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We prove that (topologically) rigid actions of surface groups on the circle by homeomorphisms are necessarily *geometric*, namely, they are semiconjugate to an embedding as a cocompact lattice in a Lie group acting transitively on S^1. This gives the converse to a theorem of the first author; thus characterizing geometric actions as the unique isolated points in the "character space" of surface group actions on S^1.

20H10, 37E10, 37E45, 57S25, 58D29

1 Introduction

Classification results in dynamical systems are often motivated by the study of special examples. Having found a system with interesting (eg stable) behavior, one seeks first to understand its properties and related examples, and then to address the broader problem of classifying all systems with such properties. As a prime example, Anosov observed that hyperbolic linear automorphisms of tori exhibit stability under perturbation, leading to the abstract definition of *Anosov diffeomorphisms*. Smale [32] observed that hyperbolic affine automorphisms of infra-nil manifolds give additional such examples; that this is an exhaustive list of all Anosov diffeomorphisms of closed manifolds up to topological conjugacy is a longstanding open conjecture.

The present work addresses the classification problem for globally rigid actions of surface groups on the circle; equivalently, for rigid, flat topological circle bundles over surfaces. Here, local rigidity, at least in the C^1 setting, already follows from the work of Anosov. A much stronger, global, C^0 rigidity phenomenon was discovered by Matsumoto [28], who proved that all representations $\pi_1\Sigma_g \to \mathrm{Homeo}^+(S^1)$ of equal, extremal Euler class are semiconjugate, in the sense of semiconjugacy for circle actions defined by Ghys [12]. These globally rigid examples are all *geometric* in the sense that they arise from embedding $\pi_1\Sigma_g$ as a cocompact lattice in a Lie subgroup of $\mathrm{Homeo}^+(S^1)$. Matsumoto's result was extended by the first author [23], who showed that, in fact *all* geometric actions of surface groups have this same global rigidity: they are characterized, up to semiconjugacy, by a finite list of rotation numbers which are constant in a neighborhood of each geometric representation. As a consequence, they descend to isolated points in the quotient of the representation space by semiconjugacy. This strong property is the definition of *rigidity* we will use throughout this article; see Section 1.2 for further discussion.

Here we solve the associated classification problem, giving a complete characterization of rigid actions of surface groups on the circle.

Theorem 1.1 *Let Σ_g be a surface of genus $g \geqslant 2$. Then every rigid representation $\pi_1\Sigma_g$ to* $\mathrm{Homeo}^+(S^1)$ *is geometric: up to semiconjugacy it is obtained by embedding $\pi_1\Sigma_g$ as a lattice in a transitive Lie group in* $\mathrm{Homeo}^+(S^1)$.

The geometric representations referenced in the theorem are easily classified; the Lie groups in question are simply the finite cyclic extensions of $\mathrm{PSL}_2(\mathbb{R})$.

The arc of our proof resembles in spirit the *convergence group theorem* of Tukia [35], Gabai [10] and Casson and Jungries [7]. Both in our case and theirs, one starts with purely dynamical information (in the convergence group case, information on the dynamics of sequences of elements; in ours merely the assumption of rigidity) and from that reconstructs the geometric–topological data of a subgroup of $\mathrm{PSL}_2(\mathbb{R})$ or one of its covers. The key in our case is to show that, under an arbitrary rigid action, elements of $\pi_1\Sigma_g$ which can be represented by nonseparating simple closed curves have the same dynamics as the geometric examples. From there, we again use rigidity to "reconstruct" the topology of the surface, recovering the intersection pattern of these curves on Σ_g.

We note also that, while the statement of Theorem 1.1 resembles Sullivan's "structural stability implies hyperbolicity" for Kleinian groups [33], our methods and conclusion are quite different: for Sullivan, structural stability is a local and C^1 condition, and the groups in consideration are convex-cocompact, acting on their limit set satisfying a hyperbolicity or local hyperbolicity condition.

1.1 Motivation

Our motivation comes from the highly influential work of Milnor, Wood and Goldman. Milnor's contribution to the *Milnor–Wood inequality* is the statement that a principal $\mathrm{PSL}_2(\mathbb{R})$ bundle over a surface admits a flat connection if and only if the *Euler number* of the bundle is bounded in absolute value by the Euler characteristic of the surface. Following this, the natural next question is to what extent the Euler number distinguishes flat bundles. This was answered by Goldman [15], who showed that it is a complete invariant of flat $\mathrm{PSL}_2(\mathbb{R})$ bundles *up to deformation*: the connected components of $\mathrm{Hom}(\pi_1\Sigma_g, \mathrm{PSL}_2(\mathbb{R}))$ are classified by the Euler numbers of the associated bundles.

Here we are interested in these same basic questions in the topological rather than the linear category. Wood [36] showed that Milnor's bound holds in the topological setting as well, demonstrating that topological S^1 bundles over Σ_g which admit a flat connection (or in this case a foliation transverse to the fibers) are precisely those whose Euler numbers are bounded by $\pm(2g-2)$. However, work of the first author [23] showed that Goldman's theorem is no longer true in this setting: there are many connected components of $\mathrm{Hom}(\pi_1\Sigma_g, \mathrm{Homeo}^+(S^1))$ consisting of bundles with the same Euler number.

In fact, the topology of the *space* of flat circle bundles, which can be thought of either as the representation space $\mathrm{Hom}(\pi_1\Sigma_g, \mathrm{Homeo}^+(S^1))$ or the associated *character space* described below, remains very mysterious. For instance, it is an open question whether either space has finitely many or infinitely many connected components. Theorem 1.1 gives the first step towards a global picture, giving a complete classification of the *isolated points* of the character space, and our hope is that the tools we develop should be useful towards the broader program.

1.2 Character spaces and rigidity

As in Goldman's work, the appropriate framing for our work is the study of character spaces. Typically these are defined algebraically, but they generalize naturally to the broad context of groups acting on manifolds.

Let Γ be any discrete group and let G be a topological group such that $G \subset \mathrm{Homeo}(X)$ for some space X. The *representation space* $\mathrm{Hom}(\Gamma, G)$, equipped with the compact-open topology, parametrizes actions of Γ on X with image in G. Typically, G is used to specify the regularity of the action — for instance, taking $G = \mathrm{Diff}(X)$ parametrizes smooth actions, while if G is a Lie group acting transitively on M these are *geometric* actions in the sense of Ehresmann. Since conjugate actions are dynamically equivalent, the appropriate moduli space of actions is the quotient $\mathrm{Hom}(\Gamma, G)/G$ under the natural conjugation action of G. However, this quotient space is typically non-Hausdorff and so in practice difficult to study.

When G is a Lie group and $\mathrm{Hom}(\Gamma, G)$ is an affine variety, algebraic geometers solve this problem by considering the quotient $\mathrm{Hom}(\Gamma, G) /\!/ G$ from geometric invariant theory. In the special case where G is a semisimple complex reductive Lie group, this GIT quotient is simply the quotient of $\mathrm{Hom}(\Gamma, G)$ by the equivalence relation $\rho_1 \sim \rho_2$ whenever *the closures* of their conjugacy classes intersect (see Luna [21; 22]); in particular, this relation makes the quotient space Hausdorff. In the well-studied case of $G = \mathrm{SL}(n, \mathbb{C})$, the GIT quotient agrees with the space of *characters* of G–representations, motivating the following terminology.

Definition 1.2 For any discrete group Γ and topological group G, the *character space* $X(\Gamma, G)$ is the largest Hausdorff quotient[1] of $\mathrm{Hom}(\Gamma, G)/G$. Two representations are *weakly conjugate* if they define the same point in $X(\Gamma, G)$.

Loosely speaking, a representation $\rho \colon \Gamma \to G$ is rigid if all deformations of $\rho(\Gamma)$ in G are trivial. This notion can be made precise in the setting of character spaces as follows.

Definition 1.3 A representation $\rho \in \mathrm{Hom}(\Gamma, G)$ is *rigid* if the image of ρ is an isolated point in the character space $X(\Gamma, G)$.

[1] Recall that the largest Hausdorff quotient X_H of a topological space X is a space with the universal property that any continuous map $f \colon X \to Y$ from X to a Hausdorff topological space factors canonically through the projection $X \to X_H$. One construction of X_H is as the quotient of X by the intersection of all equivalence relations $\sim$ such that $X/\sim$ is Hausdorff.

This is a strong condition on ρ, and less strict forms of rigidity will also be useful. In particular, we say that ρ is *path-rigid* if the path component of ρ in $\mathrm{Hom}(\Gamma, G)$ is contained in a single weak-conjugacy class.

The case of interest in this article is when $G = \mathrm{Homeo}^+(S^1)$, the group of orientation-preserving homeomorphisms of the circle, and $\Gamma = \Gamma_g = \pi_1(\Sigma_g)$ is the fundamental group of an orientable surface of genus $g \geq 2$. As we explain in Section 2.3, in this setting the character space $X(\Gamma, G)$ agrees with the space of *semiconjugacy* classes of actions in the sense of Ghys [12]. In this and related work, semiconjugacy is used to refer to an equivalence relation for group actions on the circle. However, semiconjugacy has a precise and different meaning in topological dynamics. For this reason, we will use the term "weak conjugacy" when referring to the character space $X(\Gamma, G)$, even though this terminology is not yet well established in the literature, and use the term semiconjugacy only when referencing classical results following [12].

1.3 Geometric representations

It is our philosophy that dynamical rigidity often comes from some underlying geometric or algebraic structure. This motivates the following definition.

Definition 1.4 (Mann [24]) Suppose that M is a manifold, and Γ a countable group. A representation $\rho\colon \Gamma \to \mathrm{Homeo}(M)$ is called *geometric* if it is weakly conjugate to a faithful representation with image a cocompact[2] lattice in a transitive, connected Lie group $G \subset \mathrm{Homeo}(M)$.

Indeed, the first known example of a rigid action of a surface group on the circle was a geometric one, due to Matsumoto [28]. Matsumoto's result is that the set of representations with maximal Euler number (equal to $2g-2$ by Milnor–Wood) in $X(\Gamma_g, G)$ consists of a single point — all are weakly conjugate to discrete, faithful representations into $\mathrm{PSL}_2(\mathbb{R}) \subset \mathrm{Homeo}^+(S^1)$. As the Euler number is a continuous function on $\mathrm{Hom}(\Gamma_g, G)$, this implies that representations of maximal Euler number are rigid. The same holds for representations with Euler number $-2g+2$.

While Matsumoto's proof uses maximality of the Euler number in an essential way — a theme that has been taken up in the study of "maximal representations" of surface groups in higher Teichmüller theory, see eg Burger, Iozzi and Wienhard [5] — the idea hints at a separate underlying phenomenon for rigidity, namely *geometricity*.

As hinted above, geometric representations of surface groups in $\mathrm{Homeo}^+(S^1)$ (up to weak conjugacy) all are either discrete, faithful representations into $\mathrm{PSL}_2(\mathbb{R})$, or obtained by lifting such a representation to a finite cyclic extension of $\mathrm{PSL}_2(\mathbb{R})$ (see [24]) and the main result of [23] is their rigidity.

[2] Our choice to require that Γ be cocompact here is motivated by the definition of *model geometries* in the sense of Thurston, where one is interested in compact quotients. It also simplifies the statement of rigidity theorems in low dimensions: noncocompact lattices in $\mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{PSL}_2(\mathbb{C})$ are not rigid, even in the space of representations into $\mathrm{PSL}_2(\mathbb{R})$ or $\mathrm{PSL}_2(\mathbb{C})$.

Theorem 1.5 (Mann [23]) *In the space* $\mathrm{Hom}(\Gamma_g, \mathrm{Homeo}^+(S^1))$, *all geometric representations are rigid.*

Though actually stated there in a slightly weaker form, the proof is carried out on the level of semiconjugacy (or weak-conjugacy, we show in Section 2 these notions coincide) invariants of representations, so actually shows that geometric representations are isolated points in $X(\Gamma_g, \mathrm{Homeo}^+(S^1))$.

1.4 Strategy of proof and outline of the article

The entirety of this work is devoted to the proof of Theorem 1.1, ie the converse of Theorem 1.5. Our main technical result is the following statement, which gives a stronger result for representations of nonzero Euler class.

Theorem 1.6 *Let* $\rho\colon \Gamma_g \to \mathrm{Homeo}^+(S^1)$ *be a path-rigid representation. If* ρ *is not geometric, then its Euler class is zero, and there exists a one-holed, genus* $g-1$ *subsurface* $\Sigma' \subset \Sigma_g$ *such that* $\rho|_{\pi_1\Sigma'}$ *has a finite orbit.*

The surface group representations with Euler class zero are precisely those which can be lifted to actions on the line. It is not entirely surprising that our theorem identifies these as a special case, as more complicated dynamical phenomena sometimes occur for such representations. Notably, Ghys [11] shows that an action of a surface group on S^1 by *real analytic* diffeomorphisms admits a minimal exceptional set only if it has Euler class zero. However, the condition of having a large subsurface with a finite orbit makes it very likely that such a representation could be deformed along a path; giving strong evidence for the fact that all path-rigid representations should in fact be geometric.

The main ingredient in the proof of Theorem 1.6 is the effect of *bending deformations* on the periodic sets of simple closed curves. Bending deformations are classical in (higher) Teichmüller theory (see Section 2.2.2 for a reminder); and we extend their study to representations to $\mathrm{Homeo}^+(S^1)$. While the proof of Theorem 1.6 is quite long, a much simpler argument can be carried out under the additional significant assumption that the relative Euler number on some genus 1 subsurface is equal to 1 (this is the case in particular for representations of Euler class $\geq g$). This much weaker proof is presented in our expository article [25]; the reader may find it helpful to take that work as a starting point or a companion.

We now outline the major steps.

Step 1 (local-to-global) Our proof starts by making a strong additional technical hypothesis on representations that forces them to look "locally" (ie on the level of some pairs of elements) like representations into $\mathrm{PSL}_2^k(\mathbb{R})$. Specifically, we say that the action of two elements $a, b \in \Gamma_g$ representing standard generators of a one-holed torus subsurface of Σ_g *satisfies* $S_k(a,b)$ if $\rho(a)$ and $\rho(b)$ are separately conjugate to hyperbolic elements of $\mathrm{PSL}_2^k(\mathbb{R})$, and their periodic points alternate around the circle. We show the following.

Theorem 1.7 *Let $\rho\colon \Gamma_g \to \mathrm{Homeo}^+(S^1)$ be a path-rigid, minimal representation, and suppose furthermore that there exists $k \geq 1$ such that $S_k(a,b)$ holds for all standard generators of one-holed torus subsurfaces. Then ρ is geometric.*

The proof of Theorem 1.7 uses bending deformations of ρ to move the periodic points of generators of $\pi_1\Sigma_g$; provided ρ is path-rigid, we are able to conclude the periodic points of many simple closed curves are in the same cyclic order as if ρ were geometric. In the toy version we presented in [25] — whose additional hypothesis guarantees that $k = 1$ — this same process was sufficient to demonstrate that ρ has maximal Euler number, hence is geometric. Here in the general case, we need to use a more sophisticated tool, and invoke Matsumoto's theory of *basic partitions*; see Section 3.4.

Step 2 (good and bad tori) We next make extensive use of bending deformations to prove the following result on periodic sets and rotation numbers.

Proposition 1.8 *If a representation $\Gamma_g \to G$ is path-rigid, then all nonseparating simple closed curves have rational rotation number.*

Theorem 1.9 *Suppose ρ is path-rigid and minimal. Then, for all standard generators a, b of one-holed subsurfaces, we have the implication*

$$\mathrm{Per}(\rho(a)) \cap \mathrm{Per}(\rho(b)) = \varnothing \implies S_k(a,b) \text{ for some } k.$$

The upshot of these results is that, if a path-rigid and minimal representation *fails* to be geometric, then many curves are forced to have common periodic points. Common periodic points hint at the existence of a finite orbit for ρ, so we next look for a finite orbit in order to derive a contradiction (indeed, representations with a finite orbit are easily seen to be non-path-rigid). This idea proves difficult to implement, so we search first for curves with rotation number zero, as the dynamics of these are easier to control. This search can be performed separately in every one-holed torus in the surface, where the action of the mapping class group is simple to work with. Accordingly, a one-holed torus in Σ_g is called a *good torus* if it contains a nonseparating simple loop with rotation number zero; otherwise we say it is a *bad torus*. A one-holed torus is called *very good* if its fundamental group has a finite orbit in S^1. We prove:

Proposition 1.10 *Let ρ be path-rigid. Suppose that Σ_g contains a bad torus Σ'. Then its complement $\Sigma_g \setminus \Sigma'$ contains only very good tori.*

Proposition 1.11 *Let ρ be path-rigid, and nongeometric. Then there cannot exist two disjoint good tori that are not very good.*

Theorem 1.12 *Let ρ be a path-rigid representation. Let $\Sigma_{g',1}$ be a subsurface in which all tori are very good. Then $\pi_1\Sigma_{g',1}$ has a finite orbit.*

These three last statements show that if ρ is a path-rigid and nongeometric representation, then it has a subsurface of genus $g-1$ with a finite orbit; the statement about the Euler class in Theorem 1.6 then follows easily.

Conclusion Provided $g \geq 3$, Theorem 1.12 implies that if ρ is path-rigid and nongeometric, then there exist curves a, b, generating a torus subsurface of Σ_g, such that $\rho(a)$ and $\rho(b)$ have a common fixed point. It then follows from a recent theorem of Alonso, Brum and Rivas [1] that ρ cannot be rigid. However, path-rigidity and the genus $g = 2$ case do not follow, so we prove an independent, elementary lemma on rigid representations that shows all torus subsurfaces have only finitely many finite orbits. This applies to all genera, and allows us conclude the proof of Theorem 1.1.

Roadmap The article is organized as follows. Section 2 introduces tools and frameworks that will be frequently used in the proof. We review background and prove new results on *complexes of based curves*; then prove a series of results on the movement of periodic sets under specific bending deformations; and finally discuss character spaces, semiconjugacy, and the Euler class. In Section 3 we prove Theorem 1.7. In Section 4 we prove Proposition 1.8 and Theorem 1.9. The proof of Theorem 1.6 is then completed in Section 5. Finally, in Section 6 we complete the proof of Theorem 1.1 and state some open questions and directions for further work.

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2 Preliminaries

2.1 Based curves on surfaces

This subsection should seem familiar to low-dimensional topologists, except that we will give much more attention to *based* curves than is usually present in the literature. As in the introduction, we use the notation $\Gamma_g = \pi_1 \Sigma_g$. While this notation omits mention of a basepoint, all elements of Γ_g are always assumed based. This is crucial — for example, we recall (as a warning) that the set of elements represented by *based* simple closed loops, in Γ_g, is not closed under conjugation. We now set some conventions.

Since we are interested in actions of Γ_g by homeomorphisms on the circle, we will write words in Γ_g (ie products of loops by concatenation) from right to left, in the same order as composition of homeomorphisms. We also fix the commutator notation to be $[a, b] := b^{-1}a^{-1}ba$.

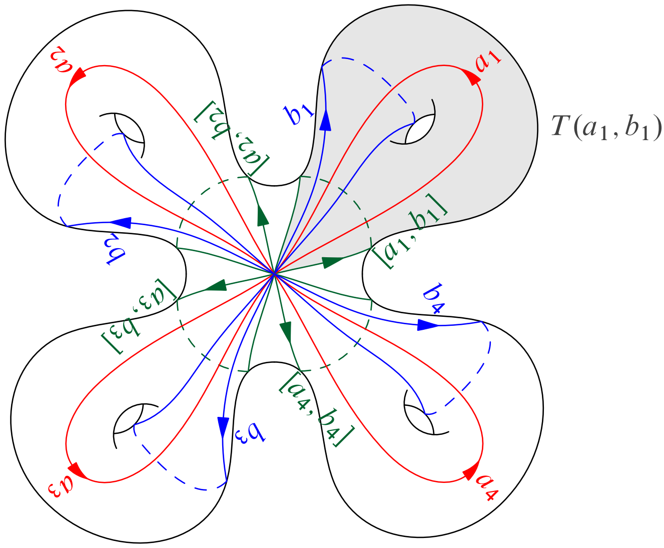

Figure 1: Standard generators on the genus g surface ($g = 4$).

The based curves $(a_1, b_1, \dots, a_g, b_g)$, depicted in Figure 1, are called a *standard system of loops*, and give the following standard presentation of Γ_g:

$$\Gamma_g = \langle a_1, b_1, \dots, a_g, b_g \mid [a_g, b_g] \cdots [a_1, b_1] = 1 \rangle.$$

We will make extensive use of systems of curves that look like those in Figure 2. Accordingly, we will say that a tuple $(\gamma_1, \dots, \gamma_k)$ of elements of Γ_g is an *oriented*, *directed k–chain* if these elements of Γ_g can be realized by differentiable based loops, $[0, 1] \to \Sigma_g$, that do not intersect outside the basepoint, and with cyclic order $(\gamma_1'(0), \gamma_2'(0), -\gamma_1'(1), \gamma_3'(0), -\gamma_2'(1), \gamma_4'(0), \dots, -\gamma_k'(1))$. In other words, an oriented, directed k–chain is a k–tuple of loops arising from an orientation-preserving embedding of the graph of Figure 2 (note that we do not require this embedding to be π_1–injective). If we do not insist that the embedding be orientation-preserving, we call it a *directed k–chain*, and, similarly, $(\gamma_1, \dots, \gamma_k)$ is simply a *k–chain* if there exist signs $\epsilon_1, \dots, \epsilon_k$ such that $(\gamma_1^{\epsilon_1}, \dots, \gamma_k^{\epsilon_k})$ is a directed k–chain. Also, we will say that a (oriented and/or directed) k–chain is *completable* if it sits in the middle of a (orientable and/or directed) $(k+2)$–chain.

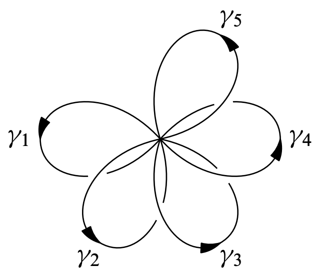

Figure 2: A directed chain of length 5.

For example, $(a_1^{-1}b_1a_1, a_1, b_1^{-1})$ is a noncompletable 3–chain in Σ_g, and the collection

$$(a_1, \delta_1, a_2, \delta_2, \dots, \delta_{g-1}, a_g, b_g^{-1})$$

(as well as its subchains), where we have set $\delta_i = a_{i+1}^{-1}b_{i+1}a_{i+1}b_i^{-1}$, forms a directed chain. Also, the family $(a_1^{-1}b_1a_1, a_1, \delta_1, a_2, b_2^{-1})$ forms a (noncompletable) 5–chain that will be handy in Section 5.3.

If two simple closed loops $a, b \in \Gamma_g$ do not intersect outside of the basepoint, we will write $i(a,b) = 1$ if (a,b) is an oriented, directed 2–chain, and we will write $i(a,b) = -1$ if $i(b,a) = 1$. Otherwise we will write $i(a,b) = 0$; if a and b are nonseparating, to say $i(a,b) = 0$ is equivalent to the existence of a curve c such that (a,c,b) is a 3–chain. Though reminiscent of the algebraic intersection number, $i(a,b)$ is an ad hoc definition, as we do not define $i(a,b)$ for most pairs (a,b) of elements of Γ_g.

Finally, if two curves $a, b \in \Gamma_g$ satisfy $i(a,b) = \pm 1$, we will denote by $T(a,b)$ the genus 1 subsurface of Σ_g defined by a and b; Figure 1 illustrates $T(a_1, b_1)$. While $T(a,b)$ is only defined up to based homotopy, it still makes sense to say, for example, that a curve γ is *disjoint* from $T(a,b)$, if $i(a,\gamma)$, $i(b,\gamma)$, $i([a,b],\gamma)$ are all defined and equal to 0.

We conclude this paragraph with some considerations on complexes of pairs of based curves.

Lemma 2.1 *Let G_0 denote the graph whose vertices are the pairs $(a,b) \in \Gamma_g^2$ with $i(a,b) = \pm 1$, with an edge between two pairs (a,b) and (b,c) whenever (a,b,c) is a 3–chain. Then G_0 is connected.*

The main results of this article do not depend on this lemma, as we will simply need to work on a connected component of this graph; our proof in the companion article [25] follows this strategy. However, the lemma is quite elementary, so here we take the honest approach of giving the proof and using the whole connected graph instead of making reference to a connected component.

