Cluster categories and rational curves

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We study rational curves on smooth complex Calabi–Yau 3–folds via noncommutative algebra. By the general theory of derived noncommutative deformations due to Efimov, Lunts and Orlov, the structure sheaf of a rational curve in a smooth CY 3–fold Y is pro-represented by a nonpositively graded dg algebra Γ. The curve is called *nc rigid* if $H^0\Gamma$ is finite-dimensional. When C is contractible, $H^0\Gamma$ is isomorphic to the contraction algebra defined by Donovan and Wemyss. More generally, one can show that there exists a Γ pro-representing the (derived) multipointed deformation (defined by Kawamata) of a collection of rational curves $C_1, \ldots, C_t$ with $\dim(\mathrm{Hom}_Y(\mathbb{O}_{C_i}, \mathbb{O}_{C_j})) = \delta_{ij}$. The collection is called *nc rigid* if $H^0\Gamma$ is finite-dimensional. We prove that Γ is a homologically smooth bimodule 3–CY algebra. As a consequence, we define a (2–CY) cluster category $\mathscr{C}_\Gamma$ for such a collection of rational curves in Y. It has finite-dimensional morphism spaces if and only if the collection is nc rigid. When $\bigcup_{i=1}^t C_i$ is (formally) contractible by a morphism $\hat{Y} \to \hat{X}$, then $\mathscr{C}_\Gamma$ is equivalent to the singularity category of $\hat{X}$ and thus categorifies the contraction algebra of Donovan and Wemyss. The Calabi–Yau structure on Y determines a canonical class $[w]$ (defined up to right equivalence) in the zeroth Hochschild homology of $H^0\Gamma$. Using our previous work on the noncommutative Mather–Yau theorem and singular Hochschild cohomology, we prove that the singularities underlying a 3–dimensional smooth flopping contraction are classified by the derived equivalence class of the pair $(H^0\Gamma, [w])$. We also give a new necessary condition for contractibility of rational curves in terms of Γ.

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1 Introduction

The study of rational curves in algebraic varieties lies at the core of birational geometry. A smooth rational curve C in a quasiprojective variety Y is called *rigid* if the component of the Hilbert scheme of curves containing C is a finite scheme. Note that this is weaker than the notion of *infinitesimally rigid*, which says that $\mathrm{Ext}^1_Y(\mathbb{O}_C, \mathbb{O}_C) = 0$. If a curve is not rigid then we call it *movable*. When Y is a smooth projective surface, a smooth rational curve $C \subset Y$ is rigid if and only if its normal bundle $N_{C/Y}$ is negative. And, if C is rigid, then it is *contractible*, ie for the formal completion $\widehat{Y}$ of Y along C there exists a birational morphism $f : \widehat{Y} \to \widehat{X}$ to a normal surface $\widehat{X}$ that contracts C. The definition of contractibility in general can be found in [Definition 2.9](#).

Here we will focus on the case when Y is a smooth complex Calabi–Yau threefold, ie ω_Y is trivial. The situation is much more complicated than the surface case. We call a rational curve $C \subset Y$ *type* (a, b) if it has normal bundle $\mathbb{O}(a) \oplus \mathbb{O}(b)$. By the adjunction formula, we have $a + b = -2$. A $(-1, -1)$–curve is contractible. The underlying singular variety $\widehat{X}$ is equivalent to the singular hypersurface $x^2 + y^2 + u^2 + v^2 = 0$. There exists a different resolution $\widehat{Y}^+ \to \widehat{X}$ and the birational map $\widehat{Y} \dashrightarrow \widehat{Y}^+$ is called the *Atiyah flop*. Reid [1983] proves that a $(0, -2)$–curve is either contractible or movable. The contractible case corresponds to the *Pagoda flops*. Laufer proves that a contractible curve is of type $(-1, -1)$, $(0, -2)$ or $(1, -3)$ (see [Pinkham 1983]). Katz and Morrison [1992] show that any simple flopping contraction (see the definition in [Section 2.5](#)) can be constructed as base change of a universal contraction. In general, it is not true that all rigid curves are contractible. A counterexample was constructed by Clemens [1989].

We study the contractibility of rational curves in Calabi–Yau 3–folds via noncommutative methods. In general, given a rational curve $C \subset Y$ the problem is two-fold:

(1) Find infinitesimal criteria for the contractibility of C.

(2) If C is contractible, determine the underlying singularity of the contraction.

Our research is motivated by a remarkable paper of Donovan and Wemyss [2016], who considered the algebra Λ that represents the noncommutative deformation functor of $\mathbb{O}_C$ for a contractible rational curve $C \subset Y$. They prove that Λ is finite-dimensional and call it the *contraction algebra*. Indeed, the contraction algebra can be defined in a more general context where Y may be neither CY nor smooth, and the birational morphism may contract a divisor containing C. However, we will focus on the special case when Y is a smooth CY 3–fold and the contraction is not divisorial. Donovan and Wemyss conjectured that the 3–dimensional simple flops are classified by the isomorphism types of the contraction algebras (see [Conjecture 5.10](#)).

In order to deal with the case of general flops where the exceptional fiber can have multiple irreducible components, Kawamata proposes to study the multipointed noncommutative deformation of a *semisimple*

collection (see the definition in Section 2)[1] of sheaves $\mathscr{E}_1, \ldots, \mathscr{E}_t$. A case of special interest is when the collection is $\mathbb{O}_{C_1}, \ldots, \mathbb{O}_{C_t}$, where the C_i are irreducible components of the reduced exceptional fiber of a contraction (see [Kawamata 2018, Example 6.5]). We consider the derived noncommutative deformation theory of $\mathscr{E} := \bigoplus_{i=1}^t \mathscr{E}_i$ for a semisimple collection of sheaves $\{\mathscr{E}_i\}_{i=1}^t$ on a smooth CY 3–fold Y. By a result of Efimov, Lunts and Orlov (see Theorem 2.2), such deformation functor is pro-represented by a nonpositively graded dg algebra Γ. We call Γ the *derived deformation algebra* of the semisimple collection $\{\mathscr{E}_i\}_{i=1}^t$. We call a semisimple collection $\{\mathscr{E}_i\}_{i=1}^t$ *nc rigid* ("nc" stands for noncommutative) if $H^0\Gamma$ is finite-dimensional. Given a collection of smooth rational curve $C_1, \ldots, C_t$ such that $\{\mathbb{O}_{C_i}\}_{i=1}^t$ form a semisimple collection, we call $\{C_i\}_{i=1}^t$ an *nc rigid* collection of rational curves if $H^0\Gamma$ is finite-dimensional. If $t = 1$ and $C = C_1$ is nc rigid rational curve, then the abelianization of $H^0\Gamma$ represents the commutative deformation functor of $\mathbb{O}_C$. Therefore, an nc rigid curve is in particular rigid. Our first result is:

Theorem A (Proposition 3.9) *Let $C_1, \ldots, C_t$ be a collection of rational curves in a smooth quasiprojective Calabi–Yau 3–fold Y such that $\{\mathbb{O}_{C_i}\}_{i=1}^t$ form a semisimple collection. The derived deformation algebra Γ of $\bigoplus_{i=1}^t \mathbb{O}_{C_i}$ is a nonpositive pseudocompact dg algebra that is*

   (1)   *homologically smooth*;

   (2)   *bimodule 3–CY.*

Moreover, Γ is exact 3–CY in either one of the following cases:

   (a)   *Y is projective.*

   (b)   *There is a (formal) contraction $\hat{f} : \hat{Y} \to \hat{X}$ such that $\mathrm{Ex}(\hat{f}) = \bigcup_{i=1}^t C_i$, where $\mathrm{Ex}(\hat{f})$ stands for the reduced exceptional fiber of $\hat{f}$.*

This theorem establishes a link between birational geometry and the theory of cluster categories. We consider the triangle quotient $\mathscr{C}_\Gamma := \mathrm{per}(\Gamma)/D_{\mathrm{fd}}(\Gamma)$ (see Section 3.3). It is Hom–finite if and only if $\{C_i\}_{i=1}^t$ is nc rigid. By a result of Amiot [2009], it is then a 2–CY category. When C is contractible by a morphism $\hat{Y} \to \hat{X}$, then $H^0\Gamma$ is isomorphic to the contraction algebra Λ defined in [Donovan and Wemyss 2016] and $\mathscr{C}_\Gamma$ is equivalent to the singularity category of $\hat{X}$ and thus categorifies the contraction algebra of Donovan and Wemyss. If Γ is exact 3–CY, Van den Bergh proved that it is quasi-isomorphic to a (complete) Ginzburg algebra $\mathfrak{D}(Q, w)$ for some finite quiver Q and a potential w. If we fix the CY structure on Y, then there is a canonical class $[w]$, defined up to right equivalence, in the zeroth Hochschild homology of $H^0\Gamma$ (see Proposition 4.8). The class $[w]$ can be viewed as the "classical shadow" of the Calabi–Yau structure on Y. Our second result is:

Theorem B (Theorem 5.11) *Let $\hat{f} : \hat{Y} \to \hat{X}$ and $\hat{f}' : \hat{Y}' \to \hat{X}'$ be two formal flopping contractions with reduced exceptional fibers $\mathrm{Ex}(\hat{f}) = \bigcup_{i=1}^t C_i$ and $\mathrm{Ex}(\hat{f}') = \bigcup_{i=1}^s C_i'$. Denote respectively by Γ*

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[1]Kawamata calls it a *simple collection*.

and Γ' the derived deformation algebras of $\bigoplus_{i=1}^{t} \mathbb{O}_{C_i}$ and $\bigoplus_{i=1}^{s} \mathbb{O}_{C'_i}$, and by $[w] \in \mathrm{HH}_0(H^0\Gamma)$ and $[w'] \in \mathrm{HH}_0(H^0\Gamma')$ the canonical classes. Suppose there is a triangle equivalence

$$? \otimes_{H^0\Gamma}^{\mathbb{L}} Z \colon D(H^0\Gamma) \xrightarrow{\sim} D(H^0\Gamma')$$

given by a dg bimodule Z such that $\mathrm{HH}_0(Z)$ (defined in [Keller 1998]) takes $[w]$ to $[w']$ in $\mathrm{HH}_0(H^0\Gamma') = \mathrm{HH}_0(\Gamma')$. Then $\hat{X}$ is isomorphic to $\hat{X}'$. In particular, s is equal to t.

For general (nonsimple) flopping contractions, there exist derived equivalent algebras $H^0\Gamma$ and $H^0\Gamma'$ that are nonisomorphic. August [2020a, Theorem 1.4] proves that the isomorphism classes of such algebras in a fixed derived equivalence class of $H^0\Gamma$ are precisely the contraction algebras for the iterated flops of $\hat{Y}$. Different contraction algebras in the same derived equivalence class are related by the iterated mutations of the tilting objects. The mutations are the homological counterpart of flops between different minimal models. We refer to [Wemyss 2018; 2023] for the general framework of the homological minimal model program.

Theorem B says that the underlying singularity type of the smooth minimal models is determined by the derived equivalence class of the pair $(H^0\Gamma, [w])$. We sketch the idea of the proof. From 3–dimensional birational geometry we know that the underlying (isolated) singularity of a smooth flopping contraction is a hypersurface (see Section 2.5). It is a classical theorem of Mather and Yau that, up to isomorphism, a germ of isolated hypersurface singularity is determined by its Tyurina algebra (see [Mather and Yau 1982] for the analytic case and [Greuel and Pham 2017] for the formal case). Next we prove that the derived equivalence class of $H^0\Gamma$ together with the canonical class $[w]$ recovers the Tyurina algebra of the singularity. We solve this problem in two steps. First, we prove that the Tyurina algebra, and therefore the isomorphism class of the hypersurface singularity, can be recovered from the ($\mathbb{Z}$–graded dg enhanced) cluster category $\mathscr{C}_\Gamma$. This result, proved in Section 5, should have independent interest. Secondly, we show that the isomorphism class of the Ginzburg algebra $\mathfrak{D}(Q, w)$ that is quasi-isomorphic to Γ can be recovered from the data $(H^0\Gamma, [w])$. The proof uses a result of Hua and Gui-song Zhou [2023] in noncommutative differential calculus of potentials with finite-dimensional Jacobi algebras. Finally, we prove (in Sections 4.4 and 4.5) that any derived Morita equivalence $D(H^0\Gamma) \simeq D(H^0\Gamma')$ preserving the canonical class yields a derived Morita equivalence $\mathrm{per}(\Gamma) \simeq \mathrm{per}(\Gamma')$.

Note that Hua and Toda [2018] gave an alternative definition of the contraction algebra associated to a flopping contraction using the category of matrix factorizations. In this definition, the contraction algebra carries an additional (compared with the definition in [Donovan and Wemyss 2016]) $\mathbb{Z}/2$–graded A_∞–structure. Hua [2018] proved that the Tyurina algebra of the singularity can be recovered from the $\mathbb{Z}/2$–graded A_∞–structure. Our proof of Theorem B shows that the $\mathbb{Z}/2$–graded A_∞–structure on the contraction algebra can be recovered from the class $[w]$. Theorem B without the condition on the preservation of the canonical class is precisely the generalization of the conjecture by Donovan and Wemyss stated by August [2020a, Conjecture 1.3]. See Conjecture 5.10 for the original conjecture of

Donovan and Wemyss, which is for simple flopping contractions. The generalized form of the conjecture has recently been proved in [Jasso and Muro 2022, Appendix A] — see also [Jasso et al. 2024] — by combining the derived Auslander–Iyama correspondence of [Jasso and Muro 2022] with our Theorem 5.8. The geometric meaning of the class $[w]$ remains to be understood. It is believed that the vanishing of $[w]$ is closely related to the condition that $\widehat{X}$ is quasihomogeneous.

Our third result is a necessary condition on the contractibility of an nc rigid rational curve in a smooth CY 3–fold. Let u be a variable of degree 2.

Theorem C (Proposition 6.9) *Let C be an nc rigid rational curve in a smooth CY 3–fold Y. If C is contractible, then its derived deformation algebra Γ is $k[u^{-1}]$–enhanced (see the definition in Section 6). Moreover, $H^0\Gamma$ is a symmetric Frobenius algebra.*

We conjecture that an nc rigid rational curve C is contractible if and only if Γ is $k[u^{-1}]$–enhanced (see Conjecture 6.8).

The paper is organized as follows. In Section 2 we review basics on derived noncommutative deformation theory, noncommutative crepant resolutions and flopping contractions. Concerning derived deformations, we complement the results of Efimov, Lunts and Orlov [Efimov et al. 2010] by explaining the link to classical deformations in abelian categories in Section 2.3. In Section 3, we discuss various notions of Calabi–Yau structures in geometry and algebra and prove Theorem A. The notion of cluster category is introduced in Section 3.3. When the curve is contractible, there are two cluster categories associated to it: one via derived deformation and the other via the NCCR. We prove that these two cluster categories are algebraically equivalent. In Section 4, we recall the definition of Ginzburg algebras and several results in noncommutative differential calculus, including the noncommutative Mather–Yau theorem and noncommutative Saito theorem. We further show that, for a contractible curve in a CY 3–fold, there exists a Ginzburg algebra weakly equivalent to the derived deformation algebra whose potential is canonically defined up to right equivalence. Then we establish a relation between the silting theory of a nonpositive dg algebra and the silting theory of its zeroth homology. In Section 5, we study the relation between the cluster category associated to the contractible curves and their underlying singularities via Hochschild cohomology. In particular, Theorem B is proved. In Section 6, we introduce the notion of $k[u^{-1}]$–enhancement of dg algebras. For derived deformation algebra, we establish a link between the existence of $k[u^{-1}]$–enhancement and contractibility of rational curve and prove Theorem C.

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2 Preliminaries

2.1 Notation and conventions

Throughout k will be a ground field unless stated otherwise. Unadorned tensor products are over k. Let V be a k–vector space. We denote its dual vector space by DV. When V is graded, DV is understood as the dual in the category of graded vector spaces. For a subspace V' of a complete topological vector space V, we denote the closure of V' in V by $(V')^c$. By definition, a *pseudocompact k–vector space* is a linear topological vector space which is complete and whose topology is generated by subspaces of finite codimension. Following [Van den Bergh 2015], we will denote the corresponding category by $\mathrm{PC}(k)$. We have inverse dualities

$$\mathbb{D}\colon \mathrm{Mod}(k) \to \mathrm{PC}(k)^{\mathrm{op}}, \qquad V \mapsto \mathrm{Hom}_k(V, k),$$

$$\mathbb{D}\colon \mathrm{PC}(k) \to \mathrm{Mod}(k)^{\mathrm{op}}, \qquad W \mapsto \mathrm{Hom}_{\mathrm{PC}(k)}(W, k),$$

where we recall that, for $V \in \mathrm{Mod}(k)$, the topology on $\mathbb{D}V$ is generated by the kernels of $\mathbb{D}V \to \mathbb{D}V'$, where V' runs through the finite-dimensional subspaces of V. Similarly, if V is graded then $\mathbb{D}$ is understood in the graded sense. For the definition of Hom–space and tensor product in $\mathrm{PC}(k)$, we refer to [Van den Bergh 2015, Section 4]. Using the tensor product in $\mathrm{PC}(k)$, we define the pseudocompact dg algebras, modules and bimodules to be the corresponding objects in the category of graded objects of $\mathrm{PC}(k)$. Let A be a pseudocompact dg k–algebra. Denote by $\mathrm{PC}(A^e)$ the category of pseudocompact A–bimodules. We will sometimes take a finite-dimensional separable k–algebra l to be the ground ring. The definition of the duality functor $\mathbb{D}$ on $\mathrm{PC}(l^e)$ requires some extra care due to the noncommutativity of l. We refer to [Van den Bergh 2015, Section 5] for detailed discussion.

Denote by $\mathrm{PCAlgc}(l)$ the category of augmented pseudocompact dg algebras A whose underlying graded algebras have their augmentation ideal equal to their Jacobson radical (see [Van den Bergh 2015, Proposition 4.3 and Section 6]). Our main interest is in the case when $l \cong ke_1 \times ke_2 \times \cdots \times ke_n$ for central orthogonal idempotents $(e_i)_i$. For an object $A \in \mathrm{PCAlgc}(l)$, we use Hom–spaces and tensor products in $\mathrm{PC}(l^e)$ to define the Hochschild and cyclic (co)homology. For details, we refer to [Van den Bergh 2015, Section 7 and Appendix A]. If A is an l–algebra in $\mathrm{PCAlgc}(l)$, we use $\mathrm{HH}_*(A)$, $\mathrm{HH}^*(A)$ and $\mathrm{HC}_*(A)$ to denote the continuous Hochschild homology, cohomology and cyclic homology of A. Because for a pseudocompact dg algebra we will only consider continuous Hochschild homology, cohomology and cyclic homology, there is no risk of confusion. By an abuse of notation, for $A \in \mathrm{PCAlgc}(l)$ we denote by $D(A)$ the pseudocompact derived category of A. Its subcategories $\mathrm{per}(A)$ and $D_{\mathrm{fd}}(A)$ are defined as

the thick subcategory generated by the free A–module A and as the full subcategory of all objects with homology of finite total dimension. Similar to the algebraic case, the notion of homological smoothness can be defined in the pseudocompact setting. We refer to [Keller and Yang 2011, Appendix] for a careful treatment. For the bar–cobar formalism and Koszul duality of pseudocompact dg algebras, we refer to [Van den Bergh 2008, Appendices A and D].

2.2 Derived deformation theory

We briefly recall the setup of derived noncommutative deformation theory of [Efimov et al. 2010]. In this section, we fix a field k. We refer to [Keller 2006] for foundational material on dg categories. For a dg category $\mathscr{A}$, we denote by $D(\mathscr{A})$ the derived category of right dg $\mathscr{A}$–modules. Fix a positive integer n and let l be the separable k–algebra $ke_1 \times \cdots \times ke_n$. An l–*algebra* A is a k–algebra together with a morphism of k–algebras $l \to A$ (note that l is *not* necessarily central in A). An equivalent datum is that of the k–category with n objects $1, \ldots, n$ whose morphism space from i to j is given by $e_j A e_i$. An l–*augmented* (dg) algebra is a (dg) l–algebra $\mathscr{R}$ together with an l–algebra morphism $\mathscr{R} \to l$ such that the composition $l \to \mathscr{R} \to l$ is the identity morphism. Its *augmentation ideal* is the kernel of the augmentation morphism $\mathscr{R} \to l$. An *artinian* l–algebra is an augmented l–algebra whose augmentation ideal is finite-dimensional and nilpotent. A dg l–algebra is *artinian* if it is an augmented dg l–algebra whose augmentation ideal is finite-dimensional and nilpotent. Denote by $\textbf{Art}_l$ and $\textbf{cArt}_l$ the categories of artinian l–algebras and of commutative artinian l–algebras. Denote by $\textbf{dgArt}_l$ the category of artinian dg algebras and by $\textbf{dgArt}_l^-$ the subcategory of $\textbf{dgArt}_l$ consisting of dg algebras concentrated in nonpositive degrees.

Fix a dg category $\mathscr{A}$ and a dg $\mathscr{A}$–module E with a decomposition $E = E_1 \oplus \cdots \oplus E_n$. We view E as an $l^{\mathrm{op}} \otimes \mathscr{A}$–module in the natural way. The *dg endomorphism l–algebra* of E is the dg endomorphism algebra over $\mathscr{A}$ of the sum E viewed as an l–algebra in the natural way. We are going to define a pseudofunctor $\mathrm{Def}(E)$ from $\textbf{dgArt}_l$ to the category $\textbf{Gpd}$ of groupoids. This pseudofunctor assigns to an artinian dg l–algebra $\mathscr{R}$ the groupoid $\mathrm{Def}_\mathscr{R}(E)$ of $\mathscr{R}$–deformations of E in the derived category $D(\mathscr{A})$. We will mostly follow the notation of [Efimov et al. 2009] and identify $\mathscr{R}$ with the dg category with n objects $1, \ldots, n$, where the morphism complex from i to j is given by $e_j \mathscr{R} e_i$. Denote the dg category $\mathscr{R}^{\mathrm{op}} \otimes \mathscr{A}$ by $\mathscr{A}_\mathscr{R}$. The augmentation $\varepsilon \colon \mathscr{R} \to l$ yields the functor of extension of scalars ε^* taking a dg $\mathscr{R}$–module S to the dg $l^{\mathrm{op}} \otimes \mathscr{A}$–module

$$\varepsilon^*(S) = l \otimes_\mathscr{R}^{\mathbb{L}} S.$$

Definition 2.1 [Efimov et al. 2009, Definition 10.1] Fix an artinian augmented dg l–algebra $\mathscr{R}$. An object of the groupoid $\mathrm{Def}_\mathscr{R}(E)$ is a pair (S, σ), where S is an object of $D(\mathscr{A}_\mathscr{R})$ and

$$\sigma \colon \varepsilon^*(S) \to E$$

is an isomorphism in $D(l^{\mathrm{op}} \otimes \mathcal{A})$. A morphism $f : (S, \sigma) \to (T, \tau)$ between two $\mathcal{R}$–deformations of E is an isomorphism $f : S \to T$ in $D(\mathcal{A}_{\mathcal{R}})$ such that

$$\tau \circ \varepsilon^*(f) = \sigma.$$

This defines the groupoid $\mathrm{Def}_{\mathcal{R}}(E)$. A homomorphism of augmented artinian dg l–algebras $\phi : \mathcal{R} \to \mathcal{Q}$ induces the functor

$$\phi^* : \mathrm{Def}_{\mathcal{R}}(E) \to \mathrm{Def}_{\mathcal{Q}}(E)$$

given by $\mathcal{Q} \otimes^{\mathbb{L}}_{\mathcal{R}} ?$. Thus we obtain a pseudofunctor

$$\mathrm{Def}(E) : \mathbf{dgArt}_l \to \mathbf{Gpd}.$$

We call $\mathrm{Def}(E)$ the *pseudofunctor of derived deformations of E*. We denote by $\mathrm{Def}_-(E)$ the restrictions of the pseudofunctor $\mathrm{Def}(E)$ to the subcategory $\mathbf{dgArt}_l^-$.

The category of augmented dg l–algebras can be naturally enhanced to a weak 2–category. We refer to [Efimov et al. 2010, Definition 11.1] for the precise definition of the 2–category structure. In particular, we denote the corresponding 2–categorical enhancements of $\mathbf{dgArt}_l$, $\mathbf{dgArt}_l^-$ and $\mathbf{Art}_l$ by $\mathbf{2\text{–}dgArt}_l$, $\mathbf{2\text{–}dgArt}_l^-$ and $\mathbf{2\text{–}Art}_l$ (in [Efimov et al. 2010], they are denoted by $\mathbf{2'\text{–}dgArt}_l$ etc.). By [Efimov et al. 2010, Proposition 11.4], there exists a pseudofunctor $\mathrm{DEF}(E)$ from $\mathbf{2\text{–}dgArt}_l$ to $\mathbf{Gpd}$ and which is an extension to $\mathbf{2\text{–}dgArt}_l$ of the pseudofunctor $\mathrm{Def}(E)$. Similarly, there exists a pseudofunctor $\mathrm{DEF}_-(E)$ extending $\mathrm{Def}_-(E)$.

The main theorem of [Efimov et al. 2010] is:

Theorem 2.2 [Efimov et al. 2010, Theorems 15.1 and 15.2] *Let $E_1, \ldots, E_n$ be a collection of objects in $D(\mathcal{A})$. Let E be the direct sum of the E_i and C the extension algebra $\mathrm{Ext}^*_{\mathcal{A}}(E, E)$ considered as a graded l–algebra. Assume that*

(a) *$C^p = 0$ for all $p < 0$;*

(b) *$C^0 = l$;*

(c) *$\dim_k C^p < \infty$ for all p and $\dim_k C^p = 0$ for all $p \gg 0$.*

Denote by $\mathcal{C}$ the dg endomorphism l–algebra of E. Let A be a strictly unital minimal model of $\mathcal{C}$. Then the pseudofunctor $\mathrm{DEF}_-(E)$ is pro-representable by the dg l–algebra $\Gamma = \mathbb{D}BA$, where B denotes the bar construction. That is, there exists an equivalence of pseudofunctors $\mathrm{DEF}_-(E) \simeq h_{\Gamma}$ from $\mathbf{2\text{–}dgArt}_l^-$ to $\mathbf{Gpd}$, where h_{Γ} denotes the groupoid of 1–morphisms 1–$\mathrm{Hom}(\Gamma, ?)$.

In the case where the dg category $\mathcal{A}$ is given by an algebra A concentrated in degree 0 and E is a 1–dimensional A–module, Booth [2018, Theorem 3.5.9] obtains an analogous pro-representability result for the set-valued framed deformation functor $\mathrm{Def}_A^{\mathrm{fr}, \leq 0}(E)$ without having to impose the finiteness condition (c).

Let Y be a smooth algebraic variety. A collection of compactly supported coherent sheaves $\mathscr{E}_1, \ldots, \mathscr{E}_t$ on Y is called *semisimple* if $\mathrm{Hom}_Y(\mathscr{E}_i, \mathscr{E}_i) \cong k$ for all i and $\mathrm{Hom}_Y(\mathscr{E}_i, \mathscr{E}_j) = 0$ for all $i \neq j$. The finiteness assumption in Theorem 2.2 is satisfied by any semisimple collection. Let $\mathscr{E}$ be the direct sum of such a collection of coherent sheaves on Y. We may denote the completion of Y along the support of $\mathscr{E}$ by $\widehat{Y}$.

Corollary 2.3 *Given a semisimple collection $\{\mathscr{E}_i\}_{i=1}^t$ in $D^b(\mathrm{coh}\,\widehat{Y})$, denote by $\mathscr{C}$ the dg endomorphism l–algebra of $\mathscr{E} := \bigoplus_{i=1}^t \mathscr{E}_i$. Let A be a strictly unital minimal model of $\mathscr{C}$ and $\Gamma = \mathbb{D}BA$. Then there is an equivalence*

$$\mathrm{DEF}_-(\mathscr{E}) \simeq h_\Gamma.$$

We call Γ the *derived deformation algebra* of the collection $\{\mathscr{E}_i\}_{i=1}^t$ in Y. When we want to emphasize the dependence on Y and $\{\mathscr{E}_i\}_{i=1}^t$, Γ is replaced by $\Gamma_{\mathscr{E}}^Y$. The semisimple collection $\{\mathscr{E}_i\}_{i=1}^t$ is called *nc rigid* if $\dim_k H^0(\Gamma_{\mathscr{E}}^Y) < \infty$. We are mainly interested in the case when $\{\mathscr{E}_i\}_{i=1}^t$ is (the structure sheaves of) a collection of smooth rational curves $C_1, \ldots, C_t$ that satisfies the condition that $\mathrm{Hom}_Y(\mathcal{O}_{C_i}, \mathcal{O}_{C_j}) = 0$ for $i \neq j$. For such a collection of rational curves $C := \{C_i\}_{i=1}^n$, where we write Γ_C^Y for $\Gamma_{\{\mathcal{O}_{C_i}\}_{i=1}^t}^Y$.

In the context of classical noncommutative deformation theory, the representability of noncommutative deformations of contractible rational curves was proved by Donovan and Wemyss:

Theorem 2.4 [Donovan and Wemyss 2016, Proposition 3.1 and Corollary 3.3] *Let $f \colon Y \to X$ be a simple flopping contraction of 3–folds (see the definition in Section 2.5) and let C be the reduced exceptional fiber of f. The functor*

$$\pi_0(\mathrm{Def}^{\mathrm{cl}}(\mathcal{O}_C)) \colon \mathbf{Art}_k \to \mathbf{Set}$$

*is representable. The artinian algebra Λ representing it is called the **contraction algebra** associated to $f \colon Y \to X$.*

The definition of the classical deformation functor $\mathrm{Def}^{\mathrm{cl}}$ is recalled in the next section (see [Donovan and Wemyss 2016, Section 2]). If Γ is the derived deformation algebra of C (with $t = 1$), it follows from the above theorem and Theorem 2.5 below that the contraction algebra Λ is isomorphic to $H^0\Gamma$. Indeed, they both represent the same deformation functor and this determines them up to (nonunique) isomorphism (see [Segal 2008, Proof of Theorem 2.14]).

