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We give a new proof for the parabolic Verlinde formula in all ranks based on a comparison of wall-crossings in geometric invariant theory and certain iterated residue functionals. On the way, we develop a tautological variant of Hecke correspondences, calculate the Hilbert polynomials of the moduli spaces, and present a new, transparent, local approach to the ρ–shift problem of the theory.

14D20, 14H60

1 Introduction

1.1 The Verlinde formula

The Verlinde formula [31] is a strikingly beautiful statement in enumerative geometry motivated by quantum physics. Our focus in this paper will be the more difficult, parabolic variant, which we briefly describe below.

Let C be a smooth, complex projective curve of genus $g \geq 1$, and fix an auxiliary point $p \in C$. We will call a vector $c = (c_1 > c_2 > \cdots > c_r) \in \mathbb{R}^r$ satisfying $\sum c_i = 0$ and $c_1 - c_r < 1$ *regular* if no nontrivial subset of its coordinates sums to an integer. For such a $c \in \mathbb{R}^r$, there exists a smooth projective moduli space $P_0(c)$ (see Seshadri [21], Mehta and Seshadri [15] and Bhosle [4]), whose points are in one-to-one correspondence with the equivalence classes of pairs (W, F_*), where $W \to C$ is a vector bundle of

rank r on C with trivial determinant, F_* is a full flag of the fiber W_p, and the pair satisfies a certain parabolic stability condition depending on c; see Section 2.1. This condition roughly states that for a proper subbundle $W' \subset W$, the degree $\deg(W')$ is strictly smaller than the sum of a subset of the coordinates of c depending on the position of W'_p with respect to F_*.

There is a natural way to associate to a positive integer k and an integer vector $\lambda \in \mathbb{Z}^r$ satisfying $\lambda_1 + \cdots + \lambda_r = 0$ a line bundle $\mathscr{L}(k; \lambda)$ on $P_0(c)$, in such a way that if $c = \lambda/k$, then $\mathscr{L}(k; \lambda)$ is ample. The *parabolic Verlinde formula* is the following expression for the Euler characteristic of the ample line bundle $\mathscr{L}(k; \lambda)$: assume $c = \lambda/k$ is regular; then

$$(1) \qquad \chi(P_0(c), \mathscr{L}(k; \lambda)) = N_{r,k} \cdot \sum \frac{(-i)^{\binom{r}{2}} \exp(2\pi i \hat{\lambda} \cdot x)}{\prod_{i<j} (2 \sin \pi (x_i - x_j))^{2g-1}},$$

where $N_{r,k} = r(r(k+r)^{r-1})^{g-1}$, $\hat{\lambda} = \lambda + \frac{1}{2}(r-1, r-3, \ldots, 1-r)$, and the sum is taken over the finite set of those points in the interior of the parallelopiped

$$\{x = (x_1, x_2, \ldots, x_r = 0) \mid 0 < x_i - x_{i+1} < 1 \text{ for } i = 1, \ldots, r-1\},$$

which satisfy the conditions

- $(k+r)x \in \mathbb{Z}^r$, and
- $x_i - x_j \notin \mathbb{Z}$ for $1 \le i < j < r$.

Remark 1.1 This finite set is a set of lattice points in the interior of $(r-1)!$ identical simplices. (These are the orange-colored points in the rhombus on Figure 1.) By symmetrizing with respect to the group of permutations of the r coordinates, one obtains the same function on each of these simplices. Using the Weyl character formula, this allows one to rewrite (1) in a more familiar form as

$$\chi(P_0(c), \mathscr{L}(k; \lambda)) = (r(k+r)^{r-1})^{g-1} \cdot \sum \frac{\chi_\lambda(x)}{\prod_{i<j} (2 \sin \pi (x_i - x_j))^{2g-2}},$$

where χ_λ is the character of the irreducible SU(r)–representation of highest weight λ, and the sum is now taken over the lattice points of the form $(k+r)x \in \mathbb{Z}^r$ in the interior of a single simplex $\{x = (1 > x_1 > x_2 > \cdots > x_{r-1} > x_r = 0)\}$.

Remark 1.2 Equality (1) remains valid in greater generality, for certain cases when λ/k is nonregular. This slightly more technical statement will be given in Theorems 4.7 and 4.8.

Notation We denote the discrete sum in (1) depending on k, λ, r and g by $\mathrm{Ver}(k, \lambda)$. In what follows, the shift $\frac{1}{2}(r-1, r-3, \ldots, 1-r)$ will be denoted by ρ, and thus we have $\hat{\lambda} = \lambda + \rho$.

Equality (1), the *parabolic Verlinde formula*, has attracted a lot of attention over the years, and there are a number of different proofs. There is a generalization of this formula associated to a simply connected compact Lie group, and the form presented here corresponds to the case of the group SU(r).
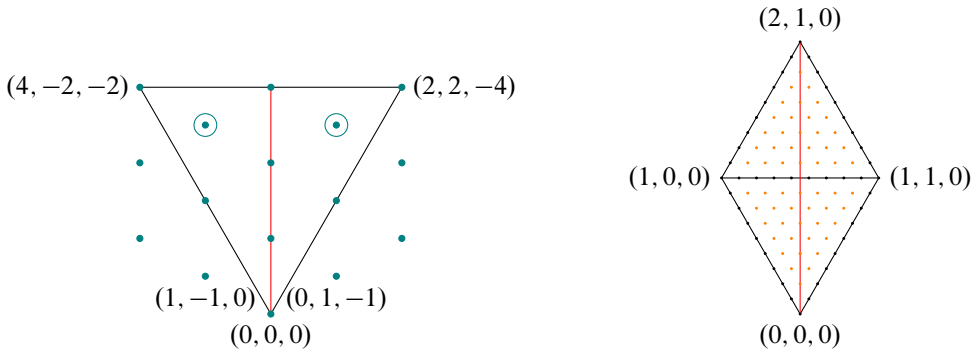
Figure 1: The set of λ vectors, left, and the finite set from (1), right, for $k = 6$, $r = 3$.

In this article, we give a novel proof of this result, which stands out with its technical simplicity. We believe the methods and ideas described in the paper will have other applications in geometric invariant theory and the study of moduli spaces.

Below, we give a quick sketch of the strategy of the proof, treating the example of the case of rank 3 in Sections 1.2–1.4. Next, in Section 1.5, we give a short guide to the contents of the paper.

Our work has a close relationship with several earlier approaches, and we describe these links in Section 1.6

1.2 The residue formula

The proof is based on 3 ideas. We will follow the arguments below for the case $r = 3$. We thus fix an integer $k > 1$ and an integer vector $\lambda = (\lambda_1 > \lambda_2 > \lambda_3)$ such that $\lambda_1 + \lambda_2 + \lambda_3 = 0$ and $\lambda_1 - \lambda_r < k$.

We start with the study of the right-hand side of (1), which, for $r = 3$, may be written in the somewhat simplified form

$$\mathrm{Ver}(k, \lambda) = N_{3,k} \sum_{0 < n_2 < n_1 < k+3} \frac{-2 \sin 2\pi \dfrac{(\lambda_1 + 1)n_1 + \lambda_2 n_2}{k+3}}{\left(8 \sin \pi \dfrac{n_1 - n_2}{k+3} \sin \pi \dfrac{n_2}{k+3} \sin \pi \dfrac{n_1}{k+3}\right)^{2g-1}},$$

where n_1 and n_2 are integers.

Remark 1.3 This is essentially the form of the formula given in Remark 1.1. Since we are considering representations with trivial central character, in this case, χ_λ is 3 identical sine terms, and this factor of 3 is hidden in $N_{3,k}$.

Using Theorem 4.7 and Remark 4.6 one can show that

$$\mathrm{Ver}(k,\lambda) = \begin{cases} p_+(k;\lambda) & \text{if } \lambda_2 > 0, \\ p_-(k;\lambda) & \text{if } \lambda_2 < 0, \end{cases}$$

where p_+ and p_- are two polynomials, given by the right-hand sides of the expressions of Example 4.11 on page 2278. We note two properties of $p_\pm(k,\lambda)$:

(A) The wall-crossing difference $p_- - p_+$ has a relatively simple form (cf Example 4.15 with $\lambda_1 + \lambda_3$ replaced by $-\lambda_2$):

$$p_-(k;\lambda) - p_+(k;\lambda) = \mathop{\mathrm{Res}}_{y=0} \mathop{\mathrm{Res}}_{x=0} \frac{(-3(k+3)^2)^g \cdot e^{(\lambda_1+1)x - \lambda_2 y}}{(1 - e^{x(k+3)}) w_\Phi(x,y)^{2g-1}} \, dx \, dy,$$

where $w_\Phi(x,y) = 2\sinh(\tfrac{1}{2}x) \cdot 2\sinh(\tfrac{1}{2}y) \cdot 2\sinh(\tfrac{1}{2}(x+y))$.

(B) An easy calculation via substitution shows that for any permutation on three elements $\sigma \in \Sigma_3$, our polynomials have the symmetries

(2) $$p_+(k; \sigma \cdot \lambda + \theta_1[k]) = (-1)^\sigma p_+(k; \lambda + \theta_1[k]),$$

(3) $$p_-(k; \sigma \cdot \lambda + \theta_{-1}[k]) = (-1)^\sigma p_-(k; \lambda + \theta_{-1}[k]),$$

where

$$\theta_1[k] = \tfrac{1}{3}k(1,1,-2) + (0,1,-1) \quad \text{and} \quad \theta_{-1}[k] = \tfrac{1}{3}k(2,-1,-1) + (1,-1,0).$$

1.3 Wall-crossings in moduli spaces

Now consider the left-hand side of (1). It is easy to check that the set of isomorphism classes of parabolic bundles in $P_0(c)$ remains unchanged as long as c_2 does not change sign. Hence, effectively, we have two moduli spaces $P_0(>)$ and $P_0(<)$, corresponding to the two chambers separated by the red ($c_2 = 0$) line in Figure 1. Introduce the notation

$$q_+(k;\lambda) = \chi(P_0(>), \mathscr{L}(k;\lambda)) \quad \text{and} \quad q_-(k;\lambda) = \chi(P_0(<), \mathscr{L}(k;\lambda))$$

for the generalized Hilbert polynomials of these two spaces.

In Section 5, we derive a simple formula (31) for the wall-crossing difference in geometric invariant theory. The formula has the form of a residue of an equivariant integral, taken with respect to the equivariant parameter. In our case, the space on which we integrate is the space of rank-3 parabolic bundles which split into a direct sum of a rank-2 and a rank-1 bundle. This equivariant integral may be evaluated using induction on the rank (cf the detailed calculation in Example 6.15 on page 2295), and the result is

(4) $$q_-(k;\lambda) - q_+(k;\lambda) = \mathop{\mathrm{Res}}_{u=0} \mathop{\mathrm{Res}}_{z=0} \frac{(-3(k+3)^2)^g \cdot e^{\lambda_1 z + \lambda_2 u + z}}{-w_\Phi(z,-u)^{2g-1}(1 - e^{(k+3)z})} \, dz \, du,$$

where u plays the role of the equivariant parameter, the generator of $H_{\mathbb{C}^*}^*(\mathrm{pt})$. This iterated residue coincides with the expression above after changing (z,u) to $(x,-y)$, and thus we have

(5) $$p_+ - p_- = q_+ - q_-.$$

1.4 Hecke correspondences, Serre duality and the symmetry argument

Hecke correspondences between moduli spaces of bundles of different degrees were introduced by Narasimhan and Ramanan in [19]. In Section 7 of our paper, we describe a "tautological" variant of this construction, which identifies the same space with several moduli spaces of parabolic bundles with different degrees and weights. Using this construction we can fiber our two moduli spaces, $P_0(>)$ and $P_0(<)$ over the moduli spaces of stable bundles (without parabolic structure) of degrees 1 and -1:

$$\text{Flag}_3 \to P_0(<) \to N_{-1} \quad \text{and} \quad N_1 \leftarrow P_0(>) \leftarrow \text{Flag}_3,$$

where the fibers are full flags of 3–dimensional vector spaces. Serre duality applied to a Flag_3–bundle implies a Σ_3–antisymmetry of the Euler characteristics of line bundles on this space, and after careful identification of these bundles, we derive the same symmetry properties for the functions $q_\pm$ as we did for the polynomials $p_\pm$: $q_+(k;\lambda)$ satisfies (2), while $q_-(k;\lambda)$ satisfies (3).

The final argument is elegant: we can rearrange equation (5) describing the equality of wall-crossings as

$$p_+ - q_+ = p_- - q_-,$$

and we introduce the notation $\Theta(k;\lambda)$ for this polynomial. Then Θ satisfies both (2) and (3), and thus it is anti-invariant with respect to an affine Weyl group action in the plane for each fixed k. This implies that $\Theta(k;\lambda)$ vanishes and this completes the proof.

1.5 Contents of the paper

There are a number of complications which arise when $r > 3$. We will highlight these in this section, and also give a brief guide to the contents of the paper.

We start with a quick introduction into the theory of parabolic bundles in Section 2. Here we describe the line bundles we are considering, as well as the chamber structure of the space of parabolic weights induced by the stability condition. The combinatorics of the iterated residue formulas mentioned in Section 1.2 above is considerably more complicated in the higher rank case, and is best treated using the notion of *diagonal bases* of hyperplane arrangements introduced in Szenes [23]; we review this construction in the special case of the A_r root arrangement in Section 3.

Using this notion, in Section 4, we present a residue formula for the Verlinde sums on the right-hand side of (1) obtained in Szenes [24] (Theorems 4.4 and 4.7). It turns out that because of a standard ρ–shift type effect in the theory, this residue formula does not have a manifestly polynomial form on our chambers, and thus, we formulate our main result, Theorem 4.8 in two parts: in part (I) we state the equality of the Euler characteristics of line bundles with a modified residue formula, which is manifestly polynomial on our chambers, and in part (II) we state the equality of the modified formula with the original residue formula from [24]. Part (II) is proved in Section 10, while the proof of part (I) takes up the rest of the paper.

At the end of Section 4, we present our wall-crossing formula for Verlinde sums in Proposition 4.18, which uses in an essential manner the yoga of diagonal bases; cf property (A) above for the case of $r = 3$.

The geometric part of our work starts in Section 5, where we derive a simple general result, formula (31), for wall-crossings in GIT. We apply this result to parabolic moduli spaces in Section 6 and, using induction on the rank, obtain Theorem 6.13, the higher-rank version of formula (4) above.

It is downhill from here: in Section 7 we describe the tautological Hecke correspondences we need in several places in the paper, and in Section 8 we derive the Weyl-symmetries of the polynomials $q_\pm$, and finish the proof along the lines sketched above.

We are essentially done, but we hit a snag when checking the beginning of our induction on the rank: our argument does not work for $r = 2$. Roughly, the reason for this is that we need our simplex of parabolic weights to have at least two regular vertices, and for $r = 2$, we have only one. The way out is to consider the moduli space with two punctures and then all the pieces fall in place. This argument is carried out in Section 9.

1.6 Historical remarks

There is a long list of proofs of the Verlinde formula, and we cannot do justice to all the approaches in this short introduction. We will thus focus on the historical lineage of our paper, and the works that are closest in spirit to what we do; see Sorger [22] for a more comprehensive overview.

The proofs of the Verlinde formula fall in two categories: proofs of the fusion rules and proofs that find some interpretation of the "Fourier-transformed" discrete sum on the right-hand side of (1); our work belongs to this second group. Another line of division concerns the model which one uses for the moduli spaces: via the Narasimhan–Seshadri correspondence, the moduli spaces of vector bundles may equally be presented as symplectic manifolds of certain types of flat connections on punctured Riemann surfaces, and this opens the way of using the methods of symplectic geometry. While these symplectic approaches lead to results equivalent to the ones coming out of the algebrogeometric setup, the fields of applications of the two approaches seem to be very different.

The idea of proving the Verlinde formula via wall-crossings appeared in the seminal paper of Michael Thaddeus [27]. He used a geometric approach and managed to prove the Verlinde formula in rank 2 by crossing walls in the moduli of stable pairs. The *master space construction*, which plays a central role in our paper, also first appeared in his work [28]. In a sense, our paper may be thought of as the completion of his program.

A paper closely related to our work is that of Jeffrey and Kirwan [13], which approaches the problem from a symplectic/cohomological point of view (see also Jeffrey and Kirwan [12]), and has a somewhat different angle form ours. This paper also uses the residue calculus introduced in [23; 24], but not quite as consistently as our work, and the parabolic case was not resolved from this point of view; see Jeffrey [11]. The geometric model used in [13] to represent the moduli spaces as quotients is rather complicated.

In a comprehensive paper covering the case of all compact groups, Bismut and Labourie [5] used a differential-geometric approach to find the generating function for the parabolic Verlinde formula. This work was the motivation for the residue formula in [24], which is also used in the present paper.

In a remarkable series of papers Alekseev, Meinrenken and Woodward [1], again approaching the subject from the symplectic point of view, gave a direct proof of (1), using reduction in infinite dimensions. A general approach related to twisted K-theory was introduced by Meinrenken in [16]. We should also mention recent work by Loizides and Meinrenken in [14], which employs the residue techniques of [24].

Finally, we drew motivation from the paper of Teleman and Woodward [26], where the Verlinde formula is put in the framework of localization in K-theory of stacks. This very impressive work is probably accessible to a small number of experts only. In the present article we demonstrate, in particular, that the sophisticated tools employed in [26], at least in this instance, may be replaced by a simple combinatorial device.

In summary, the virtues of this article are:

- A proof of the parabolic Verlinde formula which needs as background only the basics of GIT.

- The discrete sum, and the generating function giving the coefficients of the Hilbert polynomial are treated at the same time, and the ρ–shift is dealt with explicitly.

- A few technical innovations, such as an efficient wall-crossing formula in GIT (Theorem 5.7) and the tautological Hecke correspondences, keep the arguments simple, and the technical difficulties related to infinite-dimensional quotients or singularities, in our approach, are absorbed by a combinatorial device: the theory of iterated residues.

2 Parabolic bundles

2.1 Definitions

Let C be a smooth complex projective curve of genus $g \geq 2$, and fix a point $p \in C$.

- A *parabolic bundle* on C is a vector bundle W of rank r with a full flag F_* in the fiber over p,

$$W_p = F_r \supsetneq \cdots \supsetneq F_1 \supsetneq F_0 = 0,$$

and *parabolic weights* $c = (c_1, \ldots, c_r)$ assigned to $F_r, F_{r-1}, \ldots, F_1$, satisfying the conditions

$$c_1 > c_2 > \cdots > c_r \quad \text{and} \quad c_1 - c_r < 1.$$

- The *parabolic degree*[1] and the *parabolic slope* of W are defined as

$$\text{par deg } W = \deg W - \sum_{i=1}^{r} c_i \quad \text{and} \quad \text{par slope } W = \frac{\text{par deg } W}{\text{rank } W}.$$

---

[1] For technical reasons, we have chosen a sign convention opposite to that in the majority of treatments in the literature.

- A *morphism* $f : W \to W'$ of parabolic bundles is a morphism of vector bundles satisfying $f_p(F_i) \subset F'_{j-1}$ if $c_{r-i+1} < c'_{r-j+1}$. In particular, an *endomorphism* of a parabolic bundle W is a vector bundle endomorphism preserving the flag F_*.

- Denote by $\mathrm{ParHom}(W, W')$ the sheaf of parabolic morphisms from W to W'. Then there is a short exact sequence of sheaves

(6) $$0 \to \mathrm{ParHom}(W, W') \to \mathrm{Hom}(W, W') \to T_p \to 0,$$

  where T_p is a torsion sheaf supported at p. The rank of T_p is the number of pairs (i, j), such that $c_i < c'_j$; cf Boden and Hu [6].

If $W' \subset W$ is a subbundle of W, then both W' and the quotient W/W' inherit a parabolic structure from W in a natural way; cf Mehta and Seshadri [15, Definition 1.7].

- A parabolic bundle W is *stable of weight c* if any proper subbundle $W' \subset W$ satisfies $\mathrm{par\,slope}(W') < \mathrm{par\,slope}(W)$; and W is *semistable of weight c*, if the inequality is not strict.

Remark 2.1 The parabolic stability condition depends on the parabolic weights only up to adding the same constant to all weights c_i.

2.2 Construction of the moduli spaces

We start with a quick review of the construction of Mehta and Seshadri [15] of the moduli space of stable parabolic bundles. It follows from Remark 2.1 that, without loss of generality, we can assume that the parabolic weights of a rank-r degree-d bundle belong to the simplex

$$\Delta_d = \left\{ (c_1, c_2, \ldots, c_r) \,\middle|\, c_1 > c_2 > \cdots > c_r,\, c_1 - c_r < 1,\, \sum_i c_i = d \right\}.$$

Definition 2.2 We will call a vector $c = (c_1, \ldots, c_r) \in \mathbb{R}^r$ such that $\sum_i c_i \in \mathbb{Z}$ *regular* if for any nontrivial subset $\Psi \subset \{1, 2, \ldots, r\}$, we have $\sum_{i \in \Psi} c_i \notin \mathbb{Z}$.

Now choose an integer $d \gg 0$ such that $H^1(W) = 0$ and W is generated by global sections for any rank-r degree-d semistable parabolic bundle W of parabolic degree 0. Put $N = r(1-g) + d$.

- Consider the Grothendieck quot scheme $\mathrm{Quot}(N, r)$ [9] parametrizing quotients $\mathbb{O}^N \twoheadrightarrow W$, where W is a coherent sheaf of degree d and rank r.

- This space is endowed with a universal bundle UQ, and a generically free action of the group $G = \mathrm{PSL}(N)$ which does not, however, lift to UQ.

- Let $\mathrm{LFQuot} \subset \mathrm{Quot}(N, r)$ be the open subscheme consisting of locally free quotients W such that the induced map $H^0(\mathbb{O}^N) \to H^0(W)$ is an isomorphism.

- Denote by XQ the total space of the flag bundle $\mathrm{Flag}(UQ_p)$ on $\mathrm{LFQuot} \times p$. This space is endowed with the flag of vector bundles $\mathrm{Fl}_1 \subset \cdots \subset \mathrm{Fl}_{r-1} \subset \mathrm{Fl}_r = UQ_p$.

- Let $k \in \mathbb{Z}$ and $(\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^r$ be such that $\sum_{i=1}^r \lambda_i = kd$, and consider the line bundle

$$L(k; \lambda) = \det(UQ_p)^{k(1-g)} \otimes \det(\pi_* UQ)^{-k} \otimes (\mathrm{Fl}_r/\mathrm{Fl}_{r-1})^{\lambda_1} \otimes \cdots \otimes (\mathrm{Fl}_1)^{\lambda_r}$$

on XQ, which does carry a G–linearization (lift of the G–action from XQ).

- Finally, assume $c \in \Delta_d$ is regular (cf Definition 2.2 above) and define $\widetilde{P}_d(c)$, the moduli space of stable parabolic weight-c vector bundles on C, as the GIT quotient $XQ /\!/^c G$ of XQ with respect to any linearization $L(k; \lambda)$ such that $\lambda/k = c$.