The proof of Lemma 2.1 is divided into two main observations. It essentially copies the proof of Proposition 6.7 of [26], but corrects a minor mistake there, where the complex of *based* curves should have been used instead of the standard curve complex.

Observation 2.2 *Let G_1 be the graph whose vertices are the elements of Γ_g represented by simple, nonseparating curves, and with edges between a and b if and only if $i(a,b) = \pm 1$. Then G_1 is connected.*

Proof Let G_2 be the graph with the same vertices, but with edge between a and b whenever $i(a,b)$ is well defined. Let G_3 be the graph with vertex set consisting of the elements of Γ_g represented by simple curves (possibly separating), with an edge between a and b whenever $i(a,b)$ is well defined.

By drilling a puncture in Σ_g at the basepoint, G_3 can be identified with the arc graph of the surface $\Sigma_{g,1}$, which is well known to be connected; see eg [17]. Given a path in G_3 between two vertices of G_2, every time a separating curve appears we may either delete it or replace it by a nonseparating curve, producing a new path in G_2. Thus, G_2 is connected. Finally, we prove that any path in G_2 can be promoted to a path in G_1. Let $a_1 - a_2$ be an edge of G_2 which is not in G_1, ie we have $i(a_1, a_2) = 0$. Then a neighborhood of the curves a_1 and a_2 in Σ_g is a pair of pants P, with three boundary components, freely homotopic to a_1, a_2 and $a_1 a_2^{\pm 1}$. If Σ, Σ' and Σ'' are, respectively, the connected components of $\Sigma_g \setminus P$ separated from P by a_1, a_2 and $a_1 a_2^{\pm 1}$, then we cannot have $\Sigma' \neq \Sigma''$, for otherwise a_1 or a_2 would be separating. Hence, there exists a curve b such that $a_1 - b - a_2$ is a path in G_1. □

Observation 2.3 *Let a, b and a' be such that $i(a, b) = \pm 1$ and $i(a', b) = \pm 1$. Then (a, b) and (a', b) lie in the same connected component of the graph G_0 from Lemma 2.1.*

Proof Let $\sim$ denote the equivalence relation on vertices of G_0 of being in the same connected component. Let a, b, a' be as in the statement of the observation, and let N be the (geometric) minimum number of disjoint intersections, besides the basepoint, between the based curves a and a'. We will proceed by induction on N, starting with the base case $N = 0$. In this case $i(a, a') \in \{0, \pm 1\}$. If $i(a, a') = 0$, then (a, b, a') is a 3–chain and $(a, b) \sim (b, a')$. If $i(a, a') = \pm 1$, then for some $\epsilon \in \{-1, 1\}$, we have $i(b^\epsilon a, a') = 0$ (this is seen by looking at a neighborhood of the basepoint), hence $(b^\epsilon a, b, a')$ is a 3–chain and $(b^\epsilon a, b) \sim (b, a')$. Now $(b^\epsilon a, b) \sim (a, b)$, because there exists a curve c such that $(b^\epsilon a, b, c)$ and (a, b, c) are both 3–chains. This proves the base case.

Now, suppose $N \geq 1$. Orient the curves a and a' so that their tangent vectors at $t = 0$ are on the same side of b at the basepoint. Let $(x_1, \dots, x_N)$ be the intersection points of a and a', as ordered along the path a. Let a'' be the path obtained from following a' until we hit x_N (actually, any of the x_i would do), and then following the end of the path a. Then we have $i(a, b) = \pm 1$, $i(a', b) = \pm 1$, $i(a'', b) = \pm 1$ and the intersections of a and a' with a'' outside the basepoint are strictly less than N; this concludes our induction. □

Proof of Lemma 2.1 Let (a, b) and (c, d) be such that $i(a, b) = \pm 1$ and $i(c, d) = \pm 1$. There exists a path between b and c in G_1, which can be extended to a path $\gamma_1 - \gamma_2 - \cdots - \gamma_n$ in G_1 with $(a, b, c, d) = (\gamma_1, \gamma_2, \gamma_{n-1}, \gamma_n)$. By Observation 2.3, for all $j \in \{1, \dots, n-2\}$, (γ_j, γ_{j+1}) is connected to $(\gamma_{j+1}, \gamma_{j+2})$ in G_0, hence (a, b) is connected to (c, d). □

Finally, we will also use the following easy variation of Lemma 2.1.

Lemma 2.4 *Let G denote graph whose vertices are the pairs $(a, b) \in \Gamma_g^2$ with $i(a, b) = \pm 1$, with an edge between two pairs (a, b) and (b, c) whenever (a, b, c) is a* ***completable*** *3–chain. Then G is connected.*

Proof First, observe that whenever $T(a, b)$ and $T(c, d)$ are disjoint, (a, b) and (c, d) are in the same connected component of G. Now, observe that if (a, b, c) is a directed 3–chain, then it is completable if and only if ca is nonseparating. (The reader may find it helpful to draw a picture.) It follows that, if (a, b, c) is a noncompletable 3–chain in Σ_g, then there exists a pair (d, e) such that a, b, c do not enter $T(d, e)$. Hence, (a, b) and (b, c) are connected to (d, e) in G, and it follows that G is connected. □

2.2 Actions on the circle

2.2.1 Basic dynamics of circle homeomorphisms We quickly review some definitions for the purpose of setting notation. For more detailed background on this material, the reader may consult [12; 24; 13; 31] for example.

We denote by $\mathrm{Homeo}^{\mathbb{Z}}(\mathbb{R})$ the group of homeomorphisms of $\mathbb{R}$ commuting with translation by 1; we have a natural central extension

$$\mathbb{Z} \to \mathrm{Homeo}^{\mathbb{Z}}(\mathbb{R}) \to \mathrm{Homeo}^+(S^1).$$

The *translation number* (or rotation number) of an element $f \in \mathrm{Homeo}^{\mathbb{Z}}(\mathbb{R})$ is defined as

$$\widetilde{\mathrm{rot}}(f) := \lim_{n\to\infty} \frac{f^n(0)}{n} \in \mathbb{R},$$

and the Poincaré *rotation number* of an element $f \in \mathrm{Homeo}^+(S^1)$ is defined as $\mathrm{rot}(f) := \widetilde{\mathrm{rot}}(\tilde{f}) \bmod \mathbb{Z}$, where $\tilde{f}$ is any lift of f.

We assume the reader is familiar with these invariants, and with their essential properties. Those that we will use most frequently are that rot and $\widetilde{\mathrm{rot}}$ are homomorphisms when restricted to abelian (eg cyclic) subgroups, that $\mathrm{rot}(f) = p/q \in \mathbb{Q} \bmod \mathbb{Z}$ in reduced form if and only if f has a periodic orbit of period q, and that $\widetilde{\mathrm{rot}}$, and hence rot, are invariant under semiconjugacy. (The definition of semiconjugacy is recalled in Section 2.3, where we will be using it.)

We denote by $\mathrm{Per}(f) = \{x \in S^1 \mid f^n(x) = x \text{ for some } n \in \mathbb{Z} - \{0\}\}$ the set of periodic points of f. If $n = 1$, we also denote this by $\mathrm{Fix}(f)$. For $\tilde{f} \in \mathrm{Homeo}^{\mathbb{Z}}(\mathbb{R})$, we use $\mathrm{Per}(\tilde{f})$ to denote the set of all lifts of points of $\mathrm{Per}(f)$ to $\mathbb{R}$.

For $f \in \mathrm{Homeo}^+(S^1)$ with $\mathrm{Per}(f) \neq \varnothing$, let $q(f)$ denote the smallest positive integer such that $\mathrm{Fix}(f^{q(f)}) \neq \varnothing$, and let $p(f)$ be the least nonnegative integer such that f has rotation number equal to $p(f)/q(f) \bmod \mathbb{Z}$.

Define an *attracting periodic point* for f to be a point $x \in \mathrm{Per}(f)$ with a neighborhood I of x such that $f^{nq(f)}(I) \to x$ as $n \to \infty$. A *repelling* periodic point of f is defined as an attracting periodic point of f^{-1}. The sets of attracting and repelling periodic points will be denoted by $\mathrm{Per}^+(f)$ and $\mathrm{Per}^-(f)$, respectively.

2.2.2 One-parameter families and bending deformations Let $\gamma \in \Gamma_g$ be a based, simple loop. Cutting Σ_g along γ decomposes Γ_g into an amalgamated product $\Gamma_g = A *_{\langle\gamma\rangle} B$, or an HNN-extension $A*_{\langle\gamma\rangle}$, depending on whether γ is separating.

In both cases, if $\rho\colon \Gamma_g \to \mathrm{Homeo}^+(S^1)$ is a representation and if $(\gamma_t)_{t\in\mathbb{R}}$ is a continuous family of homeomorphisms commuting with $\rho(\gamma)$, we may define a deformation of ρ, as follows. If γ is separating and $\Gamma_g = A *_{\langle\gamma\rangle} B$, we define ρ_t to agree with ρ on A, while setting $\rho_t(\delta) = \gamma_t \rho(\delta)\gamma_t^{-1}$ for all $\delta \in B$. If γ is nonseparating, we may write $a_1 = \gamma$ and complete it into a standard generating system $(a_1, \dots, b_g)$, and set ρ_t to agree with ρ on all the generators except b_1, and put $\rho_t(b_1) = \gamma_t \rho(b_1)$.

In both cases, we call this deformation a *bending along* γ. These types of deformations were used by Thurston in order to parametrize *quasi-Fuchsian* representations of surface groups (he actually used more general bendings, as here we bend only along one simple curve). At the level of the representations, this is made explicit for example in [18], and this is the source of our inspiration. Some of our results involving bendings, especially in Section 4, can also be compared to the classical Baumslag's lemma [3, Proposition 1] and its usage in [4] or [20].

Most of the time (but not all) we will use these bendings with a one-parameter group γ_t, ie a morphism $\mathbb{R} \to \mathrm{Homeo}^+(S^1)$, $t \mapsto \gamma_t$, as provided by Lemma 2.7 below. In the special case when $\rho(\gamma) = \gamma_1$, then the deformation defined above, at $t = 1$, is the precomposition of ρ with $\tau_{\gamma *}$, where τ_γ is the Dehn twist along γ. However, for a Dehn twist to make sense as an automorphism of Γ (not up to inner automorphisms), we will use the following convention.

Convention 2.5 Suppose we are given a directed k–chain $(\gamma_1, \dots, \gamma_k)$, and wish to write a Dehn twist along the loop γ_i. Then we will always do so by pushing γ_i outside the basepoint in such a way that it intersects only γ_{i-1} and γ_{i+1} (if these curves exist) in a neighborhood of the chain. Accordingly, if ρ is a given representation and γ_i^t is a one-parameter family commuting with $\rho(\gamma_i)$, then the deformation leaves γ_j unchanged for $|j - i| \geq 2$ and $j = i$, and changes $\rho(\gamma_{i-1})$ into $\gamma_i^{-t}\rho(\gamma_{i-1})$ and $\rho(\gamma_{i+1})$ into $\rho(\gamma_{i+1})\gamma_i^t$.

Not all elements of $\mathrm{Homeo}^+(S^1)$ embed in a one-parameter subgroup. In fact, if $\mathrm{rot}(f)$ is irrational, then f embeds in such a subgroup if and only if the action of f is minimal, in which case f is conjugate to a minimal rotation. However, elements with rational rotation number do have large centralizers, giving us some flexibility in the use of bending deformations. We formalize this in the next lemma. Here, and later on, it will be convenient to fix a section of $\mathrm{Homeo}^+(S^1)$ in $\mathrm{Homeo}^{\mathbb{Z}}(\mathbb{R})$.

Notation 2.6 For $f \in \mathrm{Homeo}^+(S^1)$, let $\hat{f} \in \mathrm{Homeo}^{\mathbb{Z}}(\mathbb{R})$ be the (unique) lift of f with $\widetilde{\mathrm{rot}}(\hat{f}) \in [0, 1)$; we will call it the *canonical lift* of f. Later, we will also need to refer to the lift of f with translation number in $(-1, 0]$, this we denote by $\check{f}$. Note that $\hat{f}^{-1} = \overset{\smile}{f^{-1}}$.

Lemma 2.7 (positive one-parameter families) *Let $f \in \mathrm{Homeo}^+(S^1)$ have rational rotation number, and suppose $\mathrm{Per}(f) \neq S^1$. Then there exists a one-parameter group $(f_t)_{t\in\mathbb{R}}$, which commutes with f, such that for all $t \neq 0$, $\mathrm{Fix}(f_t) = \partial\mathrm{Per}(f)$, and for all $t > 0$ and $x \in \mathbb{R} \setminus \partial\mathrm{Per}(\tilde{f})$, we have $\hat{f}_t(x) > x$.*

Here and in what follows, ∂X denotes the *frontier* of a subset X of $\mathbb{R}$ or S^1.

Proof The set $S^1 \setminus \partial\mathrm{Per}(f)$ consists of a union of open intervals permuted by f. Choose a single representative interval I_α from each orbit. Note that $f^{q(f)}(I_\alpha) = I_\alpha$ for any such interval, and the restriction of $f^{q(f)}$ to $S^1 \setminus \mathrm{Per}(f)$ is either fixed-point free or the identity. Thus, we may identify each I_α with $\mathbb{R}$ such that $f^{q(f)}$, in coordinates, is $x \mapsto x + C$ for some $C \in \{-1, 0, 1\}$. Define s_t on I_α to be $x \mapsto x + t$. Since these I_α are in different orbits of the action of f on S^1, we may extend s_t equivariantly to a one-parameter family of homeomorphisms of S^1. □

In all the rest of this text, if $f \in \mathrm{Homeo}^+(S^1)$, any family f_t as in Lemma 2.7 will be called a *positive one-parameter family commuting with* f, or simply a *positive one-parameter family* if f is understood.

2.2.3 Periodic sets under deformations We now make some observations on how periodic sets change under bending deformations using positive one-parameter families. The main application of these comes in Section 5.2, but they will also make a few earlier appearances.

Let f and $g \in \mathrm{Homeo}^+(S^1)$ have rational rotation numbers. It follows immediately from the definition of canonical lift that

$$x \in \mathrm{Per}(\widehat{f}) \iff \widehat{f}^{q(f)}(x) = x + p(f).$$

Let f_t be a positive one-parameter family commuting with f. Let $g_t := f_t \circ g$, and let $\widetilde{g}_t = \widehat{f}_t \circ \widehat{g}$. Note that $\widetilde{g}_t = \widehat{g}_t$, provided the rotation number of g_t is constant as t varies.

For all $(x, t_1, \dots, t_{q(g)}) \in S^1 \times \mathbb{R}^{q(g)}$, we set

$$\Delta_{f,g}(x, t_1, \dots, t_{q(g)}) = \widetilde{g}_{t_{q(g)}} \circ \cdots \circ \widetilde{g}_{t_1}(\widetilde{x}) - \widetilde{x} - p(g),$$
$$\delta_{f,g}(x, t) = \Delta_{f,g}(x, t, \dots, t) = (\widetilde{g}_t)^{q(g)}(\widetilde{x}) - \widetilde{x} - p(g).$$

This does not depend on the lift $\widetilde{x} \in \mathbb{R}$ of x, but does depend on the choice of the one-parameter family f_t (so we are somewhat abusing notation). Further, we set

$$P(f, g) = \{x \in S^1 \mid \delta_{f,g}(x, t) = 0 \text{ for all } t \in \mathbb{R}\},$$
$$N(f, g) = \{x \in S^1 \mid \delta_{f,g}(x, t) \neq 0 \text{ for all } t \in \mathbb{R}\},$$
$$U(f, g) = \{x \in S^1 \mid \text{there exists a unique } t \in \mathbb{R} \text{ such that } \delta_{f,g}(x, t) = 0\}.$$

Unlike $\delta_{f,g}$, these sets do not depend on the choice of the positive one-parameter family (provided that it is chosen as in Lemma 2.7).

Assuming $\mathrm{rot}(g_t)$ is constant, then $P(f, g) = \bigcap_{t \in \mathbb{R}} \mathrm{Per}(g_t)$ is the set of *persistent* periodic points; $N(f, g)$ is the set of points that are *never* periodic for any g_t, and $U(f, g)$ is the set of points that lie in $\mathrm{Per}(g_t)$ for a *unique* time t.

Let $T_{f,g} : U(f, g) \to \mathbb{R}$ be the map that assigns to each $x \in U(f, g)$ the unique time $t \in \mathbb{R}$ for which $\delta_{f,g}(x, t) = 0$.

Lemma 2.8 *Suppose g_t has constant rotation number. Then we have the following properties.*

(1) *The set $P(f, g)$ is closed; moreover,*

$$P(f, g) = \mathrm{Per}(g) \cap \bigcap_{k=0}^{q(g)-1} g^k(\partial \mathrm{Per}(f));$$

in particular, if $\mathrm{rot}(f) = 0$ then every element of $P(f, g)$ has a finite orbit under the group $\langle f, g \rangle$.

(2) *The sets $P(f, g)$, $N(f, g)$ and $U(f, g)$ partition the circle.*

(3) *The set $U(f,g)$ is open, and the map $T_{f,g}\colon U(f,g)\to\mathbb{R}$ is continuous.*

(4) *For any $\varepsilon>0$, there exists t_0 such that $\mathrm{Per}(f_t\circ g)$ lies in the ε–neighborhood of $P(f,g)\cup\partial N(f,g)$ for all $t>t_0$.*

Proof By construction, the map $\Delta_{f,g}(x,\cdot)$ is (separately, in each variable t_j) constant if $\widetilde{g}_{t_{j-1}}\circ\cdots\circ\widetilde{g}_{t_1}(\widetilde{x})$ is in $\partial\mathrm{Per}(f)$, and is strictly increasing otherwise. Monotonicity implies that the subsets $\Delta_{f,g}(x,\mathbb{R}^{q(g)})$ and $\delta_{f,g}(x,\mathbb{R})$ of $\mathbb{R}$ coincide. The affirmations (1) and (2) are easy consequences of these observations. Let us prove (3). Let $x_0\in U(f,g)$, and write $t_0=T(x_0)$, so $\delta(x_0,t_0)=0$. Fix $\varepsilon>0$. Since $x_0\in U(f,g)$, we have $\delta(x_0,t_0+\varepsilon)>0$, and $\delta(x_0,t_0-\varepsilon)<0$. Since the maps $x\mapsto\delta(x,t_0+\varepsilon)$ and $x\mapsto\delta(x,t_0-\varepsilon)$ are continuous, there exists $\eta>0$ such that for all $x\in(x_0-\eta,x_0+\eta)$, we have $\delta(x,t_0+\varepsilon)>0$ and $\delta(x,t_0-\varepsilon)<0$. Thus, for each $x\in(x_0-\eta,x_0+\eta)$, the map $t\mapsto\delta(x,t)$ takes positive and negative values, hence has a (unique) zero in the interval $(t_0-\varepsilon,t_0+\varepsilon)$. In other words, $(x_0-\eta,x_0+\eta)\subset U(f,g)$, and for all $x\in(x_0-\eta,x_0+\eta)$, we have $|T(x)-T(x_0)|<\varepsilon$.

For statement (4), fix $\varepsilon>0$. Let $I_1,\dots,I_n$ denote the (finitely many) connected components of $U(f,g)$ of length $>\varepsilon$. Let $K\subset U(f,g)$ be the set of points of $U(f,g)$ that are at distance at least ε from $P\cup\partial N$. Then, $K\subset\bigcup_i I_i$, and it follows that K is compact. Since T is continuous, its restriction to K takes values in some segment $[-t_0,t_0]$, this gives the t_0 from the statement. □

The next proposition describes the topology of the sets $P(f,g)$, $N(f,g)$ and $U(f,g)$ in more detail.

Proposition 2.9 *Suppose g_t has constant rotation number. Then all accumulation points of $\partial N(f,g)$ lie in $P(f,g)$.*

The bulk of the proof of this is accomplished by the following lemma.

Lemma 2.10 *Let $x_0\in S^1\setminus\mathrm{Per}(g)$, and suppose there exists $u_k\in U(f,g)$ converging to x_0 from the right. Then there exists $\varepsilon>0$ such that $(x_0,x_0+\varepsilon)\subset U(f,g)$.*

Of course the symmetric statement, with sequences converging to x_0 from the left, holds as well, with a symmetric proof.

Proof Let $x_0\notin\mathrm{Per}(g)$, so we have $d:=d(x_0,g^{q(g)}(x_0))>0$, and suppose some sequence $u_k\in U(f,g)$ converges to x_0 from the right. First, we claim that there exists some $j\in\{1,\dots,q(g)\}$ such that $g^j(x_0)$ is *not* accumulated on the right by points of $\partial\mathrm{Per}(f)$.

To prove the claim, suppose for contradiction that for all $j\in\{1,\dots,q(g)\}$, $g^j(x_0)$ is accumulated on the right by $\partial\mathrm{Per}(f)$. Choose $z_{q(g)}\in\partial\mathrm{Per}(f)\cap\big(g^{q(g)}(x_0),g^{q(g)}(x_0)+\tfrac12 d\big)$ and, inductively for $j=q(g)-1,q(g)-2,\dots,1$, define $z_j\in\partial\mathrm{Per}(f)\cap(g^j(x_0),g^{-1}(z_{j+1}))$ for $j\in\{1,\dots,q(g)-1\}$, and set $\delta=g^{-1}(z_1)-x_0$. Then, for all $t>0$, we have $(f_tg)^j(x_0,x_0+\delta)\subset(g^j(x_0),z_j)$, hence

$$(f_tg)^{q(g)}(x_0,x_0+\delta)\subset\big(g^{q(g)}(x_0),g^{q(g)}(x_0)+\tfrac12 d\big).$$

Now let $k \geq 0$ be such that $u_k \in (x_0, x_0 + \delta)$. Choose $y_1 \in (g(x_0), g(u_k)) \cap \partial\mathrm{Per}(f)$ and, inductively for $j \geq 2$, choose $y_j \in (g^j(x_0), g(y_{j-1})) \cap \partial\mathrm{Per}(f)$. Then we have $(f_t g)^{q(g)}(u_k) \in (y_{q(g)}, z_{q(g)})$ for all $t \in \mathbb{R}$, hence $(f_t g)^{q(g)}(u_k) \in \big(g^{q(g)}(x_0), g^{q(g)}(x_0) + \frac{1}{2}d\big)$; this contradicts that $u_k \in U(f, g)$, and proves the claim.

Let j be the minimum element of $\{1, \dots, q(g)\}$ such that $g^j(x_0)$ is not accumulated on the right by points of $\partial\mathrm{Per}(f)$ (ie satisfying the claim above), and let y be such that $(g^j(x_0), y] \subset S^1 \setminus \partial\mathrm{Per}(f)$. Let k be large enough that $g \circ (f_t \circ g)^{j-1}(u_k) \in (g^j(x_0), y]$ holds for all $t \in \mathbb{R}$. (Such k exists using the argument above, since $g^i(x_0)$ is accumulated on the right by $\partial\mathrm{Per}(f)$ for all $i < j$.) Let $z \in (x_0, u_k)$. We will now show that $z \in U(f, g)$.

Since f_t acts transitively on $(g^j(x_0), y]$, for T sufficiently large we have

$$f_T \circ g \circ (f_T \circ g)^{j-1}(z) > g \circ (f_T \circ g)^{j-1}(u_k).$$

If $T > T(u_k)$, this guarantees that $\delta_{f,g}(z, T) > 0$. Similarly, if T' is small enough, we will have $f_{T'} \circ g \circ (f_{T'} \circ g)^{j-1}(z) < g \circ (f_T \circ g)^{j-1}(u_{k'})$ for any given $u_{k'} \in (x_0, z)$, and choosing $T' < T(u'_k)$ ensures that $\delta_{f,g}(z, T') < 0$. This shows that $z \in U(f, g)$, as desired. □

Proof of Proposition 2.9 Let x_0 be an accumulation point of $\partial N(f, g)$. If $x_0 \notin \mathrm{Per}(g)$, then by Lemma 2.10, on any side of x_0 containing a sequence of points in $\partial N(f, g)$, there is a neighborhood of x_0 containing no points of $U(f, g)$. Since $P(f, g)$, $N(f, g)$ and $U(f, g)$ partition S^1, it follows that there is also a sequence of points in $P(f, g)$ approaching x_0 from this side. Since $P(f, g)$ is closed, $x_0 \in P(f, g) \subset \mathrm{Per}(g)$, a contradiction.