2.3 Link to classical deformations

Let $\mathscr{A}$ be a dg category. Let $\mathscr{H} \subset D(\mathscr{A})$ be the heart of a t–structure on $D(\mathscr{A})$. We assume that $\mathscr{H}$ is *faithful*, ie the higher extension groups computed in $\mathscr{H}$ are canonically isomorphic to those computed in $D(\mathscr{A})$.

Let $\mathscr{R}$ be an augmented artinian l–algebra. By an $\mathscr{R}$–*module in* $\mathscr{H}$, we mean an object M of $\mathscr{H}$ endowed with an algebra homomorphism $\mathscr{R} \to \mathrm{End}(M)$. Given such an $\mathscr{R}$–module, we denote by $? \otimes_{\mathscr{R}} M$ the unique right exact functor $\mathrm{mod}\,\mathscr{R} \to \mathscr{H}$ extending the obvious additive functor $\mathrm{proj}\,\mathscr{R} \to \mathscr{H}$ taking $\mathscr{R}$ to M. Here we denote by $\mathrm{proj}\,\mathscr{R}$ the category of finitely generated projective (right) $\mathscr{R}$–modules and by $\mathrm{mod}\,\mathscr{R}$ the category of finitely generated $\mathscr{R}$–modules. Notice that $? \otimes_{\mathscr{R}}$ is a left Kan extension and thus unique up to unique isomorphism and functorial in M. It can be computed using projective resolutions. It is obvious how to define morphisms of $\mathscr{R}$–modules in $\mathscr{H}$.

Let E be the direct sum of a collection of n objects $E_1, \ldots, E_n$ of $\mathscr{H}$. We view E as an l–module in $\mathscr{H}$ in the natural way. For an augmented artinian l–algebra $\mathscr{R}$, we define the groupoid $\mathrm{Def}^{\mathrm{cl}}_{\mathscr{R}}(E)$ of classical deformations of E as follows: Its objects are pairs (M, μ) where M is an $\mathscr{R}$–module in $\mathscr{H}$ such that the functor $? \otimes_{\mathscr{R}} M$ is exact and $\mu \colon l \otimes_{\mathscr{R}} M \xrightarrow{\sim} E$ is an isomorphism of l–modules in $\mathscr{H}$. A morphism $(L, \lambda) \to (M, \mu)$ is an isomorphism $f \colon L \to M$ of $\mathscr{R}$–modules in $\mathscr{H}$ such that $\mu \circ (l \otimes_{\mathscr{R}} f) = \lambda$.

For an augmented l–algebra A and an augmented artinian l–algebra $\mathscr{R}$, we define $G(A, \mathscr{R})$ to be the groupoid whose objects are the morphisms $A \to \mathscr{R}$ of augmented l–algebras and whose morphisms $\phi_1 \to \phi_2$ are the invertible elements r of $\mathscr{R}$ such that $\phi_2(a) = r\phi_1(a)r^{-1}$ for all a in A.

Theorem 2.5 *Suppose that, in addition to the above assumptions, E satisfies the hypotheses of Theorem 2.2. Let Γ be the pseudocompact dg l–algebra defined there. Let $\mathscr{R}$ be an augmented artinian l–algebra. Then $H^0\Gamma$ represents the classical deformations of E in the sense that there is an equivalence of groupoids*

$$\mathrm{Def}^{\mathrm{cl}}_{\mathscr{R}}(E) \xrightarrow{\sim} G(H^0\Gamma, \mathscr{R}).$$

Proof By Theorem 2.2, we have an equivalence of groupoids

$$\mathrm{DEF}_-(E)(\mathscr{R}) \xrightarrow{\sim} 1\text{–Hom}(\Gamma, \mathscr{R}),$$

where 1–Hom denotes the groupoid of 1–morphisms in **2–dgArt**. We will show that $\mathrm{DEF}_-(E)(\mathscr{R})$ is equivalent to $\mathrm{Def}^{\mathrm{cl}}_{\mathscr{R}}(E)$ and $1\text{–Hom}(\Gamma, \mathscr{R})$ is equivalent to $G(H^0\Gamma, \mathscr{R})$. We start with the second equivalence. By [Efimov et al. 2010, Definition 11.1], an object of $1\text{–Hom}(\Gamma, \mathscr{R})$ is a pair (M, θ) consisting of

- a dg bimodule M in $D(\Gamma^{\mathrm{op}} \otimes \mathscr{R})$ such that the restriction to $\mathscr{R}$ of M is isomorphic to $\mathscr{R}$ in $D(\mathscr{R})$ and

- an isomorphism $\theta \colon M \otimes^{\mathbb{L}}_{\mathscr{R}} l \to l$ in $D(\Gamma^{\mathrm{op}})$.

A 2–morphism $f \colon (M_1, \theta_1) \to (M_2, \theta_2)$ is an isomorphism $f \colon M_1 \to M_2$ in $D(\Gamma^{\mathrm{op}} \otimes \mathscr{R})$ such that $\theta_2 \circ (f \otimes^{\mathbb{L}}_{\mathscr{R}} l) = \theta_1$. We define a functor $F \colon G(H^0\Gamma, \mathscr{R}) \to 1\text{–Hom}(\Gamma, \mathscr{R})$ as follows: Let $\phi \colon H^0(\Gamma) \to \mathscr{R}$ be a morphism of augmented l–algebras. Since Γ is concentrated in degrees ≤ 0, we have a canonical algebra morphism $\Gamma \to H^0(\Gamma)$. By composing it with ϕ we get a morphism of augmented dg l–algebras

$\Gamma \to \mathcal{R}$. It defines a structure of dg bimodule M on $\mathcal{R}$. We put $F\phi = (M, \theta)$, where $\theta \colon \mathcal{R} \otimes_{\mathcal{R}}^{\mathbb{L}} l \to l$ is the canonical isomorphism. Now let ϕ_1 and ϕ_2 be two morphisms of augmented algebras $H^0(\Gamma) \to \mathcal{R}$. Put $(M_i, \theta_i) = F\phi_i$ for $i = 1, 2$. Let r an be an invertible element of $\mathcal{R}$ such that $\phi_2(a) = r\phi_1(a)r^{-1}$ for all a in $H^0(\Gamma)$. Then it is clear that left multiplication with r defines an isomorphism of bimodules $M_1 \to M_2$ compatible with the θ_i. Recall that $\Gamma^{\mathrm{op}} \otimes \mathcal{R}$ is concentrated in degrees ≤ 0, so its derived category has a canonical t–structure. Since the M_i live in the heart of this t–structure on $D(\Gamma^{\mathrm{op}} \otimes \mathcal{R})$, it is also clear that F is fully faithful. It remains to be checked that F is essentially surjective. So let (M, θ) be given. Since M is quasi-isomorphic to $\mathcal{R}$ when restricted to $\mathcal{R}$, its homology is concentrated in degree 0. We can therefore replace M with $H^0(M)$, which is an ordinary $H^0(\Gamma)$–$\mathcal{R}$–bimodule isomorphic to $\mathcal{R}$ as a right $\mathcal{R}$–module (we also consider it as a left Γ–module via the canonical morphism $\Gamma \to H^0(\Gamma)$). In particular, M is right projective and so $M \otimes_{\mathcal{R}}^{\mathbb{L}} ? = M \otimes_{\mathcal{R}} ?$. We choose an isomorphism $f \colon M \xrightarrow{\sim} \mathcal{R}$ of right $\mathcal{R}$–modules. After multiplying f with an invertible element of l, we may assume that $f \otimes_{\mathcal{R}} l = \theta$. The left Γ–module structure on M yields an algebra morphism

$$\phi \colon H^0(\Gamma) \to \mathrm{End}_{\mathcal{R}}(M) \xrightarrow{\sim} \mathrm{End}_{\mathcal{R}}(\mathcal{R}) = \mathcal{R}.$$

It is clear that f yields an isomorphism between (M, θ) and $F\phi$.

We now construct an equivalence from $\mathrm{DEF}_-(E)(\mathcal{R})$ to $\mathrm{Def}_{\mathcal{R}}^{\mathrm{cl}}(E)$. Recall from [Efimov et al. 2010, Proposition 11.4] that the groupoid $\mathrm{DEF}_-(E)(\mathcal{R})$ equals the groupoid $\mathrm{Def}_{\mathcal{R}}(E)$ of Definition 2.1 (but DEF_- has enhanced 2–functoriality). Let $P \to E$ be a cofibrant resolution of E. Since the graded algebra $\mathrm{Ext}^*(E, E)$ has vanishing components in degree -1 and in all sufficiently high degrees, we can apply [Efimov et al. 2010, Theorem 11.8] to conclude that the groupoid $\mathrm{Def}_{\mathcal{R}}(E)$ is equivalent to the groupoid $\mathrm{Def}_{\mathcal{R}}^h(P)$ of homotopy deformations of [Efimov et al. 2009, Definition 4.1]. We now construct an equivalence F from $\mathrm{Def}_{\mathcal{R}}^h(P)$ to $\mathrm{Def}_{\mathcal{R}}^{\mathrm{cl}}(E)$. Let (S, σ) be an object of $\mathrm{Def}_{\mathcal{R}}^h(P)$. We may assume that $S = \mathcal{R} \otimes_l P$ as a graded bimodule and that σ is the canonical isomorphism $l \otimes_{\mathcal{R}} (\mathcal{R} \otimes P) \xrightarrow{\sim} P$. Let I denote the augmentation ideal of $\mathcal{R}$. Then S has a finite filtration by the dg submodules $I^p S$ for $p \geq 0$, and each subquotient is isomorphic to a summand of a finite sum of copies of $l \otimes_{\mathcal{R}} S = P$. Thus, the underlying dg $\mathcal{A}$–module M of S is isomorphic in $D(\mathcal{A})$ to a finite iterated extension of objects of $\mathrm{add}(E)$, the subcategory of direct factors of finite direct sums of copies of E. Therefore, M still lies in the heart $\mathcal{H}$. Note that, as shown in [Efimov et al. 2010, Proof of Theorem 11.8], S is cofibrant over $\mathcal{R}^{\mathrm{op}} \otimes \mathcal{A}$. Therefore, M is cofibrant over $\mathcal{A}$. The left $\mathcal{R}$–module structure on S yields an algebra homomorphism $\mathcal{R} \to \mathrm{End}(M)$. Since each object of $\mathrm{mod}\,\mathcal{R}$ is a finite iterated extension of 1–dimensional l–modules, the functor $? \otimes_{\mathcal{R}} S \colon D(\mathcal{R}) \to D(\mathcal{A})$ takes $\mathrm{mod}\,\mathcal{R}$ to $\mathcal{H}$. Since $? \otimes_{\mathcal{R}} S$ is a triangle functor, the induced functor $\mathrm{mod}\,\mathcal{R} \to \mathcal{H}$ is exact. Clearly it restricts to the natural functor $\mathrm{proj}\,\mathcal{R} \to \mathcal{H}$ and is therefore isomorphic to $? \otimes_{\mathcal{R}} M \colon \mathrm{mod}\,\mathcal{R} \to \mathcal{H}$. Finally, the isomorphism $l \otimes_{\mathcal{R}} S \xrightarrow{\sim} E$ yields an isomorphism $\mu \colon l \otimes_{\mathcal{R}} M \xrightarrow{\sim} E$. In this way, to an object (S, σ) of $\mathrm{Def}_{\mathcal{R}}^h(P)$, we have associated an object $F(S, \sigma) = (M, \mu)$ of $\mathrm{Def}_{\mathcal{R}}^{\mathrm{cl}}(E)$. Notice that, by what we have just shown, we may also describe $M \in \mathcal{H}$ as the zeroth homology $H_{\mathcal{H}}^0(S)$ with respect to the t–structure associated with $\mathcal{H}$, that we have an isomorphism $l \otimes_{\mathcal{R}} M \xrightarrow{\sim} H_{\mathcal{H}}^0(l \otimes_{\mathcal{R}}^{\mathbb{L}} S)$ and that the isomorphism $\mu \colon l \otimes_{\mathcal{R}} M \xrightarrow{\sim} E$ is induced by $\sigma \colon l \otimes_{\mathcal{R}}^{\mathbb{L}} S \xrightarrow{\sim} E$. Recall that a morphism

$(S_1, \sigma_1) \to (S_2, \sigma_2)$ of $\mathrm{Def}_{\mathcal{R}}^h(P)$ is a class of isomorphisms $S_1 \to S_2$ of dg $\mathcal{R}^{\mathrm{op}} \otimes \mathcal{A}$–modules compatible with the σ_i modulo homotopies compatible with the σ_i. Since the functor $\mathrm{Def}_{\mathcal{R}}^h(P) \to \mathrm{Def}_{\mathcal{R}}(E)$ is an equivalence, these morphisms are in bijection with the isomorphisms $S_1 \to S_2$ of $D(\mathcal{R}^{\mathrm{op}} \otimes \mathcal{A})$ compatible with the σ_i. Clearly each such morphism induces an isomorphism $(M_1, \lambda_1) \to (M_2, \lambda_2)$, where $(M_i, \lambda_i) = F(S_i, \sigma_i)$ for $i = 1, 2$. It follows from Lemma 2.6 below that this assignment is a bijection. It remains to be shown that $F \colon \mathrm{Def}_{\mathcal{R}}^h(P) \to \mathrm{Def}_{\mathcal{R}}^{\mathrm{cl}}(E)$ is essentially surjective. Since we have an equivalence $\mathrm{Def}_{\mathcal{R}}^h(P) \xrightarrow{\sim} \mathrm{Def}_{\mathcal{R}}(E)$, it suffices to lift a given object (M, μ) of $\mathrm{Def}_{\mathcal{R}}^{\mathrm{cl}}(E)$ to an object (S, σ) of $\mathrm{Def}_{\mathcal{R}}(E)$. Let A denote the dg endomorphism l–algebra $\mathbb{R}\mathrm{Hom}_{\mathcal{A}}(M, M)$. Then M becomes canonically an object of $D(A^{\mathrm{op}} \otimes \mathcal{A})$. Now, since M is in the heart of a t–structure, its negative self-extension groups vanish and we have a quasi-isomorphism $\tau_{\leq 0} A \xrightarrow{\sim} \mathrm{End}_{\mathcal{H}}(M)$. Thus, in the homotopy category of dg l–algebras, we have a morphism

$$\mathcal{R} \to \mathrm{End}_{\mathcal{H}}(M) \xrightarrow{\sim} \tau_{\leq 0} A \to A.$$

By tensoring with $\mathcal{A}$ we obtain a morphism $\mathcal{R}^{\mathrm{op}} \otimes \mathcal{A} \to A^{\mathrm{op}} \otimes \mathcal{A}$ in the homotopy category of dg categories. The associated restriction functor $D(A^{\mathrm{op}} \otimes \mathcal{A}) \to D(\mathcal{R}^{\mathrm{op}} \otimes \mathcal{A})$ sends M to an object S of $D(\mathcal{R}^{\mathrm{op}} \otimes \mathcal{A})$. By construction, the restriction of S to $\mathcal{A}$ is isomorphic to M in $D(\mathcal{A})$ and the left action of $\mathcal{R}$ on S induces the given algebra morphism $\mathcal{R} \to \mathrm{End}_{\mathcal{H}}(M)$. Since $\mathcal{H}$ is the heart of a t–structure, we have a canonical realization functor $D^b(\mathcal{H}) \to D(\mathcal{A})$ extending the inclusion $\mathcal{H} \to D(\mathcal{A})$; see [Beĭlinson et al. 1982, Section 3.1.10] or [Keller and Vossieck 1987, Section 3.2]. Moreover, since $\mathcal{H}$ is faithful, the realization functor is fully faithful. Since we only know how to compare tensor functors, we use a different construction to extend the inclusion $\mathcal{H} \to D(\mathcal{A})$ to a triangle functor $D^b(\mathcal{H}) \to D(\mathcal{A})$. Let $\mathcal{H}_{\mathrm{dg}}$ be the full subcategory of the dg category of right $\mathcal{A}$–modules formed by cofibrant resolutions of the objects of $\mathcal{H}$. We have an equivalence of k–categories $\mathcal{H} \xrightarrow{\sim} H^0(\mathcal{H}_{\mathrm{dg}})$. Since $\mathcal{H}$ is the heart of a t–structure, the homology of the dg category $\mathcal{H}_{\mathrm{dg}}$ is concentrated in degrees ≥ 0. Thus, we have quasiequivalences $\tau_{\leq 0} \mathcal{H}_{\mathrm{dg}} \xrightarrow{\sim} H^0(\mathcal{H}_{\mathrm{dg}}) \xrightarrow{\sim} \mathcal{H}$. Therefore, in the homotopy category of dg categories, we obtain a morphism

$$\mathcal{H} \to \tau_{\leq 0}(\mathcal{H}_{\mathrm{dg}}) \to \mathcal{H}_{\mathrm{dg}} \to D_{\mathrm{dg}}(\mathcal{A}),$$

where $D_{\mathrm{dg}}(\mathcal{A})$ denotes the dg category of cofibrant dg $\mathcal{A}$–modules. It gives rise to an $\mathcal{H}$–$\mathcal{A}$–bimodule R. Let $\mathcal{H}^b(\mathcal{H})$ denote the category modulo homotopy of bounded complexes of objects of $\mathcal{H}$. Using the fact that short exact sequences of $\mathcal{H}$ give rise to triangles in $D(\mathcal{A})$, one checks that the induced functor $? \otimes_{\mathcal{H}} R \colon \mathcal{H}^b(\mathcal{H}) \to D(\mathcal{A})$ vanishes on the bounded acyclic complexes and therefore induces a triangle functor $D^b(\mathcal{H}) \to D(\mathcal{A})$, still denoted by $? \otimes_{\mathcal{H}} R$. Since $\mathcal{H}$ is a faithful heart, one obtains that this triangle functor is fully faithful. We claim that we have a square of triangle functors, commutative up to isomorphism,

$$
\begin{CD}
D^b(\mathrm{mod}\,\mathcal{R}) @>{? \otimes_{\mathcal{R}} M}>> D^b(\mathcal{H}) \\
@VVV @VV{? \otimes_{\mathcal{H}} R}V \\
D(\mathcal{R}) @>{? \otimes_{\mathcal{A}}^{\mathbb{L}} S}>> D(\mathcal{A})
\end{CD}
$$

To check this, one has to check that the bimodules S and $M \otimes_{\mathcal{H}} R$ are isomorphic in $D(\mathcal{R}^{\mathrm{op}} \otimes \mathcal{A})$. This is easy using Lemma 2.6 below. Since $? \otimes_{\mathcal{R}} M : \mathrm{mod}\,\mathcal{R} \to \mathcal{H}$ is exact, the given isomorphism $l \otimes_{\mathcal{R}} M \xrightarrow{\sim} E$ yields an isomorphism $l \otimes_{\mathcal{R}}^{\mathbb{L}} M \xrightarrow{\sim} E$ in $D^b(\mathcal{H})$ and thus an isomorphism $\sigma : l \otimes_{\mathcal{R}}^{\mathbb{L}} S \xrightarrow{\sim} E$ in $D(\mathcal{A})$. It is now clear that we can recover M as $H_{\mathcal{R}}^0(S)$ and that $\mu : l \otimes_{\mathcal{R}} M \xrightarrow{\sim} E$ is the morphism induced by $\sigma : l \otimes_{\mathcal{R}}^{\mathbb{L}} S \xrightarrow{\sim} E$ in $H_{\mathcal{R}}^0$. By the description of F given above, this shows that we do have $F(S, \sigma) \xrightarrow{\sim} (M, \mu)$. $\qquad\square$

Let $\mathcal{A}$ be a dg category and B an ordinary k–algebra. Let X and Y be objects of $D(B^{\mathrm{op}} \otimes \mathcal{A})$ and let $\mathrm{res}(X)$ be the restriction of X to $\mathcal{A}$. The left action of B on X defines an algebra morphism

$$\alpha_X : B \to \mathrm{End}_{D(\mathcal{A})}(\mathrm{res}(X)).$$

Let $\mathcal{M}(X, Y)$ be the space of all morphisms $f : \mathrm{res}(X) \to \mathrm{res}(Y)$ in $D(\mathcal{A})$ such that

$$f \circ \alpha_X(b) = \alpha_Y(b) \circ f$$

for all $b \in B$. The restriction functor induces a natural map

$$\Phi : \mathrm{Hom}_{D(B^{\mathrm{op}} \otimes \mathcal{A})}(X, Y) \to \mathcal{M}(X, Y).$$

Lemma 2.6 *If*

$$\mathrm{Hom}_{D(\mathcal{A})}(\mathrm{res}(X), \Sigma^{-n}\mathrm{res}(Y)) = 0$$

for all $n > 0$, the map Φ is bijective.

Proof We adapt the argument of [Keller 2000, Section 5]. We may suppose that X is cofibrant over $B^{\mathrm{op}} \otimes \mathcal{A}$ and in particular cofibrant over $\mathcal{A}$. Then the sum total dg module of the bar resolution

$$\cdots \to B \otimes B^{\otimes p} \otimes X \to \cdots \to B \otimes B \otimes X \to B \otimes X \to 0$$

is still cofibrant over $B^{\mathrm{op}} \otimes \mathcal{A}$ and quasi-isomorphic to X. We use it to compute $\mathrm{Hom}_{D(B^{\mathrm{op}} \otimes \mathcal{A})}(X, Y)$. By applying $\mathrm{Hom}_{B^{\mathrm{op}} \otimes \mathcal{A}}(?, Y)$ to the bar resolution, we get a double complex D of the form

$$\mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{A}}(B \otimes X, Y) \to \cdots \to \mathrm{Hom}_{\mathcal{A}}(B^{\otimes p} \otimes X, Y) \to \cdots .$$

We have to compute H^0 of the product total complex $\mathrm{Tot}^{\Pi} D$. Let $D_{\geq 0}$ be the double complex obtained by applying the intelligent truncation functor $\tau_{\geq 0}$ to each column of D. Let $D_{<0}$ be the kernel of $D \to D_{\geq 0}$. We claim that the product total complex of $D_{<0}$ is acyclic. Indeed, the homology of the pth column of $D_{<0}$ in degree $-q$ is isomorphic to

$$\mathrm{Hom}_{D(\mathcal{A})}(B^{\otimes p} \otimes X, \Sigma^{-q}Y).$$

It vanishes for $-q < 0$ by our assumption. To show that the product total complex of $D_{<0}$ is acyclic, we consider the column filtration $F_p D_{<0}$. Then $D_{<0}$ is the inverse limit of the $F_p D_{<0}$. By induction on p, each $F_p D_{<0}$ has an acyclic total complex. Moreover, the transition maps $F_{p+1} D_{<0} \to F_p D_{<0}$ induce

componentwise surjections in the total complexes. It follows that the inverse limit of the total complexes of the $F_p D_{<0}$ is still acyclic and this inverse limit is the product total complex of $D_{<0}$. So it is enough to compute H^0 of $\mathrm{Tot}^{\Pi} D_{\geq 0}$. For this, let us denote by

$$\overline{\mathrm{Hom}}^0_{\mathscr{A}}(X, Y)$$

the quotient of $\mathrm{Hom}^0_{\mathscr{A}}(X, Y)$ by the nullhomotopic morphisms and similarly for $\overline{\mathrm{Hom}}^0_{\mathscr{A}}(B \otimes X, Y)$. The space we have to compute is the homology in degree 0 of the total complex of the double complex

$$
\begin{array}{ccc}
\mathrm{Hom}^1_{\mathscr{A}}(X, Y) & \longrightarrow & \mathrm{Hom}^1_{\mathscr{A}}(B \otimes X, Y) \\
\uparrow & & \uparrow \\
\overline{\mathrm{Hom}}^0_{\mathscr{A}}(X, Y) & \longrightarrow & \overline{\mathrm{Hom}}^0_{\mathscr{A}}(B \otimes X, Y)
\end{array}
$$

where the lower left corner is in bidegree $(0, 0)$. This equals the intersection of the kernels of the maps from bidegree $(0, 0)$ to bidegrees $(0, 1)$ and $(1, 0)$. The kernels of the vertical maps are respectively $\mathrm{Hom}_{D(\mathscr{A})}(X, Y)$ and $\mathrm{Hom}_{D(\mathscr{A})}(B \otimes X, Y)$ and the space we have to compute thus identifies with the kernel of the map

$$\mathrm{Hom}_{D(\mathscr{A})}(X, Y) \to \mathrm{Hom}_{D(\mathscr{A})}(B \otimes X, Y),$$

which is easily seen to take $f : X \to Y$ to $f \circ \alpha_X - \alpha_Y \circ f$. Thus, the homology to be computed is isomorphic to $\mathcal{M}(X, Y)$. $\qquad\square$

2.4 Noncommutative crepant resolutions

Definition 2.7 Let $(R, \mathfrak{m})$ be a complete commutative noetherian local Gorenstein k–algebra of Krull dimension n with isolated singularity and with residue field k. Denote the category of maximal Cohen–Macaulay (MCM) modules by CM_R and its stable category by $\underline{\mathrm{CM}}_R$. Let $N_0 = R, N_1, N_2, \ldots, N_t$ be pairwise nonisomorphic indecomposables in CM_R and $A := \mathrm{End}_R\big(\bigoplus_{i=0}^t N_i\big)$. We call A a *noncommutative resolution* (NCR) of R if it has finite global dimension. An NCR is called a *noncommutative crepant resolution* (NCCR) if A further satisfies that

(a) $A \in \mathrm{CM}_R$,

(b) $\mathrm{gldim}(A) = n$.

If A is an NCCR, we call $\bigoplus_{i=0}^t N_i$ a *tilting module*. Under the above conditions, Iyama shows that $\bigoplus_{i=1}^t N_i$ is a *cluster tilting object* (see the definition in Section 3.3) in $\underline{\mathrm{CM}}_R$. Denote $\bar{l}$ for $A/\mathrm{rad}A$ and e_0 for the idempotent given by the projection $R \oplus \bigoplus_{i=1}^t N_i \to R$. Let $S_0, S_1, \ldots, S_t$ be the simple A–modules with S_0 corresponding to the summand R of $R \oplus \bigoplus_{i=1}^t N_i$. De Thanhoffer de Völcsey and Van den Bergh prove that $\underline{\mathrm{CM}}_R$ admits an explicit dg model in this case:

Theorem 2.8 [de Thanhoffer de Völcsey and Van den Bergh 2016, Theorem 1.1] *There exists a finite-dimensional graded $\bar{l}$–bimodule V and a minimal model $(\widehat{T}_{\bar{l}}V, d) \xrightarrow{\sim} A$ for A, where $\widehat{T}_{\bar{l}}V$ is the graded completion of the ordinary tensor algebra $T_{\bar{l}}V$ with respect to the two-sided ideal generated by V. Put $\Gamma = \widehat{T}_{\bar{l}}V/\widehat{T}_{\bar{l}}Ve_0\widehat{T}_{\bar{l}}V$. Then*

$$\underline{\mathrm{CM}}_R \cong \mathrm{per}(\Gamma)/\mathbf{thick}(S_1, \ldots, S_t)$$

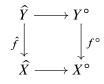*and, furthermore, Γ has the following properties:*

(1) *Γ has finite-dimensional cohomology in each degree.*

(2) *Γ is concentrated in negative degrees.*

(3) *$H^0\Gamma = A/Ae_0A$.*

(4) *As a graded algebra, Γ is of the form $\widehat{T}_l V^0$ for $V^0 = (1-e_0)V(1-e_0)$ with $l := \bar{l}/ke_0$.*

2.5 Flopping contraction

Definition 2.9 A smooth rational curve C in a normal variety Y is called *contractible* if there exists an open subscheme $Y^\circ \subset Y$ containing C and a proper birational morphism $f^\circ : Y^\circ \to X^\circ$ such that

(1) *X° is normal,*

(2) *the exceptional locus $\mathrm{Ex}(f^\circ)$ contains C,*

(3) *f° is an isomorphism in codimension 1.*

The above definition of contractibility is more restrictive than the standard one since it rules out the divisorial contraction (by the last condition). If Y is a 3–fold (which is our main interest), then $\mathrm{Ex}(f)$ must have dimension 1 by condition (3). However, it may contain other components besides C. If C is a contractible curve in Y, denote by $\widehat{X}$ the formal completion of X° along the exceptional subscheme, ie where f° is not an isomorphism. Consider the Cartesian diagram

$$\begin{array}{ccc} \widehat{Y} & \longrightarrow & Y^\circ \\ \widehat{f} \downarrow & & \downarrow f^\circ \\ \widehat{X} & \longrightarrow & X^\circ \end{array}$$

where $\widehat{Y}$ is the fiber product. We call $\widehat{f} : \widehat{Y} \to \widehat{X}$ the *formal contraction* associate to the contraction $f^\circ : Y^\circ \to X^\circ$.

The following definition is a special case of [Kollár and Mori 1998, Definition 6.10]:

Definition 2.10 Let Y be a normal variety of dimension 3. A *flopping contraction* is a proper birational morphism $f : Y \to X$ to a normal variety Y such that f is an isomorphism in codimension 1, and K_Y is f–trivial. If Y is smooth, then we call f a smooth flopping contraction.

In this paper, we only consider smooth flopping contractions unless stated otherwise. Given a 3–dimensional flopping contraction $f : Y \to X$, let D be a divisor on Y such that $-(K_Y + D)$ is f–ample. By [Kollár and Mori 1998, Theorem 6.14], there exists a D–flop $f^+ : Y^+ \to X$. To be more precise, f^+ is a proper birational morphism that is an isomorphism in codimension 1, and $K_{Y^+} + D^+$ is f^+–ample, where D^+ is the birational transform of D on Y^+. In particular, X is Gorenstein terminal. Without loss of generality, we may work locally near the exceptional fiber of f. By the classification theorem of 3–dimensional terminal singularities, X has an isolated cDV singularity (see [Kollár and Mori 1998, Corollary 5.38]). Recall that a 3–fold singularity $(X, 0)$ is called cDV if a generic hypersurface section $0 \in H \subset X$ is a Du Val singularity. Because H has embedded dimension 3, X has embedded dimension 4, ie X is a hypersurface.

