one integral Lagrangian torus in the complement of the collection of standard product integral Lagrangian tori.

53D12, 53D35

1 Introduction

In this paper we consider packings of symplectic manifolds by Lagrangian tori. Since every symplectic manifold contains infinitely many disjoint Lagrangian tori, we must set a scale in order to pose meaningful questions. We therefore restrict our attention to Lagrangian tori whose area homomorphism takes only integer values. These will be referred to as *integral Lagrangian tori*.[1] The fundamental packing question, in this setting, is the following:

> *What is the maximum number of disjoint integral Lagrangian tori contained in a given (pre)compact symplectic manifold?*

A more approachable version of this question is to consider a specific collection of disjoint integral Lagrangian tori in a symplectic manifold (M, ω), and to ask if it is a *maximal integral packing* in the sense that any other integral Lagrangian torus in M must intersect at least one torus in the collection. In this paper, we study this question in the simplest nontrivial setting.

1.1 Results

Equip the sphere S^2 with its standard symplectic form ω scaled so that $\int_{S^2} \omega = 2$. Let $L_{1,1}$ be the monotone Clifford torus (product of equators) in $(S^2 \times S^2, \pi_1^*\omega + \pi_2^*\omega)$. Our first result is the following.

Theorem 1.1 *The Clifford torus $L_{1,1}$ is a maximal integral packing of $(S^2 \times S^2, \pi_1^*\omega + \pi_2^*\omega)$.*

[1] These are also sometimes called Bohr–Sommerfeld Lagrangians.

© 2024 MSP (Mathematical Sciences Publishers). Distributed under the Creative Commons Attribution License 4.0 (CC BY). Open Access made possible by subscribing institutions via Subscribe to Open.

Consider $\mathbb{R}^4$ equipped with its standard symplectic structure ω_4. For real numbers $a, b > 0$, consider the symplectic polydisk

$$P(a,b) = \{(z_1, z_2) \in \mathbb{C}^2 \mid \pi|z_1|^2 < a,\, \pi|z_2|^2 < b\} \subset \mathbb{R}^4.$$

Identifying $L_{1,1}$ with the standard Clifford torus in $\mathbb{R}^4$, Theorem 1.1 implies that $L_{1,1}$ is a maximal integral packing of each $P(a,b)$ with $1 < a, b < 2$.

If a and b are both greater then 2, then a natural candidate for a maximal integral packing of $P(a,b)$ is the collection of integral Lagrangian tori

$$\{L_{k,l} \mid k, l \in \mathbb{N},\, k \le \lfloor a \rfloor,\, l \le \lfloor b \rfloor\},$$

where $L_{k,l}$ is the product of the circle about the origin bounding area k in the z_1–plane with the circle about the origin bounding area l in the z_2–plane. The analogous packing in dimension two is always maximal. Our second result shows that, in dimension four, this candidate always fails.

Theorem 1.2 *If* $\min(a,b) > 2$, *then* $\{L_{k,l} \mid k, l \in \mathbb{N},\, k \le \lfloor a \rfloor,\, l \le \lfloor b \rfloor\}$ *is not a maximal integral packing of* $P(a,b)$. *For every* $\epsilon > 0$, *there is an integral Lagrangian torus* L^+ *in*

$$P(2+\epsilon, 2+\epsilon) \setminus \{L_{k,l} \mid k, l \in \{1,2\}\}.$$

1.2 Overview

The first step in our proof of Theorem 1.1 is to show that any integral Lagrangian torus contained in $(S^2 \times S^2, \pi_1^*\omega + \pi_2^*\omega)$ is actually monotone. This follows from the work of Hind and Opshtein [9], and is proved in Proposition 3.2 below. Arguing by contradiction, we then assume there is a monotone Lagrangian torus $\mathbb{L}$ in $(S^2 \times S^2, \pi_1^*\omega + \pi_2^*\omega)$ that is disjoint from the Clifford torus $L_{1,1}$. The work of Ivrii [11] and Dimitroglou-Rizell, Goodman and Ivrii [5] implies that there is a finite-energy holomorphic foliation $\mathcal{F}$ of $S^2 \times S^2 \setminus (\mathbb{L} \cup L_{1,1})$ which has a normal form near $\mathbb{L}$ and $L_{1,1}$; see Section 3.5. We use $\mathcal{F}$ to establish the existence of two symplectic spheres, F and G, in $(S^2 \times S^2, \pi_1^*\omega + \pi_2^*\omega)$. These are obtained from the compactifications of the pseudoholomorphic buildings obtained in Section 3.7; see Propositions 3.20 and 3.22. Both F and G represent a homology class of the form $(1,d) \in H_2(S^2 \times S^2; \mathbb{Z}) = \mathbb{Z}^2$ for some large d. They also have special intersection properties with the leaves of $\mathcal{F}$ and with each other; see Proposition 3.24. Using the spheres F and G, together with the operations of blow-up, inflation and blow-down, we then alter the ambient symplectic manifold away from $\mathbb{L} \cup L_{1,1}$ to obtain a new monotone symplectic manifold, (X, Ω). This new manifold is symplectomorphic to $(S^2 \times S^2, (d+1)(\pi_1^*\omega + \pi_2^*\omega))$, and $\mathbb{L}$ and $L_{1,1}$ remain disjoint and monotone therein. However, the images (transforms) of the spheres F and G in (X, Ω) are now in the class $(1,0)$ and their existence implies, by the work of Cieliebak and Schwingenheuer in [4], that $\mathbb{L}$ and $L_{1,1}$ must both be Hamiltonian isotopic to the Clifford torus in (X, Ω). It then follows from standard monotone Lagrangian Floer theory (as in Oh [17]) that it is not possible for $\mathbb{L}$ and $L_{1,1}$ to be disjoint. This contradiction completes the proof of Theorem 1.1.

To prove Theorem 1.2 we construct, for every $\epsilon > 0$, an explicit embedding of the closure of $P(1,1)$ into $P(2+\epsilon, 2+\epsilon) \setminus \{L_{k,l} \mid k,l \in \{1,2\}\}$, using a time-dependent Hamiltonian flow. The desired Lagrangian, L^+, is the one on the boundary of the image.

1.3 Commentary and further questions

Given that Theorem 1.1 is reduced to the problem of detecting intersection points of two monotone Lagrangian tori, using Hind and Opshtein [9], it is natural to ask whether Lagrangian Floer theory (rigid holomorphic curves) can also be used to prove Theorem 1.1 directly. To the knowledge of the authors this is not yet possible. The following result seems to be as close to a proof of Theorem 1.1 as one can currently get using Lagrangian Floer theory.

Theorem 1.3 *Suppose that L is a monotone Lagrangian torus in $(S^2 \times S^2, \pi_1^*\omega + \pi_2^*\omega)$. If the Lagrangian Floer homology of L, with respect to some $\mathbb{C}^*$–local system, is nontrivial, then L must intersect $L_{1,1}$.*

This follows from the work of Ritter and Smith in [19].[2] In particular, Corollary 1.5 of [19] implies that the Clifford torus $L_{1,1}$ split-generates the monotone Fukaya category of $(S^2 \times S^2, \pi_1^*\omega + \pi_2^*\omega)$. It is not known whether there exist monotone Lagrangian tori in $(S^2 \times S^2, \pi_1^*\omega + \pi_2^*\omega)$ whose Lagrangian Floer homology is trivial for every choice of $\mathbb{C}^*$–local system. In [20], Vianna constructs a countably infinite collection of monotone Lagrangian tori in $(S^2 \times S^2, \pi_1^*\omega + \pi_2^*\omega)$, no two of which are Hamiltonian isotopic. Each of the tori in Vianna's collection satisfies the hypothesis of Theorem 1.3.

The following question, in the spirit of Theorem 1.1, remains unresolved.

Question 1.4 *Does every pair of monotone Lagrangian tori in $(S^2 \times S^2, \pi_1^*\omega + \pi_2^*\omega)$ intersect?*

Progress on other aspects of the study of disjoint Lagrangian tori has also recently been made in two related works by Mak and Smith [13], and by Polterovich and Shelukhin [18]. Let $\{\gamma_i\}$ be a collection of disjoint circles bounding disks of the same area, and let E be the equator in the sphere S^2. In [13] and [18] it is shown that, with respect to certain nonmonotone symplectic forms on $S^2 \times S^2$, packings of the form $\mathscr{L} = \bigsqcup \gamma_i \times E$ are maximal in the sense that any Lagrangian torus Hamiltonian isotopic to $\gamma_1 \times E$ must intersect $\mathscr{L}$. In comparison, the maximal packing given by Theorem 1.1 only includes a single torus, $L_{1,1}$, but we do not assume any other tori are Hamiltonian isotopic to it. Theorem 1.2 shows that analogous packings of the form $\bigsqcup \gamma_i \times \gamma_j$ are no longer maximal.

Below are a few of the questions suggested by Theorem 1.2, which also remain unresolved.

Question 1.5 *Is every integral Lagrangian torus in $P(2+\epsilon, 2+\epsilon) \setminus \{L_{k,l} \mid k,l \in \{1,2\}\}$ Hamiltonian isotopic to $L_{1,1}$?*

[2] We are grateful to the referee for pointing out this reference.

Question 1.6 *Suppose $2 < a, b < 3$. Are there six disjoint integral Lagrangian tori in $P(a, b)$?*

Question 1.7 *Suppose $2 < b < 3$. Are there three disjoint integral Lagrangian tori in $P(2, b)$?*

Question 1.5 has recently been answered negatively, and Questions 1.6 and 1.7 positively, by Hicks and Mak in the preprint [7]. The question of whether these domains might actually contain infinitely many disjoint integral Lagrangians remains completely open.

Acknowledgements The authors would like to thank the referee of this paper for their careful analysis and many valuable comments. We also thank Karim Boustany for helpful comments and for pointing out a few mistakes.

The authors are supported by Simons Foundation grants 633715 (Hind) and 429872 (Kerman).

2 Conventions, labels and notation

Every copy of the two-dimensional sphere S^2 will implicitly be identified with the unit sphere in $\mathbb{R}^3$ and we will label the north and south poles by ∞ and 0, respectively. In $(S^2 \times S^2, \pi_1^*\omega + \pi_2^*\omega)$, we use these points to define the four symplectic spheres $S_0 = S^2 \times \{0\}$, $S_\infty = S^2 \times \{\infty\}$, $T_0 = \{0\} \times S^2$ and $T_\infty = \{\infty\} \times S^2$. The ordered basis $\{[S_0], [T_0]\}$ of $H_2(S^2 \times S^2; \mathbb{Z})$ is used to identify it with $\mathbb{Z}^2$.

Let $L \subset (M, \Omega)$ be a Lagrangian torus in a four-dimensional symplectic manifold. A diffeomorphism ψ from $\mathbb{T}^2 = S^1 \times S^1$ to L will be referred to as a parametrization of L. It specifies a basis of $H_1(L; \mathbb{Z})$ and thus an isomorphism from $H_1(L; \mathbb{Z})$ to $\mathbb{Z}^2$. We will denote this copy of $\mathbb{Z}^2$ by $H_1^\psi(L; \mathbb{Z})$. The parametrization ψ can also be extended to a symplectomorphism Ψ from a neighborhood of the zero section in $T^*\mathbb{T}^2$ to a Weinstein neighborhood $\mathcal{U}(L)$ of L in M. We will denote the corresponding coordinates in the neighborhood $\mathcal{U}(L)$ of L by (p_1, p_2, q_1, q_2) and, for simplicity, we will assume that

$$\mathcal{U}(L) = \{|p_1| < \epsilon, |p_2| < \epsilon\} \quad \text{for some } \epsilon > 0.$$

3 Proof of Theorem 1.1

Arguing by contradiction, we begin with the following.

Assumption 1 There is an integral Lagrangian torus $\mathbb{L}$ in $(S^2 \times S^2, \pi_1^*\omega + \pi_2^*\omega)$ which is disjoint from the Clifford torus $L_{1,1}$.

We will show that Assumption 1 can be refined in three ways.

3.1 Refinement 1: we may assume that $\mathbb{L}$ is monotone

A symplectic manifold (M, Ω) is *monotone* if the Chern and area homomorphisms,

$$c_1 \colon \pi_2(M) \subset H_2(M, \mathbb{Z}) \to \mathbb{Z} \quad \text{and} \quad \Omega \colon \pi_2(M) \to \mathbb{R},$$

are positively proportional. Recall that a Lagrangian submanifold $L \subset (M, \Omega)$ is *monotone* if its Maslov and area homomorphisms,

$$\mu\colon \pi_2(M,L) \to \mathbb{Z} \quad \text{and} \quad \Omega\colon \pi_2(M,L) \to \mathbb{R},$$

are positively proportional. We will denote the constant of proportionality of L by λ.

If L is a Lagrangian torus, one can verify monotonicity by checking it for a collection of disks whose boundaries generate $H_1(L;\mathbb{Z})$.

Lemma 3.1 *Suppose that (M,Ω) is a symplectic 4–manifold which is monotone with constant $\frac{1}{2}\lambda$. A Lagrangian torus L in (M,Ω) is monotone with constant λ if there are two smooth maps $v_1, v_2\colon (D^2, S^1) \to (M, L)$ such that the boundary maps $v_1|_{S^1}$ and $v_2|_{S^1}$ determine an integral basis of $H_1(L;\mathbb{Z})$ and $\mu([v_i]) = \lambda\Omega([v_i])$ for $i = 1, 2$.*

Refinement 1 is validated by the following result.

Proposition 3.2 *Every integral Lagrangian torus L in $(S^2 \times S^2, \pi_1^*\omega + \pi_2^*\omega)$ is monotone.*

Proof By Theorem C of [5] there is a Hamiltonian diffeomorphism which displaces L from the pair of spheres $S_\infty \cup T_\infty$. Hence, L can be identified with an integral Lagrangian torus $\boldsymbol{L}$ inside the polydisk $P(2-\epsilon, 2-\epsilon) \subset (\mathbb{R}^4, \omega_4)$ for some sufficiently small $\epsilon > 0$. By Lemma 3.1, it suffices to find two smooth maps $v_1, v_2\colon (D^2, S^1) \to (\mathbb{R}^4, \boldsymbol{L})$ such that the boundary maps $v_1|_{S^1}$ and $v_2|_{S^1}$ determine an integral basis of $H_1(\boldsymbol{L};\mathbb{Z})$ and $\mu([v_i]) = 2\omega_4([v_i])$ for $i = 1, 2$. Simplifying further, we note that, for $\mathbb{R}^4$, the maps μ and ω_4 can be recast as homomorphisms

$$\mu\colon H_1(\boldsymbol{L};\mathbb{Z}) \to \mathbb{Z} \quad \text{and} \quad \omega_4\colon H_1(\boldsymbol{L};\mathbb{Z}) \to \mathbb{R}$$

and it suffices to find an integral basis $\{e_1, e_2\}$ of $H_1(\boldsymbol{L};\mathbb{Z})$ such that $\mu(e_i) = 2\omega_4(e_i)$ for $i = 1, 2$.

Since $\boldsymbol{L}$ is contained in $P(2-\epsilon, 2-\epsilon)$, it follows from [3] that there is a smooth map $f\colon (D, S^1) \to (\mathbb{R}^4, \boldsymbol{L})$ of Maslov index 2 whose symplectic area is 1. To see this we include the polydisk $P(2-\epsilon, 2-\epsilon)$ into $B^4(4-2\epsilon)$, the ball of capacity $4-2\epsilon$, and then compactify this ball to $\mathbb{C}P^2$ equipped with the Fubini–Study form rescaled by $(4-2\epsilon)/\pi$. In this setting, the proofs of Theorems 1.1 and 1.2 of [3] imply that there are three discs mapping to $\mathbb{C}P^2$, with boundary on $\boldsymbol{L}$, that each have Maslov index equal to 2 and positive symplectic areas whose sum is at most $(4-2\epsilon)$. These discs are holomorphic away from $\boldsymbol{L}$ and are obtained from a limit of spheres in the class $[\mathbb{C}P^1]$. Hence, by positivity of intersection, exactly one of the three discs intersects the line at infinity, and the other two discs, f and g, can be viewed as maps to $B^4(4-2\epsilon) \subset \mathbb{R}^4$. Since $\boldsymbol{L}$ is integral, the total symplectic area of f and g is either 2 or 3. In either case, one of them, say f, has symplectic area equal to 1. If e_1 is the element of $H_1(\boldsymbol{L};\mathbb{Z})$ represented by $f|_{S^1}$, we then have $\mu(e_1) = 2$ and $\omega_4(e_1) = 1$.

Let c be a class in $H_1(\boldsymbol{L};\mathbb{Z})$ such that $\{e_1, c\}$ is an integral basis. Since $\mu(c)$ is even, by adding integer multiples of e_1 to c, if necessary, we may assume that $\mu(c) = 2$. It remains to show that $\omega_4(c) = 1$.

Arguing by contradiction, assume that $\omega_4(c) \neq 1$. Set

$$\hat{c} = \begin{cases} c & \text{if } \omega_4(c) > 1, \\ c + 2(e_1 - c) & \text{if } \omega_4(c) < 1. \end{cases}$$

Then $\{e_1, \hat{c}\}$ is an integer basis of $H_1(\boldsymbol{L};\mathbb{Z})$ that satisfies

$$\omega_4(e_1) = 1, \quad \omega_4(\hat{c}) \geq 2 \quad \text{and} \quad \mu(e_1) = \mu(\hat{c}) = 2.$$

In [9], Hind and Opshtein prove that if a Lagrangian torus in $P(a,b)$ admits such a basis, then either $a > 2$ or $b > 2$. This contradicts the assumption that $\boldsymbol{L}$ lies in $P(2-\epsilon, 2-\epsilon)$ and we are done. □

3.2 Refinement 2: we may assume that $\mathbb{L}$ lies in the complement of $S_0 \cup S_\infty \cup T_0 \cup T_\infty$

To verify this, we utilize the relative finite-energy foliations from [5], which we now recall.

3.2.1 Foliations of $(S^2 \times S^2) \setminus L$ In [6], Gromov proves that if J is a smooth almost complex structure J on $S^2 \times S^2$ that is tamed by the symplectic form $\pi_1^*\omega + \pi_2^*\omega$, then there is a foliation of $S^2 \times S^2$ by J–holomorphic spheres in the class $(0,1)$, and another with fibers in the class $(1,0)$. For any monotone Lagrangian torus $L \subset (S^2 \times S^2, \pi_1^*\omega + \pi_2^*\omega)$, there is an analogous relative foliation theory, developed first by Ivrii [11] and then completed by Dimitroglou-Rizell, Goodman and Ivrii [5], with input from Wendl [21] and Hind and Lisi [8]. By stretching certain Gromov foliations along L and smoothing the compactifications of the limiting buildings with more than one level, they obtain symplectic S^2–foliations of $S^2 \times S^2$ that are *compatible with* L. A version of this is described below. As in [8], we focus on the curves which, after stretching, map to $S^2 \times S^2 \setminus L$.

Input Let L be a monotone Lagrangian torus in $(S^2 \times S^2, \pi_1^*\omega + \pi_2^*\omega)$. Fix a parametrization ψ of L and the corresponding Weinstein neighborhood

$$\mathcal{U}(L) = \{|p_1| < \epsilon, |p_2| < \epsilon\}.$$

Definition 3.3 A tame almost complex structure J on $(S^2 \times S^2 \setminus L, \pi_1^*\omega + \pi_2^*\omega)$ is said to be *adapted to the parametrization* ψ if, in $\mathcal{U}(L)$, we have

$$J\frac{\partial}{\partial q_i} = -\sqrt{p_1^2 + p_2^2}\,\frac{\partial}{\partial p_i}.$$

For such a J, each negative end of a finite-energy J–holomorphic curve u mapping to $S^2 \times S^2 \setminus L$ is asymptotic to a closed Reeb orbit on a copy of the flat unit cotangent bundle $S_L^*\mathbb{T}^2$ of $\mathbb{T}^2$, corresponding to L. This Reeb orbit covers a closed geodesic γ of the flat metric on $\mathbb{T}^2$. In this case, we simply say that the end of u is asymptotic to L along γ.

Output As described in Section 2.5 of [5], each J adapted to the parametrization ψ of L is part of the limit, as $\tau \to \infty$, of a standard family almost complex structures $J_{\tau \geq 0}$ on $S^2 \times S^2$ that are tame with respect to $\pi_1^* \omega + \pi_2^* \omega$. Taking the limit of the Gromov foliations for the J_τ as $\tau \to \infty$, it follows from Theorem D and Propositions 5.3 and 5.16 of [5], and the fact that L is monotone, that one obtains a foliation $\mathcal{F} = \mathcal{F}(L, \psi, J)$ of $S^2 \times S^2 \setminus L$ with the following properties.

- The foliation $\mathcal{F}$ has two kind of leaves: unbroken ones consisting of a single closed J–holomorphic sphere in $S^2 \times S^2 \setminus L$ of class $(0,1)$, and broken leaves consisting of a pair of finite-energy J–holomorphic planes in $S^2 \times S^2 \setminus L$.
- Each leaf of $\mathcal{F}$ intersects S_∞ in exactly one point. For a broken leaf this means that exactly one of its planes intersects S_∞.
- The ends of two planes of a broken leaf are asymptotic to the same geodesic, but with opposite orientations. This geodesic is embedded. We denote its homology class, equipped with the orientation determined by the plane which intersects S_∞, by $\beta \in H_1(L; \mathbb{Z})$. This class is the same for all broken leaves of $\mathcal{F}$ and is referred to as the foliation class of $\mathcal{F}$.
- Limits of the Gromov spheres in the completion of a neighborhood of L, which is a copy of $T^*\mathbb{T}^2$, are cylinders asymptotic to geodesics in the classes $\pm\beta$.
- Each point $z \in S^2 \times S^2 \setminus L$ lies in a unique leaf of $\mathcal{F}$, and each point of L lies on a unique geodesic in the foliation class β that corresponds to a unique plane of a broken leaf of $\mathcal{F}$ that intersects S_∞.
- If L is disjoint from a configuration of symplectic spheres, then we may assume these spheres are complex with respect to all J_τ. In particular, if L has been displaced from $S_0 \cup S_\infty \cup T_0 \cup T_\infty$, then we may assume this configuration of symplectic spheres is J–complex.
- Suppose L is disjoint from S_∞ and we therefore take S_∞ to be complex. Then, by positivity of intersection, there is a well-defined map $p \colon S^2 \times S^2 \to S_\infty$ which takes $z \in S^2 \times S^2 \setminus L$ to the unique intersection of its leaf with S_∞, and takes $z \in L$ to the intersection with S_∞ of the broken leaf asymptotic to the unique geodesic through z representing the foliation class. The image $p(L)$ is an embedded closed curve in S_∞. Moreover, if L is homotopic to $L_{1,1}$ in the complement of $T_0 \cup T_\infty$, then $p(T_0)$ and $p(T_\infty)$ — which are points, since T_0 and T_∞ are complex — lie on opposite sides of the closed curve $p(L)$.

Lemma 3.4 (straightening) *For all sufficiently small $\epsilon > 0$ we may assume, by perturbing J outside of $\mathcal{U}(L)$, that the unbroken leaves of $\mathcal{F}$ that intersect $\mathcal{U}(L)$ do so along the annuli*

$$\{p_1 = \delta, q_1 = \theta, -\epsilon < p_2 < \epsilon\}$$

for some $\theta \in S^1$ and nonzero $\delta \in (-\epsilon, \epsilon)$.

Proof The statement for broken leaves was established in Proposition 5.16 of [5]; see the first bullet point of the proof. This means the parts of the broken leaves lying outside of $\mathcal{U}(L)$ form two S^1–families of holomorphic disks, with boundaries $\{p_1 = 0, q_1 = \theta, p_2 = \pm\epsilon\}$. We may smoothly identify a neighborhood of such an S^1–family of holomorphic disks in the complement of $\mathcal{U}(L)$ with $(-\epsilon_0, \epsilon_0) \times S^1 \times D^2$, where the disks correspond to subsets $\{0\} \times \{\theta\} \times D^2$, and the circles $\{p_1 = \delta, q_1 = \theta, p_2 = \pm\epsilon\}$ in $\partial\mathcal{U}(L)$ match with the circles $\{\delta\} \times \{\theta\} \times \partial D^2$. Hence our S^1–families of holomorphic disks can be extended to smooth families of disjoint smoothly embedded disks $D_{\delta,\theta} = \{\delta\} \times \{\theta\} \times D^2$ with $|\delta| < \epsilon_0$ and $\theta \in S^1$. We may assume these disks extend smoothly into $\mathcal{U}(L)$ along the surfaces $\{p_1 = \delta, q_1 = \theta, -\epsilon < p_2 < \epsilon\}$. For a sufficiently small $\epsilon_0 > 0$, we may also assume that the disks $D_{\delta,\theta}$ are symplectic, since they are C^∞–close to the holomorphic disks corresponding to $\delta = 0$. Hence, we may choose a tame almost complex structure, J_0, which agrees with J inside $\mathcal{U}(L)$ but is chosen outside of $\mathcal{U}(L)$ so that the disks $D_{\delta,\theta}$ are J_0–holomorphic. With this, we replace the foliation $\mathcal{F} = \mathcal{F}(L, \psi, J)$ with the foliation $\mathcal{F}_0 = \mathcal{F}(L, \psi, J_0)$ and the neighborhood $\mathcal{U}(L)$ with

$$\mathcal{U}_0(L) = \{|p_1| < \epsilon_0, |p_2| < \epsilon\}.$$

By construction, for each annulus $\{p_1 = \delta, q_1 = \theta, -\epsilon < p_2 < \epsilon\}$ with $|\delta| < \epsilon_0$, there is a pair of J_0–holomorphic disks which join smoothly with the boundary components to form J_0–holomorphic spheres in the class $(0, 1)$. These are unbroken leaves of $\mathcal{F}_0$ for $\delta \neq 0$, and broken leaves for $\delta = 0$. Moreover, by positivity of intersection, these are the only leaves of $\mathcal{F}_0$ intersecting $\mathcal{U}_0(L)$. ◻

Example 3.5 (solid tori bounded by $L_{1,1}$) For the Clifford torus $L_{1,1} \subset S^2 \times S^2$ and a J adapted to the standard parametrization $\psi_{1,1}$ of $L_{1,1}$, we get a foliation $\mathcal{F}_{1,1}$ of $S^2 \times S^2 \setminus L_{1,1}$ with leaves in the class $(0, 1)$. As $L_{1,1}$ is disjoint from $S_0 \cup S_\infty \cup T_0 \cup T_\infty$ we may assume that these four spheres are J–complex. The broken leaves of $\mathcal{F}_{1,1}$ comprise two families of J–holomorphic planes with boundary on $L_{1,1}$, which can be labeled as follows: $\mathfrak{s}_0$, which consists of the planes intersecting S_0, and $\mathfrak{s}_\infty$, which consists of the planes intersecting S_∞. The families of holomorphic planes $\mathfrak{s}_0$ and $\mathfrak{s}_\infty$ can be seen directly for a model almost complex structure, but in fact exist for all J adapted to a parametrization of $L_{1,1}$. We will write $\mathfrak{s}_0(J)$ and $\mathfrak{s}_\infty(J)$ when we want to highlight the dependence of these families on J. Modulo reparametrization, $\mathfrak{s}_0$ and $\mathfrak{s}_\infty$ form compact moduli spaces, as they represent classes of minimal positive area in $H_2(S^2 \times S^2, L_{1,1})$. These moduli spaces are automatically regular by [21, Theorem 1].