Theorem 2.3 [21] *Assume that $c \in \Delta_d$ is a regular weight vector. Then the moduli space $\widetilde{P}_d(c)$ is a smooth projective variety of dimension $r^2(g-1) + \binom{r}{2} + 1$, whose points are in one-to-one correspondence with the set of isomorphism classes of stable parabolic bundles of weight c (cf Section 2.1).*

Remark 2.4 Via the determinant map, the moduli space $\widetilde{P}_d(c)$ fibers over the Jacobian of degree-d line bundles with isomorphic fibers, and in this paper, we will focus on the moduli space

$$P_d(c) = \{W \in \widetilde{P}_d(c) \mid \det W \simeq \mathbb{O}(dp)\},$$

which is smooth, projective and has dimension $(r^2 - 1)(g - 1) + \binom{r}{2}$.

Remark 2.5 Tensoring with the line bundle $\mathbb{O}(mp)$ induces an isomorphism

$$\otimes \mathbb{O}(mp) \colon P_d(c) \to P_{d+rm}(c),$$

so the moduli spaces $P_d(c)$, essentially, depend only on d modulo r.

2.3 The Picard group of $P_d(c)$

For a regular $c \in \Delta_d$, there exist universal bundles U over $P_d(c) \times C$ endowed with a flag $\mathscr{F}_1 \subset \cdots \subset \mathscr{F}_{r-1} \subset \mathscr{F}_r = U_p$, and satisfying the obvious tautological properties. In general, such universal bundles U, and hence the flag line bundles $\mathscr{F}_{i+1}/\mathscr{F}_i$, are unique only up to tensoring by the pullback of a line bundle from $P_d(c)$. Nevertheless, we have the following statement.

Lemma 2.6 *For $k \in \mathbb{Z}$ and $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^r$ such that $\sum_{i=1}^r \lambda_i = kd$, the line bundle*

$$(7) \qquad \mathscr{L}_d(k; \lambda) = \det(U_p)^{k(1-g)} \otimes \det(\pi_* U)^{-k} \otimes (\mathscr{F}_r/\mathscr{F}_{r-1})^{\lambda_1} \otimes \cdots \otimes (\mathscr{F}_1)^{\lambda_r}$$

on $P_d(c)$ is independent of the choice of the universal bundle U.

Proof Note that tensoring U with a pullback $\pi^* \mathscr{L}$ of a line bundle $\mathscr{L}$ on $P_0(c)$ changes $\det(U_p)$ by $\mathscr{L}^r$ and $\det(\pi_* U)$ by $\mathscr{L}^{d-r(g-1)}$. $\qquad \square$

Remark 2.7 The line bundle $L(k; \lambda)$ defined in Section 2.2 descends to the line bundle $\mathscr{L}_d(k; \lambda)$ on the GIT quotient $P_d(c)$.

Notation We will say that U is *normalized* if the line subbundle $\mathcal{F}_1 \subset U_p$ is trivial. The parameter k is often called the *level*.

Let $\omega \in H^2(C)$ be the fundamental class of our curve C, and $e_1, \ldots, e_{2g}$ a basis of $H^1(C)$ such that $e_i e_{i+g} = \omega$ for $1 \le i \le g$, and all other intersection numbers $e_i e_j$ equal 0. For a class $\delta \in H^*(P \times C)$ of a product, we introduce notation for its Künneth components (cf [32]):

$$
(8) \qquad \delta = \delta_{(0)} \otimes 1 + \sum_i \delta_{(e_i)} \otimes e_i + \delta_{(2)} \otimes \omega \in \bigoplus_{i=0}^{2} H^{*-i}(P) \otimes H^i(C).
$$

We will need the following formula.

Lemma 2.8 The equality $2\, c_1(\mathcal{L}_d(r; d, \ldots, d)) = c_2(\mathrm{End}_0(U_d))_{(2)}$ holds, where End_0 stands for *traceless endomorphisms*.

Proof Taking the first Chern class on both sides of (7), we obtain

$$
c_1(\mathcal{L}_d(r; d, \ldots, d)) = r(1-g)c_1(U_d)_{(0)} - rc_1(\pi_*(U_d)) + dc_1(U_d)_{(0)},
$$

where we evaluate the middle term using the Grothendieck–Riemann–Roch theorem, and $c_1(U_d)_{(2)} = d$:

$$
\begin{aligned}
c_1(\pi_*(U_d)) = \mathrm{ch}_1(\pi_!(U_d)) &= \pi_* \mathrm{ch}_2(U_d) - (g-1)c_1(U_d)_{(0)} \\
&= c_1(U_d)_{(0)} d - c_2(U_d)_{(2)} - (g-1)c_1(U_d)_{(0)}.
\end{aligned}
$$

This leads to the formula

$$
c_1(\mathcal{L}_d(r; d, \ldots, d)) = -d(r-1)c_1(U_d)_{(0)} + rc_2(U_d)_{(2)},
$$

which is easily seen to equal $\frac{1}{2}c_2(\mathrm{End}_0(U_0))_{(2)}$. $\qquad\square$

2.4 Walls and chambers

The central question we address in this paper is how the moduli space of stable parabolic bundles depends on the choice of parabolic weights. Let W be a vector bundle of degree d with a fixed full flag F_* of the fiber W_p, and let us try to determine the structure of the set of parabolic weights $c \in \Delta_d$ for which W is stable. Clearly, for this we need to study the set of parabolic weights $c = (c_1, c_2, \ldots c_r)$ for which one can find a proper subbundle $W' \subset W$ such that

$$
(9) \qquad \mathrm{par\,slope}(W') = \mathrm{par\,slope}(W) = 0.
$$

A subbundle $W' \subset W$ determines a short exact sequence of parabolic bundles

$$
0 \to W' \to W \to W'' \to 0,
$$

and the position of W'_p with respect to F_* gives rise to a nontrivial partition of the set $\{1, 2, \ldots, r\}$ into two sets, Π' and Π'', cf [15, Definition 1.7]; the parabolic weights of W' and W'' are then $c' = (c_i)_{i \in \Pi'}$ and $c'' = (c_i)_{i \in \Pi''}$, correspondingly. The slope condition (9) translates into a pair of equivalent equalities

$$
(10) \qquad d' = \sum_{i \in \Pi''} c_i \quad \text{and} \quad d'' = \sum_{i \in \Pi''} c_i,
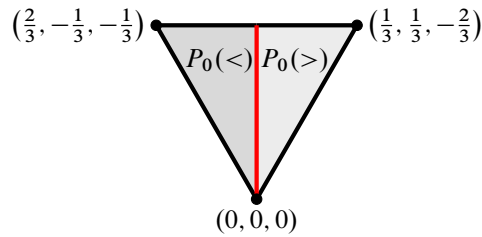$$

Figure 2: The space of admissible parabolic weights for rank $r = 3$.

where d' and $d'' = d - d'$ are the degrees of W' and W'', respectively. This means that the critical values of $c \in \Delta_d$ for which (9) is possible lie on the union of affine hyperplanes (or *walls*) defined by the equations

$$\sum_{i \in \Pi'} c_i = l, \quad \text{where } l \in \mathbb{Z}, \text{ and } \Pi' \subset \{1, 2, \ldots, r\} \text{ is nontrivial.}$$

As only finitely many of these walls intersect the simplex Δ_d, their complement is a finite union of open polyhedral *chambers*. It is easy to verify that as we vary c inside one of these chambers, the stability condition, and thus the moduli space $P_d(c)$, does not change.

Example 2.9 Consider the case of rank-3 degree-0 stable parabolic bundles with parabolic weights $c = (c_1, c_2, c_3) \in \Delta_0$. The set Δ_0 is an open triangle with vertices $(0, 0, 0)$, $\left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right)$ and $\left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right)$ (see Figure 2), and there exist only two essentially different stability conditions. The wall separating the two regimes is given by the condition $c_2 = 0$. We write $P_0(>)$ for the moduli space $P_0(c_1, c_2, c_3)$ with $c_2 > 0$, and $P_0(<)$ for $P_0(c_1, c_2, c_3)$ with $c_2 < 0$.

3 Wall-crossing in the Verlinde formula

A key component of our approach is the notion of *diagonal basis* and the associated generalized Bernoulli polynomials introduced for general hyperplane arrangements in [23]. Using this formalism, we will be able to formulate our main result, Theorem 4.8.

3.1 Notation

We begin by setting up some extra notation for the space of parabolic weights introduced in Section 2.1.

- Let $V = \mathbb{R}^r / \mathbb{R}(1, 1, \ldots, 1)$ be the $(r-1)$–dimensional vector space, obtained as the quotient of $\mathbb{R}^r$. The dual space V^* is then naturally represented as

$$V^* = \{a = (a_1, \ldots, a_r) \in \mathbb{R}^r \mid a_1 + \cdots + a_r = 0\}.$$

  Let $x_1, x_2, \ldots, x_r$ be the coordinates on $\mathbb{R}^r$; given $a \in V^*$, we will write $\langle a, x \rangle$ for the linear function $\sum_i a_i x_i$ on V. We will sometimes identify this linear function with the vector a itself.

- The vector space V^* is endowed with a lattice Λ of full rank:

$$\Lambda = \{\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^r \mid \lambda_1 + \cdots + \lambda_r = 0\}.$$

In particular, for $1 \leq i \neq j \leq r$, we can define the element $\alpha^{ij} = x_i - x_j$ in Λ.

- Our arrangement is the set of hyperplanes $\{x_i = x_j\} \subset V$, with $1 \leq i < j \leq r$. It will be convenient for us to think about this set as the set of roots of the A_{r-1} root system with the opposite roots identified:

$$\Phi = \{ \pm\alpha^{ij} \mid 1 \leq i < j \leq r\}.$$

Note that V^* carries a natural action of the permutation group Σ_r, permuting the coordinates x_j for $j = 1, \ldots, r$, and this action restricts to an action on Φ as well.

- The basic object of the theory is an *ordered* linear basis $\boldsymbol{B}$ of V^* consisting of the elements of Φ. Let us denote the set of these objects by $\mathcal{B}$:

$$\mathcal{B} = \{\boldsymbol{B} = (\beta^{[1]}, \ldots, \beta^{[r-1]}) \in \Phi^{r-1} \mid \boldsymbol{B} \text{ a basis of } V^*\}.$$

- For $\boldsymbol{B} \in \mathcal{B}$, we will write $\mathrm{Fl}(\boldsymbol{B})$ for the full flag

$$[V^* = \langle \beta^{[1]}, \beta^{[2]}, \ldots, \beta^{[r-1]} \rangle_{\mathrm{lin}}, \ldots, \langle \beta^{[r-1]}, \beta^{[r-2]} \rangle_{\mathrm{lin}}, \langle \beta^{[r-1]} \rangle_{\mathrm{lin}}],$$

where $\langle \, \cdot \, \rangle_{\mathrm{lin}}$ stands for linear span.

3.2 Diagonal bases

Definition 3.1   • For $\tau \in \Sigma_{r-1}$ and $\boldsymbol{B} \in \mathcal{B}$, we will write $\boldsymbol{B} \circlearrowleft \tau$ for the permuted sequence

$$(\beta^{[\tau(1)]}, \beta^{[\tau(2)]}, \ldots, \beta^{[\tau(r-1)]}).$$

- For two elements $\boldsymbol{B}, \boldsymbol{C} \in \mathcal{B}$ we will write $\boldsymbol{B} \dashv \boldsymbol{C}$ if for any $\tau \in \Sigma_{r-1}$, we have $\mathrm{Fl}(\boldsymbol{B} \circlearrowleft \tau) \neq \mathrm{Fl}(\boldsymbol{C})$.

- A subset $\mathcal{D} \subset \mathcal{B}$ of $(r-1)!$ elements is called a *diagonal basis* if for any two different elements $\boldsymbol{B}, \boldsymbol{C} \in \mathcal{D}$, we have $\boldsymbol{B} \dashv \boldsymbol{C}$.

Remark 3.2   This definition is motivated by a construction in [23], which associates to each diagonal basis $\mathcal{D}$ a pair of dual bases of the middle homology and the cohomology of the complexified hyperplane arrangement on $V \otimes_{\mathbb{R}} \mathbb{C}$ defined by Φ. The dimension of these (co)homology spaces is $(r-1)!$.

3.3 Combinatorial interpretation

This notion has the following purely combinatorial form.

- We can think of Φ as the edges of the complete graph on r vertices.

- Then the set $\mathcal{B}$ may be thought of as the set of spanning trees of this graph with edges enumerated from 1 to $r-1$. We will introduce the notation

$$\boldsymbol{B} \mapsto \mathrm{Tree}(\boldsymbol{B})$$

for this ordered tree.

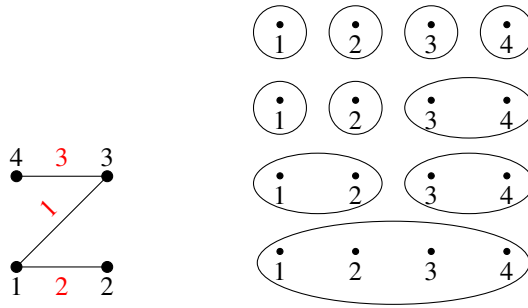Figure 3: $\boldsymbol{B} = (\alpha^{1,3}, \alpha^{1,2}, \alpha^{3,4})$.

- In this language, the flag $\mathrm{Fl}(\boldsymbol{B})$ corresponds to a sequence of r nested partitions of the vertices (starting with the total partition into one-element sets and ending with the trivial partition) associated to $\mathrm{Tree}(\boldsymbol{B})$, the jth partition being the one induced by the first $j - 1$ edges. For example, the ordered tree $[(2, 4)(1, 3), (1, 2)]$ induces the same sequence of partitions as $[(1, 4), (2, 3), (1, 2)]$; see Figure 3.

- A diagonal basis $\mathscr{D}$ is then a set of $(r - 1)!$ ordered trees such that the $(r - 1)!$ partition sequences obtained by reordering the edges of any one of the ordered trees are different from $(r - 1)! - 1$ sequences of partitions obtained from the remaining elements of $\mathscr{D}$.

3.4 Examples

There are essentially two known constructions of diagonal bases [23].

I. **The Hamiltonian basis** For each permutation $\sigma \in \Sigma_r$, we can define

$$(11) \qquad \sigma(\boldsymbol{B}) = (\alpha^{\sigma(r-1),\sigma(r)}, \alpha^{\sigma(r-2),\sigma(r-1)}, \dots, \alpha^{\sigma(1),\sigma(2)}) \in \mathscr{B}.$$

The set $\mathscr{H}_m = \{\sigma(\boldsymbol{B}) \mid \sigma \in \Sigma_r, \, \sigma(1) = m\}$ is then a diagonal basis. In the combinatorial description, this diagonal basis corresponds to the set of Hamiltonian paths starting at vertex m, and endowed with the reversed natural ordering of edges.

Example 3.3 Here are some examples of Hamiltonian bases:

- For $r = 3$: $\mathscr{H}_1 = \{(\alpha^{2,3}, \alpha^{1,2}), (\alpha^{3,2}, \alpha^{1,3})\}$.
- For $r = 4$: $\mathscr{H}_1 = \{(\alpha^{3,4}, \alpha^{2,3}, \alpha^{1,2}), (\alpha^{2,4}, \alpha^{3,2}, \alpha^{1,3}), (\alpha^{4,3}, \alpha^{2,4}, \alpha^{1,2}), (\alpha^{3,2}, \alpha^{4,3}, \alpha^{1,4}),$
$$(\alpha^{4,2}, \alpha^{3,4}, \alpha^{1,3}), (\alpha^{2,3}, \alpha^{4,2}, \alpha^{1,4})\}.$$

II. **The no-broken-circuit (nbc) bases** Let $\upsilon \colon \{1, \dots, \frac{1}{2}r(r-1)\} \to \Phi$ be a total ordering, which we will represent as an order relation $\overset{\upsilon}{<}$ on Φ. To this ordering, one can associate the so-called *noncommutative no-broken-circuit diagonal basis* [23]

$$\mathscr{D}[\upsilon] = \{(\beta^{[1]}, \dots, \beta^{[r-1]}) \in \mathscr{B} \mid \beta^{[1]} \overset{\upsilon}{<} \cdots \overset{\upsilon}{<} \beta^{[r-1]}, \text{ and}$$
$$\alpha^{ij} \overset{\upsilon}{<} \beta^{[m]} \implies (\alpha^{ij}, \beta^{[m]}, \dots, \beta^{[r-1]}) \text{ linearly independent}\}.$$

Example 3.4 Let $\alpha^{1,3} \overset{v}{<} \alpha^{1,4} \overset{v}{<} \alpha^{2,3} \overset{v}{<} \alpha^{2,4} \overset{v}{<} \alpha^{1,2} \overset{v}{<} \alpha^{3,4}$ be the ordering of the positive roots for rank $r = 4$. Then

$$\mathscr{D}[v] = \{(\alpha^{1,3}, \alpha^{1,2}, \alpha^{3,4}), (\alpha^{1,3}, \alpha^{1,4}, \alpha^{2,3}), (\alpha^{1,3}, \alpha^{1,4}, \alpha^{2,4}), (\alpha^{1,3}, \alpha^{1,4}, \alpha^{1,2}),$$
$$(\alpha^{1,3}, \alpha^{2,3}, \alpha^{3,4}), (\alpha^{1,3}, \alpha^{2,3}, \alpha^{2,4})\}$$

is the corresponding no-broken-circuit diagonal basis.

Remark 3.5 The hyperplane arrangement induced by Φ is invariant under the natural action of Σ_r on the vector space V. It follows easily from the definition that if $\mathscr{D}$ is a diagonal basis and $\sigma \in \Sigma_r$ is a permutation, then $\sigma(\mathscr{D})$ is also a diagonal basis.

4 The residue formula and the main result

In this section, we recall the residue formula from [23] for $\mathrm{Ver}(k, \lambda)$, the discrete Verlinde sum on the right-hand side of (1). The key feature of this formula is that it exposes the piecewise polynomial nature of $\mathrm{Ver}(k, \lambda)$, which is key for our wall-crossing analysis. While the objects are relatively simple, the formalism is heavy with notation, so we begin by describing the one-dimensional case.

4.1 The residue formula in dimension 1

The story begins with the Fourier series

$$(12) \qquad \frac{1}{(2\pi i)^m} \sum_{n \in \mathbb{Z} \setminus 0} \frac{\exp(2\pi i a n)}{n^m}$$

for $m \geq 2$, which is a periodic, piecewise polynomial function given by the formula

$$\operatorname*{Res}_{x=0} \frac{\exp(\{a\}x)}{1 - \exp(x)} \frac{dx}{x^m},$$

where $\{a\}$ is the fractional part of the real number a. The polynomial functions thus obtained on the interval $[0, 1]$ are called *Bernoulli polynomials*. The polynomial on the interval containing the real number $c \in \mathbb{R} \setminus \mathbb{Z}$ is given by

$$\operatorname*{Res}_{x=0} \frac{\exp((a - [c])x)}{1 - \exp(x)} \frac{dx}{x^m},$$

where $[c]$ is the integer part of c.

Now we pass to a trigonometric version of this formula, calculating finite sums of values of rational trigonometric functions over rational points with denominators equal to an integer k.

We thus replace the rational function x^{-m} by the (hyperbolic) trigonometric function

$$f(x) = \left(2 \sinh\left(\tfrac{1}{2} x\right)\right)^{-2m},$$

and introduce an integer parameter λ related to a via $ka = \lambda$. We consider the sum of values of the function f over a finite set of rational points in analogy with (12),

$$\sum_{n=1}^{k-1} \frac{\exp(2\pi i \lambda n / k)}{(2 \sin(\pi n / k))^{2m}},$$

where $\lambda, k \in \mathbb{Z}$. This sum is again periodic in $\lambda \mod k$, and for $m \geq 2$ we can evaluate it via the residue theorem as

$$(-1)^m \operatorname*{Res}_{z=1} \frac{z^{k\{\lambda/k\}}}{(z^{1/2} - z^{-1/2})^{2m}} \cdot \frac{k \, dz}{z(1 - z^k)} \stackrel{z=\exp(x/k)}{=} (-1)^m \operatorname*{Res}_{x=0} \frac{\exp(\{\lambda/k\} \cdot x)}{1 - \exp(x)} \cdot f\left(\frac{x}{k}\right) dx.$$

Again, this is a piecewise polynomial function in the pair (k, λ), which is polynomial in the cones bounded by the lines $\lambda = qk$ for $q \in \mathbb{Z}$.

Note that in these calculations, a key role is played by the Bernoulli operator

$$(13) \qquad f \mapsto \operatorname{Ber}[f](a) = \frac{f(x) \exp(ax) \, dx}{1 - \exp(x)},$$

which transforms meromorphic functions in the variable x into polynomials in a, and plays the role of a generalized Fourier operator.

4.2 The multidimensional case

We return to the setup of Section 3 with the vector space V endowed with the hyperplane arrangement Φ. We introduce the notation $\mathcal{F}_\Phi$ *for the space of meromorphic functions defined in a neighborhood of* 0 *in* $V \otimes_\mathbb{R} \mathbb{C}$ *with poles on the union of hyperplanes*

$$\bigcup_{1 \leq i < j \leq r} \{x \mid \langle \alpha^{ij}, x \rangle = 0\}.$$

In particular, the function

$$w_\Phi = \prod_{i < j} \big(2 \sinh(\pi(x_i - x_j))\big)$$

is an element of $\mathcal{F}_\Phi$.

To write down our residue formula, we need a multidimensional generalization of the notions of integer and fractional parts. Given a basis $\boldsymbol{B} = (\beta^{[1]}, \ldots, \beta^{[r-1]}) \in \mathcal{B}$ of V^*, and an element $a \in V^*$, we define $[a]_{\boldsymbol{B}}$ and $\{a\}_{\boldsymbol{B}}$ to be the unique elements of V^* satisfying

$$[a]_{\boldsymbol{B}} = a - \{a\}_{\boldsymbol{B}} \in \Lambda \quad \text{and} \quad \{a\}_{\boldsymbol{B}} \in \sum_{j=1}^{r-1} [0, 1) \beta^{[j]}.$$

This notion naturally induces a chamber structure on V^*: we will call $a \in V^*$ *regular* if a is a point of continuity for the functions $a \mapsto [a]_{\boldsymbol{B}}, \{a\}_{\boldsymbol{B}}$ for all $\boldsymbol{B} \in \mathcal{B}$, ie when $\{a\}_{\boldsymbol{B}} \in \sum_{j=1}^{r-1}(0, 1)\beta^{[j]}$. Now, for regular a and b we define the equivalence relation

$$(14) \qquad a \sim b \quad \text{when} \quad [a]_{\boldsymbol{B}} = [b]_{\boldsymbol{B}} \quad \text{for all} \quad \boldsymbol{B} \in \mathcal{B}.$$

The equivalence classes for this relation form a Λ–periodic system of *chambers* in V^*.
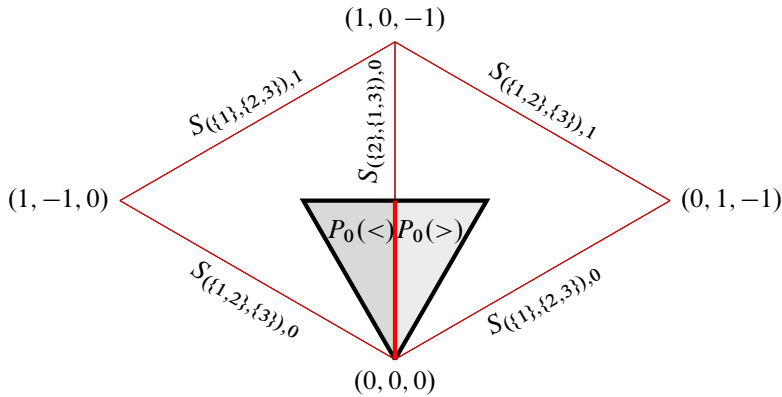
Figure 4: Chambers for rank $r = 3$.