It follows that $x_0 \in \mathrm{Per}(g)$. If also $x_0 \notin P(f, g)$, then $x_0 \in U(f, g)$ since x_0 is a periodic point for $f_0 \circ g = g$. But $U(f, g)$ is open, a contradiction. □

All the discussion above describes the variation of $\mathrm{Per}(g)$ upon deforming g by composition with f_t on the left. However, one may equally well replace g by $g f_t$ and define sets P, N and U with the same properties — indeed, replacing g by $g f_t$ is equivalent to replacing g^{-1} by $f_{-t} g^{-1}$. There is no reason to privilege left-side deformations in the definition of bending, and we will occasionally make use of deformations on the right.

2.3 The character space for $\mathrm{Homeo}^+(S^1)$

Following [12], for a group Γ, two homomorphisms ρ_1 and $\rho_2 \in \mathrm{Hom}(\Gamma, \mathrm{Homeo}^{\mathbb{Z}}(\mathbb{R}))$ are said to be *semiconjugate*[3] if there exists a monotone (possibly noncontinuous or noninjective) map $h\colon \mathbb{R} \to \mathbb{R}$ such that $h(x+1) = h(x) + 1$ for all $x \in \mathbb{R}$, and $h \circ \rho_1(\gamma) = \rho_2(\gamma) \circ h$ for all $\gamma \in \Gamma$. Similarly, ρ_1 and

[3]Note that this definition is *not* the usual notion of semiconjugacy from topological dynamical systems (eg as in [19]), which is not a symmetric relation.

$\rho_2 \in \mathrm{Hom}(\Gamma, \mathrm{Homeo}^+(S^1))$ are semiconjugate if there is such a map $h\colon \mathbb{R} \to \mathbb{R}$ such that for all γ, there are lifts $\widetilde{\rho_1(\gamma)}$ and $\widetilde{\rho_2(\gamma)} \in \mathrm{Homeo}^{\mathbb{Z}}(\mathbb{R})$ which are semiconjugate by this map h. Ghys [12] proved that, under this definition, semiconjugacy is an equivalence relation. Note that this is particular to actions on S^1 and does not agree with the usual definition of semiconjugacy from topological dynamics.

In [6, Section 1], Calegari and Walker describe an analogy between rotation numbers of elements of $\mathrm{Homeo}^+(S^1)$ and characters of linear representations. Much as characters capture the dynamics of a linear representation; rotation numbers capture representations up to semiconjugacy:

Theorem 2.11 (Ghys [12], Matsumoto [27]) *Let Γ be any group, and let S be a generating set for Γ. For $f, g \in \mathrm{Homeo}^+(S^1)$, define $\tau(f,g) := \widetilde{\mathrm{rot}}(\tilde f \tilde g) - \widetilde{\mathrm{rot}}(\tilde f) - \widetilde{\mathrm{rot}}(\tilde g)$ for any lifts $\tilde f$ and $\tilde g \in \mathrm{Homeo}^{\mathbb{Z}}(\mathbb{R})$. With this notation, two representations ρ_1 and ρ_2 in $\mathrm{Hom}(\Gamma, \mathrm{Homeo}^+(S^1))$ are semiconjugate if and only if the following two conditions hold*:

(i) $\mathrm{rot}(\rho_1(s)) = \mathrm{rot}(\rho_2(s))$ *for each $s \in S$.*

(ii) $\tau(\rho_1(a), \rho_1(b)) = \tau(\rho_2(a), \rho_2(b))$ *for all a and b in Γ.*

We observe here that one can recover Calegari and Walker's analogy from our more general definition of character spaces for arbitrary groups. For a topological group G, recall that $X(\Gamma, G)$ denotes the largest Hausdorff quotient of $\mathrm{Hom}(\Gamma, G)/G$. Let $G /\!\!/ G$ denote the space $X(\mathbb{Z}, G)$; then there is, for each $\gamma \in \Gamma$ a natural, continuous map $X(\Gamma, G) \to G /\!\!/ G$, which sends the class of a representation ρ to the class of $\rho(\gamma)$. For example, when $G = \mathrm{SL}(2, \mathbb{C})$, these are precisely the trace functions. The next proposition says that when $G = \mathrm{Homeo}^+(S^1)$, these are the *rotation numbers*, and the space $X(\Gamma, G)$ is, as a set, exactly the set of semiconjugacy classes of representations.

Proposition 2.12 *Let Γ be a group. Representations $\rho_1, \rho_2 \in \mathrm{Hom}(\Gamma, \mathrm{Homeo}^+(S^1))$ are semiconjugate if and only if they are equivalent in $X(\Gamma, \mathrm{Homeo}^+(S^1))$.*

Following this analogy, the "character variety" for $\mathrm{Homeo}^+(S^1)$ not only comes with its "ring of functions" (the rotation number functions), but with an underlying topological space as well. This gives the most natural setting to speak of rigidity, or to study the global topology of the space of representations.

We defer the proof of Proposition 2.12 in order to make some preliminary observations. The first is the important remark that Proposition 2.12 has no analog in $\mathrm{Homeo}^+(\mathbb{R})$ — a group may have many dynamically distinct actions on the line, but the character space is a single point:

Proposition 2.13 *For any discrete group Γ, the space $X(\Gamma, \mathrm{Homeo}^+(\mathbb{R}))$ consists of a single point.*

Proof Let $\rho \in \mathrm{Hom}(\Gamma, \mathrm{Homeo}^+(\mathbb{R}))$. Let S be a finite, symmetric subset of Γ. Given $\varepsilon > 0$, we will conjugate ρ so that $|\rho(s)(x) - x| < \varepsilon$ holds for all $s \in S$ and $x \in \mathbb{R}$, hence show that conjugates of ρ approach the trivial representation in the compact-open topology.

As a first case, assume also that the subgroup generated by S has no global fixed points in $\mathbb{R}$. Then define $h(0) = 0$, and iteratively, for $n \in \mathbb{Z}$ define $h\big(\frac{1}{2}n\varepsilon\big) = \max_{s\in S} s\big(h\big(\frac{1}{2}(n-1)\varepsilon\big)\big)$ if $n > 0$, and $h\big(\frac{1}{2}n\varepsilon\big) = \min_{s\in S} s\big(h\big(\frac{1}{2}(n+1)\varepsilon\big)\big)$ if $n < 0$. Extend h over the interior of each interval $\big[\frac{1}{2}n\varepsilon, \frac{1}{2}(n+1)\varepsilon\big]$ as an affine map. Since S has no global fixed point, this map h is surjective, hence it is an orientation-preserving homeomorphism. Furthermore, we have $hsh^{-1}\big(\frac{1}{2}n\varepsilon\big) \in \big[\frac{1}{2}(n-1)\varepsilon, \frac{1}{2}(n+1)\varepsilon\big]$ for all $s \in S$. Thus, $|hsh^{-1}(x) - x| < \varepsilon$ holds for all $x \in \mathbb{R}$.

If instead the subgroup generated by S does have a global fixed point, we may define h to be the identity on the set F of global fixed points, and define it as above on each connected component of $\mathbb{R} \setminus F$. □

Recall that the action of any group on S^1 is either minimal, or has a finite orbit, or has a closed, invariant set (called the *exceptional minimal set*) homeomorphic to a Cantor set, on which the restriction of the action is minimal. The following is an easy consequence of the definition of semiconjugacy, which we will use in the proof of Proposition 2.12.

Observation 2.14 *Every action ρ_1 with an exceptional minimal set is semiconjugate to a minimal action ρ_2, by a* ***continuous*** *map h satisfying $h \circ \rho_1(\gamma) = \rho_2(\gamma) \circ h$. Furthermore, if ρ_2 is minimal, and ρ_1 arbitrary, then any h satisfying this equation is necessarily continuous. In particular, a semiconjugacy h between two minimal actions is invertible, and hence a* ***conjugacy****.*

Proof of Proposition 2.12 For one direction, it suffices to prove that the quotient of the space $\mathrm{Hom}(\Gamma, \mathrm{Homeo}^+(S^1))$ by semiconjugacy is Hausdorff. This follows from Theorem 2.11, since the maps rot and τ in the theorem are continuous, well defined on semiconjugacy classes, take values in the (Hausdorff) spaces S^1 and $\mathbb{R}$, and distinguish semiconjugacy classes.

For the converse, if ρ has a finite orbit, then we can employ a similar strategy to the proof of Proposition 2.13 to conjugate it arbitrarily close to an action on the circle by rigid rotations. Hence, there is a unique element of the character space corresponding to the semiconjugacy class of ρ.

Now suppose instead that ρ has an exceptional minimal set. By Observation 2.14 there is a minimal action ρ' and continuous map h such that each $\gamma \in \Gamma$ has lifts satisfying

$$\widetilde{\rho'(\gamma)} \circ h = h \circ \widetilde{\rho(\gamma)}$$

as in the definition of semiconjugacy. Let S be a finite subset of Γ, and fix $\varepsilon > 0$. Let $\delta \in (0, \varepsilon)$ be small enough that for all $s \in S$ and all $x, y \in S^1$, $|x - y| < \delta$ implies $|\rho'(s)(x) - \rho'(s)(y)| < \varepsilon$.

Since h is continuous and commutes with $x \mapsto x + 1$, we can approximate it by a homeomorphism $h' \in \mathrm{Homeo}^{\mathbb{Z}}(\mathbb{R})$ at C^0 distance at most δ from h. Let $s \in S$ and $x \in \mathbb{R}$, and take the lifts $\widetilde{\rho'(s)}$ and $\widetilde{\rho(s)}$ as above. Then we have

$$|\widetilde{\rho'(s)}(x) - \widetilde{\rho'(s)} \circ (h \circ h'^{-1})(x)| < \varepsilon \quad \text{and} \quad |h \circ \widetilde{\rho(s)} \circ h'^{-1}(x) - h' \circ \widetilde{\rho(s)} \circ h'^{-1}(x)| < \varepsilon,$$

hence the definition of semiconjugacy and the triangle inequality gives

$$|\widetilde{\rho'(s)}(x) - h' \circ \widetilde{\rho(s)} \circ h'^{-1}(x)| < 2\varepsilon.$$

This proves that every representation without finite orbit is weakly conjugate to the minimal representation in its semiconjugacy class. □

We conclude this section with two observations and a short lemma that will be useful later on. The observations are simple consequences of Observation 2.14.

Observation 2.15 *Let $\rho_2 \in \mathrm{Hom}(\Gamma, \mathrm{Homeo}^+(S^1))$ be minimal, and let ρ_1 be any action which is semiconjugate to ρ_2* (as in Observation 2.14). *Then for any $\gamma \in \Gamma$, we have $\mathrm{Per}(\rho_2(\gamma)) = h\,\mathrm{Per}(\rho_1(\gamma))$, and hence $|\mathrm{Per}(\rho_2(\gamma))| \le |\mathrm{Per}(\rho_1(\gamma))|$.*

Observation 2.16 *Suppose that ρ is minimal and path-rigid, and let a and b satisfy $i(a,b) = -1$ and $\mathrm{rot}(\rho(b)) \in \mathbb{Q}$. Since $\rho(b^{q(b)})$ lies in a one-parameter family, there is a bending deformation replacing $\rho(a)$ with $\rho(b^{Nq(b)}a)$ for any $N \in \mathbb{Z}$, which is realized by precomposition with a Dehn twist* (see Section 2.2.2). *Thus the new representation has the same image as ρ; in particular it is minimal, hence **conjugate** to ρ.*

Lemma 2.17 *Let $f, g \in \mathrm{Homeo}^+(S^1)$ be two homeomorphisms with rational rotation number. The property that f and g share a periodic point depends only on the semiconjugacy class of $\langle f, g\rangle$.*

Proof For $f_1, \dots, f_n \in \mathrm{Homeo}^+(S^1)$, let $\tau(f_1, \dots, f_n) = \widetilde{\mathrm{rot}}(\tilde{f}_n \circ \cdots \circ \tilde{f}_1) - \sum_i \widetilde{\mathrm{rot}}(\tilde{f}_i)$, which obviously does not depend on the choices of lifts. Note that

$$\tau(f_1, \dots, f_n) = \tau(f_1, f_n \circ \cdots \circ f_2) - \sum_{j=2}^{n-1} \tau(f_j, f_n \circ \cdots \circ f_{j+1}),$$

so this function can be recovered from the two-variable τ of Theorem 2.11.

To prove the lemma, we prove the stronger statement that f and g sharing a periodic point is equivalent to the following assertion:

For any $\ell \ge 1$ and any integers $n_1, m_1, \dots, n_\ell, m_\ell$, we have

$$\tau(f^{n_1 q(f)}, g^{m_1 q(g)}, \dots, f^{n_\ell q(f)}, g^{m_\ell q(g)}) = 0.$$

Applying Theorem 2.11 gives the desired conclusion.

The assertion is clearly true if f and g share a periodic point. To prove the converse, suppose that $\mathrm{Per}(f) \cap \mathrm{Per}(g) = \varnothing$, so $S^1 \setminus (\mathrm{Per}(f) \cup \mathrm{Per}(g))$ is a union of intervals. As $\mathrm{Per}(f)$ and $\mathrm{Per}(g)$ are closed, disjoint sets, only finitely many of these complementary intervals have one boundary point in each of $\mathrm{Per}(f)$ and $\mathrm{Per}(g)$. Those bounded on the right by a point of $\mathrm{Per}(f)$ and at their left by a point of

$\mathrm{Per}(g)$ alternate with the others (with the roles of right and left reversed), in particular there are an even number of such complementary intervals. Let $I_1, \dots, I_{2\ell}$ denote these intervals, in their cyclic order on the circle, and let $I_j = (x_j, y_j)$. Up to shifting the indices cyclically, we have $x_i, y_{i+1} \in \mathrm{Per}(g)$ and $x_{i+1}, y_i \in \mathrm{Per}(f)$ for all i even.

Choose a point x in I_1. Since the interval (x_1, y_2) contains only points of $\mathrm{Per}(g)$, there exists n_1 such that $f^{n_1 q(f)}(x) \in I_2$. Similarly, there exists a power n_2 of $g^{q(g)}$ which maps $f^{n_1 q(f)}(x)$ into I_3, and so on for n_i, with $i > 2$. The last operation can be done so that the image of x, under a suitable word $g^{n_\ell q(g)} f^{n_\ell q(f)} \cdots g^{n_2 q(g)} f^{n_1 q(f)}$, lies to the right of x in I_1. Then, choosing the canonical lifts of $f^{n_i q(f)}$ and $g^{m_i q(g)}$, we observe that $\tau(f^{n_1 q(f)}, g^{m_1 q(g)}, \dots, f^{n_\ell q(f)}, g^{m_\ell q(g)}) \geq 1$. ◻

Remark 2.18 In the case $\mathrm{Per}(f) \cap \mathrm{Per}(g) = \varnothing$, the integer ℓ in the proof above also only depends on τ; in fact, it is the *minimal* integer such that there exist $m_i, n_i \in \mathbb{Z}$ with

$$\tau(f^{n_1 q(f)}, g^{m_1 q(g)}, \dots, f^{n_\ell q(f)}, g^{m_\ell q(g)}) \geq 1.$$

2.4 The Euler class

Recall that the (*integer*) *Euler class* for circle bundles is a generator e (well defined up to sign) of $H^2(\mathrm{Homeo}^+(S^1); \mathbb{Z}) \cong \mathbb{Z}$; and the *Euler number* of a representation $\rho\colon \Gamma_g \to \mathrm{Homeo}^+(S^1)$ is the integer $\langle \rho^*(e), [\Gamma_g] \rangle$, where $[\Gamma_g]$ denotes the fundamental class, ie a generator of $H_2(\Gamma_g, \mathbb{Z})$. Under the correspondence between second cohomology and central extensions, e is represented by the extension $\mathbb{Z} \to \mathrm{Homeo}^{\mathbb{Z}}(\mathbb{R}) \to \mathrm{Homeo}^+(S^1)$ described in Section 2.2.1 and hence can be seen as the obstruction to lifting a representation to $\mathrm{Homeo}^{\mathbb{Z}}(\mathbb{R})$.

Although this definition only makes sense for fundamental groups of closed surfaces — a surface with boundary has free fundamental group, and $H_2(F_n; \mathbb{Z}) = 0$ — there is a *relative* Euler number for surfaces with boundary, which is additive when such subsurfaces are glued together. This can be made precise in the language of bounded cohomology as explained in [5, Section 4.3]. (Compare also Goldman [14] and Matsumoto [28].) Following [5], we make the following definition.

Definition 2.19 (Euler number for pants) Let $P \subset \Sigma_g$ be a subsurface homeomorphic to a pair of pants; equip it with three based curves a, b and c as in Figure 3. (If P does not contain the basepoint, choose a path in Σ_g from the basepoint to a chosen point in P, and use it to define the curves a, b and c.) Let $\rho\colon \pi_1 \Sigma_g \to \mathrm{Homeo}^+(S^1)$, and let $\widetilde{\rho(a)}$, $\widetilde{\rho(b)}$ be any lifts of $\rho(a)$ and $\rho(b)$ to $\mathrm{Homeo}^{\mathbb{Z}}(\mathbb{R})$, and let $\widetilde{\rho(c)} = \bigl(\widetilde{\rho(b)}\widetilde{\rho(a)}\bigr)^{-1}$. Then the contribution of P to the Euler number of ρ is

$$\mathrm{eu}_P(\rho) := \widetilde{\mathrm{rot}}(\widetilde{\rho(a)}) + \widetilde{\mathrm{rot}}(\widetilde{\rho(b)}) + \widetilde{\mathrm{rot}}(\widetilde{\rho(c)}).$$

If the surface Σ_g is cut into pairs of pants, the Euler class of ρ is the sum of the contributions of these pants. See [5, Section 4.3] for a detailed discussion, and [25] for a short exposition and proof that this does not depend on the decomposition. Definition 2.19 extends naturally to one-holed tori: if $T = T(a, b) \subset \Sigma_g$ is a one-holed torus, cutting T along a simple closed curve (say, freely homotopic to a or b) yields the

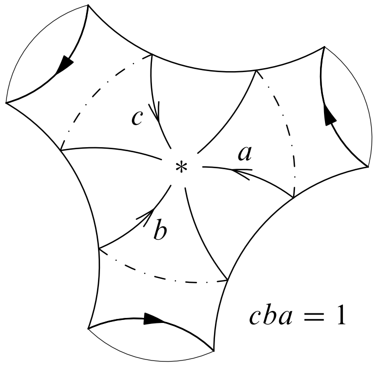

Figure 3: A pair of pants with standard generators of its fundamental group.

formula $\mathrm{eu}_T(\rho) = \widetilde{\mathrm{rot}}(\widetilde{\rho(b)}^{-1}\widetilde{\rho(a)}^{-1}\widetilde{\rho(b)}\widetilde{\rho(a)})$, which, in turn, gives Milnor's classical formula [30], $\mathrm{eu}(\rho) = \prod_{i=1}^{g}[\widetilde{\rho(a_i)}, \widetilde{\rho(b_i)}]$, where $(a_1, \dots, b_g)$ is a standard system of curves, and where the lifts are taken arbitrarily.

3 A first statement

This section proves the main theorem under a strong additional hypothesis. We will show that if ρ is path-rigid and if for every $a, b \in \Gamma_g$ with $i(a,b) = \pm 1$, $\rho(a)$ and $\rho(b)$ resemble, dynamically, a geometric representation, then ρ is in fact geometric. In other words, the local condition that ρ "looks geometric" on pairs a, b with $i(a,b) = \pm 1$ implies global geometricity. To formalize this, we introduce some definitions.

Definition 3.1 Say that an element $f \in \mathrm{PSL}_2^k(\mathbb{R})$ is *hyperbolic* if its projection to $\mathrm{PSL}_2(\mathbb{R})$ is hyperbolic. Equivalently, all its periodic points are hyperbolic in the sense of classical smooth dynamics.

Definition 3.2 Let $a, b \in \Gamma_g$ and $\rho\colon \Gamma_g \to \mathrm{Homeo}^+(S^1)$. Denote by $S_k(a,b)$ (the notation ρ is suppressed) the property that

(i) $i(a,b) = \pm 1$ and $\rho(a)$ and $\rho(b)$ are each separately conjugate to a hyperbolic element of $\mathrm{PSL}_2^k(\mathbb{R})$, and

(ii) their periodic points *alternate* around the circle, meaning that each pair of points of $\mathrm{Per}(a)$ are separated by $\mathrm{Per}(b)$, and vice versa.

If all pairs a, b with $i(a,b) = \pm 1$ have $S_k(a,b)$, then we say that ρ has property S_k.

With this notation we can state the main result of this section.

Theorem 3.3 *Let ρ be a path-rigid, minimal representation, and suppose ρ satisfies S_k for some k. Then ρ is geometric.*

Before embarking on the proof, we discuss some other variations on hyperbolicity to be used later in the section.

Let $f \in \mathrm{Homeo}^+(S^1)$. We say that an open interval $I \subset S^1$ is *attracting for* f if $f(\bar{I}) \subset I$. We say that I is *repelling for* f if it is attracting for f^{-1}. Matsumoto [28] calls homeomorphisms that do not admit attracting intervals *tame*. In line with his terminology, we call those homeomorphisms which do *savage*. More specifically, we have:

Definition 3.4 A homeomorphism $f \in \mathrm{Homeo}^+(S^1)$ is *n–savage* if there exist $2n$ open intervals with pairwise disjoint closures, indexed in cyclic order by $I_1^-, I_1^+, \dots, I_n^-, I_n^+$ such that

$$f\Bigl(S^1 \Bigm\backslash \Bigl(\bigcup_{j=1}^{n} \overline{I_j^-}\Bigr)\Bigr) = \bigcup_{j=1}^{n} I_j^+.$$

In this sense, savage means 1–savage.

The next observation is an immediate consequence of the definition; we leave the proof to the reader.

Observation 3.5 *If f is n–savage, then f^k is also n–savage for any $k \in \mathbb{Z} \setminus \{0\}$. Furthermore,* $\mathrm{rot}(f^n) = 0$ *and f has at least one periodic point in each interval I_j^+ and I_j^-.*

As a concrete example, note that if f is conjugate to a hyperbolic element in $\mathrm{PSL}_2^k(\mathbb{R})$, then f is n–savage for $n \le k$.

The intervals I_j^+ and I_j^- in the definition of savage are by no way unique, but it will be convenient to use the notation $I^+(f) := \bigcup_{j=1}^n I_j^+$ and $I^-(f) := \bigcup_{j=1}^n I_j^-$, even if these sets depend on choices. We also set $I(f) := I^+(f) \cup I^-(f)$.

Definition 3.6 Two n–savage homeomorphisms $f, g \in \mathrm{Homeo}^+(S^1)$ are *in n–Schottky position* if their respective attracting and repelling intervals $I_j^\pm$ can be chosen so that $I(f)$ and $I(g)$ have disjoint closures.

Note that, if f and g are n–Schottky, then f^{-1} and g are n–Schottky as well. Note also that the condition $S_k(a,b)$ is not equivalent to k–Schottky, although $S_k(a,b)$ does imply that a^N and b^N are k–Schottky for sufficiently large N. We will prove however that hypothesis S_k on a path-rigid representation ρ implies that a and b are indeed k–Schottky whenever $i(a,b) = \pm 1$.

3.1 Outline of proof of Theorem 3.3

We start in Section 3.2 with a series of lemmas that use rigidity and property S_k to show the cyclic order of periodic points of various nonseparating curves agrees with that of a geometric representation, and that certain pairs of curves are k–Schottky. Following this, we show in Section 3.3 that the Euler number of a path-rigid, minimal, S_k representation agrees with a geometric one, ie is equal to $\pm(2g-2)/k$. From there, we need to improve this essentially combinatorial result to the fact that the representation is actually geometric. Our main tool is existing work of Matsumoto on *basic partitions*.

We are now ready to embark on the proof. Throughout, we make the following assumption.

Assumption 3.7 For the rest of this section, ρ denotes a path-rigid minimal representation of Γ_g that satisfies S_k. To simplify notation, we often omit ρ, identifying $a \in \Gamma_g$ with $\rho(a) \in \mathrm{Homeo}^+(S^1)$. Thus, we will speak of $\mathrm{Per}(a)$, denote an attracting point of $\rho(a)$ by a^+, etc.

3.2 Order of periodic points

Property S_k makes it much easier to understand periodic points under deformations. We start with several lemmas to this effect.