Denote by $\mathrm{Ex}(f)$ the reduced exceptional fiber of f. It is well known that $\mathrm{Ex}(f)$ is a tree of rational curves

$$\mathrm{Ex}(f) = \bigcup_{i=1}^{t} C_i$$

with normal crossings such that $C_i \cong \mathbb{P}^1$ (see [Van den Bergh 2004, Lemma 3.4.1]). We call a 3–dimensional flopping contraction $f : Y \to X$ *simple* if $\mathrm{Ex}(f) \cong \mathbb{P}^1$. Let p be the singular point of X. By the remark above, $R := \widehat{\mathcal{O}}_{X,p}$ is a complete local ring of the form $k[\![x, y, u, v]\!]/(g)$. Now we set $\widehat{X} := \mathrm{Spec}\, R$ and $\widehat{Y} := \widehat{X} \times_X Y$ and denote by $\hat{f}$ the base change of f. We call the triple $(\widehat{Y}, \hat{f}, R)$ the *formal contraction* associated to the flopping contraction $f : Y \to X$. Note that $\widehat{Y}$ is Calabi–Yau.

Let $(\widehat{Y}, \hat{f}, R)$ be a formal flopping contraction. Now we consider the NCCR associated to a 3–dimensional flopping contraction $\hat{f} : \widehat{Y} \to \widehat{X}$ constructed as follows. For $i = 1, \ldots, t$, let $\widehat{\mathscr{L}}_i$ be a line bundle on $\widehat{Y}$ such that $\deg_{C_j} \widehat{\mathscr{L}}_i = \delta_{ij}$. Define $\widehat{\mathcal{N}}_i$ to be given by the maximal extension

$$(2\text{-}1) \qquad\qquad 0 \to \widehat{\mathscr{L}}_i^{-1} \to \widehat{\mathcal{N}}_i \to \mathcal{O}_{\widehat{Y}}^{\oplus r_i} \to 0$$

associated to a minimal set of r_i generators of $H^1(\widehat{Y}, \widehat{\mathscr{L}}_i^{-1})$. Set $N_i := \mathbb{R}f_* \widehat{\mathcal{N}}_i = f_* \widehat{\mathcal{N}}_i$ for $i = 1, \ldots, t$. We set

$$(2\text{-}2) \qquad\qquad A := \mathrm{End}_{\widehat{Y}}(\mathcal{O}_{\widehat{Y}} \oplus \widehat{\mathcal{N}}_1 \oplus \cdots \oplus \widehat{\mathcal{N}}_t) \cong \mathrm{End}_R(R \oplus N_1 \oplus \cdots \oplus N_t).$$

The second isomorphism can be proved as follows. It is clear that the natural morphism from A to $\mathrm{End}_R(R \oplus N_1 \oplus \cdots \oplus N_t)$ is an isomorphism away from the exceptional locus, which has codimension 3. Since both are reflexive R–modules, it must be an isomorphism.

For simplicity, we denote by $\widehat{\mathcal{N}}$ the direct sum $\bigoplus_{i=1}^{t} \widehat{\mathcal{N}}_i$ and denote by N the direct sum $\bigoplus_{i=1}^{t} N_i$.

Theorem 2.11 [Van den Bergh 2004, Corollary 3.2.10] *The functor $\widehat{F} := \mathbb{R}\mathrm{Hom}_{\widehat{Y}}(\mathcal{O}_{\widehat{Y}} \oplus \widehat{\mathcal{N}}, -)$ defines a triangle equivalence between $D^b(\mathrm{coh}\,\widehat{Y})$ and $D^b(\mathrm{Mod}\, A)$, with quasi-inverse $F^{-1} := (-) \otimes_A (\mathcal{O}_{\widehat{Y}} \oplus \widehat{\mathcal{N}})$. In addition, A is itself Cohen–Macaulay.*

It follows that A is an NCCR.

Corollary 2.12 *Let $(\hat{Y}, \hat{f}, R)$ be a 3–dimensional formal flopping contraction. Then:*

(1) *The structure sheaves of the irreducible components $\{C_i\}_{i=1}^t$ of $\mathrm{Ex}(\hat{f})$ form a semisimple collection in $\mathrm{coh}(\hat{Y})$.*

(2) *The derived deformation algebra $\Gamma_C^{\hat{Y}}$ of the collection $C := \{C_i\}_{i=1}^t$ is linked by quasi-isomorphisms to Γ in [Theorem 2.8](#).*

(3) *For any $i = 1, \ldots, t$, C_i is nc rigid.*

Proof To prove that $\mathrm{Hom}_{\hat{Y}}(\mathcal{O}_{C_i}, \mathcal{O}_{C_j}) = 0$ for $i \neq j$, we simply need to use the condition that $\mathrm{Ex}(\hat{f})$ is a tree of rational curves with normal crossings.

For any $i = 1, \ldots, t$, it is easy to check that $S_i \cong \hat{F}(\Sigma \mathcal{O}_{C_i}(-1))$. Note that $\left(\bigoplus_{i=1}^t \mathcal{O}_{C_i}\right) \otimes \left(\bigotimes_{i=1}^t \mathscr{L}_i^{-1}\right) \cong \bigoplus_{i=1}^t \mathcal{O}_{C_i}(-1)$. By [Theorem 2.11](#), $\hat{F}$ induces an isomorphism of A_∞–algebras

$$\tag{2-3} \mathrm{Ext}_{\hat{Y}}^*\left(\bigoplus_{i=1}^t \mathcal{O}_{C_i}, \bigoplus_{i=1}^t \mathcal{O}_{C_i}\right) \cong \mathrm{Ext}_A^*\left(\bigoplus_{i=1}^t S_i, \bigoplus_{i=1}^t S_i\right).$$

Let $\bar{l} = A/\mathrm{rad}(A)$ and $l = \bar{l}/ke_0$. Then there is a natural isomorphism of A–modules $l \cong \bigoplus_{i=0}^t S_i$, where S_0 is the simple A–module that corresponds to the summand R of $R \oplus \bigoplus_{i=1}^t N_i$. By [de Thanhoffer de Völcsey and Van den Bergh 2016, Lemma 4.1], the vector space V in [Theorem 2.8](#) can be chosen as $D(\Sigma \mathrm{Ext}_A^{\geq 1}(\bar{l}, \bar{l}))$. Therefore, $\Gamma := \hat{T}_{\bar{l}} V / \hat{T}_{\bar{l}} V e_0 \hat{T}_{\bar{l}} V \cong \mathbb{D}B(\mathrm{Ext}_A^*(l, l))$ represents the noncommutative deformations of semisimple collection $\{S_i\}_{i=1}^t \in D^b(\mathrm{mod}\, A)$. Part (2) follows from the isomorphism (2-3). Denote by $\Gamma_{C_i}^{\hat{Y}}$ the derived deformation algebra of $\mathcal{O}_{C_i}$. We have

$$\Gamma_{C_i}^{\hat{Y}} \cong \Gamma \Big/ \sum_{j \neq i} \Gamma e_j \Gamma \quad \text{and} \quad H^0\Gamma_{C_i}^{\hat{Y}} \cong H^0\Gamma \Big/ \sum_{j \neq i} H^0\Gamma e_j H^0\Gamma.$$

Then part (3) follows from [Theorem 2.8](#)(1). □

3 Calabi–Yau structure and cluster category

In this section, we first review several notions of Calabi–Yau property for triangulated categories, for homologically smooth dg algebras and for proper dg algebras. Then we recall geometric versions of the Calabi–Yau property and translate them into algebraic notions for endomorphism algebras of generators and for derived deformation algebras. Finally, we classify Calabi–Yau structures for 3–dimensional flopping contractions and review the cluster category.

3.1 CY structures

3.1.1 CY triangulated categories
Let $\mathscr{T}$ be a Hom–finite k–linear triangulated category.

Definition 3.1 A *right Serre functor* for $\mathcal{T}$ is a triangle functor $S\colon \mathcal{T} \to \mathcal{T}$ such that there are bifunctorial isomorphisms

$$\mathcal{T}(Y, SX) \to D\mathcal{T}(X, Y)$$

for all $X, Y \in \mathcal{T}$. It is a *Serre functor* if it is an autoequivalence.

One can show that a right Serre functor exists if and only if, for each object X of $\mathcal{T}$, the functor $D\mathcal{T}(X, ?)$ is representable and, in this case, the right Serre functor is unique up to canonical isomorphism of triangle functors [Bondal and Kapranov 1989; Van den Bergh 2008]. Let d be an integer. The triangulated category $\mathcal{T}$ is d–*Calabi–Yau* if it admits a Serre functor isomorphic to Σ^d.

3.1.2 CY smooth dg algebras A dg k–algebra Γ is called *homologically smooth* if Γ is perfect in $D(\Gamma^e)$. Then one checks that $D_{\mathrm{fd}}(\Gamma)$, the subcategory of $D(\Gamma)$ consisting of the modules whose homology is of finite total dimension, is contained in the perfect derived category $\mathrm{per}(\Gamma)$. Put

$$\Theta_\Gamma = \mathbb{R}\mathrm{Hom}_{\Gamma^e}(\Gamma, \Gamma^e).$$

Then we have a canonical isomorphism

$$\mathrm{HH}_d(\Gamma) \xrightarrow{\;\sim\;} \mathrm{Hom}_{D(\Gamma^e)}(\Theta_\Gamma, \Sigma^{-d}\Gamma).$$

Definition 3.2 The dg algebra Γ is called *bimodule d–CY* if it is homologically smooth and there is an isomorphism, in $D(\Gamma^e)$,

$$\eta\colon \Theta_\Gamma \xrightarrow{\;\sim\;} \Sigma^{-d}\Gamma.$$

A class $\eta \in \mathrm{HH}_d(\Gamma)$ is called a d–*CY structure* if the corresponding morphism $\eta\colon \Theta_\Gamma \to \Sigma^{-d}\Gamma$ is an isomorphism in $D(\Gamma^e)$. A d–CY structure η is called *exact* if there exists a class $\xi \in \mathrm{HC}_{d-1}(\Gamma)$ such that $B\xi = \eta$, where B is the Connes morphism. A choice of ξ will be called an *exact lifting* of the d–CY structure η. We call a bimodule d–CY algebra Γ an *exact d–CY algebra* if the d–CY structure is exact in addition.

Definition 3.3 The dg algebra Γ is said to satisfy the *relative d–CY property* if, for $L \in D_{\mathrm{fd}}(\Gamma)$ and $M \in \mathrm{per}(\Gamma)$, we have a bifunctorial isomorphism

$$D\mathbb{R}\mathrm{Hom}_{D(\Gamma)}(L, M) \simeq \mathbb{R}\mathrm{Hom}_{D(\Gamma)}(M, \Sigma^d L).$$

Remark 3.4 If Γ is a topologically homologically smooth pseudocompact dg algebra in $\mathrm{PCAlgc}(l)$, we call Γ a bimodule d–CY if η is an isomorphism in the pseudocompact derived category of bimodules. The isomorphism η represents a class in the continuous Hochschild homology $\mathrm{HH}_d(\Gamma)$. Exactness is defined similarly by taking the continuous cyclic homology. We call a bimodule d–CY pseudocompact dg algebra Γ in $\mathrm{PCAlgc}(l)$ an *exact d–CY algebra* if the d–CY structure is exact in addition. Similarly, for a pseudocompact algebra Γ in $\mathrm{PCAlgc}(l)$, the relative d–CY property is defined by replacing $D(\Gamma), \mathrm{per}(\Gamma)$ and $D_{\mathrm{fd}}(\Gamma)$ with their pseudocompact counterparts.

Given a homologically smooth dg algebra Γ, it follows from [Keller 2011, Lemma 3.4] that

$$\Gamma \text{ is bimodule } d\text{–CY} \implies \Gamma \text{ satisfies the relative } d\text{–CY property}$$

$$\implies D_{\mathrm{fd}}(\Gamma) \text{ is a Hom–finite } d\text{–CY triangulated category.}$$

A similar chain of implications holds in the pseudocompact case.

3.1.3 CY proper dg algebras Let A be a dg algebra. Suppose that A is *proper*, ie that its homology is of finite total dimension. Then the category $\mathrm{per}(A)$ is Hom–finite. The proper dg algebra A is called *perfectly d–CY* if there is an isomorphism

$$DA \xrightarrow{\sim} \Sigma^d A$$

in $D(A^e)$. By Lemma 3.6 below, the triangulated category $\mathrm{per}(A)$ is then d–CY.

Definition 3.5 Let k be a field of characteristic zero. Given a finite-dimensional A_∞–algebra A, a *cyclic A_∞–structure* of degree d on A is a nondegenerate symmetric bilinear form

$$(?, ?)\colon A \times A \to \Sigma^d A$$

of degree d such that

$$(m_n(a_1, \ldots, a_n), a_{n+1}) = (-1)^n (-1)^{|a_1|(|a_2|+\cdots+|a_{n+1}|)} (m_n(a_2, \ldots, a_{n+1}), a_1).$$

In this case, we have in particular an isomorphism $DA \xrightarrow{\sim} \Sigma^d A$ in the derived category of A–bimodules. Thus, if a dg algebra is quasi-isomorphic to an A_∞–algebra admitting a cyclic A_∞–structure of degree d, then it is perfectly d–CY.

Lemma 3.6 *Let $T \geq 1$ be an integer, l a product of t copies of k and A a dg algebra augmented over l and such that A belongs to the triangulated subcategory of $D(A \otimes A^{\mathrm{op}})$ generated by $l \otimes l^{\mathrm{op}}$ (in particular, the dg algebra A is proper). Let Γ be the pseudocompact dg algebra $\mathbb{D} BA$, where B denotes the bar construction over l. Then Γ is topologically homologically smooth. Moreover, if A is perfectly d–Calabi–Yau, then Γ is bimodule d–Calabi–Yau.*

Proof We refer to [Keller 2003] for a brief summary of the bar–cobar formalism that we are going to use and to [Lefèvre-Hasegawa 2003; Van den Bergh 2015, Appendix A] for in-depth treatments. Let C be the augmented dg coalgebra BA and $\tau\colon C \to A$ the canonical twisting cochain. Since τ is acyclic, the canonical morphism

$$C \to C \otimes_\tau A \otimes_\tau C$$

is a weak equivalence of dg C–bimodules (see [Keller 2003]). Recall (for example from [Van den Bergh 2015, Appendix A]) that a dg comodule is fibrant if its underlying graded comodule is cofree. Clearly, this holds for $C \otimes_\tau A \otimes_\tau C$. Since A belongs to the triangulated subcategory of the derived

category of $A \otimes A^{\mathrm{op}}$ generated by $l \otimes l$, the dg bicomodule $C \otimes_\tau A \otimes_\tau C$ belongs to the triangulated subcategory of the coderived category of $C \otimes C^{\mathrm{op}}$ generated by $C \otimes C^{\mathrm{op}}$. By applying the duality $\mathbb{D}$, we obtain that the morphism

$$\tag{3-1} \Gamma \otimes_\tau \mathbb{D}A \otimes_\tau \Gamma \to \Gamma$$

is a weak equivalence and that the object on the left is cofibrant. Moreover, we see that Γ belongs to the perfect derived category of $\Gamma \otimes \Gamma^{\mathrm{op}}$. This means that Γ is topologically homologically smooth. Now suppose that A is perfectly d–Calabi–Yau. Since A is proper, it is weakly equivalent to its pseudocompact completion $\widehat{A}$. By the Calabi–Yau property, we have an isomorphism $\mathbb{D}A \xrightarrow{\sim} \Sigma^d \widehat{A}$ in the pseudocompact derived category of dg $\widehat{A}$–bimodules. Now we compute the inverse dualizing complex of Γ using the resolution (3-1). We have isomorphisms in the pseudocompact derived category of Γ–bimodules

$$\mathbb{R}\mathrm{Hom}_{\Gamma^e}(\Gamma, \Gamma^e) = \mathrm{Hom}_{\Gamma^e}(\Gamma \otimes_\tau \mathbb{D}A \otimes_\tau \Gamma, \Gamma^e) = \mathrm{Hom}^{\tau,\tau}_{\mathrm{psc}}(\mathbb{D}A, \Gamma^e) = \mathrm{Hom}^{\tau,\tau}_{\mathrm{psc}}(\Sigma^d \widehat{A}, \Gamma^e)$$
$$= \Sigma^{-d} \Gamma \otimes_\tau \mathbb{D}A \otimes_\tau \Gamma = \Sigma^{-d} \Gamma.$$

Here $\mathrm{Hom}^{\tau,\tau}_{\mathrm{psc}}$ denotes the space of morphisms in the category of pseudocompact vector spaces twisted twice by τ. This shows that Γ is topologically homologically bimodule d–Calabi–Yau. $\qquad\square$

3.1.4 CY structures in geometry, algebraic consequences

We let $k = \mathbb{C}$ be the field of complex numbers unless specified otherwise.

Definition 3.7 Let Y be a d–dimensional smooth quasiprojective $\mathbb{C}$–variety. We call Y a d–dimensional Calabi–Yau variety if there is an isomorphism $\omega_Y := \Omega^d_Y \cong \mathbb{O}_Y$, ie there exists a nowhere-vanishing d–form. We call a nowhere-vanishing section $\eta \colon \mathbb{O}_Y \to \omega_Y$ a d–CY *structure* on Y. We call the d–CY structure *exact* if the d–form η is exact, ie there exists a $(d-1)$–form $\xi \in \Omega^{d-1}_Y$ such that $d\xi = \eta$. A choice of ξ will be called an *exact lifting* of the d–CY structure η. If $\widehat{Y}$ is a smooth formal scheme, we may define d–CY structure in a similar way by considering the de Rham complex of the formal scheme $\Omega^*_{\widehat{Y}}$.

Given a smooth quasiprojective variety Y of dimension d and a bounded complex of coherent sheaves L on Y whose cohomology has proper support, the derived endomorphism algebra $A := \mathbb{R}\mathrm{Hom}_{D(\mathrm{Qcoh}\,Y)}(L, L)$ is a proper dga. We will show that A is perfectly d–CY if Y is d–Calabi–Yau.

Lemma 3.8 *Let Y be a quasiprojective smooth CY d–fold with a fixed CY structure. Let $L \in D^b_c(\mathrm{coh}\,Y)$ be a bounded complex of coherent sheaves with proper support. Then*

$$A = \mathbb{R}\mathrm{Hom}_{D(\mathrm{Qcoh}\,Y)}(L, L)$$

is perfectly d–Calabi–Yau.

Proof Since we work over a field of characteristic zero, there exists $\overline{Y}$, a smooth projective variety that compactifies Y. Denote by $i : Y \to \overline{Y}$ the canonical embedding. Since i is an open immersion and L has proper support on Y, $\mathbb{R}\mathrm{Hom}_{D(\mathrm{Qcoh}\,\overline{Y})}(i_* L, i_* L)$ is quasi-isomorphic to A as dg algebras.

From now on, we simply assume that $A = \mathbb{R}\mathrm{Hom}_{D(\mathrm{Qcoh}\,\overline{Y})}(i_* L, i_* L)$. We adopt the notation of the appendix to write $\mathrm{Qcoh}(Y)$ for the dg category of (fibrant replacements of) unbounded complexes of quasicoherent sheaves and $\mathrm{coh}(Y)$ for its full dg subcategory of complexes with coherent cohomology and bounded cohomological amplitude. By Theorem A.1, there is a bifunctorial quasi-isomorphism

$$D\mathrm{Hom}^{\mathrm{dg}}_{\mathrm{coh}(\overline{Y})}(M, N) \simeq \mathrm{Hom}^{\mathrm{dg}}_{\mathrm{coh}(\overline{Y})}(N, M \otimes \omega_{\overline{Y}}[d]).$$

Now let $M = N = i_* L$. Then $M \otimes \omega_{\overline{Y}} \cong i_*(L \otimes \omega_Y) \cong i_* L$ using the Calabi–Yau structure on Y. By the bifunctoriality, this is an isomorphism in $D(A^e)$. $\qquad\square$

Proposition 3.9 *Let $\{\mathscr{E}_i\}_{i=1}^t$ be a semisimple collection of compactly supported sheaves in a smooth quasiprojective CY d–fold Y. Write $\Gamma := \Gamma^Y_{\mathscr{E}}$ for the derived deformation algebra of $\mathscr{E} := \bigoplus_{i=1}^t \mathscr{E}_i$. Then Γ is topologically homologically smooth and bimodule d–Calabi–Yau.*

Proof Let A be the derived endomorphism algebra of $\mathscr{E}$. Since Y is smooth and $\mathscr{E}$ has compact support, A is proper. Moreover, it can be chosen augmented. Clearly, it has its homology concentrated in nonnegative degrees and $H^0(A)$ is isomorphic to a product of t copies of k, which is l. By [Keller and Nicolás 2013, Corollary 4.1], the dg algebra A belongs to the triangulated subcategory of its derived category generated by the semisimple object l. We know that Γ is quasi-isomorphic to $\mathbb{D}BA$. Thus, by Lemma 3.6, Γ is topologically homologically smooth. Since Y is smooth and d–Calabi–Yau, A is perfectly d–Calabi–Yau by Lemma 3.8. Hence, Γ is bimodule d–Calabi–Yau by Lemma 3.6. $\qquad\square$

Remark 3.10 If Y is smooth and projective, then one can show that $\mathrm{Ext}^*_Y(\mathscr{E}, \mathscr{E})$ is a cyclic A_∞–algebra. This can be proved by reducing to the analytic case and applying the holomorphic Chern–Simons theory (see [Kontsevich and Soibelman 2009, Example 10.2.7]). An algebraic proof of the case when Y is a projective Gorenstein CY curve can be found in [Polishchuk 2020].

Proposition 3.11 *Let $\{\mathscr{E}_i\}_{i=1}^t$ be a semisimple collection of sheaves in a smooth projective CY d–fold Y. Write $\Gamma := \Gamma^Y_{\mathscr{E}}$ for the derived deformation algebra of $\mathscr{E} := \bigoplus_{i=1}^t \mathscr{E}_i$. Then Γ is an exact d–CY algebra.*

Proof As in the proof of Proposition 3.9, we see that Γ is topologically homologically smooth and bimodule d–CY. The exactness of the CY structure follows from [Van den Bergh 2015, Theorem 12.1] and the remark above. $\qquad\square$

Theorem 3.12 (Iyama and Reiten) *Let R be an equicodimensional Gorenstein normal domain of dimension d over an algebraically closed field k, and let A be an NCCR. Then A satisfies the relative d–CY property. Moreover, if R is complete local, then A is bimodule d–CY.*

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Proof The fact that A satisfies the relative d–CY property is proved in [Wemyss 2012, Theorem 4.23]. We have $A = \operatorname{End}_R(R \oplus N)$ for the Cohen–Macaulay module $N = \bigoplus_{i=1}^t N_i$ with $N_1, \ldots, N_t$ indecomposable. Denote by $\bar{l}$ the algebra k^{t+1}. By [de Thanhoffer de Völcsey and Van den Bergh 2016, Theorem 1.1], the algebra A is quasi-isomorphic to a pseudocompact dg algebra $\widetilde{\Gamma} := (\widehat{T}_{\bar{l}}(V), d)$ in $\operatorname{PCAlgc}(l)$ for a finite-dimensional graded $\bar{l}$–bimodule V concentrated in degrees ≤ 0 and a differential d taking V into the square of the kernel of the augmentation $\widehat{T}_{\bar{l}}(V) \to \bar{l}$. Since A is of finite global dimension, $\widetilde{\Gamma}$ is homologically smooth and, by the first part, we have bifunctorial isomorphisms

$$\operatorname{Hom}_{D(\widetilde{\Gamma})}(M, P) = D\operatorname{Hom}_{D(\widetilde{\Gamma})}(P, \Sigma^d M)$$

for M in $D_{\mathrm{fd}}(\widetilde{\Gamma})$ and P in $\operatorname{per}(\widetilde{\Gamma})$. Let

$$\Theta = \mathbb{R}\operatorname{Hom}_{\widetilde{\Gamma}^e}(\widetilde{\Gamma}, \widetilde{\Gamma}^e)$$

be the inverse dualizing complex of $\widetilde{\Gamma}$. By [Keller 2008, Lemma 4.1], we have bifunctorial isomorphisms

$$\operatorname{Hom}_{D(\widetilde{\Gamma})}(L \otimes_{\widetilde{\Gamma}}^{\mathbb{L}} \Theta, M) = D\operatorname{Hom}_{D(\widetilde{\Gamma})}(M, L)$$

for M in $D_{\mathrm{fd}}(\widetilde{\Gamma})$ and an arbitrary object L of $D(\widetilde{\Gamma})$. By combining these with the previous isomorphisms, we find

$$\operatorname{Hom}_{D(\widetilde{\Gamma})}(P \otimes_{\widetilde{\Gamma}}^{\mathbb{L}} \Theta, M) = \operatorname{Hom}_{D(\widetilde{\Gamma})}(\Sigma^{-d} P, M)$$

for P in $\operatorname{per}(\widetilde{\Gamma})$ and M in $D_{\mathrm{fd}}(\widetilde{\Gamma})$. Since an object L of $D(\widetilde{\Gamma})$ is perfect if and only if $\operatorname{Hom}_{D(\widetilde{\Gamma})}(L, M)$ is finite-dimensional for each M in $D_{\mathrm{fd}}(\widetilde{\Gamma})$, we see that $P \otimes_{\widetilde{\Gamma}}^{\mathbb{L}} \Theta$ is perfect for each perfect P. Now, by taking $P = \widetilde{\Gamma}$ and $M = \Sigma^n S_i$, where $n \in \mathbb{Z}$ and S_i is one of the $t + 1$ simple $\widetilde{\Gamma}$–modules, we see that, as a right $\widetilde{\Gamma}$–module, Θ is quasi-isomorphic to $\Sigma^{-d}\widetilde{\Gamma}$. For the rest of the argument, let us replace $\widetilde{\Gamma}$ by the quasi-isomorphic pseudocompact algebra $G = H^0(\widetilde{\Gamma})$, which is isomorphic to the original algebra A. Since G and $\widetilde{\Gamma}$ have canonically equivalent derived categories and derived categories of bimodules, we can view Θ as a G–bimodule complex concentrated in degree d. After replacing it with a quasi-isomorphic bimodule complex, we may assume that it is an actual G–bimodule concentrated in degree d. Moreover, we know that, as a right module, it is isomorphic to G. Thus, there is an automorphism σ of G such that $\Sigma^d \Theta$ is isomorphic to $_\sigma G$ as a bimodule. Since each object L of $D_{\mathrm{fd}}(G)$ is perfect, we have

$$\operatorname{Hom}_{D(G)}(L_\sigma, M) = \operatorname{Hom}_{D(G)}(L, M)$$

for all L and M in $D_{\mathrm{fd}}(G)$, which shows that there is a functorial isomorphism $L \xrightarrow{\sim} L_\sigma$ for each L in $D_{\mathrm{fd}}(G)$. In particular, for L, we can take the finite-dimensional quotients of G. We deduce that, in each finite-dimensional quotient of G, the automorphism σ induces an inner automorphism. Thus, σ itself is inner and $_\sigma G$ is isomorphic to G as a bimodule. This shows that Θ is quasi-isomorphic to $\Sigma^{-d}\widetilde{\Gamma}$ as a bimodule. $\qquad\square$

Corollary 3.13 *Let R be a local complete equicodimensional Gorenstein normal domain of dimension d over an algebraically closed field k of characteristic zero, and let A be an NCCR. Let Γ be the dg algebra constructed in Theorem 2.8. Then Γ is topologically homologically smooth and bimodule d–CY.*

3.2 Classification of CY structures for 3–dimensional flopping contractions

If $(\hat{Y}, \hat{f}, R)$ is a 3–dimensional formal flopping contraction, then R is a hypersurface. The natural isomorphism $\hat{f}_* \omega_{\hat{Y}} \cong \omega_R$ identifies a CY structure η on $\hat{Y}$ with a nonzero section $\hat{f}(\eta)$ of ω_R. By the Gorenstein property, $\hat{f}(\eta)$ defines an isomorphism $R \cong \omega_R$.

Theorem 3.14 *Let $(\hat{Y}, \hat{f}, R)$ be a 3–dimensional formal flopping contraction. The space of 3–CY structures can be identified with $R^\times$. Moreover, every Calabi–Yau structure on $\hat{Y}$ is exact. The space of all exact liftings of a 3–CY structure can be identified with the cohomology group $H^1(\hat{Y}, \Omega^1_{\hat{Y}})$.*

Proof Assume that $C := \mathrm{Ex}(f)$ has t irreducible components $C_1, \ldots, C_t$. Because R has rational singularities, $H^0(\hat{Y}, \Omega^3_{\hat{Y}}) \cong H^0(\hat{Y}, \mathcal{O}_{\hat{Y}}) \cong R$. Then the first claim follows. The Hodge-to-de Rham spectral sequence with E_1 term

$$E_1^{pq} = H^q(\hat{Y}, \Omega^p_{\hat{Y}})$$

converges to $H^{p+q}_{\mathrm{DR}}(\hat{Y}, \mathbb{C})$. We claim that $H^1(\Omega^1_{\hat{Y}}) \cong \mathbb{C}^t$. Because the first Chern classes of $\hat{\mathscr{L}}_i$ for $i = 1, \ldots, t$ are linearly independent, $\dim_{\mathbb{C}} H^1(\Omega^1_{\hat{Y}}) \geq t$. We write $\hat{X} := \mathrm{Spec}\, R$. By the Leray spectral sequence

$$H^p(\hat{X}, \mathbb{R}^q \hat{f}_* \Omega^1) \Rightarrow H^{p+q}(\hat{Y}, \Omega^1),$$

we have an exact sequence

$$0 \to H^0(\hat{X}, \mathbb{R}^1 f_* \Omega^1) \to H^1(\hat{Y}, \Omega^1) \to H^1(\hat{X}, f_* \Omega^1) \to 0.$$

The rightmost term vanishes since $\hat{X}$ is affine. Since $\hat{Y}$ is a small resolution of 3–dimensional Gorenstein singularities, the normal bundle of C_i is $\mathcal{O}(a) \oplus \mathcal{O}(b)$ with $(a, b) = \{(-1, -1), (0, -2), (1, -3)\}$ [Pinkham 1983, Theorem 4]. By the exact sequence

$$0 \to \mathcal{O}(-a) \oplus \mathcal{O}(-b) \to \Omega^1|_{C_i} \to \mathcal{O}(-2) \to 0,$$

we have $H^1(C_i, \Omega^1|_{C_i}) \cong H^1(C_i, \mathcal{O}(-2)) = \mathbb{C}$. By the normal crossing condition, there is a short exact sequence of sheaves

$$0 \to \Omega^1|_C \to \Omega^1|_{C_1} \oplus \Omega^1|_{\bigcup_{i=2}^t C_i} \to \Omega^1|_p \to 0,$$

where $p = C_1 \cap \left(\bigcup_{i=2}^t C_i \right)$. Then we get a surjection

$$H^1(C, \Omega^1|_C) \to H^1(C_1, \Omega^1|_{C_1}) \oplus H^1\left(\bigcup_{i=2}^t C_i, \Omega^1_{\bigcup_{i=2}^t C_i} \right).$$

By induction, $\dim_{\mathbb{C}} H^1(C, \Omega^1|_C) \leq t$. Then the conclusion follows from the formal function theorem and the Leray spectral sequence.