Convention We will think of a partition Π of $\{1, 2, \ldots, r\}$ into two nonempty sets as an ordered partition $\Pi = (\Pi', \Pi'')$ such that $r \in \Pi''$, and we will call these objects *nontrivial partitions* for short.

The following statement is straightforward.

Lemma 4.1 *The equivalence classes of the relation $\sim$ are precisely the chambers in V^* created by the walls parametrized by a nontrivial partition $\Pi = (\Pi', \Pi'')$ of the first r positive integers, and an integer l:*

$$(15) \qquad S_{\Pi,l} = \left\{ c \in V^* \ \Big| \ \sum_{j \in \Pi'} c_j = l \right\}.$$

Remark 4.2 The walls given in (10) are precisely the same as the ones given in (10) for the case $d = 0$, where they play the role of walls separating the chambers of parabolic weights c in which the parabolic moduli spaces $P_0(c)$ are naturally the same. This "coincidence" is precisely what we need for our comparative wall-crossing strategy. There is a small terminological issue here: the "chambers" in Section 2.4 are the intersections of the equivalence classes of $\sim$ defined above with the open simplex $\Delta_0 \overset{\text{def}}{=} \Delta$, where the parabolic weights live; see Figures 2 and 4. We will use the term "chamber" in both cases if this causes no confusion.

Each element $\boldsymbol{B} = (\beta^{[1]}, \ldots, \beta^{[r-1]}) \in \mathscr{B}$ defines an *iterated* version of the Bernoulli operator (13) on the space of functions $\mathscr{F}_\Phi$: interpreting the elements $a, \beta^{[j]} \in V^*$ as linear functions on V, we define

$$(16) \qquad \operatorname{iBer}_{\boldsymbol{B}}[f(x)](a) = \frac{1}{(2\pi i)^{r-1}} \int_{Z(\boldsymbol{B})} \frac{f(x) \exp\langle a, x\rangle \, d\langle \beta^{[1]}, x\rangle \wedge \cdots \wedge d\langle \beta^{[r-1]}, x\rangle}{(1 - \exp(\langle \beta^{[1]}, x\rangle)) \cdots (1 - \exp(\langle \beta^{[r-1]}, x\rangle))},$$

where the naturally oriented cycle $Z_{\boldsymbol{B}}$ is defined by

$$Z_{\boldsymbol{B}} = \{v \in V \otimes_{\mathbb{R}} \mathbb{C} \mid |\langle \beta^{[j]}, x\rangle| = \varepsilon_j \text{ for } j = 1, \ldots, r-1\} \subset V \otimes_{\mathbb{R}} \mathbb{C} \setminus \{w_\Phi(x) = 0\},$$

with real constants ε_j satisfying $0 \leq \varepsilon_{r-1} \ll \cdots \ll \varepsilon_1$. Thus again, $\operatorname{iBer}_{\boldsymbol{B}}$ is a linear operator which associates to a function in $\mathscr{F}_\Phi$ a polynomial on V^*.

Remark 4.3 Let us make a small remark about the computational aspects of the operator $\mathrm{iBer}_{\boldsymbol{B}}$. Denoting the coordinate $\langle \beta^{[j]}, x \rangle$ by y_j for $j = 1, \ldots, r-1$, and writing f and a in these coordinates: $f(x) = \hat{f}(y)$, $\langle a, x \rangle = \langle \hat{a}, y \rangle$, we can rewrite (16) as

$$\mathrm{iBer}_{\boldsymbol{B}}[f(x)](a) = \operatorname*{Res}_{y_1 = 0} \cdots \operatorname*{Res}_{y_{r-1} = 0} \frac{\hat{f}(y) \exp\langle \hat{a}, y \rangle \, dy_1 \wedge \cdots \wedge dy_{r-1}}{(1 - \exp(y_1)) \cdots (1 - \exp(y_{r-1}))},$$

where *iterating* the residues here means that we keep the variables with lower indices as unknown constants, and then use geometric series expansions of the type

$$\frac{1}{1 - \exp(y_1 - y_2)} = \frac{y_1 - y_2}{1 - \exp(y_1 - y_2)} \cdot \frac{1}{y_1 - y_2} = \frac{y_1 - y_2}{1 - \exp(y_1 - y_2)} \cdot \sum_{n=0}^{\infty} \frac{y_2^n}{y_1^{n+1}}.$$

4.3 Invariance of diagonal bases and the main results

Diagonal bases have the following key invariance property.

Theorem 4.4 [23] *Let* $f \in \mathscr{F}_\Phi$, *and* $c \in V^*$ *be regular; let* $\mathscr{D}$ *be a diagonal basis of* Φ. *Then the functional*

$$f \mapsto \sum_{\boldsymbol{B} \in \mathscr{D}} \mathrm{iBer}_{\boldsymbol{B}}[f(x)](a - [c]_{\boldsymbol{B}})$$

(see (16) *above)* *transforming a meromorphic function* $f \in \mathscr{F}_\Phi$ *into a polynomial in the variable* $a \in V^*$ *is independent of the choice of the diagonal basis* $\mathscr{D}$. *In particular, for regular* $a \in V^*$, *the functional*

$$(17) \qquad f \mapsto \sum_{\boldsymbol{B} \in \mathscr{D}} \mathrm{iBer}_{\boldsymbol{B}}[f(x)](\{a\}_{\boldsymbol{B}})$$

transforms f *into a well-defined piecewise polynomial function on* V^*, *which is polynomial in each chamber.*

As this functional is invariantly defined, it is not surprising that it is equivariant with respect to the symmetries of our hyperplane arrangement. For $\sigma \in \Sigma_r$, we define, as usual

$$(18) \qquad \sigma \cdot f(x) = f(\sigma^{-1} x).$$

This convention is consistent with (11).

Lemma 4.5 *Let* $f \in \mathscr{F}_\Phi$ *and* $\sigma \in \Sigma_r$, *and pick any diagonal basis* $\mathscr{D}$. *Then*

$$\sum_{\boldsymbol{B} \in \mathscr{D}} \mathrm{iBer}_{\boldsymbol{B}}[f(x)](\sigma \cdot a - [\sigma \cdot c]_{\boldsymbol{B}}) = \sum_{\boldsymbol{B} \in \mathscr{D}} \mathrm{iBer}_{\boldsymbol{B}}[\sigma^{-1} \cdot f(x)](a - [c]_{\boldsymbol{B}}).$$

Proof Indeed, recall that $\sigma \in \Sigma$ takes a diagonal basis to another diagonal basis (see Remark 3.5), so

$$\sum_{\boldsymbol{B} \in \mathscr{D}} \mathrm{iBer}_{\boldsymbol{B}}[f(x)](\sigma \cdot a - [\sigma \cdot c]_{\boldsymbol{B}}) = \sum_{\boldsymbol{B} \in \mathscr{D}} \mathrm{iBer}_{\sigma \boldsymbol{B}}[f(x)](\sigma \cdot a - [\sigma \cdot c]_{\sigma \boldsymbol{B}}).$$

Now we perform the linear substitution $x = \sigma(y)$, and obtain

$$\sum_{\boldsymbol{B} \in \mathscr{D}} \underset{\sigma \boldsymbol{B}}{\mathrm{iBer}}[f(x)](\sigma \cdot a - [\sigma \cdot c]_{\sigma \boldsymbol{B}}) = \sum_{\boldsymbol{B} \in \mathscr{D}} \underset{\boldsymbol{B}}{\mathrm{iBer}}[\sigma^{-1} \cdot f(y)](a - [c]_{\boldsymbol{B}}). \qquad \square$$

Remark 4.6 By picking the Hamiltonian diagonal basis $\mathscr{H}_1 = \{\sigma \cdot \boldsymbol{B}_0 \mid \sigma \in \mathrm{Stab}(1, \Sigma_r)\}$, we can turn the argument in the proof above around, and obtain the formula

$$\sum_{\boldsymbol{B} \in \mathscr{H}_1} \underset{\boldsymbol{B}}{\mathrm{iBer}}[f(x)](a - [c]_{\boldsymbol{B}}) = \sum_{\sigma \in \mathrm{Stab}(1, \Sigma_r)} \underset{\boldsymbol{B}_0}{\mathrm{iBer}}[\sigma \cdot f(x)](\sigma \cdot a - [\sigma \cdot c]_{\boldsymbol{B}})$$

$$= \underset{y_1 = 0}{\mathrm{Res}} \cdots \underset{y_{r-1} = 0}{\mathrm{Res}} \sum_{\sigma \in \mathrm{Stab}(1, \Sigma_r)} \frac{\sigma \cdot f(y) \exp\langle \sigma \cdot a - [\sigma \cdot c]_{\boldsymbol{B}}, y \rangle \, dy_1 \wedge \cdots \wedge dy_{r-1}}{(1 - \exp(y_1)) \cdots (1 - \exp(y_{r-1}))},$$

where

$$\boldsymbol{B}_0 = (y_1 = x_{r-1} - x_r, \ldots, y_{r-2} = x_2 - x_3, y_{r-1} = x_1 - x_2) \in \mathscr{B}.$$

Now we are ready to write down the residue formula for the Verlinde sums proved in [24, Theorem 4.2]. Recall that we denoted by $\mathrm{Ver}(k, \lambda)$ the finite sum on the right-hand side of (1).

Theorem 4.7 Let $g \geq 1$, $k \in \mathbb{Z}^{>0}$, $\lambda \in \Lambda$, and let $\mathscr{D}$ be any diagonal basis of Φ. Introducing the notation $\widehat{k} = k + r$ and $\widehat{\lambda} = \lambda + \rho$, we have

$$(19) \qquad \mathrm{Ver}(k, \lambda) = \widetilde{N}_{r,k} \sum_{\boldsymbol{B} \in \mathscr{D}} \underset{\boldsymbol{B}}{\mathrm{iBer}}[w_\Phi^{1-2g}(x/\widehat{k})](\widehat{\lambda}/\widehat{k} - [\widehat{c}]_{\boldsymbol{B}}),$$

where $\widetilde{N}_{r,k} = (-1)^{\binom{r}{2}(g-1)} N_{r,k}$ (see (1)) and $\widehat{c} \in V^*$ is a regular point in a chamber that contains $\widehat{\lambda}/\widehat{k}$ in its closure.

Now, if we look at our main goal (1): proving the equality

$$(20) \qquad \mathrm{Ver}(k, \lambda) = \chi(P_0(\lambda/k), \mathscr{L}_0(k; \lambda)),$$

we discover a rather embarrassing mismatch. Both sides are piecewise polynomial functions; however,

- according to the HRR theorem, $\chi(P(\lambda/k), \mathscr{L}_0(k; \lambda))$ is polynomial on the cones over the equivalence classes (see (14)) of λ/k, while
- according to (19), $\mathrm{Ver}(k, \lambda)$ is polynomial on the cones over the equivalence classes of $\widehat{\lambda}/\widehat{k}$,

and these conic partitions of $\{(k, \lambda) \mid \lambda/k \in \Delta\}$ could clearly be different; see Figure 5 for a sketch of this problem.

Thus for (1) to be true, some miracle needs to occur, and these miracles are well-known in the area of "quantization commutes with reduction" [17; 29; 30; 25]. We will return to this problem in Section 10, but for now, we will be satisfied to use (19) to write down a (conjectural for the moment) formula for $\chi(P_0(\lambda/k), \mathscr{L}_0(k; \lambda))$, which is manifestly polynomial on the cones where λ/k is in a fixed equivalence class.
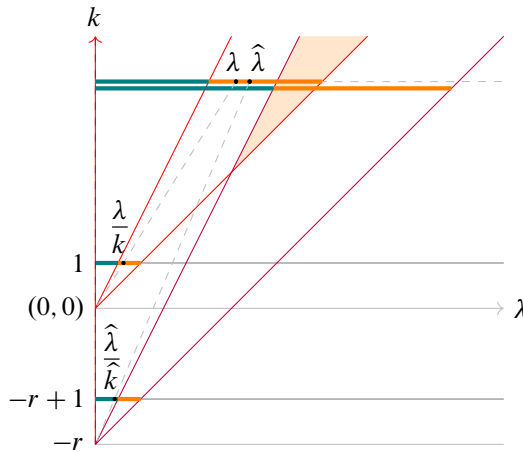
Figure 5: The vector λ/k is in the orange chamber, while $\widehat{\lambda}/\widehat{k}$ is in the green chamber.

Let us fix a regular $c \in \Delta$ marking a particular chamber in Δ. The two cones $\{(k; \lambda) \mid \lambda/k \sim c\}$ and $\{(k; \lambda) \mid \widehat{\lambda}/\widehat{k} \sim c\}$ intersect along an open cone (this cone is shaded in orange in Figure 5), and on this intersection, the expression

$$(21) \qquad \sum_{\boldsymbol{B} \in \mathcal{D}} \operatorname*{iBer}_{\boldsymbol{B}}[w_{\Phi}^{1-2g}(x/\widehat{k})](\widehat{\lambda}/\widehat{k} - [\lambda/k]_{\boldsymbol{B}})$$

coincides with the right-hand side of (19). As (21) is manifestly polynomial on each cone where λ/k is in a particular chamber in Δ, this expression will be then our main candidate for $\chi(P_0(\lambda/k), \mathscr{L}_0(k; \lambda))$.

Our plan is thus to split the proof of (20) into three parts: the first is equality (19), and the other two are given in our main theorem below. We formulated all our statements in a manner that allows us to treat the cases when λ/k or $\widehat{\lambda}/\widehat{k}$ are on a boundary separating two of our chambers in Δ.

Theorem 4.8 *Let* $\lambda \in \Lambda$ *and* $k \in \mathbb{Z}^{>0}$ *be such that* $\lambda/k \in \Delta$. *Let* ς *and* $\widehat{\varsigma} \in \Delta$ *be regular elements, specifying two chambers in* Δ, *which contain* λ/k *and* $\widehat{\lambda}/\widehat{k}$ *in their closures, correspondingly. Then for any diagonal basis* $\mathcal{D}$, *the following two equalities hold:*

$$(\mathrm{I}) \qquad \chi(P_0(\varsigma), \mathscr{L}(k; \lambda)) = \widetilde{N}_{r,k} \sum_{\boldsymbol{B} \in \mathcal{D}} \operatorname*{iBer}_{\boldsymbol{B}}[w_{\Phi}^{1-2g}(x/\widehat{k})](\widehat{\lambda}/\widehat{k} - [\varsigma]_{\boldsymbol{B}}),$$

$$(\mathrm{II}) \qquad \sum_{\boldsymbol{B} \in \mathcal{D}} \operatorname*{iBer}_{\boldsymbol{B}}[w_{\Phi}^{1-2g}(x/\widehat{k})]4\widehat{\lambda}/\widehat{k} - [\varsigma]_{\boldsymbol{B}}) = \sum_{\boldsymbol{B} \in \mathcal{D}} \operatorname*{iBer}_{\boldsymbol{B}}[w_{\Phi}^{1-2g}(x/\widehat{k})](\widehat{\lambda}/\widehat{k} - [\widehat{\varsigma}]_{\boldsymbol{B}}).$$

Remark 4.9 Part (I) of the theorem implies that if $\lambda/k \in \Delta$ is not regular, then

$$\chi(P_0(c^+), \mathscr{L}(k; \lambda)) = \chi(P_0(c^-), \mathscr{L}(k; \lambda))$$

for regular $c^{\pm} \in \Delta$ in two neighboring chambers that contain λ/k in their closure; see Proposition 10.1 and Remark 10.4.

Before we proceed, we formulate a mild generalization of part (I) of our theorem. As observed above, if we fix a generic $c \in \Delta$, and vary (λ, k) in such a way that $\lambda/k \sim c$, then both sides of the equality (I) are manifestly polynomial, and thus we can extend the validity of this equality as follows.

Corollary 4.10 *Let* $c \in \Delta$ *be a regular element, which thus specifies a chamber in* Δ *and a parabolic moduli space* $P_0(c)$ *as well. Then for a diagonal basis* $\mathscr{D}$, *an arbitrary weight* $\lambda \in \Lambda$, *and a positive integer* k, *we have*

$$(22) \qquad \chi(P_0(c), \mathscr{L}(k; \lambda)) = \tilde{N}_{r,k} \sum_{\boldsymbol{B} \in \mathscr{D}} \mathrm{iBer}[w_{\Phi}^{1-2g}(x/\hat{k})](\hat{\lambda}/\hat{k} - [c]_{\boldsymbol{B}}).$$

Example 4.11 Let us write down these formulas in case of $r = 3$ explicitly. Let $\mathscr{D}$ be the diagonal basis from Example 3.3; then using Remark 4.6, we obtain

$$\chi(P_0(<), \mathscr{L}(k; \lambda)) = (-1)^{g-1}(3(k+3)^2)^g \operatorname*{Res}_{y=0} \operatorname*{Res}_{x=0} \frac{e^{\lambda_1 x + (\lambda_1 + \lambda_2)y + x + y} - e^{\lambda_1 x + (\lambda_1 + \lambda_3)y + x}}{(1 - e^{x(k+3)})(1 - e^{y(k+3)})w_{\Phi}(x, y)^{2g-1}} \, dx \, dy,$$

$$\chi(P_0(>), \mathscr{L}(k; \lambda)) =$$
$$(-1)^{g-1}(3(k+3)^2)^g \operatorname*{Res}_{y=0} \operatorname*{Res}_{x=0} \frac{e^{\lambda_1 x + (\lambda_1 + \lambda_2)y + x + y} - e^{\lambda_1 x + (\lambda_1 + \lambda_3)y + x + y(k+3)}}{(1 - e^{x(k+3)})(1 - e^{y(k+3)})w_{\Phi}(x, y)^{2g-1}} \, dx \, dy,$$

where $w_{\Phi}(x, y) = 2 \sinh(\frac{1}{2}x) \, 2 \sinh(\frac{1}{2}y) \, 2 \sinh(\frac{1}{2}(x + y))$.

4.4 The walls

Our first step is to identify the wall-crossing terms of the residue formula (22), which originate in the discontinuities of the function $c \mapsto \{c\}_{\boldsymbol{B}}$. These discontinuities occur on "walls": the affine hyperplanes (15). The following is straightforward:

Lemma 4.12 *Let* $S_{\Pi, l}$ *be the wall defined by* (15), *and* $\boldsymbol{B} = (\beta^{[1]}, \dots, \beta^{[r-1]}) \in \mathscr{B}$ *an ordered basis of* V^*. *Then, as a function of* c, *the fractional part function* $\{c\}_{\boldsymbol{B}}$ *has a discontinuity at a generic point of the wall* $S_{\Pi, l}$ *exactly when* $\mathrm{Tree}(\boldsymbol{B})$ *(see page 2270) is a union of a tree on* Π', *a tree on* Π'' *(the enumeration of the edges is irrelevant here) and a single edge (which we will call the **link**) connecting* Π' *and* Π''.

Notation We will denote the element of $\boldsymbol{B}$ corresponding to this edge by β_{link}; this vector thus depends on $\boldsymbol{B}$ and the partition Π.

Proof We fix $\boldsymbol{B}$, and note that for our purposes, $c \in S_{\Pi, l}$ will be considered *generic* if it belongs to only this one wall in Δ; this is equivalent to the condition that the only nontrivial subsets of coordinates of c which sum up to an integer are Π' and Π''.

Note that an element

$$c = \sum_{j=1}^{r-1} b_j \beta^{[j]} \in \Delta$$

is a point of discontinuity of the fractional part function $\{\cdot\}_{\boldsymbol{B}}$ if and only if $b_j \in \mathbb{Z}$ for some $1 \leq j \leq r-1$.

Next, we express the coefficient b_j via the coordinates of c: we show that for all $1 \leq j \leq r-1$ we have

$$b_j = \sum_{i \in \Psi_j} c_i \quad \text{for some subset } \Psi_j \subset \{1, \ldots, r\}. \tag{23}$$

Now we orient the edges of Tree($\boldsymbol{B}$) in such way that they are all directed "away" from the root vertex r, and, without loss of generality (recall that $\sum c_i = 0$), we can assume that this orientation agrees with the signs of the elements $\beta^{[j]} \in \boldsymbol{B}$. It is easy to verify then that the subset

$\Psi_j = \{k \in \{1, \ldots, r\} \mid$ the unique directed path in Tree($\boldsymbol{B}$) from r to k contains the edge

$$\text{corresponding to } \beta^{[j]}\}$$

satisfies (23).

Hence we can conclude that if $c \in S_{\Pi,l}$ is generic and the coefficient b_j is an integer, then necessarily $\Psi_j = \Pi'$, and thus $\Pi'' = \{1, \ldots, r\} \setminus \Psi_j$, and cutting the edge corresponding to $\beta^{[j]}$ from Tree($\boldsymbol{B}$) results in two disjoint trees, on Π' and on Π'', respectively. $\qquad \square$

Now choose two regular elements $c^+, c^- \in V^*$ in two neighboring chambers separated by the wall $S_{\Pi,l}$, in such a way that

$$[c^+_{\Pi'}] = l \quad \text{and} \quad [c^-_{\Pi'}] = l - 1, \tag{24}$$

where

$$c_{\Pi'} \overset{\text{def}}{=} \sum_{i \in \Pi'} c_i,$$

and, as usual, $[q]$ stands for the integer part of the real number q. Now introduce the notation

$$p_{\pm}(k; \lambda) = \widetilde{N}_{r,k} \sum_{\boldsymbol{B} \in \mathscr{D}} \underset{\boldsymbol{B}}{\text{iBer}}[w_{\Phi}^{1-2g}(x/\hat{k})](\hat{\lambda}/\hat{k} - [c^{\pm}]_{\boldsymbol{B}})$$

for the two polynomial functions in (k, λ) corresponding to c^+ and c^-, respectively. We define the *wall-crossing term* in our residue formula (22) as the difference between these two polynomials:

$$p_+(k; \lambda) - p_-(k; \lambda).$$

Using Lemma 4.1 and (24), we obtain the following simple residue formula for this difference.

Lemma 4.13 *Let* (Π, l), c^+ *and* c^- *be as above, and let us fix a diagonal basis* $\mathscr{D} \subset \mathscr{B}$. *Denote by* $\mathscr{D}|\Pi$ *the subset of those elements of* $\mathscr{D}$, *which satisfy the condition described in Lemma 4.12. Then*

$$p_+(k, \lambda) - p_-(k, \lambda) = \widetilde{N}_{r,k} \sum_{\boldsymbol{B} \in \mathscr{D}|\Pi} \underset{\boldsymbol{B}}{\text{iBer}}\Big[\big(1 - \exp(\beta_{\text{link}}(x))\big) w_{\Phi}^{1-2g}(x/\hat{k})\Big](\lambda/\hat{k} - [c^+]_{\boldsymbol{B}}), \tag{25}$$

where β_{link} *is the "link" element of* $\boldsymbol{B}$ *(depending on* Π *and* $\boldsymbol{B}$*) defined after Lemma 4.12.*

Remark 4.14 The multiplication by $1 - \exp(\beta_{\text{link}}(x))$ in (25) has the effect of canceling one of the factors in the denominator in the definition (16) of the operation iBer.