Lemma 3.8 *Let $i(a,b)=1$, let $F \subset S^1$ be a countable set, and let b_t be a positive one-parameter family commuting with $b=\rho(b)$. Then for some $t \in \mathbb{R}$, we have $\mathrm{Per}(b_t\rho(a)) \cap F = \varnothing$.*

Proof We use the notation from Section 2.2.3. Path-rigidity of ρ implies that $\mathrm{rot}(b_t a)$ is constant, and Property S_k and Lemma 2.8 implies that $P(b,a)=\varnothing$, so we need only worry about points in $U=U(b,a)$. Thus, provided $t \notin T_{b,a}(F)$, we have $\mathrm{Per}(b_t a) \cap F = \varnothing$. □

Lemma 3.9 (disjoint curves have disjoint Per) *Let (a,b,c) be a completable directed 3–chain. Then $\mathrm{Per}(a) \cap \mathrm{Per}(c) = \varnothing$. In fact, $\mathrm{Per}(c) \cap b^n(\mathrm{Per}(a)) = \varnothing$ for all $n \in \mathbb{Z}$.*

Proof Fix $n \in \mathbb{N}$. Complete (a,b,c) to a directed 4–chain (a,b,c,d), and apply a bending deformation replacing c with $d_t c$ (leaving the action of a and b unchanged, hence $b^n\,\mathrm{Per}(a)$ unchanged), for a positive family d_t. By Lemma 3.8, there is some t such that $\mathrm{Per}(d_t c) \cap b^n\,\mathrm{Per}(a) = \varnothing$. Now the conclusion follows from path-rigidity of ρ, together with Lemma 2.17. □

Note that, if $i(a,b)=\pm 1$, then for any $n \in \mathbb{Z}$ we also have $i(b^n a, b) = \pm 1$, hence $S_k(b^n a, b)$ holds. The next lemma describes the position of the periodic points of $S_k(b^n a, b)$ for large n. This is particularly useful since there exist bending deformations replacing the pair a, b with $b^n a, b$ provided that $q(b)$ divides n; see Observation 2.16.

Lemma 3.10 (movement of Per by bending) *Suppose $i(a,b)=\pm 1$. Then as $N \to +\infty$, the points of $\mathrm{Per}^+(b^N a)$ approach $\mathrm{Per}^+(b)$, and $\mathrm{Per}^-(b^N a)$ approaches $a^{-1}\,\mathrm{Per}^-(b)$; similarly, as $N \to -\infty$, $\mathrm{Per}^+(b^N a)$ approaches $\mathrm{Per}^- b$ and $\mathrm{Per}^-(b^N a)$ approaches $a^{-1}\,\mathrm{Per}^+(b)$.*

Proof When $a^{-1}\,\mathrm{Per}(b) \cap \mathrm{Per}(b) = \varnothing$, the conclusion of the lemma is an easy exercise. We claim that path-rigidity of ρ implies this extra provision. To see this, suppose for example that $i(a,b)=1$, and let (c,a,b) be a completable directed 3–chain. By Lemma 3.9, $\mathrm{Per}(c) \cap \mathrm{Per}(b) = \varnothing$. Thus, we can make a positive bending deformation replacing a with ac_t, until $(ac_t)^{-1}\,\mathrm{Per}(b) \cap \mathrm{Per}(b) = \varnothing$. □

Notation 3.11 Let f and g be homeomorphisms of S^1. When talking about cyclic order of periodic points, we use the notation $((f^+, g^+, g^-, f^-))_k$ to mean that, in cyclic order, there is one attracting point for f, followed by an attracting point for g, followed by a repelling point for g, followed by an attracting point for f, with this pattern repeating k times. The notation $f^{\pm}$ means any point from $\mathrm{Per}(f)$. We also use other obvious variations, such as $((f^{\pm}, g^{-}, f^{\pm}, g^{+}))_k$, and extend this naturally to periodic points of three or more homeomorphisms.

When such a cyclic order is given, we call an interval $I \subset S^1$ of *type* (f^{+}, g^{-}) if it is bounded on the left (proceeding anticlockwise, using the natural orientation of S^1) by a point of $\mathrm{Per}^{+}(f)$ and on the right by a point of $\mathrm{Per}^{-}(g)$, and if it does not contain a proper subinterval with this property. We also use other obvious variations.

Lemma 3.12 (periodic points of 3–chains) *Let (a, b, c) be a completable directed 3–chain. Then, up to reversing the orientation of the circle, the periodic points of a, b and c come in the cyclic order*

$$((a^{-}, b^{-}, a^{+}, c^{\pm}, b^{+}, c^{\pm}))_k.$$

Proof Up to reversing orientation of S^1, we may suppose that the cyclic order of points in $\mathrm{Per}(a) \cup \mathrm{Per}(b)$ is $((a^{-}, b^{-}, a^{+}, b^{+}))_k$. Choose two consecutive points of $\mathrm{Per}(b)$ (in cyclic order), and denote these by b^{-} and b^{+}. Let a^{+} be the point of $\mathrm{Per}(a)$ between b^{-} and b^{+}, and let $c^{\pm}$ be the periodic point of c in this interval (there is exactly one by hypothesis S_k). The points of $\mathrm{Per}(a)$ in the interval (b^{-}, b^{+}) are in cyclic order $(b^{-}, a^{+}, b^{q(b)}(a^{+}), b^{+})$.

By Lemma 3.9, $c^{\pm}$ cannot be equal to a^{+} or $b^{q(b)}(a^{+})$. Suppose for contradiction that $c^{\pm}$ lies in the interval (b^{-}, a^{+}), or in the interval $(b^{q(b)}(a^{+}), b^{+})$. Then the closed segment $[a^{+}, b^{q(b)}(a^{+})]$ does not contain any periodic point of c. Let $(c_t)_{t \in \mathbb{R}}$ be a positive one-parameter family commuting with c, and use this to perform a bending along c as in Section 2.2.3. Using the notation from this section, we have $\delta_{c,b}(a^{+}, 0) > 0$, but for t sufficiently negative, we have $\Delta_{c,b}(a^{+}, 0, \dots, 0, t) < 0$. Thus, for some $t_0 < 0$, we have $\delta_{c,b}(a^{+}, t_0) = 0$, ie $a^{+} \in \mathrm{Per}(c_{t_0} b) \cap \mathrm{Per}(a)$. This, together with Lemma 2.17 and the path-rigidity of ρ, yields a contradiction.

The same argument applies to an interval of the form (b^{+}, b^{-}), where b^{+} and b^{-} denote two other consecutive points of $\mathrm{Per}(b)$. In that case, the argument shows that the (unique) periodic point of c in this interval lies between points of the form $b^{q(b)}(a^{-})$ and a^{-}, proving the lemma. □

In particular, for all pairs $a, c \in \Gamma_g$ such that there exists a completable 3–chain (a, b, c), Lemma 3.12 provides information about the periodic sets of a and c.

Corollary 3.13 *Let a and c be two nonseparating curves with $i(a, c) = 0$, and suppose c is not conjugate to a or a^{-1}. Then their periodic points are in cyclic order $((a^{\pm}, a^{\pm}, c^{\pm}, c^{\pm}))_k$.*

Proposition 3.14 *Since c is not conjugate to $a^{\pm 1}$, we may find b such that (a, b, c) is a completable directed 3–chain. Then, up to reversing the orientation of the circle, the periodic points of a, b and c and the b–preimages of $\mathrm{Per}(c)$ are in cyclic order*

$$((a^{-}, b^{-1}(c^{\pm}), b^{-}, b^{-1}(c^{\pm}), a^{+}, c^{\pm}, b^{+}, c^{\pm}))_k.$$

Proof of Proposition 3.14 Apply a bending deformation of ρ replacing b with $c^{Nq(c)}b$, and leaving the action of c and a unchanged. By Lemma 3.10, for N sufficiently large, $\mathrm{Per}^-(c^{Nq(c)}b)$ approaches $b^{-1}\,\mathrm{Per}^-(c)$, and $\mathrm{Per}^-(c^{-Nq(c)}b)$ approaches $b^{-1}\,\mathrm{Per}^+(c)$. Since ρ is path-rigid, the cyclic order of periodic points is invariant under these deformations, hence the points $b^{-1}(c^{\pm})$ all must lie in intervals of type (a^-, a^+).

Now up to replacing c with c^{-1} (its orientation is unimportant in this proof) we may assume that the order of periodic points given by Lemma 3.12 is $((a^-, b^-, a^+, c^+, b^+, c^-))_k$. Then $b^{-1}\,\mathrm{Per}^-(c)$ lies in the intervals of type (b^+, b^-), as b preserves these intervals. Thus, points of $b^{-1}\,\mathrm{Per}^-(c)$ are between consecutive points of $\mathrm{Per}^-(a)$ and $\mathrm{Per}^-(b)$. Similarly, the points $b^{-1}(c^+)$ are between consecutive points of the form b^- and a^+. □

The following variation is proved using the same style of argument.

Lemma 3.15 *Let $a, b, c \in \Gamma_g$ be three nonseparating curves such that $i(a,b) = -1$ and c is disjoint from $T(a,b)$. Up to reversing the orientation of S^1, we may suppose that the periodic points of a and b are in the order $((a^-, b^+, a^+, b^-))_k$. Then the periodic points of c all lie in intervals of type (b^-, a^-).*

Note that the order in which we prefer to take the periodic points of a and b is different here than in the two preceding statements, because here $i(a,b) = -1$.

Proof Similar to the proof of Proposition 3.14, we perform bending deformations. Since ρ is path-rigid, the cyclic order of periodic points does not change after the bending deformation replacing b with $a^{Nq(a)}b$ (leaving a and c unchanged). The effect of these deformations is to push $\mathrm{Per}^+(b)$ as close as we want to either $\mathrm{Per}^+(a)$ or $\mathrm{Per}^{-1}(a)$. Applying Lemma 3.10 as in the proof of Proposition 3.14 shows that periodic points of c cannot be in the intervals of type (a^-, b^+) or (b^+, a^+); as the argument is entirely analogous, we omit the details. The same argument again using the deformation replacing a by $b^{Nq(b)}a$ shows that the periodic points of c cannot be in the intervals of type (a^+, b^-), either. □

Proposition 3.16 *Let a and c be two nonseparating curves with $i(a,c) = 0$, and suppose c is not conjugate to a or a^{-1}. Then $\rho(a)$ and $\rho(c)$ are in k–Schottky position.*

Proof Up to changing the orientation of c, we may choose nonseparating curves b and d such that (a, b, c, d) is the beginning of a standard basis of $\pi_1\Sigma_g$.

Using a deformation as in Lemma 3.9, path-rigidity of ρ implies that the points of $\mathrm{Per}^-(d)$, $c^{-1}\,\mathrm{Per}^+(d)$, $\mathrm{Per}^-(b)$ and $a^{-1}\,\mathrm{Per}^+(b)$ are all distinct. Fix small disjoint neighborhoods U^+ of $\mathrm{Per}^-(d)$, U^- of $c^{-1}\,\mathrm{Per}^+(d)$, and also V^+ of $\mathrm{Per}^-(b)$, and V^- of $a^{-1}\,\mathrm{Per}^+(b)$.

By Lemma 3.10, $d^{-nq(d)}c(S^1 \setminus U^-) \subset U^+$ and $b^{-nq(b)}a(S^1 \setminus V^-) \subset V^+$ if n is large enough, so we may find $2k$ disjoint attracting and repelling intervals for $d^{-nq(d)}c$ and $b^{-nq(b)}a$ as in the definition of

k–Schottky. Now there exists a bending deformation that replaces c with $d^{-nq(d)}c$ and a with $b^{-nq(b)}a$, and it follows from Observation 2.16 that this deformation is conjugate to the original action. Thus, a and c are k–Schottky. □

Proposition 3.17 *Let a and c be two nonseparating curves with $i(a,c)=\pm 1$. Then $\rho(a)$ and $\rho(c)$ are in k–Schottky position.*

Proof Choose b and d so that (b,a,c,d) is a 4–chain. Now follow the proof above. □

From Proposition 3.16 we deduce an enhanced version of Lemma 3.12.

Proposition 3.18 *Let (a,b,c) be a completable directed 3–chain. Then, up to reversing the orientation of the circle, the periodic points of a, b and c are in cyclic order $((a^-,b^-,a^+,c^-,b^+,c^+))_k$.*

Proof By Lemma 3.15, we need only discard the possibility that the order is $((a^-,b^-,a^+,c^+,b^+,c^-))_k$. Suppose for contradiction that this order does hold. By Proposition 3.16, we know that a and c each have $2k$ intervals as in Definition 3.4, with pairwise disjoint closures. As $|\mathrm{Per}(a)|=|\mathrm{Per}(c)|=2k$, each of these intervals contains exactly one periodic point, so their cyclic order is specified by the order of periodic points given above.

Note that ca is nonseparating, as the 3–chain (a,b,c) is completable. Also, $\rho(ca)$ is k–savage, and we may take $I^-(ca)\subset I^-(a)$ and $I^+(ca)\subset I^+(c)$. With the same argument as above, $\rho(ca)$ has exactly one repelling periodic point in each interval of $I^-(ca)$, and one attracting periodic point in each interval of $I^+(ca)$.

If $\mathrm{Per}(b)$ is disjoint from $I^-(a)\cup I^+(c)$, then this is enough to imply that the periodic points of ca and b alternate, contradicting Lemma 3.12, since $i(ca,b)=0$. Thus, it only remains to prove that $\mathrm{Per}(b)$ can be made disjoint from $I^-(a)\cup I^+(c)$ to finish the proof. This can be done in the same manner as that of Proposition 3.16. First, complete (a,b,c) into a directed 5–chain (α,a,b,c,γ). Then, consider a bending deformation of ρ, where b is unchanged but the action of a is replaced by that of $a\alpha^{Nq(\alpha)}$ and the action of c by $\gamma^{Nq(\gamma)}c$ for N large. By Observation 2.16 this new action is conjugate to ρ. Now, provided N is large enough, we can choose our Schottky intervals to be as narrow as we want, around the points α^-, $a(\alpha^+)$, γ^+ and $c^{-1}(\gamma^-)$ which, using Lemma 3.9, are disjoint from $\mathrm{Per}(b)$. □

3.3 Euler number

As a consequence of the work in the previous section, we show that the Euler number of ρ agrees with a geometric representation.

Theorem 3.19 *Let ρ be path-rigid, minimal and satisfy S_k. Then $|\mathrm{eu}(\rho)|=(2g-2)/k$.*

In fact, we will show the following stronger statement, which implies Theorem 3.19 by additivity of the Euler number on subsurfaces.

Theorem 3.20 *Up to changing the orientation of the circle, for every pair-of-pants subsurface $P \subset \Sigma_g$, the relative Euler class of ρ on P is $-1/k$.*

Definition 3.21 Let $i(a,b)=1$. We say that the ordered pair (a,b) is of type $+$ if the periodic points of a and b are in the cyclic order $((a^-,b^-,a^+,b^+))_k$. Otherwise, we say that (a,b) is of type $-$.

As a consequence of Proposition 3.18, for every oriented, completable directed 3–chain (a,b,c), the pairs (a,b) and (b,c) have the same type. Thus, Lemma 2.4 implies that all one-holed tori have the same type. Thus, up to conjugating ρ by an orientation-reversing homeomorphism, we may suppose the type is always $+$.

Proof of Theorem 3.20 We begin by proving the claim for a pair of pants P such that at least two boundary components of P are nonseparating. Denote by a^{-1}, c^{-1} and ac the three boundary components of P, with the convention of Figure 3, and suppose that a and c are nonseparating. With these choices of orientations, the Euler number of ρ on P will be equal to $\widetilde{\mathrm{rot}}(\hat{a}\hat{c})-\widetilde{\mathrm{rot}}(\hat{a})-\widetilde{\mathrm{rot}}(\hat{c})$, and there exists a curve b such that (a,b,c) is an oriented, completable, directed 3–chain — the end of the proof of Observation 2.2 justifies the existence of such a curve b.

Since (a,b) is of type $+$, it follows from Proposition 3.18 that the periodic points of a and c are in cyclic order $((a^-,a^+,c^-,c^+))_k$; and by Proposition 3.16, they are in k–Schottky position, with Schottky intervals $I_j^\pm(a)$ and $I_j^\pm(c)$. Lift these to intervals $\tilde{I}_j^\pm(a)$ and $\tilde{I}_j^\pm(c)\subset\mathbb{R}$, indexed by integers, and in order

$$\dots\tilde{I}_j^-(a),\tilde{I}_j^+(a),\tilde{I}_j^-(c),\tilde{I}_j^+(c),\tilde{I}_{j+1}^-(a),\dots$$

such that the projection to S^1 is given by taking indices mod k. It follows easily from the definition of Savage (see also Observation 3.5) that $\hat{a}(\tilde{I}_j^+(a))\subset\tilde{I}_{j+\ell}^+(a)$ for some ℓ (which depends on a) and in this case $\ell/k=\widetilde{\mathrm{rot}}(\hat{a})$. An analogous statement holds also for c; let m/k denote its translation number.

Since a and c are in k–Schottky position, their product ac is k–savage, and we can take $I^-(ac)=I^-(c)$ and $I^+(ac)\subset I^+(a)$. Note that each of the k intervals of $I^+(ac)$ is contained in a different interval of $I^+(a)$. We now track images of intervals to compare translation numbers. Set the indexing of the intervals $\tilde{I}^\pm(ac)$ so that $\tilde{I}_1^+(a)=\tilde{I}_1^+(ac)$. This lies between $\tilde{I}_0^+(c)$ and $\tilde{I}_1^-(c)$, so we have

$$c(\tilde{I}_1^+(ac))\subset\tilde{I}_m^+(c),$$

and similarly, since $\tilde{I}_m^+(c)$ lies between $\tilde{I}_m^+(a)$ and $\tilde{I}_{m+1}^-(a)$, we have

$$ac(\tilde{I}_1^+(ac))\subset a(\tilde{I}_m^+(c))\subset\tilde{I}_{m+\ell}^+(a)=\tilde{I}_{m+\ell}^+(ac).$$

Thus, $k\cdot\widetilde{\mathrm{rot}}(\hat{a}\hat{c})=m+\ell-1=k\cdot\widetilde{\mathrm{rot}}(\hat{a})+k\cdot\widetilde{\mathrm{rot}}(\hat{c})-1$ and hence $k(\widetilde{\mathrm{rot}}(\hat{a}\hat{c})-\widetilde{\mathrm{rot}}(\hat{a})-\widetilde{\mathrm{rot}}(\hat{c}))=-1$, as desired.

This implies Theorem 3.19, as we can cut the surface Σ_g into pairs of pants whose boundary components are all nonseparating.

Now, if P is a pair of pants with possibly more than one separating boundary component, then $\Sigma_g \setminus P$ admits a pants decomposition whose pants all have at most one separating boundary component. The fact that the contribution of P to the Euler class of ρ is $-1/k$ is then a consequence of Theorem 3.19 and the additivity of the Euler class. □

3.4 Basic partitions and combinations

Fix disjoint, nonseparating curves $C_1, \dots, C_{3g-3}$ so that $\Sigma_g \setminus \left(\bigcup_i C_i\right)$ is a disjoint union of pairs of pants $P_1, \dots, P_{2g-2}$. For concreteness, the reader may use the decomposition suggested in Figure 4.

We briefly part from the convention for the presentation of $\pi_1\Sigma_g$ that was given in Section 2.1, and instead present $\pi_1\Sigma_g$ as the fundamental group of a graph of groups. Choose basepoints $x_i \in P_i$ and $y_j \in C_j$, identifying x_1 with the basepoint of $\pi_1\Sigma_g$. Also, choose paths in P_i from x_i to each basepoint of each boundary component of P_i. This collects all the basepoints of the pants and curves as the vertices of a graph G embedded in Σ_g; fix an orientation for each of its edges, and a spanning tree $T \subset G$. This data gives a *graph of groups*: the vertex (resp. edge) groups are the fundamental groups of the based pairs of pants (resp. curves), and for each edge C_j, the chosen paths define monomorphisms ϕ_j and ψ_j from $\pi_1 C_j \simeq \mathbb{Z}$ to the fundamental groups of the two adjacent (initial and final endpoints of the edge, respectively) pairs of pants. The Seifert–Van Kampen theorem then identifies $\pi_1\Sigma_g$ with the fundamental group of this graph of groups; this is the group generated by the union of the $\pi_1 P_i$, as well as one extra generator e_j for each edge that is not in T, subject to the relations that for each edge C_j (in T or not), and each $\gamma \in \pi_1 C_j$, we have $\phi_j(\gamma) = e_j^{-1}\psi_j(\gamma)e_j$ (taking $e_j = 1$ for the edges in T).

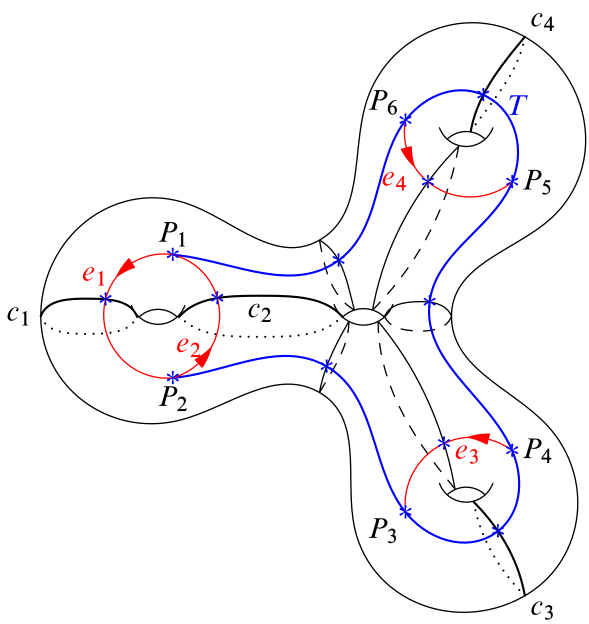

Figure 4: A decomposition of $\pi_1\Sigma_4$ into a graph of groups.

Our representation ρ gives rise to a representation of each $\pi_1(P_i)$, by using the spanning tree T to identify based curves in P_i with based curves in Σ_g. Similarly, each additional edge generator e_j can be identified with a closed, based loop in Σ_g, hence to an element $\rho(e_j)$.

We now define a geometric representation that will be our candidate for a representation semiconjugate to ρ. As a consequence of Theorem 3.20, $(2g-2)/k$ is an integer, hence a Fuchsian representation of Euler class $2g-2$ can be lifted to $\mathrm{PSL}_2^k(\mathbb{R})$. The choice of such a lift amounts to the choice of rotation numbers (in $(1/k)\mathbb{Z} \bmod \mathbb{Z}$) for the elements of a homology basis of $\pi_1\Sigma_g$. Let $c_1,\dots,c_g,e_1,\dots,e_g$ be the homology basis depicted in Figure 4, with c_j a generator of $\pi_1(C_j)$. Thus, as just observed, there exists a geometric representation ρ_0 with the same Euler class as ρ, and with $\mathrm{rot}(\rho_0(\gamma)) = \mathrm{rot}(\rho(\gamma))$ for each γ in $\{c_1,\dots,c_g,e_1,\dots,e_g\}$. This also holds for each $\gamma \in \{c_{g+1},\dots,c_{3g-3}\}$. Indeed, the contribution of the Euler class of ρ and ρ_0 on each pairs of pants are equal, and they are sums of rotation numbers, so we can propagate these equalities to the whole family of cutting curves.

To show ρ and ρ_0 are semiconjugate, thereby concluding the proof, we use (an adaptation of) Matsumoto's theory of basic partitions and combinations.

Definition 3.22 (Matsumoto [29]) Let Γ be a group generated by a finite symmetric set S, and let $\rho\colon \Gamma \to \mathrm{Homeo}^+(S^1)$. A *basic partition* (*BP*) for $\rho(\Gamma)$ is a collection P of disjoint closed intervals of S^1 such that

(i) for each $I \in P$, there is a unique $s_I \in S$ such that $\rho(s_I)(I)$ is a union of $m = m(I)$ elements of P and $m-1$ complementary intervals to P,

(ii) for any $s \neq s_I$ in S, the image $\rho(s)(I)$ is a proper subset of an element of P, and

(iii) for any complementary interval J to P and $s \in S$, either $\rho(s)(I)$ is contained in the interior of P, or is a complementary interval to P.