The term E_2^{30} of the Hodge-to-de Rham spectral sequence is the quotient $H^0(\hat{Y}, \Omega^3)/dH^0(\hat{Y}, \Omega^2)$. Recall that $H^1(\hat{Y}, \Omega^1)$ admits a $\mathbb{C}$–basis by c_1 of the line bundles $\hat{\mathscr{L}}_i$ with $i = 1, \ldots, t$. Because $(1, 1)$ classes are d–closed, the differential $H^1(\hat{Y}, \Omega^1) \to H^0(\hat{Y}, \Omega^3)/dH^0(\hat{Y}, \Omega^2)$ is zero. Moreover, since $H^2(\mathcal{O}_{\hat{Y}}) = 0$, we have $E_r^{30} = E_2^{30}$ for $r \geq 2$. Because $H^3_{\mathrm{DR}}(\hat{Y}, \mathbb{C}) \cong H^3_{\mathrm{DR}}(C, \mathbb{C}) = 0$, E_2^{30} must vanish. Therefore, all 3–forms are exact.

Denote by $\Omega_{\hat{Y}}^{\leq i}$ the stupid truncation $\sigma_{\leq i}\Omega_{\hat{Y}}^{*}$ of the de Rham complex. There is a long exact sequence of hypercohomology

$$\cdots \xrightarrow{B} H^{i-n}(\hat{Y}, \Omega_{\hat{Y}}^{i}) \xrightarrow{I} \mathbb{H}^{2i-n}(\hat{Y}, \Omega_{\hat{Y}}^{\leq i}) \xrightarrow{S} \mathbb{H}^{2i-n}(\hat{Y}, \Omega_{\hat{Y}}^{\leq i-1}) \xrightarrow{B} H^{i-n+1}(\hat{Y}, \Omega_{\hat{Y}}^{i}) \to \cdots .$$

Take $i = 3$ and $n = 4$. The leftmost term vanishes and $\mathbb{H}^{2}(\hat{Y}, \Omega_{\hat{Y}}^{\leq 3}) = H_{\mathrm{DR}}^{2}(\hat{Y}) \cong H^{1}(\hat{Y}, \Omega_{\hat{Y}}^{1})$. So the last claim is proved. $\qquad\qquad\square$

Corollary 3.15 *Let R be a complete local equicodimensional Gorenstein normal domain of dimension d over an algebraically closed field of characteristic zero, and let A be an NCCR. Then every d–CY structure on A is exact.*

Proof This is an immediate consequence of Theorems 3.12 and 2.11 and [Van den Bergh 2015, Corollary 9.3]. $\qquad\qquad\square$

The following proposition follows immediately from the Hochschild–Kostant–Rosenberg theorem:

Proposition 3.16 *Let $(\hat{Y}, \hat{f}, R)$ be a 3–dimensional formal flopping contraction. Let $A = \mathrm{End}_{R}(R \oplus N)$ be the corresponding NCCR. Then there is a bijective correspondence between the space of 3–CY structures (resp. exact 3–CY structures) on $\hat{Y}$ and that of A.*

3.3 Cluster category

Let Γ be a dg k–algebra. Suppose that Γ has the following properties:

(1) Γ is homologically smooth, ie Γ is a perfect Γ^{e}–module.

(2)   For each $p > 0$, the space $H^{p}\Gamma$ vanishes.

(3) $H^{0}\Gamma$ is finite-dimensional.

(4) Γ satisfies the relative 3–CY property.

By property (1), $D_{\mathrm{fd}}(\Gamma)$ is a subcategory of the perfect derived category $\mathrm{per}(\Gamma)$. The *generalized cluster category* $\mathscr{C}_{\Gamma}$ is defined to be the triangle quotient $\mathrm{per}(\Gamma)/D_{\mathrm{fd}}(\Gamma)$. We denote by π the canonical projection functor $\pi \colon \mathrm{per}(\Gamma) \to \mathscr{C}_{\Gamma}$. For simplicity, we will omit the adjective "generalized" and call $\mathscr{C}_{\Gamma}$ the cluster category associated to Γ. An object $T \in \mathscr{C}_{\Gamma}$ is called a *cluster-tilting object* if:

(1) $\mathrm{Ext}_{\mathscr{C}_{\Gamma}}^{1}(T, T) = 0$.

(2)   For any object X such that $\mathrm{Ext}_{\mathscr{C}_{\Gamma}}^{1}(T, X) = 0$, one has $X \in \mathrm{add}(T)$.

Amiot [2009, Theorem 2.1] has proved that $\pi(\Gamma)$ is a cluster-tilting object in $\mathscr{C}_{\Gamma}$.

We call $H^{0}\Gamma$ the *CY tilted algebra* associated to the cluster category $\mathscr{C}_{\Gamma}$; see [Reiten 2010].

Remark 3.17 If Γ is a pseudocompact dg l–algebra in $\mathrm{PCAlgc}(l)$, we may define a continuous version of cluster category. Condition (1) is replaced by

(1′) Γ is topologically homologically smooth,

and the *topological* cluster category $\mathscr{C}_\Gamma$ is defined to be the triangle quotient $\mathrm{per}(\Gamma)/D_{\mathrm{fd}}(\Gamma)$, where $\mathrm{per}(\Gamma)$ and $D_{\mathrm{fd}}(\Gamma)$ are considered as subcategories of the pseudocompact derived category. We refer to [Keller and Yang 2011, Appendix] for the details.

Remark 3.18 If we drop the assumption that $H^0\Gamma$ is finite-dimensional, the quotient category $\mathscr{C}_\Gamma = \mathrm{per}(\Gamma)/D_{\mathrm{fd}}(\Gamma)$ is no longer Hom–finite. The Calabi–Yau property only holds when one restricts to suitable subcategories; see [Plamondon 2011, Proposition 2.16].

Theorem 3.19 [Amiot 2009, Theorem 2.1][2] *Let Γ be a dg k–algebra with the above properties. Then the cluster category $\mathscr{C}_\Gamma$ is Hom–finite and 2–CY as a triangulated category. Moreover, the object $\pi(\Gamma)$ is a cluster tilting object. Its endomorphism algebra is isomorphic to $H^0\Gamma$.*

Definition 3.20 Let $\{C_i\}_{i=1}^t$ be a collection of smooth rational curves in a smooth quasiprojective CY 3–fold Y with fixed CY structure $\eta\colon \mathcal{O}_Y \xrightarrow{\sim} \omega_Y$ such that $\{\mathcal{O}_{C_i}\}$ form a semisimple collection. Denote by $\mathscr{C}(Y, \{C_i\}_{i=1}^t, \eta)$ for the (topological) cluster category associated to the derived deformation algebra of $\bigoplus_{i=1}^t \mathcal{O}_{C_i}$. We call $\mathscr{C}(Y, \{C_i\}_{i=1}^t, \eta)$ *the cluster category associated to the triple* $(Y, \eta, \{C_i\}_{i=1}^t)$.

Definition 3.21 Let R be a complete local equicodimensional Gorenstein normal domain of dimension 3 over an algebraically closed field k of characteristic zero, and let A be the NCCR associated to the collection of indecomposables $R, N_1, \ldots, N_t$. Fix a 3–CY structure $\eta \in \mathrm{HH}_3(A, A)$. Denote by $\mathscr{C}(R, \{N_i\}_{i=1}^t, \eta)$ the cluster category associated to the dg algebra Γ constructed in Theorem 2.8, and call it *the cluster category associated to the triple* $(R, \{N_i\}_{i=1}^t, \eta)$.

A priori, the dg algebra Γ constructed in Theorem 2.8 is pseudocompact. However, if $D(\Gamma)$ denotes the ordinary derived category and $D_{\mathrm{pc}}(\Gamma)$ the pseudocompact derived category, then the natural functor $D_{\mathrm{pc}}(\Gamma) \to D(\Gamma)$ induces equivalences in the perfect derived categories and in the subcategories of objects with finite-dimensional total homology. Therefore, the two candidates for the cluster category are equivalent.

The following result is an immediate consequence of Corollary 2.12:

Corollary 3.22 *Let $(\widehat{Y}, \widehat{f}, R)$ be a 3–dimensional formal flopping contraction, and let A be the NCCR associated to the collection of indecomposables $R, N_1, \ldots, N_t$ constructed in Section 2.5. Fix a 3–CY structure η on $\widehat{Y}$ and denote its counterpart on A by the same symbol. Then there is a triangle equivalence between the cluster categories*

$$\mathscr{C}(Y, \{C_i\}_{i=1}^t, \eta) \simeq \mathscr{C}(R, \{N_i\}_{i=1}^t, \eta).$$

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[2]The original statement of [Amiot 2009] assumed that Γ is bimodule 3–CY. However, the proof is still valid under the weaker assumption that Γ satisfies the relative 3–CY property.

4 Ginzburg algebras

In this section we introduce the notion of Ginzburg (dg) algebra and prove several properties of it. The cluster category can be defined via the Ginzburg algebra, which provides an effective tool to do computations.

4.1 Definitions

Fix a commutative ring k. Let Q be a finite quiver, possibly with loops and 2–cycles. Denote by Q_0 and Q_1 the set of nodes and arrows of Q, respectively. Denote by kQ the path algebra and by $\widehat{kQ}$ the complete path algebra with respect to the two-sided ideal generated by arrows. For each $a \in Q_1$, we define the cyclic derivative D_a with respect to a as the unique linear map

$$D_a : kQ/[kQ, kQ] \to kQ$$

which takes the class of a path p to the sum $\sum_{p=uav} vu$ taken over all decompositions of the path p. The definition extends to $\widehat{kQ}_{\mathrm{cyc}} := \widehat{kQ}/[\widehat{kQ}, \widehat{kQ}]^c$, where the superscript c stands for the completion with respect to the adic topology defined above. An element w in $\widehat{kQ}/[\widehat{kQ}, \widehat{kQ}]^c$ is called a *potential* on Q. It is given by a (possibly infinite) linear combination of cycles in Q.

Definition 4.1 (Ginzburg) Let Q be a finite quiver and w a potential on Q. Let $\overline{Q}$ be the graded quiver with the same vertices as Q and whose arrows are

- the arrows of Q (of degree 0),
- an arrow $a^* : j \to i$ of degree -1 for each arrow $a : i \to j$ of Q,
- a loop $t_i : i \to i$ of degree -2 for each vertex i of Q.

The (complete) Ginzburg (dg) algebra $\mathfrak{D}(Q, w)$ is the dg k–algebra whose underlying graded algebra is the completion (in the category of graded vector spaces) of the graded path algebra $k\overline{Q}$ with respect to the two-sided ideal generated by the arrows of $\overline{Q}$. Its differential is the unique linear endomorphism homogeneous of degree 1 satisfying the Leibniz rule, and which takes the following values on the arrows of $\overline{Q}$:

- $da = 0$ for $a \in Q_1$;
- $d(a^*) = D_a w$ for $a \in Q_1$;
- $d(t_i) = e_i \left(\sum_{a \in Q_1} [a, a^*] \right) e_i$ for $i \in Q_0$, where e_i is the idempotent associated to i.

Denote by l the product $\prod_{i \in Q_0} k e_i$. Then $\widehat{kQ}$ is isomorphic to the complete tensor algebra $\hat{T}_l V$ with V being the vector space spanned by arrows of Q.

Remark 4.2 In most references, the above definition corresponds to the *complete Ginzburg algebra* while the algebra without taking the graded completion is called the *Ginzburg algebra*. The complete Ginzburg algebra $\mathfrak{D}(Q, w)$ is considered as an object of PCAlgc(l). Because we only consider complete Ginzburg algebra in this paper, we will call it the Ginzburg algebra for simplicity.

Definition 4.3 Let Q be a finite quiver and w a potential on Q. The *Jacobi algebra* $\Lambda(Q, w)$ is the zeroth homology of $\mathfrak{D}(Q, w)$, which is the quotient algebra

$$\widehat{kQ}/((D_a w \mid a \in Q_1))^c,$$

where $((D_a w \mid a \in Q_1))^c$ is the closed two-sided ideal generated by $D_a w$. A Ginzburg algebra $\mathfrak{D}(Q, w)$ is called *Jacobi-finite* if $\dim_k \Lambda(Q, w) < \infty$.

Theorem 4.4 (Van den Bergh [Keller 2011, Appendix]) *Let Q be a finite quiver and w be a potential. Then $\mathfrak{D}(Q, w)$ is topologically homologically smooth and bimodule 3–CY.*

The above theorem was first proved in the algebraic setting, but the same proof can be adapted to the pseudocompact case (see [Van den Bergh 2015]). Given a Jacobi-finite Ginzburg algebra $\Gamma := \mathfrak{D}(Q, w)$, there is an associated cluster category $\mathscr{C}_\Gamma := \operatorname{per}(\Gamma)/D_{\mathrm{fd}}(\Gamma)$.

Remark 4.5 There exists a canonical exact CY structure on $\Gamma = \mathfrak{D}(Q, w)$. We follow the notation of [Van den Bergh 2015] to write $M_l := M/[l, M]$ for a l–bimodule M. Because the reduced cyclic homology of Γ is equal to the homology of $(\Gamma/l + [\Gamma, \Gamma])_l$ (see [Van den Bergh 2008, Proof of Theorem 11.2.1]), a class of $\mathrm{HC}_2(\Gamma, \Gamma)$ is represented by a degree -2 element χ of Γ such that $d\chi \in l + [\Gamma, \Gamma]$. By the definition of d of Γ, $\chi := \sum_{i \in Q_0} t_i$ represents a class in $\mathrm{HC}_2^{\mathrm{red}}(\Gamma, \Gamma)$. Because Γ is cofibrant, by [Van den Bergh 2008, Proposition 7.2.1], the Hochschild chain complex of Γ is quasi-isomorphic to the mapping cone of

$$\Omega_l^1 \Gamma/[\Gamma, \Omega_l^1 \Gamma] \xrightarrow{\partial_1} \Gamma/[l, \Gamma]$$

with differential defined by $\partial_1(a\, Db) = [a, b]$, where $Db = 1 \otimes b - b \otimes 1$. In other words, a class in $\mathrm{HH}_3(\Gamma, \Gamma)$ is represented by a pair of elements (ω, a) of degree $(-2, -3)$ satisfying $\partial_1(\omega) = da$ and $d\omega = 0$. Because d and D commute, $(D\chi, 0)$ represents a class in $\mathrm{HH}_3(\Gamma, \Gamma)$, which is the image of χ under the Connes map. In [Van den Bergh 2008], a class in $\mathrm{HH}_d(\Gamma, \Gamma)$ is called *nondegenerate* if the corresponding morphism $\Theta_\Gamma \to \Sigma^{-d}\Gamma$ is an isomorphism. By [Van den Bergh 2008, Lemma 11.1.2], $(D\chi, 0)$ is nondegenerate.

For a Ginzburg algebra $\Gamma = \mathfrak{D}(Q, w)$, denote $\Lambda(Q, w)$ by Λ for short. The image of w under the canonical map $\mathrm{HH}_0(\widehat{kQ}, \widehat{kQ}) = \widehat{kQ}_{\mathrm{cyc}} \to \mathrm{HH}_0(\Lambda, \Lambda) = \Lambda_{\mathrm{cyc}}$, denoted by $[w]$, is a canonical class associated to the Ginzburg algebra $\mathfrak{D}(Q, w)$. Therefore, we see that, starting from a Ginzburg algebra $\Gamma = \mathfrak{D}(Q, w)$, we get not only a triangulated category $\mathscr{C}_\Gamma$ but an additional piece of information that is a canonical class $[w]$ in the zeroth Hochschild homology of the CY tilted algebra. We will show in the next section that this class is determined by the CY structure up to right equivalences.

4.2 Existence and uniqueness of potential

The definition of Ginzburg algebra is not homotopically invariant. It is important to know when a bimodule 3–CY dg algebra admits a model given by a Ginzburg algebra.

Theorem 4.6 [Van den Bergh 2015, Theorem 10.2.2] *Let k be a field and l be a finite-dimensional commutative separable k–algebra. Assume that Γ is a pseudocompact dg l–algebra in $\mathrm{PCAlgc}(l)$ concentrated in nonpositive degrees. Then the following are equivalent:*

(1) Γ *is exact* 3–CY.

(2) Γ *is weakly equivalent to a Ginzburg algebra* $\mathfrak{D}(Q, w)$ *for some finite quiver Q with w containing only cubic terms and higher.*

The following result provides a lot of examples of dg algebras whose 3–CY structures can be lifted to exact ones:

Theorem 4.7 [Van den Bergh 2015, Corollary 9.3] *Assume that k has characteristic zero and let Γ be a pseudocompact dg algebra in $\mathrm{PCAlgc}(l)$ concentrated in degree 0. Then Γ is bimodule d–CY if and only if it is exact d–CY.*

By putting the above two theorems together, we see that, if Γ is a pseudocompact dg l–algebra in $\mathrm{PCAlgc}(l)$ concentrated in degree 0 that is bimodule 3–CY, then it is quasi-isomorphic to a Ginzburg algebra $\mathfrak{D}(Q, w)$ for some finite quiver Q and potential w.

Given a pseudocompact l–algebra A, two classes $[w]$ and $[w']$ in $A/[A, A]^c$ are called *right equivalent* if there exists an automorphism γ of A such that $\gamma_*[w] = [w']$. Now we assume that a bimodule 3–CY dg algebra Γ in $\mathrm{PCAlgc}(l)$ is exact. So it admits a model given by $\mathfrak{D}(Q, w)$. Note that being bimodule CY and exact CY are homotopically invariant properties. The next proposition shows that the right equivalence class of $[w]$ in $\mathrm{HH}_0(\Lambda(Q, w))$ for such a dg algebra is indeed a homotopy invariant. The proof is implicitly contained in Van den Bergh's proof of Theorem 4.6 (see [Van den Bergh 2015, Proof of Theorem 11.2.1]). We recall it for completeness. See Remark 4.9 for a conceptional explanation of Van den Bergh's result.

Proposition 4.8 (Van den Bergh) *Let k be a field of characteristic zero and $l = k e_1 \times \cdots \times k e_n$. Let Γ be a pseudocompact 3–CY dg l–algebra in $\mathrm{PCAlgc}(l)$ with a fixed exact 3–CY structure $\tau \in \mathrm{HH}_3(\Gamma, \Gamma)$. Suppose there are two pairs (Q, w) and (Q', w') such that Γ is quasi-isomorphic to $\mathfrak{D}(Q, w)$ and $\mathfrak{D}(Q', w')$, respectively. Assume that, under these quasi-isomorphisms, we have $B\chi$ and $B\chi'$ identified with τ, where χ and χ' are the canonical classes in HC_2 defined in Remark 4.5. Then $Q = Q'$ and w is right equivalent to w'. In particular, the classes $[w], [w'] \in H^0\Gamma/[H^0\Gamma, H^0\Gamma]^c$ are right equivalent.*

Proof Van den Bergh [2008, Theorem 11.2.1] proved that there is a weak equivalence $(\widehat{T}_l V, d) \to \Gamma$, where $V = \Sigma^{-1}(D\mathrm{Ext}^*_\Gamma(l, l))_{\leq 0}$, such that:

(1) $V = V_c + lz$ with $V_c = \Sigma^{-1}(D\mathrm{Ext}^1_\Gamma(l, l) \oplus D\mathrm{Ext}^2_\Gamma(l, l))$ and z being an l–central element of degree -2.

(2) $dz = \sigma'\eta\sigma''$ with $\eta \in (V_c \otimes_l V_c)_l$ being a nondegenerate and antisymmetric element. Here $\sigma' \otimes \sigma'' \in l \otimes_k l$ is the Casimir element that corresponds to the trace $l \to k$ (see [Van den Bergh 2008, Section 5]).

Since Q and Q' depend only on the l–bimodule structure on $\mathrm{Ext}^1_\Gamma(l, l)$ (see [Van den Bergh 2004, Section 10.3]), we have $Q = Q'$. Note that the perfect pairing on V_c is determined by the bimodule 3–CY structure τ but does not depend on the exact lifting. Using the perfect pairing, any nondegenerate and antisymmetric element in $(V_c \otimes_l V_c)_l$ can be reduced to a canonical form by choosing an appropriate basis on $\mathrm{Ext}^1_\Gamma(l, l)$. The element η defines a bisymplectic form ω_η of degree -1 on $\widehat{T}_l V$ (see the definition in [Van den Bergh 2008, Section 10.1]). By [Van den Bergh 2008, Lemma 11.3.1], there exists a potential $w \in \widehat{T}_l V / [\widehat{T}_l V, \widehat{T}_l V]$ of degree 0 such that, for any $f \in \widehat{T}_l V$,

$$df = \{w, f\}_{\omega_\eta},$$

where $\{-, -\}_{\omega_\eta}$ is the Poisson bracket associated to the bisymplectic form ω_η. Since w does not have constant terms, it is uniquely determined by the derivation $d = \{w, -\}_{\omega_\eta}$. In other words, any two potentials w and w' without constant terms satisfying the above equation differ by an automorphism $\widehat{T}_l V \to \widehat{T}_l V$. Moreover, since w and w' are of degree 0, and therefore do not involve variables in $\mathrm{Ext}^{\geq 2}_\Gamma(l, l)$, this automorphism is precisely a formal change of variables on $\mathrm{Ext}^1_\Gamma(l, l)$. Such a formal change of variables induces an isomorphism from $\mathfrak{D}(Q, w)$ to $\mathfrak{D}(Q, w')$ (see [Hua and Zhou 2023, Theorem 4.3]), and therefore an automorphism of the complete path algebra of Q and an automorphism of $H^0\Gamma$. $\square$

Remark 4.9 Theorem 11.2.1 of [Van den Bergh 2008] can be viewed as a Darboux–Weinstein theorem in noncommutative formal symplectic geometry. On $\mathrm{Ext}^*_\Gamma(l, l)$, the cyclic A_∞–structure can be interpreted as a symplectic structure. The symplectic structure restricts to the truncation $\mathrm{Ext}^1_\Gamma(l, l) \oplus \mathrm{Ext}^2_\Gamma(l, l)$ so that $\mathrm{Ext}^1_\Gamma(l, l)$ is a (graded) Lagrangian. Then [Van den Bergh 2008, Theorem 11.2.1] says that there exists a coordinate on $\mathrm{Ext}^1_\Gamma(l, l)$ under which the symplectic form can be normalized so that it has constant coefficients, which is in particular exact. The differential d of Γ can be interpreted as a homological vector field of degree 1. Then the contraction of the normalized symplectic form by d is the exterior derivation of a potential w of degree 0. Note that a different choice of Darboux coordinates can only differ by a change of variables on $\mathrm{Ext}^1_\Gamma(l, l)$, which leads to the above proposition.

Corollary 4.10 *Let $(\widehat{Y}, \hat{f}, R)$ be a 3–dimensional formal flopping contraction with reduced exceptional fiber $\mathrm{Ex}(f) = \bigcup_{i=1}^t C_i$, and let $A = \mathrm{End}_R\big(\bigoplus_{i=1}^t N_i \oplus R\big)$ be the NCCR associated to it. Fix a 3–CY structure η on $\widehat{Y}$, and therefore on A. Denote the CY tilted algebra of $\mathscr{C}(Y, \{C_i\}_{i=1}^t, \eta) \simeq \mathscr{C}(R, \{N_i\}_{i=1}^t, \eta)$ by Λ. Then there exists a canonical class, defined up to right equivalence, on $\mathrm{HH}_0(\Lambda) = \Lambda / [\Lambda, \Lambda]^c$ represented by a potential.*

The canonical class $[w]$ in the zeroth Hochschild homology of $H^0\Gamma$ is part of the "classical shadow" of the CY structure. The class plays a crucial role in the geometric applications. When Γ is weakly

equivalent to a Jacobi-finite Ginzburg algebra $\mathfrak{D}(\widehat{F}, w)$ for a complete free algebra $\widehat{F} = k \langle\!\langle x_1, \ldots, x_n \rangle\!\rangle$, this class vanishes if and only if w is right equivalent to a weighted-homogeneous noncommutative polynomial (see Theorem 4.16). Therefore, the quasihomogeneity of a potential is indeed a homotopy invariant of the CY algebra. This motivates the following definition:

Definition 4.11 Let k be a field and l be a finite-dimensional commutative separable k–algebra, and let Γ be a pseudocompact dg l–algebra in PCAlgc(l) concentrated in nonpositive degrees. Assume that Γ is exact 3–CY. Then Γ is called *quasihomogeneous* if the canonical class $[w]$ is (right equivalent to) zero.

The notion of quasihomogeneity is expected to be independent of choices of CY structure. In the case of simple flopping contractions, the first author and Gui-song Zhou have conjectured that this notion of quasihomogeneity is indeed equivalent to the quasihomogeneity of the underlying hypersurface singularity R (see [Hua and Zhou 2023, Conjecture 4.18]).

4.3 Properties of Jacobi-finite Ginzburg algebras

In this section, we collect several results about Jacobi-finite Ginzburg algebras. We take k to be the field of complex numbers, though some of the results are valid more generally.

Theorem 4.12 [Hua and Zhou 2023, Theorem 3.16] *Let Q be a finite quiver and w be a potential in $\widehat{\mathbb{C}Q}_{\mathrm{cyc}}$. Assume that the Jacobi algebra $\Lambda(Q, w)$ is finite-dimensional. Then w is right equivalent to a formal series with only finitely many nonzero terms.*

As a consequence, we may assume the potential is a noncommutative polynomial to begin with if the Jacobi algebra is known to be finite-dimensional.

Theorem 4.13 (noncommutative Mather–Yau theorem [Hua and Zhou 2023, Theorem 3.5]) *Let Q be a finite quiver and let $w, w' \in \widehat{\mathbb{C}Q}_{\mathrm{cyc}}$ be two potentials with only cubic terms and higher. Suppose that the Jacobi algebras $\Lambda(Q, w)$ and $\Lambda(Q, w')$ are both finite-dimensional. Then the following two statements are equivalent*:

(1) *There is an algebra isomorphism $\gamma : \Lambda(Q, w) \cong \Lambda(Q, w')$ such that $\gamma_*([w]) = [w']$ in $\Lambda(Q, w')_{\mathrm{cyc}}$.*

(2) *w and w' are right equivalent in $\widehat{\mathbb{C}Q}_{\mathrm{cyc}}$.*

The noncommutative Mather–Yau theorem has an immediate application to Ginzburg algebras.

Corollary 4.14 [Hua and Zhou 2023, Theorem 4.3] *Fix a finite quiver Q. Let $w, w' \in \widehat{\mathbb{C}Q}_{\mathrm{cyc}}$ be two potentials, with only cubic terms and higher, such that the Jacobi algebras $\Lambda(Q, w)$ and $\Lambda(Q, w')$ are both finite-dimensional. Assume there is an algebra isomorphism $\gamma : \Lambda(Q, w) \to \Lambda(Q, w')$ such that $\gamma_*([w]) = [w']$. Then there exists a dg algebra isomorphism*

$$\beta : \mathfrak{D}(Q, w) \xrightarrow{\cong} \mathfrak{D}(Q, w')$$

such that $\beta(t_i) = t_i$ for any $i \in Q_0$.

Definition 4.15 Fix $\widehat{F}$ to be the complete free associative algebra $\mathbb{C}\langle\!\langle x_1, \ldots, x_n \rangle\!\rangle$. Let $(r_1, \ldots, r_n)$ be a tuple of rational numbers. A potential $w \in \widehat{F}_{\mathrm{cyc}} := \widehat{F}/[\widehat{F}, \widehat{F}]^c$ is said to be *weighted-homogeneous of type* $(r_1, \ldots, r_n)$ if it has a representative which is a linear combination of monomials $x_{i_1} x_{i_2} \cdots x_{i_p}$ such that $r_{i_1} + r_{i_2} + \cdots + r_{i_p} = 1$.

Theorem 4.16 (noncommutative Saito theorem [Hua and Zhou 2021, Theorem 1.2]) *Let* $w \in \widehat{F}_{\mathrm{cyc}}$ *be a potential with only cubic terms and higher such that the Jacobi algebra associated to w is finite-dimensional. Then $[w] = 0$ if and only if w is right equivalent to a weighted-homogenous potential of type* $(r_1, \ldots, r_n)$ *for some rational numbers $r_1, \ldots, r_n$ which lie strictly between 0 and $\frac{1}{2}$. Moreover, in this case, all such types $(r_1, \ldots, r_n)$ agree with each other up to permutations on the indices $1, \ldots, n$.*

Recall that $\mathscr{C}_\Gamma$ is constructed as the Verdier quotient of the perfect derived category of Γ by its full subcategory of dg modules whose homology is of finite total dimension. The category of perfect dg Γ–modules has a canonical dg enhancement and we obtain a natural dg enhancement $(\mathscr{C}_\Gamma)_{\mathrm{dg}}$ for $\mathscr{C}_\Gamma$ using the Drinfeld quotient of the dg category of strictly perfect dg Γ–modules by its full subcategory on the dg modules whose homology is of finite total dimension.