Example 4.15 Calculating the difference of two polynomials from Example 4.11, we obtain the wall-crossing term for the rank-3 case:

$$p_-(k; \lambda) - p_+(k; \lambda) = (-3(k+3)^2)^g \operatorname*{Res}_{y=0} \operatorname*{Res}_{x=0} \frac{e^{\lambda_1 x + (\lambda_1 + \lambda_3)y + x}}{(1 - e^{x(k+3)}) w_\Phi(x, y)^{2g-1}} \, dx \, dy.$$

4.5 Wall-crossing and diagonal bases

Now we pass to the study of the combinatorial object $\mathscr{D}|\Pi$ defined in Lemma 4.13. One thing we will discover is that even though each diagonal basis consists of $(r-1)!$ elements and the right-hand side of (25) does not depend on the choice of $\mathscr{D}$, the number of elements in $\mathscr{D}|\Pi$ might vary with $\mathscr{D}$.

First we look at the case of the Hamiltonian basis $\mathscr{H}_1$. Form now on, we will use the notation $|\Pi'| = r'$ and $|\Pi''| = r''$ for a nontrivial partition $\Pi = (\Pi', \Pi'')$ — recall the convention $r \in \Pi''$. The following statement is easy to verify.

Lemma 4.16 *Let $\Pi = (\Pi', \Pi'')$ be a nontrivial partition such that $1 \in \Pi'$ (the other case is analogous). Then*

$$\mathscr{H}_1|\Pi = \{\sigma(\boldsymbol{B}) \mid \sigma(1) = 1 \text{ and } \sigma(\Pi') \in \Pi'\}.$$

In particular, $\left|\mathscr{H}_1|\Pi\right| = (r'-1)! \cdot r''!$.

It turns out that for our geometric applications, instead of $\mathscr{H}_1$, we will need to choose a particular nbc basis, where the ordering is chosen to be consistent with Π.

To simplify our terminology, we will use the language of graphs and edges introduced in Section 3.3, and we will think of $\alpha^{ij} \in \Phi$ as an edge in the complete graph on r vertices. To define the ordering υ, we need to choose an edge between Π', Π''; the choice is immaterial, but for simplicity we settle for $m \stackrel{\text{def}}{=} \max\{i \in \Pi'\}$ and $r \in \Pi''$, and set $\beta_{\text{link}} = \alpha^{m,r}$ to be the smallest element according to υ.

The υ–ordered list of edges thus starts with β_{link}, and then continues with the remaining $r' \cdot r'' - 1$ edges connecting Π' and Π''. Next we list the $\frac{1}{2}r'(r'-1)$ edges connecting vertices in Π' in any order, and finally, we list the remaining edges, those connecting vertices in Π''.

Notation We introduce the natural notation Φ' and Φ'' for the $A_{r'}$ and $A_{r''}$ root systems corresponding to Π' and Π'', and we denote by $\mathscr{D}[\upsilon]$, $\mathscr{D}'[\upsilon]$ and $\mathscr{D}''[\upsilon]$ the diagonal nbc bases induced by the ordering υ on Φ, Φ' and Φ'', respectively.

The following is easy to verify.

Lemma 4.17 *Given elements $\boldsymbol{B}' \in \mathscr{D}'[\upsilon]$ and $\boldsymbol{B}'' \in \mathscr{D}''[\upsilon]$, we can define an element of $\mathscr{D}[\upsilon]$ as follows: we start with β_{link}, then append $\boldsymbol{B}'$, and then continue with $\boldsymbol{B}''$. This construction creates a one-to-one correspondence*

$$(26) \qquad\qquad \mathscr{D}'[\upsilon] \times \mathscr{D}''[\upsilon] \to \mathscr{D}[\upsilon]|\Pi.$$

In particular, $\left|\mathscr{D}[\upsilon]|\Pi\right| = (r'-1)! \cdot (r''-1)!$.

Finally, putting Lemmas 4.13 and 4.17 together, we arrive at the following elegant statement.

Proposition 4.18 *Let* (Π, l), c^+ *and* c^- *be as in Lemma 4.13, and let* $\mathscr{D}'$ *and* $\mathscr{D}''$ *be diagonal bases of* Φ' *and* Φ'', *correspondingly. Then*

$$(27) \quad p_+(k; \lambda) - p_-(k; \lambda) = (k+r) \widetilde{N}_{r,k} \sum_{\boldsymbol{B}' \in \mathscr{D}'} \sum_{\boldsymbol{B}'' \in \mathscr{D}''} \operatorname*{Res}_{\beta_{\text{link}}=0} \operatorname*{iBer}_{\boldsymbol{B}'} \operatorname*{iBer}_{\boldsymbol{B}''} [w_{\Phi}^{1-2g}(x/\widehat{k})](\widehat{\lambda}/\widehat{k} - [c^+]_{\boldsymbol{B}}) \, d\beta_{\text{link}},$$

where $\operatorname{Res}_{\beta_{\text{link}}=0} \operatorname{iBer}_{\boldsymbol{B}'} \operatorname{iBer}_{\boldsymbol{B}''} d\beta_{\text{link}}$ *is simply* $\operatorname{iBer}_{\boldsymbol{B}}$ (see (16)) *with* $\boldsymbol{B}$ *obtained by appending* $\boldsymbol{B}'$, *and then* $\boldsymbol{B}''$ *to* β_{link}, *and with the factor* $(1 - \exp\langle \beta_{\text{link}}, x \rangle)$ *removed from the denominator.*

Remark 4.19 The expression

$$\operatorname*{Res}_{\beta_{\text{link}}=0} \operatorname*{iBer}_{\boldsymbol{B}'} \operatorname*{iBer}_{\boldsymbol{B}''} [w_{\Phi}^{1-2g}(x/\widehat{k})](\widehat{\lambda}/\widehat{k} - [c^+]_{\boldsymbol{B}}) \, d\beta_{\text{link}}$$

may equally be interpreted as follows. We write

$$\Lambda \ni \widehat{\lambda}/\widehat{k} - [c^+]_{\boldsymbol{B}} = m_{\text{link}}\beta_{\text{link}} + n' + n''$$

according to the splitting of $\boldsymbol{B}$, think of $w(x/\widehat{k})$ as a function in $\mathscr{F}_{\Phi''}$ with some fixed values of the parameters from $\boldsymbol{B}'$ and β_{link}, and then calculate

$$\operatorname*{iBer}_{\boldsymbol{B}''} [w_{\Phi}^{1-2g}(x/\widehat{k})](n'').$$

The result will be a rational function Q in the variables from $\boldsymbol{B}'$ and β_{link}, and we proceed to calculate $\operatorname{iBer}_{\boldsymbol{B}'}[Q](n')$ to obtain a function F in the variable β_{link}, and finally the answer is

$$\operatorname*{Res}_{\beta_{\text{link}}=0} \exp(\beta_{\text{link}}) F(\beta_{\text{link}}) \, d\beta_{\text{link}}.$$

We observe that since the trees $\operatorname{Tree}(\boldsymbol{B}')$ and $\operatorname{Tree}(\boldsymbol{B}'')$ are disjoint, the order of the application of the operations $\operatorname{iBer}_{\boldsymbol{B}'}$ and $\operatorname{iBer}_{\boldsymbol{B}''}$ is immaterial.

5 Wall-crossing in master space

Master spaces were introduced by Thaddeus in [28] in order to understand the dependence of GIT quotients on their linearizations. Following his footsteps, in this section we describe a simple but very effective method to control the changes in the Euler characteristics of line bundles when crossing a wall in the space of linearizations. (Similar results appeared in [8].)

5.1 Wall-crossing and holomorphic Euler characteristics

We begin by recalling the basic notions of geometric invariant theory.

Let X be a smooth projective variety over $\mathbb{C}$, and G a reductive group acting on X. A *linearization* of this action is a line bundle L on X with a lifting of the G–action to a linear action on L. An ample linearization is G–effective if L^n has a nonzero G–invariant section for some $n > 0$; the space of such linearizations $\operatorname{Cone}_G(X)$ is called the G–*effective ample cone*.

For $L \in \mathrm{Cone}_G(X)$, we define the invariant-theoretic quotient $M_L = X /\!/^L G$ as the Proj of the graded ring of invariant sections of the powers of L:

$$M_L = \mathrm{Proj} \bigoplus_n H^0(X, L^n)^G.$$

According to Mumford's geometric invariant theory [18], there is a partition of X (depending on L)

(28) $$X = X^{\mathrm{s}}[L] \cup X^{\mathrm{sss}}[L] \cup X^{\mathrm{us}}[L]$$

into the set of stable, strictly semistable, and unstable points such that there is a surjective map

$$(X^{\mathrm{s}}[L] \cup X^{\mathrm{sss}}[L])/G \to M_L.$$

When $X^{\mathrm{sss}}[L]$ is empty, this map is a bijection, and the quotient $M_L = X^{\mathrm{s}}[L]/G$ is a smooth orbifold.

In [7], Dolgachev and Hu studied the dependence of the GIT quotient $M_L = X /\!/^L G$ on L. They showed that $\mathrm{Cone}_G(X)$ is divided by hyperplanes, called walls, into finitely many convex chambers such that when L varies within a chamber, the partition (28) and thus the GIT quotient M_L remains unchanged. Moreover, an ample effective linearization lies on a wall precisely when it possesses a strictly semistable point.

Now let us consider two neighboring chambers, with smooth GIT quotients M_+ and M_-. We pick an arbitrary linearization $\mathscr{L}$ of the G–action on X, which descends to M_+ and M_-. This last condition means that if $S \subset G$ is the stabilizer of a generic point in X, then S acts trivially on the fibers of $\mathscr{L}$. We will call such linearizations *descending*.

Thus, given such a descending linearization $\mathscr{L}$ of the G–action on X, we obtain two line bundles: one on M_+ and one on M_-, which, by abuse of notation, we will denote by the same letter $\mathscr{L}$. Via taking Chern classes, this construction creates a correspondence between classes in $H^2(M_+, \mathbb{Z})$ and $H^2(M_-, \mathbb{Z})$, which we will assume to be an isomorphism of free $\mathbb{Z}$–modules. We will thus identify these lattices, and introduce the notation Γ for them:

$$\Gamma = H^2(M_+, \mathbb{Z}) \simeq H^2(M_-, \mathbb{Z}).$$

The walls mentioned above can be thought of as hyperplanes in $\Gamma_{\mathbb{R}} = \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$.

Our goal in this section is to compare the holomorphic Euler characteristics $\chi(M_+, \mathscr{L})$ and $\chi(M_-, \mathscr{L})$, which are given by the Hirzebruch–Riemann–Roch theorem:

$$\chi(M_\pm, \mathscr{L}) = \int_{M_\pm} \exp(c_1(\mathscr{L})) \, \mathrm{Todd}(M_\pm).$$

As this expression is manifestly polynomial in $c_1(\mathscr{L})$, we obtain thus two polynomials on Γ, and our goal is to calculate their difference, the *wall-crossing term*

(29) $$\chi(M_+, \mathscr{L}) - \chi(M_-, \mathscr{L}).$$

5.2 The master space construction

To simplify our setup, we will make some additional assumptions.

Assumptions 5.1   (a) The generic stabilizer of X is trivial.

(b) Let L_+ and L_- be two ample linearizations of the G–action on X from the adjacent chambers corresponding to the quotients M_+ and M_-. Without loss of generality, we can assume that the linearization $L_0 = L_+ \otimes L_-$ lies on the single wall separating the two chambers, and that the interval connecting $c_1(L_+)$ and $c_1(L_-)$ in $\Gamma_{\mathbb{R}} = \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ does not intersect any other walls.

(c) Let X^0 be the set of those semistable points $x \in X^{\mathrm{ss}}[L_0]$ which are not stable for $L_{\pm}$:

$$X^0 := X^{\mathrm{ss}}[L_0] \setminus (X^{\mathrm{s}}[L_+] \cup X^{\mathrm{s}}[L_-])$$

We assume that X^0 is smooth, and that for $x \in X^0$ the stabilizer subgroup $G_x \subset G$ is isomorphic to $\mathbb{C}^*$.

(d) Assume that there is a linearization $\vec{L}$ of the G–action on X such that $L_+ = L_- \otimes \vec{L}^n$ for some positive integer n, and such that for each $x \in X^0$, the stabilizer subgroup G_x acts freely on $\vec{L}_x \setminus 0$.

Now we introduce the *master space* construction of Thaddeus [28]. Consider the variety $Y = \mathbb{P}(\mathbb{O} \oplus \vec{L})$, which is a $\mathbb{P}^1$–bundle over X endowed with the additional $\mathbb{C}^*$–action $(1, t^{-1})$. As Y is a projectivization of a vector bundle on X, it comes equipped with $\mathbb{O}(1)$, which is the standard $G \times \mathbb{C}^*$–equivariant line bundle. To simplify our notation, we will denote the same way the linearizations of the G–action on X and their pullbacks (with tautological G–action) to Y.

The *master space* Z then is the GIT quotient of Y with respect to the linearization $L_-(n) = L_- \otimes \mathbb{O}(n)$:

$$Z = Y /\!/^{L_-(n)} G,$$

which inherits a $\mathbb{C}^*$–action from Y. Some additional notation:

- We denote this copy of $\mathbb{C}^*$ by T.

- We denote the projection $Y \to X$ by π, and the quotient map $Y^{\mathrm{s}} \to Z$ by ψ.

- Introduce the notation $Y(0 : \cdot)$ and $Y(\cdot : 0)$ for the two copies of X in Y, corresponding to the two poles of the projective line; then Y is partitioned into three sets

$$Y = Y(0 : \cdot) \sqcup Y(\cdot : 0) \sqcup \vec{L}^\circ,$$

where $\vec{L}^\circ$ is the line bundle $\vec{L}$ with the zero-section removed. We will write $\pi_\circ$ for the restriction of π to $\vec{L}^\circ$. We collect our maps in the following diagram:

(30)

$$\vec{L}^\circ \lhook\joinrel\longrightarrow Y = \mathbb{P}(\mathbb{O} \oplus \vec{L}) \supset Y^{\mathrm{s}} \xrightarrow{\ \psi\ } Z$$
$$\searrow{\scriptstyle \pi_\circ} \qquad \downarrow{\scriptstyle \pi}$$
$$X$$

Proposition 5.2 (i) *There are embeddings*

$$\iota_-\colon M_- \to Z \quad \text{and} \quad \iota_+\colon M_+ \to Z$$

obtained as the quotients $Y^s \cap Y(\cdot:0)/G$ and $Y^s \cap Y(0:\cdot)/G$, correspondingly.

(ii) *The strictly semistable locus of Y with respect to the linearization $L_-(n)$ is empty, and the GIT quotient $Z = Y^s/G$ is smooth.*

(iii) *There is an embedding $\iota_0\colon X^0/G \to Z$, obtained via $\psi(\pi_\circ^{-1}(X^0))$. We denote the image of ι_0 by Z^0.*

(iv) *The fixed point locus Z^T is the disjoint union of $\iota_+(M_+)$, $\iota_-(M_-)$ and Z^0.*

Proof Statements (i)–(iii) follow from [28, 4.2, 4.3]. To prove (iv), first note that $Y(\cdot:0)$ and $Y(0:\cdot)$ are fixed by T, so we immediately obtain that $M_\pm \subset Z$ are fixed components. Also the G–action on Y commutes with the T–action, so a point $\psi(y) \in \psi(\pi_\circ^{-1}(X))$ is fixed by T if and only if the T–orbit $T \cdot y \subset \pi_\circ^{-1}(X)$ is contained in the G–orbit $G \cdot y \subset \pi_\circ^{-1}(X)$. Since $T \cdot y \subset \pi_\circ^{-1}(x)$ for some $x \in X$, we need $y \in \pi_\circ^{-1}(X^0)$. Moreover, for any $y \in \pi_\circ^{-1}(x) \subset \pi_\circ^{-1}(X^0)$, $T \cdot y = \pi_\circ^{-1}(x) = G_x \cdot y$, so a point $\psi(y) \in \psi(\pi_\circ^{-1}(X))$ is fixed by T if and only if $\psi(y) \in \psi(\pi_\circ^{-1}(X^0)) = Z^0$. $\square$

Construction Given a G–equivariant vector bundle E on X, we can construct a T–equivariant vector bundle $\zeta(E) \to Z$ on Z by first pulling E back from X to Y, and endowing the resulting bundle π^*E with the trivial action of T, and the action of G pulled back from X. We then obtain $\zeta(E) \to Z$ by descending π^*E to Z.

Before we formulate our wall-crossing formula, we need one more ingredient: the identification of the normal bundles of the fixed-point components of Z.

Lemma 5.3 (i) *The normal bundle on the component M_+ of Z^T is $\zeta(\vec{L}^{-1})|_{M_+}$, and the normal bundle of M_- is $\zeta(\vec{L})|_{M_-}$.*

(ii) *Let $x \in X^0$, denote by G_x the stabilizer of x in the group G, and consider the point $\iota_0(x) \in Z^0$ (see Proposition 5.2(iii)). Then the normal vector space of $Z^0 \subset Z$ at the point $\iota_0(x)$ is canonically T–equivariantly isomorphic to the T–vector space $\vec{L}_x^\circ \times_{G_x} N_x X^0$, where $N_x X^0$ is the vector space normal to $X^0 \subset X$ at x, and the $T \simeq \mathbb{C}^*$–action is induced by left multiplication by t^{-1} on $\vec{L}_x$.*

Proof Part (i) immediately follows from the formula for the tangent space of the projective line: $T\mathbb{P}(V) \simeq \operatorname{Hom}(S, Q)$, where $S \to V \to Q$ is the tautological sequence on $\mathbb{P}(V)$, the projectivization of the vector space V.

For part (ii), consider diagram (30); our goal is to identify the descent to Z_0 of the normal bundle $N_{\pi_\circ^{-1} X^0}$ to $\pi_\circ^{-1} X^0$ in $\vec{L}^\circ$. We only need to observe that this bundle may be identified with the pullback $\pi_\circ^* N X^0$ of the normal bundle to X^0 in X, endowed with the natural G–action and a T–action, which is trivial on the fibers. $\square$

Remark 5.4 Restricting the operator ζ to X^0, we can construct a T–equivariant vector bundle on Z^0 from a G–equivariant vector bundle on X^0. Then the normal bundle N_{Z^0} of $Z^0 \subset Z$ may be also described as $\zeta|_{X^0}(N X^0)$. The T–weights of the action may be computed by fixing $x \in X^0$, identifying the stabilizer subgroup $G_x \subset G$ with T via its action on the fiber $\vec{L}_x$, and then considering the action of G_x on $N_x X^0$.

Lemma 5.5 *The restriction of the line bundle $\zeta(\vec{L})$ to Z^0 is trivial with T–weight 1.*

Proof Note that $\pi_\circ^* \vec{L}$ admits a G–equivariant tautological nonvanishing section. For calculating the weight, we observe that while T acts on $\vec{L}_x$ with weight -1, the T–weight of $\vec{L}_x^\circ \times_{G_x} \vec{L}$ is $+1$. $\qquad\square$

Definition 5.6 Given a T–vector bundle V on a manifold on which T acts trivially, the T–equivariant *K-theoretical Euler class* of V^*, which we denote by $E_t(V)$, may be described as follows: let $x_1, \ldots, x_n$ be the Chern roots of V, and $l_1, \ldots l_n \in \mathbb{Z}$ be the corresponding T–weights. Then

$$E_t(V) = \prod_{j=1}^{n} (1 - t^{-l_j} \exp(-x_j)).$$

Now we are ready to write down our wall-crossing formula for (29). A key role will be played by the following notion: given a rational differential one-form on the Riemann sphere, let us denote taking the sum of residues at 0 and at infinity by $\mu \mapsto \operatorname{Res}_{t=0,\infty} \mu$:

$$\operatorname*{Res}_{t=0,\infty} \stackrel{\text{def}}{=} \operatorname*{Res}_{t=0} + \operatorname*{Res}_{t=\infty}.$$

Theorem 5.7 *Let $\mathcal{L}$ be a linearization of the G–action on X and denote by $\zeta(\mathcal{L})$, as above, the T–equivariant line bundle on Z obtained by pullback to Y and descent to Z. If Assumptions 5.1 hold, then*

$$(31) \qquad \chi(M_+, \mathcal{L}) - \chi(M_-, \mathcal{L}) = \operatorname*{Res}_{t=0,\infty} \int_{Z^0} \frac{\operatorname{ch}_t(\zeta(\mathcal{L})|_{Z^0})}{E_t(N_{Z^0})} \operatorname{Todd}(Z^0) \frac{dt}{t},$$

where N_{Z^0} is the T–equivariant bundle on Z^0 described in Lemma 5.3, ch_t is the T–equivariant Chern character, and $E_t(N_{Z^0})$ is the K-theoretical Euler class of $N_{Z^0}^*$.

Proof The Atiyah–Bott fixed-point formula [2] applied to the line bundle $\mathcal{L}$ on our master space Z yields

$$(32) \qquad \chi_t(Z, \mathcal{L}) = \sum_{F \subset Z^T} \int_F \frac{\operatorname{ch}_t(\zeta(\mathcal{L})|_F)}{E_t(N_F)} \operatorname{Todd}(F),$$

where the sum is taken over the connected components of the fixed-point locus Z^T.

In [Proposition 5.2](#), we identified these components as M_+, M_- and Z^0. [Lemma 5.3](#) identifies the equivariant normal bundles of M_+ and M_-, and thus the corresponding contributions are

$$\int_{M_+} \frac{\operatorname{ch}(\mathcal{L}) \operatorname{Todd}(M_+)}{1 - t^{-1} \exp(c_1(\vec{L}))} \quad \text{and} \quad \int_{M_-} \frac{\operatorname{ch}(\mathcal{L}) \operatorname{Todd}(M_-)}{1 - t \exp(-c_1(\vec{L}))}.$$

We observe that $\chi_t(Z, \mathcal{L})$ is a Laurent polynomial in t since it is the alternating sum of T–characters of finite-dimensional vector spaces. Thus, as a function of t, $\chi_t(Z, \mathcal{L})$ has poles only at $t = 0, \infty$, and by the residue theorem, we have

$$\operatorname*{Res}_{t=0,\infty} \chi_t(Z, \mathcal{L}) \frac{dt}{t} = 0.$$

On the other hand, since

$$\operatorname*{Res}_{t=0,\infty} \frac{A}{1 - t^{-1} B} \frac{dt}{t} = -A \quad \text{and} \quad \operatorname*{Res}_{t=0,\infty} \frac{A}{1 - t B} \frac{dt}{t} = A,$$

we have

$$\operatorname*{Res}_{t=0,\infty} \int_{M_+} \frac{\operatorname{ch}(\mathcal{L}) \operatorname{Todd}(M_+)}{1 - t^{-1} \exp(c_1(\vec{L}))} \frac{dt}{t} = -\chi(M_+, L),$$

$$\operatorname*{Res}_{t=0,\infty} \int_{M_-} \frac{\operatorname{ch}(\mathcal{L}) \operatorname{Todd}(M_-)}{1 - t \exp(-c_1(\vec{L}))} \frac{dt}{t} = \chi(M_-, L).$$

Now, applying the functional $\operatorname{Res}_{t=0,\infty}$ to the two sides of [(32)](#) multiplied by dt/t, we obtain the desired result [(31)](#). $\qquad\square$

6 Wall-crossings in parabolic moduli spaces

In this section, we apply [Theorem 5.7](#) to wall-crossings in the moduli space of parabolic bundles.