For each J as above, there is an analogous foliation of $S^2 \times S^2 \setminus L_{1,1}$ with leaves in the class $(1, 0)$. The broken leaves in this case yield two families of J–holomorphic planes, $\mathfrak{t}_0$ and $\mathfrak{t}_\infty$, which consist of the planes intersecting T_0 and T_∞, respectively.

The following result establishes Refinement 2. The proof is based on that of Corollary E in [5].

Proposition 3.6 *Suppose that $\mathbb{L}$ is a monotone Lagrangian torus in $(S^2 \times S^2, \pi_1^*\omega + \pi_2^*\omega)$ that is disjoint from $L_{1,1}$. Then there is a Hamiltonian diffeomorphism ϕ of $S^2 \times S^2$ which displaces $\mathbb{L}$ from $S_0 \cup S_\infty \cup T_0 \cup T_\infty$ and is supported away from $L_{1,1}$. Moreover, $\phi(\mathbb{L})$ is homotopic to $L_{1,1}$ in the complement of $T_0 \cup T_\infty$ and also in the complement of $S_0 \cup S_\infty$.*

Proof We first displace $\mathbb{L}$ from S_∞ in the complement of $L_{1,1}$, or equivalently S_∞ from $\mathbb{L}$. Let J_0 be an almost complex structure on $S^2 \times S^2 \setminus L_{1,1}$ that is adapted to the standard parametrization $\psi_{1,1}$ of $L_{1,1}$ and such that S_∞ is J_0–complex. We deform J_0 through almost complex structures J_τ to an almost complex structure J_∞ which is also adapted to a parametrization of $\mathbb{L}$. For each τ we have a finite-energy J_τ–holomorphic foliation in the class $(1,0)$ including the broken leaves $\mathfrak{t}_0$ and $\mathfrak{t}_\infty$ as in Example 3.5. We can find a smooth family of holomorphic spheres H_τ in the class $(1,0)$, that is, unbroken leaves of the corresponding foliations, with $H_0 = S_\infty$.

In the limit $\tau \to \infty$, the moduli spaces of J_τ–holomorphic disks $\mathfrak{t}_0(J_\tau)$ and $\mathfrak{t}_\infty(J_\tau)$ both converge. Indeed, their limits $\mathfrak{t}_0(J_\infty)$ and $\mathfrak{t}_\infty(J_\infty)$ still represent classes of minimal positive area in $H_2(S^2 \times S^2, L_{1,1} \cup \mathbb{L})$, where all classes still have integral area; see also Lemma 3.23 for this. Moreover, as the union of broken spheres with respect to J_∞ still has codimension 1, we may assume the H_τ converge to an unbroken sphere, H_∞. As the H_τ are all disjoint from $L_{1,1}$ and H_∞ is disjoint from $\mathbb{L}$, we can find a Hamiltonian isotopy supported away from $L_{1,1}$ displacing S_∞ from $\mathbb{L}$, as required.

The remainder of the argument follows similar lines. We may assume that each of the spheres S_0, S_∞, T_0 and T_∞ are J_0–complex. Let $\mathcal{F}_0$ be the corresponding J_0–holomorphic foliation of $S^2 \times S^2 \setminus L_{1,1}$ in the class $(0,1)$. Let $p_0\colon S^2 \times S^2 \to S_\infty$ be the projection map from the (sixth bullet point of the) description of $\mathcal{F}$ above. We may assume that the points $p_0(T_0)$ and $p_0(T_\infty)$ lie in different components of $S_\infty \setminus p_0(L_{1,1})$.

Fix a parametrization of $\mathbb{L}$ and let (P_1, P_2, Q_1, Q_2) be the corresponding local coordinates on the Weinstein neighborhood $\mathcal{U}(\mathbb{L})$ of $\mathbb{L}$. We may also assume that $\mathcal{U}(\mathbb{L})$ is disjoint from the Weinstein neighborhood $\mathcal{U}(L_{1,1})$ corresponding to $\psi_{1,1}$. Consider a smooth family $J_{t \in [0,1]}$ of almost complex structures on $S^2 \times S^2 \setminus L_{1,1}$ such that each J_t is equal to J_0 in $\mathcal{U}(L_{1,1})$, and in $\mathcal{U}(\mathbb{L})$ we have

$$J_1 \frac{\partial}{\partial Q_i} = -\frac{\partial}{\partial P_i}.$$

We can then smoothly extend the family J_t to $t > 1$ to stretch (to length t) along a small sphere bundle in $\mathcal{U}(\mathbb{L})$, as in [1]. This yields a family of foliations $\mathcal{F}_t$ of $S^2 \times S^2 \setminus L_{1,1}$. Since the planes of the broken leaves of $\mathcal{F}_0$ have minimal area they persist under the deformation to yield the planes of the broken leaves of $\mathcal{F}_t$. This yields a family of maps $p_t\colon S^2 \times S^2 \to S_\infty$.

Lemma 3.7 *The sets $p_t(\mathbb{L})$ in S_∞ converge in the Hausdorff topology to a subset of a circle $C_\infty \in S_\infty$ as $t \to \infty$.*

Proof Let J_∞ be the limiting almost complex structure on $S^2 \times S^2 \setminus (\mathbb{L} \cup L_{1,1})$. The circle C_∞ is the intersection with S_∞ of the broken leaves of the J_∞ foliation which are asymptotic to $\mathbb{L}$. Now, $p_t(\mathbb{L})$ consists of the intersection with S_∞ of J_t–holomorphic spheres which intersect $\mathbb{L}$. Hence a sequence of points $z_t \in p_t(\mathbb{L})$ corresponds to a sequence of J_t–holomorphic curves in the class $(0,1)$ which all intersect $\mathbb{L}$. Up to taking a subsequence, this sequence of curves converges to a broken curve asymptotic to $\mathbb{L}$ and hence the z_t converge to a point in C_∞. □

Lemma 3.8 *If we denote the projection with respect to the fully stretched almost complex structure by p_∞, then $C_\infty = p_\infty(\mathbb{L})$ is disjoint from $p_\infty(L_{1,1})$.*

Proof This follows from the fact that the original planes of the broken leaves have area 1 and so cannot degenerate further. Indeed, since $\mathbb{L}$ is monotone, any holomorphic curve asymptotic to $\mathbb{L}$ must have integral area, and in particular curves in the class $(0,1)$ cannot converge to buildings with more than two top level curves. □

The results above imply that there is an $N > 0$ such that $p_t(L_{1,1})$ is disjoint from C_∞ for all $t \geq N$. With this we can choose two continuous curves $\gamma_0, \gamma_\infty \colon [0,\infty) \to S_\infty$ with the following properties:

- $\gamma_0(0) = p_0(T_0)$ and $\gamma_\infty(0) = p_0(T_\infty)$,
- $\gamma_0(t)$ and $\gamma_\infty(t)$ are disjoint from $p_t(L_{1,1})$ for all $t \in [0,\infty)$,
- for some $N > 0$, both $\gamma_0(t)$ and $\gamma_\infty(t)$ are disjoint from C_∞, and C_∞ is disjoint from $p_t(L_{1,1})$ for all $t \geq N$,
- C_∞ separates $\gamma_0(N)$ and $\gamma_\infty(N)$ in S_∞.

For each $t \in [0,\infty)$, both $p_t^{-1}(\gamma_0(t))$ and $p_t^{-1}(\gamma_\infty(t))$ are J_t–holomorphic spheres in the class $(0,1)$ disjoint from $L_{1,1}$. The family of spheres

$$\{p_t^{-1}(\gamma_0(t))\}_{t\in[0,N]}$$

forms a symplectic isotopy, which displaces T_0 from $\mathbb{L}$ in the complement of $L_{1,1}$. Similarly, the family of spheres

$$\{p_t^{-1}(\gamma_\infty(t))\}_{t\in[0,N]}$$

forms a symplectic isotopy which displaces T_∞ from $\mathbb{L}$ in the complement of $L_{1,1}$. Moreover, these isotopies can be generated by a single Hamiltonian flow on $S^2 \times S^2$ that fixes $L_{1,1}$. The inverse flow displaces $\mathbb{L}$ from $T_0 \cup T_\infty$. The final separation condition is enough to guarantee the homotopy condition in the theorem.

By considering also the J_t–holomorphic foliation in the class $(1,0)$ (see Example 3.5), we can displace $\mathbb{L}$ from $S_0 \cup S_\infty$ too. After adjusting the isotopy of $S_0 \cup S_\infty$, we may assume that it fixes $T_0 \cup T_\infty$; see Corollary 3.7 of [5]. Hence the inverse flow will not reintroduce intersections with T_0 or T_∞. □

3.3 Refinement 3: we may assume that $\mathbb{L}$ is homologically trivial in $(S^2 \times S^2) \setminus (S_0 \cup S_\infty \cup T_0 \cup T_\infty)$

To see this, note that $(S^2 \times S^2) \setminus (S_0 \cup S_\infty \cup T_0 \cup T_\infty)$ can be identified with a subset of the cotangent bundle of $\mathbb{T}^2$ in which $L_{1,1}$ is identified with the zero section. In this setting we can invoke the following.

Theorem 3.9 [5, Theorem 7.1] *A homologically nontrivial Lagrangian torus L in $(T^*\mathbb{T}^2, d\lambda)$ is Hamiltonian isotopic to a constant section. In particular, if L is exact then it is Hamiltonian isotopic to the zero section.*

If our monotone Lagrangian $\mathbb{L}$ was homologically nontrivial in $(S^2\times S^2)\setminus(S_0\cup S_\infty\cup T_0\cup T_\infty)$ it would then follow from Theorem 3.9 and Section 2.3.B$''_4$ of [6] that $\mathbb{L}\cap L_{1,1}\neq\varnothing$, which would contradict our original assumption.

3.4 A path to the proof of Theorem 1.1

By the three refinements established above, it suffices to show that the following assumption is false.

Assumption 2 There is a monotone Lagrangian torus $\mathbb{L}$ in the set

$$Y=(S^2\times S^2)\setminus(S_0\cup S_\infty\cup T_0\cup T_\infty)$$

which is disjoint from the Clifford torus $L_{1,1}$ and is homologically trivial in Y.

A path to a contradiction To obtain a contradiction to Assumption 2, we will show, using a sequence of blow-ups, inflations and blow-downs, that it implies the existence of two disjoint monotone Lagrangian tori in a new (monotone) copy of $S^2\times S^2$, which are both Hamiltonian isotopic to the Clifford torus therein, and hence cannot be disjoint.

To perform the necessary sequence of blow-ups, inflations and blow-downs, we must first establish the existence of a special collection of symplectic spheres and disks in our current model; see Proposition 3.24. These spheres and discs must be well placed with respect to a holomorphic foliation of $S^2\times S^2\setminus(\mathbb{L}\cup L_{1,1})$ which we introduce below in Section 3.5. They are obtained from special holomorphic buildings, whose existence we establish in Section 3.7. These existence results rely on the analysis of a general stretching scenario that is contained in Section 3.6.

Remark 3.10 To falsify Assumption 2, we must use it to build and analyze a complicated set of secondary objects in order to derive a contradiction. The reader is asked to bear in mind that many of the results established in the remainder of this section hold in a setting which will later be shown to be impossible.

3.5 Straightened holomorphic foliations of $S^2\times S^2\setminus(\mathbb{L}\cup L_{1,1})$, under Assumption 2

Let $\mathbb{L}$ be a Lagrangian torus as in Assumption 2. Here we describe the holomorphic foliations of $S^2\times S^2\setminus(\mathbb{L}\cup L_{1,1})$ that are implied by the existence of $\mathbb{L}$.

Let ψ be a parametrization of $\mathbb{L}$ and $\psi_{1,1}$ be the standard parametrization of $L_{1,1}$. Consider a tame almost complex structure J on $(S^2\times S^2\setminus(\mathbb{L}\cup L_{1,1}), \pi_1^*\omega+\pi_2^*\omega)$ which is adapted to both ψ and $\psi_{1,1}$. We will always make the following assumption.

(A1) J is equal to the standard product complex structure near S_0, S_∞, T_0 and T_∞. In particular, T_0 and T_∞ are unbroken leaves of the foliation.

Let J_τ be the family of almost complex structures on $S^2 \times S^2$ that are determined by J as in [5, Section 2.5]. Taking the limit of the Gromov foliations in the class $(0,1)$ with respect to the J_τ as $\tau \to \infty$, and arguing as in [5], we get a J–holomorphic foliation

$$\mathcal{F} = \mathcal{F}(\mathbb{L}, L_{1,1}, \psi, \psi_{1,1}, J)$$

of $S^2 \times S^2 \setminus (\mathbb{L} \cup L_{1,1})$. The features of this foliation are described below and are illustrated in Figure 1.

Each leaf of $\mathcal{F}$ still intersects S_∞ in exactly one point, but there are now three types of leaves. The first are unbroken leaves consisting of a single closed J–holomorphic sphere in $S^2 \times S^2 \setminus (\mathbb{L} \cup L_{1,1})$ of class $(0,1)$. The second type of leaves are broken and consist of a pair of finite-energy J–holomorphic planes in $S^2 \times S^2 \setminus (\mathbb{L} \cup L_{1,1})$ that are asymptotic to $L_{1,1}$ along the same embedded geodesic with opposite orientations. As in Example 3.5, the collection of planes like this which intersect S_∞ comprise a one-dimensional family, $\mathfrak{s}_\infty$, and their companion planes comprise a family $\mathfrak{s}_0$. The third class of leaves are also broken, but consist of pairs of finite-energy J–holomorphic planes in $S^2 \times S^2 \setminus (\mathbb{L} \cup L_{1,1})$ that are asymptotic to $\mathbb{L}$. These pairs also have matching ends. We denote by $\mathfrak{r}_\infty$ the set of all the J–holomorphic planes of broken leaves that are asymptotic to $\mathbb{L}$ and intersect S_∞. The collection of their companion planes will be denoted by $\mathfrak{r}_0$. As established below in Lemma 3.11, the planes of $\mathfrak{r}_\infty$ intersect both S_∞ and S_0 while the planes of $\mathfrak{r}_0$ intersect neither of these spheres. Since curves in the class $(0,1)$ have area 2, and by monotonicity planes asymptotic to our Lagrangians have integral area, no more complicated degenerations are possible.

Note that there are now two foliation classes, $\beta_{\mathbb{L}}$ and $\beta_{L_{1,1}}$, determined by each of the two classes of broken leaves.

The foliation $\mathcal{F}$ also defines a projection map

$$p\colon S^2 \times S^2 \to S_\infty.$$

In this setting, the images $p(L_{1,1})$ and $p(\mathbb{L})$ are disjoint embedded circles in S_∞ which, by Proposition 3.6, are disjoint from $T_0 \cup T_\infty$ and are homotopic in the complement. Therefore, without loss of generality, there are disjoint closed disks $A_0 \subset S_\infty$ with boundary $p(\mathbb{L})$ and $A_\infty \subset S_\infty$ with boundary $p(L_{1,1})$, such that $p(T_0) \in A_0$ and $p(T_\infty) \in A_\infty$. Denote the closed annulus defined by the closure of $S_\infty \setminus (A_0 \cup A_\infty)$ by B. These features of $\mathcal{F}$ are all represented in Figure 1.

Lemma 3.11 *The planes in $\mathfrak{r}_\infty$ intersect both S_0 and S_∞. Equivalently, the planes in $\mathfrak{r}_0$ are disjoint from $S_0 \cup S_\infty$.*

Proof We define a relative homology class $\Sigma \in H_2(S^2 \times S^2, (S_0 \cup S_\infty \cup T_0 \cup T_\infty)$ by first choosing an embedded path $\gamma\colon [0,1] \to S_\infty$ with $\gamma(0) = T_0 \cap S_\infty$ and $\gamma(1) = T_\infty \cap S_\infty$. Then choose a family of embedded paths σ_t in each $p^{-1}(\gamma(t))$ going from S_∞ to S_0. The union of the σ_t define Σ. By Proposition 3.6 we may assume that γ intersects $p(\mathbb{L})$ in a single point $\gamma(t_0)$. The intersection

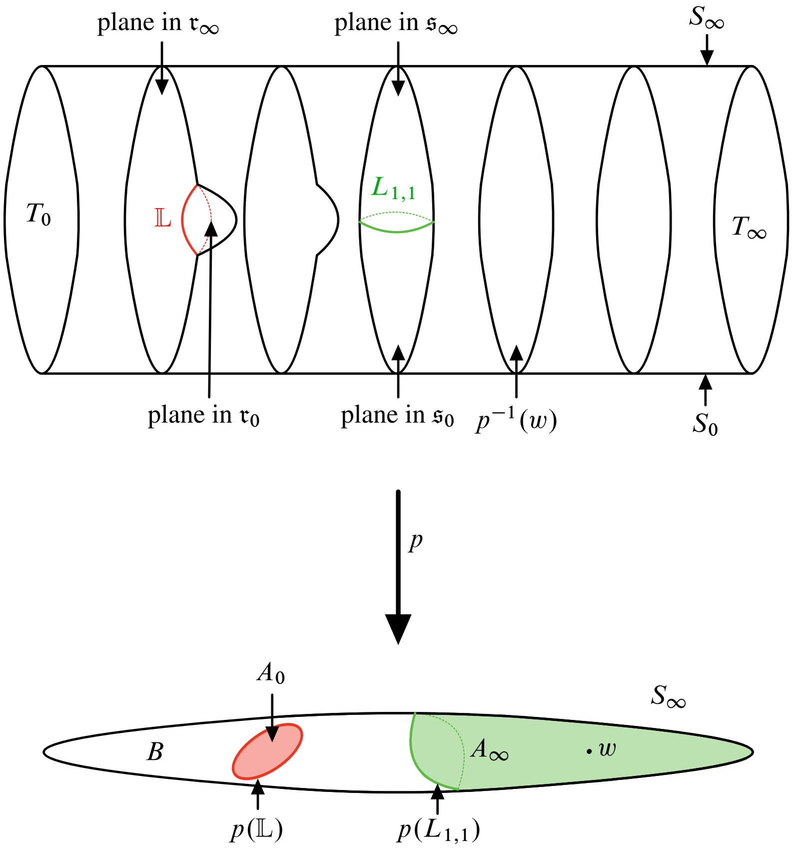

Figure 1: The foliation $\mathcal{F}$ of $S^2 \times S^2 \setminus (\mathbb{L} \cup L_{1,1})$.

$\mathbb{L} \cap p^{-1}(\gamma(t_0))$ is an embedded circle, bounding disks from $\mathfrak{r}_0$ and $\mathfrak{r}_\infty$. The disks in $\mathfrak{r}_\infty$ intersect S_∞ by definition, so arguing by contradiction, if $\mathfrak{r}_0$ happens to intersect S_0 then our circle $\mathbb{L} \cap p^{-1}(\gamma(t_0))$ separates S_0 and S_∞, and thus must intersect the path σ_{t_0}. This is the only intersection between Σ and $\mathbb{L}$, and so we would conclude that $\Sigma \bullet \mathbb{L}$ is nontrivial, contradicting Refinement 3. □

Straightening Let (P_1, P_2, Q_1, Q_2) be coordinates in the neighborhood $\mathcal{U}(\mathbb{L})$ of $\mathbb{L}$ determined by ψ, and let (p_1, p_2, q_1, q_2) be coordinates in the neighborhood $\mathcal{U}(L_{1,1})$ of $L_{1,1}$ determined by $\psi_{1,1}$. As in Lemma 3.4, where we had only one Lagrangian torus, we may assume that the leaves of $\mathcal{F}$ are straight in both $\mathcal{U}(\mathbb{L})$ and $\mathcal{U}(L_{1,1})$. In particular, we may assume that the unbroken leaves of $\mathcal{F}$ that intersect $\mathcal{U}(\mathbb{L})$ do so along the annuli $\{P_1 = \delta \neq 0, Q_1 = \theta, |P_2| < \epsilon\}$, the planes of $\mathfrak{r}_\infty$ intersect $\mathcal{U}(\mathbb{L})$ along the annuli $\{P_1 = 0, Q_1 = \theta, 0 < P_2 < \epsilon\}$, and the planes of $\mathfrak{r}_0$ intersect $\mathcal{U}(\mathbb{L})$ along the annuli $\{P_1 = 0, Q_1 = \theta, -\epsilon < P_2 < 0\}$. Similarly, we may assume that the unbroken leaves of $\mathcal{F}$ that intersect $\mathcal{U}(L_{1,1})$ do so along the annuli $\{p_1 = \delta \neq 0, q_1 = \theta, |p_2| < \epsilon\}$, the planes of $\mathfrak{s}_\infty$ intersect $\mathcal{U}(L_{1,1})$ along the annuli $\{p_1 = 0, q_1 = \theta, 0 < p_2 < \epsilon\}$, and the planes of $\mathfrak{s}_0$ intersect $\mathcal{U}(L_{1,1})$ along the annuli $\{p_1 = 0, q_1 = \theta, -\epsilon < p_2 < 0\}$.

The map p can also be described simply in these Weinstein neighborhoods. In $\mathcal{U}(\mathbb{L})$, we may assume that the region $\{P_1 < 0\} \subset \mathcal{U}(\mathbb{L})$ is mapped by p into the interior of A_0, and $\{P_1 > 0\} \subset \mathcal{U}(\mathbb{L})$ is mapped by p into the interior of B. Similarly, we may assume that in $\mathcal{U}(L_{1,1})$ the region $\{p_1 > 0\} \subset \mathcal{U}(L_{1,1})$ is mapped by p into the interior of A_∞ and $\{p_1 < 0\} \subset \mathcal{U}(L_{1,1})$ is mapped by p into the interior of B.

Using some of the freedoms available in the choice of ψ and $\psi_{1,1}$, we can add the following additional assumption:

(A2) The foliation class $\beta_{\mathbb{L}}$ is equal to $(0,-1) \in H_1^{\psi}(\mathbb{L};\mathbb{Z})$, and the foliation class $\beta_{L_{1,1}}$ is equal to $(0,-1) \in H_1^{\psi_{1,1}}(L_{1,1};\mathbb{Z})$.

3.6 Stretching scenario for class $(1, d)$, under Assumption 2

Recall that for each nonnegative integer d and a generic tame almost complex structure J on $S^2 \times S^2$ there exists a smooth J–holomorphic sphere $u\colon S^2 \to S^2 \times S^2$ representing the class $(1,d)$. Moreover, this curve is unique, up to reparametrization, if we impose $2d+1$ constraint points. To see this, note that for the integrable product complex structure such a curve can be written explicitly as the graph of a degree d rational map, and this implies that the Gromov–Witten invariant associated to the homology class and point constraints is 1. Hence, nodal curves will exist for all tame almost complex structures and away from a codimension 2 subset of almost complex structures we will have smooth curves. The uniqueness in the assertion above follows because spheres in the class $(1,d)$ have self-intersection number $2d$, so distinct spheres cannot satisfy the same $2d+1$ point constraints.

Let J_τ, for $\tau \geq 0$, be the family of almost complex structures on $S^2 \times S^2$ used in Section 3.5 to obtain the foliation $\mathcal{F}$. For a sequence $\tau_k \to \infty$, let $u_{k,d}\colon S^2 \to S^2 \times S^2$ be a sequence of J_{τ_k}–holomorphic curves in the class $(1,d)$ that converges to a holomorphic building $\boldsymbol{F}_d$ as in [1]. The limit $\boldsymbol{F}_d$ consists of genus zero holomorphic curves in three levels. The *top level* curves map to $S^2 \times S^2 \setminus (\mathbb{L} \cup L_{1,1})$ and are J–holomorphic. The *middle level* curves map to one of two copies of $\mathbb{R} \times S^*\mathbb{T}^2$, the symplectization of the unit cotangent bundle of the flat torus. These copies correspond to $\mathbb{L}$ and $L_{1,1}$ and the identifications are defined by the parametrizations ψ and $\psi_{1,1}$. It follows from the definition of the family J_τ that these middle level curves are all J_{cyl}–holomorphic where J_{cyl} is a fixed cylindrical almost complex structure. Similarly, the *bottom level* curves of the limiting building map to one of two copies of $T^*\mathbb{T}^2$ and are J_{std}–holomorphic, where J_{std} is a standard complex structure.