Theorem 4.17 *Let Q be a finite quiver and w a Jacobi-finite potential on Q. Let Γ be the complete Ginzburg algebra associated with (Q, w). Denote by Λ_{dg} the dg endomorphism algebra of Γ in the canonical dg enhancement of the cluster category $\mathscr{C}_\Gamma$. Then there is a canonical isomorphism in the homotopy category of dg algebras*

$$\Gamma \xrightarrow{\ \sim\ } \tau_{\leq 0}\Lambda_{\mathrm{dg}}.$$

Proof There is a canonical morphism

$$\Gamma = \mathbb{R}\mathrm{Hom}_\Gamma(\Gamma, \Gamma) \to (\mathscr{C}_\Gamma)_{\mathrm{dg}}(\Gamma, \Gamma) = \Lambda_{\mathrm{dg}},$$

where the right-hand side denotes the dg endomorphism algebra of Γ in the canonical dg enhancement $(\mathscr{C}_\Gamma)_{\mathrm{dg}}$ of $\mathscr{C}_\Gamma$. It suffices to show that the canonical map

$$H^{-p}(\Gamma) = \mathrm{Hom}_{\mathrm{per}(\Gamma)}(\Gamma, \Sigma^{-p}\Gamma) \to \mathrm{Hom}_{\mathscr{C}_\Gamma}(\Gamma, \Sigma^{-p}\Gamma)$$

is invertible for $p \geq 0$. By [Amiot 2009, Proposition 2.8],

$$\mathrm{Hom}_{\mathscr{C}_\Gamma}(\Gamma, \Sigma^{-p}\Gamma) = \operatorname*{colim}_n \mathrm{Hom}_{\mathrm{per}(\Gamma)}(\tau_{\leq n}\Gamma, \tau_{\leq n}(\Sigma^{-p}\Gamma)).$$

We have

$$\mathrm{Hom}_{\mathrm{per}(\Gamma)}(\tau_{\leq n}\Gamma, \tau_{\leq n}(\Sigma^{-p}\Gamma)) = \mathrm{Hom}_{\mathrm{per}(\Gamma)}(\tau_{\leq n}\Gamma, \Sigma^{-p}\Gamma).$$

Consider the canonical triangle

$$\tau_{\leq n}\Gamma \to \Gamma \to \tau_{>n}\Gamma \to \Sigma(\tau_{\leq n}\Gamma).$$

Recall that, by [Amiot 2009, Lemma 2.5], the spaces $H^p(\Gamma)$ are finite-dimensional for all $p \in \mathbb{Z}$. Therefore, the object $\tau_{>n}\Gamma$ belongs to $D_{\mathrm{fd}}(\Gamma)$ and $\Sigma^{-p}\Gamma$ belongs to $\mathrm{per}(\Gamma)$.

By the 3–Calabi–Yau property,

$$\mathrm{Hom}_{\mathrm{per}(\Gamma)}(\Sigma^{-1}\tau_{>n}\Gamma, \Sigma^{-p}\Gamma) = D\mathrm{Hom}_{\mathrm{per}(\Gamma)}(\Gamma, \Sigma^{p+2}\tau_{>n}\Gamma)),$$

which vanishes because $\tau_{>n}\Gamma$ has no homology in degrees > 0. Similarly,

$$\mathrm{Hom}_{\mathrm{per}(\Gamma)}(\tau_{>n}\Gamma, \Sigma^{-p}\Gamma) = D\mathrm{Hom}_{\mathrm{per}(\Gamma)}(\Gamma, \Sigma^{p+3}(\tau_{>n}\Gamma)),$$

which vanishes for the same reason. Thus,

$$\mathrm{Hom}_{\mathrm{per}(\Gamma)}(\tau_{\leq n}\Gamma, \Sigma^{-p}\Gamma) = \mathrm{Hom}_{\mathrm{per}(\Gamma)}(\Gamma, \Sigma^{-p}\Gamma). \qquad \square$$

Corollary 4.18 *Let Q be a quiver with one node and arbitrary number of loops and $\Gamma = \mathfrak{D}(Q, w)$ a Jacobi-finite Ginzburg algebra. Then $H^0\Gamma$ is self-injective and there is an isomorphism*

$$\Sigma^2\Gamma \xrightarrow{\sim} \tau_{\leq -1}\Gamma$$

in the derived category of dg Γ–modules. In particular, $H^i(\Gamma) = 0$ for odd i and $H^i(\Gamma) \cong H^0(\Gamma)$ for even $i \leq 0$.

Proof By [Amiot 2009], the cluster category $\mathscr{C}_\Gamma$ is a Hom–finite 2–Calabi–Yau category and the image T of Γ in $\mathscr{C}_\Gamma$ is a cluster-tilting object in $\mathscr{C}_\Gamma$. By [Adachi et al. 2014, Theorem 4.1], the cluster-tilting objects of $\mathscr{C}_\Gamma$ are in bijection with the support τ–tilting modules over $\mathrm{End}(T)$. Since $\mathrm{End}(T) = H^0(\Gamma)$ is local, the only support τ–tilting modules over $\mathrm{End}(T)$ are 0 and $\mathrm{End}(T)$, by [loc. cit., Example 6.1]. Thus, the only cluster-tilting objects of $\mathscr{C}_\Gamma$ are T and ΣT. In particular, $\Sigma^2 T$ has to be isomorphic to T (since $\mathrm{Hom}(\Sigma T, \Sigma^2 T) = 0$ and $\Sigma^2 T$ must be a cluster-tilting object). This implies that $H^0\Gamma = \mathrm{End}(T)$ is self-injective, since, by the 2–Calabi–Yau property, we have an isomorphism of right $\mathrm{End}(T)$–modules

$$D\mathrm{Hom}(T, T) = \mathrm{Hom}(T, \Sigma^2 T) = \mathrm{Hom}(T, T).$$

Let $\phi \colon \Sigma^2\Gamma \to \Gamma$ be a lift of an isomorphism $\Sigma^2 T \to T$ in $\mathscr{C}_\Gamma$. Let $p \geq 2$. In the commutative square

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{per}(\Gamma)}(\Sigma^p\Gamma, \Sigma^2\Gamma) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}_\Gamma}(\Sigma^p T, \Sigma^2 T) \\ \downarrow & & \downarrow{\scriptstyle \phi_*} \\ \mathrm{Hom}_{\mathrm{per}(\Gamma)}(\Sigma^p\Gamma, \Gamma) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}_\Gamma}(\Sigma^p T, T) \end{array}$$

the horizontal arrows are isomorphisms by Theorem 4.17 and the right vertical arrow ϕ_* is an isomorphism. Thus, the morphism $\phi \colon \Sigma^2\Gamma \to \Gamma$ induces isomorphisms in H^i for $i \leq -2$. Moreover, $H^{-1}(\Gamma) = \mathrm{Hom}(T, \Sigma^{-1}T) = \mathrm{Hom}(T, \Sigma T) = 0$ since $\Sigma^{-1}T$ is isomorphic to ΣT. It follows that ϕ induces an isomorphism

$$\Sigma^2\Gamma \xrightarrow{\sim} \tau_{\leq -1}\Gamma. \qquad \square$$

Remark 4.19 For the pair (Q, w) associated to 3–dimensional flopping contractions, one can show that $H^0\Gamma$ is indeed symmetric (see Proposition 6.5). In the context of a general contraction with 1–dimensional fiber, Kawamata [2018, Proposition 6.3] has proved that the classical (multipointed) deformation algebra of the reduced exceptional fiber is always self-injective. So, in particular, it is Gorenstein. This result overlaps with the above corollary in the case of simple flopping contractions. For a general finite quiver Q, the zeroth homology of a Jacobi-finite Ginzburg algebra $\mathfrak{D}(Q, w)$ is not self-injective. Moreover, Kawamata [2020, Corollary 6.3] proves that the deformation algebra is always isomorphic to its opposite algebra.

Corollary 4.20 *Let Γ be the Ginzburg algebra of a Jacobi-finite quiver with potential. Let T be the image of Γ in the cluster category $\mathscr{C}_\Gamma$.*

(a) *$H^0\Gamma$ is selfinjective if and only if $H^{-1}\Gamma$ vanishes if and only if T is isomorphic to $\Sigma^2 T$ in $\mathscr{C}_\Gamma$.*

(b) *If the identity functor of $\mathscr{C}_\Gamma$ is isomorphic to Σ^2, then $H^0\Gamma$ is symmetric and there is an isomorphism of graded algebras $H^0(\Gamma) \otimes k[u^{-1}] \xrightarrow{\sim} H^*(\Gamma)$, where u is of degree 2.*

Proof (a) By Theorem 4.17, the space $H^{-1}\Gamma$ is isomorphic to $\mathscr{C}_\Gamma(T, \Sigma^{-1}T)$ and $H^0\Gamma$ is isomorphic to the endomorphism algebra of T. By [Iyama and Oppermann 2013, Proposition 3.6], the endomorphism algebra is selfinjective if and only if $\mathscr{C}_\Gamma(T, \Sigma^{-1}T)$ vanishes if and only if T is isomorphic to $\Sigma^2 T$ in $\mathscr{C}_\Gamma$.

(b) By combining the functorial isomorphism from T to $\Sigma^2 T$ with the Calabi–Yau property we get an isomorphism of bimodules over the endomorphism algebra of T

$$\mathscr{C}_\Gamma(T, T) \xrightarrow{\sim} \mathscr{C}_\Gamma(T, \Sigma^2 T) \xrightarrow{\sim} D\mathscr{C}_\Gamma(T, T).$$

Since $H^0\Gamma$ is in particular selfinjective, the space $H^{-1}\Gamma$ vanishes by (a). We get an isomorphism of graded algebras

$$k[u, u^{-1}] \otimes_k \mathscr{C}_\Gamma(T, T) \xrightarrow{\sim} \bigoplus_{p \in \mathbb{Z}} \mathscr{C}_\Gamma(T, \Sigma^p T),$$

where u is of degree 2, by sending u to the functorial isomorphism $T \xrightarrow{\sim} \Sigma^2 T$. Thanks to Theorem 4.17, by truncation, we get an isomorphism of graded algebras

$$k[u^{-1}] \otimes_k H^0\Gamma \xrightarrow{\sim} H^*\Gamma. \qquad \square$$

4.4 Silting theory for a nonpositive dg algebra and its zeroth homology

Let $\mathscr{T}$ be a triangulated category. Recall that a *tilting object* for $\mathscr{T}$ is a classical generator T of $\mathscr{T}$ such that $\mathscr{T}(T, \Sigma^p T)$ vanishes for all $p \neq 0$. A *silting object* [Keller and Vossieck 1988] for $\mathscr{T}$ is a classical generator T of $\mathscr{T}$ such that $\mathscr{T}(T, \Sigma^p T)$ vanishes for all $p > 0$. The advantage of silting objects over tilting objects is that (under suitable finiteness assumptions) they are stable under mutation [Aihara and Iyama 2012].

We recall fundamental definitions and results from [Aihara and Iyama 2012]. Assume from now on that $\mathcal{T}$ is k–linear and Hom–finite and has split idempotents. In particular, it is a Krull–Schmidt category, ie indecomposables have local endomorphism rings and each object is a finite direct sum of indecomposables (which are then unique up to isomorphism and permutation). An object of $\mathcal{T}$ is *basic* if it is a direct sum of pairwise nonisomorphic indecomposables. If X is an object of $\mathcal{T}$ and $\mathcal{U}$ a full additive subcategory stable under retracts, a *left $\mathcal{U}$–approximation of X* is a morphism $f: X \to U$ to an object of $\mathcal{U}$ such that each morphism $X \to V$ to an object of $\mathcal{U}$ factors through $f: X \to U$. It is *minimal* if each endomorphism $g: U \to U$ such that $g \circ f = f$ is an isomorphism. Notice that the morphism $f: X \to U$ is a minimal left $\mathcal{U}$–approximation if and only if the morphism $f^*: \mathcal{U}(U, ?) \to \mathcal{T}(X, ?)|_{\mathcal{U}}$ is a projective cover in the category of left $\mathcal{U}$–modules. In particular, minimal left approximations are unique up to nonunique isomorphism when they exist. Existence is automatic if $\mathcal{U}$ has finitely many indecomposables $U_1, \ldots, U_n$ (which is the case in our applications) because then the functor $\mathcal{T}(X, ?)|_{\mathcal{U}}$ corresponds to a finite-dimensional left module over the finite-dimensional endomorphism algebra of the sum of the U_i. A (*minimal*) *right $\mathcal{U}$–approximation* is defined dually. For an object X of $\mathcal{T}$, we denote by add X the full subcategory formed by all direct factors of finite direct sums of copies of X.

Let M be a basic silting object of $\mathcal{T}$ and X an indecomposable direct summand of M. Denote by M/X the object such that $M \cong X \oplus M/X$. By definition, the *left mutation $\mu_X(M)$ of M at X* is the silting object $M/X \oplus Y$, where Y is defined by a triangle

$$X \to E \to Y \to \Sigma X$$

and $X \to E$ is a minimal left add(M/X)–approximation. It is not hard to show that then $E \to Y$ is a minimal right add(M/X)–approximation, which implies that Y is indecomposable. Indeed, let us recall the argument: Let $M' = M/X$. Since M is silting, we have an exact sequence of End(M')–modules

$$\mathrm{Hom}(M', E) \to \mathrm{Hom}(M', Y) \to 0,$$

where $\mathrm{Hom}(M', E)$ is projective over End(M'). Saying that $E \to Y$ is a minimal right add(M/X)–approximation is equivalent to saying that $\mathrm{Hom}(M', E) \to \mathrm{Hom}(M', Y)$ is a projective cover. If Y was decomposable, the morphism $\mathrm{Hom}(M', E) \to \mathrm{Hom}(M', Y)$ would therefore split into a direct sum of two surjective morphisms and this splitting would be induced by a splitting of the morphism $E \to Y$ as a direct sum of two nontrivial morphisms $E' \to Y'$ and $E'' \to Y''$. But then X would be decomposable, a contradiction. The right mutation $\mu_X^-(M)$ is defined dually. The right mutation of $\mu_X(M)$ at Y is isomorphic to M.

Example 4.21 Suppose that Γ is a Jacobi-finite Ginzburg algebra associated with a finite quiver and a potential not containing cycles of length ≤ 2. Then $A = \Gamma$ satisfies our assumptions and Γ is a basic silting object in per(Γ). Let M be a silting object in per(Γ) and Γ' the derived endomorphism algebra of M. Then the homologies $H^p(\Gamma')$ are finite-dimensional and vanish in degrees $p > 0$. Since M generates per(Γ), the Γ'–Γ–bimodule M yields an algebraic triangle equivalence $D(\Gamma') \xrightarrow{\sim} D(\Gamma)$. Conversely, if we start from a dg algebra Γ' whose homologies are finite-dimensional and vanish in degrees > 0 and from

an algebraic triangle equivalence $D(\Gamma') \xrightarrow{\sim} D(\Gamma)$, then the image M of Γ' in per(Γ) is a silting object. In any case, the dg algebra Γ' is an exact bimodule 3–Calabi–Yau and has its homology concentrated in degrees ≤ 0. By Van den Bergh's theorem [2015], the dg algebra Γ' is again a Jacobi-finite Ginzburg algebra (up to weak equivalence). In particular, for M we can take the mutation $M' = \mu_X \Gamma$, where $X = e_i \Gamma$ for a vertex i of the quiver of Γ. We define the associated Ginzburg algebra Γ' to be the *left mutation of* Γ *at* i. Notice that, by construction, we have a canonical derived equivalence from Γ' to Γ. In the same way, we can define the right mutation Γ'' of Γ at i using the right mutation $M'' = \mu_X^-(\Gamma)$ of Γ at X. The right mutation Γ'' turns out to be quasi-isomorphic to the left mutation Γ'. Indeed, by Theorem 4.17, these algebras are the $\tau_{\leq 0}$–truncations of the derived endomorphism algebras of the images $\pi(M')$ and $\pi(M'')$ in the cluster category $\mathscr{C}_\Gamma$. Now we have $\pi(M') \cong \pi(M'')$ because they are the left and right mutations in the sense of [Iyama and Yoshino 2008] of the cluster-tilting object $\pi(\Gamma)$ at $\pi(X)$ and, for cluster-tilting objects in 2–Calabi–Yau triangulated categories, right and left mutation coincide up to isomorphism.

Now let A be a dg k–algebra whose homologies $H^p A$ vanish in all degrees $p > 0$. An object of per(A) is called 2–*term* if it is the cone over a morphism of add(A). We write 2–per(A) for the full subcategory of per(A) formed by the 2–term objects. We write 2silt(A) for the set of isomorphism classes of 2–*term silting objects*. Our aim is to compare 2silt(A) with 2silt$(H^0 A)$. Note that, by our assumption on A, we have a canonical morphism $A \to H^0 A$ in the homotopy category of dg algebras. We write $F : \mathrm{per}(A) \to \mathrm{per}(H^0 A)$ for the derived tensor product over A with $H^0 A$.

Proposition 4.22 (a) [Brüstle and Yang 2013] *The functor F induces a bijection* 2silt$(A) \xrightarrow{\sim}$ 2silt$(H^0 A)$.

(b) *Suppose that $H^{-1}(A) = 0$. Then the functor F restricts to an equivalence*

$$2\text{–}\mathrm{per}(A) \xrightarrow{\sim} 2\text{–}\mathrm{per}(H^0 A).$$

In particular, for each 2–term object T, the functor F induces an isomorphism

$$\mathrm{End}_A(T) \xrightarrow{\sim} \mathrm{End}_{H^0 A}(FT).$$

Proof Part (a) is [Brüstle and Yang 2013, Proposition A.3]. For part (b), using the assumption and the five lemma, we check successively that F induces the following bijections:

(1) For $P, Q \in \mathrm{add}(A)$ and $p \geq -1$,

$$\mathrm{Hom}(P, \Sigma^p Q) \xrightarrow{\sim} \mathrm{Hom}(FP, \Sigma^p FQ).$$

(2) For $P \in \mathrm{add}(A)$, $M \in 2\text{–}\mathrm{per}(A)$ and $p \in \{-1, 0\}$,

$$\mathrm{Hom}(P, \Sigma^p M) \xrightarrow{\sim} \mathrm{Hom}(FP, \Sigma^p FM).$$

(3) For $M, M' \in 2\text{–}\mathrm{per}(A)$,

$$\mathrm{Hom}(M, M') \xrightarrow{\sim} \mathrm{Hom}(FM, FM'). \qquad \square$$

Now let A be a pseudocompact dg algebra in $\mathrm{PCAlgc}(l)$ strictly concentrated in degrees ≤ 0. Let e be an idempotent of $H^0 A$ and A' the derived endomorphism algebra of the image of A in the Verdier quotient of $\mathrm{per}(A)$ by the thick subcategory generated by eA. Then A' is concentrated in degrees ≤ 0 and we have a canonical morphism $A \to A'$ in the homotopy category of $\mathrm{PCAlgc}(l)$. If A is of the form $(\hat{T}_l(V), d)$ for a pseudocompact l–bimodule V concentrated in degrees ≤ 0, where $\hat{T}_l(V)$ is the completed tensor algebra, then A' is quasi-isomorphic to the quotient of A by the two-sided closed ideal generated by e (see [Braun et al. 2018]). Put $A_0 = H^0 A'$; so we have a canonical morphism $p \colon A \to A_0$. Let B and B_0 be pseudocompact dg algebras in $\mathrm{PCAlgc}(l)$, and $X \in D(A^{\mathrm{op}} \otimes B)$ such that X_B is perfect and $Q \in D(B^{\mathrm{op}} \otimes B_0)$ such that Q_{B_0} is perfect.

Proposition 4.23 *Suppose that $eX \otimes_B^{\mathbb{L}} Q$ vanishes and that the object $X \otimes_B^{\mathbb{L}} Q$ of $D(B_0)$ has no selfextensions in degrees $p < 0$. Then there is an object Y of $D(A_0^{\mathrm{op}} \otimes B_0)$, unique up to isomorphism, such that we have an isomorphism*

$$X \otimes_B^{\mathbb{L}} Q \xrightarrow{\ \sim\ } A_0 \otimes_{A_0}^{\mathbb{L}} Y$$

in $D(A^{\mathrm{op}} \otimes B_0)$. Thus, the square

$$
\begin{array}{ccc}
D(A) & \xrightarrow{\ X\ } & D(B) \\
{\scriptstyle A_0}\Big\downarrow & & \Big\downarrow{\scriptstyle Q} \\
D(A_0) & \xrightarrow{\ Y\ } & D(B_0)
\end{array}
$$

is commutative up to isomorphism, where we write dg bimodules instead of derived tensor products by dg bimodules.

Remark 4.24 In our applications in this article, the idempotent e will be 0. We state and prove the proposition in the general case because it provides an alternative approach to the problem of relating the tilting theory of maximal modification algebras [Wemyss 2018] to that of the associated contraction algebras as treated by August [2020b]. Let R be a complete local cDV singularity and M a maximal basic rigid object in the category of Cohen–Macaulay modules over R containing R as a direct summand. We can take $A = \mathrm{End}_R(M)$ and e the idempotent corresponding to the projection on R. Then $A_0 = H^0 A'$ is isomorphic to the stable endomorphism algebra of M, ie the contraction algebra associated with M. Let N be another maximal basic rigid object containing R as a direct summand, B its endomorphism algebra and B_0 the associated contraction algebra. Then $X = \mathrm{Hom}_R(N, M)$ yields a derived equivalence $? \otimes_A^{\mathbb{L}} X \colon D(A) \xrightarrow{\ \sim\ } D(B)$ taking eA to eB. Moreover, the complex $X \otimes_B^{\mathbb{L}} B_0$ is a 2–term silting object of $\mathrm{per}(B_0)$ (as follows from silting reduction [Aihara and Iyama 2012] combined with Proposition 4.22(a)) and hence a tilting object since B_0 is symmetric. Thus, the hypotheses of the proposition hold and there is a canonical two-sided tilting complex Y in $D(A_0^{\mathrm{op}} \otimes B_0)$. Clearly, the construction is compatible with compositions via derived tensor products.

Proof of Proposition 4.23 Put $U = X \otimes^{\mathbb{L}}_B Q$ viewed as an object in $D(A^{\mathrm{op}} \otimes B_0)$. The morphism $A \to A'$ induces the Verdier quotient

$$\mathrm{per}(A) \to \mathrm{per}(A') = \mathrm{per}(A)/\mathbf{thick}(eA)$$

and is therefore a dg quotient. By the universal property of the dg quotient, there is an object Z in $D(A'^{\mathrm{op}} \otimes B_0)$, unique up to isomorphism, such that the restriction of Z along $A^{\mathrm{op}} \otimes B_0 \to A'^{\mathrm{op}} \otimes B_0$ is isomorphic to U in $D(A^{\mathrm{op}} \otimes B_0)$. We have a canonical morphism in the homotopy category of dg algebras

$$A' \to \mathbb{R}\mathrm{Hom}_{B_0}(Z, Z).$$

Since, by assumption, $\mathbb{R}\mathrm{Hom}_{B_0}(Z, Z)$ is concentrated in degrees ≥ 0 and A' in degrees ≤ 0, this morphism factors uniquely through a morphism

$$A_0 = H^0 A' \to \mathbb{R}\mathrm{Hom}_{B_0}(Z, Z)$$

in the homotopy category of dg algebras. Let us show how to refine this argument so as to obtain an object Y of $D(A_0^{\mathrm{op}} \otimes B_0)$ which restricts to $Z \in D(A'^{\mathrm{op}} \otimes B_0)$. We may and will assume that A' is cofibrant and strictly concentrated in degrees ≤ 0. We may and will also assume that Z is cofibrant as a dg A'–B_0–bimodule. The left A'–module structure on Z then yields a morphism of dg algebras

$$A' \to \mathrm{Hom}_{B_0}(Z, Z).$$

Since A' is strictly concentrated in nonpositive degrees, it factors uniquely through a morphism of dg algebras

$$A' \to \tau_{\leq 0} \mathrm{Hom}_{B_0}(Z, Z).$$

Since Z_{B_0} has no negative selfextensions, we have a surjective quasi-isomorphism of dg algebras

$$\tau_{\leq 0} \mathrm{Hom}_{B_0}(Z, Z) \to H^0 \mathrm{Hom}_{B_0}(Z, Z).$$

The composition
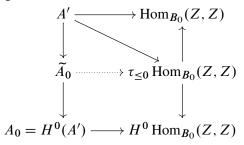
$$A' \to \tau_{\leq 0} \mathrm{Hom}_{B_0}(Z, Z) \to H^0 \mathrm{Hom}_{B_0}(Z, Z)$$

uniquely factors through an algebra morphism $A_0 = H^0(A') \to H^0 \mathrm{Hom}_{B_0}(Z, Z)$. We thus obtain a commutative square of dg algebra morphisms

$$
\begin{array}{ccc}
A' & \longrightarrow & \mathrm{Hom}_{B_0}(Z, Z) \\
\downarrow & \searrow & \uparrow \\
 & \tau_{\leq 0} \mathrm{Hom}_{B_0}(Z, Z) & \\
\downarrow & & \downarrow \\
A_0 = H^0(A') & \longrightarrow & H^0 \mathrm{Hom}_{B_0}(Z, Z)
\end{array}
$$

We factor the morphism $A' \to A_0$ as the composition $A' \to \widetilde{A}_0 \to A_0$ of an acyclic fibration with a cofibration. We consider the diagram

$$
\begin{array}{ccc}
A' & \longrightarrow & \mathrm{Hom}_{B_0}(Z, Z) \\
\downarrow & \searrow & \uparrow \\
\widetilde{A}_0 & \dashrightarrow \tau_{\leq 0} \mathrm{Hom}_{B_0}(Z, Z) & \\
\downarrow & & \downarrow \\
A_0 = H^0(A') & \longrightarrow & H^0 \mathrm{Hom}_{B_0}(Z, Z)
\end{array}
$$

Here the morphism represented by a dotted arrow exists, so the diagram becomes commutative because $A' \to \widetilde{A}_0$ is a cofibration and $\tau_{\leq 0} \mathrm{Hom}_{B_0}(Z, Z) \to H^0 \mathrm{Hom}_{B_0}(Z, Z)$ is an acyclic fibration. Thus, we obtain a structure of a dg $\widetilde{A}_0 – B_0$–bimodule on Z which restricts to the given structure of a dg $A' – B_0$–bimodule. Since we have the quasi-isomorphism $\widetilde{A}_0 \to A_0$, we can find a bimodule Y in $D(A_0^{\mathrm{op}} \otimes B_0)$ unique up to isomorphism and which restricts (up to isomorphism) to Z in $D(A'^{\mathrm{op}} \otimes B_0)$.

Now consider a second object Y' in $D(A_0^{\mathrm{op}} \otimes B_0)$ which becomes isomorphic to Z in $D(A'^{\mathrm{op}} \otimes H^0 B)$. We have a chain of isomorphisms

$$
\begin{aligned}
\mathrm{Hom}_{D(A'^{\mathrm{op}} \otimes B_0)}(Y, Y') &= \mathrm{Hom}_{D(A'^e)}(A', \mathbb{R}\mathrm{Hom}_{B_0}(Y, Y')) \\
&= \mathrm{Hom}_{D(A'^e)}(H^0 A', H^0 \mathbb{R}\mathrm{Hom}_{B_0}(Y, Y')) \\
&= \mathrm{Hom}_{H^0(A')^e}(H^0 A', H^0 \mathbb{R}\mathrm{Hom}_{B_0}(Y, Y')) \\
&= \mathrm{Hom}_{D(A_0^e)}(A_0, H^0 \mathbb{R}\mathrm{Hom}_{B_0}(Y, Y')) \\
&= \mathrm{Hom}_{D(A_0^e)}(A_0, \mathbb{R}\mathrm{Hom}_{B_0}(Y, Y')) \\
&= \mathrm{Hom}_{D(A_0^{\mathrm{op}} \otimes B_0)}(Y, Y').
\end{aligned}
$$

Clearly, the composition

$$
\mathrm{Hom}_{D(A_0^{\mathrm{op}} \otimes B_0)}(Y, Y') \xrightarrow{\sim} \mathrm{Hom}_{D(A'^{\mathrm{op}} \otimes B_0)}(Y, Y')
$$

of these isomorphisms is given by the restriction-of-scalars functor

$$
D(A^0 \otimes H^0 B) \to D(A' \otimes H^0 B).
$$

Now any restriction-of-scalars functor reflects isomorphisms because it is compatible with the forgetful functors to the derived category of vector spaces. Thus, isomorphisms are preserved. This shows the uniqueness.   □

Let B be a dg k–algebra whose homologies are finite-dimensional and vanish in degrees > 0. Let C be a finite-dimensional basic k–algebra (ie C is basic as a right module over itself) and let Z be an object of $D(C^{\mathrm{op}} \otimes H^0 B)$ such that

$$
? \otimes_C^{\mathbb{L}} Z : D(C) \to D(H^0 B)
$$

is an equivalence. Notice that Z_{H^0B} is a tilting object in $\mathrm{per}(H^0B)$ and in particular a silting object, which is basic by our assumption on C.