From now on, we assume that $d = 0$, and we write Δ for the corresponding set of admissible parabolic weights Δ_0. Recall from [Section 2.2](#) that for regular $c \in \Delta$, the moduli space of stable parabolic bundles $P_0(c)$ is the GIT quotient $XQ /\!/^c \operatorname{PSL}(\chi)$, where XQ is a subspace of the total space of a flag bundle over the Quot scheme. Let us fix a partition $\Pi = (\Pi', \Pi'')$ and an integer l, and introduce the notation Δ'_l and Δ''_{-l} for the simplices of parabolic weights of Π' and Π''. Let $\phi \in \Sigma_r$ be the unique permutation which sends $\{1, \ldots, r'\}$ to Π' preserving the order of the first r' and the last r'' elements. We choose $c^0 = (c_1^0, \ldots, c_r^0) \in S_{\pi,l}$ and two regular elements $c^+, c^- \in \Delta$ in two neighboring chambers separated by the wall $S_{\Pi,l}$, such that

$$c^{\pm} = c^0 \pm \epsilon(\ldots, 0, 1, 0, \ldots, 0, -1)$$

for some positive $\epsilon \in \mathbb{Q}$, where 1 and -1 are on the $\phi(r')^{\text{th}}$ and r^{th} places, respectively. Let

$$c' = \sum_{i \in \Pi'} c_i^0 x_i \in \Delta'_l \quad \text{and} \quad c'' = \sum_{i \in \Pi''} c_i^0 x_i \in \Delta''_{-l}.$$

For $(k, \lambda) \in \mathbb{Z} \times \Lambda$, consider the polynomials

$$q_{\pm}(k, \lambda) = \chi(P_0(c^{\pm}), \mathcal{L}_0(k; \lambda)).$$

Our goal is to calculate the difference of these two polynomials.

Notation To simplify our notation, from now on, we omit the index t from the symbols for equivariant characteristic classes.

6.1 The master space construction

We construct the master space Z from Section 5.2 using the following data:

- A smooth variety $X = XQ$; see Section 2.2.
- Linearizations $L^{\pm} = L(k; \lambda^{\pm})$ of the G–action on X (see Section 2.2) such that $\lambda^{\pm}/k = c^{\pm}$.
- The linearization $\vec{L} = L(0; x_{\phi(r')} - x_r)$ of the G–action on X.

The following statement is easy to verify.

Lemma 6.1 [6, Section 3.2] *The subset $X^0 \subset X$ is the set of points representing vector bundles W on C such that W splits as a direct sum $W' \oplus W''$, where W' and W'' are, respectively, c'– and c''–stable parabolic bundles. Therefore, we have the following description of the locus Z^0:*

$$Z^0 = \{W = W' \oplus W'' \mid W' \in \widetilde{P}_l(c'), W'' \in \widetilde{P}_{-l}(c''), \det(W) \simeq \mathcal{O}\}.$$

Remark 6.2 The locus Z^0 is fibered over Jac^l with fiber $P_l(c') \times P_{-l}(c'')$ by the determinant map $\widetilde{P}_l(c') \to \text{Jac}^l$, and

$$(33) \qquad\qquad H^*(Z^0, \mathbb{Q}) \simeq H^*(P_l(c') \times P_{-l}(c''), \mathbb{Q}) \otimes H^*(\text{Jac}^l, \mathbb{Q}).$$

Remark 6.3 If the rank of the vector bundle $W \in \widetilde{P}_l(c)$ is 1, then $c = l$ and $\widetilde{P}_l(l)$ is isomorphic to Jac^l, while $P_l(l)$ is a point.

Now we need to verify the hypotheses of Theorem 5.7. Note that in our present construction X is not projective; however, it contains all semisimple points of the flag bundle over the open subscheme of the Quot scheme parametrizing locally free quotients (see Section 2.2) for all possible polarizations, and hence the missing points of the Quot scheme have no effect on any of our constructions (a similar argument appeared in [28]).

Assumptions 5.1(a)–(b) are trivially satisfied, so we study the action of the stabilizer $G_x \subset \text{PSL}(N)$ of point $x \in X$ on the fiber $\vec{L}_x \setminus 0$.

- For a general point $x \in X$ the stabilizer of x is the center $\mathbb{Z}_N \subset \text{SL}(N)$, which acts trivially on the fiber $\vec{L}_x \setminus 0$.

- For $x \in X^0$, any element of the stabilizer of x induces an automorphism of the corresponding vector bundle $W = W' \oplus W''$, so the stabilizer of x in $\mathrm{GL}(N)$ is isomorphic to $\mathbb{C}^* \times \mathbb{C}^* \subset \mathrm{GL}(N)$. An element $(t_1, t_2) \in \mathbb{C}^* \times \mathbb{C}^*$ is in $\mathrm{SL}(N)$ if and only if $t_1^{N'} t_2^{N''} = 1$, where $N' = \chi(W')$ and $N'' = \chi(W'')$. Note that (t_1, t_2) acts on $\vec{L}_x$ as $t_1 t_2^{-1}$, and we need $t_1 = t_2$ (hence $t_1^N = 1$) for this action to be trivial, so the stabilizer of any point in $\vec{L}_x \setminus 0$ is the center $\mathbb{Z}_N \subset \mathrm{SL}(N)$.

Then the action of $G = \mathrm{PSL}(N)$ is free on $Y \setminus (Y(0 : \cdot) \cup Y(\cdot : 0))$, and the action of $G_x \subset \mathrm{PSL}(N)$ on $\vec{L}_x \setminus 0$ induces an isomorphism $G_x \simeq \mathbb{C}^* \simeq T$.

Now by Theorem 5.7, the wall-crossing polynomial $q_-(k; \lambda) - q_+(k; \lambda)$ is equal to

$$(34) \qquad \operatorname*{Res}_{t=0,\infty} \int_{Z^0} \frac{\mathrm{ch}(\mathscr{L}_0(k; \lambda)|_{Z^0})}{E(N_{Z^0})} \, \mathrm{Todd}(Z^0) \, \frac{dt}{t}.$$

Note that in our case, the T–action on Z is free outside the fixed locus Z^T, so as a function in $t \in T$, the integral in (34) may have poles only at $t = 0, 1, \infty$. Then, using the residue theorem and substituting $t = e^u$, we conclude that (34) equals

$$(35) \qquad -\operatorname*{Res}_{u=0} \int_{Z^0} \frac{\mathrm{ch}(\mathscr{L}_0(k; \lambda)|_{Z^0})}{E(N_{Z^0})} \, \mathrm{Todd}(Z^0) \, du,$$

and thus our goal is to calculate this integral.

Our first step is to identify the characteristic classes under the integral sign; see Proposition 6.11 for the result.

We start with the study of the restriction of the line bundle $\mathscr{L}_0(k; \lambda)$ to the fixed locus $Z^0 \subset Z$. First, we describe a parametrization of the factor $H^*(\mathrm{Jac}^l, \mathbb{Q})$ in (33). Let $\mathscr{J}$ be the Poincaré bundle over $\mathrm{Jac} \times C$ such that $c_1(\mathscr{J})_{(0)} = 0$; define $\eta \in H^2(\mathrm{Jac})$ by $(\sum_i c_1(\mathscr{J})_{(e_i)} \otimes e_i)^2 = -2\eta \otimes \omega$ (see Section 2.3), then (see [33]) for any $m \in \mathbb{Z}$,

$$(36) \qquad \int_{\mathrm{Jac}} e^{\eta m} = m^g.$$

As Z^0 is a connected component of the fixed locus of the T–action on Z, its equivariant cohomology factors: $H_T^*(Z^0) \simeq H^*(Z^0) \otimes \mathbb{C}[u]$. In particular, there are canonical embeddings $H^*(Z^0) \hookrightarrow H_T^*(Z^0)$ and $\mathbb{C}[u] \hookrightarrow H_T^*(Z^0)$.

Remark 6.4 It follows from Lemma 5.5 that $c_1(\zeta(\vec{L})|_{Z^0}) = u$.

Recall that for a parabolic weight $c = (c_1, \ldots, c_r) \in \Delta$, we have set $c_{\Pi'} = \sum_{i \in \Pi'} c_i$.

Lemma 6.5 Let $\lambda = (\lambda_1, \ldots, \lambda_r) \in \Lambda$, $k \in \mathbb{Z}^{>0}$ and let $\Pi = (\Pi', \Pi'')$ be a nontrivial partition with $r \in \Pi''$. Let

$$\lambda' = \sum_{i \in \Pi'} \lambda_i x_i \quad \text{and} \quad \lambda'' = \sum_{i \in \Pi''} \lambda_i x_i,$$

and define δ by $(\lambda/k)_{\Pi'} = l + \delta$. Then

$$\mathrm{ch}(\mathscr{L}_0(k;\lambda)|_{Z^0}) = e^{k\delta u} \exp\left(\frac{\eta k}{r'} + \frac{\eta k}{r''}\right) \mathrm{ch}(\mathscr{L}_l(k;\lambda'_1,\ldots,\lambda'_{r'} - k\delta) \boxtimes \mathscr{L}_{-l}(k;\lambda''_1,\ldots,\lambda''_{r''} + k\delta)),$$

where $\boxtimes$ denotes the external tensor product of line bundles on $P_l(c') \times P_{-l}(c'')$.

Proof First, note that

$$\mathrm{ch}(\mathscr{L}_0(0;\lambda)|_{Z^0}) = e^{k(l+\delta)u} \mathrm{ch}(\mathscr{L}_l(0;\lambda'_1,\ldots,\lambda'_{r'} - kl - k\delta) \boxtimes \mathscr{L}_{-l}(0;\lambda''_1,\ldots,\lambda''_{r''} + kl + k\delta)),$$

and thus it will be sufficient to identify the restriction of $\mathscr{L}(k;0)$. It follows from Lemma 2.8 that

$$c_1(\mathscr{L}_0(k;0)) = \frac{k}{2r} c_2(\mathrm{End}_0(U))_{(2)}.$$

Note that

$$c_2(\mathrm{End}_0(U))_{(2)} = -k\,\mathrm{ch}_2(U)_{(2)} + c_1^2(U)_{(2)} = -k\,\mathrm{ch}_2(U)_{(2)},$$

and thus

$$c_1(\mathscr{L}_0(k;0)) = -k\,\mathrm{ch}_2(U)_{(2)}.$$

Denote by $\widetilde{U}'$ and $\widetilde{U}''$ the normalized (see Section 2.3) universal bundles over $\widetilde{P}_l(c') \times C$ and $\widetilde{P}_{-l}(c'') \times C$, respectively. Since

$$\mathrm{ch}_2(U|_{Z^0})_{(2)} = \mathrm{ch}_2(\widetilde{U}' \otimes \zeta(\vec{L})|_{Z^0})_{(2)} + \mathrm{ch}_2(\widetilde{U}''|_{Z^0})_{(2)},$$

we have (see Remark 6.4)

$$(37) \qquad c_1(\mathscr{L}_0(k;0)|_{Z^0}) = -k\,\mathrm{ch}_2(\widetilde{U}')_{(2)} - k\,u\,c_1(\widetilde{U}')_{(2)} - k\,\mathrm{ch}_2(\widetilde{U}'')_{(2)}$$

$$= \frac{k}{2r'} c_2(\widetilde{U}')_{(2)} - \frac{k}{2r'} c_1^2(\widetilde{U}')_{(2)} + \frac{k}{2r''} c_2(\widetilde{U}'')_{(2)} - \frac{k}{2r''} c_1^2(\widetilde{U}'')_{(2)} - klu.$$

Now, since

$$c_1^2(\widetilde{U}')_{(2)} = 2l\,c_1(U') - 2\eta \quad \text{and} \quad c_1^2(\widetilde{U}'')_{(2)} = -2l\,c_1(U'') - 2\eta,$$

by Lemma 2.8 we have

$$c_1(\mathscr{L}_0(k;0)|_{Z^0}) = \frac{k}{r'} c_1(\mathscr{L}_l(r';l,\ldots,l)) - \frac{kl}{r'} c_1(U') + \eta\frac{k}{r'} + \frac{k}{r''} c_1(\mathscr{L}_{-l}(r';-l,\ldots,-l))$$

$$+ \frac{kl}{r''} c_1(U'') + \eta\frac{k}{r''} - klu$$

$$= c_1(\mathscr{L}_l(k;(0,\ldots,0,kl))) + c_1(\mathscr{L}_{-l}(k;(0,\ldots,0,-kl))) + \eta\left(\frac{k}{r'} + \frac{k}{r''}\right) - klu,$$

and this completes the proof. $\qquad\square$

Lemma 6.6 Denote by $\widetilde{U}'$ and $\widetilde{U}''$ the normalized (see Section 2.3) universal bundles over $\widetilde{P}_l(c') \times C$ and $\widetilde{P}_{-l}(c'') \times C$, and denote by π the projections along C. Then the T–equivariant normal bundle to the fixed locus $Z^0 \subset Z$ is

$$(38) \qquad N_{Z^0} = R^1_T \pi_*(\mathrm{ParHom}(\widetilde{U}',\widetilde{U}'')) \oplus R^1_T \pi_*(\mathrm{ParHom}(\widetilde{U}'',\widetilde{U}')),$$

where the $T \simeq \mathbb{C}^*$–action has weights -1 and $+1$ on the two summands, respectively.

Remark 6.7 As we are working with fixed determinant moduli spaces, the pushforwards in (38) are to be taken along the curve C in the part of $\widetilde{P}_l(c') \times \widetilde{P}_{-l}(c'') \times C$, where $\det(W') \det(W'') \simeq \mathbb{O}$; see Lemma 6.1.

Proof According to Lemma 5.3, for any point $x \in X^0$, the normal bundle N_{Z^0} at the point $\iota_0(x) \in Z^0$ may be identified with the T–vector space $\vec{L}_x^{\circ} \times_{G_x} N_x X^0$, where $N_x X^0$ is the normal bundle to $X^0 \subset X$ at x, with the T–action induced by left multiplication by t^{-1} on $\vec{L}_x$.

Denote by UQ the universal bundle over X, which descends to the normalized universal bundles on $P_0(c^{\pm})$. Recall that any point $x \in X^0$ represents a vector bundle which splits as a direct sum of two subbundles, hence we have $UQ_x = U_x^+ \oplus U_x^-$, and

$$N_x X^0 = H^1(C, \mathrm{ParHom}(U_x^+, U_x^-)) \oplus H^1(C, \mathrm{ParHom}(U_x^-, U_x^+)).$$

(See [20, Proposition 1.13] for the description of the deformation space of parabolic bundles.) A simple calculation (see Remark 5.4 and Lemma 5.5) shows we have a T–module isomorphism

$$\vec{L}_x^{\circ} \times_{G_x} H^1(C, \mathrm{ParHom}(U_x^+, U_x^-)) \simeq \vec{L}_x \otimes H^1(C, \mathrm{ParHom}(U_x^+, U_x^-))$$

with T–weight -1 induced by multiplication on $\vec{L}_x$ and trivial action on U_x^+ and U_x^-; applying the projection formula we obtain that

$$\vec{L}_x \otimes H^1(C, \mathrm{ParHom}(U_x^+, U_x^-)) \simeq H_T^1(C, \mathrm{ParHom}(U_x^+ \otimes \vec{L}_x^{-1}, U_x^-)).$$

Similarly, we have

$$\vec{L}_x^{\circ} \times_{G_x} H^1(C, \mathrm{ParHom}(U_x^-, U_x^+)) \simeq \vec{L}_x^{-1} \otimes H^1(C, \mathrm{ParHom}(U_x^-, U_x^+))$$
$$\simeq H_T^1(C, \mathrm{ParHom}(U_x^-, U_x^+ \otimes \vec{L}_x^{-1}))$$

with T–action of weight 1.

Finally, we observe that according to our normalizations, the bundles $U^+ \otimes \vec{L}^{-1}$ and U^- descend to the normalized universal bundles $\widetilde{U}'$ and $\widetilde{U}''$ over $\widetilde{P}_l(c') \times C$ and $\widetilde{P}_{-l}(c'') \times C$, respectively, and this completes the proof. $\qquad\square$

6.2 Calculation of the characteristic classes of N_{Z^0}

Before we calculate the equivariant K-theoretical Euler class of the conormal bundle $N_{Z^0}^*$, we need to introduce some notation. Recall that for $1 \leq i, j \leq r$, the differences $x_i - x_j \in V^*$ are linear functions on V, and the function $x_i - x_j$ corresponds to the linearization $L_0(0; x_i - x_j)$ on X, which descends to the line bundle $\mathscr{L}_0(0; x_i - x_j)$ on the moduli space $P_0(c)$; see Section 2.2. As in Section 5.2, we denote by $\zeta(L_0(0; x_i - x_j))$ the line bundle on Z obtained by the pullback and then descent. This way, we obtain a correspondence between the linear functions $x_i - x_j$ and the T–equivariant line bundles on Z.

Recall the definition of the permutation $\phi \in \Sigma_r$ given at the beginning of this chapter: ϕ takes the first r' numbers to Π', preserving the order of the first r' and the last r'' elements. We introduce the symbols

(39)
$$
\begin{aligned}
z_i' - z_j' &= c_1\big(\zeta(L_0(0; x_{\phi(i)} - x_{\phi(j)}))|_{Z^0}\big) && \text{for } 1 \le i, j \le r', \\
z_i'' - z_j'' &= c_1\big(\zeta(L_0(0; x_{\phi(r'+i)} - x_{\phi(r'+j)}))|_{Z^0}\big) && \text{for } 1 \le i, j \le r'', \\
u = (z_{r'}' - z_r'') &= c_1\big(\zeta(L_0(0; x_{\phi(r')} - x_r))|_{Z^0}\big)
\end{aligned}
$$

for the equivariant cohomology classes in $H_T^2(Z^0)$. The last equalities are consistent with Lemma 5.5.

Remark 6.8 Letting $\mathscr{F}_i'$ and $\mathscr{F}_i''$ denote the flag bundles (see Section 2.3) on $P_0(c')$ and $P_0(c'')$, correspondingly, we have (see Remark 6.2)

$$
\begin{aligned}
z_i' - z_j' &= c_1(\mathscr{F}_{r-i+1}'/\mathscr{F}_{r-i}' \otimes (\mathscr{F}_{r-j+1}'/\mathscr{F}_{r-j}')^*) \in H^2(P_l(c')), \\
z_i'' - z_j'' &= c_1(\mathscr{F}_{r-i+1}''/\mathscr{F}_{r-i}'' \otimes (\mathscr{F}_{r-j+1}''/\mathscr{F}_{r-j}'')^*) \in H^2(P_{-l}(c'')).
\end{aligned}
$$

Taking into account these identifications, functions on V give rise to equivariant cohomology classes on Z^0. To make the splitting $H_T^*(Z^0) \simeq H^*(Z^0) \otimes \mathbb{C}[u]$, explicit, however, we will write these classes in the form $f_u(z', z'')$, thinking of them as functions of the differences of the z_i's and the differences of the z_i'', depending on the parameter u. With this convention, we introduce the notation

$$
w_u^\times(z', z'') = \prod_{\substack{i,j \\ \phi(i)<\phi(r'+j)}} 2\sinh(z_i' - z_j'') \prod_{\substack{i,j \\ \phi(r'+j)<\phi(i)}} 2\sinh(z_j'' - z_i'),
$$

$$
\rho_u^\times(z', z'') = \frac{1}{2} \sum_{\substack{i,j \\ \phi(i)<\phi(r'+j)}} (z_i' - z_j'') + \frac{1}{2} \sum_{\substack{i,j \\ \phi(r'+j)<\phi(i)}} (z_j'' - z_i'),
$$

where, according to (39),

$$
z_i' - z_j'' = (z_i' - z_{r'}') + u - (z_j'' - z_r'') = c_1\big(\zeta(\mathscr{L}_0(0; x_{\phi(i)} - x_{\phi(r'+j)}))|_{Z^0}\big) \in H_T^2(Z^0).
$$

Proposition 6.9 *The K-theoretical Euler class $E(N_{Z^0})$ (see Definition 5.6 with $t = e^u$) is given by the formula*

$$
E(N_{Z^0})^{-1} = (-1)^{lr+r'r''(g-1)} e^{-rlu} \exp\left(\frac{\eta r}{r'} + \frac{\eta r}{r''}\right) w_u^\times(z', z'')^{1-2g} \exp(\rho_u^\times(z', z''))
$$
$$
\cdot \operatorname{ch}(\mathscr{L}_l(r''; -l, \ldots, -l, -l+rl) \boxtimes \mathscr{L}_{-l}(r'; l, \ldots, l, l-rl)).
$$

Proof It follows from the short exact sequence (6) for parabolic morphisms that

$$
\operatorname{ch}\big(-\pi_!(\operatorname{ParHom}(\tilde{U}'', \tilde{U}'))\big) = -\operatorname{ch}\big(\pi_!(\operatorname{Hom}(\tilde{U}'', \tilde{U}'))\big) + \sum_{\substack{i,j \\ \phi(i)<\phi(r'+j)}} e^{z_i'-z_j''},
$$

$$
\operatorname{ch}\big(-\pi_!(\operatorname{ParHom}(\tilde{U}', \tilde{U}''))\big) = -\operatorname{ch}\big(\pi_!(\operatorname{Hom}(\tilde{U}', \tilde{U}''))\big) + \sum_{\substack{i,j \\ \phi(r'+j)<\phi(i)}} e^{z_j''-z_i'},
$$

so by Lemma 6.6,

$$(40) \quad \mathrm{ch}(N_{Z^0}) =$$
$$\mathrm{ch}\big(-\pi_!(\mathrm{Hom}(\tilde{U}'', \tilde{U}')) \oplus -\pi_!(\mathrm{Hom}(\tilde{U}'', \tilde{U}')^*)\big) + \sum_{\substack{i,j \\ \phi(i) < \phi(r'+j)}} e^{z_i' - z_j''} + \sum_{\substack{i,j \\ \phi(r'+j) < \phi(i)}} e^{z_j'' - z_i'}.$$

Let $f(x)$ be a power series in one variable, and W a vector bundle of rank r with (equivariant) Chern roots $y_1, \ldots, y_r$. Then we denote by $[f(x)]^W$ the multiplicative (equivariant) characteristic class of W given by the function $f(x)$ in Chern roots of W:

$$[f(x)]^W = \prod_{j=1}^{r} f(y_j).$$

Lemma 6.10 *Let P be a smooth variety, and let S be a T–vector bundle on $P \times C$ with T–weight 1; pick a point $p \in C$ and denote by $\pi: P \times C \to P$ the projection along the curve. Then*

$$E(-\pi_! S \oplus -\pi_! S^*)^{-1} = (-1)^{\mathrm{rk}(-\pi_! S)} \frac{\exp(-\mathrm{ch}_2(S)_{(2)})}{\big[\big(2 \sinh(\frac{1}{2}x)\big)^{2g-2}\big]^{S_p}}.$$

Proof Note that

$$E(-\pi_! S)^{-1} = \left[\frac{1}{1 - t^{-1} e^{-x}}\right]^{-\pi_! S} = \left[\frac{-t e^x}{1 - t e^x}\right]^{-\pi_! S},$$

$$E(-\pi_! S^*)^{-1} = \left[\frac{1}{1 - t e^{-x}}\right]^{-\pi_! S^*} = \left[\frac{1}{1 - t e^x}\right]^{(-\pi_! S^*)^*}.$$

Applying Serre duality and the Grothendieck–Riemann–Roch theorem we obtain

$$\mathrm{ch}(-\pi_! S) + \mathrm{ch}((-\pi_! S^*)^*) = \mathrm{ch}(-\pi_! S) + \mathrm{ch}(\pi_!(S \otimes K_C)) = \mathrm{ch}(-\pi_! S) + \pi_*(\mathrm{ch}(S \otimes K_C) \, \mathrm{Todd}(C))$$
$$= \mathrm{ch}(-\pi_! S) + \mathrm{ch}(\pi_! S) + (2g-2) \, \mathrm{ch}(S_p) = (2g-2) \, \mathrm{ch}(S_p),$$

where K_C is the canonical sheaf on the curve C, hence

$$\left[\frac{1}{1 - t e^x}\right]^{-\pi_! S \oplus (-\pi_! S^*)^*} = \left[\frac{1}{(1 - t e^x)^{2g-2}}\right]^{S_p} = \frac{\exp(-c_1(S_p)(g-1))}{\big[\big(2 \sinh(\frac{1}{2}x)\big)^{2g-2}\big]^{S_p}}.$$

Since

$$[-t e^x]^{-\pi_! S} = (-1)^{\mathrm{rk}(-\pi_! S)} \exp(c_1(-\pi_! S)),$$

and, by the Grothendieck–Riemann–Roch theorem,

$$\mathrm{ch}_1(-\pi_! S) = \mathrm{ch}_1(S_p)(g-1) - \mathrm{ch}_2(S)_{(2)},$$

we conclude that

$$[-t e^x]^{-\pi_! S} = (-1)^{\mathrm{rk}(-\pi_! S)} \exp(c_1(S_p)(g-1)) \exp(-\mathrm{ch}_2(S)_{(2)}),$$

which finishes the proof of Lemma 6.10. □

Note that the last two terms in (40) are the sums of Chern characters of line bundles, so they contribute the multiplicative factor

$$\frac{\exp(\rho_u^\times(z', z''))}{w_u^\times(z', z'')}$$

to the equivariant class $E(N_{Z^0})^{-1}$; and using Lemma 6.10 with $S = \mathrm{Hom}(\widetilde{U}'', \widetilde{U}')$, we obtain that the inverse of the K-theoretical Euler class of the first term in (40) is

$$(-1)^{lr + r'r''(g-1)} w_u^\times(z', z'')^{2-2g} \exp\left(-\mathrm{ch}_2(\mathrm{Hom}(\widetilde{U}'', \widetilde{U}'))_{(2)}\right).$$

Note that

$$-\mathrm{ch}_2(\mathrm{Hom}(\widetilde{U}'', \widetilde{U}'))_{(2)} = \tfrac{1}{2} c_2(\mathrm{End}_0(\widetilde{U}' \oplus \widetilde{U}''))_{(2)} - \tfrac{1}{2} c_2(\mathrm{End}_0(\widetilde{U}'))_{(2)} - \tfrac{1}{2} c_2(\mathrm{End}_0(\widetilde{U}''))_{(2)}$$

$$= c_1(\mathscr{L}(r; 0)|_{Z^0} \otimes \mathscr{L}_l(-r'; -l, \dots, -l) \boxtimes \mathscr{L}_{-l}(-r''; l, \dots, l)).$$

The latter equality follows from Lemma 2.8. Finally, using Lemma 6.5 to calculate the Chern character of $\mathscr{L}(r; 0)|_{Z^0}$, we obtain the formula for the class $E(N_{Z^0})^{-1}$, and the proof of the lemma is complete. $\square$

6.3 The wall-crossing formula

Putting Lemma 6.5 and Proposition 6.9 together, we obtain the following.