Following the last condition, we may put the complementary intervals to P into a directed graph, with an edge from J_1 to J_2 if there is a generator sending J_1 to J_2. A basic partition is called *pure* if this graph consists of disjoint nontrivial cycles.

Applying this to our context, for each pair of pants P_i, choose two "preferred" boundary components as generators for $\pi_1 P_i$ (identified with a subgroup of Γ_g via T). Let a_i^{-1} and c_i^{-1} denote these elements, and consider their images under ρ. The proof of Theorem 3.20 shows that the periodic points of a_i, c_i, $c_i a_i$ and $a_i c_i$ are in the cyclic order

$$((a_i^-, a_i^+, (a_i c_i)^+, (a_i c_i)^-, c_i^-, c_i^+, (c_i a_i)^+, (c_i a_i)^-))_k$$

and that the $4k$ intervals of types $(a_i^+, (a_i c_i)^+)$, $((a_i c_i)^-, c_i^-)$, $(c_i^+, (c_i a_i)^+)$ and $((c_i a_i)^-, a_i^-)$ form a *pure basic partition* for the action of $\pi_1 P_i$ on the circle with respect to the symmetric generating set $(a_i, c_i, a_i^{-1}, c_i^{-1})$. This conclusion rested only upon rigidity and the hypothesis S_k, and the combinatorics of the BP (the images of intervals and complementary intervals following conditions (i)–(iii) of the definition) depends only on the rotation numbers of the generators. Thus, ρ_0 admits a basic partition with the same combinatorics as ρ, ie there exists a cyclic-order-preserving map sending the basic partition of one to the other, which intertwines the two actions. In this case, [29, Theorem 4.7] states that the restrictions of ρ and ρ_0 to $\pi_1 P_i$ are semiconjugate.

It remains to improve this to a global semiconjugacy between ρ and ρ_0. With the notation above, in a pair of pants P_i, let J_a (resp. J_c, resp. J_{ac}) denote the union of all intervals of type (a_-, a_+) (resp. (c_-, c_+), resp. $((ac)_+, (ac)_-)$. These are called the *entries* of the basic partition described above; their stabilizers in $\pi_1 P_i$ are the cyclic groups generated by a, c and ac, respectively.

Now consider an edge e_j of G (in T or not). It serves to conjugate one generator of $\pi_1 P_i$ for some i, a^{-1}, c^{-1} or ac, into *the inverse* of the corresponding generator of this boundary component on the adjacent pair of pants. It follows that if, say, a_i and $a_{i'}$ are the generators of $\pi_1 P_i$ and $\pi_1 P_{i'}$ on each side of an edge e_j, then the sets J_{a_i} and $\rho(e_j)(J_{a_{i'}})$ form a partition of S^1, up to the finitely many periodic points of a_i. In this situation, Matsumoto says that the two entrances J_{a_i} and $J_{a_{i'}}$ are *combinable*. More generally, given a graph of groups decomposition of a group Γ as ours, and pure basic partitions for each vertex group that have combinable entrances for every edge, Matsumoto says the collection of all basic partitions for the vertex groups form a *basic configuration* for the action $\rho(\Gamma)$ on the circle. (Matsumoto works with trees of groups; but this definition generalizes immediately to the graph setting.)

As we already argued for the $\pi_1 P_i$, the equalities between rotation numbers of ρ and ρ_0 on the curves C_i and on the edge elements e_j imply that they admit basic configurations with the same combinatorics; in other words there exists a cyclic-order-preserving bijection which maps the basic partitions of ρ to those of ρ_0, intertwining the actions.

Matsumoto's main result [29, Theorem 6.7] is that a cyclic-order-preserving bijection between basic configurations can be promoted to a semiconjugacy between ρ and ρ_0. We comment briefly on the proof. To produce a semiconjugacy, it suffices to show that some orbit of ρ and some orbit of ρ_0 are in the same cyclic order. Matsumoto's proof strategy begins by showing this property holds for elements of vertex groups (ie of some $\pi_1 P_i$) — this is the content of [29, Theorem 4.7] cited above. He then proceeds with elements of the form $\gamma_i e_j \gamma_{i'}$ (where $\gamma_i \in P_i$ and $\gamma_{i'} \in P_{i'}$ belong to adjacent pairs of pants), then of the form $\gamma_{i_3} e_{j_2} \gamma_{i_2} e_{j_1} \gamma_{i_1}$, and so on, inductively. While his proof is not carried out in the language of Bass–Serre theory, and the context is specialized to a tree of groups decompositions of $\pi_1 \Sigma_g$, the arguments adapt without modification.

4 Periodic considerations

The content of this section is the proof of the following two statements.

Proposition 4.1 *If a representation $\Gamma_g \to G$ is path-rigid, then all nonseparating simple closed curves have rational rotation number.*

Theorem 4.2 *Suppose ρ is path-rigid and minimal. Then, for all a, b with $i(a,b) = \pm 1$, we have the implication*

$$\mathrm{Per}(a) \cap \mathrm{Per}(b) = \varnothing \implies S_k(a,b) \text{ for some } k.$$

Proof of Proposition 4.1 Suppose for contradiction that there exists a nonseparating simple curve a with $\rho(a) \notin \mathbb{Q}$. After semiconjugacy, we may assume that ρ is minimal. If $\rho(a)$ is conjugate into SO(2), then it lies in a one-parameter subgroup a_t of rotations, and for any b with $i(a,b) = 1$, the bending deformation $a_t\rho(b)$ has nonconstant rotation number, contradicting rigidity. Thus, $\rho(a)$ has an invariant minimal Cantor set, which we denote by K. We next show that K is $\rho(b)$–invariant, for any curve b with $i(a,b) = 1$. This suffices to prove the proposition since Γ_g is generated by $\{a\} \cup \{b \mid i(a,b) = 1\}$, whence K is $\rho(\Gamma_g)$–invariant, contradicting minimality of ρ.

To show invariance, suppose for contradiction that $\rho(b)(K) \not\subset K$; the case where $\rho(b^{-1})(K) \not\subset K$ is analogous. Let $K' \subset K$ be the set of two-sided accumulation points of K. Since $\overline{K'} = K$, there exists $x \in K'$ such that $\rho(b)(x) \notin K$. Let I be the connected component of $S^1 \setminus K$ containing $\rho(b)(x)$. Minimality of the action of $\rho(a)$ on K implies there exists $N \in \mathbb{Z}$ such that $\rho(a)^N(I) \subset \rho(b)^{-1}(I)$, and in particular $\mathrm{rot}(\rho(a^N b)) = 0$. We work now with the pair $(a, a^N b)$ with intersection number ± 1. Let β_t be a positive one-parameter family commuting with $\rho(a^N b)$. Since $\rho(a^N b)$ does not preserve K, we can find a connected component J of $S^1 \setminus \mathrm{Fix}(\rho(a^N b))$ such that $J \cap K' \neq \varnothing$, and then find $M \in \mathbb{Z}$ such that $\rho(a)^M(J) \cap J \neq \varnothing$.

Let $\tilde{x} \in \mathbb{R}$ be a lift of a point in $\rho(a)^M(J) \cap J$. Adapting the notation from Section 2.2.3, set

$$\Delta(\tilde{x}, t_1, \dots, t_M) = \widehat{\beta_{t_M}\rho(a)} \circ \cdots \circ \widehat{\beta_{t_1}\rho(a)}(\tilde{x}) - \tilde{x} - k,$$

where k is chosen so that $\widehat{\rho(a)}^M(\tilde{J}) \cap (\tilde{J} + k) \neq \varnothing$ for any lift of J, and we set $\delta(\tilde{x}, t) = \Delta(\tilde{x}, t, \dots, t)$. Up to reversing orientation, we can suppose that $\delta(\tilde{x}, 0) > 0$. Since $\tilde{J}$ contains both $\tilde{x}$ and $\widehat{\rho(a)}^M(\tilde{x})$, there exists $t < 0$ such that $\Delta(\tilde{x}, 0, \dots, 0, t) < 0$, hence $\delta(\tilde{x}, t) < 0$. Thus, there exists t_0 such that $\delta(\tilde{x}, t_0) = 0$, hence $\mathrm{rot}(\rho_{t_0}(a)) = k/M \in \mathbb{Q}$, contradicting rigidity. □

4.1 Proof of Theorem 4.2

For this subsection, we assume ρ is path-rigid, $i(a,b) = \pm 1$, and $\mathrm{Per}(a) \cap \mathrm{Per}(b) = \varnothing$. Recall from Proposition 4.1 that $\mathrm{Per}(a)$ and $\mathrm{Per}(b)$ are nonempty. We will first establish some properties that do not use minimality, so are robust under deformations of ρ. We add the hypothesis that ρ is minimal only at the end of the proof.

Borrowing notation from the previous section, say that a connected component of $S^1 \setminus (\mathrm{Per}(a) \cup \mathrm{Per}(b))$ is of type (x, y) if it is bounded to the left by a point of $\mathrm{Per}(x)$ and to the right by a point of $\mathrm{Per}(y)$, for $x, y \in \{a, b\}$.

Definition 4.3 Let X_a denote the set of connected components of $S^1 \setminus \mathrm{Per}(a)$ that contain points of $\mathrm{Per}(b)$. We say an element I of X_a is *positive* if $a^{q(a)}$ is increasing on the interval I, and *negative* otherwise.

The set X_b and its positive and negative elements are defined by reversing the roles of a and b above. Since each (a,b) interval in $S^1 \setminus (\mathrm{Per}(a) \cap \mathrm{Per}(b))$ is followed by a collection — perhaps empty — of (b,b) intervals, and then a (b,a) interval, and $\mathrm{Per}(a)$ and $\mathrm{Per}(b)$ are disjoint closed sets, there exists an integer $m = m(\rho) \geq 1$ such that S^1 contains exactly m intervals of type (a,b) and m intervals of type (b,a), alternating around the circle, and thus $|X_a| = |X_b| = m(\rho)$. By Remark 2.18, m depends only on the semiconjugacy class of ρ.

Lemma 4.4 *The set X_a is $\rho(a)$–invariant, and the subset of positive (resp. negative) intervals in X_a is also $\rho(a)$–invariant.*

Proof Let $I \in X_a$ be a positive interval; we show that its image under a is another positive interval in X_a. The negative case is analogous. Since $a(I)$ is an interval between two consecutive points of $\mathrm{Per}(a)$ on which $a^{q(a)}$ is increasing, we need only show that $a(I) \cap \mathrm{Per}(b) \neq \varnothing$.

Suppose for contradiction that $a(I) \cap \mathrm{Per}(b) = \varnothing$. Then $a(\bar{I}) \subset J$ for some $J \in X_b$. Let b_t be a positive one-parameter family commuting with b, let $x \in I \cap \mathrm{Per}(b)$, and take lifts $\tilde{x} \in \tilde{I}$ of x and I to $\mathbb{R}$. Positivity implies $\delta_{b,a}(x,0) > 0$. If $t < 0$ is negative enough that $b_t(a(I)) \cap a(I) = \varnothing$, then we have $\widehat{b_t}(\hat{a}(\tilde{x})) < \hat{a}(\tilde{I})$; it follows that $\delta_{b,a}(x,t) < 0$. Therefore, there exists $t_0 \in \mathbb{R}$ such that $\delta_{b,a}(x,t_0) = 0$, ie $x \in \mathrm{Per}(b_{t_0}a) \cap \mathrm{Per}(b)$. This contradicts path-rigidity via Lemma 2.17. □

Obviously, reversing the roles of a and b above shows the positive and negative intervals of X_b are b–invariant. The next lemma shows X_a and X_b are invariant under particular bending deformations.

Lemma 4.5 *Let b_t be a positive one-parameter family commuting with b. For $t \in \mathbb{R}$, let $X_b(t)$ denote the set of connected components I of $S^1 \setminus \mathrm{Per}(b)$ such that $I \cap \mathrm{Per}(b_t a) \neq \varnothing$. Then $X_b(t) = X_b(0)$ for all t.*

Proof Let $X_b(t)$ be as in the statement of the lemma and let $X_a(t)$ denote the set of connected components of $S^1 \setminus \mathrm{Per}(b_t a)$ containing points of $\mathrm{Per}(b)$. By our discussion above, path-rigidity of ρ implies that the cardinality of $X_b(t)$ is constant. Let $K_a = \{(x,t) \in S^1 \times \mathbb{R} \mid x \in \mathrm{Per}(b_t a)\}$, and $K_b = \mathrm{Per}(b) \times \mathbb{R}$. These are closed, disjoint sets, and their intersections with each horizontal slice $S^1 \times \{t\}$ are the periodic sets of $b_t a$ and b, respectively.

For each connected component $I \subset S^1 \setminus \mathrm{Per}(b)$, we set

$$T_I = \{t \in \mathbb{R} \mid I \in X_b(t)\} = \{t \in \mathbb{R} \mid I \cap \mathrm{Per}(b_t a) \neq \varnothing\}.$$

Note that T_I is the projection of $K_a \cap (\bar{I} \times \mathbb{R})$ onto the $\mathbb{R}$–factor, so in particular is closed. We claim T_I is also open. To see this, let $t_0 \in T_I$, and let $I_2, \dots, I_m$ be the other components of $S^1 \setminus \mathrm{Per}(b)$ such that $t_0 \in T_{I_j}$. If $d > 0$ is the distance (for the product metric) between the disjoint compact sets $(S^1 \times [t_0-1, t_0+1]) \cap K_a$ and $(S^1 \times [t_0-1, t_0+1]) \cap K_b$, let $I_{m+1}, \dots, I_N$ be the remaining connected components of $S^1 \setminus \mathrm{Per}(b)$ of length $\geq d$. Any component J of shorter length tautologically satisfies $T_J \cap [t_0-1, t_0+1] = \varnothing$. Since the sets T_{I_j} are closed, there exists $\varepsilon > 0$ such that $(t_0-\varepsilon, t_0+\varepsilon) \cap T_{I_j} = \varnothing$ for all $j \geq m+1$, hence $(t_0-\varepsilon, t_0+\varepsilon) \subset T_I$, for otherwise $|X_b(t)|$ would fail to be constant. This proves that T_I is open, hence equal to $\varnothing$ or $\mathbb{R}$, and the intervals in $X_b(t)$ do not depend on t. □

The next two lemmas establish some properties of a and b which are, in particular, held by pairs of homeomorphisms semiconjugate to hyperbolic elements of $\mathrm{PSL}_2^k(\mathbb{R})$ satisfying $S_k(a,b)$. Of course, both lemmas also hold with the roles of a and b exchanged.

Lemma 4.6 *Any two consecutive intervals of X_a have opposite sign. In particular, $m(\rho) = 2k$ for some $k \geq 1$.*

Proof Let b_t be a positive one-parameter family commuting with $\rho(b)$. Suppose for contradiction that X_a has two successive positive intervals I_1 and I_2 (the negative case is analogous). Let $I \in X_b$ be the interval such that $I_1 \cap I \neq \varnothing$ and $I_2 \cap I \neq \varnothing$. Take $x \in I_1 \setminus I$ such that $a^{q(a)}(x) \in I$. For t large enough, we have $a^{q(a)} b_t a^{q(a)}(x) \in I_2 \setminus I$. Since b_t has positive dynamics, it follows that $(b_t a^{q(a)})^2$ moves every point of I to the right; thus, $\Delta_{b,a}(y, 0, \dots, 0, t) > 0$ for all $y \in I$, and $\mathrm{Per}(b_t a) \cap I = \varnothing$ for t large enough. But this contradicts Lemma 4.5. □

Lemma 4.7 *Let $I \in X_b$ have left endpoint in a positive interval of X_a. Then $a(I) \subset J$ for some $J \in X_b$. If, instead, $I \in X_b$ has left endpoint in a negative interval of X_a, then $a^{-1}(I) \subset J$ for some $J \in X_b$.*

Note that Lemma 4.6 implies that, in both cases, J is a positive interval of X_b if and only if I is.

Proof Let $x_1, x_2, \dots, x_6$ be points in cyclic order such that (x_1, x_3) and (x_4, x_6) are consecutive (positive and negative, respectively) intervals in X_a, and $I = (x_2, x_5) \in X_b$. Let $y_i = a(x_i)$ for $i = 1, 3, 4, 6$. Then (y_1, y_3) and (y_4, y_6) are in X_a, and both intersect some interval of X_b, say (y_2, y_5). The statement of the lemma is that $a(x_5) \leq y_5$ and $a(x_2) \geq y_2$.

Similar to the proof of Lemma 4.4, we assume the contrary and find a deformation with a common periodic point for a and b. Suppose $a(x_5) > y_5$ (the proof of the other inequality is symmetric), and choose a positive one-parameter family b_t commuting with b. Since $a^{-1}(y_5) \in (x_2, x_5)$, there is $t \in \mathbb{R}$ with $b_t a^{-1}(y_5) \in (x_1, x_3)$. As (y_1, y_3) is $a^{q(a)}$–invariant, it follows that $a^{-q(a)+1} b_t a^{-1}(y_5) < y_5$, ie $\Delta_{b,a}(y_5, 0, \dots, 0, t, 0) > 0$. On the other hand, as (y_4, y_6) is a negative interval of X_a, we have $\delta_{b,a}(y_5, 0) < 0$. Thus, there exists $t_0 \in \mathbb{R}$, such that $y_5 \in \mathrm{Per}(b_{t_0} a)$. Since $y_5 \in \mathrm{Per}(b)$, this contradicts path-rigidity by Lemma 2.17. The statement concerning $\rho(a)^{-1}$ is symmetric, and proved in the same manner. □

Now we state a lemma of purely technical nature, that will allow us to compress the periodic sets in each interval of X_a or of X_b to singletons. In the statement and proof, we let $\tau_t : \mathbb{R} \to \mathbb{R}$ denote the translation $x \mapsto x + t$.

Lemma 4.8 *Let $n \geq 1$, and for all $i = 1, \dots, n$, let f_i be an increasing homeomorphism from $\mathbb{R}$ to some interval $(a_i, b_i) \subset \mathbb{R}$. Assume that $a_i > -\infty$ for at least one i, and that $b_j < +\infty$ for at least one j. For all $t \in \mathbb{R}$, we set $F_t = \tau_t \circ f_n \circ \cdots \circ \tau_t \circ f_1$. Then there exists a subset $N \subset \mathbb{R}$ of finite Lebesgue measure and consisting of a countable union of segments, such that for all $t \notin N$, the map F_t admits a unique fixed point in $\mathbb{R}$.*

The statement of this lemma came from our attempt to better understand the argument in the first four lines of [9, page 644]. In particular, the case $n = 1$ gives an alternative end to the proof of [9, Lemma 2.7]. We defer the proof of Lemma 4.8 to the next paragraph, and use it now to finish the proof of Theorem 4.2.

Proof of Theorem 4.2 Assume now that ρ is minimal. Let b_t be a positive one-parameter family commuting with b. We will first find t such that $b_t a$ has exactly $2k$ periodic points; the conclusion will then follow easily.

Let X_a^+ denote the set of positive intervals of X_a. As observed in Lemma 4.4, $\rho(a)$ induces a permutation of X_a^+; which has, say, ℓ orbits, all of cardinality $n = k/\ell$. Fix an interval $I_0 \in X_b$ whose left endpoint lies in an element of X_a^+. Successive applications of Lemma 4.7, for $j = 1, 2, \dots n-1$, gives $\rho(a)^j(I_0) \subset I_j$ for some $I_j \in X_b$. Also, $\rho(a)^n(I_0) \subset I_0$ because $\rho(a)^n$ fixes X_a^+. Note that there exists some j such that $\rho(a)(I_{j-1}) \subset I_j$ is a strict inclusion at the left endpoint (and similarly, another for the right endpoint) as otherwise some endpoint of I_0 would lie in $\mathrm{Per}(a) \cap \mathrm{Per}(b)$.

For each j, let $\phi_j : I_j \to \mathbb{R}$ be a homeomorphism such that $\phi_j \circ b_t \circ \phi_j^{-1} = \tau_t$, and for $j \in \{1, \dots, n\}$ set $f_j = \phi_{j+1} \circ a \circ \phi_j^{-1}$, using cyclic notation for the last index. Then Lemma 4.8 applies, giving a set $N_{I_0} \subset \mathbb{R}$ of finite Lebesgue measure, such that for all $t \notin N_{I_0}$, $(b_t a)^n = \phi_1^{-1} \circ F_t \circ \phi_1$ has a unique fixed point in I_0.

We repeat this procedure for each element I of X_b, using a^{-1}, instead of a for the intervals of X_b whose left endpoint lies in an element of X_a^-. The resulting, finitely many, sets N_I, each of finite Lebesgue measure, cannot cover $\mathbb{R}$, hence there exists $t \in \mathbb{R}$ such that each element of X_b intersects $\mathrm{Per}(b_t a)$ as a singleton. By Lemma 4.5, $\mathrm{Per}(b_t a) \subset X_b$, hence $b_t a$ has exactly $2k$ periodic points. As $b_t a$ is obtained by a bending deformation that does not change the dynamics of a, by Lemma 4.6 these $2k$ periodic points have alternating attracting and repelling dynamics. One may now repeat the same procedure reversing the roles of a and b, to obtain a further deformation where b has exactly $2k$ periodic points, alternately attracting and repelling. Minimality of ρ and Observation 2.15 implies the original action of $\rho(a)$ and $\rho(b)$ also had this dynamics. □

Proof of Lemma 4.8 We suggest the reader take $n = 1$ at first reading, as the argument is less technical in that case. We will show that there exists a countable union of segments, $N_+ \subset \mathbb{R}_+$, of finite Lebesgue measure, such that F_t has a unique fixed point for all $t \in \mathbb{R}_+ \setminus N_+$. The case for $t < 0$ is symmetric and left to the reader.

Let j be an index such that $b_j < +\infty$. Let $A_t = \tau_t \circ f_j \circ \cdots \circ \tau_t \circ f_1$, and let $B_t = \tau_t \circ f_n \circ \cdots \circ \tau_t \circ f_{j+1}$. For fixed t, both maps A_t and B_t are homeomorphisms to their images so $F_t = B_t \circ A_t$ has a unique fixed point x if and only if $A_t \circ B_t$ has a unique fixed point (namely, $B_t(x)$). In other words, we may suppose without loss of generality that $j = n$.

Let $G(t,x) = F_t(x) - t$. Then G is strictly increasing in x, and increasing (strictly, if $n \geq 2$) in t. The monotonicity of G, and the assumptions $\sup(a_j) > -\infty$ and $b_n < +\infty$, imply that the range of the map $G\colon \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$ is a bounded interval, say (a_0, b_0), where $b_0 = b_n$.

If $x \geq b_0$, the map $t \mapsto F_t(x)$ is a homeomorphism between $\mathbb{R}_{\geq 0}$ and $[F_0(x), +\infty)$, and

$$F_0(x) = G(0,x) < b_0.$$

Hence, there is a unique $t = T(x)$ such that $F_t(x) = x$. This defines a function $T\colon [b_0, +\infty) \to (0, +\infty)$.

Sublemma 4.9 *The map T satisfies the following inequalities*:

(T1) *For every $x \in [b_0, +\infty)$, we have $a_0 < x - T(x) < b_0$.*

(T2) *For all $x_1, x_2 \in [b_0, +\infty)$ such that $x_1 < x_2$, we have*

$$f_1(x_1) - f_1(x_2) < T(x_2) - T(x_1) < x_2 - x_1.$$

In particular, T is continuous, at bounded distance from the identity, and its rate of increase is bounded above by 1.

The proof of Sublemma 4.9 is a straightforward consequence of the definition of T, the defining identity $F_{T(x)}(x) = x$, and monotonicity of G. We leave it as an exercise, noting for (T2) that the first inequality is trivially satisfied if $T(x_2) \geq T(x_1)$, and the second if $T(x_2) \leq T(x_1)$.

For the next step, define a map $H\colon \mathbb{R}_{\geq b_0} \to [T(b_0), +\infty)$ by

$$H(x) = \sup\{T(x') \mid x' \leq x\}.$$

The reader may verify that H is continuous, surjective, and for all $A \geq T(b_0)$, the set $H^{-1}(A)$ is a segment of the form $[a,b]$ (possibly $a = b$), with $T(a) = T(b) = A$.