Theorem 4.25 *Assume that Z_{H^0B} is a 2–term silting object and $H^{-1}(B) = 0$. Then there is a dg algebra A whose homologies H^pA are finite-dimensional and whose components vanish in degrees $p > 0$, a derived equivalence $? \otimes_A^{\mathbb{L}} X \colon D(A) \xrightarrow{\sim} D(B)$, an isomorphism of algebras $\phi \colon H^0A \xrightarrow{\sim} C$ and an isomorphism*

$$_\phi Z \xrightarrow{\sim} X \otimes_B^{\mathbb{L}} H^0B$$

in $D(A^{\mathrm{op}} \otimes H^0B)$, where the left A–module structure on $_\phi Z$ is defined via the composition $A \to H^0A \to C$. In particular, we have a diagram, commutative up to isomorphism,

$$
\begin{array}{ccc}
& D(A) \xrightarrow{\;X\;} D(B) & \\
{\scriptstyle H^0A}\nearrow & & \downarrow{\scriptstyle H^0B} \\
D(H^0A) \xrightarrow{\;\phi C\;} D(C) \xrightarrow{\;Z\;} D(H^0B) &
\end{array}
$$

where we write dg bimodules instead of derived tensor products by dg bimodules.

Proof By [Proposition 4.22](a), there is a 2–term silting object M of $\mathrm{per}(B)$ such that $M \otimes_B^{\mathbb{L}} H^0B$ is isomorphic to Z_{H^0B}. Since M is silting, its derived endomorphism algebra has its homology concentrated in nonpositive degrees and we define

$$A = \tau_{\leq 0}\mathbb{R}\mathrm{Hom}(M, M).$$

We let $X \in D(A^{\mathrm{op}} \otimes B)$ be the dg bimodule given by M with its canonical left A–action. Since $X \otimes_B^{\mathbb{L}} H^0B$ is isomorphic to the tilting object Z_{H^0B}, it has no self-extensions in degree < 0. Therefore, [Proposition 4.23](b) yields an object Y of $D(H^0(A)^{\mathrm{op}} \otimes H^0(B))$ and an isomorphism

$$\psi \colon X \otimes_B^{\mathbb{L}} H^0B \xrightarrow{\sim} Y|_{A^{\mathrm{op}} \otimes H^0B}$$

in $D(A^{\mathrm{op}} \otimes H^0B)$, where the left A–module structure on Y comes from the canonical morphism $A \to H^0(A)$. By [Proposition 4.22](b), we have an isomorphism $\mathrm{End}(M) \xrightarrow{\sim} \mathrm{End}(M \otimes_B^{\mathbb{L}} H^0B)$. By construction, we have an isomorphism $H^0A \xrightarrow{\sim} \mathrm{End}(M)$ or equivalently $H^0A \xrightarrow{\sim} \mathrm{End}(X_B)$. Thus, the composition of $? \otimes_B^{\mathbb{L}} H^0B$ with $? \otimes_A^{\mathbb{L}} X$ induces an isomorphism

$$H^0A \xrightarrow{\sim} \mathrm{End}_{D(H^0B)}(X \otimes_B^{\mathbb{L}} H^0B).$$

Via ψ, we get an isomorphism

$$H^0A \xrightarrow{\sim} \mathrm{End}_{D(H^0B)}(Y)$$

given by the left action of $H^0 A$ on Y. We choose an isomorphism $Y_{H^0 B} \xrightarrow{\sim} Z_{H^0 B}$ in $D(H^0 B)$ and define $\phi \colon H^0 A \xrightarrow{\sim} C$ so as to make the following square commutative:

$$
\begin{array}{ccc}
H^0 A & \longrightarrow & \mathrm{End}(Y_{H^0 B}) \\
\phi \downarrow & & \downarrow \\
C & \longrightarrow & \mathrm{End}(Z_{H^0 B})
\end{array}
$$

By [Lemma 2.6](), the chosen isomorphism $Y_{H^0 B} \xrightarrow{\sim} Z_{H^0 B}$ lifts to an isomorphism $Y \xrightarrow{\sim}_\phi Z$ in $D(H^0(A)^{\mathrm{op}} \otimes H^0(B))$, whence a composed isomorphism

$$
X \otimes_B^{\mathbb{L}} H^0 B \xrightarrow{\sim} Y|_{A^{\mathrm{op}} \otimes H^0 B} \xrightarrow{\sim}_\phi Z|_{A^{\mathrm{op}} \otimes H^0 B}. \qquad \square
$$

4.5 Cyclic homology and preservation of the canonical class

Let k be a field of characteristic 0 and l a finite product of copies of k. Let V be a pseudocompact l–bimodule and d a continuous differential on the completed tensor algebra $\widehat{T}_l(V)$. Put $A = (\widehat{T}_l(V), d)$. We define $\Omega_l A$ by the short exact sequence

$$
0 \to \Omega^1 A \to A \otimes_l A \xrightarrow{\mu} A \to 0,
$$

where μ is the multiplication of A. Then the morphism

$$
A \otimes_l V \otimes_l A \to \Omega^1 A
$$

taking $a \otimes v \otimes b$ to $av \otimes b - a \otimes vb$ is an isomorphism of graded l–bimodules; see [Quillen 1989, Example 3.10]. We can describe the induced differential on $A \otimes_l V \otimes_l A$ as follows (see [Keller 2011, Proposition 3.7]): Let $D \colon A \to A \otimes_l V \otimes_l A$ be the unique continuous bimodule derivation which restricts to the map $v \mapsto 1 \otimes v \otimes 1$ on V. We have

$$
D(v_1 \cdots v_n) = 1 \otimes v_1 \otimes (v_2 \cdots v_n) + \sum_{i=2}^{n-1} v_1 \cdots v_{i-1} \otimes v_i \otimes v_{i+1} \cdots v_n + (v_1 \cdots v_{n-1}) \otimes v_n \otimes 1.
$$

Then the induced differential on $A \otimes_l V \otimes_l A$ sends $a \otimes v \otimes b$ to

$$
(-1)^{|a|} a D(dv) b + (da) \otimes v \otimes b + (-1)^{|v|+|a|} a \otimes v \otimes (db).
$$

For an l–bimodule M, we write M_l for the coinvariant module $M/[l, M]$. For an A–bimodule M, we let $M_\natural$ be the coinvariant module $M/[A, M]$. We have an isomorphism of graded A–modules

$$
(A \otimes_l V \otimes_l A)_\natural \xrightarrow{\sim} (V \otimes_l A)_l
$$

taking $a \otimes v \otimes b$ to $(-1)^{|a|(|v|+|b|)} v \otimes ba$. The induced differential on the right-hand side is given as follows: if $D(dv) = \sum_i a_i \otimes v_i \otimes b_i$, then

$$
d(v \otimes a) = (-1)^{|v|} v \otimes (da) + \sum_i (-1)^{|a_i|(|v_i|+|b_i|+|a|)} v_i \otimes b_i a a_i.
$$

Following [Quillen 1989, Section 3], we define morphisms of complexes

$$\partial_1 \colon (V \otimes_l A)_l \to A_l \quad \text{and} \quad \partial_0 \colon A_l \to (V \otimes_l A)_l$$

as follows: ∂_1 sends $v \otimes a$ to $va - (-1)^{|v||a|}av$ and ∂_0 sends $v_1 \cdots v_n$ to

$$\sum_i \pm v_i \otimes v_{i+1} \cdots v_n v_1 \cdots v_{i-1},$$

where the sign is determined by the Koszul sign rule. We then have $\partial_0\partial_1 = 0 = \partial_1\partial_0$.

The (continuous) Hochschild homology of A is computed by the total complex of

$$(V \otimes_l A)_l \xrightarrow{\partial_1} A_l$$

and the (continuous) cyclic homology of A is computed by the product total complex of

$$\cdots \to (V \otimes_l A)_l \xrightarrow{\partial_1} A_l \xrightarrow{\partial_0} (V \otimes_l A)_l \xrightarrow{\partial_1} A_l.$$

Since k is of characteristic 0, the morphism $A_l \to A/([A, A] + l)$ induces a quasi-isomorphism from this complex to $A/([A, A] + l)$. The ISB sequence

$$\cdots \to \mathrm{HH}_n \xrightarrow{I} \mathrm{HC}_n \xrightarrow{S} \mathrm{HC}_{n-2} \xrightarrow{B} \mathrm{HH}_{n-1} \to \cdots$$

is induced by the sequence

Notice that the first three rows form a short exact sequence and that the composition of the last vertical morphism with the second-last vertical morphism is only homotopic to zero.

Now let Q be a finite quiver, l the product over the ke_i, where i runs through the vertices of Q, w a potential on Q and $A = \Gamma$ the associated complete Ginzburg algebra with generators the arrows α of Q in degree 0, the reversed arrows α^* in degree -1 and the loops t_i in degree -2. Then A is of the form $(\widehat{T}_l(V), d)$, where V is the l–bimodule with basis given by the arrows α, α^* and t_i. Let t be the sum of the t_i. By definition,

$$d(t) = \sum_\alpha [\alpha, \alpha^*].$$

Thus, t defines an element in $\mathrm{HC}_2(A)$.

Lemma 4.26 *The image of the class of t under $S\colon \mathrm{HC}_2(A) \to \mathrm{HC}_0(A)$ is the canonical class $[w]$, ie the image of w under the projection $\mathrm{HC}_0(T_l V) \to \mathrm{HC}_0(A)$.*

Proof We compute $S(t)$ using the above description of S. We need to lift t to an element of the total complex computing cyclic homology. We have

$$d(t) = \partial_1 \left(\sum_\alpha \alpha \otimes \alpha^* \right).$$

We have

$$d\left(\sum_\alpha \alpha \otimes \alpha^* \right) = \sum_\alpha \alpha \otimes d(\alpha^*) = \sum_\alpha \alpha \otimes D_\alpha(w).$$

Thus,

$$d\left(\sum_\alpha \alpha \otimes \alpha^* \right) = \partial_0(w)$$

and $S(t)$ is the image of w in $H^0\big(A/([A,A]+l)\big) = \mathrm{HC}_0(\Gamma)$. Notice that $BS(t) = B(w)$ is indeed a boundary in the Hochschild complex: it is the differential of

$$\sum_\alpha \alpha \otimes \alpha^* - t. \qquad \square$$

Corollary 4.27 *Let $\Gamma' = \Gamma(Q', w')$ be a Ginzburg algebra and A a pseudocompact dg algebra in $\mathrm{PCAlgc}(l)$ concentrated in degrees ≤ 0. Let X be a dg A–Γ'–bimodule such that $? \otimes_A^{\mathbb{L}} X\colon D(A) \to D(\Gamma')$ is an equivalence. Then there is a quiver with potential (Q'', w'') and a weak equivalence $s\colon \Gamma(Q'', w'') \to A$ such that, for the restriction ${}_s X$ along s, the isomorphism $\mathrm{HC}_0({}_s X)$ takes the class $[w'']$ to $[w']$.*

Proof We know that the class $[t'] \in \mathrm{HC}_2(\Gamma')$ is nondegenerate in the sense that $B[t'] \in \mathrm{HH}_3(\Gamma')$ defines an isomorphism $\Sigma^3 \Theta_{\Gamma'} \xrightarrow{\sim} \Gamma'$ in $D(\Gamma'^e)$, where $\Theta_{\Gamma'}$ is the inverse dualizing complex. Thus, the image τ of $[t']$ under $\mathrm{HC}_2(X)^{-1}$ is a nondegenerate element of $\mathrm{HC}_2(A)$. The proof of [Van den Bergh 2015, Theorem 10.2.2] then shows that there is a quiver Q'', a potential w'' and a weak equivalence $s\colon \Gamma(Q'', w'') \to A$ which takes $[t'']$ to τ. Thus, the composition $\mathrm{HC}_2(X) \circ \mathrm{HC}_2(s) = \mathrm{HC}_2({}_s X)$ takes $[t'']$ to $[t']$ and the isomorphism $\mathrm{HC}_0({}_s X)$ takes $[w''] = S[t'']$ to $[w'] = S[t']$. $\qquad \square$

5 CY tilted algebras and singularities

5.1 Basics on Hochschild cohomology

Let k be a commutative ring and A be a unital k–algebra projective over k. Denote by $\bar{A}$ the quotient $A/k \cdot 1$. Define the *normalized bar complex* associated to A to be the complex $B_k A := A \otimes_k T\Sigma \bar{A} \otimes_k A$ with differential $\sum_{i=0}^{n-1} (-1)^i b_i \colon A \otimes \bar{A}^{\otimes n-1} \otimes A \to A \otimes \bar{A}^{\otimes n-2} \otimes A$ given by

$$b_i(a_0, \ldots, a_n) = (a_0, \ldots, a_i a_{i+1}, \ldots, a_n).$$

It is a projective bimodule resolution of A. Let M be an A–bimodule. The *Hochschild cochain complex with coefficients in the bimodule M* is defined to be the complex $C^*(A, M) := \operatorname{Hom}_{A^e}(B_k(A), M)$ with differential

$$\delta(f) = -(-1)^n f \circ b$$

for $f : A \otimes \bar{A}^{\otimes n} \otimes A \to M$. The ith *Hochschild cohomology of the algebra A with coefficients in the bimodule M* is defined to be $\operatorname{HH}^i(A, M) := H^i(C^*(A, M), \delta)$.

Let A be an augmented dg k–algebra. Denote by $\bar{A}$ the kernel of the augmentation. Then the bar complex $B_k A$ is equipped with a second differential induced from the differential d_A on A. Given an A–bimodule M, the Hochschild cochain complex $C^*(A, M)$ is equipped with a second differential d induced by d_A and the internal differential d_M on M. The ith *Hochschild cohomology of the dg algebra A with coefficients in the bimodule M* is defined to be $\operatorname{HH}^i(A, M) := H^i(C^*(A, M), d + \delta)$.

It is well known that $\operatorname{HH}^i(A, M)$ is isomorphic to $\operatorname{Ext}^i_{A^e}(A, M)$. When A is a smooth commutative k–algebra, $\operatorname{HH}^*(A, A)$ is isomorphic to the polyvector fields on $\operatorname{Spec} A$ by the Hochschild–Kostant–Rosenberg theorem. For nonsmooth algebras, there exist different variants of Hochschild cohomology.

Let A be an associative k–algebra projective over k. Define the module of *Kähler differentials* Ω_A to be the kernel of the multiplication map $\mu : A \otimes A \to A$. Clearly, Ω_A inherits a bimodule structure from $A \otimes A$. It is easy to show that Ω_A is generated as a bimodule by the elements of the form $x \, dy := xy \otimes 1 - x \otimes y$. The left and right module structures are given by

$$a(x \, dy) = (ax) \, dy, \quad (x \, dy)a = x \, d(ya) - xy \, da.$$

Define the module of n–forms to be the n–fold tensor product

$$\Omega_A^n := \Omega_A \otimes_A \Omega_A \otimes \cdots \otimes_A \Omega_A.$$

Using the above identities, one can check that Ω_A^n is generated as a bimodule by the elements of the form $a_0 \, da_1 \, da_2 \cdots da_n$. There is an isomorphism of bimodules $\Omega_A^n \cong A \otimes_k \bar{A}^{\otimes n}$ defined by

$$a_0 \, da_1 \, da_2 \cdots da_n \mapsto a_0 \otimes a_1 \otimes \cdots \otimes a_n.$$

Set $\Omega_A^0 = A$ and $\Omega_A^1 = \Omega_A$. Write Ω_A^* for $\bigoplus_{n \geq 0} \Omega_A^n$. The bimodule structure on Ω_A naturally extends to an associative algebra structure on Ω_A^*. The obvious differential

$$D : a_0 \, da_1 \, da_2 \cdots da_n \mapsto da_0 \, da_1 \, da_2 \cdots da_n$$

makes Ω_A^* into a differential graded algebra.

For $m \in \mathbb{Z}$,

$$C^m(A, \Sigma^n \Omega_A^n) = \operatorname{Hom}_k(\bar{A}^{\otimes(n+m)}, A \otimes_k \bar{A}^{\otimes n}).$$

Consider the chain maps $\theta_n : C^*(A, \Sigma^n \Omega_A^n) \to C^*(A, \Sigma^{n+1} \Omega_A^{n+1})$ between the Hochschild cochain complexes defined by $f \mapsto f \otimes \operatorname{Id}_{\Sigma \bar{A}}$.

Definition 5.1 Let A be an associative k–algebra. Then the *singular Hochschild cochain complex* of A, denoted by $C_{\mathrm{sg}}^*(A, A)$, is defined as the colimit of the inductive system in the category of cochain complexes of k–modules

$$0 \to C^*(A, A) \xrightarrow{\theta_0} C^*(A, \Sigma\Omega_A^1) \xrightarrow{\theta_1} \cdots \to C^*(A, \Sigma^n\Omega_A^n) \xrightarrow{\theta_n} \cdots .$$

Namely, $C_{\mathrm{sg}}^*(A, A) := \operatorname{colim}_n C^*(A, \Sigma^n\Omega_A^n)$. Its cohomology groups are denoted by $\mathrm{HH}_{\mathrm{sg}}^*(A, A)$.

By construction, we have a natural chain morphism from $C^*(A, A)$ to $C_{\mathrm{sg}}^*(A, A)$, which induces a natural morphism from $\mathrm{HH}^*(A, A)$ to $\mathrm{HH}_{\mathrm{sg}}^*(A, A)$.

Let A be a noetherian k–algebra. Define $D_{\mathrm{sg}}(A)$ to be the Verdier quotient of $D^b(A)$ by the subcategory $\mathrm{per}(A)$. We denote the extension group in $D_{\mathrm{sg}}(A)$ by $\underline{\mathrm{Ext}}_A^i(?, ?)$. The singular Hochschild cohomology groups are related to the extension groups in $D_{\mathrm{sg}}(A^e)$.

Proposition 5.2 [Wang 2021, Theorem 3.6] *Let A be a noetherian k–algebra. Then there exists a natural isomorphism*

$$\mathrm{HH}_{\mathrm{sg}}^*(A, A) \xrightarrow{\sim} \underline{\mathrm{Ext}}_{A^e}^*(A, A)$$

such that the following diagram commutes:

$$
\begin{array}{ccc}
\mathrm{Ext}_{A^e}^*(A, A) & \longrightarrow & \underline{\mathrm{Ext}}_{A^e}^*(A, A) \\
\cong \uparrow & & \cong \uparrow \\
\mathrm{HH}^*(A, A) & \longrightarrow & \mathrm{HH}_{\mathrm{sg}}^*(A, A)
\end{array}
$$

From Wang's result, we see that the singular Hochschild cohomology admits a structure of a graded commutative algebra. Notice that this is not immediate from generalities about monoidal triangulated categories because the singularities category of A^e does not have any obvious monoidal structure.

5.2 Hochschild cohomology of Gorenstein algebras

A (not necessarily commutative) noetherian ring A is called *Gorenstein* if it has finite injective dimension both as a left and right A–module. As in the commutative case, we denote by CM_A the category of maximal Cohen–Macaulay (left) A–modules and denote by $\underline{\mathrm{CM}}_A$ its stable category. Buchweitz proved that if A is Gorenstein, then $\underline{\mathrm{CM}}_A$ is equipped with a structure of triangulated category and $\underline{\mathrm{CM}}_A \cong D_{\mathrm{sg}}(A)$.

We recall a fundamental result on extension groups in the stable category over Gorenstein rings due to Buchweitz.

Proposition 5.3 [Buchweitz 1986, Corollary 6.3.4] *Let A be a Gorenstein ring and let X and Y be objects in $D^b(A)$. There exists a positive integer m, depending on A, X and Y, such that the natural morphism $\mathrm{Ext}_A^i(X, Y) \to \underline{\mathrm{Ext}}_A^i(X, Y)$ is surjective for $i = m$ and is an isomorphism for $i > m$.*

Combining Propositions 5.2 and 5.3, we obtain the following result:

Corollary 5.4 *Let R be a commutative noetherian Gorenstein k–algebra. If $R \otimes R$ is noetherian, there exists a positive integer m such that, for $i > m$, the natural morphism*

$$\mathrm{HH}^i(R, R) \to \mathrm{HH}^i_{\mathrm{sg}}(R, R)$$

is an isomorphism.

Proof By Definition 5.1, we have a morphism $\mathrm{HH}^i(R, R) \to \mathrm{HH}^i_{\mathrm{sg}}(R, R)$ for all i. In order to apply Proposition 5.3 to show that it is an isomorphism, we need to check that $R \otimes R$ is noetherian. This follows from [Tousi and Yassemi 2003, Theorem 1.6]. □

A commutative local complete Gorenstein k–algebra $\widehat{R}$ is called a *hypersurface algebra* if

$$\widehat{R} \cong k[\![x_1, \dots, x_n]\!]/(g).$$

We say that $\widehat{R}$ is a hypersurface algebra with isolated singularities if g has an isolated critical point.

Theorem 5.5 [Guccione et al. 1992, Theorem 3.2.7] *Let $R = k[x_1, \dots, x_n]/(g)$ be a hypersurface algebra with isolated singularities. Denote by M_g the Milnor algebra $k[x_1, \dots, x_n]/(\partial g/\partial x_1, \dots, \partial g/\partial x_n)$, and by K_g and T_g the kernel and cokernel of the endomorphism of M_g defined by multiplication with g. Then, for $r \geq n$, there is an isomorphism of R–modules*

$$\mathrm{HH}^r(R, R) \cong \begin{cases} T_g & \text{if } r \text{ is even,} \\ K_g & \text{if } r \text{ is odd.} \end{cases}$$

Proof The proof in [Guccione et al. 1992] shows that, in degrees $r \geq n$, the Hochschild cohomology $\mathrm{HH}^r(R, R)$ is isomorphic to the homology in degree r of the complex

$$k[u] \otimes K\left(R, \frac{\partial g}{\partial x_1}, \dots, \frac{\partial g}{\partial x_n}\right),$$

where u is of degree 2 and K denotes the Koszul complex. Put $P = k[x_1, \dots, x_n]$. Since R is quasi-isomorphic to $K(P, g)$ and the $\partial g/\partial x_i$ form a regular sequence in P, the Koszul complex is quasi-isomorphic to $K(M_g, g)$. □

Note that T_g is the *Tyurina algebra* $k[x_1, \dots, x_n]/(g, \partial g/\partial x_1, \dots, \partial g/\partial x_n)$. Since K_g is the kernel of the multiplication map $g: M_g \to M_g$, it is a module over T_g.

Lemma 5.6 *Let A be a commutative k–algebra such that A and A^e are noetherian. Let $S \subset A$ be a multiplicative subset. If M is a finitely generated A–module and L an A–module, we have a canonical isomorphism*

$$\mathbb{R}\mathrm{Hom}_{A^e}(L, M)_S \xrightarrow{\sim} \mathbb{R}\mathrm{Hom}_{A^e_S}(L_S, M_S).$$

Proof Since L is finitely generated over A, it is finitely generated over A^e. Since A^e is noetherian, we have a projective resolution $P \to L$ with finitely generated components. This implies that we have isomorphisms

$$\mathbb{R}\mathrm{Hom}_{A^e}(L, M)_S = \mathrm{Hom}_{A^e}(P, M)_S = \mathrm{Hom}_{A^e}(P, M_S).$$

Since $A_S \otimes_A P \otimes_A A_S \to L_S$ is a projective resolution over A_S^e, we find

$$\mathrm{Hom}_{A^e}(P, M_S) = \mathrm{Hom}_{A_S^e}(A_S \otimes_A P \otimes_A A_S, M_S) = \mathbb{R}\mathrm{Hom}_{A_S^e}(L_S, M_S). \qquad \square$$

Remark 5.7 In the setting of Theorem 5.5, assume that g has isolated singularities and that the origin is a singular point of the vanishing locus of g. If we denote by $\mathfrak{m}$ the maximal ideal $(x_1, \ldots, x_n) \subset k[x_1, \ldots, x_n]$, then $g \in \mathfrak{m}$. Denote by $M_{g,\mathfrak{m}}$, $T_{g,\mathfrak{m}}$ and $K_{g,\mathfrak{m}}$ the localizations of M_g, T_g and K_g. It follows from the lemma that Theorem 5.5 still holds if one replaces R by $R_{\mathfrak{m}} := k[x_1, \ldots, x_n]_{\mathfrak{m}}/(g)$ and replaces T_g and K_g by $T_{g,\mathfrak{m}}$ and $K_{g,\mathfrak{m}}$.

For a noetherian k–algebra A, the derived category of singularities $D_{\mathrm{sg}}(A)$ is equipped with a canonical dg enhancement, obtained from its construction as a Verdier quotient of two canonically enhanced triangulated categories [Keller 1999; Drinfeld 2004]. Instead of $\mathrm{HH}^*_{\mathrm{sg}}(A, A)$, one may also consider the Hochschild cohomology of the dg category $D_{\mathrm{sg}}(A)$, which we will denote by $\mathrm{HH}^*(D_{\mathrm{sg}}(A))$.

Theorem 5.8 [Keller 2018] *There is a canonical morphism of graded algebras*

$$\mathrm{HH}^*_{\mathrm{sg}}(A, A) \to \mathrm{HH}^*(D_{\mathrm{sg}}(A)).$$

It is invertible if $D^b_{\mathrm{dg}}(\mathrm{mod}\, A)$ is smooth.

Now we establish the main result of the subsection.

Theorem 5.9 *Let $\hat{R} = k[\![x_1, \ldots, x_n]\!]/(g)$ be a hypersurface algebra with isolated singularity. Denote by $\hat{T}_g$ the Tyurina algebra of g. Then there is an isomorphism of k–algebras*

$$\mathrm{HH}^0(D_{\mathrm{sg}}(\hat{R})) \cong \hat{T}_g.$$

Moreover, if $\hat{R}' = k[\![x_1, \ldots, x_n]\!]/(g')$ is another hypersurface algebra with isolated singularity such that $D_{\mathrm{sg}}(\hat{R}')$ is quasiequivalent with $D_{\mathrm{sg}}(\hat{R})$ as dg categories, then $\hat{R}'$ is isomorphic to $\hat{R}$.

Proof Because g has an isolated critical point, we may assume that g is a polynomial without loss of generality. Denote by R the algebra $k[x_1, \ldots, x_n]_{\mathfrak{m}}/(g)$. Notice that R has an isolated singularity at the origin and that its completion identifies with $\hat{R}$. By [Dyckerhoff 2011, Theorem 5.7], the triangulated category $D_{\mathrm{sg}}(\hat{R})$ is the Karoubi envelope of $D_{\mathrm{sg}}(R)$. Therefore, the two dg categories have equivalent derived categories and there is a natural isomorphism $\mathrm{HH}^*(D_{\mathrm{sg}}(R)) \xrightarrow{\sim} \mathrm{HH}^*(D_{\mathrm{sg}}(\hat{R}))$. Orlov [2004] proved that $D_{\mathrm{sg}}(R)$ is triangle equivalent with the homotopy category of matrix factorizations $\mathrm{MF}(k[x_1, \ldots, x_n]_{\mathfrak{m}}, g)$.

The triangle equivalence is lifted to an equivalence of dg categories by the work of Blanc, Robalo, Toën and Vezzosi [Blanc et al. 2018]. Therefore, the dg category $D_{\text{sg}}(R)$ is 2–periodic and so is its Hochschild cohomology. So there exists a natural isomorphism of $\hat{R}$–modules

$$\text{HH}^0(D_{\text{sg}}(R)) \xrightarrow{\sim} \text{HH}^{2r}(D_{\text{sg}}(R))$$

for all $r \in \mathbb{Z}$. By [Elagin et al. 2020, Theorem B], the bounded dg derived category $D_{\text{dg}}^b(\text{mod } R)$ is smooth. Thus, by Theorem 5.8,

$$\text{HH}^{2r}(D_{\text{sg}}(R)) \xrightarrow{\sim} \text{HH}_{\text{sg}}^{2r}(R, R).$$

By Corollary 5.4, for $r \gg 0$,

$$\text{HH}^{2r}(R, R) \xrightarrow{\sim} \text{HH}_{\text{sg}}^{2r}(R, R),$$

and, by Theorem 5.5 and Remark 5.7,

$$\text{HH}^{2r}(R, R) \xrightarrow{\sim} T_g.$$

Because g has an isolated critical point, there is an isomorphism

$$\hat{T}_g \cong T_g.$$

Then the first claim follows. The second claim follows from the formal version of the Mather–Yau theorem (see [Greuel and Pham 2017, Theorem 1.1]). □

5.3 Classification of 3–dimensional smooth flops

Let $(\hat{Y}, \hat{f}, R)$ be a 3–dimensional formal flopping contraction with $\text{Ex}(f) = \bigcup_{i=1}^{t} C_i$, and let $A = \text{End}_R\left(\bigoplus_{i=1}^{t} N_i \oplus R\right)$ be the NCCR associated to it. We have associated to it an exact 3–CY algebra: the derived deformation algebra Γ of the semisimple collection $\mathbb{O}_{C_1}, \ldots, \mathbb{O}_{C_t}$, and the cluster category $\mathscr{C}_\Gamma$. There are two relaxations of the above context.

First we take Y a CY 3–fold with a semisimple collection of rational curves $\mathbb{O}_{C_1}, \ldots, \mathbb{O}_{C_t}$. We may still define the derived deformation algebra Γ and the cluster category $\mathscr{C}_\Gamma$. However, in general, Γ is only bimodule CY and $\mathscr{C}_\Gamma$ may not be Hom–finite. If we assume that Y is projective, then Γ will be exact. However, most $\hat{Y}$ are not expected to have CY compactifications. The second relaxation is to take a 3–dimensional hypersurface ring R with isolated singularities. Associated to it is the derived category of singularities $D_{\text{sg}}(R)$. This is a Hom–finite CY category. One may ask when is it possible to express it as a cluster category of a CY algebra.