Proposition 6.11 *The wall-crossing term* (35) *is equal to*

$$K \operatorname*{Res}_{u=0} e^{(k\delta - rl)u} \int_{P_l(c') \times P_{-l}(c'')} \left[(w_u^\times(z', z''))^{1-2g} \exp(\rho_u^\times(z', z'')) \right.$$
$$\cdot \mathrm{ch}\left(\mathscr{L}_l(k + r''; \lambda_1' - l, \dots, \lambda_{r'-1}' - l, \lambda_{r'}' - l - k\delta + rl)\right.$$
$$\boxtimes \mathscr{L}_{-l}(k + r'; \lambda_1'' + l, \dots, \lambda_{r''-1}'' + l, \lambda_{r''}'' + l + k\delta - rl))$$
$$\left. \mathrm{Todd}(P_l(c') \times P_{-l}(c'')) \right] du,$$

where δ is a parameter depending on λ and the wall $S_{\Pi, l}$ (see Lemma 6.5) and K is the constant

$$(-1)^{lr + r'r''(g-1)} \frac{(r(k+r))^g}{(r'r'')^g}.$$

Now all that is left to do is to perform the integral, using an induction on the rank based on Corollary 4.10. We will begin with the case $l = 0$, as it is simpler. For $l = 0$, the integral from Proposition 6.11 has the form

$$(41) \quad \int_{P_0(c') \times P_0(c'')} \left[w_u^\times(z', z'')^{1-2g} e^{\rho_u^\times(z', z'')} \mathrm{Todd}(P_0(c')) \mathrm{Todd}(P_0(c'')) \right.$$
$$\left. \cdot \mathrm{ch}(\mathscr{L}_0(k + r''; \lambda_1', \dots, \lambda_{r'-1}', \lambda_{r'}' - k\delta) \boxtimes \mathscr{L}_0(k + r'; \lambda_1'', \dots, \lambda_{r''-1}'', \lambda_{r''}'' + k\delta)) \right].$$

The inductive hypothesis (22) may be cast in the form

$$(42) \quad \int_{P_0(c)} \mathrm{ch}(\mathscr{L}_0(k; \lambda)) \mathrm{Todd}(P_0(c)) = \widetilde{N}_{r,k} \sum_{\boldsymbol{B} \in \mathscr{D}} \operatorname*{iBer}_{\boldsymbol{B}} [\exp\langle \lambda, x/\widehat{k}\rangle \cdot w_\Phi(x/\widehat{k})^{1-2g}](\rho/\widehat{k} - [c]_{\boldsymbol{B}}).$$

Now let us fix k, and allow λ to vary. We can extend this equality by linearity to arbitrary linear combinations of Chern characters of line bundles of the form

$$\sum_i \mathrm{ch}(\mathcal{L}_0(k;\lambda^i)) = \mathrm{ch}(\mathcal{L}_0(k;0)) \cdot \sum_i \mathrm{ch}(\mathcal{L}_0(0;\lambda^i)).$$

Since any polynomial on V up to a fixed degree may be represented as a linear combination of exponential functions of the form $\exp\langle \lambda, x/\widehat{k} \rangle$, formula (42) may be generalized in the following way.

Lemma 6.12 *Let $G(x)$ be a formal power series on V, and denote by $G(z)$ the characteristic class in $H^*(P_0(c))$ obtained by the identification of functions on V and cohomology classes of $P_0(c)$, described before the equation* (39). *Then we have*

$$(43) \qquad \int_{P_0(c)} \mathrm{ch}(\mathcal{L}_0(k;0)) G(z) \, \mathrm{Todd}(P_0(c)) = \widetilde{N}_{r,k} \sum_{\boldsymbol{B} \in \mathscr{D}} \underset{\boldsymbol{B}}{\mathrm{iBer}}[G(x/\widehat{k}) \cdot w_\Phi^{1-2g}(x/\widehat{k})](\rho/\widehat{k} - [c]_{\boldsymbol{B}}).$$

Finally, let $\mathscr{D}'$ and $\mathscr{D}''$ be Hamiltonian bases (see Section 4.5). Since

$$w_{\Phi'}(x/\widehat{k}) w_{\Phi''}(x/\widehat{k}) w_u^\times(x/\widehat{k}) = w_\Phi(x/\widehat{k}) \quad \text{and} \quad \rho'(x/\widehat{k}) \rho''(x/\widehat{k}) \rho_u^\times(x/\widehat{k}) = \rho(x/\widehat{k}),$$

where $w_{\Phi'}, w_{\Phi''}$ and ρ', ρ'' are naturally defined for the root systems Φ' and Φ'' (see Section 4.5), the integral (41) is equal to

$$\widetilde{N}_{r',k+r''} \widetilde{N}_{r'',k+r'}$$

$$\cdot \sum_{\boldsymbol{B}' \in \mathscr{D}'} \sum_{\boldsymbol{B}'' \in \mathscr{D}''} \underset{\boldsymbol{B}'}{\mathrm{iBer}} \underset{\boldsymbol{B}''}{\mathrm{iBer}}[w_\Phi(x/\widehat{k})^{1-2g} e^{\rho(x/\widehat{k})}]$$
$$\cdot ((\lambda_1', \dots, \lambda_{r'-1}', \lambda_{r'}' - k\delta)/\widehat{k} - [c']_{\boldsymbol{B}'} + (\lambda_1'', \dots, \lambda_{r''-1}'', \lambda_{r''}'' + k\delta)/\widehat{k} - [c'']_{\boldsymbol{B}''}).$$

Identifying u (see (39)) with the "link" element of the diagonal basis $\mathscr{D} = (\alpha^{\phi(r'),r} \mathscr{D}' \mathscr{D}'')$ (see Section 4.5), and moving the factor $e^{k\delta u}$ from Proposition 6.11 inside the argument of iBer, we obtain the proof of the following theorem for $l = 0$.

Theorem 6.13 *Let $c^\pm \in \Delta$ be in the neighboring chambers; then the wall-crossing term*

$$\chi(P_0(c^+), \mathcal{L}_0(k;\lambda)) - \chi(P_0(c^-), \mathcal{L}_0(k;\lambda))$$

is equal to

$$(k+r)\widetilde{N}_{r,k} \sum_{\boldsymbol{B}' \in \mathscr{D}'} \sum_{\boldsymbol{B}'' \in \mathscr{D}''} \underset{\alpha^{\phi(r'),r}=0}{\mathrm{Res}} \underset{\boldsymbol{B}'}{\mathrm{iBer}} \underset{\boldsymbol{B}''}{\mathrm{iBer}}[w_\Phi(x/\widehat{k})^{1-2g}](\widehat{\lambda}/\widehat{k} - [c^+]_{\boldsymbol{B}}) \, d\alpha^{\phi(r'),r},$$

where $\mathscr{D}'$ and $\mathscr{D}''$ are the diagonal bases of Φ' and Φ'' (see Section 4.5) correspondingly.

Remark 6.14 This wall-crossing term coincides with the one from Proposition 4.18.

Example 6.15 It follows from Example 2.9 that in case of rank 3, the permutation $\phi \in \Sigma_3$ sends $(1, 2, 3)$ to $(1, 3, 2)$. Then $u = c_1(\mathscr{F}'_1 \otimes \mathscr{F}''^*_1)$ and let $z = z''_1 - z''_2 = c_1(\mathscr{F}''_2 / \mathscr{F}''_1 \otimes \mathscr{F}''^*_1)$. Then the inverse of the K-theoretical Euler class of the conormal bundle is (see Proposition 6.9)

$$\mathrm{ch}(\mathscr{L}) e^{9\eta/2} e^{z/2} \big(2 \sinh\big(\tfrac{1}{2} u\big) \, 2 \sinh\big(\tfrac{1}{2}(z - u)\big) \big)^{1-2g},$$

where $\mathscr{L} = \mathscr{L}_0(2; 0, 0)$ is a line bundle on the moduli space P_0 of rank-2 degree-0 stable parabolic bundles. The Chern character of the restriction of the line bundle $\mathscr{L}_0(k; \lambda_1, \lambda_2, \lambda_3)$ to Σ is

$$e^{3k\eta/2} \, \mathrm{ch}(\mathscr{L}_0^k) e^{\lambda_1 z + \lambda_2 u}.$$

Hence the wall-crossing term

$$\chi(P_0(<), \mathscr{L}_0(k, \lambda)) - \chi(P_0(>), \mathscr{L}_0(k, \lambda))$$

is equal to

$$-\big(\tfrac{3}{2}(k+3)\big)^g \operatorname*{Res}_{u=0} \frac{e^{\lambda_2 u}}{\big(2 \sinh\big(\tfrac{1}{2} u\big)\big)^{2g-1}} \cdot \int_{P_0} \frac{\mathrm{ch}\big(\mathscr{L}_0\big(k+1; \lambda_1 + \tfrac{1}{2}, -\lambda_1 - \tfrac{1}{2}\big)\big)}{\big(2 \sinh\big(\tfrac{1}{2}(z-u)\big)\big)^{2g-1}} \, \mathrm{Todd}(P_0) \, du.$$

The integral is the Euler characteristics of a line bundle on a moduli space of degree-0 rank-2 stable parabolic bundles, so we can calculate it using the induction by rank. It is equal to

$$(-1)^{g-1} (2(k+3))^g \operatorname*{Res}_{z=0} \frac{e^{(\lambda_1+1)z}}{\big(2 \sinh\big(\tfrac{1}{2}(z-u)\big) \, 2 \sinh\big(\tfrac{1}{2}z\big)\big)^{2g-1} \big(1 - e^{(k+3)z}\big)} \, dz,$$

so the wall-crossing term is

$$(-3(k+3)^2)^g \operatorname*{Res}_{u=0} \operatorname*{Res}_{z=0} \frac{e^{\lambda_1 z + \lambda_2 u + z}}{\widetilde{w}_\phi(z, u)^{2g-1} \big(1 - e^{(k+3)z}\big)} \, dz \, du,$$

where $\widetilde{w}_\phi(z, u) = 2 \sinh\big(\tfrac{1}{2}(z - u)\big) \, 2 \sinh\big(\tfrac{1}{2} u\big) \, 2 \sinh\big(\tfrac{1}{2} z\big)$. Note that this is exactly the same polynomial as in Example 4.15, after changing (z, u) to $(x, -y)$.

7 Tautological Hecke correspondences

If $l \neq 0$, then we need one more step in our proof, which uses the Hecke correspondence to calculate the wall-crossing term (35).

7.1 The Hecke correspondence

Given a rank-r degree-d vector bundle W with a full flag $0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_r = W_p$ at p, one can obtain a rank-r degree-$(d-1)$ vector bundle W' with a full flag $0 \subsetneq G_1 \subsetneq \cdots \subsetneq G_r = W'_p$ using the tautological Hecke correspondence construction as follows.

The evaluation map $W \to W_p$ induces the short exact sequence of the associated sheaves of sections

(44)
$$0 \to \mathcal{W}' \xrightarrow{\tilde{\alpha}} \mathcal{W} \to W_p / F_{r-1} \to 0$$

on curve C. Since $\mathcal{W}'$ is a kernel of $\tilde{\alpha}$, it is a locally free sheaf, thus gives a rank-r vector bundle W' over C with $\det(W') \simeq \det(W) \otimes \mathcal{O}(-p)$. The image of the associated morphism of vector bundles α at the point p is $F_{r-1} \subset W_p$, so $\alpha_p \colon W'_p \to W_p$ has a one-dimensional kernel $G_1 \subset W'_p$. Moreover, compositions of α_p with the quotient morphisms $F_{r-1} \to F_{r-1}/F_i$ induce a full flag of the corresponding kernels $G_1 \subsetneq \cdots \subsetneq G_{r-1} \subsetneq G_r = W'_p$ in W'_p.

Denote this operator between the sets of isomorphism classes of degree-d and $d-1$ vector bundles with a flag at p by

$$\mathcal{H} \colon (W, F_*) \mapsto (W', G_*).$$

Similarly, for any $m \geq 0$, one can define the operator $\mathcal{H}^m$ between the sets of isomorphism classes of degree-d and $d-m$ vector bundles with a flag at the point p by iterating the above construction m times. Clearly, these maps are independent of the parabolic weights.

Proposition 7.1 *Let $c \in \Delta$ be a regular* (see page 2266) *point. Then the operator $\mathcal{H}$ induces an isomorphism between the moduli spaces $P_d(c_1, \ldots, c_r)$ and $P_{d-1}(c_2, \ldots, c_r, c_1 - 1)$.*

Proof First, we need to show that if $W \in P_d(c_1, \ldots, c_r)$ is a parabolic stable bundle with parabolic weights $(c_1, \ldots, c_r)$, then W', its image under the Hecke operator $\mathcal{H}$, is parabolic stable with respect to parabolic weights $(c_2, \ldots, c_r, c_1 - 1)$. For this, consider the subbundle $V' \subset W'$ and let $\alpha(V') = V \subset W$ (see (44)) be its image. Since W is parabolic stable,

$$\mathrm{par\,slope}(V) < \mathrm{par\,slope}(W) = \mathrm{par\,slope}(W').$$

We need to prove that $\mathrm{par\,slope}(V') < \mathrm{par\,slope}(W')$. There are two possible cases:

- If α maps V' to V isomorphically, then $\deg(V') = \deg(V)$ and $V_p \subset F_{r-1}$, hence $\mathrm{par\,slope}(V') = \mathrm{par\,slope}(V) < \mathrm{par\,slope}(W')$.

- Otherwise, $\deg(V') = \deg(V) - 1$, and V_p is not contained in F_{r-1}, so one of the parabolic weights of V' is $c_1 - 1$. Then, as in the previous case, $\mathrm{par\,slope}(V') = \mathrm{par\,slope}(V)$, and the result follows.

To show that the map $\mathcal{H}$ is an isomorphism, note that $\mathcal{H}^r$ maps

(45)
$$P_d(c_1, c_2, \ldots, c_r) \to P_{d-r}(c_1 - 1, c_2 - 1, , \ldots, c_r - 1).$$

It is easy to check that given $\mathcal{W}$ and iterating the associated morphism of locally free sheaves of sections (44) r times, we obtain a subsheaf $\mathcal{W}' \subset \mathcal{W}$ of sections of W which vanishes at the point p. So the map (45) is just tensoring by $\mathcal{O}(-p)$, and hence it is an isomorphism. □

Now we can define an operator $\mathscr{H}^m$ for any $m \in \mathbb{Z}$, taking the inverse map if necessary. We will need the following statement, which follows from Proposition 7.1 and the construction of $\mathscr{H}^m$.

Corollary 7.2 *Let $m \geq 0$. Then under the isomorphism $\mathscr{H}^m$ the line bundle $\mathscr{L}_d(k; \lambda_1, \ldots, \lambda_r)$ corresponds to the line bundle $\mathscr{L}_{d-m}(k; \lambda_{r-m+1}, \ldots, \lambda_r, \lambda_1 - k, \ldots, \lambda_{r-m} - k)$.*

7.2 The effect of the Hecke correspondence on the integral

Recall that our goal is to calculate the wall-crossing term from Proposition 6.11. For simplicity, we assume that l is positive (the other case is analogous). We apply the Hecke operators $\mathscr{H}^l$ and $\mathscr{H}^{-l}$ to the moduli spaces $P_l(c')$ and $P_{-l}(c'')$ to obtain

$$P_0' = P_0(c_{l+1}', \ldots, c_{r'}', c_1' - 1, \ldots, c_l' - 1) \simeq P_l(c'),$$
$$P_0'' = P_0(c_{r''-l+1}'' + 1, \ldots, c_{r''}'' + 1, c_1'', \ldots, c_{r''-l}'') \simeq P_{-l}(c'').$$

Recall (see page 2270) that there is a natural action of the group Σ_r on V^*, and hence (see page 2291) on $H^2(P_l(c') \times P_{-l}(c''))$. Let $\tau' \in \Sigma_{r'}$ and $\tau'' \in \Sigma_{r''}$ be the cyclic permutations defined by

$$\tau' \cdot (c_1' - 1, \ldots, c_l' - 1, c_{l+1}', \ldots, c_{r'}') = (c_{l+1}', \ldots, c_{r'}', c_1' - 1, \ldots, c_l' - 1),$$
$$\tau'' \cdot (c_1'', \ldots, c_{r''-l}'', c_{r''-l+1}'' + 1, \ldots, c_r'' + 1) = (c_{r''-l+1}'' + 1, \ldots, c_{r''}'' + 1, c_1'', \ldots, c_{r''-l}'').$$

Now set $\tau = (\tau', \tau'') \in \Sigma_{r'} \times \Sigma_{r''} \subset \Sigma_r$. Note that

$$\tau' \cdot (-l + r', \ldots, -l + r', -l, \ldots, -l) = \tau' \cdot \rho' - \rho' \quad \text{and} \quad \tau'' \cdot (l, \ldots, l, l - r'', \ldots, l - r'') = \tau'' \cdot \rho'' - \rho'',$$

so applying the Hecke operator $\mathscr{H}^l \times \mathscr{H}^{-l}$ to the wall-crossing term from Proposition 6.11 and using Corollary 7.2, we obtain that the wall-crossing term (35) is equal to

$$(46) \quad K \operatorname*{Res}_{u=0} e^{(k\delta - rl)u} \int_{P_0' \times P_0''} \Big(\tau \cdot w_u^\times(z', z'')^{1-2g} e^{\tau \cdot \rho_u^\times(z', z'')}$$
$$\cdot \operatorname{ch}\big(\mathscr{L}_0(k + r''; \tau' \cdot (\lambda_1' - \widehat{k}, \ldots, \lambda_l' - \widehat{k}, \lambda_{l+1}', \ldots, \lambda_{r'-1}', \lambda_{r'}' - k\delta + rl))\big)$$
$$\cdot \operatorname{ch}\big(\mathscr{L}_0(k + r'; \tau'' \cdot (\lambda_1'', \ldots, \lambda_{r'-l}'', \lambda_{r''-l+1}'' + \widehat{k}, \ldots, \lambda_{r''}'' + \widehat{k} + k\delta - rl))\big)$$
$$e^{\tau'' \cdot \rho''(z', z'') - \rho''(z', z'')} e^{\tau' \cdot \rho'(z', z'') - \rho'(z', z'')} \operatorname{Todd}(P_0') \operatorname{Todd}(P_0'') \Big) du.$$

As in Section 6.3, according to Lemma 6.12, we can calculate this integral using the induction on rank. Let $\mathscr{D}'$ and $\mathscr{D}''$ be two Hamiltonian diagonal bases. Then $\tau'(\mathscr{D}')$ and $\tau''(\mathscr{D}'')$ are also Hamiltonian diagonal bases (see Remark 3.5) and the integral in (46) is equal to

$$(47) \quad (-1)^{lr} \widetilde{N}_{r', k+r''} \widetilde{N}_{r'', k+r'}$$
$$\sum_{\substack{\boldsymbol{B}' \in \tau'(\mathscr{D}') \\ \boldsymbol{B}'' \in \tau''(\mathscr{D}'')}} \operatorname*{iBer}_{\boldsymbol{B}'} \operatorname*{iBer}_{\boldsymbol{B}''} [\tau \cdot w_u^\times(x/\widehat{k})^{1-2g} (w_{\Phi'}(x/\widehat{k}) w_{\Phi''}(x/\widehat{k}))^{1-2g} e^{\tau \cdot \rho(x/\widehat{k})}]$$
$$\big(\tau' \cdot (\lambda_1' - \widehat{k}, \ldots, \lambda_l' - \widehat{k}, \lambda_{l+1}', \ldots, \lambda_{r'-1}', \lambda_{r'}' - k\delta + rl)/\widehat{k}$$
$$- [\tau' \cdot (c_1' - 1, \ldots, c_l' - 1, c_{l+1}', \ldots, c_{r'}')]_{\boldsymbol{B}'}$$
$$+ \tau'' \cdot (\lambda_1'', \ldots, \lambda_{r'-l}'', \lambda_{r''-l+1}'' + \widehat{k}, \ldots, \lambda_{r''-1}'' + \widehat{k}, \lambda_{r''}'' + \widehat{k} + k\delta - rl)/\widehat{k}$$
$$- [\tau'' \cdot (c_1'', \ldots, c_{r'-l}'', c_{r''-l+1}'' + 1, \ldots, c_{r''}'' + 1)]_{\boldsymbol{B}''}\big).$$

To arrive at Theorem 6.13, we need to make additional transformations of formula (47): first, we shift λ' and λ'', and then we apply Lemma 4.5 to eliminate the cyclic permutation τ.