Each top level curve of $\boldsymbol{F}_d$ can be compactified to yield a map from a surface of genus zero with boundary to $(S^2 \times S^2, \mathbb{L} \cup L_{1,1})$. The components of the boundary correspond to the negative punctures of the curve. They are mapped to the closed geodesics on $\mathbb{L}$ or $L_{1,1}$ underlying the Reeb orbits to which the corresponding puncture is asymptotic. The middle and bottom level curves can be compactified to yield maps to either $\mathbb{L}$ or $L_{1,1}$ with the same type of boundary conditions. These compactified maps can all be glued together to form a map $\bar{\boldsymbol{F}}_d\colon S^2 \to S^2 \times S^2$ in the class $(1,d)$.

Definition 3.12 A J–holomorphic curve u in $S^2 \times S^2 \setminus (\mathbb{L} \cup L_{1,1})$ is said to be *essential* (*with respect to the foliation* $\mathcal{F}$) if the map $p \circ u$ is injective.

Definition 3.13 Let u be a J–holomorphic curve in $S^2 \times S^2 \setminus (\mathbb{L} \cup L_{1,1})$. A puncture of u is said to be of *foliation type with respect to* $\mathbb{L}$ $(L_{1,1})$ if it is asymptotic to a closed Reeb orbit which lies on the copy of $S^*\mathbb{T}^2$ that corresponds to $\mathbb{L}$ $(L_{1,1})$ and covers a closed geodesic in an integer multiple of the foliation class $\beta_{\mathbb{L}}$ $(\beta_{L_{1,1}})$. The puncture is of *positive* (*negative*) *foliation type* if this integer is positive (negative).

Lemma 3.14 *Let u be a J–holomorphic curve in $S^2 \times S^2 \setminus (\mathbb{L} \cup L_{1,1})$ with a puncture. Let $\{c_l\}$ be a sequence of circles in the domain of u which lie in a standard neighborhood of the puncture, wind once around it, and converge to it in the Hausdorff topology. If the puncture is of foliation type with respect to $\mathbb{L}$ $(L_{1,1})$, then the sets $p(u(c_l))$ converge to a point on $p(\mathbb{L})$ $(p(L_{1,1}))$. Moreover each $p(u(c_l))$ either maps into the point (in which case u covers a plane in a broken leaf) or it winds nontrivially around the point. If the puncture is not of foliation type then the sets $p(u(c_l))$ converge to $p(\mathbb{L})$ $(p(L_{1,1}))$.*

Proof This follows from the exponential convergence theorem from [10]. □

Corollary 3.15 *If u is an essential J–holomorphic curve in $S^2 \times S^2 \setminus (\mathbb{L} \cup L_{1,1})$, then its punctures on $\mathbb{L}$ are either all of foliation type or none of them are, and similarly for the punctures on $L_{1,1}$. If u has no punctures of foliation type, then it is either a J–holomorphic plane or cylinder.*

Assume u has no punctures of foliation type. If u is a plane, then the closure of the image of $p \circ u$ is A_0 or A_∞ or the closure of their complements in S_∞. If u is a cylinder, then the closure of the image of $p \circ u$ is B.

Proof Lemma 3.14 implies that if u has punctures of both foliation type and not of foliation type on $\mathbb{L}$ or $L_{1,1}$, then $p \circ u$ will not be injective. Indeed, the projection of a small circle around a puncture of foliation type on $\mathbb{L}$ (resp. $L_{1,1}$) will intersect any circle sufficiently close to $p(\mathbb{L})$ (resp. $P(L_{1,1})$), including the projections of small circles around punctures not of foliation type.

Essential curves with all punctures not of foliation type project onto connected subsets of S_∞ with boundary components equal to $\mathbb{L}$ or $L_{1,1}$. Checking possibilities, the second half of the statement follows. □

The following result can be proved in the same way as Lemma 6.2 in [8].

Lemma 3.16 *Let u be an essential curve whose punctures on $\mathbb{L}$ are all of foliation type. Then these punctures are either all positive or all negative* (see Definition 3.13).

Similarly, let v be an essential curve whose punctures on $L_{1,1}$ are all of foliation type. Then the punctures on $L_{1,1}$ are either all positive or all negative.

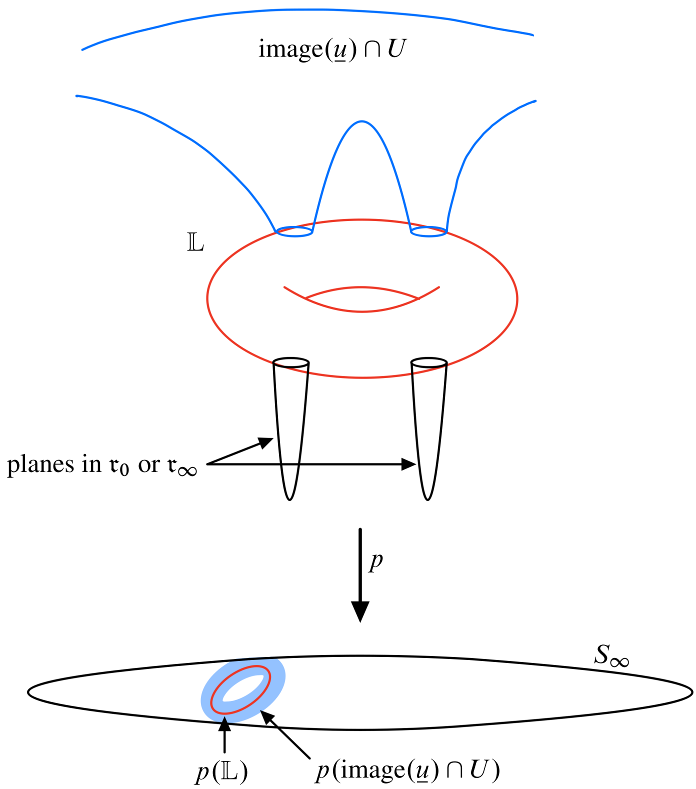

Figure 2: Images of curves of a limit $\boldsymbol{F}_d$ of Type 2b in a neighborhood U of $p^{-1}(p(\mathbb{L}))$. If one replaces $\underline{u}$ with u_d, then this picture also works for limits $\boldsymbol{F}_d$ of Type 1.

Let $u_{k,d}$ be a sequence converging to $\boldsymbol{F}_d$ as in the *stretching scenario for class* $(1,d)$. Positivity of intersection implies that the curves $u_{k,d}$ must intersect each leaf of $\mathscr{F}$ exactly once. This fact imposes several important restrictions on $\boldsymbol{F}_d$ in relation to the foliation $\mathscr{F}$, allowing us to identify a handful of possible limit types.

Proposition 3.17 *Let $\boldsymbol{F}_d$ be a limit as in the* ***stretching scenario for class*** $\boldsymbol{(1,d)}$*. Then the building $\boldsymbol{F}_d$ is of one of the following types.*

- ***Type 0*** *$\boldsymbol{F}_d$ is a (possibly nodal) J–holomorphic sphere in $S^2\times S^2\setminus(\mathbb{L}\cup L_{1,1})$ in the class $(1,d)$, where one (essential) sphere lies in the class $(1,j)$ for some $1\le j\le d$, and any remaining top level curves are either spheres covering unbroken leaves of the foliation, or pairs of planes covering broken leaves of the foliation. Any middle and bottom level curves are cylinders asymptotic to Reeb orbits in multiples of the foliation class.*
- ***Type 1*** *$\boldsymbol{F}_d$ has a unique essential curve u_d. The punctures of u_d are all of foliation type, and along $\mathbb{L}$, and also $L_{1,1}$, are either all positive or all negative. The image of $p\circ u_d$ is S_∞ minus finitely many points on $p(\mathbb{L})\cup p(L_{1,1})$. The other top level curves of $\boldsymbol{F}_d$ either cover unbroken*

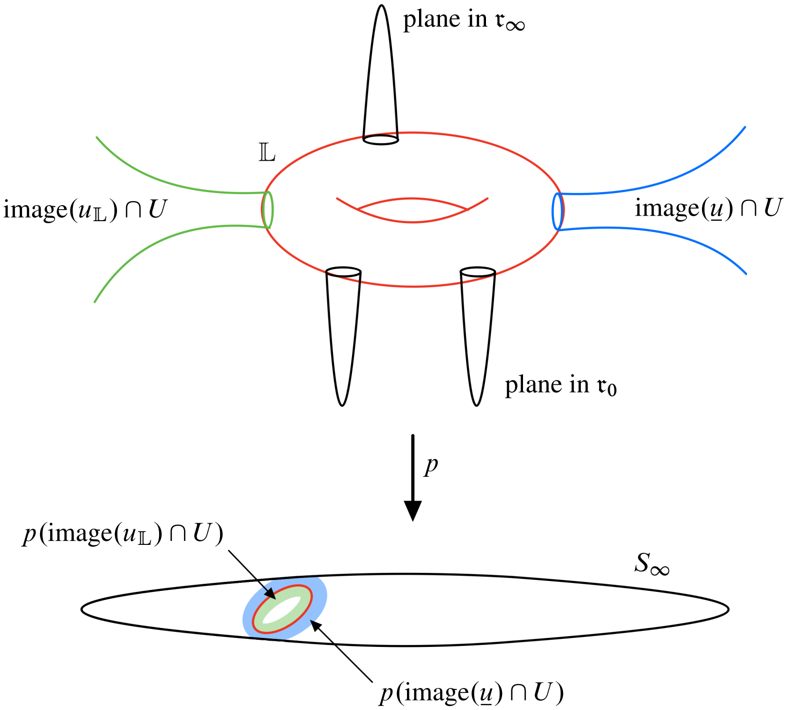

Figure 3: Images of curves of a limit $\boldsymbol{F}_d$ of Type 2a or 3 in a neighborhood U of $p^{-1}(p(\mathbb{L}))$.

leaves of the foliation, or they are J–holomorphic planes covering one of the planes of a broken leaf of the foliation. Any middle and bottom level curves cover cylinders asymptotic to Reeb orbits in multiples of the foliation class.

- ***Type 2a*** $\boldsymbol{F}_d$ *has exactly two essential curves, $u_{\mathbb{L}}$ and $\underline{u}$. The closures of the images of the maps $p \circ u_{\mathbb{L}}$ and $p \circ \underline{u}$ are A_0 and $B \cup A_\infty$, respectively. Any punctures of $\underline{u}$ on $L_{1,1}$ are all of foliation type and are either all positive or all negative. The other top level curves of $\boldsymbol{F}_d$ cover (broken or unbroken) leaves of $\mathcal{F}$. Any middle and bottom level curves in the copy of $T^*\mathbb{T}^2$ corresponding to $L_{1,1}$ cover cylinders asymptotic to Reeb orbits in multiples of the foliation class.*
- ***Type 2b*** $\boldsymbol{F}_d$ *has exactly two essential curves, $\underline{u}$ and $u_{L_{1,1}}$. The closures of the images of the maps $p \circ \underline{u}$ and $p \circ u_{L_{1,1}}$ are $A_0 \cup B$ and A_∞, respectively. Any punctures of $\underline{u}$ on $\mathbb{L}$ are all of foliation type and are either all positive or all negative. The other top level curves of $\boldsymbol{F}_d$ cover (broken or unbroken) leaves of $\mathcal{F}$. Any middle and bottom level curves in the copy of $T^*\mathbb{T}^2$ corresponding to $\mathbb{L}$ cover cylinders asymptotic to Reeb orbits in multiples of the foliation class.*
- ***Type 3*** $\boldsymbol{F}_d$ *has exactly three essential curves, $u_{\mathbb{L}}$, $\underline{u}$ and $u_{L_{1,1}}$. The closures of the images of the maps $u_{\mathbb{L}}$, $\underline{u}$ and $u_{L_{1,1}}$ are A_0, B and A_∞, respectively. The other top level curves of $\boldsymbol{F}_d$ again cover (broken or unbroken) leaves of $\mathcal{F}$.*

Limits of Type 2b and Type 3 are partially illustrated in Figures 2 and 3, respectively.

Proof of Proposition 3.17 We begin with the following result, which allows us to use essential curves to sort the limit structures.

Lemma 3.18 *Let $\boldsymbol{F}_d$ be a limit as in the* ***stretching scenario for class (1, d)****. If u is a top level curve of $\boldsymbol{F}_d$, then it is either essential or else the image of $p \circ u$ is a point. The essential curves have disjoint images under p, which are open sets, and these images include the complement of $p(\mathbb{L}) \cup p(L_{1,1})$.*

Proof The curves of $\boldsymbol{F}_d$ can be compactified and glued together to form a map $\bar{\boldsymbol{F}}_d \colon S^2 \to S^2 \times S^2$ in the class $(1, d)$. Let T be an unbroken leaf of the foliation. Since $(1, d) \bullet T = (1, d) \bullet (0, 1) = 1$, we see that T can only intersect one top level curve with $p \circ u$ nonconstant. If u is a top level curve such that the map $p \circ u$ is constant, then u covers part of a (possibly broken) leaf of our foliation and contributes intersection number 0 with all unbroken leaves.

Assume then that u is a top level curve such that $p \circ u$ is nonconstant. By the discussion above, u intersects any unbroken leaf T either once or not at all, and therefore if $p \circ u$ has any double points they must lie in $p(\mathbb{L}) \cup p(L_{1,1})$. But positivity of intersection again implies that the nonconstant map $p \circ u$ is an open mapping and this implies that the double points of $p \circ u$ form an open set. We conclude that there are no double points and u is essential. To see that the essential curves have disjoint images under p we can apply the same argument to a union $u \cup v$. The intersection number also implies that all unbroken fibers intersect at least one essential curve. □

Lemma 3.18 implies that there is an essential curve u of $\boldsymbol{F}_d$ that intersects T_0. The closure of the image of $p \circ u$ must contain A_0. By Corollary 3.15 the following cases are exhaustive.

Case 1 (u has no punctures) In this case, $p \circ u$ must be a bijection onto S_∞. Hence, u is a J–holomorphic sphere in a class of the form $(1, j)$ for j in $[0, d]$. By Lemma 3.18 all the other top level curves of $\boldsymbol{F}_d$ must cover leaves of the foliation. This also implies that middle and lower level curves cover cylinders asymptotic to multiples of the foliation class.

The top level curves of $\boldsymbol{F}_d$ which cover fibers fit together to form a possibly disconnected curve in the class $(0, d - j)$. If $j = d$ then $\boldsymbol{F}_d$ consists only of the curve u. Either way, the building is of Type 0.

Case 2 (u has punctures and they are all of foliation type) In this case we claim that $\boldsymbol{F}_d$ is of Type 1. By Lemma 3.14, the image of the map $p \circ u$ includes points in each component of the complement of $p(\mathbb{L}) \cup p(L_{1,1})$, and so by Lemma 3.18 we have that $p \circ u$ is a bijection onto S_∞ minus a finite set of points on $p(\mathbb{L}) \cup p(L_{1,1})$. The other top level curves of $\boldsymbol{F}_d$ must either cover unbroken leaves of $\mathcal{F}$ or they are J–holomorphic planes covering one of the planes of a broken leaf of $\mathcal{F}$. The statement about positivity or negativity of punctures is Lemma 3.16.

Case 3 (u has at least one puncture not of foliation type) Since u intersects the leaf T_0, the closure of the image of $p \circ u$ is either A_0 or $A_0 \cup B$. In either case, u has exactly one puncture not of foliation type and does not intersect T_∞.

Suppose that the closure of the image of $p \circ u$ is A_0. By Lemma 3.18, there is an essential curve v of $\boldsymbol{F}_d$ that intersects T_∞, and the images of $p \circ u$ and $p \circ v$ cannot intersect. Hence the closure of the image of $p \circ v$ is either A_∞ or $B \cup A_\infty$. In the first case, $\boldsymbol{F}_d$ is of Type 3 with $u_{\mathbb{L}} = u$ and $u_{L_{1,1}} = v$, where the third curve, $\underline{u}$, exists by Lemma 3.18. In the second case, $\boldsymbol{F}_d$ is of Type 2a with $u_{\mathbb{L}} = u$ and $\underline{u} = v$.

If, instead, the closure of the image of $p \circ u$ is $A_0 \cup B$, a similar argument implies that $\boldsymbol{F}_d$ is of Type 2b.

This completes the proof of Proposition 3.17. □

3.7 The existence of special buildings, under Assumption 2

In this section we will establish the existence of two special limits of the *stretching scenario for class* $(1, d)$ when d is sufficiently large. The following result will be used to exploit the large d limit.

Lemma 3.19 *There exists an* $\epsilon > 0$ *such that*

$$\operatorname{area}(u) \geq \epsilon u \bullet (S_0 \cup S_\infty)$$

for all J*–holomorphic curves* u *in* $S^2 \times S^2 \setminus (\mathbb{L} \cup L_{1,1})$.

Proof Fix an open neighborhood of S_∞ of the form $\mathcal{N}_\epsilon = S_\infty \times D^2(\epsilon)$, where $D^2(\epsilon)$ is the open disc of area ϵ. We may assume that the closure of $\mathcal{N}_\epsilon$ is disjoint from $\mathbb{L} \cup L_{1,1}$ and, by (A1), we may assume that J restricts to $\mathcal{N}_\epsilon$ as the standard split complex structure. Let $\pi_2 : S_\infty \times D^2(\epsilon) \to D^2(\epsilon)$ be projection and set

$$u_{\epsilon,\infty} = u|_{u^{-1}(\mathcal{N}_\epsilon)}.$$

By perturbing ϵ if needed we may assume that $u^{-1}(\mathcal{N}_\epsilon)$ is a smooth manifold. We have

$$\operatorname{degree}(\pi_2 \circ u_{\epsilon,d}) = u \bullet S_\infty.$$

This implies

$$\operatorname{area}(u_{\epsilon,\infty}) \geq \int_{u^{-1}(\mathcal{N}_\epsilon)} u^*_{\epsilon,\infty}(\omega \oplus \omega) \geq \int_{(\pi_2 \circ u_{\epsilon,\infty})^{-1}(D^2(\epsilon))} (\pi_2 \circ u_{\epsilon,\infty})^* \omega = \left(\int_{D^2(\epsilon)} \omega \right) u \bullet S_\infty = \epsilon u \bullet S_\infty.$$

A similar calculation for S_0 gives the result. □

Proposition 3.20 *For all sufficiently large* d*, there exists a limiting building* $\boldsymbol{F}$ *as in the* ***stretching scenario for class*** $\boldsymbol{(1, d)}$ *such that* $\boldsymbol{F}$ *is of Type* 3*. The building consists of its three essential top level curves,* $u_{\mathbb{L}}$*,* $\underline{u}$ *and* $u_{L_{1,1}}$*, together with* $d-1$ *planes in* $\mathfrak{r}_0 \cup \mathfrak{r}_\infty$ *and* d *planes in* $\mathfrak{s}_0 \cup \mathfrak{s}_\infty$*.*

Proof Fix $d+1$ points on $L_{1,1}$ and d points on $\mathbb{L}$. For $\tau \geq 0$, let J_τ be the family of almost complex structures on $S^2 \times S^2$ from Section 3.5. It follows from the discussion in Section 3.6 and the compactness result from [1] that for a sequence $\tau_k \to \infty$ there is a sequence $u_k : S^2 \to S^2 \times S^2$ of J_{τ_k}–holomorphic curves in the class $(1, d)$ that pass through the $2d+1$ constraint points and converge in the sense of [1]. Their limit, $\boldsymbol{F}$, is the desired building.

To see this we first note that the point constraints already preclude the possibility that $\boldsymbol{F}$ is of Type 0. Indeed, top level essential curves are disjoint from the point constraints, so these must be satisfied by curves of $\boldsymbol{F}$ inside copies of $T^*\mathbb{T}^2$ (corresponding to neighborhoods of $\mathbb{L}$ or $L_{1,1}$). In the Type 0 case, the nonessential curves fit together to form a union of spheres in the class $(0, d-j)$ for some $0 \leq j \leq d$. These intersect $\mathbb{L} \cup L_{1,1}$ in a finite set of geodesics, and any middle or lower level curves in our $T^*\mathbb{T}^2$ cover cylinders asymptotic to these geodesics. As there are at most d such cylinders they cannot satisfy the $2d+1$ point constraints. (The holomorphic cylinders in $T^*\mathbb{T}^2$ are described explicitly by Lemma 4.2 in [5].)

If $\boldsymbol{F}$ was of Type 1, then punctures of its essential curve on $L_{1,1}$ would all be of the foliation type. The remaining top level curves of $\boldsymbol{F}$ asymptotic to $L_{1,1}$ would cover broken planes, and all the curves of $\boldsymbol{F}$ mapping to the copy of $T^*\mathbb{T}^2$ corresponding to $L_{1,1}$ would cover cylinders over geodesics in the foliation class $\beta_{1,1}$. To satisfy the $d+1$ point constraints on $L_{1,1}$ the essential curve of $\boldsymbol{F}$ must have at least $d+1$ punctures on $L_{1,1}$, matching with at least $d+1$ cylinders in the copy of $T^*\mathbb{T}^2$. But then $\boldsymbol{F}$ would have $d+1$ curves covering planes in $\mathfrak{s}_0 \cup \mathfrak{s}_\infty$. By Lemma 3.16, the punctures of the essential curve on $L_{1,1}$ are either all positive or all negative. Hence these $d+1$ curves either all lie in $\mathfrak{s}_0$ or all lie in $\mathfrak{s}_\infty$. This contradicts the fact that (the compactification of) $\boldsymbol{F}$ has intersection number d with both S_0 and S_∞. The same argument precludes the possibility that $\boldsymbol{F}$ has Type 2a.

It remains to show that $\boldsymbol{F}$ does not have Type 2b. Assuming that $\boldsymbol{F}$ has Type 2b, we will show that it must include a collection of curves of total area equal to two, that intersect $S_0 \cup S_\infty$ d times. If d is sufficiently large, this contradicts Lemma 3.19 above.

Claim 1 *If $\boldsymbol{F}$ has Type 2b, then it includes at least d planes in $\mathfrak{s}_0 \cup \mathfrak{s}_\infty$.*

To see this, consider the subbuilding $\boldsymbol{F}_{1,1}$ of $\boldsymbol{F}$ consisting of its middle and bottom level curves mapping to the copies of $\mathbb{R} \times S^*\mathbb{T}^2$ and $T^*\mathbb{T}^2$ that correspond to $L_{1,1}$. Since it is connected and has genus zero, it follows from Proposition 3.3 of [8] that

$$\operatorname{index}(\boldsymbol{F}_{1,1}) = 2(s-1),$$

where s is the number of positive ends of $\boldsymbol{F}_{1,1}$. Since $\boldsymbol{F}_{1,1}$ passes through the $d+1$ generic point constraints on $L_{1,1}$, and the Fredholm index in these manifolds is nondecreasing under multiple covers, we must also have

$$\operatorname{index}(\boldsymbol{F}_{1,1}) \geq 2(d+1).$$

Hence, $\boldsymbol{F}_{1,1}$ has at least $d+2$ positive ends. Under the assumption that $\boldsymbol{F}$ has Type 2b, two of these positive ends match with the two essential top level curves of $\boldsymbol{F}$. This leaves at least d positive ends of $\boldsymbol{F}_{1,1}$ that match with top level curves of $\boldsymbol{F}$ that cover planes in $\mathfrak{s}_0 \cup \mathfrak{s}_\infty$.

Remark 3.21 The same argument implies that if $\boldsymbol{F}$ has Type 3, then again it must include at least d planes in $\mathfrak{s}_0 \cup \mathfrak{s}_\infty$.

Claim 2 *If F has Type 2b, then it includes d planes in $\mathfrak{r}_0$ and none in $\mathfrak{r}_\infty$.*

The d constraint points on $\mathbb{L}$ imply that, if F is of Type 2b, it must contain d planes covering broken planes asymptotic to $\mathbb{L}$. These planes match, via cylinders in $T^*\mathbb{T}^2$, with asymptotic ends of an essential curve, and by Lemma 3.16 these ends are either all positive or all negative. Hence we have d planes either all in $\mathfrak{r}_0$ or all in $\mathfrak{r}_\infty$. To show that these planes cannot be in $\mathfrak{r}_\infty$, we consider intersections with $S_0 \cup S_\infty$. Overall, the top level curves of F must intersect $S_0 \cup S_\infty$ exactly $2d$ times. The planes of F asymptotic to $L_{1,1}$ from Claim 1 account for at least d of these intersections.

Since $\mathbb{L}$ is homologically trivial in Y, by Lemma 3.11 each plane of $\mathfrak{r}_\infty$ must intersect both S_0 and S_∞, while the planes in $\mathfrak{r}_0$ intersect neither of these spheres. If the d planes of F asymptotic to $\mathbb{L}$ were in $\mathfrak{r}_\infty$ then they would contribute another $2d$ intersections with $S_0 \cup S_\infty$. By positivity of intersection, this cannot happen, so these planes must belong to $\mathfrak{r}_0$ as claimed.

To complete the argument, we now balance areas. The total area of all the curves in F is $2(d+1)$. If F has Type 2b, then the planes from Claim 1 and Claim 2 have total area at least $2d$. Its essential curves must then have total area equal to 2. Also, they must contribute the remaining d intersections with $S_0 \cup S_\infty$. It follows from Lemma 3.19, that this is impossible for all d sufficiently large. Hence F cannot be of Type 2b, and must instead be of Type 3. Arguing as above, it follows that in addition to its three essential top level curves, F must then have d planes in $\mathfrak{s}_0 \cup \mathfrak{s}_\infty$ and $d-1$ planes in $\mathfrak{r}_0 \cup \mathfrak{r}_\infty$. □

Proposition 3.22 *For all sufficiently large d, there exists a limiting building G as in the* ***stretching scenario for class*** $(1,d)$ *such that G is of Type 3. In addition to its three essential curves it consists of d planes in $\mathfrak{r}_0 \cup \mathfrak{r}_\infty$ and $d-1$ planes in $\mathfrak{s}_0 \cup \mathfrak{s}_\infty$.*

Proof Here we fix d points on $L_{1,1}$ and $d+1$ points on $\mathbb{L}$, and for J_τ as in Proposition 3.20 consider the limit, G, of a convergent sequence of J_{τ_k}–holomorphic spheres, for $\tau_k \to \infty$, that represent the class $(1,d)$ and pass through the $2d+1$ constraint points. The point constraints imply that G is not of Type 0.