Now let $W = \{w \in [T(b_0), +\infty) \mid H^{-1}(w) \text{ is not a singleton}\}$, and for all $w \in W$ denote $H^{-1}(w)$ by $[a_w, b_w]$. Since these segments are disjoint and of positive length, W is countable. By definition of H, we have $F_w(a_w) = a_w$, ie $G(w, a_w) + w = a_w$; and the same holds for b_w in place of a_w. Thus, the segment $[G(w,a_w), G(w,b_w)]$ has the same length $b_w - a_w$. The reader may now easily deduce from monotonicity of G that these segments are disjoint; as they are contained in $[a_0, b_0]$, this implies $\sum_{w \in W} b_w - a_w \leq b_0 - a_0$.

Finally, for all $w \in W$, define $N_w := [w - (b_w - a_w), w]$, and define

$$N_+ = [0, b_0 - a_0] \cup \bigcup_{w \in W} N_w.$$

This may not be a disjoint union, but the remarks above imply this countable union of segments has finite Lebesgue measure. Hence, the proof of Lemma 4.8 amounts to the following sublemma.

Sublemma 4.10 *For all $t \in \mathbb{R}_{\geq 0} \setminus N_+$, the map F_t has a* ***unique*** *fixed point.*

Proof Let $t > b_0 - a_0$ be such that F_t has at least two distinct fixed points, say x_1, x_2 with $x_1 < x_2$. By definition, these satisfy $G(t, x_i) + t = x_i$. Since $G(t, x) > a_0$ for all x, and $t > b_0 - a_0$, this implies $x_1, x_2 \in [b_0, +\infty)$. By definition of T, we have $T(x_1) = T(x_2) = t$. Let $x_0 = \min\{x \leq x_2 \mid T(x) = H(x_2)\}$. Then $x_0 < x_2$. Indeed, if $H(x_2) = t$ then $x_0 \leq x_1$, and if $H(x_2) > t$ then the maximum $H(x_2)$ is reached at some point to the left of x_2. Thus, $x_0 = a_w$ for some $w \in W$, and we also have $b_w \geq x_2$.

We claim now that $t \in N_w$. Since $x_2 \leq b_w$, by definition of H we have $w = H(b_w) \geq t = T(x_2)$. Applying inequality (T2) to x_2 and b_w now gives $w - t \leq b_w - x_2$, so $w - t \leq b_w - a_w$, hence $t \geq w - (b_w - a_w)$. Thus we indeed have $t \in N_w$. □

This concludes the proof of Lemma 4.8. □

5 Proof of Theorem 1.6

In this section we finish the proof of the main result for path-rigid representations, showing that a path-rigid representation ρ of Γ_g is either geometric, or has Euler class zero and a genus $g-1$ subsurface whose fundamental group has finite orbit under ρ. (We believe the latter case cannot actually occur.) As in Section 3, we will frequently drop the notation ρ when the context is clear, using a to denote $\rho(a)$.

Recall from the introduction that, if ρ is a given representation and $T \subset \Sigma_g$ is a one-holed torus, we say that T is a *good torus* if it contains a nonseparating simple closed curve a with $\operatorname{rot}(a) = 0$, and that T is *bad* otherwise. We say T is *very good* if $\pi_1(T)$ has a finite orbit in S^1.

Note that very good implies good: if $T(a, b)$ is very good, then $\operatorname{rot}\colon \pi_1(T) \to \mathbb{R}/\mathbb{Z}$ is a homomorphism onto a finite subgroup, so if $0 \neq |\operatorname{rot}(a)| \leq |\operatorname{rot}(b)| < 1$, one may find n such that $|\operatorname{rot}(a^n b)| < |\operatorname{rot}(a)|$. Iterating this process produces a simple closed curve with rotation number zero.

Assumption 5.1 For the remainder of this section, we assume $\rho\colon \Gamma_g \to \operatorname{Homeo}^+(S^1)$ is path-rigid.

5.1 Bad tori

This subsection contains the proof of Proposition 1.10: under Assumption 5.1 we show that if Σ_g contains a bad torus T, then $\Sigma_g \setminus T$ contains only very good tori.

Definition 5.2 Let $f, g \in \operatorname{Homeo}^{\mathbb{Z}}(\mathbb{R})$. We say that g *dominates* f, and write $f < g$, if $f(x) < g(x)$ for all $x \in \mathbb{R}$.

Note that $<$ is a left- and right-invariant partial order on $\mathrm{Homeo}^{\mathbb{Z}}(\mathbb{R})$, and satisfies the following obvious properties:

(1) For all $f, g \in \mathrm{Homeo}^{\mathbb{Z}}(\mathbb{R})$, $f > g \iff f^{-1} < g^{-1}$.

(2) For all $f \in \mathrm{Homeo}^+(S^1)$, $\hat{f} > \mathrm{Id} \iff \mathrm{rot}(f) \neq 0$.

(3) For all $f, g \in \mathrm{Homeo}^{\mathbb{Z}}(\mathbb{R})$,

$$f < g \implies \widetilde{\mathrm{rot}}(f) \leq \widetilde{\mathrm{rot}}(g) \quad \text{and} \quad (f < g \text{ or } g < f) \iff \widetilde{\mathrm{rot}}(f^{-1}g) \neq 0.$$

Property (2) uses the notation $\hat{f}$ from Notation 2.6, which is also adopted throughout this section. The following easy observation will be handy; it follows directly from property (3) above.

Observation 5.3 *Let $f, g \in \mathrm{Homeo}^{\mathbb{Z}}(\mathbb{R})$. Suppose that $\widetilde{\mathrm{rot}}(f) < \widetilde{\mathrm{rot}}(g)$ and also that $\widetilde{\mathrm{rot}}(g^{-1}f) \neq 0$. Then $f < g$.*

Building on this observation, we have the following.

Lemma 5.4 *Let (a, b) be standard generators of a bad torus T. Then there exist integers m and n, unique and well defined modulo $q(a)$, with $(n-m)p(a) = 1 \mod q(a)$, and such that for all j not divisible by $q(a)$, we have $\widehat{a^n b} < \widehat{a^j}$, and $\widecheck{a^j} < \widecheck{a^m b}$. Moreover, if $p(a) = 1$, then we have $\widehat{a^n b}^2 < \hat{a}$, or $\widecheck{a^{n-1}b^{-2}} < \hat{a}$, or both.*

Assumption 5.1 is used in the proof only to guarantee that all nonseparating simple closed curves have rational rotation number (Proposition 4.1).

Proof Let F be a finite orbit of a. If there exists some point $x \in F \cap b^{-1}(F)$, then there exists $N \geq 0$ such that $\rho(a)^N \rho(b)(x) = x$, thus $\mathrm{rot}(a^N b) = 0$, contradicting the fact that T was bad. Thus, $F \cap b^{-1}(F) = \varnothing$.

Now we claim that F and $b^{-1}(F)$ alternate. Suppose for contradiction that some connected component $I = (x_1, x_2)$ of $S^1 \setminus F$ contains at least two points of $b^{-1}(F)$. Let $y_1 \in b^{-1}(F)$ be the leftmost point of $b^{-1}(F)$ in I, and y_2 be the second leftmost such point. Then there exists $N > 0$ such that $a^N b(y_1) = x_1$. It follows that $a^N b(y_2) = x_2$ and $(a^N b)^{-1}(I) = (y_1, y_2) \subset I$, so $\mathrm{rot}(a^N b) = 0$, giving the desired contradiction.

Now that we know these sets alternate, choose $x \in b^{-1}(F)$, and let $y_\ell, y_r \in F$ be the left and right endpoints of the component of $S^1 \setminus F$ containing x. Then there exists a unique pair $(n, m) \in \{0, \dots, q(a)-1\}^2$ such that $a^n b(x) = y_r$ and $a^m b(x) = y_\ell$. In particular, $(n-m)p(a) = 1 \mod q(a)$. These m, n are obviously the only candidates, modulo $q(a)$, for the dominations $\widehat{a^n b} < \widehat{a^j}$ and $\widecheck{a^m b} > \widecheck{a^{-j}}$, for an integer j such that $a^j(y_\ell) = y_r$. (This shows m and n do not depend on F.) We claim that this pair (n, m) satisfies the statement of the lemma.

To see this, lift F to $\widetilde{F} \subset \mathbb{R}$ and let $x_1 < x_2 < \cdots < x_{q(a)}$ be consecutive points of $\widetilde{F}$. Then $\widehat{a^n b}(x_i) \leq x_{i+1}$ for all i, hence $\widetilde{\mathrm{rot}}(\widehat{a^n b}^{q(a)}) \leq 1$ and $\widetilde{\mathrm{rot}}(\widehat{a^n b}) \leq 1/q(a)$. Also, for any integer j not divisible by $q(a)$ we have $\widetilde{\mathrm{rot}}(\widehat{a^n b}) \le \widetilde{\mathrm{rot}}(\widehat{a^j})$. Since T is bad, $\widetilde{\mathrm{rot}}(\widehat{a^j}^{-1}\widehat{a^n b}) \neq 0$, so we must have $\widehat{a^n b} < \widehat{a^j}$ by Observation 5.3. An essentially identical argument shows that $\widetilde{a^m b} > \widetilde{a^j}$.

It remains only to prove the statement regarding the case $p(a) = 1$, where $n-1 = m \bmod q(a)$. We know that $\widehat{a} > \widehat{a^n b}$ and $\widehat{a} > \widehat{b^{-1}a^{1-n}} = \widetilde{a^{n-1}b^{-1}}$, and this immediately implies $\widehat{a} = \widehat{a^n b} \cdot \widehat{b^{-1}a^{1-n}}$. As $(a, a^n b)$ and hence $(b^{-1}a^{1-n}, a^n b)$ are also standard generating sets of $\pi_1(T)$, we must either have $\widehat{b^{-1}a^{1-n}} > \widehat{a^n b}$, or $\widehat{b^{-1}a^{1-n}} < \widehat{a^n b}$, otherwise the nonseparating simple closed curve $a^{n-1}ba^n b$ would have rotation number zero. The statement follows. □

As a consequence, we have the following.

Proposition 5.5 *Let (a,b) be a standard generating set for a bad torus. Let $(a_k, b_k)_{k\ge 0}$ be the sequence of standard generating sets, defined inductively as follows.*

- *Define $(a_0, b_0) = (a, b)$.*
- *If k is even, let $a_{k+1} = a_k$ and $b_{k+1} = a_k^{n(k)} b_k$, where $0 \le n(k) \le q(a_k) - 1$ is the integer given by Lemma 5.4 applied to the generators (a_k, b_k).*
- *If k is odd, let $b_{k+1} = b_k$ and $a_{k+1} = b_k^{n(k)} a_k$, where $0 \le n(k) \le q(a_k) - 1$ is obtained, similarly, by inputting (b_k, a_k) into Lemma 5.4.*

Then for all $k \ge 0$ even, we have $\widehat{a_{k+1}} > \widehat{b_{k+1}}$, and for $k \ge 0$ odd, we have $\widehat{a_{k+1}} < \widehat{b_{k+1}}$.

Moreover, for all $k \ge 0$, we have $\widehat{a_k} > \widehat{a_{k+2}}^2$, and $\widehat{b_k} > \widehat{b_{k+2}}^2$. In particular, both sequences $(\mathrm{rot}(a_k))_{k\ge 0}$ and $(\mathrm{rot}(b_k))_{k\ge 0}$ converge to zero.

Note that the sequence (a_k, b_k) is built so that both $\mathrm{rot}(a_k)$ and $\mathrm{rot}(b_k)$ converge to zero from above. This choice is arbitrary.

Proof The first consideration follows immediately from the first statement of Lemma 5.4. Let us prove the second. Let $k \ge 0$ be even. If $p(a_k) \ge 2$, let $n = n(k) \ge 0$ be such that $np(a_k) = 1 \bmod q(a_k)$. Then $\mathrm{rot}(a_k^n) = 1/q(a_k)$, and $\widehat{a_k^n}^{p(a_k)} = \widehat{a_k}$. By a direct application of Lemma 5.4 we conclude that $\widehat{b_{k+1}} < \widehat{a_k^n}$, hence $\widehat{b_{k+1}}^{p(a_k)} < \widehat{a_k}$, and $\widehat{a_{k+2}}^2 < \widehat{a_k}$.

Otherwise, $p(a_k) = 1$, and again we take $n(k)$ as in Lemma 5.4. If $\widehat{a_k^{n(k)} b_k}^2 < \widehat{a_k}$, then we may conclude as above. Otherwise, $\widehat{b_k^{-1} a_k^{1-n}}^2 < \widehat{a_k}$, ie $\widehat{b_{k+1}^{-1} a_{k+1}}^2 < \widehat{a_k}$. Thus, either $n(k+1)$ is equal to -1 modulo $q(b_{k+1})$, or not; in which case we have

$$\widetilde{\mathrm{rot}}\big(\widehat{b_{k+1}^{n(k+1)} a_{k+1}}\big) < \widetilde{\mathrm{rot}}\big(\widehat{b_{k+1}^{-1} a_{k+1}}\big),$$

and then $\widehat{b_{k+1}^{n(k+1)} a_{k+1}} < \widehat{b_{k+1}^{-1} a_{k+1}}$. In either case we conclude that $\widehat{a_{k+2}}^2 < \widehat{a_k}$.

The argument is symmetric for k odd, and for b_k instead of a_k. In particular, $\widehat{a_{k+2}}^2 < \widehat{a_k}$ implies that $0 < \widetilde{\mathrm{rot}}(\widehat{a_{k+2}}) < \frac{1}{2}\,\widetilde{\mathrm{rot}}(\widehat{a_k})$, hence the sequences $(\widetilde{\mathrm{rot}}(\widehat{a_k}))$ and $(\widetilde{\mathrm{rot}}(\widehat{b_k}))$ converge to zero from above. □

Let $T = T(a,b)$ be a bad torus, and let (a_k, b_k) be the sequence furnished by Proposition 5.5. Let $x \in S^1$, and let $\tilde{x} \in \mathbb{R}$ be a lift of x. Then, by Proposition 5.5, the sequence $(\widehat{a_k}(\tilde{x}))_k$ is decreasing, bounded below by $\tilde{x}$, hence it converges to some real number that we denote by $\tilde{x} + j_T(x)$. Note that $j_T(x)$ does not depend on the choice of the lift of x. We define

$$\mathcal{A}_T := \{x \in S^1 \mid j_T(x) = 0\}.$$

The reader should interpret this as the set of points that are moved arbitrarily small distances by elements of $\{a_k\}$. Although the notation (a,b) is suppressed, $\mathcal{A}_T$ as defined is dependent on the generating set we started with. (But see Step 1 of the proof of Proposition 5.7 below.) As usual, we let $\widetilde{\mathcal{A}_T}$ denote the preimage of $\mathcal{A}_T$ in $\mathbb{R}$. The following proposition may be viewed as an algorithmic proof (as it runs essentially on the Euclidean algorithm as introduced in Proposition 5.5) of Hölder's classical result that any group acting freely on the circle is abelian.

Proposition 5.6 (properties of $\mathcal{A}_T$) (1) *$\mathcal{A}_T$ is a nonempty, proper subset of S^1, with no isolated points, hence is infinite.*

(2) *For every $x \in S^1$, we have $\min\{\widetilde{\mathcal{A}_T} \cap [\tilde{x}, \infty)\} = \tilde{x} + j_T(x)$. In particular, $x + j_T(x) \in \mathcal{A}_T$ for all x.*

(3) *The commutator $[a,b]$ fixes $\mathcal{A}_T$ pointwise.*

Proof Let $x \in \mathbb{R}$. For all $k \geq 0$ we have $\widehat{a_k}(x) > x + j_T(x)$, from which follows $\widehat{a_k}^2(x) > x + j_T(x) + j_T(x + j_T(x))$. But $\widehat{a_{k-2}}(x) > \widehat{a_k}^2(x)$, and, by definition, $\widehat{a_{k-2}}(x)$ converges to $x + j_T(x)$. This proves that $x + j_T(x) \in \mathcal{A}_T$ and thus $\mathcal{A}_T$ is nonempty. Further, if the open interval $(x, x + j_T(x))$ contained a point $y \in \widetilde{\mathcal{A}_T}$, then for large k we would have $x + j_T(x) > \widehat{a_k}(y) > y > x$, contradicting that a_k preserves orientation. This proves property (2).

To prove property (3), let $x \in \widetilde{\mathcal{A}_T}$ and observe, as above, that the sequence $\widehat{a_k}^4(x)$ also converges to x. Fix $\varepsilon > 0$, and let k be even, and large enough that $x_1 = x$, $x_2 = \widehat{a_k}(x)$, $x_3 = \widehat{a_k}^2(x)$ and $x_4 = \widehat{a_k}^3(x)$ all lie in the interval $[x, x + \varepsilon]$. By Lemma 5.4, $a_{k+1} = a_k$ and $\widehat{b_{k+1}}$ is dominated by $\widehat{a_{k+1}}$. Thus, $\widehat{b_{k+1}}(x_3) \in (x_3, x_4)$, and $\widehat{b_{k+1}}^{-1}(x_2, x_3) \subset (x_1, x_3)$. It follows that $[a_{k+1}, b_{k+1}] = [a,b]$ maps the point x_2 into the interval (x_1, x_3), hence, for all $\varepsilon > 0$, $[a,b]$ maps a point of $[x, x + \varepsilon]$ in $[x, x + \varepsilon]$, whence $[a,b](x) = x$.

It remains to prove that $\mathcal{A}_T \neq S^1$, and $\mathcal{A}_T$ has no isolated point. If $\mathcal{A}_T = S^1$, then $[a,b] = \mathrm{id}$ and the restriction of ρ to $\langle a, b\rangle$ would have abelian image; this contradicts the fact that T is bad. Finally if x were an isolated point of $\mathcal{A}_T$, we could take $x_0 \in S^1$ such that $[x_0, x) \cap \mathcal{A}_T = \varnothing$. Let x_1 be the next point of $\mathcal{A}_T$ to the right of x. Then $x_0 + j_T(x_0) = x$, so for all $k \geq 0$, we have $\widehat{a_k}(x_0) > x$. But then x_1 is the next point of $\mathcal{A}_T$ to the right of $\widehat{a_k}(x_0)$, so $\widehat{a_k}^2(x_0) > x_1$ holds, and hence, also, $\widehat{a_{k-2}}(x_0) > x_1$. As this is true for all k, it contradicts the fact that $\widehat{a_{k-2}}(x_0)$ converges to x as $k \to \infty$. □

Using j_T, we now prove the following major step towards Proposition 1.10.

Proposition 5.7 *There cannot exist two disjoint bad tori in Σ_g.*

Proof By contradiction, let $T = T(a,b)$ and $T' = T(a',b')$ be two disjoint bad tori. Up to re-indexing and reversing some of these curves, we may suppose that (a,b,a',b') is the beginning of a standard basis of $\pi_1\Sigma_g$.

Step 1 *We have $j_T = j_{T'}$.*

We proceed by contradiction. Suppose for some $x_0 \in S^1$ we have $j_T(x_0) \neq j_{T'}(x_0)$; without loss of generality assume $j_T(x_0) < j_{T'}(x_0)$. Let $(a_k,b_k)_{k\geq 0}$ and $(a'_k,b'_k)_{k\geq 0}$ be the sequences of generators of T and T' furnished by Proposition 5.5. For k large enough, we have $\widehat{a_k}(x_0) < x_0 + j_{T'}(x_0)$. Let m be as in Lemma 5.4 applied to (a_k,b_k), and put $\alpha = a_k$, and $\beta = a_k^m b_k$. Then (α,β) is a standard generating set for T, and $\hat{\alpha} > \widehat{\beta^{-1}}$. Since $\operatorname{rot}(b'_\ell) \to 0$, for $\ell \geq 0$ large enough we have $\widetilde{\operatorname{rot}}(\widehat{b'_\ell}) < \widetilde{\operatorname{rot}}(\widehat{\beta^{-1}})$. But $\widehat{b'_\ell}(x_0) > x_0 + j_{T'}(x_0)$ (indeed, $\widehat{b'_\ell}$ dominates $\widehat{a'_{\ell+1}}$, by construction of the sequences in Proposition 5.5); hence $\widehat{a_k}$ does not dominate $\widehat{b'_\ell}$. We now prove a sublemma to derive a contradiction; this will conclude the proof of Step 1.

Sublemma 5.8 *Let $T(a,b)$ be a bad torus, and let b' be a nonseparating simple curve outside $T(a,b)$ such that $b'^{-1}a$ and bb' are simple. Suppose that $\hat{a} > \widehat{b^{-1}}$ and $\widetilde{\operatorname{rot}}(\widehat{b^{-1}}) > \widetilde{\operatorname{rot}}(\widehat{b'})$. Then $\hat{a}$ dominates $\widehat{b'}$.*

Proof Suppose that $\hat{a}$ does not dominate $\widehat{b'}$. Then $\widehat{b^{-1}}$ does not dominate $\widehat{b'}$ either. Observation 5.3 then asserts that $\operatorname{rot}(b'^{-1}a) = \operatorname{rot}(bb') = 0$. Now $i(b'^{-1}a, bb') = \pm 1$, and $b'^{-1}a$ lies in a one-parameter family, so, as in Observation 2.16, there is a path-deformation of ρ replacing the action of bb' with $b'^{-1}a \cdot bb'$. Hence,

$$\operatorname{rot}(bb') = 0 = \operatorname{rot}(b'^{-1}a \cdot bb') = \operatorname{rot}(ab).$$

This contradicts that $T(a,b)$ is bad. □

Step 2 *We can deform the representation so that $j_T \neq j_{T'}$.*

As shown in the proof of Proposition 5.6, $[a,b] \neq \mathrm{id}$, but $\mathcal{A}_T \subset \operatorname{Fix}([a,b])$. Let $x \in S^1 \setminus \operatorname{Fix}([a,b])$, so then $j_T(x) > 0$. Let $y = x + j_T(x)$, let I be the connected component of $S^1 \setminus \operatorname{Fix}([a,b])$ containing x, and let c_t be a one-parameter family of homeomorphisms commuting with $[a,b]$, and with support equal to $\bar{I}$.

Then the distance between $c_t(x)$ and $c_t(y)$ varies, in a nonconstant way, with t: it goes to zero as $t \to \infty$ if $y \in I$, and simply changes if $y \notin I$. Now, consider a bending deformation of ρ defined by $\rho_t(\gamma) = \rho(\gamma)$ for all curves outside T, and $\rho_t(\gamma) = c_t\rho(\gamma)c_{-t}$ for $\gamma \in \langle a,b\rangle$. This deformation changes the value of $j_T(x)$, without changing the value of $j_{T'}(x)$. In particular, after this path-deformation, Step 1 no longer holds! This gives a contradiction. □

Supposing again that $T(a,b)$ is a bad torus, it remains to show that any torus in $\Sigma_g \setminus T(a,b)$ is not only good, but *very good*. The next lemma will allow us to easily achieve this goal.

Lemma 5.9 *Let $T = T(a,b)$ be a bad torus, and let γ be a nonseparating simple closed curve outside of T, with $\mathrm{rot}(\gamma) = 0$. Then $\mathcal{A}_T \subset \mathrm{Fix}(\gamma)$.*

Proof Let $(a_k, b_k)_{k\geq 0}$ be the sequence given by Proposition 5.5, and orient γ so that $\gamma^{-1}a_k$ is also a (nonseparating) simple curve. Fix $k \geq 0$, and let $\alpha = a_k$ and $\beta = a_k^m b_k$, as in Lemma 5.4. Then, by Sublemma 5.8, we have $\widehat{a_k} > \hat{\gamma}$. This holds for all $k \geq 0$; hence, for all $x \in \mathbb{R}$, we have $\hat{\gamma}(x) \leq x + j_T(x)$. In particular, if $x \in \widetilde{\mathcal{A}}_T$, we have $\hat{\gamma}(x) \leq x$.