Note that given an ordered basis $\boldsymbol{B} \in \mathscr{B}$ and an element $v \in V^*$ such that $\{v\}_{\boldsymbol{B}} = 0$, for any weight $\lambda \in \Lambda$ and positive integer k we have

$$(48) \qquad (\lambda + \widehat{k}v)/\widehat{k} - [c+v]_{\boldsymbol{B}} = \lambda/\widehat{k} - [c]_{\boldsymbol{B}}.$$

In particular, to perform the shift of λ' in (47), we use the following equality for any $\boldsymbol{B}' \in \mathscr{D}'$:

$$(49) \quad (\lambda'_1 - \widehat{k}, \ldots, \lambda'_l - \widehat{k}, \lambda'_{l+1}, \ldots, \lambda'_{r'-1}, \lambda'_{r'} - k\delta + rl)/\widehat{k}$$
$$- [(c'_1, \ldots, c'_{r'-1}, c'_{r'} - l) - (1, \ldots, 1, 0, \ldots 0, -l)]_{\boldsymbol{B}'}$$
$$= (\lambda'_1, \ldots, \lambda'_{r'-1}, \lambda'_{r'} - k\delta + rl - l\widehat{k})/\widehat{k} - [(c'_1, \ldots, c'_{r'-1}, c'_{r'} - l)]_{\boldsymbol{B}'},$$

which clearly remains true after changing $\mathscr{D}'$ to $\tau'(\mathscr{D}')$ and applying τ' to both sides of the equation. Similarly, shifting the last terms of (47) by $\tau''(0, \ldots, 0, -1, \cdots - 1, -1 + l)$, we can rewrite (47) as

$$(50) \quad (-1)^{lr} \widetilde{N}_{r',k+r''} \widetilde{N}_{r'',k+r'}$$
$$\sum_{\substack{\boldsymbol{B}' \in \tau'(\mathscr{D}') \\ \boldsymbol{B}'' \in \tau''(\mathscr{D}'')}} \operatorname*{iBer}_{\boldsymbol{B}'} \operatorname*{iBer}_{\boldsymbol{B}''} [\tau \cdot w_u^{\times}(x/\widehat{k})^{1-2g} (w_{\Phi'}(x/\widehat{k}) w_{\Phi''}(x/\widehat{k}))^{1-2g} e^{\tau \cdot \rho(x/\widehat{k})}]$$
$$\cdot \big(\tau' \cdot (\lambda'_1, \ldots, \lambda'_{r'-1}, \lambda'_{r'} - k\delta + rl - l\widehat{k})/\widehat{k} - [\tau' \cdot (c'_1, \ldots, c'_{r'-1}, c'_{r'} - l)]_{\boldsymbol{B}'}$$
$$+ \tau'' \cdot (\lambda''_1, \ldots, \lambda''_{r''-1}, \lambda''_{r''} + k\delta - rl + l\widehat{k})/\widehat{k} - [\tau'' \cdot (c''_1, \ldots, c''_{r''-1}, c''_{r''} + l)]_{\boldsymbol{B}''}\big).$$

Finally, identifying u (see (39)) with the "link" element of the diagonal basis

$$\tau(\mathscr{D}) = (\alpha^{\tau\phi(r'),\tau(r)} \tau'(\mathscr{D}') \tau''(\mathscr{D}''))$$

(see Section 4.5) and

- moving the factor $e^{(k\delta - rl)u}$ from (46) inside the argument of $\operatorname{iBer}_{\boldsymbol{B}}$, where
$$\boldsymbol{B} = (\alpha^{\tau\phi(r'),\tau(r)} \boldsymbol{B}' \boldsymbol{B}''),$$

- applying (48) with $\boldsymbol{B} = (\alpha^{\tau\phi(r'),\tau(r)} \boldsymbol{B}' \boldsymbol{B}'')$ and $v = l\alpha^{\tau\phi(r'),\tau(r)}$,

- applying Lemma 4.5, and

- using the fact that
$$\tau^{-1} \cdot (w_{\Phi'}(x/\widehat{k}) w_{\Phi''}(x/\widehat{k})) = (-1)^{lr} w_{\Phi'}(x/\widehat{k}) w_{\Phi''}(x/\widehat{k}),$$

we obtain the formula of Theorem 6.13 for arbitrary $l \in \mathbb{Z}$.

8 Affine Weyl symmetry and the proof of Theorem 4.8(I)

In this section, we prove certain symmetry properties of our Hilbert polynomials on the left-hand side of equation (1), and we finish the proof of Theorem 4.8(I). We start with the basic instance of symmetry of Hilbert polynomials: relative Serre duality.

8.1 Serre duality

Proposition 8.1 *Let $\mathscr{E} \to X$ be a rank-2 vector bundle over a smooth variety X, let $\pi\colon Y = \mathbb{P}(\mathscr{E}) \to X$ be its projectivization and $\omega_{X/Y}$ the relative cotangent line bundle. Then*

$$\chi(Y, \pi^*\mathscr{L} \otimes \omega_{X/Y}^m) = -\chi(Y, \pi^*\mathscr{L} \otimes \omega_{X/Y}^{-m+1})$$

for any line bundle $\mathscr{L} \in \mathrm{Pic}(X)$.

Proof By Serre duality for families of curves [10, Chapter III, Sections 7–8], for any integer n,

$$(51) \qquad \chi(Y, \pi^*\mathscr{L} \otimes \mathcal{O}(n)) = -\chi(Y, \pi^*(\mathscr{L} \otimes (\wedge^2\mathscr{E})^{n+1}) \otimes \mathcal{O}(-n-2)).$$

Denote by $\Omega_{X/Y}$ the sheaf of relative differentials on Y; the short exact sequence

$$0 \to \Omega_{X/Y} \otimes \mathcal{O}_X \to \pi^*\mathscr{E}(-1) \to \mathcal{O}_X \to 0$$

implies that

$$\omega_{X/Y} = \wedge^2(\pi^*\mathscr{E}(-1)) = \pi^*(\wedge^2\mathscr{E}) \otimes \mathcal{O}(-2).$$

Then the statement follows from (51) by substituting $n = -2m$. $\qquad\square$

Now we can generalize this statement to the case of flag bundles.

Proposition 8.2 *Let $\pi\colon Y = \mathrm{Flag}(\mathscr{E}) \to X$ be a rank-r flag bundle over X. Let $\mathscr{L}$ be a line bundle on X, and $\mathscr{F}_1, \mathscr{F}_2/\mathscr{F}_1, \ldots, \mathscr{F}_r/\mathscr{F}_{r-1}$ the standard flag line bundles on Y. For $k \in \mathbb{Z}$ and $\lambda = (\lambda_1, \ldots, \lambda_r) \in \Lambda$, write*

$$\mathscr{L}(k; \lambda) = (\pi^*\mathscr{L})^k \otimes (\mathscr{F}_r/\mathscr{F}_{r-1})^{\lambda_1} \otimes (\mathscr{F}_{r-1}/\mathscr{F}_{r-2})^{\lambda_2} \otimes \cdots \otimes \mathscr{F}_1^{\lambda_r}.$$

Consider the polynomial

$$q(k; \lambda_1, \lambda_2, \ldots, \lambda_r) = \chi(Y, \mathscr{L}(k; \lambda_1, \lambda_2, \ldots, \lambda_r))$$

in $(k, \lambda) \in \mathbb{Z} \times \Lambda$, and extend this definition to $\mathbb{R} \times V^*$. Then $q(k; \lambda - \rho)$ is anti-invariant under the permutations of $\lambda_1, \lambda_2, \ldots, \lambda_r$.

Proof For $1 \leq k < r$, let $\mathrm{Flag}_{\hat{k}}(\mathscr{E}) \to X$ be the flag bundle over X obtained from Y by forgetting the k–dimensional subspace. Then $Y \simeq \mathbb{P}(\mathscr{F}_{k+1}/\mathscr{F}_{k-1}) \to \mathrm{Flag}_{\hat{k}}(\mathscr{E})$ is a $\mathbb{P}^1$–bundle over $\mathrm{Flag}_{\hat{k}}(\mathscr{E})$, and thus applying Proposition 8.1 we obtain

$$\chi(Y, \mathscr{L}(k; \lambda_1, \ldots, \lambda_{r-k}, \lambda_{r-k+1}, \ldots, \lambda_r)) = -\chi(Y, \mathscr{L}(k; \lambda_1, \ldots, \lambda_{r-k+1} - 1, \lambda_{r-k} + 1, \ldots, \lambda_r)),$$

and the result follows. $\qquad\square$

8.2 The Weyl antisymmetry of the functions q_1 and q_{-1}

Armed with this statement, we are ready to take on the symmetries of the Hilbert polynomial of our parabolic moduli spaces. We note that the two sets $\Delta_{\pm 1}$ of weights for degree-± 1 stable parabolic bundles are simplices with one of their vertices at $(1/r, \ldots, 1/r)$ and $(-1/r, \ldots, -1/r)$, correspondingly; see Section 2.2.

Denote by $N_{\pm 1}$ the moduli spaces of rank-r degree-± 1 stable vector bundles and by UN any universal bundle over $N_{\pm 1} \times C$; see eg [3].

Lemma 8.3 *Let $c = (c_1, \ldots, c_r)$ be a parabolic weight from the chamber in Δ_1, which has as one of its vertices the (regular) point $(1/r, \ldots, 1/r)$. Then the moduli space $P_1(c)$ of rank-r degree-1 stable parabolic bundles is isomorphic to the flag bundle $\mathrm{Flag}(UN_p)$ over N_1. An analogous statement holds in the case of degree -1 and the point $(-1/r, \ldots, -1/r) \in \Delta_{-1}$.*

Proof A simple calculation shows that a point $(c_1, \ldots, c_r) \in \Delta_1$ such that all $c_i > 0$ lies inside the chamber in Δ_1 with the vertex $(1/r, \ldots, 1/r)$. Hence it is enough to prove the first statement for the moduli space $P_1(c_1, \ldots, c_r)$ with positive parabolic weights.

Moreover, it is sufficient to show that if (W, F_*) is a parabolic stable vector bundle which represents a point in $P_1(c_1, \ldots, c_r)$, then W is stable as an ordinary bundle. Assume that W admits a proper subbundle W' with $slope(W') \geq slope(W) = 1/r$, then $\deg(W') \geq 1$. Since all parabolic weights of W are positive, this implies that $\mathrm{par\,slope}(W') > 0 = \mathrm{par\,slope}(W)$, and therefore W is parabolic unstable. The proof for degree-(-1) bundles is analogous. $\qquad\square$

Denote the moduli spaces described above by $P_1(>)$ and $P_{-1}(<)$, correspondingly, and their images under the Hecke isomorphisms $\mathcal{H}$ and $\mathcal{H}^{-1}$ by $P_0(>)$ and $P_0(<)$.

The following statement is straightforward; see Lemma 2.8.

Lemma 8.4 *The line bundles $\mathscr{L}_1(r; 1, \ldots, 1)$ and $\mathscr{L}_{-1}(r; -1, \ldots, -1)$ on $P_1(>)$ and $P_{-1}(<)$ defined in Lemma 2.6 may be obtained as pullbacks of the ample generators of the Picard groups $\mathrm{Pic}(N_{\pm 1})$.*

Example 8.5 In the case of rank-3 parabolic bundles the moduli space $P_1(c_1, c_2, c_3)$ with $2c_3 > c_1 + c_2 - 1$ is a flag bundle over N_1 and it is isomorphic to the moduli space $P_0(>)$ from Example 2.9, while the moduli space $P_{-1}(c_1, c_2, c_3)$ with $2c_1 < c_2 + c_3 + 1$ is a flag bundle over N_{-1} and it is isomorphic to $P_0(<)$.

Now we establish the Weyl antisymmetry of the polynomials

$$q_{-1}(k; \lambda_1, \ldots, \lambda_r) = \chi(P_0(<), \mathscr{L}_0(k; \lambda_1, \ldots, \lambda_r)),$$
$$q_1(k; \lambda_1, \ldots, \lambda_r) = \chi(P_0(>), \mathscr{L}_0(k; \lambda_1, \ldots, \lambda_r)),$$

defined on $\mathbb{R} \times \Lambda$, as in Proposition 8.2. Let $\tau \in \Sigma_r$ be the cyclic permutation such that $\tau \cdot (c_1, \ldots, c_r) = (c_2, \ldots, c_r, c_1)$, and consider two points in V^*:

$$
\begin{aligned}
\theta_1[k] &= \frac{k+r}{r} \cdot (1, 1, \ldots, 1) - (k+r)x_r - \rho = \tau \cdot \left(\frac{k}{r} - k, \frac{k}{r}, \ldots, \frac{k}{r} \right) - \tau \cdot (\rho) \\
&= \left(\frac{k}{r} - \tfrac{1}{2}(r-1) + 1, \frac{k}{r} - \tfrac{1}{2}(r-1) + 2, \ldots, \frac{k}{r} - \tfrac{1}{2}(r-1) + r - 1, -k + \frac{k}{r} - \tfrac{1}{2}(r-1) \right), \\
\theta_{-1}[k] &= -\frac{k+r}{r} \cdot (1, 1, \ldots, 1) + (k+r)x_1 - \rho = \tau^{-1} \cdot \left(-\frac{k}{r}, \ldots, -\frac{k}{r}, -\frac{k}{r} + k \right) - \tau^{-1} \cdot (\rho) \\
&= \left(k - \frac{k}{r} + \tfrac{1}{2}(r-1), -\frac{k}{r} - \tfrac{1}{2}(r-1), -\frac{k}{r} - \tfrac{1}{2}(r-1) + 1, \ldots, -\frac{k}{r} - \tfrac{1}{2}(r-1) + r - 2 \right).
\end{aligned}
$$

Proposition 8.6 *The polynomials $q_1(k; \lambda + \theta_1[k])$ and $q_{-1}(k; \lambda + \theta_{-1}[k])$ are anti-invariant under the action of the Weyl group by permutations of $\lambda_1, \ldots, \lambda_r$.*

Proof Recall that the moduli space $P_0(>)$ is isomorphic to the flag bundle $P_1(>)$ over N_1 under the Hecke isomorphism $\mathcal{H}^{-1}$. Then using Corollary 7.2, Proposition 8.2 and Lemma 8.4, for any permutation $\sigma \in \Sigma_r$ we obtain

$$
\begin{aligned}
q_1(k; \sigma \cdot \lambda + \theta_1[k]) &\overset{\text{def}}{=} \chi\big(P_0(>), \mathcal{L}_0(k; \sigma \cdot \lambda + \theta_1[k]) \big) \\
&= \chi\left(P_1(>), \mathcal{L}_1\left(k; \tau^{-1} \cdot \sigma \cdot \lambda + \left(\frac{k}{r}, \ldots, \frac{k}{r} \right) - \rho \right) \right) \\
&= (-1)^\sigma \chi\left(P_1(>), \mathcal{L}_1\left(k; \tau^{-1} \cdot \lambda + \left(\frac{k}{r}, \ldots, \frac{k}{r} \right) - \rho \right) \right) \\
&= (-1)^\sigma \chi\big(P_0(>), \mathcal{L}_0(k; \lambda + \theta_1[k]) \big) \\
&\overset{\text{def}}{=} (-1)^\sigma q_1(k; \lambda + \theta_1[k]),
\end{aligned}
$$

where the second and fourth equalities hold by Corollary 7.2, and the third equality follows from Proposition 8.2 and Lemma 8.4. The proof for q_{-1} is similar. $\qquad\square$

The two group actions in Proposition 8.6 may be combined in the following manner. For $k \geq 0$, we define an action of the *affine Weyl group* $\Sigma \ltimes \Lambda$ on $\Lambda \times \mathbb{Z}$, which acts trivially on the second factor, the level, and the action at level k is given by setting

$$
\sigma.\lambda = \sigma \cdot (\lambda + \rho) - \rho \quad \text{and} \quad \gamma.\lambda = \lambda + (k+r)\gamma \quad \text{for } \sigma \in \Sigma, \gamma \in \Lambda.
$$

We denote the resulting group of affine-linear transformations of V^* by $\widetilde{\Sigma}[k]$, and note that the action is defined in such a way that

$$
(52) \qquad\qquad \sigma.\lambda + \rho = \sigma \cdot (\lambda + \rho) \quad \text{and} \quad (\gamma.\lambda + \rho)/\widehat{k} = \gamma + (\lambda + \rho)/\widehat{k}.
$$

It is easy to verify that the stabilizer subgroup

$$
\Sigma_r^+ \overset{\text{def}}{=} \text{Stab}(\theta_1[k], \widetilde{\Sigma}[k]) \subset \widetilde{\Sigma}[k]
$$

is generated by the transpositions $s_{i,i+1}$ for $1 \le i \le r-2$, and the reflection $\alpha^{r-1,r} \circ s_{r-1,r}$; similarly,

$$\Sigma_r^- \overset{\text{def}}{=} \text{Stab}(\theta_{-1}[k], \widetilde{\Sigma}[k]) \subset \widetilde{\Sigma}[k]$$

is generated by $s_{i,i+1}$ for $2 \le i \le r-1$, and the reflection $\alpha^{1,2} \circ s_{1,2}$.

Then Proposition 8.6 maybe recast in the following form: the polynomial $q_1(k;\lambda)$ is anti-invariant with respect to the copy Σ_r^+ of the symmetric group Σ_r, while $q_{-1}(k;\lambda)$ is anti-invariant with respect to the copy Σ_r^- of the symmetric group Σ_r.

The following statement is straightforward:

Lemma 8.7 *Both subgroups $\Sigma_r^\pm$ are isomorphic to Σ_r, and for $r > 2$, the two subgroups generate the affine Weyl group $\widetilde{\Sigma}[k]$.*

8.3 The Weyl antisymmetry of the polynomials p_1 and p_{-1}

Following (22), we define the two polynomials

$$p_{\pm 1}(k;\lambda) = \sum_{\boldsymbol{B} \in \mathscr{D}} \underset{\boldsymbol{B}}{\text{iBer}}[w_\Phi^{1-2g}(x/\widehat{k})](\widehat{\lambda}/\widehat{k} - [\theta_{\pm 1}]_{\boldsymbol{B}}),$$

where

$$\theta_1 = \frac{1}{r} \cdot (1,1,\ldots,1) - x_r \quad \text{and} \quad \theta_{-1} = -\frac{1}{r} \cdot (1,1,\ldots,1) + x_1.$$

Proposition 8.8 *The polynomial $p_1(k;\lambda)$ is anti-invariant with respect to Σ_r^+, and $p_{-1}(k;\lambda)$ is anti-invariant with respect to Σ_r^-.*

Proof The points $\theta_{\pm 1}[k]$ are the fixed points of the actions of $\Sigma^\pm$, and clearly $\lim_{k\to\infty} \theta_{\pm 1}[k]/k = \theta_{\pm 1}$. This means that we can fix a small open ball $D \subset V^*$ centered at θ_1 such that

(53) $\lambda/k \in D \implies (\sigma.\lambda + \rho)/\widehat{k} \sim \theta_1$ for all $\sigma \in \Sigma^+$.

Then for $\lambda/k \in D$ we have

$$p_1(k;\lambda) = \sum_{\boldsymbol{B} \in \mathscr{D}} \underset{\boldsymbol{B}}{\text{iBer}}[w_\Phi^{1-2g}(x/\widehat{k})](\{\widehat{\lambda}/\widehat{k}\}_{\boldsymbol{B}}).$$

Now let us consider a generator of Σ^+ of the type $\sigma = s_{i,i+1}$ for some $1 \le i \le r-2$. Using (52) and Lemma 4.5, and the fact that $\sigma \cdot w_\Phi = -w_\Phi$, we obtain

$$p_1(k;\sigma.\lambda) = \sum_{\boldsymbol{B} \in \mathscr{D}} \underset{\boldsymbol{B}}{\text{iBer}}[w_\Phi^{1-2g}(x/\widehat{k})](\sigma \cdot \{\widehat{\lambda}/\widehat{k}\}_{\boldsymbol{B}}) = \sum_{\boldsymbol{B} \in \mathscr{D}} \underset{\boldsymbol{B}}{\text{iBer}}[(-w_\phi)^{1-2g}(x/\widehat{k})](\{\widehat{\lambda}/\widehat{k}\}_{\boldsymbol{B}}) = -p_1(k;\lambda).$$

The case of the last generator $\alpha^{r-1,r} \circ s_{r-1,r}$ is similar, but after the substitution we need to use the equality $\{\alpha^{r-1,r} + \widehat{\lambda}/\widehat{k}\}_{\boldsymbol{B}} = \{\widehat{\lambda}/\widehat{k}\}_{\boldsymbol{B}}$ to obtain $p_1(k;\widehat{k}\alpha^{r-1,r} + s_{r-1,r}.\lambda) = -p_1(k;\lambda)$. □

8.4 Proof of Theorem 4.8(I)

Recall that in Lemma 4.1 we introduced a chamber structure on $\Delta \subset V^*$ created by the walls $S_{\Pi,l}$, where $\Pi = (\Pi', \Pi'')$ is a nontrivial partition, and $l \in \mathbb{Z}$. Before we proceed, we introduce some extra notation. Denote by

$$\check{\Delta} = \{(k; a) \mid a/k \in \Delta\} \subset \mathbb{R}^{>0} \times V^*$$

the cone over $\Delta \subset V^*$, and let

$$\check{\Delta}^{\mathrm{reg}} = \{(k; a) \mid a/k \in \Delta \text{ is regular}\} \subset \check{\Delta}$$

be the set of its regular points. Denote by $\check{S}_{\Pi,l} \subset \check{\Delta}$ the cone over the wall $S_{\Pi,l} \subset \Delta$; then $\check{\Delta}^{\mathrm{reg}}$ is the complement of the union of walls $\check{S}_{\Pi,l}$ in $\check{\Delta}$. Finally, denote by $\check{\Delta}^{\mathrm{reg}}_{\Lambda}$ the intersection of the lattice $\mathbb{Z}^{>0} \times \Lambda$ with $\check{\Delta}^{\mathrm{reg}}$.

By substituting $\varsigma = \lambda/k$, we can consider the left-hand side and the right-hand side of formula (I) of Theorem 4.8 as functions in $(k, \lambda) \in \check{\Delta}^{\mathrm{reg}}_{\Lambda}$. We denote by $q(k; \lambda)$ and $p(k; \lambda)$ the left-hand side and the right-hand side, correspondingly.

We showed that $q(k; \lambda)$ and $p(k; \lambda)$ are *polynomials* on the cone over each chamber in Δ; see Theorem 4.4, Section 2.4. We proved that the wall-crossing terms — ie the differences between polynomials on neighboring chambers — for $q(k; \lambda)$ (see Theorem 6.13) and for $p(k; \lambda)$ (see Proposition 4.18) coincide, hence there exists a polynomial $\Theta(k; \lambda)$ on $\mathbb{Z}^{>0} \times \Lambda$ such that the restriction of $\Theta(k; \lambda)$ to $\check{\Delta}^{\mathrm{reg}}_{\Lambda}$ is equal to the difference $p(k; \lambda) - q(k; \lambda)$.