If G was of Type 1, the point constraints would imply that G includes at least d planes, which by Lemma 3.16 either all lie in $\mathfrak{s}_0$ or all lie in $\mathfrak{s}_\infty$, and at least $d+1$ planes either all in $\mathfrak{r}_0$ or all in $\mathfrak{r}_\infty$. From this it follows that the essential curve of G would have area 1. Recalling Lemma 3.11, since L is homologically trivial, the planes of $\mathfrak{r}_\infty$ each intersect $S_0 \cup S_\infty$ twice. Arguing as in Claim 2 from the proof of Proposition 3.20, if the planes asymptotic to $\mathbb{L}$ lie in $\mathfrak{r}_\infty$ then the broken planes will contribute a total of $d+2(d+1)$ intersections with $S_0 \cup S_\infty$, a contradiction as there are only $2d$ such intersections. On the other hand, if these planes all lie in $\mathfrak{r}_0$ then the essential curve must contribute d intersections with $S_0 \cup S_\infty$. As this essential curve has area 1, this contradicts Lemma 3.19 when d is sufficiently large. Hence, G is not of Type 1.

Next we show that G cannot be of Type 2b. Assume that it is. Then G includes $d+1$ planes either all in $\mathfrak{r}_0$ or all in $\mathfrak{r}_\infty$. Counting intersections as above, G must have $d+1$ planes in $\mathfrak{r}_0$.

Arguing as in Claim 1 above, we consider the subbuilding $\boldsymbol{G}_{1,1}$ of $\boldsymbol{G}$ consisting of its middle and bottom level curves that map to the copies of $\mathbb{R} \times S^*\mathbb{T}^2$ and $T^*\mathbb{T}^2$ that correspond to $L_{1,1}$. Since $\boldsymbol{G}_{1,1}$ is connected and has genus zero, we have

$$\operatorname{index}(\boldsymbol{G}_{1,1}) = 2(s-1),$$

where s is the number of positive ends of $\boldsymbol{G}_{1,1}$. Since $\boldsymbol{G}_{1,1}$ passes through the d generic point constraints on $\mathbb{L}$, we also have

$$\operatorname{index}(\boldsymbol{G}_{1,1}) \geq 2d.$$

Hence, $\boldsymbol{G}_{1,1}$ has at least $d+1$ positive ends. Two of these positive ends match with negative ends of the two essential curves of $\boldsymbol{G}_{1,1}$. It follows that $\boldsymbol{G}$ must have at least $d-1$ planes in $\mathfrak{s}_0 \cup \mathfrak{s}_\infty$. This means the planes covering broken leaves then have area at least $2d$. As the limiting building has total area $2d+2$ and also includes two essential curves, we see that the essential curves each have area 1 and there are exactly $d-1$ planes in $\mathfrak{s}_0 \cup \mathfrak{s}_\infty$. As the planes in $\mathfrak{r}_0$ are disjoint from $S_0 \cup S_\infty$, the essential curves of $\boldsymbol{G}$ must have $d+1$ intersections with $S_0 \cup S_\infty$. Lemma 3.19 again implies that this is impossible for all sufficiently large d.

Finally we show that $\boldsymbol{G}$ cannot be of Type 2a. In this case $\boldsymbol{G}$ includes d planes in $\mathfrak{r}_0 \cup \mathfrak{r}_\infty$ and d planes all in either $\mathfrak{s}_0$ or all in $\mathfrak{s}_\infty$. The planes asymptotic to $L_{1,1}$ thus account for all intersections with either S_0 or S_∞ and so the planes asymptotic to $\mathbb{L}$ therefore all lie in $\mathfrak{r}_0$. The essential curves have total area 2 and must together account for all intersections with either S_0 or S_∞. This contradicts Lemma 3.19 as before. □

Lemma 3.23 *All curves in the limiting buildings $\boldsymbol{F}$ and $\boldsymbol{G}$ that map to $S^2 \times S^2 \setminus (\mathbb{L} \cup L_{1,1})$ have area 1, and in particular are simply covered.*

Proof To see this, first observe that classes in $H_2(S^2 \times S^2, \mathbb{L} \cup L_{1,1})$ all have integral area. Indeed, adding classes which lie only in $H_2(S^2 \times S^2, \mathbb{L})$ or $H_2(S^2 \times S^2, L_{1,1})$, which have integral area by monotonicity, any relative class can be completed to an integral area absolute homology class.

Note that since $\boldsymbol{F}$ is of Type 3, it has its three essential curves together with $2d-1$ other top level curves that cover leaves of the foliation. Since $\boldsymbol{F}$ has total area $2d+2$ and all curves have integral area, the result for $\boldsymbol{F}$ follows.

The same argument applies to $\boldsymbol{G}$. □

3.8 A collection of symplectic spheres and disks, under Assumption 2

Let J be a tame almost complex structure on $S^2 \times S^2 \setminus (\mathbb{L} \cup L_{1,1})$ that is adapted to parametrizations ψ and $\psi_{1,1}$ of $\mathbb{L}$ and $L_{1,1}$, respectively. Recall that for the projection $p : S^2 \times S^2 \to S_\infty$, defined by the foliation $\mathcal{F}$ corresponding to J, the images $p(\mathbb{L})$ and $p(L_{1,1})$ are disjoint circles. There are also disjoint disks $A_0 \subset S_\infty$ with boundary $p(\mathbb{L})$ and $A_\infty \subset S_\infty$ with boundary $p(L_{1,1})$ such that $p(T_0) \in A_0$ and $p(T_\infty) \in A_\infty$; see Figure 1. In this section we will prove the following result.

Proposition 3.24 *For sufficiently large d there exist embedded symplectic spheres*

$$F, G\colon S^2 \to S^2 \times S^2 \setminus (\mathbb{L} \cup L_{1,1})$$

in the class $(1, d)$, and embedded symplectic disks

$$\mathbb{E}\colon (D^2, S^1) \to (S^2 \times S^2, \mathbb{L}) \quad \text{and} \quad E_{1,1}\colon (D^2, S^1) \to (S^2 \times S^2, L_{1,1})$$

of Maslov index 2, such that:

1. *F, G, $\mathbb{E}$ and $E_{1,1}$ are all J–holomorphic away from arbitrarily small neighborhoods of a collection of Lagrangian tori whose elements are near to, and Lagrangian isotopic to, either $\mathbb{L}$ or $L_{1,1}$.*
2. *The class of $\mathbb{E}|_{S^1}$ and the foliation class $\beta_{\mathbb{L}}$ form an integral basis of $H_1(\mathbb{L} : \mathbb{Z})$.*
3. *The class of $E_{1,1}|_{S^1}$ and the foliation class $\beta_{L_{1,1}}$ form an integral basis of $H_1(L_{1,1} : \mathbb{Z})$.*
4. *Exactly one of F and G intersects the planes of $\mathfrak{r}_0$ and the other intersects the planes of $\mathfrak{r}_\infty$.*
5. *Exactly one of F and G intersects the planes of $\mathfrak{s}_0$ and the other intersects the planes of $\mathfrak{s}_\infty$.*
6. *$F \bullet \mathbb{E} + G \bullet \mathbb{E} = d$.*
7. *$F \bullet E_{1,1} + G \bullet E_{1,1} = d$.*
8. *$F \bullet G = 2d$.*
9. *The set $p(F \cap G)$ consists of d points in A_0 and d points in A_∞.*

Remark 3.25 This proposition is the key to our result. Following Theorem 1.1, the spheres F and G will eventually be transformed to form axes of a new copy of $S^2 \times S^2$ in which $\mathbb{L}$ and $L_{1,1}$ must intersect. A natural approach to finding spheres in the complement of the Lagrangians may have been to fix constraint points in the complement of $\mathbb{L} \cup L_{1,1}$ and then take a limit of holomorphic spheres through these points as we stretch along $\mathbb{L} \cup L_{1,1}$. Indeed, generically this does give holomorphic spheres in the complement, but it seems difficult to obtain in this way a pair of spheres where one intersects the family $\mathfrak{r}_0$ and the other the family $\mathfrak{r}_\infty$.

Our alternative approach is to start with the Type 3 curves given by Propositions 3.20 and 3.22, and this is why we need to assume d is large. The Type 3 buildings intersect our Lagrangians in such a way that they can be deformed, by a diffeomorphism supported near the Lagrangians, into cycles which are disjoint from $\mathbb{L}$ and $L_{1,1}$ and have the required intersections. The process is illustrated in Figure 4. In the figure, the blue curves correspond to curves in the building $\boldsymbol{F}$ and the black curves correspond to the deformed cycle contained in the complement of $\mathbb{L}$. The curves running vertically correspond to broken leaves of the foliation, and those running horizontally to essential curves.

More precisely, we will deform the building $\boldsymbol{F}$ constructed in Proposition 3.20 to a building $\boldsymbol{F}(\{\boldsymbol{v}_1, \boldsymbol{w}_1\})$ containing curves asymptotic to Lagrangians $\mathbb{L}(\boldsymbol{v}_1)$ and $L_{1,1}(\boldsymbol{w}_1)$. In local coordinates $\mathbb{L}(\boldsymbol{v}_1)$ will be a translation of $\mathbb{L}$ and $L_{1,1}(\boldsymbol{w}_1)$ a translation of $L_{1,1}$. Similarly, the building $\boldsymbol{G}$ constructed in Proposition 3.22 is deformed to a building $\boldsymbol{G}(\{\boldsymbol{v}_2, \boldsymbol{w}_2\})$ containing curves asymptotic to Lagrangians $\mathbb{L}(\boldsymbol{v}_2)$

and $L_{1,1}(\boldsymbol{w}_2)$, where again, in local coordinates, $\mathbb{L}(\boldsymbol{v}_2)$ will be a translation of $\mathbb{L}$ and $L_{1,1}(\boldsymbol{w}_2)$ a translation of $L_{1,1}$.

Our proof proceeds by describing these deformations carefully and then remarking that the images of the buildings can be smoothed to form our symplectic spheres. This smoothing occurs only near $\mathbb{L}(\boldsymbol{v}_1)$ and $L_{1,1}(\boldsymbol{w}_1)$ for $\boldsymbol{F}(\{\boldsymbol{v}_1, \boldsymbol{w}_1\})$, and only near $\mathbb{L}(\boldsymbol{v}_2)$ and $L_{1,1}(\boldsymbol{w}_2)$ for $\boldsymbol{G}(\{\boldsymbol{v}_2, \boldsymbol{w}_2\})$. As the six Lagrangian tori $\mathbb{L}$, $\mathbb{L}(\boldsymbol{v}_1)$, $\mathbb{L}(\boldsymbol{v}_2)$, $L_{1,1}$, $L_{1,1}(\boldsymbol{w}_1)$ and $L_{1,1}(\boldsymbol{w}_2)$ are disjoint, and in fact disjoint from any intersections between different buildings, this smoothing does not affect our intersection pattern calculation.

Proof of Proposition 3.24 In what follows, $\boldsymbol{F}$ and $\boldsymbol{G}$ will be limiting J–holomorphic buildings of Type 3 as established in Proposition 3.20 and Proposition 3.22, respectively. We assume that they are in the same class $(1, d)$ for some large d.

The top level curves of $\boldsymbol{F}$ are

$$\{u_{\mathbb{L}}, \underline{u}, u_{L_{1,1}}, u_1, \dots, u_{d-1}, \mathfrak{u}_1, \dots, \mathfrak{u}_d\}.$$

Here, $u_{\mathbb{L}}$, $\underline{u}$ and $u_{L_{1,1}}$ are the essential curves of $\boldsymbol{F}$. For some nonnegative integer $\alpha_0 \leq d-1$, the curves $u_1, \dots, u_{\alpha_0}$ belong to $\mathfrak{r}_0$ and the curves $u_{\alpha_0+1}, \dots, u_{d-1}$ belong to $\mathfrak{r}_\infty$. For some nonnegative integer $\beta_0 \leq d$, the curves $\mathfrak{u}_1, \dots, \mathfrak{u}_{\beta_0}$ belong to $\mathfrak{s}_0$ and the curves $\mathfrak{u}_{\beta_0+1}, \dots, \mathfrak{u}_d$ belong to $\mathfrak{s}_\infty$.

Similarly, the top level curves of $\boldsymbol{G}$ are

$$\{v_{\mathbb{L}}, \underline{v}, v_{L_{1,1}}, v_1, \dots, v_d, \mathfrak{v}_1, \dots, \mathfrak{v}_{d-1}\},$$

where for some nonnegative integer $\gamma_0 \leq d$ the curves $v_1, \dots, v_{\gamma_0}$ belong to $\mathfrak{r}_0$ and $v_{\gamma_0+1}, \dots, v_d$ belong to $\mathfrak{r}_\infty$, and for some nonnegative integer $\delta_0 \leq d-1$, the curves $\mathfrak{v}_1, \dots, \mathfrak{v}_{\delta_0}$ belong to $\mathfrak{s}_0$ and the curves $\mathfrak{v}_{\delta_0+1}, \dots, \mathfrak{v}_{d-1}$ belong to $\mathfrak{s}_\infty$.

3.8.1 Deformations near $\mathbb{L}$ Consider the coordinates (P_1, Q_1, P_2, Q_2) in the Weinstein neighborhood of $\mathbb{L}$,

$$\mathcal{U}(\mathbb{L}) = \{|P_1| < \epsilon, |P_2| < \epsilon\}.$$

For each *translation vector* $\boldsymbol{v} = (a, b) \in (-\epsilon, \epsilon) \times (-\epsilon, \epsilon)$, there is a corresponding nearby Lagrangian torus

$$\mathbb{L}(\boldsymbol{v}) = \mathbb{L}(a, b) = \{P_1 = a, P_2 = b\} \subset \mathcal{U}(\mathbb{L}).$$

Note that the parametrization ψ of $\mathbb{L}$ determines an obvious parametrization, $\psi(\boldsymbol{v}) = \psi(a, b)$ of $\mathbb{L}(a, b)$, and a canonical isomorphism from $H_1^{\psi}(L; \mathbb{Z})$ to $H_1^{\psi(a,b)}(L(a, b); \mathbb{Z})$.

Given a finite, nonempty collection of translation vectors,

$$\boldsymbol{V} = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\} = \{(a_1, b_1), \dots, (a_k, b_k)\},$$

let $J_{\boldsymbol{V}}$ be an almost complex structure on the complement of the collection of Lagrangians

$$\mathbb{L}(\boldsymbol{V}) = \bigcup_{i=1}^{k} \mathbb{L}(\boldsymbol{v}_i),$$

which coincides with J outside $\mathcal{U}(\mathbb{L})$ and inside has the form

$$J_V \frac{\partial}{\partial Q_i} = -\rho_V \frac{\partial}{\partial P_i}, \tag{3-1}$$

where ρ_V is a positive function away from $\mathbb{L}(V)$ and in a neighborhood of each $\mathbb{L}(v_i)$ has the form

$$\rho_V = \sqrt{(P_1 - a_i)^2 + (P_2 - b_i)^2}.$$

In this case, we say that J_V is *adapted to* $\mathbb{L}(V)$ *with respect to* ψ. The set of all such almost complex structures adapted to some nontrivial collection $\mathbb{L}(V) \subset \mathcal{U}(\mathbb{L})$ will be denoted by $\mathcal{J}_{\mathcal{U}(\mathbb{L})}$.

Following Section 2.5 of [5], for each J_V in $\mathcal{J}_{\mathcal{U}(\mathbb{L})}$ one can construct, for $\tau \geq 0$, a standard family of almost complex structures $J_{V,\tau}$ on $S^2 \times S^2$ that are tame with respect to $\pi_1^*\omega + \pi_2^*\omega$, such that the limit $\tau \to \infty$ corresponds to the process of stretching the neck along small sphere bundles surrounding each of the components of $\mathbb{L}(V)$; see [1]. The structure J_V is the part of the limit of the $J_{V,\tau}$ corresponding to $S^2 \times S^2 \setminus (\mathbb{L}(V) \cup L_{1,1})$. The limit of the Gromov foliations for the $J_{V,\tau}$, in class $(0,1)$, yields a foliation $\mathcal{F}(V)$ of $S^2 \times S^2 \setminus (\mathbb{L}(V) \cup L_{1,1})$. For example, for $V = \{(0,0)\}$ we have $J_V = J$ and $\mathcal{F}(V) = \mathcal{F}$.

Lemma 3.26 *Leaves of the foliation $\mathcal{F}(V)$ intersect $\mathcal{U}(\mathbb{L})$ along the annuli $\{P_1 = \delta, Q_1 = \theta, |P_2| < \epsilon\}$. A leaf of $\mathcal{F}(V)$ that intersects $\mathcal{U}(\mathbb{L})$ along the annulus $\{P_1 = \delta, Q_1 = \theta, |P_2| < \epsilon\}$ is broken if and only if the collection V contains an element of the form (δ, b_i).*

Proof It follows from equation (3-1) that these annuli are J_V–holomorphic. By assuming J satisfies the conclusions of Lemma 3.4, they also extend to J_V–holomorphic spheres in the class $(0,1)$. By positivity of intersection, these spheres, and indeed any holomorphic sphere in the class $(0,1)$, are leaves of the foliation $\mathcal{F}(V)$. □

First deformation process Our first deformation process allows us to deform a regular J–holomorphic curve so that its ends on $\mathbb{L}$ become ends on a nearby Lagrangian $\mathbb{L}(v)$.

Lemma 3.27 (Fukaya's trick) *Let u be a regular J–holomorphic curve with $k \geq 0$ ends on $\mathbb{L}$ and $l \geq 0$ ends on $L_{1,1}$. For all $v = (a,b)$ with $\|v\|^2 = a^2 + b^2$ sufficiently small, there is a regular J_v–holomorphic curve $u(v)$ with k ends on $\mathbb{L}(v)$ and l ends on $L_{1,1}$. Moreover, the ends of $u(v)$ on $\mathbb{L}(v)$ represent the identical classes in $H_1^{\psi(v)}(L,\mathbb{R})$, as do those of u in $H_1^{\psi}(L,\mathbb{R})$. The classes corresponding to the ends of $u(v)$ on $L_{1,1}$ are also identical to those of u.*

Proof For $\|v\|$ sufficiently small, the Lagrangian isotopy $t \mapsto \mathbb{L}(tv)$ for $0 \leq t \leq 1$ is contained in $\mathcal{U}(\mathbb{L})$. Let $f_{t,v}$ be a family of diffeomorphisms of $S^2 \times S^2$ such that

- $f_{0,v}$ is the identity map,
- $f_{t,v}(\mathbb{L}) = \mathbb{L}(tv)$ for all $t \in [0,1]$,
- each $f_{t,v}$ is equal to the identity map outside of $\mathcal{U}(\mathbb{L})$, and
- $\|f_{t,v}\|_{C^1}$ is of order 1 in $\|v\|$.

Let $J_{t\boldsymbol{v}}$ be a family of tame almost complex structures in $\mathscr{J}_{\mathcal{U}(\mathbb{L})}$ such that each $J_{t\boldsymbol{v}}$ is adapted to $\mathbb{L}(t\boldsymbol{v})$ with respect to ψ. Set

$$\widetilde{J}_{t\boldsymbol{v}} = (f_{t,\boldsymbol{v}}^{-1})_* J_{t\boldsymbol{v}}.$$

For $\|\boldsymbol{v}\|$ sufficiently small, $\widetilde{J}_{t\boldsymbol{v}}$ is a tame almost complex structure on $S^2 \times S^2 \setminus (\mathbb{L} \cup L_{1,1})$ for all $t \in [0,1]$. Since u is regular, for sufficiently small $\|\boldsymbol{v}\|$ the curve u persists to yield a regular $\widetilde{J}_{\boldsymbol{v}}$–holomorphic curve $\widetilde{u}(\boldsymbol{v})$ with the same asymptotic behavior as u. By our choice of $\widetilde{J}_{t\boldsymbol{v}}$, the curve

$$u(\boldsymbol{v}) = f_{1,\boldsymbol{v}} \circ \widetilde{u}(\boldsymbol{v})$$

is then regular, $J_{\boldsymbol{v}}$–holomorphic and has k ends on $\mathbb{L}(\boldsymbol{v})$ instead of $\mathbb{L}$. □

Applying Lemma 3.27 to $\boldsymbol{F}$ and $\boldsymbol{G}$ To apply Lemma 3.27 to the top level curves of $\boldsymbol{F}$ and $\boldsymbol{G}$ we need these curves to be regular. Lemma 3.23 implies that the top level curves of the buildings $\boldsymbol{F}$ and $\boldsymbol{G}$ are somewhere injective. Since they are the limits of embedded curves, they are actually embedded and hence regular for generic choice of J. The work of Wendl in [21] implies that they are regular for all J.

Lemma 3.28 *For any tame almost complex structure J on*

$$(S^2 \times S^2 \setminus (\mathbb{L} \cup L_{1,1}), \pi_1^*\omega + \pi_2^*\omega)$$

that is adapted to both ψ and $\psi_{1,1}$, the top level curves of the buildings $\boldsymbol{F}$ and $\boldsymbol{G}$ are all regular.

Proof By [21, Theorem 1], any embedded J–holomorphic curve u mapping to $S^2 \times S^2 \setminus (\mathbb{L} \cup L_{1,1})$ is regular if its Fredholm index is greater than or equal to the number of its asymptotic ends. In our setting, we have

$$\operatorname{index}(u) = s - 2 + 2c_1(u^*T(S^2 \times S^2), u^*TL),$$

where s is the number of ends and c_1 is the relative first Chern class. It suffices to show that for each top level curve u of the buildings $\boldsymbol{F}$ and $\boldsymbol{G}$, we have $c_1(u^*T(S^2 \times S^2), u^*TL) \geq 1$.

Since $\boldsymbol{F}$ and $\boldsymbol{G}$ both have Type 3, it follows from Lemma 3.23 that each top level curve u is either a J–holomorphic plane or cylinder. If u is a plane, then $2c_1(u^*T(S^2 \times S^2), u^*TL)$ is just the Maslov class. By monotonicity, this is equal to 2 since, by Lemma 3.23, our top level curves all have area 1.

If u is a J–holomorphic cylinder, we can then produce a disk v from it by compactifying the ends of u and smoothly gluing a disk w to one of them. If the disk w has area A then, by monotonicity, it has Maslov index $2A$. By additivity of the area and the Chern class, v has area $A+1$ and Maslov index $2c_1(u^*T(S^2 \times S^2), u^*TL) + 2A$. Since the area is $A+1$, by monotonicity v must have Maslov index $2(A+1)$. This implies that $2c_1(u^*T(S^2 \times S^2), u^*TL) = 2$, as required. □

For $\boldsymbol{v} = (a,b)$ with $\|\boldsymbol{v}\|$ sufficiently small we now define the deformed building $\boldsymbol{F}(\boldsymbol{v})$ as follows. The top level curves of $\boldsymbol{F}(\boldsymbol{v})$ are obtained by applying Lemma 3.27 to those top level curves of $\boldsymbol{F}$ with ends on $\mathbb{L}$, and leaving the others unchanged. That is, the top level curves of $\boldsymbol{F}(\boldsymbol{v})$ are

$$\{u_{\mathbb{L}}(\boldsymbol{v}), \underline{u}(\boldsymbol{v}), u_{L_{1,1}}, u_1(\boldsymbol{v}), \ldots, u_{d-1}(\boldsymbol{v}), \mathrm{u}_1, \ldots, \mathrm{u}_d\}.$$

The middle and bottom level curves of $\boldsymbol{F}(\boldsymbol{v})$ are the same as those of $\boldsymbol{F}$ except they are now considered to map to copies of $\mathbb{R} \times S^*\mathbb{T}^2$ and $T^*\mathbb{T}^2$ that correspond to $\mathbb{L}(\boldsymbol{v})$ rather than $\mathbb{L}$.

Note that $\boldsymbol{F}(\boldsymbol{v})$ still has a continuous compactification $\bar{\boldsymbol{F}}(\boldsymbol{v})\colon S^2 \to S^2 \times S^2$, which can be deformed arbitrarily close to $\mathbb{L}(\boldsymbol{v})$ to obtain a smooth sphere $F = F(\boldsymbol{v})\colon S^2 \to S^2 \times S^2$ which is $J_{\boldsymbol{v}}$–holomorphic away from a small neighborhood of $\mathbb{L}(\boldsymbol{v})$.

Lemma 3.29 *Set $\boldsymbol{v} = (a, b)$ and $\boldsymbol{V} = \{(0,0), \boldsymbol{v}\}$ and suppose that $\|\boldsymbol{v}\|$ is small enough for $\boldsymbol{F}(\boldsymbol{v})$ to exist. If a and b are both nonzero and $|a|$ is sufficiently small with respect to $|b|$, then each top level curve of $\boldsymbol{F}(\boldsymbol{v})$ is $J_{\boldsymbol{V}}$–holomorphic for some $J_{\boldsymbol{V}}$ in $J_{\mathcal{U}(\mathbb{L})}$.*

Proof By Lemma 3.26, for any adapted almost complex structure, the leaves of the corresponding foliation intersect $\mathcal{U}(\mathbb{L})$ in the annuli $\{P_1 = \delta, Q_1 = \theta, |P_2| < \epsilon\}$. Hence if $b \neq 0$ and $a = 0$, the preimages of the regions A_0 and B for $\mathbb{L}(\boldsymbol{v})$ intersect $\mathcal{U}(\mathbb{L})$ in the subsets $\{P_1 < 0\}$ and $\{P_1 > 0\}$, respectively (since they consist of the leaves which are not broken along $\mathbb{L}(\boldsymbol{v})$). It follows that the closures of the essential curves of $\boldsymbol{F}(\boldsymbol{v})$ are disjoint from $\mathbb{L}$: the curves themselves are disjoint since they project to the regions A_0, B or A_∞, and they are compactified by circles in $\mathbb{L}(0, b)$ or $L_{1,1}$.