In the case of flopping contractions, these two relaxations are related by Theorem 2.8. By [Van den Bergh 2004], $\hat{Y}$ admits a tilting bundle, and Γ is exact and has finite-dimensional cohomology. By Corollary 2.12 and Theorems 4.6 and 4.7, Γ is weakly equivalent to a Ginzburg algebra $\mathfrak{D}(Q, w)$ with t nodes. On the other hand, if R admits an NCCR A, then $D_{\text{sg}}(R)$ is equivalent to $\mathscr{C}_\Gamma$ for the CY algebra Γ. By Theorem 2.8, A admits a minimal model $(\hat{T}_{\hat{l}} V, d)$ with the dual space of $\Sigma^{-1}\text{Ext}_A^{\geq 1}\left(\bigoplus_{i=0}^{t} S_i, \bigoplus_{i=0}^{t} S_i\right)$

(see [de Thanhoffer de Völcsey and Van den Bergh 2016, Section 4]). Since A is Calabi–Yau, d can be derived from a potential, ie $(\widehat{T}_{\bar{l}}V, d)$ is a Ginzburg algebra (see [de Thanhoffer de Völcsey and Van den Bergh 2016, Proposition 1.2]). By the derived equivalence, Theorem 2.11, of Van den Bergh, the derived deformation algebra of the semisimple collection $\mathbb{O}_{C_1}, \ldots, \mathbb{O}_{C_t}$ is isomorphic to the quotient $\widehat{T}_{\bar{l}}V/\widehat{T}_{\bar{l}}Ve_0\widehat{T}_{\bar{l}}V$ in Theorem 2.8. It is natural to expect that the deformation theory of the exceptional curves and the singularity theory of R should determine each other since both are governed by the CY algebra Γ.

Recall that the CY tilted algebra $\Lambda := H^0\Gamma$ is isomorphic to $\mathrm{End}_{\mathscr{C}_\Gamma}(\pi(\Gamma))$ (see Theorem 3.19). Donovan and Wemyss conjectured that Λ alone can already determine the analytic type of R:

Conjecture 5.10 Let $(\widehat{Y}, \hat{f}, R)$ and $(\widehat{Y}', \hat{f}', R')$ be two 3–dimensional simple formal flopping contractions with associated CY tilted algebras Λ and Λ'. Then the following are equivalent:

(1) R is isomorphic to R'.

(2) Λ is isomorphic to Λ'.

Donovan and Wemyss have extended this conjecture to the case of not necessarily simple formal flopping contractions by replacing (2) with

(2′) Λ is derived equivalent to Λ';

see [August 2020a, Conjecture 1.3]. In this situation, the implication from (1) to (2) is known to be true by iterating a construction of Dugas [2015]. The implication from (2′) to (1) is one of the main open problems in the homological minimal model program for 3–folds. In this section, we will prove a slightly weaker version of this implication.

The exactness of Γ poses a strong constraint on Λ, ie the relations of Λ can be written as cyclic derivatives of a potential w by Theorem 4.6. If we fix the exact CY structure, then w is uniquely determined up to right equivalences (Proposition 4.8).

Theorem 5.11 Let $(\widehat{Y}, \hat{f}, R)$ and $(\widehat{Y}', \hat{f}', R')$ be formal flopping contractions. Given exact CY structures η and η' on $\widehat{Y}$ and $\widehat{Y}'$, respectively, denote by $(\Lambda, [w])$ and $(\Lambda', [w'])$ the associated CY tilted algebras and the canonical classes of potentials. If there exists a derived equivalence from Λ to Λ' given by a bimodule complex Z such that the induced map $\mathrm{HH}_0(Z)$ takes $[w]$ to $[w']$, then R is isomorphic to R'.

The proof of this theorem will take up the rest of the section. Here we highlight the major components of the proof. First we prove that the cluster category $\mathscr{C}_\Gamma$ is dg equivalent to the $D_{\mathrm{sg}}(R)$ with its canonical $\mathbb{Z}$–graded dg structure (Lemma 5.12). The second step is to establish that the analytic type of a isolated hypersurface singularity (with fixed embedded dimension) is determined by its $\mathbb{Z}$–graded dg category of singularities (Theorem 5.9). In the last step, we prove that Γ can be reconstructed from the CY tilted algebra Λ together with the class $[w] \in \mathrm{HH}_0(\Lambda)$ represented by the potential.
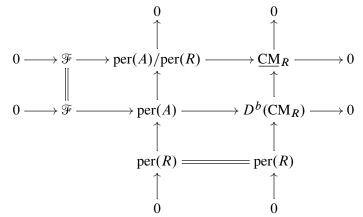
Let $(R, \mathfrak{m})$ be a complete commutative noetherian local Gorenstein k–algebra of Krull dimension n with isolated singularity and with residue field k. Suppose that R admits an NCCR; then $D_{\mathrm{sg}}(R)$ has another dg model via the triangle equivalence $D_{\mathrm{sg}}(R) \simeq \mathscr{C}_\Gamma$ (see Theorem 2.8). We first prove that these two models are dg quasiequivalent.

Lemma 5.12 *In the homotopy category of dg categories, there is an isomorphism between* $\mathscr{C}_\Gamma = \mathrm{per}(\Gamma)/D_{\mathrm{fd}}(\Gamma)$ *and the category of singularities* $D_{\mathrm{sg}}(R) = D^b(R)/K^b(\mathrm{proj}(R))$, *both equipped with their canonical dg enhancements.*

Proof Let $\mathscr{A}$ and $\mathscr{B}$ be two pretriangulated dg categories. We call a triangle functor $F \colon H^0(\mathscr{A}) \to H^0(\mathscr{B})$ algebraic if there is a dg $\mathscr{A}$–$\mathscr{B}$–bimodule X such that we have a square of triangle functors, commutative up to isomorphism,

$$
\begin{array}{ccc}
H^0(\mathscr{A}) & \xrightarrow{\ F\ } & H^0(\mathscr{B}) \\
\downarrow & & \downarrow \\
D(\mathscr{A}) & \xrightarrow{\ ?\otimes_{\mathscr{A}}^{\mathbb{L}} X\ } & D(\mathscr{B})
\end{array}
$$

where the vertical arrows are induced by the Yoneda functors. We know from [Toën 2007] that morphisms $\mathscr{A} \to \mathscr{B}$ in the homotopy category of dg categories are in bijection with isomorphism classes of right quasirepresentable $\mathscr{A}$–$\mathscr{B}$–bimodules in the derived category of bimodules. Thus, it suffices to show that the triangle equivalence $\mathscr{C}_\Gamma \xrightarrow{\sim} D_{\mathrm{sg}}(R)$ is algebraic. We use the notation of Section 2.4 and put $N = N_1 \oplus \cdots \oplus N_t$. Let $\mathscr{F}$ denote the thick triangulated subcategory of $\mathrm{per}(A)$ generated by the simples $S_1, \ldots, S_t$. Let us recall from [Palu 2009, Proposition 3] that we have a diagram of triangle functors, commutative up to isomorphism and whose rows and columns are exact sequences of triangulated categories,

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & \mathscr{F} & \longrightarrow & \mathrm{per}(A)/\mathrm{per}(R) & \longrightarrow & \underline{\mathrm{CM}}_R & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathscr{F} & \longrightarrow & \mathrm{per}(A) & \longrightarrow & D^b(\mathrm{CM}_R) & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \mathrm{per}(R) & = & \mathrm{per}(R) & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Here the category $D^b(\mathrm{CM}_R)$ is the bounded derived category of the exact category CM_R, the functor $\mathrm{per}(R) \to D^b(\mathrm{CM}_R)$ is induced by the inclusion $\mathrm{proj}(R) \to \mathrm{CM}_R$, the functor $\mathrm{per}(A) \to D^b(\mathrm{CM}_R)$ is induced by $? \otimes_A^{\mathbb{L}} (R \oplus N)$, and $\mathrm{per}(R) \to \mathrm{per}(A)$ is induced by the inclusion

$$
\mathrm{add}(R) \to \mathrm{add}(R \oplus N) = \mathrm{proj}(A),
$$

where the categories $\mathrm{add}(R)$ and $\mathrm{add}(R \oplus N)$ are full subcategories of CM_R and the last equality denotes the equivalence given by the functor $\mathrm{Hom}(R \oplus N, ?)$. We endow $\mathrm{per}(A)/\mathrm{per}(R)$ with the dg enhancement given by the dg quotient [Keller 1999; Drinfeld 2004]. It is then clear that the triangle functors of the middle row and of the middle column are algebraic. Let us show that the functor $D^b(\mathrm{CM}_R) \to \underline{\mathrm{CM}}_R$ is algebraic. The canonical dg enhancement of $\underline{\mathrm{CM}}_R$ is given by the triangle equivalence from the homotopy category $H^0(\mathscr{A})$ of the dg category $\mathscr{A}$ of acyclic complexes over $\mathrm{proj}(R)$ to $\underline{\mathrm{CM}}_R$ taking an acyclic complex P to its zero cycles $Z^0(P)$. Let $\mathscr{B}$ be the dg enhanced derived category $D^b_{\mathrm{dg}}(\mathrm{CM}_R)$. We define a $\mathscr{B}$–$\mathscr{A}$–bimodule X by putting

$$X(P, M) = \mathrm{Hom}(P, \Sigma M),$$

where P is an acyclic complex of finitely generated projective R–modules and M a bounded complex over CM_R. For $M \in \mathscr{B}$, denote by $M^\wedge$ the representable dg functor $\mathscr{B}(?, M)$. We have $M^\wedge \otimes_\mathscr{B} X = X(?, M)$. If M is concentrated in degree 0, we have canonical isomorphisms

$$H^0 X(P, M) = \mathrm{Hom}_{\underline{\mathrm{CM}}_R}(Z^0(P), M) \quad \text{and} \quad H^p X(P, M) = \mathrm{Hom}_{\underline{\mathrm{CM}}_R}(Z^0(P), \Sigma^p M),$$

which shows that $X(?, M)$ is quasirepresentable by a complete resolution of M. By dévissage, $X(?, M)$ is quasirepresentable for any bounded complex M and one checks easily that the (derived = nonderived) tensor product with X induces the canonical triangle functor $D^b(\mathrm{CM}_R) \to \underline{\mathrm{CM}}_R$. It follows that, at the level of dg categories, $\underline{\mathrm{CM}}_R$ identifies with the dg quotient of $D^b(\mathrm{CM}_R)$ by $\mathrm{per}(R)$. In other words, the canonical equivalence

$$D^b(\mathrm{CM}_R)/\mathrm{per}(R) \xrightarrow{\sim} \underline{\mathrm{CM}}_R$$

is algebraic. Therefore, the induced functor $\mathrm{per}(A)/\mathrm{per}(R) \to \underline{\mathrm{CM}}_R$ is algebraic. Thus, the whole diagram is made up of algebraic functors. Now notice that the inclusion $\mathrm{CM}_R \subset \mathrm{mod}\,R$ induces algebraic equivalences $D^b(\mathrm{CM}_R) \xrightarrow{\sim} D^b(\mathrm{mod}\,R)$, so we get algebraic equivalences

$$\underline{\mathrm{CM}}_R \xleftarrow{\sim} D^b(\mathrm{CM}_R)/\mathrm{per}(R) \xrightarrow{\sim} D^b(\mathrm{mod}\,R)/\mathrm{per}(R).$$

Thus, it will suffice to prove that the equivalence between the cluster category of Γ and the stable category $\underline{\mathrm{CM}}_R$ is algebraic.

Now, using the notation of Theorem 2.8, put $\widetilde{\Gamma} = (\hat{T}_l V, d)$, so we have a quasi-isomorphism $\widetilde{\Gamma} \xrightarrow{\sim} A$. It induces an algebraic equivalence $\mathrm{per}(\widetilde{\Gamma}) \xrightarrow{\sim} \mathrm{per}(A)$. This equivalence induces an algebraic equivalence $\mathrm{tria}(e_0 \widetilde{\Gamma}) \to \mathrm{per}(R)$, where $\mathrm{tria}(e_0 \widetilde{\Gamma})$ is the triangulated subcategory generated by the $\widetilde{\Gamma}$–module $e_0 \widetilde{\Gamma}$. The quotient map $\widetilde{\Gamma} \to \Gamma$ induces an algebraic triangle functor $\mathrm{per}(\widetilde{\Gamma}) \to \mathrm{per}(\Gamma)$ and we know from [Kalck and Yang 2018, Lemma 7.2] that it is a localization with kernel $\mathrm{tria}(e_0 \widetilde{\Gamma})$. We obtain a diagram of triangle functors, commutative up to isomorphism, whose vertical arrows are equivalences and whose rows are exact:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{tria}(e_0 \widetilde{\Gamma}) & \longrightarrow & \mathrm{per}(\widetilde{\Gamma}) & \longrightarrow & \mathrm{per}(\Gamma) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{per}(R) & \longrightarrow & \mathrm{per}(A) & \longrightarrow & \mathrm{per}(A)/\mathrm{per}(R) & \longrightarrow & 0
\end{array}
$$

By passage to the dg quotient, the rightmost vertical arrow is an algebraic triangle equivalence. By composing the algebraic inclusion $\mathscr{F} \to \mathrm{per}(A)/\mathrm{per}(R)$ with a quasi-inverse of the algebraic equivalence $\mathrm{per}(\Gamma) \xrightarrow{\sim} \mathrm{per}(A)/\mathrm{per}(R)$, we obtain an algebraic inclusion $\mathscr{F} \to \mathrm{per}(\Gamma)$ whose image identifies with the thick subcategory of $\mathrm{per}(\Gamma)$ generated by the simple $H^0(\Gamma)$–modules. This subcategory equals $D_{\mathrm{fd}}(\Gamma)$. Indeed, clearly it is contained in $D_{\mathrm{fd}}(\Gamma)$ and, conversely, it contains the category $\mathrm{mod}\, H^0(\Gamma)$ of finite-dimensional $H^0(\Gamma)$–modules since $H^0(\Gamma)$ is finite-dimensional, so every finite-dimensional $H^0(\Gamma)$–module is a finite iterated extension of simples. Since $\mathrm{mod}\, H^0(\Gamma)$ is the heart of a bounded t–structure on $D_{\mathrm{fd}}(\Gamma)$, the image of $\mathscr{F}$ equals $D_{\mathrm{fd}}(\Gamma)$. This yields the first row of the following diagram with exact rows, whose vertical arrows are equivalences:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{F} & \longrightarrow & \mathrm{per}(\Gamma) & \longrightarrow & \mathscr{C}_\Gamma & \longrightarrow & 0 \\
& & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{F} & \longrightarrow & \mathrm{per}(A)/\mathrm{per}(R) & \longrightarrow & \underline{\mathrm{CM}}_R & \longrightarrow & 0
\end{array}
$$

Again by passage to the dg quotient, the rightmost vertical arrow is an algebraic triangle equivalence. $\square$

Proof of Theorem 5.11 Let (Q, w) and (Q', w') be the quivers with potential constructed from the formal flopping contractions and Γ and Γ' the associated Ginzburg dg algebras. Let $\Lambda = H^0(\Gamma)$ and $\Lambda' = H^0(\Gamma')$ be the associated contraction algebras. Recall that these algebras are symmetric, so tilting objects coincide with silting objects in their derived categories. We will construct a quiver with potential (Q'', w'') with associated Ginzburg algebra Γ'', a dg Γ''–Γ'–bimodule $\widetilde{Z}$ and an isomorphism ψ from $\Lambda'' = H^0(\Gamma'')$ to C such that the square

$$
\begin{array}{ccc}
\Gamma'' & \xrightarrow{\ \widetilde{Z}\ } & \Gamma' \\
{\scriptstyle \Lambda''} \downarrow & & \downarrow {\scriptstyle \Lambda'} \\
\Lambda'' & \xrightarrow[\ \psi Z\ ]{} & \Lambda'
\end{array}
$$

is commutative and the isomorphism $\mathrm{HC}_0(\widetilde{Z})$ takes the class $[w'']$ to $[w']$. Here we write dg bimodules instead of derived tensor products, algebras instead of their derived categories, and the top and bottom arrows are equivalences. Notice that, by [Keller 1998], Hochschild homology is functorial with respect to right perfect dg bimodules, so the notation $\mathrm{HC}_0(\widetilde{Z})$ does make sense.

To construct the above square, we may assume that Z is a 2–term silting object since, by [August 2020b, Theorem 7.2(3)], the standard derived equivalence given by Z is the composition of equivalences given by 2–term tilting complexes and their inverses. Let A be the dg algebra obtained by applying Theorem 4.25 to $B = \Gamma'$, $H^0(B) = \Lambda'$, $C = \Lambda$ and $Z = Z$. Its homologies $H^p(A)$ are finite-dimensional and vanish for $p > 0$. Thus, it is quasi-isomorphic to a dg algebra of the form $(\widehat{T}_l V, d)$, where V is a graded bimodule whose components vanish in degrees > 0 and are finite-dimensional in all degrees ≤ 0. So we may assume that A is in $\mathrm{PCAlgc}(l)$. By Theorem 4.25, there is moreover a dg A–Γ'–bimodule X yielding

an equivalence $? \otimes_A^{\mathbb{L}} X : D(A) \xrightarrow{\sim} D(\Gamma')$ and an isomorphism $\phi : H^0(A) \to \Lambda$ such that we have an isomorphism

$$_\phi Z \xrightarrow{\sim} X \otimes_{\Gamma'}^{\mathbb{L}} \Lambda'$$

in the derived category $D(A^{\mathrm{op}} \otimes \Lambda')$ and in particular a square, commutative up to isomorphism,

$$
\begin{array}{ccc}
A & \xrightarrow{\quad X \quad} & \Gamma' \\
{\scriptstyle H^0 A}\downarrow & & \downarrow{\scriptstyle \Lambda'} \\
H^0(A) \xrightarrow[\phi]{\sim} \Lambda & \xrightarrow{\quad Z \quad} & \Lambda'
\end{array}
$$

By Corollary 4.27, there is a quiver with potential (Q'', w'') and a weak equivalence $s : \Gamma'' \to A$ from the associated Ginzburg algebra Γ'' to A such that the isomorphism $\mathrm{HC}_0(_s X)$ takes the class $[w'']$ to $[w']$. We define $\widetilde{Z} = {_s X}$ and $\psi = \phi \circ H^0(s)$ to obtain the diagram

$$
\begin{array}{ccccc}
\Gamma'' & \xrightarrow{\quad s \quad} & A & \xrightarrow{\quad X \quad} & \Gamma' \\
\downarrow & & {\scriptstyle H^0 A}\downarrow & & \downarrow{\scriptstyle \Lambda'} \\
H^0(\Gamma'') \xrightarrow[H^0 s]{\sim} & H^0(A) & \xrightarrow[\phi]{\sim} \Lambda & \xrightarrow{\quad Z \quad} & \Lambda'
\end{array}
$$

So, by construction, the isomorphism $\mathrm{HH}_0(_s X) = \mathrm{HH}_0(_\psi Z)$ takes $[w'']$ to $[w']$. We may assume all potentials contain no cycles of length ≤ 2 and then it follows that $\psi = \phi \circ H^0(s)$ induces an isomorphism of quivers $Q'' \xrightarrow{\sim} Q$. Indeed, it induces an isomorphism in the Jacobian algebras and the vertices i of the quiver Q are in bijection with the isomorphism classes of simple modules S_i of the pseudocompact algebra $\Lambda = H^0(\Gamma)$ and the number of arrows from i to j equals the dimension of the space of extensions $\mathrm{Ext}^1_\Lambda(S_j, S_i)$. By Corollary 4.14, there is an isomorphism $\beta : \Gamma'' \to \Gamma$. The dg bimodule $_{s\beta^{-1}} X$ yields an algebraic triangle equivalence $\mathrm{per}(\Gamma) \xrightarrow{\sim} \mathrm{per}(\Gamma')$. Such an equivalence induces an equivalence between the subcategories of dg modules with finite-dimensional homologies because their objects M are characterized as those for which $\mathrm{Hom}(P, M)$ is finite-dimensional for any object P. Thus, the algebraic triangle equivalence $\mathrm{per}(\Gamma) \xrightarrow{\sim} \mathrm{per}(\Gamma')$ induces an algebraic triangle equivalence in the cluster categories. Hence, by Lemma 5.12, it induces an algebraic triangle equivalence $D_{\mathrm{sg}}(R) \xrightarrow{\sim} D_{\mathrm{sg}}(R')$ and therefore an algebra isomorphism $\mathrm{HH}^0(D_{\mathrm{sg}}(R)) \cong \mathrm{HH}^0(D_{\mathrm{sg}}(R'))$. By Theorem 5.9, we get an isomorphism $R \cong R'$. □

6 Contractibility of rational curve

6.1 Dg $k[u^{-1}]$–algebras

In this section, we define dg $k[u^{-1}]$–algebras and study their properties. All the definitions and results can be adapted to the pseudocompact case, with objects appropriately replaced by their pseudocompact counterparts.

Definition 6.1 Let S be a commutative dg algebra. Denote by $\mathscr{C}(S)$ the category of complexes of S–modules with the monoidal structure given by the tensor product over S. A dg S–algebra is an algebra in $\mathscr{C}(S)$, ie a dg S–module with an S–bilinear multiplication and a unit.

Let k be a field, S a commutative dg k–algebra and Γ a dg S–algebra. By restriction, each dg Γ–module becomes a dg S–module and the morphism complexes between dg Γ–modules are naturally dg S–modules. Thus, the derived category $D(\Gamma)$ is naturally enriched over $D(S)$.

Definition 6.2 Let $S = k[u^{-1}]$ be the commutative dg algebra with $\deg(u) = 2, \deg(u^{-1}) = -2$ and zero differential. We call a dg k–algebra A $k[u^{-1}]$–*enhanced* if A is isomorphic to a dg $k[u^{-1}]$–algebra in the homotopy category of dg k–algebras.

Let us put $S = k[u^{-1}]$ and $K = k[u, u^{-1}]$. For a dg S–module M, we have a canonical isomorphism

$$H^*(M \otimes_S K) = H^*(M) \otimes_S K.$$

We call M a *torsion module* if $M \otimes_S K$ is acyclic. This happens if and only if $H^*(M)$ is a torsion module, ie for each m in $H^*(M)$, there exists a $p \gg 0$ such that $mu^{-p} = 0$.

Let A be a dg S–algebra. The functor taking a dg A–module M to the dg $A \otimes_S K$–module $M \otimes_S K$ preserves quasi-isomorphisms. Thus, it induces a functor $? \otimes_S K \colon D(A) \to D(A \otimes_S K)$. The kernel of this functor consists of the dg A–modules which are torsion as dg S–modules. We write $D(A)_{u^{-1}\text{–tor}}$ for the kernel and $\operatorname{per}(A)_{u^{-1}\text{–tor}}$ for its intersection with the perfect derived category $\operatorname{per}(A)$.

Lemma 6.3 *We have exact sequences of triangulated categories*

$$0 \to D(A)_{u^{-1}\text{–tor}} \to D(A) \to D(A \otimes_S K) \to 0$$

and

$$0 \to \operatorname{per}(A)_{u^{-1}\text{–tor}} \to \operatorname{per}(A) \to \operatorname{per}(A \otimes_S K) \to 0.$$

Proof The restriction along $A \to A \otimes_S K$ induces a fully faithful right adjoint to $? \otimes_S K \colon D(A) \to D(A \otimes_S K)$. Thus, the latter functor is a localization functor. By definition, its kernel is $D(A)_{u^{-1}\text{–tor}}$, so we obtain the first sequence. To deduce the second one, it suffices to show that the kernel of $D(A) \to D(A \otimes_S K)$ is compactly generated. Indeed, let P be the cone over the morphism

$$A \to \Sigma^{-2} A$$

given by multiplication by u^{-1}. Clearly P is compact and we claim that it generates the kernel. For this, it suffices to show that the right orthogonal of P in the kernel vanishes. Indeed, let M be in the kernel. If $\mathbb{R}\operatorname{Hom}_A(P, M)$ vanishes, then the morphism $\Sigma^2 M \to M$ given by multiplication by u^{-1} is a quasi-isomorphism. Thus, u^{-1} acts in $H^*(M)$ by an isomorphism. But, on the other hand, $H^*(M)$ is torsion. So $H^*(M)$ vanishes and M is acyclic, as was to be shown. □

A dg S–algebra A concentrated in nonpositive degrees is *nondegenerate* if the morphism $A \to \Sigma^{-2} A$ given by multiplication by u^{-1} induces isomorphisms $H^n(A) \xrightarrow{\sim} H^{n-2}(A)$ for all $n \leq 0$.

Lemma 6.4 *Let A be a dg k–algebra concentrated in nonpositive degrees. Assume that A is homologically smooth, that $H^n(A)$ is finite-dimensional for each $n \in \mathbb{Z}$ and that A admits a nondegenerate $k[u^{-1}]$–enhancement. Then the subcategory $D_{\mathrm{fd}}(A) \subset \mathrm{per}(A)$ coincides with $\mathrm{per}(A)_{u^{-1}\text{–tor}}$.*

Proof Since A is homologically smooth, $D_{\mathrm{fd}}(A)$ is contained in $\mathrm{per}(A)$ and clearly it consists of torsion modules. Conversely, we know from the proof of Lemma 6.3 that $\mathrm{per}(A)_{u^{-1}\text{–tor}}$ is the thick subcategory of $D(A)$ generated by the cone P over the morphism $A \to \Sigma^{-2}A$ given by multiplication by u^{-1}. Since A is nondegenerate, the object P lies in $D_{\mathrm{fd}}(A)$. $\qquad\square$

Proposition 6.5 *Let Q be a finite quiver with potential w such that the associated Ginzburg algebra $\Gamma = \mathfrak{D}(Q, w)$ has finite-dimensional Jacobi algebra. Assume that Γ is equipped with a nondegenerate $k[u^{-1}]$–enhancement. Then $\mathscr{C}_\Gamma$ is a $\mathbb{Z}/2$–graded 0–CY triangulated category, equivalent with the category of perfect modules over $\Gamma \otimes_{k[u^{-1}]} k[u, u^{-1}]$ as $k[u, u^{-1}]$–enhanced triangulated categories. In particular, the Jacobi algebra $H^0(\Gamma)$ is a symmetric Frobenius algebra.*

Proof We may assume that Γ itself is a differential graded $k[u^{-1}]$–algebra. Multiplication by u^{-1} yields a functorial morphism $\mathrm{Id} \to \Sigma^{-2}$ of triangle functors $D(\Gamma) \to D(\Gamma)$ and $\mathrm{per}(\Gamma) \to \mathrm{per}(\Gamma)$. This induces a functorial morphism of triangle functors $\mathrm{Id} \to \Sigma^{-2}$ in the cluster category $\mathscr{C}_\Gamma$. To check that it is invertible, it is enough to check that its action on the cluster-tilting object $\Gamma \in \mathscr{C}_\Gamma$ is invertible (since $\mathscr{C}_\Gamma$ equals its thick subcategory generated by Γ). Now, by our assumption, the morphism $u^{-1} \colon \Sigma^2 \Gamma \to \Gamma$ induces isomorphisms in H^n for $n \leq -2$ and the 0–map for $n \geq -1$. Thus, it induces a quasi-isomorphism $\Sigma^2 \Gamma \to \tau_{\leq -2} \Gamma$. We claim that the canonical morphism $\tau_{\leq -2} \Gamma \to \Gamma$ becomes invertible in the cluster category. Indeed, the homology of its cone is of finite total dimension since $H^p \Gamma$ is of finite dimension for all integers p by [Amiot 2009, Lemma 2.5]. It follows that u induces an isomorphism $\Sigma^2 \Gamma \xrightarrow{\sim} \Gamma$ in $\mathscr{C}_\Gamma$. The rest follows because we have isomorphisms of $H^0(\Gamma)$–bimodules

$$H^0(\Gamma) \xrightarrow{\sim} \mathscr{C}_\Gamma(\Gamma, \Gamma) \xrightarrow{\sim} \mathscr{C}_\Gamma(\Gamma, \Sigma^2 \Gamma) \xrightarrow{\sim} D\mathscr{C}_\Gamma(\Gamma, \Gamma). \qquad\square$$

6.2 $k[u^{-1}]$–structures on Ginzburg algebras associated to contractible curves

Recall that, for a Jacobi-finite Ginzburg algebra $\Gamma := \mathfrak{D}(Q, w)$ associated to the quiver Q with only one vertex (and multiple loops), $H^0(\Gamma)$ is self-injective. On the other hand, if Γ is nondegenerately $k[u^{-1}]$–enhanced, then $H^0(\Gamma)$ is symmetric.

The simplest case is when $\Gamma = k[t]$ with zero differential. It is $k[u^{-1}]$–enhanced by setting $u^{-1} = t$. This is the derived deformation algebra for $\mathbb{O}_C$ of a $(-1, -1)$–curve, which is always contractible.