Now for $r > 2$, we can conclude that

$$\Theta(k; \lambda) = p_1(k; \lambda) - q_1(k; \lambda) = p_{-1}(k; \lambda) - q_{-1}(k; \lambda),$$

where $p_{\pm 1}(k; \lambda)$ and $q_{\pm 1}(k; \lambda)$ are the restrictions of $p(k; \lambda)$ and $q(k; \lambda)$ to two specific chambers defined in Sections 8.3 and 8.2. Then, according to Propositions 8.6 and 8.8, the polynomial $\Theta(k; \lambda)$ is anti-invariant with respect to the action of the subgroups $\Sigma^{\pm}_r$, and hence by Lemma 8.7, it is anti-invariant under the action of the entire affine Weyl group $\widetilde{\Sigma}[k]$. It is easy to see that any such polynomial function has to vanish, and thus $p(k; \lambda) = q(k; \lambda)$, and this completes the proof of part (I) of Theorem 4.8 for the case when $\lambda/k \in \Delta$ is regular.

As in Corollary 4.10, we can extend $p(k; \lambda)$ from the interior of each chamber to its boundary by polynomiality. Clearly, to prove part (I) of Theorem 4.8 for the cases when λ/k is not regular, it is sufficient to show that these extensions from the chambers containing λ/k in their closure give the same value on $(k; \lambda)$. It follows from Remark 10.4 that this is the case, and this completes the proof of part (I) of Theorem 4.8; see Remark 4.9.

9 Rank 2, two points

Unfortunately, the argument above does not work for $r = 2$ because in this case, $\theta_1[k] = \theta_{-1}[k]$, the groups Σ_r^- and Σ_r^+ coincide, and thus they do not generate the entire affine Weyl group. The way out is to pass to the 2–punctured case.

9.1 Wall-crossing

We will thus fix two points $p, s \in C$, and study the moduli space of rank-2, stable parabolic bundles W with fixed determinant isomorphic to $\mathbb{O}(pd)$, with parabolic structure given by a line $F_1 \subset W_p$ with weight $(c, -c)$, and a line $G_1 \subset W_s$ with weight $(a, -a)$.

Now we need to repeat the analysis of our work so far in this somewhat simpler case; some details will thus be omitted.

Set $d = 0$; then the space of admissible weights (see Figure 6) is a square

$$\square = \{(c, a) \mid 1 > 2c > 0,\, 1 > 2a > 0\},$$

which has two adjacent chambers defined by the conditions

$$c > a \quad \text{and} \quad c < a.$$

Denote the corresponding moduli spaces by $P_0(c > a)$ and $P_0(c < a)$. Again, we have universal bundles over $P_0(c > a) \times C$ and $P_0(c < a) \times C$, which we will denote by the same symbol U; this bundle is endowed with two flags, $\mathscr{F}_1 \subset \mathscr{F}_2 = U_p$ and $\mathscr{G}_1 \subset \mathscr{G}_2 = U_s$. For $\mu, \lambda \in \mathbb{Z}$, we introduce the line bundle

$$\mathscr{L}(k; \lambda, \mu) = \det(U_p)^{k(1-g)} \otimes \det(\pi_*(U))^{-k} \otimes (\mathscr{F}_2/\mathscr{F}_1)^\lambda \otimes (\mathscr{F}_1)^{-\lambda} \otimes (\mathscr{G}_2/\mathscr{G}_1)^\mu \otimes (\mathscr{G}_1)^{-\mu}.$$

We repeat the construction of the master space from Section 5.2, choosing a point (c^0, c^0) on the wall and two points

$$(c, a)^\pm = (c^0, c^0) \pm \epsilon(1, 0) \in \square, \quad \text{where } \epsilon \in \mathbb{Q}_{>0},$$

from the adjacent chambers. We can identify the fixed-point set Z^0 as follows.
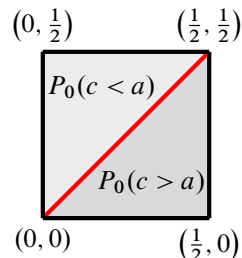
Figure 6: The space of admissible weights in the case of rank $r = 2$, two points.

Lemma 9.1 *The locus Z^0 defined in [Proposition 5.2](#) is*

$$Z^0 \simeq \mathrm{Jac}^0 \simeq \{V = L \oplus L^{-1} \mid L_p = F_1, L_s^{-1} = G_1\}.$$

As in [Section 6.1](#), denote by $\mathcal{J}$ the universal bundle over $\mathrm{Jac}^0 \times C$ normalized in such a way that $c_1(\mathcal{J})_{(0)} = 0$; see [(8)](#). Define

$$\eta \in H^2(\mathrm{Jac}) \quad \text{by} \quad \left(\sum_i c_1(\mathcal{J})_{(e_i)} \otimes e_i\right)^2 = -2\eta \otimes \omega.$$

We have then $\int_{\mathrm{Jac}} e^{\eta m} = m^g$ for $m \in \mathbb{Z}$.

Let $\pi \colon \mathrm{Jac}^0 \times C \to \mathrm{Jac}^0$ be the projection and N_{Z^0} the equivariant normal bundle to Z^0 in Z. Then, as in [Lemma 6.6](#), [Proposition 6.9](#) and [Lemma 6.5](#), we obtain the identifications

- $N_{Z^0} = R_T^1 \pi_*(\mathrm{ParHom}(\mathcal{J}, \mathcal{J}^{-1})) \oplus R_T^1 \pi_*(\mathrm{ParHom}(\mathcal{J}^{-1}, \mathcal{J}))$, where $T \simeq \mathbb{C}^*$–action has weights $(-1, 1)$,

- $E(N_{Z^0})^{-1} = (-1)^g \left(2\sinh\left(\tfrac{1}{2}u\right)\right)^{-2g} \exp(4\eta)$,

- $\mathrm{ch}_T(\mathcal{L}(k; \lambda, \mu)|_{Z^0}) = \exp(2k\eta) \exp(u(\lambda - \mu))$.

Now we define the polynomials

$$h_>(k; \lambda, \mu) \overset{\text{def}}{=} \chi(P_0(c > a), \mathcal{L}(k; \lambda, \mu)) \quad \text{and} \quad h_<(k; \lambda, \mu) \overset{\text{def}}{=} \chi(P_0(c < a), \mathcal{L}(k; \lambda, \mu)).$$

Applying [Theorem 5.7](#), we obtain the following expression for their difference.

Lemma 9.2 *The wall-crossing term equals*

$$h_>(k; \lambda, \mu) - h_<(k; \lambda, \mu) = (-1)^g (2k + 4)^g \operatorname*{Res}_{u=0} \frac{\exp(u(\lambda - \mu))}{\left(2\sinh\left(\tfrac{1}{2}u\right)\right)^{2g}} \, du.$$

9.2 Symmetry

Denote by $P_{-1}(c > a)$ the image of the moduli space $P_0(c > a)$ under the Hecke isomorphism $\mathcal{H}$ (see [Section 7](#)) at the point p, and by $P_{-1}(c < a)$ the image of the moduli space $P_0(c < a)$ under the Hecke isomorphism $\mathcal{H}$ at the point s.

We have the following analogue of [Lemma 8.3](#).

Lemma 9.3 *Denote by N_{-1} the moduli space of rank-2 degree-(-1) stable bundles on C, and by UN any universal bundle over $N_{-1} \times C$. Then the moduli spaces $P_{-1}(c > a)$ and $P_{-1}(c < a)$ are isomorphic to the bundle $\mathbb{P}(UN_p) \times \mathbb{P}(UN_s)$ over N_{-1}.*

Denote by $\mathcal{T}[p]$ and $\mathcal{T}[s]$ the vertical tangent lines of $\mathbb{P}(UN_p)$ and $\mathbb{P}(UN_s)$, respectively, and by $\mathcal{L}_{-1}$ the pullback of the ample generator of the Picard group of N_{-1} to $\mathbb{P}(UN_p) \times \mathbb{P}(UN_s)$; see [Lemma 8.4](#). Then a simple calculation shows the following.
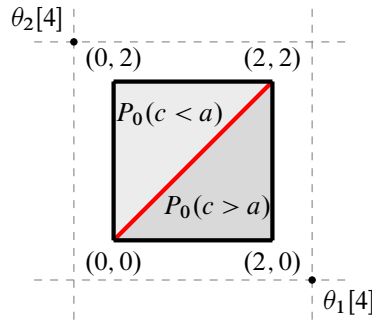
Figure 7: The space of admissible weights in the case of $k = 4$, $r = 2$, two points.

Lemma 9.4 *Under the Hecke isomorphism $\mathcal{H}$ at p, the line bundle $\mathcal{L}(2k; \lambda, \mu)$ on $P_0(c > a)$ corresponds to the line bundle $\mathcal{L}^k_{-1} \otimes \mathcal{T}[p]^{-\lambda+k} \otimes \mathcal{T}[s]^\mu$ on $P_{-1}(c > a)$.*

Under the Hecke isomorphism $\mathcal{H}$ at the point s, $\mathcal{L}(2k; \lambda, \mu)$ on $P_0(c < a)$ corresponds to the line bundle $\mathcal{L}^k_{-1} \otimes \mathcal{T}[p]^\lambda \otimes \mathcal{T}[s]^{-\mu+k}$ on $P_{-1}(c < a)$.

As in Section 8.2, applying Serre duality for families of curves (see Proposition 8.2) to the line bundles on the two $\mathbb{P}^1 \times \mathbb{P}^1$ bundles over N_{-1}, we obtain that the polynomials $h_>(k; \lambda, \mu)$ and $h_<(k; \lambda, \mu)$ are anti-invariant under the action of the Weyl group $\Sigma_2 \times \Sigma_2$ with the center at $\theta_1[k] = \left(\frac{1}{2}(k+1), -\frac{1}{2}\right)$ and $\theta_2[k] = \left(-\frac{1}{2}, \frac{1}{2}(k+1)\right)$, correspondingly; see Figure 7. In other words, we obtain the following four identities.

Lemma 9.5
$$h_>(k; \lambda, \mu) = -h_>(k; \lambda, -\mu - 1) = -h_>(k; -\lambda + k + 1, \mu),$$
$$h_<(k; \lambda, \mu) = -h_<(k; -\lambda - 1, \mu) = -h_<(k; \lambda, -\mu + k + 1).$$

Now define the polynomials
$$\widetilde{h}_>(k; \lambda, \mu) = (-1)^{g-1}(2k+4)^g \operatorname*{Res}_{u=0} \frac{\exp(u(\lambda + \mu + 1)) - \exp(u(\lambda - \mu))}{\left(2\sinh\left(\frac{1}{2}u\right)\right)^{2g}(1 - e^{u(k+2)})} \, du,$$

$$\widetilde{h}_<(k; \lambda, \mu) = (-1)^{g-1}(2k+4)^g \operatorname*{Res}_{u=0} \frac{\exp(u(\lambda + \mu + 1)) - \exp(u(\lambda - \mu + k + 2))}{\left(2\sinh\left(\frac{1}{2}u\right)\right)^{2g}(1 - e^{u(k+2)})} \, du,$$

and from here we can follow the logic of the proof of part (I) of Theorem 4.8.

Proposition 9.6 *The polynomials introduced above, in fact, coincide:*
$$h_>(k; \lambda, \mu) = \widetilde{h}_>(k; \lambda, \mu) \quad \text{and} \quad h_<(k; \lambda, \mu) = \widetilde{h}_<(k; \lambda, \mu).$$

Proof It is a simple exercise to show that $\widetilde{h}_>(k; \lambda, \mu)$ and $\widetilde{h}_<(k; \lambda, \mu)$ satisfy the identities appearing in Lemmas 9.2 and 9.5, and hence the polynomial
$$\Theta(k; \lambda, \mu) = h_>(k; \lambda, \mu) - \widetilde{h}_>(k; \lambda, \mu) = h_<(k; \lambda, \mu) - \widetilde{h}_<(k; \lambda, \mu)$$

satisfies all four Σ_2–symmetries listed in Lemma 9.5. These groups together generate a double action of the affine Weyl group $\widetilde{\Sigma}$ in λ and μ separately, and this implies the vanishing of Θ. □

As $P_0(c > a)$ is a $\mathbb{P}^1$–bundle over the moduli space of rank-2 degree-0 stable parabolic bundles $P_0(c, -c)$, substituting $\mu = 0$ in $\widetilde{h}_>$, we obtain the Verlinde formula for rank 2.

Corollary 9.7 $\chi(P_0(c, -c), \mathscr{L}_0(k; \lambda)) = (-1)^{g-1}(2k+4)^g \operatorname{Res}_{u=0} \dfrac{\exp\left(u\left(\lambda+\frac{1}{2}\right)\right)}{\left(2\sinh\left(\frac{1}{2}u\right)\right)^{2g-1}(1-e^{u(k+2)})} \, du.$

10 The combinatorics of the $[Q, R] = 0$

In this section, we give a proof of the second part of Theorem 4.8. Let $\lambda/k \in \Delta$, and fix a regular element $\varsigma \in \Delta$ in a chamber containing λ/k in its closure, and another regular element $\widehat{\varsigma} \in \Delta$ containing $\widehat{\lambda}/\widehat{k}$ in its closure. Our goal is to prove the equality $p_\varsigma(k; \lambda) = p_{\widehat{\varsigma}}(k; \lambda)$, where we define

$$(54) \qquad p_c(k; \lambda) = \widetilde{N}_{r,k} \sum_{\boldsymbol{B} \in \mathscr{D}} \operatorname{iBer}_{\boldsymbol{B}}[w_\Phi^{1-2g}(x/\widehat{k})](\widehat{\lambda}/\widehat{k} - [c]_{\boldsymbol{B}})$$

for a regular $c \in \Delta$ and diagonal basis $\mathscr{D}$. This is a subtle statement, which is a combinatorial-geometric projection of the idea of quantization commutes with reduction (or $[Q, R] = 0$ for short; see [17; 25]).

If $\lambda/k \sim \widehat{\lambda}/\widehat{k}$, ie when λ/k and $\widehat{\lambda}/\widehat{k}$ are regular elements in the same chamber in Δ, then $p_\varsigma(k; \lambda) = p_{\widehat{\varsigma}}(k; \lambda)$ is a tautology. We assume thus that this is not the case, and denote by $\mathscr{S}(k, \lambda)$ the set of walls separating ς and $\widehat{\varsigma}$, or containing either λ/k or $\widehat{\lambda}/\widehat{k}$ or both. Equivalently, the wall $S_{\Pi,l}$ belongs to $\mathscr{S}(k, \lambda)$ if

$$(\lambda/k)_{\Pi'} \geq l \geq (\widehat{\lambda}/\widehat{k})_{\Pi'} \quad \text{or} \quad (\lambda/k)_{\Pi'} \leq l \leq (\widehat{\lambda}/\widehat{k})_{\Pi'},$$

where $c_{\Pi'} = \sum_{i \in \Pi'} c_i$ for an element $c = (c_1, \ldots, c_r) \in V^*$. Clearly, there is a path in Δ connecting ς and $\widehat{\varsigma}$, which intersects only walls from $\mathscr{S}(k, \lambda)$ in a generic points. Then to prove the equality $p_\varsigma(k; \lambda) = p_{\widehat{\varsigma}}(k; \lambda)$, it is enough to show the following, at first sight somewhat surprising, fact.

Proposition 10.1 *Assume $g \geq 1$, $\lambda/k \in \Delta$, $S_{\Pi,l} \in \mathscr{S}(k, \lambda)$ and let $c^\pm \in \Delta$ be two points in two neighboring chambers separated by the wall $S_{\Pi,l}$. Then*

$$(55) \qquad p_{c^+}(k; \lambda) = p_{c^-}(k; \lambda).$$

Proof The difference of the two sides of (55) is expressed as a residue in (34). The integral in (34) is a rational expression in the variable t, and our plan is to show by degree count in t and t^{-1} that its residues at zero and at ∞ vanish. We define the degree of the quotient of two polynomials $R = P/Q$ of the variable t as $\deg_t(R) = \deg_t(P) - \deg_t(Q)$, and we set $\deg_{t^{-1}}(R) = \deg_t(R(t^{-1}))$. Then, clearly,

$$\deg_t(R) < 0 \implies \operatorname{Res}_{t=\infty} R \frac{dt}{t} = 0 \quad \text{and} \quad \deg_{t^{-1}}(R) < 0 \implies \operatorname{Res}_{t=0} R \frac{dt}{t} = 0.$$

A convenient expression for (34) will be (46), where we change variables via $t = e^u$. In what follows, we will always tacitly assume this substitution, and we will write, for example, $\deg_{t \pm 1}(1/(e^u - e^{-u})) = -1$. We thus obtain a formula of the form $\mathrm{Res}_{t=0,\infty} f(t)\, dt/t$, and to show that this is zero, it is sufficient to show that $\deg_t(f) < 0$ and $\deg_{t^{-1}}(f) < 0$.

Now we observe that the variable u occurs only in the first line of (46), and thus, calculating the degrees in t and t^{-1} separately, we obtain the formula

$$(56) \qquad \deg_{t \pm 1}(f) = \pm(k\delta - rl) + (1 - 2g)\deg_{t \pm 1}(\tau \cdot w_u^\times) + \deg_{t \pm 1}(\exp(\tau \cdot \rho_u^\times)).$$

Recall that here, δ represents the distance of λ/k from the wall $S_{\Pi,l}$, while $w_u^\times$ and $\rho_u^\times$, represent the parts of the Weyl denominator and the ρ–shift corresponding to roots connecting Π' and Π'', respectively.

We begin the study of this expression with some simple remarks. We recall that the permutation τ preserves the partition $\Pi = (\Pi', \Pi'')$, and thus we have

$$\deg_{t \pm 1}(\tau \cdot w_u^\times) = \deg_{t \pm 1}(w_u^\times) = \tfrac{1}{2} r' r''.$$

Using, in addition, that $\rho_u^\times$ is linear in u, we obtain

$$\deg_t(\exp(\tau \cdot \rho_u^\times)) = -\deg_{t^{-1}}(\exp(\tau \cdot \rho_u^\times)) = \deg_t(\exp(\rho_u^\times)).$$

Combining these equalities, and assuming $g \geq 1$, we arrive at the following conclusion.

Lemma 10.2 *The inequality*

$$(57) \qquad |(k\delta - rl) + \deg_t(\exp(\rho_u^\times))| < \tfrac{1}{2} r' r''$$

implies the vanishing of the wall-crossing term: *equality* (55).

Before we proceed, we introduce some notation. Denote by

$$\mathrm{Inv}(\Pi) = \{(i, j) \mid \Pi' \ni i > j \in \Pi''\}$$

the set of "inverted" pairs of elements of the partition Π. The number of these pairs $|\mathrm{Inv}(\Pi)|$ coincides with the standard notion of length of the shuffle permutation $\phi \in \Sigma_r$ introduced in Section 6.

Each pair (i, j) which is not inverted contributes $+\tfrac{1}{2} u$ to $\rho_u^\times$, while each inverted pair contributes $-\tfrac{1}{2} u$, and thus we have

$$(58) \qquad \deg_t(\exp(\rho_u^\times)) = \tfrac{1}{2} r' r'' - |\mathrm{Inv}(\Pi)|.$$

Also, recall the notation $c_{\Pi'} = \sum_{i \in \Pi'} c_i$ for an element $c = (c_1, \dots, c_r) \in V^*$; in particular, we have $(\lambda/k)_{\Pi'} = l + \delta$ and

$$\rho_{\Pi'} = \sum_{i \in \Pi'} \tfrac{1}{2}(r + 1) - i.$$

The following is a simple exercise, whose proof will be omitted:

$$(59) \qquad \deg_t(\exp(\rho_u^\times)) = \rho_{\Pi'}.$$

Now we come to a key point of our argument.

Lemma 10.3 *If the intersection of the wall $S_{\Pi,l}$ with Δ is nonempty, then*

$$(60) \qquad -\tfrac{1}{2}r'r'' < lr - \rho_{\Pi'} < \tfrac{1}{2}r'r''.$$

Proof Pick a point $c = (c_1, \ldots, c_r)$ in the intersection $S_{\Pi,l} \cap \Delta$, and recall that for any $1 \leq i < j \leq r$, we have $0 < c_i - c_j < 1$, and

$$\sum_{i \in \Pi'} c_i = -\sum_{i \in \Pi''} c_i = l.$$

Then

$$-|\mathrm{Inv}(\Pi)| < \sum_{(i,j) \in \mathrm{Inv}(\Pi)} (c_i - c_j) \leq \sum_{i \in \Pi'} \sum_{j \in \Pi''} (c_i - c_j),$$

and, similarly,

$$\sum_{i \in \Pi'} \sum_{j \in \Pi''} (c_i - c_j) < r'r'' - |\mathrm{Inv}(\Pi)|.$$

Now, since

$$\sum_{i \in \Pi'} \sum_{j \in \Pi''} (c_i - c_j) = r'' \sum_{i \in \Pi'} c_i - r' \sum_{j \in \Pi''} c_j = lr,$$

we can conclude

$$-|\mathrm{Inv}(\Pi)| < lr < r'r'' - |\mathrm{Inv}(\Pi)|.$$

In view of (58) and (59), these inequalities are equivalent to (60), and this completes the proof. $\qquad\square$

Now we are ready to prove (55). The condition $S_{\Pi,l} \in \mathcal{S}(k, \lambda)$, ie that $S_{\Pi,l}$ separates λ/k and $\hat{\lambda}/\hat{k}$ or contains λ/k or $\hat{\lambda}/\hat{k}$, may occur in two ways.

- $(\lambda/k)_{\Pi'} \geq l \geq (\hat{\lambda}/\hat{k})_{\Pi'}$, which is equivalent to the two inequalities $\delta \geq 0$ and $lk + lr \geq \lambda_{\Pi'} + \rho_{\Pi'}$. After canceling lk and reordering the terms, we can rewrite these as

$$(61) \qquad 0 \geq k\delta - lr + \rho_{\Pi'} \geq \rho_{\Pi'} - lr.$$

  Using Lemma 10.3 then we can conclude that

$$0 \geq k\delta - lr + \rho_{\Pi'} > -\tfrac{1}{2}r'r'',$$

  which, in view of the equality (59), implies the necessary estimate (57).

- The second case is similar: $(\lambda/k)_{\Pi'} \leq l \leq (\hat{\lambda}/\hat{k})_{\Pi'}$ is equivalent to $\delta \leq 0$ and $lk + lr \leq \lambda_{\Pi'} + \rho_{\Pi'}$. This leads to

$$(62) \qquad 0 \leq k\delta - lr + \rho_{\Pi'} \leq \rho_{\Pi'} - lr,$$

  which, in turn, implies

$$0 \leq k\delta - lr + \rho_{\Pi'} < \tfrac{1}{2}r'r'',$$

  and hence (57).

This completes the proof of Proposition 10.1: indeed, a simple calculation shows that if $\lambda/k \in \Delta$ then $\widehat{\lambda}/\widehat{k} \in \Delta$, so the conditions of Lemma 10.3 hold. We have just shown that this implies (57), and according to Lemma 10.2, we can conclude the vanishing of the wall-crossing term (55). $\qquad\square$

Remark 10.4 If $\lambda/k \in \Delta$ is nonregular, then it belongs to some wall from the set $\mathscr{S}(k,\lambda)$. Hence Proposition 10.1 implies that the right-hand side of formula (I) of Theorem 4.8 is a well-defined function on the cone over Δ:

$$\{(k,\lambda) \in \mathbb{Z}^{>0} \times \Lambda \mid \lambda/k \in \Delta\}.$$

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