For the reverse inequality, first note the conditions $\breve{a} < \widetilde{b^{-1}}$ and $\widetilde{\mathrm{rot}}(\widetilde{b^{-1}}) < \widetilde{\mathrm{rot}}(\breve{\gamma})$ imply the domination $\breve{a} < \breve{\gamma}$ (this is exactly the statement of Sublemma 5.8 after reversing the orientation of $\mathbb{R}$), and $\breve{\gamma} = \hat{\gamma}$ since $\mathrm{rot}(\gamma) = 0$. Let $x \in \widetilde{\mathcal{A}}_T$, and fix $\varepsilon > 0$. For k large enough, the sequence (a_k, b_k) from Proposition 5.5 satisfies $\widehat{a_k}(x) < x + \varepsilon$. Let $(a', b') = (a_k, b_k)$ for such a large k, and define $b'' = b'$ and $a'' = (b')^m a'$ and then $\alpha = a''$ and $\beta = (a'')^n b''$, where m, and then n, are given by Lemma 5.4 with these two successive pairs. Then, we have $\widetilde{\mathrm{rot}}(\breve{\alpha}) < \widetilde{\mathrm{rot}}(\widetilde{\beta^{-1}}) < \widetilde{\mathrm{rot}}(\breve{\gamma})$, hence, $\breve{\alpha} < \breve{\gamma}$, ie $\widehat{\alpha^{-1}}$ dominates $\hat{\gamma}^{-1}$. It follows that $x \leq \hat{\gamma}(x + \varepsilon)$. This shows $\breve{\gamma}(x) \geq x$, as desired. □

End of the proof of Proposition 1.10 Suppose that $T = T(a,b)$ is a bad torus, and let T' be a torus disjoint from T. By Sublemma 5.8, T' is good and we may take $T' = T(a', b')$, where $\mathrm{rot}(a') = 0$. Then we have $\mathrm{Fix}(a') \supset \mathcal{A}_T$ by Lemma 5.9. This is also true after replacing a' with a deformation $b'_t a'$, so $\mathrm{Per}(b') \supset \mathcal{A}_T$, or equivalently, $\mathrm{Fix}((b')^{q(b')}) \supset \mathcal{A}_T$. Since this is also true after replacing b' with any deformation $a'_t b'$, we conclude $\mathcal{A}_T \subset P(a', b')$. By Lemma 2.8(1), this means that $\langle a', b'\rangle$ has a finite orbit in S^1. □

5.2 Good tori

In this section, we prove Proposition 1.11: if ρ is path-rigid and nongeometric, then there cannot exist two disjoint good tori which are both not very good. In the course of the proof, we will develop some tools that will be used again in Section 6 for the proof of Theorem 1.1.

To motivate the first step, observe that if ρ has two disjoint good tori $T(a,b)$ and $T(d,e)$ with $\mathrm{rot}(a) = \mathrm{rot}(e) = 0$, and if neither of these tori are very good, then $P(a,b) = P(e,d) = \varnothing$. We can also find c so that (a,b,c,d,e) is a 5–chain. This is the set-up of the next proposition.

Proposition 5.10 *Let ρ be path-rigid minimal and let (a,b,c,d,e) be a 5–chain. Suppose that both $P(a,b)$ and $P(e,d)$ are empty. Then we have $S_k(b,c)$ for some $k \geq 1$.*

Proof After changing orientations of these curves, we may suppose that (a,b,c,d,e) is a directed 5–chain. By Theorem 4.2, it suffices to show that $\mathrm{Per}(b) \cap \mathrm{Per}(c) = \varnothing$. Since $P(a,b) = \varnothing$, Proposition 2.9 says that $\partial N(a,b)$ is finite. Choose a positive one-parameter family $(e_t)_{t\in\mathbb{R}}$, commuting with $\rho(e)$. Since $P(e,d) = \varnothing$, we have $\mathrm{Per}(e_t d) \subset U(e,d)$ for all t, so the sets $\mathrm{Per}(e_t d)$, for varying t, are pairwise disjoint and we can choose t_0 so that $\mathrm{Per}(e_{t_0} d) \cap \partial N(a,b) = \varnothing$. Abusing notation, we now replace d

with $e_{t_0}d$ (we will not further use e). With this change in notation, we now have $\partial N(a,b)\cap P(d,c)=\varnothing$. The remaining step will be a useful tool later in Section 6, so we split it off to a separate statement (Lemma 5.11), proved below. □

Lemma 5.11 *Let ρ be path-rigid, and let (a,b,c,d) be a 4–chain. Suppose that $P(a,b)=\varnothing$ and $\partial N(a,b)\cap P(d,c)=\varnothing$. Then $\mathrm{Per}(b)\cap\mathrm{Per}(c)=\varnothing$.*

Proof Orient the curves so that (a,b^{-1},c,d) is a directed 4–chain. Let a_t and d_t be positive one-parameter families commuting with a and d, respectively. By Lemma 2.17, it suffices to find t and s such that $\mathrm{Per}(a_tb)\cap\mathrm{Per}(d_sc)=\varnothing$.

Let $F_0=\partial N(a,b)\cap\partial N(d,c)$. Since $P(a,b)=\varnothing$, Proposition 2.9 says $\partial N(a,b)$ is finite. Hence, F_0 is finite. Let $F_1=\partial N(a,b)\setminus F_0$ and $F_2=(P(d,c)\cup\partial N(d,c))\setminus F_0$. By construction, the F_i are disjoint closed sets; let $\varepsilon>0$ be smaller than the minimum distance between any two of them. Fix t large, so that (by Lemma 2.8), $\mathrm{Per}(a_tb)$ is contained in the ε–neighborhood of $F_0\cup F_1$, hence disjoint from F_2. Since $F_0\subset N(a,b)$, it is also disjoint from $\mathrm{Per}(a_tb)$, ie $\mathrm{Per}(a_tb)\cap(F_0\cup F_2)=\varnothing$. Now let $\eta>0$ be smaller than the distance between $F_0\cup F_2$ and $\mathrm{Per}(a_tb)$. By Lemma 2.8 again, for s large enough, the set $\mathrm{Per}(d_sc)$ is in the η–neighborhood of $F_0\cup F_2$. Hence, $\mathrm{Per}(a_tb)$ and $\mathrm{Per}(d_sc)$ are disjoint, as desired. □

Our next goal is to propagate $S_k(\cdot,\cdot)$ to other curves. For this, we define two stronger properties.

Definition 5.12 (strengthenings of S_k) Say that two curves a and b satisfy $S_k^+(a,b)$ if they satisfy $S_k(a,b)$ and if additionally $a(\mathrm{Per}(b))\cap\mathrm{Per}(b)=\varnothing$. Say that a and b satisfy $S_k^{++}(a,b)$ if they satisfy both $S_k^+(a,b)$ and $S_k^+(b,a)$.

Property $S_k^+(\cdot,\cdot)$ allows one to move families of periodic points continuously by twist deformations, as described in the following lemma.

Lemma 5.13 *Let a and b be any curves with $i(a,b)=-1$ satisfying $S_k^+(a,b)$. There exists a continuous family a_t commuting with a such that $\mathrm{Per}(a_tb)\cap\mathrm{Per}(a_sb)=\varnothing$ for all $s\neq t$, and $|\mathrm{Per}(a_tb)|=2k$ for all t.*

Since property $S_k(a,b)$ immediately implies that $\mathrm{Per}(b)\subset U(a,b)$, the nontrivial part of this lemma is controlling the cardinality of $\mathrm{Per}(a_tb)$. This requires a special construction of one-parameter family a_t, which is, for once, not a one-parameter group.

Proof With Lemma 4.7, the assumption $a\,\mathrm{Per}(b)\cap\mathrm{Per}(b)=\varnothing$ completely prescribes the cyclic order on the set $\bigcup_n a^n(\mathrm{Per}(b))$; it follows that we may choose a neighborhood V of $\mathrm{Per}(b)$, consisting of $2k$ open intervals, such that $a^n(V)\cap a^m(V)=\varnothing$ for all $n,m\in\mathbb{Z}$. We now construct a continuous family of homeomorphisms a_t commuting with a, supported on $\bigcup_{n\in\mathbb{Z}}a^nV$.

Choose one point in each of the periodic orbits of b; let $x_1, x_2, \dots, x_m$ denote these points. Parametrize S^1 so that, for each x_i, b agrees with a rigid rotation by $p(b)/q(b)$ on a small neighborhood of $b^k(x_i)$ for $k = 0, 1, \dots, q(b) - 2$ and so that b maps a neighborhood of $b^{q(b)-1}(x_i)$ to a neighborhood of $x_i = b^{q(b)}(x_i)$ by the map $x \mapsto 2x$ or $x \mapsto \frac{1}{2}x$, in coordinates, depending on whether the orbit of x_i is repelling or attracting.

Let $V_{i,k}$ denote the connected component of V containing $b^k(x_i)$. Define a_t to be the identity on $V_{i,k}$ for $k = 0, 1, \dots, q(b) - 2$ and all i. On $V_{i,q(b)-1}$, using the local coordinates in which b is linear, define a_t to agree in a neighborhood of 0 with the translation $x \mapsto x + t$, and extend a_t equivariantly (with respect to a) over S^1. This all can be done continuously in t. After shrinking the $V_{i,k}$ if needed, by construction, each $(a_t b)^{q(b)}$ has a unique fixed point in each $V_{i,k}$, and these vary continuously. Additionally, for t sufficiently small, no new fixed points will be introduced; this proves the lemma. □

The next lemma and proposition allow one to propagate S_k^{++} along chains.

Lemma 5.14 *Let (a, b, c) be a completable 3–chain. Then $S_k^+(a, b)$ implies $S_k(b, c)$.*

Proposition 5.15 *Let (a, b, c) be a completable 3–chain. Suppose that $S_k^{++}(a, b)$ holds. Then $S_k^{++}(b, c)$ holds as well.*

To prove these two statements, we will need a quick sublemma.

Sublemma 5.16 (Per has empty interior) *Let a and b be any curves with $i(a, b) = \pm 1$, and let b_t be a positive one-parameter family commuting with b. Then, for all but countably many t, the set $\mathrm{Per}(b_t a)$ has empty interior.*

Proof Let $X = S^1 \setminus P(b, a)$. Then for $t \neq s$, we have $\mathrm{Per}(b_t a) \cap \mathrm{Per}(b_s a) \cap X = \varnothing$. In particular, the set $T = \{t : \mathrm{Per}(b_t a) \cap X \text{ contains a nonempty open set}\}$ is countable. Also if $U \subset \mathrm{Per}(b_t a)$ is nonempty and open, then $U \cap X = U \setminus P(b, a)$ is nonempty and open since $P(b, a)$ is closed with empty interior, hence $t \in T$. It follows that for all $t \notin T$, $\mathrm{Per}(b_t a)$ has empty interior. □

Proof of Lemma 5.14 Complete (a, b, c) to a 4–chain (a, b, c, d), and let $(d_t)_{t \in \mathbb{R}}$ be a positive one-parameter family commuting with d. By Sublemma 5.16, $\mathrm{Per}(d_{t_0} c)$ has empty interior for some $t_0 \in \mathbb{R}$. Now, by Lemma 5.13, there exists a continuous family $(a_s)_{s \in \mathbb{R}}$, an interval $I \subset \mathbb{R}$ and $2k$ maps, $\phi_j : I \to S^1$, each a homeomorphism to its image, such that for all $s \in I$, the $2k$ periodic points of $\mathrm{Per}(a_s b)$ are precisely $\phi_1(s), \dots, \phi_{2k}(s)$. The set $\bigcap \phi_j^{-1}(\mathrm{Per}(d_{t_0} c))$ then has empty interior in I, hence there exists $s_0 \in I$ such that $\mathrm{Per}(a_{s_0} b) \cap \mathrm{Per}(d_{t_0} c) = \varnothing$, and $\mathrm{Per}(b) \cap \mathrm{Per}(c) = \varnothing$ by Lemma 2.17. We conclude by using Theorem 4.2. □

Proof of Proposition 5.15 Complete the 3–chain into a 5–chain, (e, a, b, c, d), and apply Lemma 5.14 to the 3–chains (a, b, c) and (e, a, b) to conclude $S_k(b, c)$ and $S_k(a, e)$. By Lemma 3.8, we may then use a bending deformation of a along e to move the periodic set of a disjoint from any finite set, so in particular $\mathrm{Per}(a) \cap \mathrm{Per}(c) = \varnothing$. Let a_t be a positive one-parameter family, commuting with a. Then $\mathrm{Per}(a) \cap \mathrm{Per}(c) = \varnothing$, and $a_{-t}\,\mathrm{Per}(c)$ moves continuously in t, so there exists some t such that $b\,\mathrm{Per}(c) \cap a_{-t}\,\mathrm{Per}(c) = \varnothing$. Thus, $a_t b\,\mathrm{Per}(c) \cap \mathrm{Per}(c) = \varnothing$; hence, by Lemma 2.17, $b\,\mathrm{Per}(c) \cap \mathrm{Per}(c) = \varnothing$. Thus, we conclude that $S_k^+(b,c)$ holds. By Lemma 5.14, this implies that $S_k(c,d)$ holds as well. In particular, $\mathrm{Per}(d)$ is finite. We can now apply Lemma 3.8 and use a bending deformation so that $\mathrm{Per}(a_t b) \cap \mathrm{Per}(d) = \varnothing$, which implies that $\mathrm{Per}(b) \cap \mathrm{Per}(d) = \varnothing$, and repeat the argument above (with d and c playing the roles of a and b) to conclude $S_k^+(c,b)$ holds as well. □

Proposition 5.15, Theorem 3.3, and the connectedness of the graph in Lemma 2.4 immediately gives:

Corollary 5.17 *Let ρ be a path-rigid, minimal representation, and suppose there exists (a,b) such that $S_k^{++}(a,b)$ holds. Then ρ is geometric.*

This consequence is strong enough to imply the main result of the companion article [25]. We explain this now, as it will be used again in Section 6.

Corollary 5.18 *Let ρ be a path-rigid, minimal representation, and suppose that there is some torus $T(a,b)$ such that the relative Euler number of $T(a,b)$ is ± 1. Then ρ is semiconjugate to a Fuchsian representation.*

Proof Since $T(a,b)$ has Euler number 1, it follows from [29] that the restriction of ρ to $\langle a,b\rangle$ is semiconjugate to a geometric representation in $\mathrm{PSL}_2(\mathbb{R})$. (This is not difficult: that $\widetilde{\mathrm{rot}}([\widehat{\rho(a)},\widehat{\rho(b)}]) = \pm 1$ easily implies that $\rho(a)$ and $\rho(b)$ are 1–Schottky, hence are semiconjugate to a geometric representation in $\mathrm{PSL}_2(\mathbb{R})$. See the beginning of [29, Section 3].) In particular, property $S_1^{++}(a,b)$ holds, and Corollary 5.17 implies that ρ is geometric. □

Given Corollary 5.17, the main goal of this section reduces to the following.

Proposition 5.19 *Let (a,b,c,d,e) be a 5–chain, and suppose that $P(a,b) = P(e,d) = \varnothing$. Then we have $S_k^{++}(b,c)$.*

Proof Suppose $P(a,b) = P(e,d) = \varnothing$. By Proposition 5.10, we have $S_k(b,c)$ and $S_k(c,d)$ for some $k \geq 1$. Since $P(e,d) = \varnothing$ and $\mathrm{Per}(b)$ is finite, we have a bending deformation $e_t d$ such that $\mathrm{Per}(b) \cap \mathrm{Per}(e_t d) = \varnothing$; hence $\mathrm{Per}(b) \cap \mathrm{Per}(d) = \varnothing$. Hence, $\mathrm{Per}(b) \cap d_t c\,\mathrm{Per}(b) = \varnothing$ for some t, so we have $\mathrm{Per}(b) \cap c\,\mathrm{Per}(b) = \varnothing$, ie $S_k^+(c,b)$ holds. By Lemma 5.14, this gives $S_k(a,b)$. In particular, $\mathrm{Per}(a)$ is finite, and so there exists a bending deformation replacing c with $d_t c$ such that $\mathrm{Per}(a) \cap \mathrm{Per}(d_t c) = \varnothing$, and hence $\mathrm{Per}(a) \cap \mathrm{Per}(c) = \varnothing$. Repeating the argument above, we conclude $S_k^+(b,c)$ holds. □

The main result of this section is now a quick corollary. We restate it here for convenience and to summarize our work.

Corollary 5.20 *Let ρ be a path-rigid, minimal representation. Suppose ρ admits two disjoint good tori that are not very good. Then ρ is geometric.*

Proof Let $T(a,b)$ and $T(d,e)$ be two disjoint good tori. Since they are good, we may suppose $\mathrm{rot}(a)=\mathrm{rot}(e)=0$. Since they are not very good, we have $P(a,b)=\varnothing$ and $P(e,d)=\varnothing$. We may find a curve c such that (a,b,c,d,e) is a 5–chain, and then Proposition 5.19 and Corollary 5.17 imply that ρ is geometric. □

5.3 Finite orbits

The goal of this section is the proof of the following proposition.

Proposition 5.21 *Let $\rho\colon \Gamma_g \to \mathrm{Homeo}^+(S^1)$ be a path-rigid representation, and let $\Sigma=\Sigma_{g-1,1}$ be a subsurface containing only very good tori. Then $\rho_{|\pi_1\Sigma}$ has a finite orbit.*

If $T(a,b)$ is very good, then a and b act with a finite orbit, so $\mathrm{rot}(ab)=\mathrm{rot}(a)+\mathrm{rot}(b)$. Thus, in a subsurface where all tori are very good, rotation number is additive on any pair of curves with intersection number ± 1. This motivates the following proposition, which gives our first step.

Proposition 5.22 *Let Σ be a one-holed surface of genus ≥ 2. Suppose that $\pi_1\Sigma$ acts on the circle in such a way that all nonseparating simple curves have rational rotation number, and that for all γ_1,γ_2 with $i(\gamma_1,\gamma_2)=\pm 1$, we have $\mathrm{rot}(\gamma_1\gamma_2)=\mathrm{rot}(\gamma_1)+\mathrm{rot}(\gamma_2)$.*

Then, there exist two curves γ_1,γ_2 with $i(\gamma_1,\gamma_2)=\pm 1$ and $\mathrm{rot}(\gamma_1)=\mathrm{rot}(\gamma_2)=0$.

Proof Let $(a_1,\dots,b_g)$ be a standard generating set of $\pi_1\Sigma$, and consider the noncompletable directed 5–chain $(\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5)=(a_1^{-1}b_1a_1,a_1,\delta_1,a_2,b_2^{-1})$, with the notation of Section 2.1.

Let $r_i=\mathrm{rot}(\gamma_i)$ and let τ_i denote the map on rotation numbers induced by the Dehn twist along γ_i. Then $\tau_i(r_1,r_2,r_3,r_4,r_5)=(r_1',\dots,r_5')$, where $r_{i-1}'=r_{i-1}-r_i$ and $r_{i+1}'=r_{i+1}+r_i$, and $r_j'=r_j$. As Dehn twists preserve chains, the proof of the proposition is reduced to showing that the operations τ_i can be iterated to transform any vector in $(\mathbb{Q}/\mathbb{Z})^5$ to a vector of the form $(0,0,r_3,r_4,r_5)$. This is a straightforward exercise (and should be familiar to anyone familiar with the symplectic group $\mathrm{Sp}(2g,\mathbb{Z})$); we leave the details to the reader. □

Proposition 5.22 is useful because it is much easier to control the dynamics of two curves if their rotation numbers are zero, as in the next proposition.

Proposition 5.23 *Suppose $\mathrm{rot}(a)=\mathrm{rot}(b)=0$. Then for every $\varepsilon>0$, there exists a one-parameter family $(a_t)_{t\in\mathbb{R}}$ commuting with a, an interval $J\subset\mathbb{R}$, and a finite collection of homeomorphisms $\phi_i : J\to S^1$ with disjoint images, such that for all $t\in J$,*

$$\mathrm{Fix}(a_tb)\cap\big(S^1\setminus V_\varepsilon(P(a,b))\big)=\{\phi_1(t),\dots,\phi_n(t)\}.$$

In other words, for all $t \in J$, the fixed points of $a_t b$ at distance $\geq \varepsilon$ to $P(a,b)$ are finite in number and move continuously in t. Compare with Lemma 5.13. Note that we do not require a_t to be a *positive* family.

Proof Fix a positive one-parameter family α_t commuting with a. We will modify α_t to obtain the desired family a_t.

When $\mathrm{rot}(a) = \mathrm{rot}(b) = 0$, we have $P(a,b) = \mathrm{Fix}(b) \cap \partial\mathrm{Fix}(a)$, and the set $U(a,b)$ has a very simple description: $x \in U(a,b)$ if and only if x and $b(x)$ are in the same connected component of $S^1 \setminus \partial\mathrm{Fix}(a)$. Thus, $U(a,b) = \bigcup_I (I \cap b^{-1}(I))$, where I ranges over the connected components of $S^1 \setminus \partial\mathrm{Fix}(a)$. As each connected component I is a–invariant, we may define a_t separately on each connected component, affecting only $\mathrm{Fix}(a_t b) \cap I$.

For every connected component I of $S^1 \setminus \partial\mathrm{Fix}(a)$, let $U(I)$ denote $I \cap b^{-1}(I)$. By definition, each endpoint of $U(I)$ lies in $\partial N(a,b) \cup P(a,b)$. Thus, by Proposition 2.9, all but finitely many intervals $U(I)$ lie in $V_\varepsilon(P(a,b))$. On all the corresponding connected components I of $S^1 \setminus \partial\mathrm{Fix}(a)$ we set $a_t = \alpha_t$.

Now we treat the remaining (finitely many) intervals I of $S^1 \setminus \mathrm{Fix}(a)$ such that $U(I)$ is nonempty, considering the configuration of I and $b^{-1}(I)$. As a first case, suppose that I and $b^{-1}(I)$ share an endpoint, ie a point in $P(a,b)$. If this is the right endpoint, define $a_t = \alpha_t$ on I. If the left endpoint is shared, take instead $a_t = \alpha_{-t}$. If $I = b(I)$, either choice will work. In each case, for all s sufficiently large, we have

$$\mathrm{Fix}(a_s b) \cap I \subset V_\varepsilon(P(a,b)). \tag{5-1}$$

As a second case, suppose b shifts I. If the shift is to the right, ie $I = (x_1, x_3)$ and $b(I) = (x_2, x_4)$ with x_1, x_2, x_3, x_4 in cyclic order, define $a_t = \alpha_t$ on I, and if the shift is to the left, set $a_t = \alpha_{-t}$. In either case, for all s sufficiently large, we have

$$\mathrm{Fix}(a_s b) \cap I = \varnothing. \tag{5-2}$$

We are left with the case where either $b(\bar{I}) \subset I$ or $\bar{I} \subset b(I)$. Suppose the first holds, as the second can be dealt with by a symmetric argument. Note that (using α_t and b) we are in the case $n = 1$ of Lemma 4.8 of the preceding section. Thus, there exists $s \in \mathbb{R}$ such that $\alpha_s b$ has a unique fixed point in I. Moreover, $b(\bar{I}) \subset I$ implies that this unique fixed point is an attracting point, ie we may take local coordinates so that the map $\alpha_s b$ agrees with $x \mapsto \frac{1}{2}x$ at the origin. After reparametrization of α_t on I, we may assume that this time s is sufficiently large to satisfy (5-1) and (5-2) above. Working in coordinates, let $(-\delta, \delta)$ be a neighborhood of 0 contained in a fundamental domain for a. Let τ_t be a smooth family of bump functions supported on $(-\delta, \delta)$ and agreeing with $x \mapsto x + t$ on an even smaller (fixed) neighborhood of 0, for all $t < \delta' < \delta$. Extend τ_t a–equivariantly to a homeomorphism of I. Now define a_t on I to agree with α_t for $t < s$, to agree with $\tau_{t-s}\alpha_s$ for $s \leq t \leq s + \delta'$, and arbitrarily (for example, constant in t) for $t \geq s + \delta'$. Varying t in $J := (s, s + \delta')$, the homeomorphism $a_t b$ has a unique fixed point in I that moves continuously with t, as desired. Of course, we can choose parametrizations of a_t on each of these (finitely many) intervals so that J does not depend on I. This proves the lemma. □

Using this tool, we can propagate finite orbits over chains.

Proposition 5.24 *Let $a, \gamma_1, \gamma_2, \gamma_3, \dots \gamma_k$ be a chain. Suppose that $\mathrm{Per}(a)$ has empty interior, $\mathrm{rot}(\gamma_i) = 0$ for all i, the subgroup $\langle a, \gamma_1 \rangle$ has a finite orbit and $\langle \gamma_i, \gamma_{i+1} \rangle$ has a global fixed point. Then $\langle a, \gamma_i, \dots, \gamma_k \rangle$ has a finite orbit.*

Proof Inductively, suppose the statement holds for chains of length k and take a chain of length $k+1$ of the form $a, \gamma_1, \dots, \gamma_k$. By inductive hypothesis the group generated by the first k elements $\langle a, \gamma_1, \dots, \gamma_{k-1} \rangle$ has a finite orbit, ie there is a periodic orbit of a contained in $\bigcap_{i=1}^{k-1} \mathrm{Fix}(\gamma_i)$.