By continuity, these essential curves remain disjoint from $\mathbb{L}$ also for sufficiently small a when we deform using Lemma 3.27. Therefore, for all $|a|$ sufficiently small, the essential curves of $\boldsymbol{F}(\boldsymbol{v})$ are $J_{\boldsymbol{V}}$–holomorphic for any $J_{\boldsymbol{V}}$ which only differs from $J_{\boldsymbol{v}}$ in a small enough neighborhood of $\mathbb{L}$. Meanwhile, any top level curves of $\boldsymbol{F}(\boldsymbol{v})$ that cover broken leaves intersect $\mathcal{U}(\mathbb{L})$ in annuli lying in $\{P_1 = a\}$, and these annuli are holomorphic for any adapted almost complex structure. □

Arguing in a similar fashion we can assert that the top level curves of $\boldsymbol{F}(\boldsymbol{v})$ are $J_{\boldsymbol{V}}$–holomorphic for more general collections $\boldsymbol{V}$. For example, we have the following statement.

Lemma 3.30 *Set $\boldsymbol{v}_1 = (a_1, b_1)$, $\boldsymbol{v}_2 = (a_2, b_2)$ and $\boldsymbol{V} = \{(0,0), \boldsymbol{v}_1, \boldsymbol{v}_2\}$, and suppose that $\|\boldsymbol{v}_1\|$ is small enough for $\boldsymbol{F}(\boldsymbol{v}_1)$ to exist. If a_1 and b_1 are both nonzero, $|a_1|$ is sufficiently small with respect to $|b_1|$, and $\|\boldsymbol{v}_2\|$ is sufficiently small with respect to $|a_1|$, then each top level curve of $\boldsymbol{F}(\boldsymbol{v}_1)$ is $J_{\boldsymbol{V}}$–holomorphic for some $J_{\boldsymbol{V}}$ in $J_{\mathcal{U}(\mathbb{L})}$.*

The deformed building $\boldsymbol{G}(\boldsymbol{v})$ is defined analogously, and satisfies the analogues of Lemmas 3.29 and 3.30.

Second deformation process Consider $\boldsymbol{V} = \{(0,0), (a_1, b_1), (a_2, b_2)\}$ with b_1 and b_2 nonzero. For suitable choices of a_i and b_i, our second deformation process deforms the essential J–holomorphic curve $u_{\mathbb{L}}$ of $\boldsymbol{F}$ into a curve, $u_{\mathbb{L}}^{\boldsymbol{V}}$, which has the same asymptotics as $u_{\mathbb{L}}$ but is $J_{\boldsymbol{V}}$–holomorphic for some $J_{\boldsymbol{V}}$ that is adapted to $\mathbb{L}(\boldsymbol{V})$ with respect to ψ.

The primary deformation result is as follows.

Lemma 3.31 *Set $\boldsymbol{v} = (0, b)$, $\boldsymbol{V} = \{(0, 0), \boldsymbol{v}\}$ and suppose that $0 < |b| < \epsilon$. For $s \in [0, 1]$, let J_s be a smooth family of almost complex structures in $\mathcal{J}_{\mathcal{U}(\mathbb{L})}$ such that*

- *$J_0 = J$,*
- *J_s is adapted to $\mathbb{L}$, with respect to ψ, for all $s \in [0, 1)$, and*
- *J_1 is adapted to $\mathbb{L}(\boldsymbol{V})$ with respect to ψ.*

Then the essential curve $u_{\mathbb{L}}$ of $\boldsymbol{F}$ belongs to a smooth family of J_s–holomorphic planes $u_{\mathbb{L}}(s)$ for $s \in [0, 1]$. Moreover, the $J_{\boldsymbol{V}}$–holomorphic plane

$$u_{\mathbb{L}}(1)\colon \mathbb{C} \to S^2 \times S^2 \setminus (\mathbb{L}(\boldsymbol{V}) \cup L_{1,1})$$

is disjoint from the region $\{P_1 > 0\}$ and is essential with respect to $\mathcal{F}$, and the closure of the image of $p \circ u_{\mathbb{L}}(1)$ is A_0.

Proof By Lemma 3.23, the initial curve $u_{\mathbb{L}}$ has area equal to 1. Since $\mathbb{L}$ is monotone, no degenerations are possible until $s = 1$. In other words, the family of deformed curves $u_{\mathbb{L}}(s)$ exists for all $s \in [0, 1)$ and it suffices to show that it extends to $s = 1$. To prove the first assertion of Lemma 3.31 we argue by contradiction, and assume that there is a sequence $s_j \to 1$ such that the curves $u_{\mathbb{L}}(s_j)$ converge to a nontrivial $J_{\boldsymbol{V}}$–holomorphic building $\boldsymbol{H}$ which includes curves with punctures asymptotic to $\mathbb{L}(\boldsymbol{v})$. We will show that this implies that, unlike $u_{\mathbb{L}}$, none of the curves of $\boldsymbol{H}$ intersect T_0, a contradiction.

Claim 1 *Let v be a $J_{\boldsymbol{V}}$–holomorphic curve of $\boldsymbol{H}$. Any puncture of v asymptotic to $\mathbb{L}(\boldsymbol{v})$ must cover a closed geodesic in a class $(k, l) \in H_1(\mathbb{L}(\boldsymbol{v}); \mathbb{Z})$ with $k \leq 0$.*

Since the closure of $p \circ u_{\mathbb{L}}$ is A_0, by our choice of coordinates in Section 3.5, $u_{\mathbb{L}}$ is disjoint from the leaves of $\mathcal{F}$ which intersect $\mathcal{U}(\mathbb{L})$ in the region $\{P_1 > 0\}$. The same is true of the curves $u_{\mathbb{L}}(s)$ for all $s < 1$. Hence, v must also be disjoint from these leaves. The curve v can be extended smoothly to the oriented blow-up of the relevant puncture, such that the resulting map $\bar{v}$ acts on the corresponding boundary circle as

$$\theta \mapsto (0, b, Q_1 + k\theta, Q_2 + l\theta)$$

for some $Q_1, Q_2 \in S^1$. The tangent space to the image of $\bar{v}$ at a boundary point on the circle is spanned by $\{k\, \partial/\partial Q_1 + l\, \partial/\partial Q_2, k\, \partial/\partial P_1 + l\, \partial/\partial P_2\}$. If k were positive, this would contradict the fact that v is disjoint from the leaves through $\{P_1 > 0\}$ since $\boldsymbol{v} = (0, b)$. This proves Claim 1.

Claim 2 *Let v be a $J_{\boldsymbol{V}}$–holomorphic curve with a puncture that is asymptotic to $\mathbb{L}(\boldsymbol{v})$ along a geodesic in a class which is a multiple of the foliation class, ie of the form $(0, l) \in H_1^{\psi(\boldsymbol{v})}(\mathbb{L}(\boldsymbol{v}); \mathbb{Z})$. Then v must cover a plane or cylinder of a twice broken leaf of the foliation $\mathcal{F}(\boldsymbol{V})$.*

This follows, as in [8, Lemma 6.2], from the asymptotic properties of holomorphic curves and the fact that v lies in $\{P_1 \le 0\}$. Let w be a broken plane asymptotic (modulo taking to covers) to the same Reeb orbit as an end of v. Then if v does not cover w it must intersect all nearby leaves of the foliation, including those which lie in the region $\{P_1 > 0\}$. This gives a contradiction as in Claim 1, proving Claim 2.

We can now complete the proof of the first assertion of Lemma 3.31. Let $\boldsymbol{H}_{\text{top}}$ denote the collection of top level curves of $\boldsymbol{H}$, let $\boldsymbol{H}_{\mathbb{L}}$ be the subbuilding consisting of the middle and bottom level curves of $\boldsymbol{H}$ that map to the copies of $\mathbb{R} \times S^*\mathbb{T}^2$ and $T^*\mathbb{T}^2$ corresponding to $\mathbb{L}$, and let $\boldsymbol{H}_{\boldsymbol{v}}$ be the subbuilding consisting of the middle and bottom level curves of $\boldsymbol{H}$ that map to the copies of $\mathbb{R} \times S^*\mathbb{T}^2$ and $T^*\mathbb{T}^2$ corresponding to $\mathbb{L}(\boldsymbol{v})$.

Consider the classes $(k_1, l_1), \dots, (k_m, l_m) \in H_1(\mathbb{L}(\boldsymbol{v}); \mathbb{Z})$ of the geodesics determined by all of the punctures of top level curves of $\boldsymbol{H}$ that are asymptotic to $\mathbb{L}(\boldsymbol{v})$. These constitute the boundary of the cycle in $\mathbb{L}(\boldsymbol{v})$ that is obtained by gluing together the compactifications of the curves of $\boldsymbol{H}_{\boldsymbol{v}}$. Hence, the sum of the classes $(k_1, l_1), \dots, (k_m, l_m)$ must be $(0,0)$ and, by Claim 1, each k_i must be zero. It then follows from Claim 2 that any curve of $\boldsymbol{H}$ with an end on $\mathbb{L}(\boldsymbol{v})$ must cover a plane or cylinder of a broken leaf of $\mathcal{F}(\boldsymbol{v})$.

Partition the curves of $\boldsymbol{H}_{\text{top}} \cup \boldsymbol{H}_{\boldsymbol{v}} = \boldsymbol{H} \setminus \boldsymbol{H}_{\mathbb{L}}$ into connected components based on the matching of their ends in the copies of $\mathbb{R} \times S^*\mathbb{T}^2$ and $T^*\mathbb{T}^2$ corresponding to $\mathbb{L}(\boldsymbol{v})$. Denote these components by $\boldsymbol{H}_1, \dots, \boldsymbol{H}_k$. The compactification of each $\boldsymbol{H}_j$ is a cycle representing a class in $\pi_2(S^2 \times S^2, \mathbb{L})$. By monotonicity, the symplectic area of this cycle is a positive integer. Since the area of $u_{\mathbb{L}}$ is one, we must have $k = 1$ and the area of the cycle determined by $\boldsymbol{H}_1$ must be one. Assuming the limit is a building including curves asymptotic to $\mathbb{L}(\boldsymbol{v})$, by definition all curves of $\boldsymbol{H}_1$ have ends on $\mathbb{L}(\boldsymbol{v})$. By the discussion above, this implies that all the curves of $\boldsymbol{H}_1$ must cover a plane or cylinder of a broken leaf of $\mathcal{F}(\boldsymbol{V})$ through $\mathbb{L}(\boldsymbol{v})$. None of these leaves intersect T_0, and neither do the curves of $\boldsymbol{H}_{\mathbb{L}}$. Hence, no curve of $\boldsymbol{H} = \boldsymbol{H}_1 \cup \boldsymbol{H}_{\mathbb{L}}$ intersects T_0, which is the desired contradiction.

The remaining assertions of Lemma 3.31 follow easily from positivity of intersection. To see that $u_{\mathbb{L}}(1)$ is disjoint from the region $\{P_1 > 0\}$ note that the initial curve $u_{\mathbb{L}}$ is disjoint from the leaves of the foliation that intersect this region since its image under p is A_0 and our choice of coordinates has $\{P_1 > 0\}$ projecting to B. Positivity of intersection implies that no new intersections of the $u_{\mathbb{L}}(s)$ with these fibers can appear during the deformation.

Finally, since $u_{\mathbb{L}}(1)$ does not cover a leaf of the foliation, it also follows from positivity of intersection that $u_{\mathbb{L}}(1)$ is disjoint from the hypersurface $\{P_1 = 0\}$. In particular, any intersection with leaves in $\{P_1 = 0\}$ would imply intersections with the region $\{P_1 > 0\}$. Hence the closure of $p \circ u_{\mathbb{L}}(1)$ is equal to A_0. □

Translating the Lagrangian tori of Lemma 3.31 slightly in the P_1–direction, we get the following generalization.

Corollary 3.32 *Let $u_{\mathbb{L}}$ be the essential curve of $\boldsymbol{F}$ which is mapped by p onto A_0. Choose nonzero constants b_1 and b_2 in $(-\epsilon, \epsilon)$. If $\delta > 0$ is sufficiently small, then for any a_1 and a_2 in $(-\delta, \delta)$ and*

$$\boldsymbol{V} = \{(0,0), (a_1, b_1), (a_2, b_2)\},$$

there is a $J_{\boldsymbol{V}}$–holomorphic curve

$$u_{\mathbb{L}}^{\boldsymbol{V}} : \mathbb{C} \to S^2 \times S^2 \setminus (\mathbb{L}(\boldsymbol{V}) \cup L_{1,1})$$

in the class of $u_{\mathbb{L}}$ such that $u_{\mathbb{L}}^{\boldsymbol{V}}$ is disjoint from the region $\{P_1 > 0\}$ and is essential with respect to $\mathscr{F}$, and the closure of the image of $p \circ u_{\mathbb{L}}^{\boldsymbol{V}}$ is A_0.

Proof This has been established in the case when $a_1 = a_2 = 0$. But then if the a_i are sufficiently small we can appeal to Lemma 3.27 to see that still there are no degenerations. □

Intersections near $\mathbb{L}$ Consider translation data

$$\boldsymbol{V} = \{\boldsymbol{0}, \boldsymbol{v}_1, \boldsymbol{v}_2\} = \{(0,0), (a_1, b_1), (a_2, b_2)\}.$$

In what follows we will always assume that $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are distinct and the a_i and b_i are as small as necessary but not zero. If $\|\boldsymbol{v}_1\|$ is sufficiently small then, as described in Lemma 3.30, the deformed building $\boldsymbol{F}(\boldsymbol{v}_1)$ is well defined and its top level curves

$$\{u_{\mathbb{L}}(\boldsymbol{v}_1), \underline{u}(\boldsymbol{v}_1), u_{L_{1,1}}, u_1(\boldsymbol{v}_1), \dots, u_{d-1}(\boldsymbol{v}_1), \mathfrak{u}_1, \dots, \mathfrak{u}_d\}$$

are all $J_{\boldsymbol{V}}$–holomorphic for some $J_{\boldsymbol{V}}$ in $J_{\mathfrak{U}(\mathbb{L})}$.

We also assume that Corollary 3.32 holds for $\boldsymbol{V}$. This yields a $J_{\boldsymbol{V}}$–holomorphic curve $u_{\mathbb{L}}^{\boldsymbol{V}}$ which is disjoint from the region $\{P_1 > 0\}$ and intersects the leaves of $\mathscr{F}(\boldsymbol{V})$ that pass through the planes $\{P_1 = c < 0,\ Q_1 = \theta\}$ exactly once.

The intersection number between each top level curve of $\boldsymbol{F}(\boldsymbol{v}_1)$ and the curve $u_{\mathbb{L}}^{\boldsymbol{V}}$ is well defined since the curves of $\boldsymbol{F}(\boldsymbol{v}_1)$ are disjoint from $\mathbb{L}$, as established in Lemma 3.30; see Figure 4. We denote the total of these intersection numbers by $\boldsymbol{F}(\boldsymbol{v}_1) \bullet u_{\mathbb{L}}^{\boldsymbol{V}}$.

Similarly, the intersection number of each top level curve of $\boldsymbol{F}(\boldsymbol{v}_1)$ with any of the planes in either $\mathfrak{r}_0$ or $\mathfrak{r}_\infty$ is well defined and all such intersections are positive. Since this number is the same for any plane in the family, we denote these numbers by $\boldsymbol{F}(\boldsymbol{v}_1) \bullet \mathfrak{r}_0$ and $\boldsymbol{F}(\boldsymbol{v}_1) \bullet \mathfrak{r}_\infty$, respectively.

Let $\bar{\boldsymbol{F}}(\boldsymbol{v}_1) \colon S^2 \to S^2 \times S^2$ be the compactification of $\boldsymbol{F}(\boldsymbol{v}_1)$, let $\mathbb{E} \colon (D^2, S^1) \to (S^2 \times S^2, \mathbb{L})$ be the compactification of the curve $u_{\mathbb{L}}^{\boldsymbol{V}}$, and let $\bar{\mathfrak{r}}_0$ and $\bar{\mathfrak{r}}_\infty$ be the solid tori obtained by compactifying the planes of $\mathfrak{r}_0$ and $\mathfrak{r}_\infty$. Deforming $\bar{\boldsymbol{F}}(\boldsymbol{v}_1)$ in a neighborhood of $\mathbb{L}(\boldsymbol{v}_1)$, we obtain a smooth map $F = F(\boldsymbol{v}_1) \colon S^2 \to S^2 \times S^2$ such that

$$F \bullet \mathbb{E} = \bar{\boldsymbol{F}}(\boldsymbol{v}_1) \bullet \mathbb{E} = \boldsymbol{F}(\boldsymbol{v}_1) \bullet u_{\mathbb{L}}^{\boldsymbol{V}}, \tag{3-2}$$

$$F \bullet \bar{\mathfrak{r}}_* = \bar{\boldsymbol{F}}(\boldsymbol{v}_1) \bullet \bar{\mathfrak{r}}_* = \boldsymbol{F}(\boldsymbol{v}_1) \bullet \mathfrak{r}_* \quad \text{for } * = 0, \infty, \tag{3-3}$$

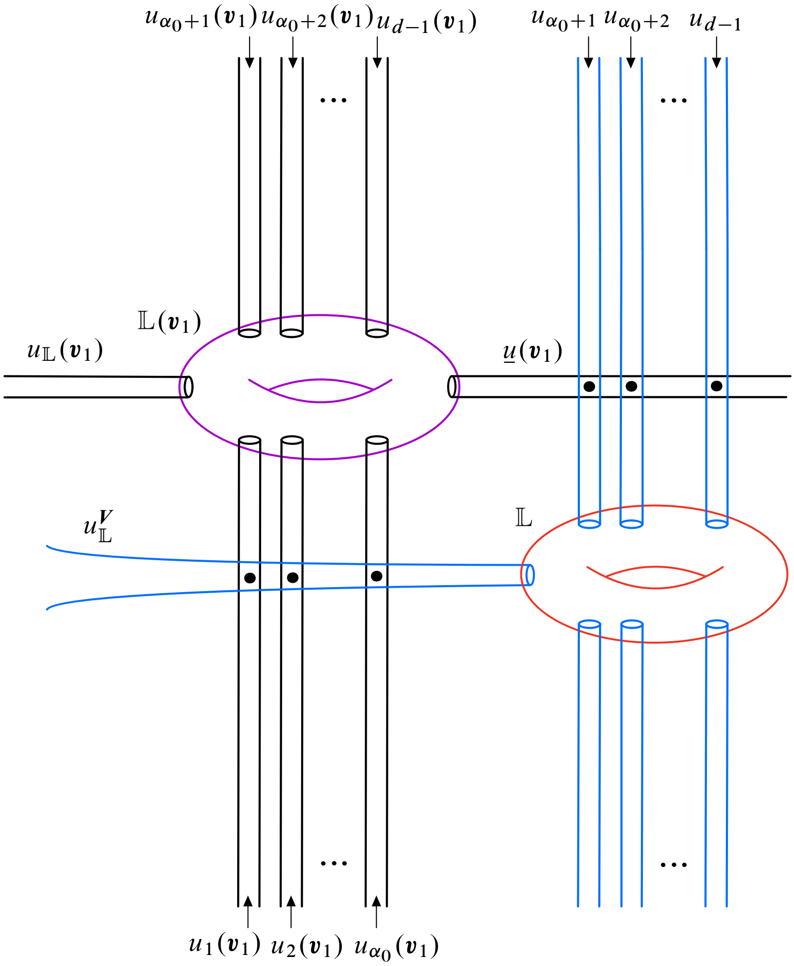

Figure 4: The intersection pattern of Lemma 3.33 for the case $b_1 > 0$. The large dots in the figure represent the isolated intersections points, in $\mathcal{U}(\mathbb{L})$, of the relevant pairs of curves.

where $\bar{\boldsymbol{F}}(\boldsymbol{v}_1) \bullet \bar{\mathfrak{r}}_*$ denotes the intersection number with any disk in the family. Moreover, the intersection points that determine the equal intersection numbers in (3-2) and (3-3) are identical.

Recall that α_0 is the number of top level curves of $\boldsymbol{F}$ lying in $\mathfrak{r}_0$. Hence, by Proposition 3.20, there are $d-1-\alpha_0$ top level curves lying in $\mathfrak{r}_\infty$.

Lemma 3.33 *Consider $V = \{\boldsymbol{0}, \boldsymbol{v}_1, \boldsymbol{v}_2\} = \{(0,0), (a_1, b_1), (a_2, b_2)\}$ such that $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are distinct, a_1 is negative, and b_1 and b_2 are nonzero. Suppose that $|a_1|$ is sufficiently small with respect to $|b_1|$.*

If $b_1 > 0$, then $F \bullet \bar{\mathfrak{r}}_0 = 0$, $F \bullet \bar{\mathfrak{r}}_\infty = 1$ and $F \bullet \mathbb{E} = \alpha_0$.

If $b_1 < 0$, then $F \bullet \bar{\mathfrak{r}}_0 = 1$, $F \bullet \bar{\mathfrak{r}}_\infty = 0$ and $F \bullet \mathbb{E} = d - 1 - \alpha_0$.

Proof Here we give the proof of the case when b_1 is positive. The proof for $b_1 < 0$ is identical and is left to the reader.

The situation for $b_1 > 0$ is illustrated in Figure 4, where the black curves can be deformed near $\mathbb{L}(\boldsymbol{v}_1)$ to form our sphere F. As the figure suggests, the contribution to $F \bullet \bar{\mathfrak{r}}_0$ from intersections in $\mathcal{U}(\mathbb{L})$ is zero, the contribution to $F \bullet \bar{\mathfrak{r}}_\infty$ from intersections in $\mathcal{U}(\mathbb{L})$ is one, and the contribution to $F \bullet \mathbb{E}$ from intersections in $\mathcal{U}(\mathbb{L})$ is α_0. These assertions are proven below along with the fact that there are no other contributions to these numbers.

The map F represents the class $(1, d)$. For each disk in $\bar{\mathfrak{r}}_0$ there is a companion disc in $\bar{\mathfrak{r}}_\infty$ such that the pair can be glued together, along $\mathbb{L}$, to form a sphere in the class $(0, 1)$. Hence,

$$F \bullet \bar{\mathfrak{r}}_0 + F \bullet \bar{\mathfrak{r}}_\infty = 1.$$

Since all intersections are positive, in order to prove that $F \bullet \bar{\mathfrak{r}}_0 = 0$, and $F \bullet \bar{\mathfrak{r}}_\infty = 1$, it suffices to prove that $F \bullet \bar{\mathfrak{r}}_\infty \geq 1$. In particular, it suffices to show that for the curve $\underline{u}(\boldsymbol{v}_1)$ of $\boldsymbol{F}(\boldsymbol{v}_1)$, we have $\underline{u}(\boldsymbol{v}_1) \bullet \mathfrak{r}_\infty \geq 1$.

The curve $\underline{u}(\boldsymbol{v}_1)$ is essential and projects under p to the region in S_∞ bounded by $p(\mathbb{L}(\boldsymbol{v}_1))$ and $p(L_{1,1})$. Thus it intersects $\mathcal{U}(\mathbb{L})$ in the region $\{P_1 > a_1\}$ and intersects all leaves of the foliation which meet $\mathcal{U}(\mathbb{L})$ in this set. Also, if $\mathcal{U}(\mathbb{L})$ is sufficiently small, it intersects $\mathcal{U}(\mathbb{L})$ inside $\{P_2 > 0\}$. This is true when $a_1 = 0$ because $b_1 > 0$, and remains true for small a_1 by continuity. As $\mathfrak{r}_\infty$ intersects $\mathcal{U}(\mathbb{L})$ in the region $\{P_1 = 0,\ P_2 > 0\}$ and $\mathfrak{r}_0$ in the region $\{P_1 = 0,\ P_2 < 0\}$, we see that $\underline{u}(\boldsymbol{v}_1)$ intersects the planes in $\mathfrak{r}_\infty$ rather than those in $\mathfrak{r}_0$, as required.

It remains to prove that $F \bullet \mathbb{E} = \alpha_0$ when $|a_1|$ is sufficiently small with respect to $|b_1|$. By (3-2), and the fact that the top level curves of $\boldsymbol{F}(\boldsymbol{v}_1)$ are

$$\{u_{\mathbb{L}}(\boldsymbol{v}_1), \underline{u}(\boldsymbol{v}_1), u_{L_{1,1}}, u_1(\boldsymbol{v}_1), \dots, u_{d-1}(\boldsymbol{v}_1), \mathfrak{u}_1, \dots, \mathfrak{u}_d\},$$

it suffices to prove that for $|a_1|$ sufficiently small with respect to $|b_1|$, we have

$$u_i(\boldsymbol{v}_1) \bullet u_{\mathbb{L}}^{\boldsymbol{V}} = 1 \quad \text{for } 1 \leq i \leq \alpha_0, \tag{3-4}$$

and $u_{\mathbb{L}}^{\boldsymbol{V}}$ is disjoint from all the other top level curves of $\boldsymbol{F}(\boldsymbol{v}_1)$.

By Corollary 3.32, the curve $u_{\mathbb{L}}^{\boldsymbol{V}}$ is essential for $\mathcal{F}$, and the closure of the image of $p \circ u_{\mathbb{L}}^{\boldsymbol{V}}$ is A_0. So if w is another curve in $S^2 \times S^2$ and $p \circ w$ is disjoint from A_0, then $u_{\mathbb{L}}^{\boldsymbol{V}}$ is disjoint from w. This observation implies that $u_{\mathbb{L}}^{\boldsymbol{V}}$ is disjoint from $u_{L_{1,1}}$ and the $\mathfrak{u}_j$ for $j = 1, \dots, d$, since these curves all project into A_∞.