Proposition 6.6 *Let $F = k\langle\!\langle x \rangle\!\rangle$ be the complete free algebra of one generator and $w \in F$ be an element with no constant term. Then $\Gamma := \mathfrak{D}(F, w)$ is $k[u^{-1}]$–enhanced.*

Proof A general element $w \in F$ is of the form

$$w = x^{n+1} + \text{higher-order terms}.$$

When $n = 1$, we are in the case of a $(-1, -1)$–curve. We assume that $n \geq 2$. The Jacobi algebra of $\mathfrak{D}(F, w)$ is isomorphic to $k[\![x]\!]/(x^n)$. It is always finite-dimensional. Because $w = x^{n+1} \cdot u$ for some unit $u \in k[\![x]\!]$, $[w] = 0$ in $k[\![x]\!]/(x^n)$. By Theorem 4.16, w is right equivalent to x^{n+1}. Without loss of generality, we may assume $w = x^{n+1}/(n + 1)$ to begin with. Then the Ginzburg algebra $\mathfrak{D}(F, w)$ is isomorphic to $k\langle\!\langle x, \theta, t \rangle\!\rangle$ with $dt = [x, \theta]$ and $d\theta = x^n$. It is easy to check that the two-sided differential ideal $(t, [x, \theta])$ is acyclic. As a consequence, the quotient morphism

$$\Gamma = (k\langle\!\langle x, \theta, t \rangle\!\rangle, d) \to \Gamma' := \big(k\langle\!\langle x, \theta, t \rangle\!\rangle/(t, [x, \theta]), d\big)$$

is a quasi-isomorphism of dg algebras. Note that Γ' is isomorphic to the complex

$$\cdots \to k[\![x]\!]\theta^3 \xrightarrow{d} k[\![x]\!]\theta^2 \xrightarrow{d} k[\![x]\!]\theta \xrightarrow{d} k[\![x]\!] \to 0,$$

where

$$d(\theta^{2k}) = 0, \quad d(\theta^{2k+1}) = x^n \theta^{2k}.$$

Define the action of u^{-1} on Γ' by multiplication by θ^2. It is easy to check it makes Γ' a dg $k[u^{-1}]$–algebra. $\qquad \square$

From the above proposition, we see that the Ginzburg algebras associated to the "one-loop quiver" are essentially classified by the dimension of their Jacobi algebras. Moreover, they all admit $k[u^{-1}]$–enhancements. If $\dim_k(H^0\Gamma) = n$ for $n > 1$, then Γ is equivalent to the derived deformation algebra of a floppable $(0, -2)$–curve of width n (see [Reid 1983] for the geometric definition of width). The following corollary can be viewed as a noncommutative counterpart of the classification theorem of Reid [1983]:

Corollary 6.7 *Let C be a rational curve in a quasiprojective smooth CY 3–fold Y with normal bundle $\mathbb{O}_C \oplus \mathbb{O}_C(-2)$. Denote its derived deformation algebra by Γ. Then:*

(1) *C is movable if and only if Γ has infinite-dimensional Jacobi algebra.*

(2) *If C is rigid, then it is contractible. The dimension of $H^0\Gamma$ is equal to n for $n > 1$ if and only if the underlying singularity is isomorphic to the germ of the hypersurface $x^2 + y^2 + u^2 + v^{2n} = 0$ at the origin.*

It is proved by Laufer (see [Pinkham 1983]) that a contractible rational curve in a CY 3–fold must have normal bundle of type $(-1, -1)$, $(0, -2)$ or $(1, -3)$. Donovan and Wemyss [2019, Example 6.4] give an example of a rigid rational curve of type $(1, -3)$ that is not nc rigid. In their example, there exists a birational morphism that contracts a divisor containing the $(1, -3)$–curve. Kawamata [2020, Question 6.6] asked whether it is true that C is contractible if it is nc rigid. We formulate a conjecture in terms of the derived deformation algebra:

Conjecture 6.8 *Let $C \subset Y$ be an nc rigid rational curve in a smooth quasiprojective CY 3–fold. Denote its associated derived deformation algebra by Γ_C^Y. Then C is contractible if and only if Γ_C^Y is $k[u^{-1}]$–enhanced.*

Note that one direction of the conjecture follows from our Theorem 4.17:

Proposition 6.9 *Let $C \subset Y$ be a contractible rational curve in a smooth quasiprojective CY 3–fold. Then Γ_C^Y is $k[u^{-1}]$–enhanced.*

Proof Denote by R the ring of formal functions on the singularity underlying the contraction. For simplicity, we denote the derived deformation algebra Γ_C^Y by Γ. By Lemma 5.12, $\mathscr{C}_\Gamma$ is quasiequivalent to $D_{sg}(R)$ as dg categories. Under the equivalence, the projection image of Γ is identified with the Cohen–Macaulay module $N \in D_{sg}(R) \cong \underline{CM}_R$. By Theorem 4.17, Γ is isomorphic to $\tau_{\leq 0}\Lambda_{dg}$, where Λ_{dg} is the dg endomorphism algebra of N in $\underline{CM}_R$. Because R is a hypersurface ring, the dg category $\underline{CM}_R$ carries a canonical $\mathbb{Z}/2$–graded structure (equivalently a $k[u, u^{-1}]$–structure) by Eisenbud's theorem [1980]. Therefore, Γ is $k[u^{-1}]$–enhanced. $\square$

We have already seen that, in the $(-1, -1)$ and $(0, -2)$ cases, the $k[u^{-1}]$–structure on Γ can be computed explicitly. However, we don't have any explicit construction for the $(1, -3)$ case even though we know it must exist. We do have an explicit formula for the symmetric Frobenius structure on the CY tilted algebra $H^0\Gamma$ in terms of the residue map of matrix factorizations (see [Hua and Toda 2018]).

Appendix   Serre duality for sheaves and modules

In this appendix, we give two proofs of the link between the inverse dualizing sheaf on a smooth quasiprojective variety Y and the inverse dualizing bimodule for the derived endomorphism algebra of any perfect generator of $D(\mathrm{Qcoh}(Y))$.

The proof in Section A.1 is based on [Gaitsgory 2013; Gaitsgory and Rozenblyum 2017]. The proof in Section A.2 is essentially taken from [Kinjo and Masuda 2023, Example 2.7]. It combines Grothendieck duality with the results of Ben-Zvi, Francis and Nadler [Ben-Zvi et al. 2010]. Both proofs rely on the foundational work of Toën, Joyal, Lurie and many others.

A.1   Dg Serre duality, after Gaitsgory

We recall a result of Gaitsgory and Gaitsgory–Rozenblyum which could be viewed as Serre duality for the dg category of coherent sheaves. They use the category of ind-coherent sheaves, which behaves better on singular spaces. Though we only need the smooth case, we recall the basics of ind-coherent sheaves and their properties. We will follow [Gaitsgory and Rozenblyum 2017; Gaitsgory 2013].

We consider (quasi)coherent sheaves on quasicompact separated noetherian schemes. We write Qcoh for the dg category of (fibrant replacements of) unbounded complexes of quasicoherent sheaves and coh for its full dg subcategory of complexes with coherent cohomology and bounded cohomological amplitude. Let

IndCoh be the ind-completion of coh. Thus, the dg category IndCoh is quasiequivalent to the dg enhanced derived category of the (essentially) small dg category coh. There is a natural functor $\Psi\colon \mathrm{IndCoh} \to \mathrm{Qcoh}$ which commutes with coproducts and, restricted to coh, becomes the inclusion $\mathrm{coh} \to \mathrm{Qcoh}$. If X is smooth then $\Psi_X\colon \mathrm{IndCoh}(X) \to \mathrm{Qcoh}(X)$ is an equivalence [Gaitsgory and Rozenblyum 2017, Chapter 4, Lemma 1.1.3]. Let $f\colon X \to Y$ be a proper morphism. There exists a continuous pushforward functor $f_*^{\mathrm{IndCoh}}\colon \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(Y)$ with a commutative diagram

$$
\begin{array}{ccc}
\mathrm{IndCoh}(X) & \xrightarrow{\;f_*^{\mathrm{IndCoh}}\;} & \mathrm{IndCoh}(Y) \\
\Psi_X \downarrow & & \downarrow \Psi_Y \\
\mathrm{Qcoh}(X) & \xrightarrow{\;f_*\;} & \mathrm{Qcoh}(Y)
\end{array}
$$

See [Gaitsgory and Rozenblyum 2017, Chapter 4, Proposition 2.1.2]. Moreover, since f is proper, f_*^{IndCoh} sends $\mathrm{coh}(X)$ to $\mathrm{coh}(Y)$. The above commutative diagram is compatible with tensor products by quasicoherent complexes [Gaitsgory and Rozenblyum 2017, Chapter 4, Proposition 2.1.4]. The pushforward functor f_*^{IndCoh} admits a *continuous* right adjoint functor $f^!\colon \mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(X)$ [Gaitsgory and Rozenblyum 2017, Chapter 4, 5.1.5] and it is compatible with the tensor product with quasicoherent complexes [Gaitsgory and Rozenblyum 2017, Chapter 4, 5.1.7]. To distinguish $f^!$ from the right adjoint of f_* in Qcoh, we denote the latter by $f^{\mathrm{Qcoh},!}$. We have the following comparison theorem between these two functors [Gaitsgory and Rozenblyum 2017, Chapter 4, Lemma 5.1.9]: there is a commutative diagram

(A-1)
$$
\begin{array}{ccc}
\mathrm{IndCoh}(X)^+ & \xrightarrow{\;\Psi_X\;} & \mathrm{Qcoh}(X)^+ \\
f^! \uparrow & & \uparrow f^{\mathrm{Qcoh},!} \\
\mathrm{IndCoh}(Y)^+ & \xrightarrow{\;\Psi_Y\;} & \mathrm{Qcoh}(Y)^+
\end{array}
$$

where the superscript $+$ refers to the subcategory consisting of objects whose cohomological amplitude is bounded below. Note that the similar diagram with $+$ removed is not commutative (see [Gaitsgory and Rozenblyum 2017, Chapter 4, Remark 5.1.10]). This result shows that we can compute $f^!\mathscr{E}$ via $f^{\mathrm{Qcoh},!}\mathscr{E}$ for $\mathscr{E} \in \mathrm{coh}(Y)$. The left adjoint of f_*, denoted by $f^{\mathrm{IndCoh},*}$, exists. It is compatible with tensor products by quasicoherent complexes and satisfies a similar commutative diagram as diagram (A-1) with $+$ removed [Gaitsgory and Rozenblyum 2017, Chapter 4, Proposition 3.1.6].

For a scheme X of finite type, by [Gaitsgory and Rozenblyum 2017, Chapter 5, Theorem 4.2.5], there is a canonical equivalence

$$
\boldsymbol{D}_X\colon \mathrm{IndCoh}(X)^\vee \simeq \mathrm{IndCoh}(X)
$$

such that

$$
\mathrm{IndCoh}(X)^{\vee\vee} \simeq \mathrm{IndCoh}(X)^\vee \simeq \mathrm{IndCoh}(X).
$$

The dual category $\mathscr{C}^\vee$ of a dualizable dg category $\mathscr{C}$ can be identified with the category of continuous dg functors from $\mathscr{C}$ to dg_k, where dg_k is the category of dg k–modules.

Theorem A.1 *Let X be a smooth and proper k–scheme of dimension d for $k = \mathbb{C}$. Let M and N be objects of* $\mathrm{coh}(X)$. *There is a bifunctorial quasi-isomorphism*

$$D\mathrm{Hom}^{\mathrm{dg}}_{\mathrm{coh}(X)}(M, N) \simeq \mathrm{Hom}^{\mathrm{dg}}_{\mathrm{coh}(X)}(N, M \otimes \omega_X[d]).$$

Proof   We will interpret

$$M^\vee \boxtimes N \mapsto D\mathrm{Hom}^{\mathrm{dg}}_{\mathrm{coh}(X)}(M, N) \quad \text{and} \quad M^\vee \boxtimes N \mapsto \mathrm{Hom}^{\mathrm{dg}}_{\mathrm{coh}(X)}(N, M \otimes \omega_Y[d])$$

as dg functors from $\mathrm{coh}(X \times X)$ to dg_k and show there is a natural isomorphism between them. By [Gaitsgory and Rozenblyum 2017, Chapter 3, Proposition 3.1.7], we have an equivalence $\mathrm{Qcoh}(X \times X) \simeq \mathrm{Qcoh}(X) \otimes \mathrm{Qcoh}(X)$. Therefore, we obtain a $\mathrm{coh}(X)^{\mathrm{op}} \otimes \mathrm{coh}(X)$–module structure by letting M and N vary in $\mathrm{coh}(X)$.

Let $f\colon X \to X \times X$ be the diagonal map and $p\colon X \to \mathrm{Spec}\, k$ be the counit map. By [Gaitsgory and Rozenblyum 2017, Chapter 5, Theorem 4.2.5], there is a commutative diagram of dg functors (both the upper and the lower square commute; see [Gaitsgory 2013, 9.2.3])

$$
\begin{array}{ccc}
\mathrm{IndCoh}(k)^\vee & \xrightarrow{\; \boldsymbol{D}_k \;} & \mathrm{IndCoh}(k) \\
{\scriptstyle (p^!)^\vee} \big\uparrow & & \big\uparrow {\scriptstyle p_*} \\
\mathrm{IndCoh}(X)^\vee & \xrightarrow{\; \boldsymbol{D}_X \;} & \mathrm{IndCoh}(X) \\
{\scriptstyle f_*^\vee} \big\uparrow & & \big\uparrow {\scriptstyle f^!} \\
\mathrm{IndCoh}(X \times X)^\vee & \xrightarrow{\boldsymbol{D}_{X \times X}} & \mathrm{IndCoh}(X \times X)
\end{array}
$$

By [Gaitsgory 2013, Corollary 9.5.9], if we restrict to $\mathrm{coh}(X)$, $\mathrm{coh}(k)$ and $\mathrm{coh}(X \times X)$, we get a commutative diagram

$$
\begin{array}{ccc}
\mathrm{coh}(k)^{\mathrm{op}} & \xrightarrow{\; \boldsymbol{D}_k \;} & \mathrm{coh}(k) \\
{\scriptstyle (p_*)^{\mathrm{op}}} \big\uparrow & & \big\uparrow {\scriptstyle p_*} \\
\mathrm{coh}(X)^{\mathrm{op}} & \xrightarrow{\; \boldsymbol{D}_X \;} & \mathrm{coh}(X) \\
{\scriptstyle (f^*)^{\mathrm{op}}} \big\uparrow & & \big\uparrow {\scriptstyle f^!} \\
\mathrm{coh}(X \times X)^{\mathrm{op}} & \xrightarrow{\boldsymbol{D}_{X \times X}} & \mathrm{coh}(X \times X)
\end{array}
$$

Then we have a natural isomorphism

$$p_* \circ f^! \circ \boldsymbol{D}_{X \times X} \simeq \boldsymbol{D}_k \circ (p_*)^{\mathrm{op}} \circ (f^*)^{\mathrm{op}}.$$

Since X is smooth, coherent sheaves are dualizable. In this case,

$$\boldsymbol{D}_X(E) = E^\vee \otimes \omega_X[d]$$

for $E \in \mathrm{coh}(X)$ [Gaitsgory 2013, Lemma 9.5.5].[3]

---

[3]Gaitsgory [2013] denotes by ω_X the !–pullback $p^! k$ for $p\colon X \to k$, which differs from the standard notion of dualizing complex by $[d]$.

Given an object $M^\vee \boxtimes N$ in $\mathrm{coh}(X \times X)$, we may compute

$$p_* \circ f^! \circ \boldsymbol{D}_{X \times X}(M^\vee \boxtimes N) \cong p_*\big(f^!\big((M \otimes \omega_X[d]) \boxtimes (N^\vee \otimes \omega_X[d]))\big)\big)$$

$$\cong p_*(f^*(M \boxtimes N^\vee) \otimes \omega_X[d])$$

$$\cong \mathrm{Hom}^{\mathrm{dg}}_{\mathrm{coh}(X)}(N, M \otimes \omega_X[d])$$

In the last step, we use the condition that X is smooth and proper. On the other hand,

$$\boldsymbol{D}_k \circ (p_*)^{\mathrm{op}} \circ (f^*)^{\mathrm{op}}(M^\vee \boxtimes N) \cong \boldsymbol{D}_k \mathrm{Hom}^{\mathrm{dg}}_{\mathrm{coh}(X)}(M, N) \cong D\mathrm{Hom}^{\mathrm{dg}}_{\mathrm{coh}(X)}(M, N). \qquad \square$$

A.2 Inverse dualizing sheaves and bimodules, after Ben-Zvi, Francis and Nadler

The following proposition is a consequence of [Ben-Zvi et al. 2010, Theorem 4.7 and Corollary 4.8] combined with Grothendieck duality. The proof we give is an elaboration on [Kinjo and Masuda 2023, Example 2.7].

Proposition A.2 [Kinjo and Masuda 2023] *Let Y be a smooth quasiprojective variety of dimension d. Let G be a perfect generator of $D(\mathrm{Qcoh}\, Y)$ and $B = \mathbb{R}\mathrm{Hom}(G, G)$. Then there is a canonical equivalence*

$$D(\mathrm{Qcoh}(Y \times Y)) \to D(B \otimes B^{\mathrm{op}})$$

taking $\Delta_*(\Sigma^{-d}\omega_Y^{-1})$ to the inverse dualizing complex $\Theta_B = \mathrm{Hom}_{B^e}(B, B^e)$ of B and $\Delta_*(\mathbb{O}_Y)$ to the identity bimodule B. In particular, if Y is d–Calabi–Yau, then B is bimodule d–Calabi–Yau.

Remark A.3 One could make further use of the results of [Ben-Zvi et al. 2010] to show more precisely that the derived endomorphism algebra of $G \boxtimes G^\vee$ in $D(Y \times Y)$ is quasi-isomorphic to B^e and that the canonical equivalence of the theorem is given by

$$\mathbb{R}\mathrm{Hom}(G \boxtimes G^\vee)\colon D(Y \times Y) \xrightarrow{\sim} D(B^e).$$

Proof Let $k = \mathbb{C}$. We mostly work in the ∞–category $\mathscr{S}t$ of k–linear stable presentable ∞–categories whose 1–morphisms are cocontinuous k–linear exact ∞–functors (or equivalently left adjoints of k–linear exact ∞–functors). We recommend [Kinjo and Masuda 2023, Section 2] for a concise but readable introduction to this setting. Each (large) pretriangulated dg k–category $\mathscr{A}$ such that $H^0(\mathscr{A})$ has arbitrary (set-indexed) coproducts and is compactly generated gives rise to an object of $\mathscr{S}t$; each dg functor $F\colon \mathscr{A} \to \mathscr{B}$ between two such categories such that $H^0(F)$ commutes with arbitrary coproducts gives rise to a 1–morphism in $\mathscr{S}t$ and this 1–morphism is an equivalence if and only if $H^0(F)$ is an equivalence. To make these statements more precise, let us denote by $\mathrm{dgcat}_k^{\mathrm{pretr},\oplus}$ the category whose

- objects are the pretriangulated dg categories $\mathscr{A}$ such that $H^0(\mathscr{A})$ has arbitrary coproducts and is compactly generated, and

- morphisms are the dg functors F such that $H^0(F)$ commutes with arbitrary coproducts.

We then have a canonical ∞–functor (where on the left we write the ordinary category instead of its nerve)

$$\mathrm{can}\colon \mathrm{dgcat}_k^{\mathrm{pretr},\oplus} \to \mathscr{S}t.$$

Each object $\mathscr{X}$ of $\mathscr{S}t$ has an underlying ∞–category $\mathscr{X}_\infty$ (obtained by forgetting the k–linear structure). It is a stable presentable ∞–category and its 1–categorical truncation $\tau(\mathscr{X}_\infty)$ is naturally a triangulated category with arbitrary coproducts; see [Lurie 2017, Section 1.4.4]. Similarly, if $f\colon \mathscr{X} \to \mathscr{Y}$ is a 1–morphism of $\mathscr{S}t$, then its underlying ∞–functor $f_\infty\colon \mathscr{X}_\infty \to \mathscr{Y}_\infty$ is exact and cocontinuous and its 1–categorical truncation $\tau(f_\infty)\colon \tau(\mathscr{X}_\infty) \to \tau(\mathscr{Y}_\infty)$ is naturally a triangulated functor and commutes with arbitrary coproducts. To state these facts more precisely, let us denote by $\mathrm{Tria}^\oplus$ the category whose objects are the triangulated categories with arbitrary coproducts and whose morphisms are the triangle functors which commute with arbitrary coproducts; let us denote by $\mathscr{P}r_{\mathrm{st}}^L$ Lurie's ∞–category of presentable stable ∞–categories whose 1–morphisms are the exact left adjoint ∞–functors. Then we have ∞–functors

$$\mathrm{dgcat}_k^{\mathrm{pretr},\oplus} \xrightarrow{\ \mathrm{can}\ } \mathscr{S}t \xrightarrow{\ ?_\infty\ } \mathscr{P}r_{\mathrm{st}}^L \xrightarrow{\ \tau\ } \mathrm{Tria}^\oplus.$$

The ∞–category $\mathscr{P}r_{\mathrm{st}}^L$ is by definition a subcategory of the ∞–category $\mathscr{P}r_{\mathrm{st}}$ of presentable stable ∞–categories whose 1–morphisms are all exact ∞–functors. We will use the following facts:

(a) If $f\colon \mathscr{X} \to \mathscr{Y}$ is a 1–morphism of $\mathscr{P}r_{\mathrm{st}}$, then the values of f and of $\tau(f)$ on objects are equal in $\tau(\mathscr{Y})$.

(b) A 1–morphism $f\colon \mathscr{X} \to \mathscr{Y}$ of $\mathscr{P}r_{\mathrm{st}}$ belongs to $\mathscr{P}r_{\mathrm{st}}^L$ if and only if its truncation $\tau(f)\colon \tau(\mathscr{X}) \to \tau(\mathscr{Y})$ is a (triangle) functor which commutes with arbitrary coproducts; see [Lurie 2017, Proposition 1.4.4.1(2)].

(c) A 1–morphism $f\colon \mathscr{X} \to \mathscr{Y}$ of $\mathscr{P}r_{\mathrm{st}}$ admits a left adjoint $f_\lambda\colon \mathscr{Y} \to \mathscr{X}$ in $\mathscr{P}r_{\mathrm{st}}$ if and only if the (triangle) functor $\tau(f)$ admits a left adjoint $\tau(f)_\lambda$. By definition, in this case, the 1–morphism f_λ belongs to $\mathscr{P}r_{\mathrm{st}}^L$. Moreover, the functor $\tau(f_\lambda)$ is isomorphic to $\tau(f)_\lambda$ since truncation preserves adjunctions.

For a dg algebra A, we denote by $\mathscr{D}(A)$ the object of $\mathscr{S}t$ corresponding to the dg category of K–projective dg A–modules (ie dg A–modules which are homotopy equivalent to cofibrant dg A–modules). Notice that $\mathscr{D}(A)$ denotes an object of $\mathscr{S}t$ whereas $D(A)$ denotes the ordinary derived category (with its triangulated structure). The ordinary derived category $D(A)$ is naturally equivalent to the truncation $\tau(\mathscr{D}(A)_\infty)$.

Let X be a quasiprojective variety (or, more generally, a quasiprojective separated scheme). Let $\mathscr{D}(X)$ denote the object of $\mathscr{S}t$ corresponding to the dg category of fibrant complexes of quasicoherent sheaves on X. Then the truncation $\tau(\mathscr{D}(X)_\infty)$ identifies with the derived category $D(\mathrm{Qcoh}(X))$. Let T be a compact generator of the unbounded derived category of quasicoherent sheaves on X. Suppose that T is a fibrant complex of quasicoherent sheaves and let $A = \mathrm{Hom}(T, T)$ be its dg endomorphism algebra. The homotopy category of fibrant complexes of quasicoherent sheaves on X is compactly generated by T and so the dg A–module $\mathrm{Hom}(T, I)$ is K–projective for all fibrant complexes I of

quasicoherent sheaves on X. Moreover, the dg functor $\mathrm{Hom}(T, ?)$ induces a quasiequivalence between the homotopy categories of fibrant complexes on X and K–projective dg A–modules. Thus, the dg functor $\mathrm{Hom}(T, ?)$ induces a 1–morphism $\mathscr{D}(X) \to \mathscr{D}(A)$ in $\mathscr{S}t$. Its truncation $\tau(\mathrm{Hom}(T, ?)_\infty)$ identifies with $\mathbb{R}\mathrm{Hom}(T, ?) \colon D(\mathrm{Qcoh}(X)) \to D(\mathrm{Qcoh}(Y))$, which is an equivalence by our assumption on T. Thus, we have an equivalence $\mathscr{D}(X) \to \mathscr{D}(A)$ in $\mathscr{S}t$.

As detailed in [Lurie 2017, Section 4.8; Kinjo and Masuda 2023, Section 2.2], the ∞–category $\mathscr{S}t$ is endowed with a symmetric monoidal structure. The unit of the monoidal structure on $\mathscr{S}t$ is the k–linear (symmetric monoidal) ∞–category $\mathscr{D}(k)$. If A_1 and A_2 are two dg algebras, then, by [Ben-Zvi et al. 2010, Proposition 4.1(2)], we have a canonical equivalence

$$\mathscr{D}(A_1) \otimes \mathscr{D}(A_2) \xrightarrow{\ \sim\ } \mathscr{D}(A_1 \otimes A_2)$$

induced by the dg functor taking a pair (L, M) of cofibrant dg modules to the cofibrant $A_1 \otimes A_2$–module $L \otimes M$.

For any dg algebra A, the object $\mathscr{D}(A)$ is dualizable in the monoidal ∞–category $\mathscr{S}t$ and its dual is equivalent to $\mathscr{D}(A^{\mathrm{op}})$; see [Ben-Zvi et al. 2010, Proposition 4.3(3)]. More precisely, the duality between $\mathscr{D}(A)$ and $\mathscr{D}(A^{\mathrm{op}})$ is given by the evaluation morphism

$$\mathscr{D}(A^{\mathrm{op}}) \otimes \mathscr{D}(A) \to \mathscr{D}(k)$$

and the coevaluation morphism

$$\mathscr{D}(k) \to \mathscr{D}(A) \otimes \mathscr{D}(A^{\mathrm{op}}).$$

The composition of the coevaluation morphism with the canonical morphism $\mathscr{D}(A) \otimes \mathscr{D}(A^{\mathrm{op}}) \xrightarrow{\ \sim\ } \mathscr{D}(A \otimes A^{\mathrm{op}})$ is the morphism

$$\mathscr{D}(k) \to \mathscr{D}(A \otimes A^{\mathrm{op}})$$

induced by the dg functor taking a complex V to $V \otimes A$, where A is the identity bimodule. The evaluation morphism is obtained as the composition

$$\mathscr{D}(A^{\mathrm{op}}) \otimes \mathscr{D}(A) \xrightarrow{\ \sim\ } \mathscr{D}(A^{\mathrm{op}} \otimes A) \xrightarrow{\ \mathrm{ev}^A\ } \mathscr{D}(k),$$

where the second morphism is induced by the dg functor taking a cofibrant dg bimodule M to $M \otimes_{A^e} A$. Let us abbreviate ev^A_∞ by Ev^A. Then the truncation $\tau(\mathrm{Ev}^A)$ identifies with the derived functor

$$? \otimes^{\mathbb{L}}_{A^e} A \colon D(A^{\mathrm{op}} \otimes A) \to D(k).$$

When A is smooth, this functor is isomorphic to $\mathbb{R}\mathrm{Hom}_{A^e}(\Theta, ?)$, where $\Theta = \Theta_A = \mathbb{R}\mathrm{Hom}_{A^e}(A, A^e)$ is the inverse dualizing bimodule. Thus, it admits the left adjoint

$$? \otimes \Theta \colon D(k) \to D(A^{\mathrm{op}} \otimes A).$$

Hence, by fact (c) above, the ∞–functor Ev^A admits a left adjoint Ev^A_λ and $\tau(\mathrm{Ev}^A_\lambda)$ is isomorphic to $? \otimes^{\mathbb{L}}_{A^e} \Theta$. In particular, the left adjoint Ev^A_λ takes k to Θ (fact (a) above).

Let X_1 and X_2 be quasiprojective varieties (or, more generally, quasicompact, separated schemes). They are in particular perfect stacks in the sense of [Ben-Zvi et al. 2010]. By [loc. cit., Theorem 4.7], we have a canonical equivalence

$$\mathscr{D}(X_1) \otimes \mathscr{D}(X_2) \xrightarrow{\sim} \mathscr{D}(X_1 \times X_2).$$

Fix a quasiprojective variety X. By [Ben-Zvi et al. 2010, Corollary 4.8] and the first three lines of its proof, the object $\mathscr{D}(X)$ becomes its own dual and the evaluation morphism is given by a composition

$$\mathscr{D}(X) \otimes \mathscr{D}(X) \xrightarrow{\sim} \mathscr{D}(X \times X) \xrightarrow{\mathrm{ev}^X} \mathscr{D}(k),$$

where the truncation $\tau(\mathrm{ev}^X_\infty)$ identifies with the composition of derived functors

$$D(X \times X) \xrightarrow{\Delta^*} D(X) \xrightarrow{\pi_*} D(k).$$

The coevaluation morphism is the composition

$$\mathscr{D}(k) \to \mathscr{D}(X \times X) \xleftarrow{\sim} \mathscr{D}(X) \otimes \mathscr{D}(X),$$

where the first morphism is induced by the dg functor taking a complex V to $V \otimes I$, where I is an injective resolution of $\Delta_*(\mathbb{O}_X)$.

Now consider the smooth quasiprojective variety Y of the claim. In this case, by Grothendieck duality, the functor $\Delta^* \colon D(Y \times Y) \to D(Y)$ admits the left adjoint $\Delta_*(?) \otimes \Sigma^{-d} \omega_Y^{-1}$. Thus, the composed functor

$$D(Y \times Y) \xrightarrow{\Delta^*} D(Y) \xrightarrow{\pi_*} D(k)$$

admits the left adjoint

$$\Delta_*(\pi^*(?) \otimes \Sigma^{-d} \omega_Y^{-1}),$$

which takes k to $\Delta_*(\Sigma^{-d} \omega_Y^{-1})$. As above, it follows that the ∞–functor $\mathrm{Ev}^Y = \mathrm{ev}^Y_\infty$ admits a left adjoint Ev^Y_λ and that this left adjoint takes k to $\Delta_*(\Sigma^{-d} \omega_Y^{-1})$.

As we have seen above, under our hypotheses, the dg functor taking a fibrant complex of quasicoherent sheaves C to $\mathrm{Hom}(G, C)$ induces an equivalence

$$\mathscr{D}(Y) \xrightarrow{\sim} \mathscr{D}(B).$$

We deduce an equivalence

$$\mathscr{D}(Y \times Y) \xleftarrow{\sim} \mathscr{D}(Y) \otimes \mathscr{D}(Y) \xrightarrow{\sim} \mathscr{D}(Y) \otimes \mathscr{D}(Y)^\vee \xrightarrow{\sim} \mathscr{D}(B) \otimes \mathscr{D}(B^{\mathrm{op}}) \xrightarrow{\sim} \mathscr{D}(B \otimes B^{\mathrm{op}}).$$

The intrinsic descriptions above show that, under the 1–categorical truncation of this equivalence, the object $\Delta_*(\Sigma^{-d} \omega_Y^{-1})$ corresponds to the inverse dualizing complex $\Theta_B = \mathrm{Hom}_{B^e}(B, B^e)$ of B. Moreover, the object $\Delta_*(\mathbb{O}_Y)$ corresponds to the identity bimodule $B \in D(B^e)$, as we see by examining the coevaluation morphisms. Thus, an isomorphism $\omega_Y \xrightarrow{\sim} \mathbb{O}_Y$ yields an isomorphism $\Theta \xrightarrow{\sim} \Sigma^{-d} B$ in $D(B^e)$. $\qquad \square$

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