Since $\mathrm{Per}(a)$ has empty interior, for any $n \in \mathbb{N}$, we can use Proposition 5.23 to produce a homeomorphism $c(n)$ lying in a one-parameter family commuting with γ_k such that

$$\mathrm{Fix}(c(n)\gamma_{k-1}) \cap \mathrm{Per}(a) \subset V_{1/n}(P(\gamma_{k-1}, \gamma_k)).$$

Indeed, with the notation of that proposition, there exists $t \in J$ such that $\phi_j(t) \notin \mathrm{Per}(a)$ for all j, because $\bigcap_j \phi_j^{-1}(\mathrm{Per}(a))$ has empty interior in J. Do this for each $n \in \mathbb{N}$; we do not require that the $c(n)$ all belong to a common one-parameter family, all that is important is that they are each obtainable by a bending deformation, hence give a semiconjugate representation.

The result is a sequence of bending deformations $c(n)\gamma_{k-1}$ of γ_{k-1} such that

$$\mathrm{Fix}(c(n)\gamma_{k-1}) \cap \mathrm{Per}(a) \subset V_{1/n}(\mathrm{Fix}(\gamma_{k-1}) \cap \mathrm{Fix}(\gamma_k)).$$

Since $\langle a, \gamma_1, \dots, \gamma_{k-1} \rangle$ has a finite orbit, and this property is stable under semiconjugacy, it follows that, for every n, $\bigcap_{i=1}^{k-2} \mathrm{Fix}(\gamma_i) \cap \mathrm{Fix}(c(n)\gamma_{k-1})$ contains a full orbit of a. For each n, choose one such full orbit, and denote it by $\mathbb{O}_n$. After passing to a subsequence, the sets $\mathbb{O}_n$ converge pointwise to a finite subset of $\bigcap_{i=1}^{k-2} \mathrm{Fix}(\gamma_i) \cap \mathrm{Per}(a)$ that is invariant under a (as these are both closed conditions) so the limit is a full orbit. Moreover, this orbit is contained in every open neighborhood of $\mathrm{Fix}(\gamma_{k-1}) \cap \mathrm{Fix}(\gamma_k)$, so also lies in $\mathrm{Fix}(\gamma_{k-1}) \cap \mathrm{Fix}(\gamma_k)$. This gives a periodic orbit of a in $\bigcap_{i=1}^{k} \mathrm{Fix}(\gamma_i)$, as desired. □

We now prove the main result advertised at the beginning of this section.

Proof of Proposition 5.21 Let $\Sigma_{g-1,1}$ be a surface with one boundary component, in which all tori are very good. Recall that our goal is to show that ρ has a finite orbit. Since all tori are very good, we may use Proposition 5.22 to find a standard system of generators $a_1, b_1, \dots, a_{g-1}, b_{g-1}$ where $\mathrm{rot}(a_i) = \mathrm{rot}(b_i) = 0$ for all $i = 2, 3, \dots, g-1$. Since $T(a_1, b_1)$ is good, we may also assume that $\mathrm{rot}(b_1) = 0$.

Let $\delta_i = a_{i+1}^{-1} b_{i+1} a_{i+1} b_i^{-1}$ as in Section 2.1, so that $(a_1, \delta_1, a_2, \delta_2, \dots, \delta_{g-2}, a_{g-1}, b_{g-1})$ forms a chain. For each i, we can use Sublemma 5.16 in order to assume without loss of generality that $\mathrm{Per}(\delta_i)$ has empty interior, and then apply Proposition 5.24 to the chain (δ_i, a_i, b_i). It follows that $\langle \delta_i, b_i \rangle$ has a finite orbit, hence

$$\mathrm{rot}(\delta_i) + \mathrm{rot}(b_i) = \mathrm{rot}(a_{i+1}^{-1} b_{i+1} a_{i+1}) = \mathrm{rot}(b_{i+1}).$$

Thus, $\mathrm{rot}(\delta_i) = 0$ for all i.

Sublemma 5.16 implies that, after a deformation, we may assume that $\mathrm{Per}(a_1)$ has empty interior. We can apply Proposition 5.24 to the chain $(a_1, \delta_1, a_2, \delta_2, \dots \delta_{g-2}, a_{g-1}, b_{g-1})$ to conclude that the subgroup generated by these elements has a finite orbit. As this subgroup is equal to $\pi_1(\Sigma_{g-1,1})$, this proves the proposition. □

5.4 Proof of Theorem 1.6

Theorem 1.6 is now a quick consequence of Proposition 5.21 and Corollary 5.18.

Proof of Theorem 1.6 Let $\rho\colon \pi_1(\Sigma_g) \to \mathrm{Homeo}_+(S^1)$ be a path-rigid representation, and suppose that ρ is not geometric. If Σ contains a bad torus T, then by Proposition 1.11, $\Sigma \setminus T$ contains only very good tori. If Σ contains no bad torus, but some torus T' that is not very good, then Proposition 1.11 implies that $\Sigma \setminus T'$ contains only very good tori. In either case, there is a genus $g-1$ subsurface $\Sigma_{g-1,1}$ containing only very good tori, hence by Proposition 5.21 the restriction of ρ to $\Sigma_{g-1,1}$ has a finite orbit. In particular, the boundary curve of this subsurface has zero rotation number, and the restriction of ρ to this subsurface has relative Euler number zero.

It follows that the Euler number of the remaining (not very good) torus is either 0 or ± 1. By Corollary 5.18, if it is ± 1, then ρ is geometric. Thus, the remaining torus has Euler number 0, and by additivity the Euler number of ρ is zero. □

The proof of Proposition 5.21 also shows the following, which will be useful to us in the next section of this work.

Corollary 5.25 *Suppose ρ is a path-rigid representation such that Σ has only very good tori. Then ρ has a finite orbit.*

Proof To show this, one simply runs the proof of Proposition 5.21 for a genus g surface (rather than a genus $g-1$ surface with boundary), finding a standard system of generators $a_1, b_1, \dots, a_g, b_g$ and ignoring the extra relation. The remainder of the proof applies verbatim, with g replacing $g-1$. □

6 Proof of Theorem 1.1 and last comments

6.1 Proof of Theorem 1.1

Here is where we use the stronger hypothesis of rigidity. Our proof relies on the following observation, inspired by work in the recent article [1].

Lemma 6.1 *Let ρ be a rigid, minimal representation. Let $T = T(a,b)$ be a very good torus. Then only finitely many points of S^1 have a finite orbit under $\langle a, b\rangle$. In particular, if $\mathrm{rot}(a) = 0$, then $P(a,b)$ is a finite set.*

This lemma is the *only* place where we use rigidity instead of path-rigidity.

Proof Let $F(a,b)$ denote the set of points whose orbit under $\langle a,b\rangle$ is finite. To simplify the exposition of the proof, fix a metric on S^1 so that a and b act on $F(a,b)$ by rigid rotations. Given any $\varepsilon > 0$, let $J_1, J_2, \dots$ denote the (finitely many) connected components of $S^1 \setminus F(a,b)$ consisting of intervals of length greater than ε — by our choice of metric, this is a $\langle a,b\rangle$–invariant set. If $F(a,b)$ is finite, and ε small enough, then $\bigcup_i \bar{J}_i = S^1$. Otherwise (even in the case where $\bigcup_i \bar{J}_i = \varnothing$), we may divide $S^1 \setminus \bigcup_i \bar{J}_i$ into finitely many disjoint open intervals $I_1, I_2, \dots$ each of length at most ε and with endpoints in $F(a,b)$, such that these intervals are permuted by $\langle a,b\rangle$, and such that $S^1 = \bigl(\bigcup_i \bar{J}_i\bigr) \cup \bigl(\bigcup_i \bar{I}_i\bigr)$.

Since T is very good, we can suppose without loss of generality that $\mathrm{rot}(a) = 0$. We claim that there exist $a', b' \in \mathrm{Homeo}^+(S^1)$, agreeing with a and b on $S^1 \setminus \bigcup_i I_i$, such that $[a',b'] = [a,b]$ holds globally, and such that $\mathrm{Per}(b') \cap \bigcup I_i = \varnothing$.

Let $c = [a,b]$. As $\bigcup_i I_i$ is a,b–invariant, constructing a' and b' amounts to solving the equation $b'c = a'^{-1}ba'$ on $\bigcup_i J_i$. That this can be solved is shown in [9, Lemma 2.7]; as their notation and context is slightly different, we explain the strategy. Take coordinates identifying each J_i with $\mathbb{R}$. If b' is defined on some J_i (with image in J_j) to increase sufficiently quickly (as a homomorphism $\mathbb{R} \to \mathbb{R}$), then $b'c$ will also be strictly increasing, hence conjugate to b'. One then defines a' to be this conjugacy.

Let ρ' be the representation obtained from ρ by replacing (a,b) by (a',b'). As $\varepsilon > 0$ is arbitrary, this ρ' can be taken arbitrarily close to ρ in $\mathrm{Hom}(\Gamma_g, \mathrm{Homeo}^+(S^1))$. Rigidity implies that, for small enough ε, ρ' is semiconjugate to ρ. Minimality implies that there is a *continuous* semiconjugacy $h\colon S^1 \to S^1$ such that $h \circ \rho' = \rho \circ h$. Let

$$F' := \{x \in S^1 \mid x \text{ has finite orbit under } \langle \rho'(a), \rho'(b)\rangle\}.$$

By construction of ρ', this set is finite. However, $h(F') = F(a,b)$. It follows that $F(a,b)$ was finite as well. □

To apply this to the proof of Theorem 1.1, let ρ be a rigid, minimal representation, and assume for contradiction that ρ is nongeometric. If ρ has a bad torus T, then by Theorem 1.6 any torus $T(a,b)$ disjoint from T is very good. In particular, we can take such a torus where $\mathrm{rot}(a) = 0$. Lemma 5.9 implies then that $A_T \subset \mathrm{Fix}(a)$. Since the same holds after replacing a with a deformation $b_t a$, we conclude that $A_T \subset P(a,b)$. However, Proposition 5.6 states that A_T is infinite, contradicting Lemma 6.1. We conclude that ρ has no bad tori.

In order to derive a contradiction, we will show that all good tori are actually very good. We pursue this with an argument in the spirit of Proposition 5.10.

Lemma 6.2 *Suppose $P(a,b) = \varnothing$. Then $\partial N(a,b) \subset \partial\mathrm{Per}(a) \cup b^{-1}(\partial\mathrm{Per}(a))$.*

Proof Assume $P(a,b) = \varnothing$ and let $x \in \partial N(a,b)$. Since $P(a,b) = \varnothing$, the set $N(a,b)$ is closed, hence $x \in N(a,b) \cap \overline{U(a,b)}$.

Suppose that $x \notin (\partial\mathrm{Per}(a) \cup b^{-1}(\partial\mathrm{Per}(a))$. Then there exists two intervals I, J, neighborhoods of x, with $I \subset S^1 \setminus \partial\mathrm{Per}(a)$ and $J \subset S^1 \setminus b^{-1}(\partial\mathrm{Per}(a))$. As $x \in \overline{U(a,b)}$, there exists $u \in U(a,b) \cap I \cap J$. Let a_t be a positive one-parameter family commuting with a. Since $b(J)$ contains $b(x)$ and $b(u)$ and $b(J) \cap \partial\mathrm{Per}(a) = \varnothing$, there exists $t_0 \in \mathbb{R}$ such that $a_{t_0} b(x) = b(u)$. Similarly, there exists $t_1 \in \mathbb{R}$ such that $a_{t_1}(u) = x$. Thus, $\Delta_{a,b}(x, t_1 + T(u), T(u), \dots, T(u), T(u) + t_0) = 0$, and it now follows easily that $x \in U(a,b)$. This proves the lemma. □

Lemma 6.3 *Suppose* $\mathrm{rot}(a) = 0$ *and that* $\langle a, b\rangle$ *has no finite orbit. Choose a positive one-parameter group* b_t *that commutes with* b*. Then for all* $x \in S^1$*, there exist at most two values of* t *such that* $x \in \partial N(b_t a, b)$*.*

Proof Since $\langle a, b\rangle$ has no finite orbit, $P(a,b) = \varnothing$ and hence $P(b_t a, b) = \varnothing$ for all t. Let $x \in S^1$; we will apply Lemma 6.2 to the pairs $(b_t a, b)$. If $x \in \mathrm{Per}(b)$, then $x \notin N(b_t a, b)$, and in particular $x \notin \partial N(b_t a, b)$ for all $t \in \mathbb{R}$. Thus, suppose $x \notin \mathrm{Per}(b)$.

By Lemma 6.2, if $x \in \partial N(b_t a, b)$, then $x \in \partial\mathrm{Per}(b_t a) \cup b^{-1}(\partial\mathrm{Per}(b_t a))$. Note that x cannot be in $P(b,a)$, as $x \notin \mathrm{Per}(b)$. Hence, if there exists some $t \in \mathbb{R}$ such that $x \in \mathrm{Per}(b_t a)$, then $x \in U(b,a)$, and this t is unique. Similarly, if there exists some $t \in \mathbb{R}$ such that $b(x) \in \mathrm{Per}(b_t a)$, then $b(x) \in U(b,a)$, and this t is unique. This concludes the proof. □

Using these tools, we will now show that ρ (always assumed rigid and minimal) satisfies hypothesis S_k.

Lemma 6.4 *Let* (a,b,c,d) *be a 4–chain, and suppose* $\mathrm{rot}(a) = \mathrm{rot}(d) = 0$ *holds. Suppose that* $T(a,b)$ *is good but not very good. Then we have* $S_k(b,c)$*.*

Proof If $T(d,c)$ is good but not very good, then $P(d,c)$ is empty. Otherwise, it is very good and so by Lemma 6.1, the set $P(d,c)$ is finite. In either case, using Lemma 6.3, we can first deform a to some $b_t a$, so that $\partial N(a,b)$ does not intersect $P(d,c)$. Then by Lemma 5.11, we have $\mathrm{Per}(b) \cap \mathrm{Per}(c) = \varnothing$, and so Theorem 4.2 says that $S_k(b,c)$ holds. □

Lemma 6.5 *Let* (a,b,c,d) *be a 4–chain, and suppose* $S_k(a,b)$ *and* $\mathrm{rot}(d) = 0$ *hold. Then we have* $S_k(b,c)$*.*

Proof Similarly to the previous lemma, in this case we may again use Lemma 6.1 to conclude that the set $P(d,c)$ is finite. By Lemma 3.8 in the torus $T(a,b)$, the set $\mathrm{Per}(b)$ is disjoint from $P(d,c)$.

Hence, $\mathrm{Per}(b) \subset U(d,c) \cup N(d,c)$, and $\mathrm{Per}(b)$ is finite. Thus, for all but finitely many t, we have $\mathrm{Per}(b) \cap \mathrm{Per}(d_t c) = \varnothing$. Hence $\mathrm{Per}(b) \cap \mathrm{Per}(c) = \varnothing$ by Lemma 2.17. □

Now we can complete the proof of the theorem.

Proof of Theorem 1.1 Let ρ be a rigid, minimal representation. As remarked above, ρ has no bad torus. If all tori are very good, then by Corollary 5.25, we know that ρ admits a finite orbit, a contradiction.

Thus, ρ admits a good torus, $T(a,b)$, which is not very good. We may suppose $\mathrm{rot}(a)=0$. As all tori are good, we may choose a curve d outside $T(a,b)$ with $\mathrm{rot}(d)=0$, and we may form a 4–chain (a,b,c,d). By Lemma 6.4, we have $S_k(b,c)$ for some k.

Now rename (b,c) into (a,b), and forget about the other curves, remembering only that we have two curves a,b with $S_k(a,b)$. Since all tori are good, we may choose a curve d outside $T(a,b)$ such that $\mathrm{rot}(d)=0$, and such that there exists a standard generating system beginning with (a,b,d,γ). Define $u=\gamma a^{-1}b^{-1}a$ and $v=\gamma a^{-1}$. Then (u,a,b,v), (d,u,a,b) and (a,b,v,d) are 4–chains; we encourage the reader to refer to Figure 1 and draw these curves u and v for him/herself. Apply Lemma 6.5 to the 4–chain (a,b,v,d). This proves that $S_k(b,v)$ holds. The same lemma applied to the 4–chain (d,u,a,b) implies $S_k(u,a)$. Hence, the 4–chain (u,a,b,v) satisfies $S_k(u,a)$, $S_k(a,b)$ and $S_k(b,v)$. We can deform a along u, thanks to Lemma 3.8, in such a way that $\mathrm{Per}(a)\cap\mathrm{Per}(v)=\varnothing$, hence we have $S_k^+(b,a)$, and we can deform b along v, in such a way that $\mathrm{Per}(b)\cap\mathrm{Per}(u)=\varnothing$, hence we have $S_k^+(a,b)$. Finally, this proves $S_k^{++}(a,b)$, and thus ρ is geometric by Corollary 5.17. □

6.2 Comments and further questions

We conclude this paper by discussing some natural questions and directions for further work.

6.2.1 Path-rigidity Given Theorem 1.6, we expect that path-rigidity should suffice to imply that a representation is geometric. The most obvious route to this result would be through an improvement of Lemma 6.1, as it is the only place where we use the stronger hypothesis of rigidity.

Question 6.6 *Does Lemma 6.1 hold when "rigid" is replaced by "path-rigid"?*

This question also arises naturally out of the work of Alonso, Brum and Rivas in [1]. Their main result is the following.

Theorem 6.7 (Alonso–Brum–Rivas) *Let ρ be in $\mathrm{Hom}(\Gamma_g,\mathrm{Homeo}^+(S^1))$ or $\mathrm{Hom}(\Gamma_g,\mathrm{Homeo}^+(\mathbb{R}))$. In any neighborhood U of ρ, there exists a representation ρ' without global fixed points.*

Since it is unknown whether these representation spaces are locally connected, their result does not imply that there is a *path-deformation* of ρ without global fixed points. Thus, the obvious problem arising out of their work is to upgrade this result to path-deformations. A first step in this direction would be to attempt to reprove [1, Lemma 3.9, 3.10]. These lemmas show that, in any neighborhood of ρ, there exists a representation ρ' whose fixed points are isolated and either attracting or repelling points. Can ρ' be attained by deforming along a path? If so, can this be generalized to finite orbits, rather than fixed points, for actions on S^1?

6.2.2 The commutator equation More general than Question 6.6 above, the following basic problem appears to be essential in understanding the topology of $\mathrm{Hom}(\Gamma_g, \mathrm{Homeo}^+(S^1))$.

Probem 6.8 *For fixed* $h \in \mathrm{Homeo}^+(S^1)$, *describe the topology of the set*

$$\nu_h := \{f, g \in \mathrm{Homeo}^+(S^1) \times \mathrm{Homeo}^+(S^1) \mid [f, g] = h\}.$$

As it stands, remarkably little is known about this space. If $\mathrm{rot}(h) \in \mathbb{Q} \setminus \{0\}$, then it is known that ν_h is not connected; however, we do not know the number of connected components, nor do we know in any circumstances whether ν_h is locally connected or not.

Probem 6.8 is strongly related to the following major problem.

Probem 6.9 *Classify the connected components of* $X(\Gamma_g, \mathrm{Homeo}^+(S^1))$.

As was mentioned in the introduction, it is still unknown whether $X(\Gamma_g, \mathrm{Homeo}^+(S^1))$ (or equivalently, $\mathrm{Hom}(\Gamma_g, \mathrm{Homeo}^+(S^1))$) has finitely many or infinitely many connected components. The relationship with Probem 6.8 comes through the analogy with Goldman's work on $\mathrm{Hom}(\Gamma_g, \mathrm{PSL}_2(\mathbb{R}))$. Indeed, Goldman's classification of connected components of $\mathrm{Hom}(\Gamma_g, \mathrm{PSL}_2(\mathbb{R}))$ given in [15] is built upon a complete understanding of the space $\nu_h \cap (\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R}))$. This is of course a much easier problem, as $\mathrm{PSL}_2(\mathbb{R})$ is a finite-dimensional Lie group, and the commutator map is smooth. The result of the first author in [23] — that Euler number does not classify connected components of $\mathrm{Hom}(\Gamma_g, \mathrm{Homeo}^+(S^1))$, unlike the $\mathrm{PSL}_2(\mathbb{R})$ case — may also serve as warning that the topology of ν_h should be more complicated than its intersection with $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$.

Throughout this paper, we navigated within ν_h by making bending deformations. This raises a few obvious questions, such as the following.

Question 6.10 *Let* $h \in \mathrm{Homeo}^+(S^1)$, *and let* (f, g) *and* (f', g') *be in the same path-component of* ν_h. *Identifying* f, g *with the image of generators of a one-holed torus, is there a path from* (f, g) *to* (f', g') *consisting of a sequence of bending deformations? More generally, given* ρ *and* ρ' *in the same path-component of* $\mathrm{Hom}(\Gamma_g, \mathrm{Homeo}^+(S^1))$, *is there a path from* ρ *to* ρ' *using bending deformations in simple closed curves on* Σ_g*?*

This question is reminiscent of Thurston's earthquake theorem [34] for Teichmüller space. It also calls to mind work of Goldman and Xia [16], who use the analogous (positive) result for bending deformations in connected components of classical character varieties in order to studying the action of the mapping class group on these varieties. As well as justifying our use of bending deformations alone, a positive answer to Question 6.10 would give another analogy between classical character varieties and $\chi(\Gamma_g, \mathrm{Homeo}^+(S^1))$.

6.2.3 Bad tori In Section 5, we used a long series of lemmas to prove that a path-rigid representation cannot contain two disjoint bad tori. However, we do not know any example of a path-rigid representation with even a single bad torus. Besides being an interesting question in itself, the question of existence of bad tori could provide a means of showing path-rigid representations are geometric: if one showed that path-rigid representations of Γ_g have no bad tori, an enhanced version of Lemma 5.11 would complete the proof.

However, we were somewhat surprised to be unable to tackle the following even more basic question.

Question 6.11 *Let $T(a,b)$ be a one-holed torus. Does there exist a representation*

$$\rho\colon \pi_1(T) \to \mathrm{Homeo}^+(S^1)$$

such that the rotation number of every nonseparating simple closed curve is rational, but nonzero?

This is obviously related to understanding mapping class group actions on character varieties, as we are asking for a nonseparating simple closed curve.

By contrast, relaxing the condition that curves be simple gives a problem already solved by a classical result of Antonov. See [31, Exercise 2.3.24] for an outline of the proof. An equivalent statement can be found in [8, Proposition 5.2].

Theorem 6.12 (Antonov [2]) *Let $\rho\colon \langle a,b\rangle \to \mathrm{Homeo}^+(S^1)$ be a minimal action. Either ρ has abelian image and is conjugate to an action by rotations, or — up to taking a quotient of S^1 by a finite-order rotation commuting with ρ — the probability that the rotation number of the image of a random word of length N in $\{a,b,a^{-1},b^{-1}\}$ (with respect to some nondegenerate distribution on the set) is zero tends to 1 as N tends to ∞.*

In the case where ρ commutes with a finite order rotation, say of order n, but does not have image conjugate into SO(2), the rotation numbers of random words equidistribute in $\{0, 1/n, \dots, (n-1)/n\}$. Thus, for any such action, almost all words have rational rotation number.

6.2.4 Local versus global rigidity Thus far, we have discussed rigidity and path-rigidity of representations; rigidity being the natural notion to study from our interest in character spaces, and path-deformations being easier to work with in practice. However, from a dynamical perspective, it is also interesting to study *local rigidity* or *stability* of actions.

Definition 6.13 ([24, Definition 3.1]; see also [1]) A representation ρ is *locally rigid* if it has a neighborhood *in the representation space* $\mathrm{Hom}(\Gamma_g, \mathrm{Homeo}^+(S^1))$ containing only representations semiconjugate to ρ.

In many circumstances, this condition is much easier to satisfy than rigidity or path-rigidity. For example, a savage element $g \in \mathrm{Homeo}^+(S^1)$ (as in Definition 3.4 above), thought of as a representation of $\mathbb{Z}$, is easily seen to be locally rigid, but it is semiconjugate to the identity. We do not know if this phenomenon generalizes to representations of Γ_g.

Question 6.14 *Is there a representation $\rho \in \mathrm{Hom}(\Gamma_g, \mathrm{Homeo}^+(S^1))$ that is locally rigid, but not rigid?*

Again, a natural first step to this question could be to study the local topology of the sets ν_h defined above.

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