Another consequence of $u_{\mathbb{L}}^{\boldsymbol{V}}$ being essential with respect to $\mathcal{F}$ is that it intersects any fiber of $\mathcal{F}$ either once or not at all. The curve $u_{\mathbb{L}}^{\boldsymbol{V}}$ intersects $\mathcal{U}(\mathbb{L})$ in the region $\{P_1 < 0\}$ and has an end asymptotic to a circle in $\mathbb{L} = \{P_1 = P_2 = 0\}$. Since $b_1 > 0$, this implies that for all $a_1 < 0$ such that $|a_1|$ is sufficiently small with respect to b_1, $u_{\mathbb{L}}^{\boldsymbol{V}}$ must intersect the annuli of the form $\{P_1 = a_1, Q_1 = \theta, P_2 < b_1\}$ exactly once. Now the planes $u_i(\boldsymbol{v}_1)$ all belong to broken fibers of $\mathcal{F}$ that intersect $\mathcal{U}(\mathbb{L})$. For $1 \leq i \leq \alpha_0$, the curves $u_i(\boldsymbol{v}_1)$ intersect $\mathcal{U}(\mathbb{L})$ in annuli of the form $\{P_1 = a_1, Q_1 = \theta, P_2 < b_1\}$. For $i > \alpha_0$, the $u_i(\boldsymbol{v}_1)$ intersect $\mathcal{U}(\mathbb{L})$ in annuli of the form $\{P_1 = a_1, Q_1 = \theta, P_2 > b_1\}$. Hence, for $1 \leq i \leq \alpha_0$, $u_{\mathbb{L}}^{\boldsymbol{V}}$ intersects the fiber of $\mathcal{F}$ containing $u_i(\boldsymbol{v}_1)$ at a point on $u_i(\boldsymbol{v}_1)$. This yields equation (3-4). On the other hand, for $i > \alpha_0$, $u^V_{\mathbb{L}}$ intersects the fiber of $\mathcal{F}$ containing $u_i(\boldsymbol{v}_1)$ at a point in the complement of $u_i(\boldsymbol{v}_1)$. Hence, $u^V_{\mathbb{L}}$ is disjoint from these curves.

Next we show that, when $|a_1|$ is sufficiently small with respect to $|b_1|$, $u^V_{\mathbb{L}}$ is disjoint from $\underline{u}(\boldsymbol{v}_1)$. Considering projections, it is clear that the part of $\underline{u}(\boldsymbol{v}_1)$ in the complement of $\mathcal{U}(\mathbb{L})$ is disjoint from $u^V_{\mathbb{L}}$ since its projection is contained in the interior of $B \cup A_\infty$.

Suppose that $a_1 = 0$. Then $\underline{u}((0, b_1)) \cap \mathcal{U}(\mathbb{L})$ is contained in $\{P_1 > 0\}$ and is asymptotic to $\mathbb{L}(0, b_1)$. This is disjoint from $u^V_{\mathbb{L}} \cap \mathcal{U}(\mathbb{L})$, which is contained in $\{P_1 < 0\}$ and is asymptotic to $\mathbb{L}$. By continuity, $\underline{u}((a_1, b_1)) \cap \mathcal{U}(\mathbb{L})$ is then disjoint from $u^V_{\mathbb{L}} \cap \mathcal{U}(\mathbb{L})$ for all $a_1 < 0$ with $|a_1|$ sufficiently small with respect to $|b_1|$.

Lastly, we must prove that

$$u_{\mathbb{L}}(\boldsymbol{v}_1) \bullet u^V_{\mathbb{L}} = 0$$

when $|a_1|$ is sufficiently small with respect to $|b_1|$. Following Lemma 3.31 the compactifications of $u^V_{\mathbb{L}}$ and $u_{\mathbb{L}}$ are homotopic in the space of smooth maps $(D^2, S^1) \to (p^{-1}(A_0), \mathbb{L})$, so for a_1 sufficiently small it suffices to show that

$$u_{\mathbb{L}}(\boldsymbol{v}_1) \bullet u_{\mathbb{L}} = 0.$$

Let $\bar{u}_{\mathbb{L}}(\boldsymbol{v}_1)$ and $\bar{u}_{\mathbb{L}}$ be compactifications of $u_{\mathbb{L}}(\boldsymbol{v}_1)$ and $u_{\mathbb{L}}$. We claim that $u_{\mathbb{L}}(\boldsymbol{v}_1) \bullet u_{\mathbb{L}} = 0$ is equivalent to the fact that the Maslov index of $\bar{u}_{\mathbb{L}}$ is equal to 2. To see this we recall that

$$\mu(\bar{u}_{\mathbb{L}}) = 2c_1(\bar{u}_{\mathbb{L}}), \tag{3-5}$$

where $c_1(\bar{u}_{\mathbb{L}})$ is the relative Chern number of $\bar{u}_{\mathbb{L}}$, which is equal to the number of zeros of a generic section ξ of $\bar{u}^*_{\mathbb{L}}(\Lambda^2(T(S^2 \times S^2)))$ such that $\xi|_{S^1}$ is nonvanishing and is tangent to $\Lambda^2(T\mathbb{L})$.

Let $\nu(\bar{u}_{\mathbb{L}})$ be the normal bundle to the embedding $\bar{u}_{\mathbb{L}}$ and fix an identification of $\bar{u}^*_{\mathbb{L}}(T(S^2 \times S^2))$ with the Whitney sum $\nu(\bar{u}_{\mathbb{L}}) \oplus T(D^2)$. For polar coordinates (r, θ) on D^2 consider the section $r\,\partial/\partial\theta$ of $\bar{u}^*_{\mathbb{L}}(T(S^2 \times S^2))$. The restriction $r\,\partial/\partial\theta|_{S^1}$ is nonvanishing and tangent to $T\mathbb{L}$.

Replacing $\boldsymbol{v}_1$ by $t\boldsymbol{v}_1$ for some small $t > 0$, if necessary, we may assume that $\bar{u}_{\mathbb{L}}(\boldsymbol{v}_1)$ is close enough $\bar{u}_{\mathbb{L}}$, in the C^1–topology, to be identified with a section, $\sigma_{\mathbb{L}}(\boldsymbol{v}_1)$, of $\nu(\bar{u}_{\mathbb{L}}) \subset \bar{u}^*_{\mathbb{L}}(T(S^2 \times S^2))$. The restriction $\sigma_{\mathbb{L}}(\boldsymbol{v}_1)|_{S^1}$ is roughly parallel to the vector field $\partial/\partial P_2$. By rotating in the normal bundle this section is homotopic through nonvanishing sections of the normal bundle to a section of $T\mathbb{L}$ along ∂D^2 which is orthogonal to $\partial/\partial\theta$.

Set $\xi = r\,\partial/\partial\theta \wedge \sigma_{\mathbb{L}}(\boldsymbol{v}_1)$. It follows from the discussion above that $\xi|_{S^1}$ is nonvanishing and is tangent to $\Lambda^2(T\mathbb{L})$. Moreover, the zeroes of ξ correspond to the union of the zeros of $r\,\partial/\partial\theta$ and $\sigma_{\mathbb{L}}(\boldsymbol{v}_1)$. Since $\bar{u}_{\mathbb{L}}$ is embedded, the zeros of $\sigma_{\mathbb{L}}(\boldsymbol{v}_1)$ exactly correspond to the intersections $u_{\mathbb{L}}(\boldsymbol{v}_1) \bullet u_{\mathbb{L}}$. By (3-5), we have

$$\mu(\bar{u}_{\mathbb{L}}) = 2(1 + u_{\mathbb{L}}(\boldsymbol{v}_1) \bullet u_{\mathbb{L}}).$$

As $\mu(\bar{u}_{\mathbb{L}}) = 2$ (as it has area 1 by Lemma 3.23, and $\mathbb{L}$ is monotone) we have $u_{\mathbb{L}}(\boldsymbol{v}_1) \bullet u_{\mathbb{L}} = 0$. □

Assuming that $\boldsymbol{v}_2 = (a_2, b_2)$ is sufficiently small, we can deform the building $\boldsymbol{G}$ to obtain a new building $\boldsymbol{G}(\boldsymbol{v}_2)$ with top level curves

$$\{v_{\mathbb{L}}(\boldsymbol{v}_2), \underline{v}(\boldsymbol{v}_2), u_{L_{1,1}}, v_1(\boldsymbol{v}_2), \dots, v_d(\boldsymbol{v}_2), \mathfrak{v}_1, \dots, \mathfrak{v}_{d-1}\}.$$

Let $\bar{\boldsymbol{G}}(\boldsymbol{v}_2)\colon S^2 \to S^2 \times S^2$ be the compactification of $\boldsymbol{G}(\boldsymbol{v}_2)$. Again we can deform $\bar{\boldsymbol{G}}(\boldsymbol{v}_2)$, arbitrarily close to $\mathbb{L}(\boldsymbol{v}_2)$, to get a smooth map $G = G(\boldsymbol{v}_2)\colon S^2 \to S^2 \times S^2$ such that

$$G \bullet \mathbb{E} = \bar{\boldsymbol{G}}(\boldsymbol{v}_2) \bullet \mathbb{E} = \boldsymbol{G}(\boldsymbol{v}_2) \bullet u_{\mathbb{L}}^{\boldsymbol{V}},$$
$$G \bullet \bar{\mathfrak{r}}_* = \bar{\boldsymbol{G}}(\boldsymbol{v}_2) \bullet \bar{\mathfrak{r}}_* = \boldsymbol{G}(\boldsymbol{v}_2) \bullet \mathfrak{r}_* \quad \text{for } * = 0, \infty.$$

Arguing as in the proof of Lemma 3.33 we get the following.

Lemma 3.34 *Consider $\boldsymbol{V} = \{\boldsymbol{0}, \boldsymbol{v}_1, \boldsymbol{v}_2\} = \{(0,0), (a_1, b_1), (a_2, b_2)\}$ such that a_2 is negative, and b_1 and b_2 are nonzero. Suppose that $|a_2|$ is sufficiently small with respect to $|b_2|$.*

If $b_2 > 0$, then

$$G \bullet \mathbb{E} = \gamma_0 + v_{\mathbb{L}}(\boldsymbol{v}_2) \bullet u_{\mathbb{L}}^{\boldsymbol{V}}, \quad G \bullet \bar{\mathfrak{r}}_0 = 0 \quad \textit{and} \quad G \bullet \bar{\mathfrak{r}}_\infty = 1.$$

If $b_2 < 0$, then

$$G \bullet \mathbb{E} = d - \gamma_0 + v_{\mathbb{L}}(\boldsymbol{v}_2) \bullet u_{\mathbb{L}}^{\boldsymbol{V}}, \quad G \bullet \bar{\mathfrak{r}}_0 = 1 \quad \textit{and} \quad G \bullet \bar{\mathfrak{r}}_\infty = 0.$$

The term $v_{\mathbb{L}}(\boldsymbol{v}_2) \bullet u_{\mathbb{L}}^{\boldsymbol{V}}$ is not necessarily equal to zero. Instead we have the following identity.

Lemma 3.35 *For $\boldsymbol{V} = \{\boldsymbol{0}, \boldsymbol{v}_1, \boldsymbol{v}_2\} = \{(0,0), (a_1, b_1), (a_2, b_2)\}$, where b_1 and b_2 have opposite signs, and a_1 and a_2 are sufficiently small relative to b_1 and b_2, we have*

$$v_{\mathbb{L}}(\boldsymbol{v}_2) \bullet u_{\mathbb{L}}^{\boldsymbol{V}} = v_{\mathbb{L}}(\boldsymbol{v}_2) \bullet u_{\mathbb{L}}(\boldsymbol{v}_1).$$

Proof First we consider the case when $a_1 = a_2 = 0$. The image of the map $v_{\mathbb{L}}(\boldsymbol{v}_2)$ projects to A_0 and its boundary lies in $\mathbb{L}(\boldsymbol{v}_2)$. Hence, using our assumption on sign, the family of Lagrangians $\mathbb{L}(t\boldsymbol{v}_1)$ for $0 \le t \le 1$ are disjoint from the compactification of $v_{\mathbb{L}}(\boldsymbol{v}_2)$. It then follows from the proof of Lemma 3.27 that the compactification of $u_{\mathbb{L}}$ is connected to that of $u_{\mathbb{L}}(\boldsymbol{v}_1)$ by a path of smooth maps $u_t\colon (D^2, S^1) \to (S^2 \times S^2, \mathbb{L}(t\boldsymbol{v}_1))$. Therefore we have, as required,

$$v_{\mathbb{L}}(\boldsymbol{v}_2) \bullet u_{\mathbb{L}} = v_{\mathbb{L}}(\boldsymbol{v}_2) \bullet u_{\mathbb{L}}(\boldsymbol{v}_1).$$

For the general case we use the fact that the maps vary continuously with the parameters and so the intersection numbers remain unchanged for a_1 and a_2 sufficiently small. □

When $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are distinct, with $b_1 \neq b_2$ and $a_1 \neq a_2$ sufficiently small, the intersection numbers of the top level curves of $\boldsymbol{F}(\boldsymbol{v}_1)$ and $\boldsymbol{G}(\boldsymbol{v}_2)$ are also well defined. The following results concerning these intersections will be useful.

Lemma 3.36 *For $\boldsymbol{v}_1 = (a_1, b_1)$ and $\boldsymbol{v}_2 = (a_2, b_2)$, suppose that $a_1 < a_2 < 0$, with a_1 and a_2 sufficiently small.*

If $b_1 > b_2$, then

$$u_i(\boldsymbol{v}_1) \bullet v_{\mathbb{L}}(\boldsymbol{v}_2) = 1 \quad \text{for } i = 1, \dots, \alpha_0,$$
$$v_i(\boldsymbol{v}_2) \bullet \underline{u}(\boldsymbol{v}_1) = 1 \quad \text{for } i = \gamma_0 + 1, \dots, d.$$

If $b_1 < b_2$, then

$$u_i(\boldsymbol{v}_1) \bullet v_{\mathbb{L}}(\boldsymbol{v}_2) = 1 \quad \text{for } i = \alpha_0 + 1, \dots, d - 1,$$
$$v_i(\boldsymbol{v}_2) \bullet \underline{u}(\boldsymbol{v}_1) = 1 \quad \text{for } i = 1, \dots, \gamma_0.$$

Moreover, all the intersection points here project to A_0.

Proof Since the curves $\underline{u}(\boldsymbol{v}_1)$ and $v_{\mathbb{L}}(\boldsymbol{v}_2)$ are essential with respect to $\mathcal{F}$, they intersect a leaf of the foliation either once or not at all. Hence it suffices to detect a single intersection of the relevant pairs of curves listed. We detect an intersection for the first type of pair above and leave the other cases to the reader. For $1 \le i \le \alpha_0$ the planes $u_i(\boldsymbol{v}_1)$ intersect $\mathcal{U}(\mathbb{L})$ in annuli $\{P_1 = a_1, Q_1 = \theta, P_2 < b_1\}$. As $v_{\mathbb{L}}((0, b_2))$ is asymptotic to $\mathbb{L}((0, b_2)) = \{P_1 = 0, P_2 = b_2\}$ it intersects $u_i(\boldsymbol{v}_1)$ provided a_1 is sufficiently small (since the boundary of $v_{\mathbb{L}}((0, b_2))$ intersects all annuli $\{P_1 = 0, Q_1 = \theta, P_2 < b_1\}$). For a_2 sufficiently small, the plane $v_{\mathbb{L}}(\boldsymbol{v}_2)$ is a deformation of $v_{\mathbb{L}}((0, b_2))$ and so the intersection persists. As $v_{\mathbb{L}}(\boldsymbol{v}_2)$ intersects fibers at most once, the intersection number is equal to 1. Since $a_1 < 0$, the intersection point projects to A_0. □

Corollary 3.37 *For $\boldsymbol{v}_1 = (a_1, b_1)$ and $\boldsymbol{v}_2 = (a_2, b_2)$, suppose that $a_1 < a_2 < 0$, with a_1 and a_2 sufficiently small.*

If $b_1 > b_2$, then $F \cap G$ contains at least $\alpha_0 + d - \gamma_0$ points in $\mathcal{U}(\mathbb{L})$ that project to A_0.

If $b_1 < b_2$, then $F \cap G$ contains at least $d - 1 - \alpha_0 + \gamma_0$ points in $\mathcal{U}(\mathbb{L})$ that project to A_0.

Remark 3.38 It follows from Lemma 3.35 that any *excess* intersection points between F and G in $p^{-1}(A_0)$, that is, more than described by Corollary 3.37, correspond to intersection points between G and $\mathbb{E}$, at least if the b_i have opposite sign and the a_i are sufficiently small.

Adding deformations near $L_{1,1}$ To completely resolve the intersections of F and G we must also apply deformations in the Weinstein neighborhood

$$\mathcal{U}(L_{1,1}) = \{|p_1| < \epsilon, |p_2| < \epsilon\}.$$

Here we consider nearby Lagrangian tori of the form

$$L_{1,1}(\boldsymbol{w}) := \{p_1 = c, p_2 = d\} \quad \text{for } \boldsymbol{w} = (c, d) \in (-\epsilon, \epsilon) \times (-\epsilon, \epsilon).$$

The space of almost complex structures that are adapted to collections of these translated Lagrangian tori near $L_{1,1}$, with respect to $\psi_{1,1}$, is defined analogously to $\mathcal{J}_{\mathcal{U}(\mathbb{L})}$ and is denoted by $\mathcal{J}_{\mathcal{U}(L_{1,1})}$.

Given nontrivial collections

$$\boldsymbol{V}=\{\boldsymbol{v}_1,\dots,\boldsymbol{v}_k\}=\{(a_1,b_1),\dots,(a_k,b_k)\} \quad \text{and} \quad \boldsymbol{W}=\{\boldsymbol{w}_1,\dots,\boldsymbol{w}_l\}=\{(c_1,d_1),\dots,(c_l,d_l)\},$$

set $\boldsymbol{X}=\{\boldsymbol{V},\boldsymbol{W}\}$. Let $J_{\boldsymbol{X}}$ denote the corresponding (doubly) adapted almost complex structures in $\mathcal{J}_{\mathcal{U}(\mathbb{L})}\cap\mathcal{J}_{\mathcal{U}(L_{1,1})}$.

Lemma 3.27 generalizes to this setting as follows.

Lemma 3.39 *Let u be a regular J–holomorphic curve with $k\geq 0$ ends on $\mathbb{L}$ and $l\geq 0$ ends on $L_{1,1}$. For all $\boldsymbol{x}=\{\boldsymbol{v},\boldsymbol{w}\}=\{(a,b),(c,d)\}$ with $\|\boldsymbol{x}\|$ sufficiently small, there is a $J_{\boldsymbol{x}}$–holomorphic curve $u(\boldsymbol{x})$ that represents the class in $\pi_2(S^2\times S^2,\mathbb{L}(\boldsymbol{v})\cup L_{1,1}(\boldsymbol{w}))$ and which corresponds to the class $[u]$ in $\pi_2(S^2\times S^2,\mathbb{L}\cup L_{1,1})$ under the obvious identification. The curve $u(\boldsymbol{x})$ has k ends on $\mathbb{L}(\boldsymbol{v})$ and these represent the identical classes in $H_1^{\psi(\boldsymbol{v})}(\mathbb{L};\mathbb{Z})$ as do those of u in $H_1^{\psi}(\mathbb{L};\mathbb{Z})$. The curve also has l ends on $L_{1,1}(\boldsymbol{w})$ which represent the identical classes in $H_1^{\psi_{1,1}(\boldsymbol{w})}(L_{1,1};\mathbb{Z})$ as do those of u in $H_1^{\psi_{1,1}}(L_{1,1};\mathbb{Z})$.*

Corollary 3.32 generalizes as follows.

Lemma 3.40 *Let $u_{\mathbb{L}}$ and $u_{L_{1,1}}$ be the essential curves of a building $\boldsymbol{F}$ as in Proposition 3.20. Let $\boldsymbol{X}=\{\boldsymbol{V},\boldsymbol{W}\}$, where*

$$\boldsymbol{V}=\{(0,0),(a_1,b_1),(a_2,b_2)\} \quad \textit{and} \quad \boldsymbol{W}=\{(0,0),(c_1,d_1),(c_2,d_2)\}.$$

If b_1, b_2, d_1 and d_2 are in $(-\epsilon,\epsilon)$ and a_1, a_2, c_1 and c_2 are in $(-\delta,\delta)$, then for all sufficiently small δ there is a $J_{\boldsymbol{X}}$–holomorphic curve

$$u^{\boldsymbol{X}}_{\mathbb{L}}\colon \mathbb{C}\to S^2\times S^2\setminus(\mathbb{L}(\boldsymbol{V})\cup L_{1,1}(\boldsymbol{W}))$$

in the class of $u_{\mathbb{L}}$ such that $u^{\boldsymbol{X}}_{\mathbb{L}}$ is disjoint from the region $\{P_1>0\}$, the closure of the image of $p\circ u^{\boldsymbol{X}}_{\mathbb{L}}$ is A_0, and $u^{\boldsymbol{X}}_{\mathbb{L}}$ intersects, exactly once, the leaves of $\mathcal{F}(\boldsymbol{X})$ that pass through the planes $\{P_1=c<0,\ Q_1=\theta\}$.

There is also a $J_{\boldsymbol{X}}$–holomorphic curve

$$u^{\boldsymbol{X}}_{L_{1,1}}\colon \mathbb{C}\to S^2\times S^2\setminus(L(\boldsymbol{V})\cup L_{1,1}(\boldsymbol{W}))$$

in the class of $u_{L_{1,1}}$ such that $u^{\boldsymbol{X}}_{L_{1,1}}$ is disjoint from the region $\{p_1<0\}$, the closure of the image of $p\circ u^{\boldsymbol{X}}_{L_{1,1}}$ is A_∞, and $u^{\boldsymbol{X}}_{L_{1,1}}$ intersects, exactly once, the leaves of $\mathcal{F}(\boldsymbol{X})$ that pass through the planes $\{p_1=c>0, q_1=\theta\}$.

Completion of the proof of Proposition 3.24 Let $\boldsymbol{F}$ be a building as in Proposition 3.20 and let $\boldsymbol{G}$ be a building as in Proposition 3.22. Set

$$\boldsymbol{x}_1=\{\boldsymbol{v}_1,\boldsymbol{w}_1\}=\{(a_1,b_1),(c_1,d_1)\}, \qquad \boldsymbol{x}_2=\{\boldsymbol{v}_2,\boldsymbol{w}_2\}=\{(a_2,b_2),(c_2,d_2)\},$$

$$\boldsymbol{V}=\{\boldsymbol{0},\boldsymbol{v}_1,\boldsymbol{v}_2\}=\{(0,0),(a_1,b_1),(a_2,b_2)\}, \quad \boldsymbol{W}=\{\boldsymbol{0},\boldsymbol{w}_1,\boldsymbol{w}_2\}=\{(0,0),(c_1,d_1),(c_2,d_2)\},$$

and set

$$\boldsymbol{X}=\{\boldsymbol{V},\boldsymbol{W}\}.$$

We assume that $\|\boldsymbol{x}_1\|$ and $\|\boldsymbol{x}_2\|$ are small enough for Lemma 3.39 to yield the deformed buildings $\boldsymbol{F}(\boldsymbol{x}_1)$ and $\boldsymbol{G}(\boldsymbol{x}_2)$. We also assume that $|a_1|^2+|a_2|^2+|c_1|^2+|c_2|^2$ is small enough with respect to $|b_1|^2+|b_2|^2+|d_1|^2+|d_2|^2$ for Lemma 3.40 to yield the deformations $u_{\mathbb{L}}^{\boldsymbol{X}}$ and $u_{L_{1,1}}^{\boldsymbol{X}}$.

Let $\mathbb{E}\colon (D^2,S^1)\to(S^2\times S^2,\mathbb{L})$ be the compactification of $u_{\mathbb{L}}^{\boldsymbol{X}}$, and $E_{1,1}\colon (D^2,S^1)\to(S^2\times S^2,L_{1,1})$ be the compactification of $u_{L_{1,1}}^{\boldsymbol{X}}$. Since the homology classes represented by the ends of $u_{\mathbb{L}}^{\boldsymbol{X}}$ and $u_{L_{1,1}}^{\boldsymbol{X}}$ are identical to those of the essential curves $u_{\mathbb{L}}$ and $u_{L_{1,1}}$, the maps $\mathbb{E}$ and $E_{1,1}$ satisfy conditions (2) and (3) of Proposition 3.24.

Consider compactifications $\bar{\boldsymbol{F}}(\boldsymbol{x}_1)\colon S^2\to S^2\times S^2$ of $\boldsymbol{F}(\boldsymbol{x}_1)$, and $\bar{\boldsymbol{G}}(\boldsymbol{x}_2)\colon S^2\to S^2\times S^2$ of $\boldsymbol{G}(\boldsymbol{x}_2)$. Arguing as before, we can perturb these maps, arbitrarily close to the Lagrangians $\mathbb{L}(\boldsymbol{v}_1)$, $L_{1,1}(\boldsymbol{w}_1)$, $\mathbb{L}(\boldsymbol{v}_2)$ and $L_{1,1}(\boldsymbol{w}_2)$, to obtain smooth spheres F and G such that condition (1) of Proposition 3.24 holds.

It remains to verify the conditions (4) through (9) of Proposition 3.24, which involve intersections.

In the current setting, Lemma 3.33 holds as stated and the proof is unchanged.

Lemma 3.41 *Suppose a_1 is negative, and b_1 and b_2 are nonzero. Suppose that $|a_1|$ is sufficiently small with respect to $|b_1|$.*

If $b_1>0$, then $F\bullet\bar{\mathfrak{r}}_0=0$, $F\bullet\bar{\mathfrak{r}}_\infty=1$ and $F\bullet\mathbb{E}=\alpha_0$.

If $b_1<0$, then $F\bullet\bar{\mathfrak{r}}_0=1$, $F\bullet\bar{\mathfrak{r}}_\infty=0$ and $F\bullet\mathbb{E}=d-1-\alpha_0$.

Lemmas 3.34 and 3.35 and Corollary 3.37 change only in notation, and yield the following.

Lemma 3.42 *Suppose that a_2 is negative, b_1 and b_2 are nonzero, and $|a_2|$ is sufficiently small with respect to $|b_2|$.*

If $b_2>0$, then $G\bullet\bar{\mathfrak{r}}_0=0$, $G\bullet\bar{\mathfrak{r}}_\infty=1$ and

$$G\bullet\mathbb{E}=\gamma_0+v_{\mathbb{L}}(\boldsymbol{x}_2)\bullet u_{\mathbb{L}}^{\boldsymbol{X}}.$$

If $b_2<0$, then $G\bullet\bar{\mathfrak{r}}_0=1$, $G\bullet\bar{\mathfrak{r}}_\infty=0$ and

$$G\bullet\mathbb{E}=d-\gamma_0+v_{\mathbb{L}}(\boldsymbol{x}_2)\bullet u_{\mathbb{L}}^{\boldsymbol{X}}.$$

Lemma 3.43 *If b_1 and b_2 have opposite sign, and a_1 and a_2 are sufficiently small, then*

$$v_{\mathbb{L}}(\boldsymbol{x}_2)\bullet u_{\mathbb{L}}^{\boldsymbol{X}}=v_{\mathbb{L}}(\boldsymbol{x}_2)\bullet u_{\mathbb{L}}(\boldsymbol{x}_1).$$

Lemma 3.44 *Suppose that $a_1<a_2<0$, and a_1 and a_2 are sufficiently small.*

If $b_1>b_2$, then $F\cap G$ contains at least $\alpha_0+d-\gamma_0$ points in $\mathcal{U}(\mathbb{L})$ that project to A_0.

If $b_1<b_2$, then $F\cap G\cap\mathcal{U}(\mathbb{L})$ contains at least $d-1-\alpha_0+\gamma_0$ points in $\mathcal{U}(\mathbb{L})$ that project to A_0.

The following analogous results follow from similar arguments.

Lemma 3.45 *Suppose c_1 is positive, d_1 and d_2 are nonzero, and $|c_1|$ is sufficiently small with respect to $|d_1|$.*

If $d_1 > 0$, then $F \bullet \bar{\mathfrak{s}}_0 = 0$, $F \bullet \bar{\mathfrak{s}}_\infty = 1$ and $F \bullet E_{1,1} = \beta_0$.

If $d_1 < 0$, then $F \bullet \bar{\mathfrak{s}}_0 = 1$, $F \bullet \bar{\mathfrak{s}}_\infty = 0$ and $F \bullet E_{1,1} = d - \beta_0$.

Lemma 3.46 *Suppose c_2 is positive, d_1 and d_2 are nonzero, and $|c_2|$ is sufficiently small with respect to $|d_2|$.*

If $d_2 > 0$, then $G \bullet \bar{\mathfrak{s}}_0 = 0$, $G \bullet \bar{\mathfrak{s}}_\infty = 1$ and

$$G \bullet E_{1,1} = \delta_0 + v_{L_{1,1}}(\boldsymbol{x}_2) \bullet u^{\boldsymbol{X}}_{L_{1,1}}.$$

If $d_2 < 0$, then $G \bullet \bar{\mathfrak{s}}_0 = 1$, $G \bullet \bar{\mathfrak{s}}_\infty = 0$ and

$$G \bullet E_{1,1} = d - 1 - \delta_0 + v_{L_{1,1}}(\boldsymbol{x}_2) \bullet u^{\boldsymbol{X}}_{L_{1,1}}.$$

Lemma 3.47 *If d_1 and d_2 have opposite sign, and c_1 and c_2 are sufficiently small, then*

$$v_{L_{1,1}}(\boldsymbol{x}_2) \bullet u^{\boldsymbol{X}}_{L_{1,1}} = v_{L_{1,1}}(\boldsymbol{x}_2) \bullet u_{L_{1,1}}(\boldsymbol{x}_1).$$

Lemma 3.48 *Suppose that $c_1 > c_2 > 0$, and c_1 and c_2 are sufficiently small.*

If $d_1 > d_2$, then $F \cap G$ contains at least $\beta_0 + d - 1 - \delta_0$ points in $\mathcal{U}(L_{1,1})$ that project to A_∞.

If $d_1 < d_2$, then $F \cap G$ contains at least $d - \beta_0 + \delta_0$ points in $\mathcal{U}(L_{1,1})$ that project to A_∞.

With F and G fixed as above, the remaining analysis can be organized using the following two alternatives:

- **Alternative 1** Either $\alpha_0 \geq \gamma_0$ or $\gamma_0 \geq \alpha_0 + 1$.
- **Alternative 2** Either $\beta_0 \geq \delta_0 + 1$ or $\delta_0 \geq \beta_0$.

Case 1 ($\alpha_0 \geq \gamma_0$ and $\beta_0 \geq \delta_0 + 1$) In this case, we choose our translations so that

$$a_1 < a_2 < 0, \quad b_2 < 0 < b_1, \quad 0 < c_2 < c_1, \quad d_2 < 0 < d_1.$$

For these conditions on b_1 and b_2, Lemmas 3.41 and 3.42 yield $F \bullet \mathfrak{r}_0 = 0$, $F \bullet \mathfrak{r}_\infty = 1$, $G \bullet \mathfrak{r}_0 = 1$ and $G \bullet \mathfrak{r}_\infty = 0$. This implies condition (4) of Proposition 3.24.

Similarly, for these conditions on d_1 and d_2, Lemmas 3.45 and 3.46 imply that $F \bullet \mathfrak{s}_0 = 0$, $F \bullet \mathfrak{s}_\infty = 1$, $G \bullet \mathfrak{s}_0 = 1$ and $G \bullet \mathfrak{s}_\infty = 0$. This gives condition (5) of Proposition 3.24.

The maps F and G both represent the class $(1, d)$ in $H_2(S^2 \times S^2; \mathbb{Z})$, so $F \bullet G = (1, d) \bullet (1, d) = 2d$. On the other hand, for the choices above, Lemmas 3.44 and 3.48 imply that

$$F \bullet G \geq (\alpha_0 + d - \gamma_0) + (\beta_0 + d - 1 - \delta_0).$$

In the current case, with $\alpha_0 \geq \gamma_0$ and $\beta_0 \geq \delta_0 + 1$, these two summands are each at least d, and so we must have

$$\alpha_0 = \gamma_0, \tag{3-6}$$

$$\beta_0 = 1 + \delta_0. \tag{3-7}$$

It follows that $F \cap G$ consists of exactly $2d$ points, d of which are contained in $\mathcal{U}(\mathbb{L})$ and project to A_0, and d of which are contained in $\mathcal{U}(L_{1,1})$ and project to A_∞. This yields conditions (8) and (9) of Proposition 3.24.

Since $F \bullet G = \boldsymbol{F}(\boldsymbol{x}_1) \bullet \boldsymbol{G}(\boldsymbol{x}_2)$, it follows from the equalities above that there can be no intersections between the essential curves of $\boldsymbol{F}(\boldsymbol{x}_1)$ and those of $\boldsymbol{G}(\boldsymbol{x}_2)$. In particular, we must have

$$v_{\mathbb{L}}(\boldsymbol{x}_2) \bullet u_{\mathbb{L}}(\boldsymbol{x}_1) = 0, \tag{3-8}$$

$$v_{L_{1,1}}(\boldsymbol{x}_2) \bullet u_{L_{1,1}}(\boldsymbol{x}_1) = 0. \tag{3-9}$$

Equation (3-8) and Lemma 3.43 imply that

$$v_{\mathbb{L}}(\boldsymbol{x}_2) \bullet u_{\mathbb{L}}^{\boldsymbol{X}} = 0.$$

By Lemmas 3.41 and 3.42 and equation (3-6), we then have

$$F \bullet \mathbb{E} + G \bullet \mathbb{E} = \alpha_0 + d - \gamma_0 = d,$$

which yields condition (6) of Proposition 3.24.

Similarly, Lemmas 3.45, 3.46 and 3.47, together with equations (3-7) and (3-9), imply that

$$F \bullet E_{1,1} + G \bullet E_{1,1} = d$$

and hence condition (7) of Proposition 3.24. This completes the proof of Proposition 3.24 in the present case.

Other cases The proofs in the other cases follow along identical lines. For the sake of completeness we list the inequalities for the components of the translations that lead to the desired intersection patterns of Proposition 3.24, in the remaining scenarios. For the case $\alpha_0 \geq \gamma_0$ and $\delta_0 \geq \beta_0$, we choose

$$a_1 < a_2 < 0, \quad b_2 < 0 < b_1, \quad 0 < c_2 < c_1, \quad d_1 < 0 < d_2.$$

For $\gamma_0 \geq \alpha_0 + 1$ and $\beta_0 \geq \delta_0 + 1$, we choose

$$a_1 < a_2 < 0, \quad b_1 < 0 < b_2, \quad 0 < c_2 < c_1, \quad d_2 < 0 < d_1.$$

Finally, for the case $\gamma_0 \geq \alpha_0 + 1$ and $\delta_0 \geq \beta_0$, we choose

$$a_1 < a_2 < 0, \quad b_1 < 0 < b_2, \quad 0 < c_2 < c_1, \quad d_1 < 0 < d_2.$$

To complete the proof of Proposition 3.24, we remark that the smoothings F and G can be replaced by smooth symplectic spheres without changing the various intersection numbers. To do this, it is enough to replace F and G by symplectic spheres which coincide with F and G away from neighborhoods of $\mathbb{L}(\boldsymbol{v}_1)$ and $L_{1,1}(\boldsymbol{w}_1)$, respectively $\mathbb{L}(\boldsymbol{v}_2)$ and $L_{1,1}(\boldsymbol{w}_2)$; that is, the new spheres differ only away from all intersection points.

Now, we know that the asymptotic ends of the top level curves of $\boldsymbol{F}(\boldsymbol{x}_1)$ and $\boldsymbol{G}(\boldsymbol{x}_2)$ are simply covered, either because the curves are essential, or for covers of leaves by applying Lemma 3.23. Generically the asymptotic limits are distinct. Then, for small perturbations, we may assume that the top level curves restricted to a neighborhood of the Lagrangians are symplectically isotopic to the corresponding top level curves of our original buildings $\boldsymbol{F}$ and $\boldsymbol{G}$. (In the case of $\boldsymbol{F}(\boldsymbol{x}_1)$ the isotopy maps $\mathbb{L}(\boldsymbol{v}_1)$ and $L_{1,1}(\boldsymbol{w}_1)$ to $\mathbb{L}$ and $L_{1,1}$, respectively.) Finally, recall that the buildings $\boldsymbol{F}$ and $\boldsymbol{G}$ are limits of sequences of smooth embedded holomorphic spheres as our almost complex structures are stretched along the Lagrangians. Therefore, after a small perturbation, we may assume the top level curves of these buildings restricted to a compact subset of the complement of $\mathbb{L} \cup L_{1,1}$ extend to smooth symplectic spheres in $S^2 \times S^2$. Combining the isotopies and these extensions gives our symplectic spheres as required.

3.9 Scene change

Consider $(S^2 \times S^2, \pi_1^*\omega + \pi_2^*\omega)$ with our disjoint Lagrangian tori $\mathbb{L}$ and $L_{1,1}$ and the various symplectic spheres and disks constructed in Proposition 3.24: F, G, $\mathbb{E}$ and $E_{1,1}$.

To prepare for the proof of our main theorem, we specify our choice of almost complex structure. Near their various intersection points, the listed spheres and disks are already complex for a suitable almost complex structure. We can correct this almost complex structure, without perturbing F or G, to an almost complex structure J which is compatible with our symplectic form at the intersection points (not just tame) and extends to make our symplectic spheres and planes (the interiors of the disks) complex. Also we may assume J remains adapted to the parametrizations ψ and $\psi_{1,1}$ of $\mathbb{L}$ and $L_{1,1}$, respectively. As the spheres and planes from Proposition 3.24 were already holomorphic near the axes T_0, T_∞, S_0 and S_∞, and also near the broken planes in $\mathfrak{r}_0$, $\mathfrak{r}_\infty$, $\mathfrak{s}_0$ and $\mathfrak{s}_\infty$, we may assume that these curves all remain complex. In other words the only correction from the J used in Proposition 3.24 occurs near the intersection points to ensure compatibility, and near the regions where the F and G are symplectic but not complex.

Given this choice of J we have an associated foliation $\mathcal{F}$ and projection $p\colon S^2 \times S^2 \to S_\infty$. As the broken curves are the same as in Proposition 3.24, the subsets A_0, B and A_∞ of S_∞ are the same as in the proposition, and in particular property (9) continues to hold.

Let H be a sphere of $\mathcal{F}$ which is disjoint from $F \cap G$, and H_i for $1 \le i \le 2d$ the spheres of $\mathcal{F}$ intersecting the $2d$ points $\{p_1, \dots, p_{2d}\}$ of $F \cap G$. We note that these H_i are distinct since F and G both represent a homology class $(1, d)$ and so intersect fibers of $\mathcal{F}$ each in a single point.

One other small perturbation is required. We may choose Darboux charts about each p_i mapping an open set B_i to the round open ball about the 0 of capacity ϵ in $\mathbb{R}^4 = \mathbb{C}^2$, such that J is pushed forward to

	F	G	$\mathbb{E}$	$E_{1,1}$	H
F	$2d$				
G	$2d$	$2d$			
$\mathbb{E}$	k	$d-k$	$*$		
$E_{1,1}$	l	$d-l$	0	$*$	
H	1	1	$*$	$*$	0

	$\pi_1^*\omega+\pi_2^*\omega$–area
F	$2+2d$
G	$2+2d$
$\mathbb{E}$	1
$E_{1,1}$	1
H	2

Table 1: Initial intersection numbers, left, and initial symplectic areas, right.

the standard complex structure at 0 (this requires compatibility of J). We may assume these charts are disjoint from H. In these charts, F, G and H_i intersect the origin and are tangent to distinct complex planes. Making ϵ smaller if necessary, we are able to perturb our symplectic spheres so that they actually coincide with their tangent plane in the open chart. Finally we adjust J so that it is pushed forward to the standard structure on the whole ball, while F, G and H_i remain complex.

Given this, we proceed with the main proof. We start with the intersection pattern and area profile in Table 1.

For pairs of distinct curves the intersection numbers here just denote a signed count of intersection points with multiplicity. The self intersection number of closed spheres are defined as usual. The asterisks denote undefined quantities. The numbers come from the fact that F and G represent the class $(1,d)$, and from the properties listed in Proposition 3.24. The integers $0\le k\le d$ and $0\le l\le d$ are undetermined.

We now alter $(S^2\times S^2,\pi_1^*\omega+\pi_2^*\omega)$, away from $\mathbb{L}$ and $L_{1,1}$, to obtain a new ambient symplectic manifold in which the disjointness of these Lagrangians is a contradiction.

Step 1 Blow up the balls B_i of capacity ϵ around each of the $2d$ points p_i in $F\cap G$.

Denote the new manifold by (W,Ω_1). It follows from the analysis of the blow-up procedure from [15], see also Proposition 9.3.3 of [16], that (W,Ω_1) contains $2d$ exceptional divisors $\mathcal{E}_i$ each of area ϵ. Since the H_i intersect the balls B_i in J–holomorphic planes, (W,Ω_1) also contains the proper transforms of the H_i. These are denoted here by $\widehat{H}_i$ and are symplectic spheres of area $2-\epsilon$. By property (9) of Proposition 3.24, d of the $\widehat{H}_i$ intersect $\mathbb{E}$ once, and the other d of the $\widehat{H}_i$ intersect $E_{1,1}$ once.

Again, since they intersect the B_i in planes, the proper transforms of F and G, denoted by $\widehat{F}$ and $\widehat{G}$, are also well defined. These are spheres of area $2d+2-2d\epsilon$ which are now disjoint. The sphere H of $\overline{\mathcal{F}}$ is disjoint from the balls, but we denote its image in the blow up by $\widehat{H}$, which remains of area 2. After this, the relevant intersection numbers and areas are as in Table 2.

Step 2 Inflate both $\widehat{F}$ and $\widehat{G}$ by adding a tubular neighborhood of capacity d.

Here we recall that since $\widehat{F}$ and $\widehat{G}$ are symplectic spheres of self intersection 0 they have tubular neighborhoods which can be identified symplectically with $S^2\times D^2(\delta)$, where S^2 is a sphere of area

	$\widehat{F}$	$\widehat{G}$	$\mathbb{E}$	$E_{1,1}$	$\widehat{H}$	$\mathscr{E}_i$	$\widehat{H}_i$
$\widehat{F}$	0						
$\widehat{G}$	0	0					
$\mathbb{E}$	k	$d-k$	$*$				
$E_{1,1}$	l	$d-l$	0	$*$			
$\widehat{H}$	1	1	$*$	$*$	0		
$\{\mathscr{E}_i\}$	$2d$	$2d$	0	0	0	-1	
$\{\widehat{H}_i\}$	0	0	d	d	0	1	-1

	Ω_1–area
$\widehat{F}$	$2+2d-2d\epsilon$
$\widehat{G}$	$2+2d-2d\epsilon$
$\mathbb{E}$	1
$E_{1,1}$	1
$\widehat{H}$	2
$\mathscr{E}_i$	ϵ
$\widehat{H}_i$	$2-\epsilon$

Table 2: Intersection numbers after Step 1, left, and areas after Step 1, right.

$2+2d-2d\epsilon$ and $D^2(\delta)$ a disk of area δ. In this case inflation means replacing the symplectic form Ω_1 on this neighborhood by another one, Ω_2, such that $\Omega_2 - \Omega_1$ is a compactly supported area form of total area d on the disk factor, $D^2(\delta)$.

Applying the inflation result from [12], we may assume, by Lemma 3.1 in [14], that J is also tame with respect to Ω_2. This means that all of our J–holomorphic curves which intersect $\widehat{F}$ and $\widehat{G}$, namely $\mathbb{E}$, $E_{1,1}$, $\widehat{H}$ and $\mathscr{E}_i$, remain J–holomorphic and, in particular, symplectic.

The inflation procedure does not change the intersection pattern, and the Ω_2–area of curves increases, from the previous step, by d times the sum of the intersection numbers with $\widehat{F}$ and $\widehat{G}$ leaving us with the area profile in Table 3, left.

Step 3 Apply the negative inflation procedure from [2], of size ϵ, to each $\mathscr{E}_i$.

This negative inflation procedure yields a new symplectic form, Ω_3, such that the Ω_3–area of each $\mathscr{E}_i$ is less than its Ω_2–area by ϵ. That is, the Ω_3–area of each $\mathscr{E}_i$ is $2d$. One way to visualize this is to blow-down the $\mathscr{E}_i$ giving balls of capacity $\epsilon + 2d$ and then blow-up slightly smaller balls of capacity $2d$.

Negative inflation by ϵ also increases the area of homology classes by ϵ times the sum of the intersection numbers with the $\mathscr{E}_i$. The Ω_3 area profile is given in Table 3, right.

	Ω_2–area
$\widehat{F}$	$2+2d-2d\epsilon$
$\widehat{G}$	$2+2d-2d\epsilon$
$\mathbb{E}$	$1+d^2$
$E_{1,1}$	$1+d^2$
$\widehat{H}$	$2+2d$
$\mathscr{E}_i$	$\epsilon+2d$
$\widehat{H}_i$	$2-\epsilon$

	Ω_3–area
$\widehat{F}$	$2+2d$
$\widehat{G}$	$2+2d$
$\mathbb{E}$	$1+d^2$
$E_{1,1}$	$1+d^2$
$\widehat{H}$	$2+2d$
$\mathscr{E}_i$	$2d$
$\widehat{H}_i$	2

Table 3

	$\widehat{F}$	$\widehat{G}$	$\mathbb{E}^X$	$E^X_{1,1}$	$\widehat{H}$	$\mathcal{H}_i$
$\widehat{F}$	0					
$\widehat{G}$	0	0				
$\mathbb{E}^X$	k	$d-k$	$*$			
$E^X_{1,1}$	l	$d-l$	0	$*$		
$\widehat{H}$	1	1	$*$	$*$	0	
$\{\mathcal{H}_i\}$	$2d$	$2d$	d	d	0	0

	Ω–area
$\widehat{F}$	$2+2d$
$\widehat{G}$	$2+2d$
$\mathbb{E}^X$	$1+d^2+2d$
$E^X_{1,1}$	$1+d^2+2d$
$\widehat{H}$	$2+2d$
$\mathcal{H}_i$	$2+2d$

Table 4: Intersection numbers after Step 4, left, and areas after Step 4, right.

Step 4 Blow down each $\widehat{H}_i$.

We denote the symplectic manifold resulting from this final step by (X,Ω). Each of the exceptional divisors $\mathcal{E}_i$ in (W,Ω_3) is transformed, by Step 4, into a sphere $\mathcal{H}_i$ in X which has Ω–area equal to $2d+2$ and now lies in the same class as $\widehat{H}$. The disks $\mathbb{E}$ and $E_{1,1}$ each intersect d of the $\widehat{H}_i$ and so are transformed by Step 4 into disks $\mathbb{E}^X$ and $E^X_{1,1}$, whose symplectic areas have each been increased by $2d$. See Table 4.

Lemma 3.49 (X,Ω) *is symplectomorphic to*

$$(S^2\times S^2,(d+1)\omega\oplus(d+1)\omega).$$

Proof The presence of the embedded symplectic spheres $\widehat{F}$ and $\widehat{H}$, with the same Ω–area and satisfying

$$\widehat{F}\bullet\widehat{F}=\widehat{H}\bullet\widehat{H}=0 \quad\text{and}\quad \widehat{F}\bullet\widehat{H}=1,$$

implies that either (X,ω) is symplectomorphic to

$$(S^2\times S^2,(d+1)\omega\oplus(d+1)\omega),$$

or there are finitely many symplectically embedded spheres with self-intersection number -1 in the complement of $\widehat{F}$ and $\widehat{H}$ in X, and X can be blown down to a copy of $S^2\times S^2$. This follows from the proof of Theorem 9.4.7 of [16]. As a consequence, if $H_2(X;\mathbb{Z})$ has rank 2 then X is symplectomorphic to $S^2\times S^2$.

A simple analysis of the construction of (X,Ω) from $(S^2\times S^2,\pi_1^*\omega+\pi_2^*\omega)$ allows us to compute this rank. The $2d$ blow ups in Step 1 imply that the rank of $H_2(W;\mathbb{Z})$ is $2+2d$. The subsequent $2d$ blow down operations in Step 4 imply that the rank of $H_2(X;\mathbb{Z})$ is 2, as required. □

Henceforth, we may identify (X,Ω) with $(S^2\times S^2,(d+1)\omega\oplus(d+1)\omega)$. The Lagrangian tori $\mathbb{L}$ and $L_{1,1}$ are untouched, as submanifolds, by the four steps above. They remain Lagrangian and disjoint in (X,Ω). Note that $L_{1,1}$ is not equal to the Clifford torus in (X,Ω) with respect to the identification above. In what follows we denote the Clifford torus in (X,Ω) by L_X.

The manifold (X, Ω) also inherits an almost complex structure, denoted here by $\hat{J}$, which equals J away from the collection $\{\hat{\mathcal{H}}_i\}$. In particular, $\hat{J}$ is adapted to the original parametrizations ψ and $\psi_{1,1}$ of L and $L_{1,1}$. As in Section 3.5, $\hat{J}$ determines a straightened foliation $\hat{\mathcal{F}} = \mathcal{F}(\mathbb{L}, L_{1,1}, \psi, \psi_{1,1}, \hat{J})$ of $X \setminus (\mathbb{L} \cup L_{1,1})$. The original collections of planes $\mathfrak{s}_0$, $\mathfrak{s}_\infty$, $\mathfrak{r}_0$ and $\mathfrak{r}_\infty$ still comprise the broken leaves of this new foliation. The symplectic spheres $\hat{F}$ and $\hat{G}$ now represent the class $(1,0) \in H_2(X;\mathbb{Z}) = H_2(S^2 \times S^2;\mathbb{Z})$. As in Proposition 3.24, it is still true that exactly one of $\hat{F}$ and $\hat{G}$ intersects the planes of $\mathfrak{s}_0$ and the other intersects the planes of $\mathfrak{s}_\infty$, and exactly one of $\hat{F}$ and $\hat{G}$ intersects the planes of $\mathfrak{r}_0$ and the other intersects the planes of $\mathfrak{r}_\infty$.

Lemma 3.50 *The Lagrangian tori $\mathbb{L}$ and $L_{1,1}$ are both monotone in (X, Ω).*

Proof Let $D_\infty \colon (D^2, S^1) \to (S^2 \times S^2, \mathbb{L})$ be a compactification of one of the planes of $\mathfrak{r}_\infty$. The disk D_∞ has Maslov index equal to 2 and symplectic area equal to 1 with respect to $\pi_1^*\omega + \pi_2^*\omega$. The map $D_\infty|_{S^1}$ represents the foliation class $\beta_{\mathbb{L}}$. The image of the map D_∞ is unaffected by the four steps defining the passage from $(S^2 \times S^2, \pi_1^*\omega + \pi_2^*\omega)$ to (X, Ω). Viewed as a map from (D^2, S^1) to $(X, \mathbb{L})$, D_∞ still has Maslov index 2, and $D_\infty|_{S^1}$ still represents $\beta_{\mathbb{L}}$. The Ω–area of D_∞, as a map into (X, Ω), is $d+1$. This follows from the fact that exactly one of F and G intersect D_∞ and so the inflations in Step 2 increase the symplectic area by d.

By assertion (4) of Proposition 3.24, the boundary $\mathbb{E}|_{S^1}$ represents a class which, together with $\beta_{\mathbb{L}}$, forms an integral basis of $H_1(\mathbb{L};\mathbb{Z})$. The same holds for $\mathbb{E}^X|_{S^1}$. To prove that $\mathbb{L}$ is a monotone Lagrangian torus in (X, Ω) it then suffices, by Lemma 3.1, to prove that the Maslov index of $\mathbb{E}^X \colon (D^2, S^1) \to (X, \mathbb{L})$ is equal to

$$\frac{2}{d+1}(1 + d^2 + 2d) = 2d + 2,$$

where $1 + d^2 + 2d$ is the area of $\mathbb{E}^X$. This follows from the fact that, in (W, Ω_3), $\mathbb{E}$ has Maslov index 2, intersects exactly d of the $\hat{H}_i$, and each of the corresponding intersection numbers is 1. In blowing down the $\hat{H}_i$, and passing from $\mathbb{E}$ to $\mathbb{E}^X$, each of these intersection points yields an increase of 2 in the Maslov index.

The proof that $L_{1,1}$ is monotone in (X, Ω) is identical. □

Lemma 3.51 *The Lagrangians $\mathbb{L}$ and $L_{1,1}$ are both Hamiltonian isotopic to the Clifford torus L_X in (X, Ω).*

Proof This follows from the main result of Cieliebak and Schwingenheuer in [4]. In the language of that paper, the compactification of the straightened foliation $\hat{\mathcal{F}} = \mathcal{F}(\mathbb{L}, L_{1,1}, \psi, \psi_{1,1}, \hat{J})$ yields a fibering of $\mathbb{L}$ and a fibering of $L_{1,1}$. For the fibering of $\mathbb{L}$, the spheres $\hat{F}$ and $\hat{G}$ are disjoint sections in the class $(1,0)$ and exactly one of them intersects the (compactification of the) planes of $\mathfrak{r}_0$ and the other intersects the those of $\mathfrak{r}_\infty$. The main theorem of [4] then implies that L is Hamiltonian isotopic to the Clifford torus L_X in (X, Ω). An identical argument holds for $L_{1,1}$. □

With this, the contradiction to Assumption 2 becomes apparent. Since the Lagrangian Floer homology of L_X is nontrivial by [17], any Lagrangian tori Hamiltonian isotopic to L_X must intersect nontrivially. Hence, $\mathbb{L}$ and $L_{1,1}$ cannot be disjoint in (X, Ω).

Remark 3.52 The assumption that $\mathbb{L}$ and $L_{1,1}$ are disjoint is used twice in the proof of Theorem 1.1: at the very end, and in the proof of Refinement 3 in Section 3.3.

Remark 3.53 The fact that $L_{1,1}$ is the Clifford torus (and not just another monotone Lagrangian torus) is crucial (only) in the proof of the existence results in Propositions 3.20 and 3.22.

Remark 3.54 There is an alternative to the argument used at the end of the proof of Theorem 1.1 that avoids appealing to Lagrangian Floer homology. Instead, one can use the fact that the symplectomorphism in Lemma 3.49 can be chosen to map $\widehat{F}$, $\widehat{G}$ and the transforms $\widehat{T}_0$ and $\widehat{T}_\infty$ to the axes $S^2 \times \{0\}$, $S^2 \times \{\infty\}$, $\{0\} \times S^2$ and $\{\infty\} \times S^2$, respectively. The complement of these axes in $S^2 \times S^2$ can be identified with a domain in T^*T^2 in which the Clifford torus is identified with the zero section. We can check that $\mathbb{L}$ and $L_{1,1}$ are homologically nontrivial in this copy of T^*T^2 and so, by Theorem 3.9, are Hamiltonian isotopic to constant sections. The monotonicity condition then implies the constant section must be the zero section. Finally Gromov's intersection theorem for exact Lagrangians in cotangent bundles, from Section $2.3.B_4''$ of [6], implies that they must intersect.

4 Proof of Theorem 1.2

It suffices to prove the following.

Theorem 4.1 *For any $\epsilon > 0$, there is a $\delta > 0$ and a symplectic embedding of the polydisk $P(1+\delta, 1+\delta)$ into $P(2+\epsilon, 2+\epsilon)$ whose image is disjoint from the product Lagrangians $L_{k,l}$ for $k, l \in \{1, 2\}$.*

The desired additional integral Lagrangian torus L^+ is the one on (the image of) the boundary of $P(1,1) \subset P(1+\delta, 1+\delta)$.

4.1 Proof of Theorem 4.1

We will use rescaled polar coordinates θ_i, R_i on $\mathbb{R}^4 = \mathbb{C}^2$, where $R_i = \pi|z_i|^2$ and $\theta_i \in \mathbb{R}/\mathbb{Z}$. In these coordinates the standard symplectic form is

$$\omega = \sum_{i=1}^{2} dR_i \wedge d\theta_i,$$

and $L_{k,l} = \{(\theta_1, k, \theta_2, l)\}$.

4.1.1 A polydisk For $\epsilon > 0$ fixed, choose positive numbers ℓ, w such that

$$2 < \ell < 2+\epsilon, \quad w < 2, \quad \frac{1}{\ell} + \frac{1}{w} < 1.$$

Then choose positive constants σ and δ such that

$$\ell+\sigma<2+\epsilon, \quad w+\sigma<2, \quad \frac{1+\delta}{\ell}+\frac{1+\delta}{w}<1.$$

Set

$$S=\{\sigma<R_1<\ell+\sigma, \sigma<R_2<w+\sigma\} \quad \text{and} \quad T=\left\{0<\theta_1<\frac{1+\delta}{\ell}, 0<\theta_2<\frac{1+\delta}{w}\right\}.$$

Note that $S\times T$ is a subset of $P(2+\epsilon,2+\epsilon)$ and is symplectomorphic to $P(1+\delta,1+\delta)$. Both $L_{1,1}$ and $L_{2,1}$ intersect $S\times T$, while $L_{1,2}$ and $L_{2,2}$ do not.

4.1.2 The plan To prove Theorem 4.1 it suffices to find a Hamiltonian diffeomorphism of $P(2+\epsilon,2+\epsilon)$ that displaces $S\times T$ from the $L_{k,l}$. Equivalently, we construct a Hamiltonian diffeomorphism Ψ of $P(2+\epsilon,2+\epsilon)$ such that each of the images $\Psi(L_{k,l})$ is disjoint from $S\times T$.

To construct Ψ we use Hamiltonian functions which are of the form $F(\theta_1,\theta_2)$. The Hamiltonian flow ϕ_F^t of such a function preserves θ_1 and θ_2 and generates a Hamiltonian vector field parallel to the R_1R_2–plane. In particular, the only points of $\phi_F^t(L_{k,l})$ which could possibly intersect $S\times T$ are those whose (θ_1,θ_2) coordinates lie in T.

Since we only need to control the images of the $L_{k,l}$, we can cut off autonomous functions like F in (moving) neighborhoods of $\phi_F^t(L_{k,l})$ for specific values of k and l. After this cutting off, the new Hamiltonian will depend on all variables and be time dependent. In general, for a closed subset V, we denote the function obtained by cutting of F along $\phi_F^t(V)$ by $F_{[V]}$. Note that

$$\phi^t_{F_{[V]}}(v)=\phi_F^t(v) \quad \text{for all } v\in V \text{ and } t\in[0,1].$$

Also, each map $\phi^t_{F_{[V]}}$ is equal to the identity away from an arbitrarily small neighborhood of

$$\bigcup_{t\in[0,1]}\phi_F^t(V).$$

4.1.3 A diagonal move Let $g:\mathbb{R}/\mathbb{Z}\to\mathbb{R}$ be a smooth function such that for some positive real number $c(g)>0$ we have

$$g'(s)=c(g) \quad \text{for } s\in\left[0, \frac{1+\delta}{\ell}+\frac{1+\delta}{w}\right],$$

$\max(g')=c(g)$, and $\min(g')$ is less than and arbitrarily close to

$$-c(g)\left(\frac{\frac{1+\delta}{\ell}+\frac{1+\delta}{w}}{1-\frac{1+\delta}{\ell}-\frac{1+\delta}{w}}\right).$$

Letting $G(\theta_1,\theta_2)=g(\theta_1+\theta_2)$, we have

$$\phi_G^t(\theta_1,R_1,\theta_2,R_2)=(\theta_1,R_1+tg'(\theta_1+\theta_2),\theta_2,R_2+tg'(\theta_1+\theta_2)). \tag{4-1}$$

The image $\phi_G^1(L_{1,2})$ is well defined as long as

$$c(g) < \frac{1 - \frac{1+\delta}{\ell} - \frac{1+\delta}{w}}{\frac{1+\delta}{\ell} + \frac{1+\delta}{w}}, \tag{4-2}$$

and is contained in $P(2+\epsilon, 2+\epsilon)$ as long as $c(g) < \epsilon$. Henceforth, we will assume that ℓ, w and δ have been chosen such that the first constraint on $c(g)$ implies the second.

It follows from (4-1) and (4-2) that $\phi_G^t(L_{1,2})$ is contained in

$$\{R_1 \le 1 + c(g)\} \cap \{R_2 > 1\}$$

for all $t \in [0,1]$. Hence, each image $\phi_G^t(L_{1,2})$ is disjoint from the other $L_{k,l}$. Since $g' = c(g) > 0$ on T, each $\phi_G^t(L_{1,2})$ is also disjoint from $S \times T$.

4.1.4 A vertical move Let $h \colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ be a smooth function such that for some positive real number $0 < c(h) < \sigma$ we have

$$h'(s) = -(1 - c(h)) \quad \text{for } s \in \left[0, \frac{1+\delta}{w}\right],$$

$\min(h') = -(1 - c(h))$ and $\max(h')$ is greater than and arbitrarily close to

$$\frac{(1 - c(h))\frac{1+\delta}{w}}{1 - \frac{1+\delta}{w}} = \frac{1 - c(h)}{\frac{w}{1+\delta} - 1},$$

which is greater than one since $w + \sigma < 2$ and $c(h) < \sigma$.

Letting $H(\theta_1, \theta_2) = h(\theta_2)$, we have

$$\phi_H^t(\theta_1, R_1, \theta_2, R_2) = (\theta_1, R_1, \theta_2, R_2 + t h'(\theta_2)). \tag{4-3}$$

Clearly, $L_{2,1}$ and $L_{2,2}$ are disjoint from $\phi_H^t(L_{1,1})$ for all $t \in [0,1]$. Moreover, for θ_2 in $[0, (1+\delta)/w]$ we have

$$\phi_H^1(\theta_1, 1, \theta_2, 1) = (\theta_1, 1, \theta_2, c(h)).$$

So $\phi_H^1(L_{1,1})$ is disjoint from $T \times S$ by our choice of $c(h)$.

Some points of $L_{1,1}$, with values of θ_2 in $((1+\delta)/w, 1)$, are mapped by ϕ_H^1 to points having R_2 coordinate greater than and arbitrarily close to

$$1 + \frac{1 - c(h)}{\frac{w}{1+\delta} - 1} > 2.$$

Choosing w sufficiently close to 2 and δ sufficiently small ensures that $\phi_H^1(L_{1,1})$ lies in $P(2+\epsilon, 2+\epsilon)$.

4.1.5 A time delay The Hamiltonian diffeomorphism $\phi^1_{H_{[L_{1,1}]}}$ cannot be used to move $L_{1,1}$ off of $S \times T$ while leaving $L_{1,2}$ undisturbed. For, as described in the discussion above, $\phi^1_{H_{[L_{1,1}]}}(L_{1,2})$ will intersect $S \times T$.

The Hamiltonian diffeomorphism

$$\phi^1_{H_{[L_{1,1}]}} \circ \phi^1_{G_{[L_{1,2}]}}$$

has the same problem. By (4-1) and (4-3), the image of $(\theta_1, 1, \theta_2, 1) \in L_{1,1}$ under ϕ^t_H belongs to $\phi^1_G(L_{1,2})$ if and only if $g'(\theta_1+\theta_2) = 0$ and $th'(\theta_2) = 1$. Since $\max(h') > 1$, these intersections occur and so the map above will again push $L_{1,2}$ into $S \times T$.

We can fix this by adding a time delay. The first intersection between $\phi^t_H(L_{1,1})$ and $\phi^1_G(L_{1,2})$ occurs at $t = (\max(h'))^{-1}$. Let τ be less than and arbitrarily close to $(\max(h'))^{-1}$. Hence, τ is also less than and arbitrarily close to

$$\frac{\frac{w}{1+\delta} - 1}{1 - c(h)}.$$

Consider the Hamiltonian diffeomorphism

$$\widetilde{\Psi} = \phi^{1-\tau}_{H_{[\phi^\tau_H(L_{1,1}) \cup \phi^1_G(L_{1,2})]}} \circ \phi^\tau_{H_{[L_{1,1}]}} \circ \phi^1_{G_{[L_{1,2}]}}.$$

It follows from the analysis above that the map $\widetilde{\Psi}$ is compactly supported in $P(2+\epsilon, 2+\epsilon)$. In fact, it is supported in an arbitrarily small neighborhood of the subset $\{R_1 \le 1 + c(g)\}$. Hence, $\widetilde{\Psi}(L_{2,1}) = L_{2,1}$ and $\widetilde{\Psi}(L_{2,2}) = L_{2,2}$. By the definitions of τ and the cut-off operation, we have $\widetilde{\Psi}(L_{1,1}) = \phi^1_H(L_{1,1})$ and thus $\widetilde{\Psi}(L_{1,1})$ is disjoint from $S \times T$. In addition, we now have the following.

Lemma 4.2 *The image $\widetilde{\Psi}(L_{1,2})$ is disjoint from $S \times T$ when $c(h)$ is sufficiently close to σ and δ is sufficiently small.*

Proof By construction, for $(\theta_1, \theta_2) \in T$ we have

$$\begin{aligned}\widetilde{\Psi}(\theta_1, 1, \theta_2, 2) &= (\theta_1, 1 + g'(\theta_1+\theta_2), \theta_2, 2 + g'(\theta_1+\theta_2) + (1-\tau)h'(\theta_2)) \\ &= \big(\theta_1, 1 + c(g), \theta_2, 2 + c(g) - (1-\tau)(1-c(h))\big).\end{aligned}$$

It suffices to show that we can choose $c(g)$ and $c(h)$ so that

$$2 + c(g) - (1-\tau)(1-c(h)) > w + \sigma. \tag{4-4}$$

Since τ is less than and arbitrarily close to

$$\frac{\frac{w}{1+\delta} - 1}{1 - c(h)},$$

is also suffices to show that we can choose $c(g)$ and $c(h)$ so that

$$c(g) > w\Big(1 - \frac{1}{1+\delta}\Big) + (\sigma - c(h)).$$

The right-hand side can be made arbitrarily small by taking $c(h)$ to be close to σ and δ to be small. Since the choice of $c(g)$ is independent of the choice of $c(h)$ and the constraint (4-2) on $c(g)$ relaxes as δ goes to zero, we are done. □

Henceforth, we will assume that the conditions of Lemma 4.2 hold.

4.1.6 A final (horizontal) adjustment The images $\widetilde{\Psi}(L_{1,1})$, $\widetilde{\Psi}(L_{1,2})$ and $\widetilde{\Psi}(L_{2,2})$ are disjoint from $S \times T$ but $\widetilde{\Psi}$ still fixes $L_{1,2}$, which intersects $S \times T$. Since $L_{1,2}$ is close to the boundary of $S \times T$, we can make a simple adjustment to obtain the desired map, Ψ, which moves $L_{1,2}$ off of $S \times T$ as well.

Let $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ be a smooth function such that for some positive real number $c(f)$ greater than and arbitrarily close to $\ell + \sigma - 2$ we have

$$f'(s) = c(f) \quad \text{for } s \in \left[0, \frac{1+\delta}{\ell}\right],$$

$\max(f') = c(f)$ and $\min(f')$ is less than and arbitrarily close to

$$-\frac{c(f)}{\dfrac{\ell}{1+\delta} - 1}.$$

Setting $F(\theta_1, \theta_2) = f(\theta_1)$, we have

$$\phi_F^t(\theta_1, R_1, \theta_2, R_2) = (\theta_1, R_1 + tf'(\theta_1), \theta_2, R_2).$$

Our lower bound for $c(f)$ implies that $\phi_F^1(L_{2,1})$ is disjoint from $S \times T$. Looking at the R_2–component, it is clear that $\phi_F^1(L_{2,1})$ is disjoint from $L_{2,2} = \widetilde{\Psi}(L_{2,2})$. To prove that $\phi_F^1(L_{2,1})$ is also disjoint from $\widetilde{\Psi}(L_{1,1})$ and $\widetilde{\Psi}(L_{1,2})$, it suffices to prove the following.

Lemma 4.3 *The sets $\{R_1 \le 1 + c(g)\}$ and $\phi_F^1(L_{2,1})$ are disjoint.*

Proof It suffices to prove that

$$2 - \frac{c(f)}{\dfrac{\ell}{1+\delta} - 1} > 1 + c(g)$$

or, even more, that

$$1 > c(g) + \frac{\ell + \sigma - 2}{\dfrac{\ell}{1+\delta} - 1}.$$

The latter inequality clearly holds for all sufficiently small values of $c(g)$ and $\ell + \sigma - 2$. □

The Hamiltonian diffeomorphism

$$\Psi = \phi^1_{F_{[L_{2,1}]}} \circ \phi^{1-\tau}_{H_{[\phi^\tau_H(L_{1,1}) \cup \phi^1_G(L_{1,2})]}} \circ \phi^\tau_{H_{[L_{1,1}]}} \circ \phi^1_{G_{[L_{1,2}]}}$$

now has all the desired properties. With its construction, the proof of Theorem 4.1 is complete.

Question 4.4 *Can Ψ, or any other Hamiltonian diffeomorphism which displaces the $L_{k,l}$ from $S \times T$, be generated by an autonomous Hamiltonian?*

References

[1] **F Bourgeois**, **Y Eliashberg**, **H Hofer**, **K Wysocki**, **E Zehnder**, *Compactness results in symplectic field theory*, Geom. Topol. 7 (2003) 799–888 MR Zbl

[2] **O Buşe**, *Negative inflation and stability in symplectomorphism groups of ruled surfaces*, J. Symplectic Geom. 9 (2011) 147–160 MR Zbl

[3] **K Cieliebak**, **K Mohnke**, *Punctured holomorphic curves and Lagrangian embeddings*, Invent. Math. 212 (2018) 213–295 MR Zbl

[4] **K Cieliebak**, **M Schwingenheuer**, *Hamiltonian unknottedness of certain monotone Lagrangian tori in* $S^2 \times S^2$, Pacific J. Math. 299 (2019) 427–468 MR Zbl

[5] **G Dimitroglou Rizell**, **E Goodman**, **A Ivrii**, *Lagrangian isotopy of tori in* $S^2 \times S^2$ *and* $\mathbb{C}P^2$, Geom. Funct. Anal. 26 (2016) 1297–1358 MR Zbl

[6] **M Gromov**, *Pseudo holomorphic curves in symplectic manifolds*, Invent. Math. 82 (1985) 307–347 MR Zbl

[7] **J Hicks**, **C Y Mak**, *Some cute applications of Lagrangian cobordisms towards examples in quantitative symplectic geometry*, preprint (2022) arXiv 2208.14498

[8] **R Hind**, **S Lisi**, *Symplectic embeddings of polydisks*, Selecta Math. 21 (2015) 1099–1120 MR Zbl Correction in 23 (2017) 813–815

[9] **R Hind**, **E Opshtein**, *Squeezing Lagrangian tori in dimension* 4, Comment. Math. Helv. 95 (2020) 535–567 MR Zbl

[10] **H Hofer**, **K Wysocki**, **E Zehnder**, *Properties of pseudoholomorphic curves in symplectisations, I: Asymptotics*, Ann. Inst. H. Poincaré C Anal. Non Linéaire 13 (1996) 337–379 MR Zbl

[11] **A Ivrii**, *Lagrangian unknottedness of tori in certain symplectic* 4*–manifolds*, PhD thesis, Stanford University (2003) Available at `https://www.proquest.com/docview/305300870`

[12] **F Lalonde**, **D McDuff**, J*–curves and the classification of rational and ruled symplectic* 4*–manifolds*, from "Contact and symplectic geometry" (C B Thomas, editor), Publ. Newton Inst. 8, Cambridge Univ. Press (1996) 3–42 MR Zbl

[13] **C Y Mak**, **I Smith**, *Non-displaceable Lagrangian links in four-manifolds*, Geom. Funct. Anal. 31 (2021) 438–481 MR Zbl

[14] **D McDuff**, *Symplectomorphism groups and almost complex structures*, from "Essays on geometry and related topics, II" (É Ghys, P de la Harpe, V F R Jones, V Sergiescu, T Tsuboi, editors), Monogr. Enseign. Math. 38, Enseign. Math., Geneva (2001) 527–556 MR Zbl

[15] **D McDuff**, **L Polterovich**, *Symplectic packings and algebraic geometry*, Invent. Math. 115 (1994) 405–434 MR Zbl

[16] **D McDuff**, **D Salamon**, J*–holomorphic curves and symplectic topology*, Amer. Math. Soc. Colloq. Publ. 52, Amer. Math. Soc., Providence, RI (2004) MR Zbl

[17] **Y-G Oh**, *Addendum to: "Floer cohomology of Lagrangian intersections and pseudo-holomorphic disks, I"*, Comm. Pure Appl. Math. 48 (1995) 1299–1302 MR Zbl

[18] **L Polterovich**, **E Shelukhin**, *Lagrangian configurations and Hamiltonian maps*, Compos. Math. 159 (2023) 2483–2520 MR Zbl

[19] **A F Ritter**, **I Smith**, *The monotone wrapped Fukaya category and the open-closed string map*, Selecta Math. 23 (2017) 533–642 MR Zbl

[20] **R Vianna**, *Infinitely many monotone Lagrangian tori in del Pezzo surfaces*, Selecta Math. 23 (2017) 1955–1996 MR Zbl

[21] **C Wendl**, *Automatic transversality and orbifolds of punctured holomorphic curves in dimension four*, Comment. Math. Helv. 85 (2010) 347–407 MR Zbl

Department of Mathematics, University of Notre Dame
Notre Dame, IN, United States

Department of Mathematics, University of Illinois at Urbana-Champaign
Urbana, IL, United States

hind.1@nd.edu, ekerman@illinois.edu

Proposed: Leonid Polterovich — Received: 15 February 2022
Seconded: Dmitri Burago, Yakov Eliashberg — Revised: 27 October 2022

GEOMETRY & TOPOLOGY

msp.org/gt

MANAGING EDITOR

András I Stipsicz — Alfréd Rényi Institute of Mathematics, stipsicz@renyi.hu

BOARD OF EDITORS

Mohammed Abouzaid	Stanford University, abouzaid@stanford.edu
Dan Abramovich	Brown University, dan_abramovich@brown.edu
Ian Agol	University of California, Berkeley, ianagol@math.berkeley.edu
Arend Bayer	University of Edinburgh, arend.bayer@ed.ac.uk
Mark Behrens	University of Notre Dame, mbehren1@nd.edu
Mladen Bestvina	University of Utah, bestvina@math.utah.edu
Martin R Bridson	University of Oxford, bridson@maths.ox.ac.uk
Jim Bryan	University of British Columbia, jbryan@math.ubc.ca
Dmitri Burago	Pennsylvania State University, burago@math.psu.edu
Tobias H Colding	Massachusetts Institute of Technology, colding@math.mit.edu
Simon Donaldson	Imperial College, London, s.donaldson@ic.ac.uk
Yasha Eliashberg	Stanford University, eliash-gt@math.stanford.edu
Benson Farb	University of Chicago, farb@math.uchicago.edu
David M Fisher	Rice University, davidfisher@rice.edu
Mike Freedman	Microsoft Research, michaelf@microsoft.com
David Gabai	Princeton University, gabai@princeton.edu
Stavros Garoufalidis	Southern U. of Sci. and Tech., China, stavros@mpim-bonn.mpg.de
Cameron Gordon	University of Texas, gordon@math.utexas.edu
Jesper Grodal	University of Copenhagen, jg@math.ku.dk
Misha Gromov	IHÉS and NYU, Courant Institute, gromov@ihes.fr
Mark Gross	University of Cambridge, mgross@dpmms.cam.ac.uk
Rob Kirby	University of California, Berkeley, kirby@math.berkeley.edu
Bruce Kleiner	NYU, Courant Institute, bkleiner@cims.nyu.edu
Sándor Kovács	University of Washington, skovacs@uw.edu
Urs Lang	ETH Zürich, urs.lang@math.ethz.ch
Marc Levine	Universität Duisburg-Essen, marc.levine@uni-due.de
Ciprian Manolescu	University of California, Los Angeles, cm@math.ucla.edu
Haynes Miller	Massachusetts Institute of Technology, hrm@math.mit.edu
Tomasz Mrowka	Massachusetts Institute of Technology, mrowka@math.mit.edu
Aaron Naber	Northwestern University, anaber@math.northwestern.edu
Peter Ozsváth	Princeton University, petero@math.princeton.edu
Leonid Polterovich	Tel Aviv University, polterov@post.tau.ac.il
Colin Rourke	University of Warwick, gt@maths.warwick.ac.uk
Roman Sauer	Karlsruhe Institute of Technology, roman.sauer@kit.edu
Stefan Schwede	Universität Bonn, schwede@math.uni-bonn.de
Natasa Sesum	Rutgers University, natasas@math.rutgers.edu
Gang Tian	Massachusetts Institute of Technology, tian@math.mit.edu
Ulrike Tillmann	Oxford University, tillmann@maths.ox.ac.uk
Nathalie Wahl	University of Copenhagen, wahl@math.ku.dk
Anna Wienhard	Universität Heidelberg, wienhard@mathi.uni-heidelberg.de

See inside back cover or msp.org/gt for submission instructions.

The subscription price for 2024 is US $805/year for the electronic version, and $1135/year (+$70, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Geometry & Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Geometry & Topology (ISSN 1465-3060 printed, 1364-0380 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

GT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

© 2024 Mathematical Sciences Publishers

GEOMETRY & TOPOLOGY

Volume 28 Issue 5 (pages 1995–2482) 2024