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Limits of manifolds with a Kato bound on the Ricci curvature

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We study the structure of Gromov–Hausdorff limits of sequences of Riemannian manifolds $\{(M_\alpha^n, g_\alpha)\}_{\alpha\in A}$ whose Ricci curvature satisfies a uniform Kato bound. We first obtain Mosco convergence of the Dirichlet energies to the Cheeger energy, and show that tangent cones of such limits satisfy the $\mathrm{RCD}(0,n)$ condition. Under a noncollapsing assumption, we introduce a new family of monotone quantities, which allows us to prove that tangent cones are also metric cones. We then show the existence of a well-defined stratification in terms of splittings of tangent cones. We finally prove volume convergence to the Hausdorff n–measure.

53C21; 53C23, 31C25, 58J35

Introduction

We study the structure of Gromov–Hausdorff or measured Gromov–Hausdorff limits of manifolds whose Ricci curvature satisfies a Kato type bound. Our results extend previous results proven by J Cheeger and T Colding for the limits of the manifolds carrying a uniform lower bound on the Ricci curvature.

Gromov–Hausdorff convergence of manifolds In the 1980s, M Gromov showed a compactness result for Riemannian manifolds satisfying a uniform lower bound on the Ricci curvature. It states that, if $\{(M^n_\alpha, g_\alpha, o_\alpha)\}_\alpha$ is a sequence of pointed complete Riemannian manifolds satisfying uniformly

$$\mathrm{Ric} \geq K \tag{1}$$

for some $K \in \mathbb{R}$, then up to extracting a subsequence, the sequence $\{(M^n_\alpha, \mathsf{d}_{g_\alpha}, o_\alpha)\}_\alpha$ converges in the pointed Gromov–Hausdorff topology to a complete proper metric space (X, d, o). A natural question was then to describe the structure of such metric spaces arising as limits of smooth manifolds. In the 1990s, a series of results by Cheeger and Colding [31; 23; 24; 25; 26] made it possible to better understand this problem and their work launched a vast research program. Many recent results have led to a significant understanding of the so-called "Ricci limit spaces"; see Cheeger [22], Cheeger and Naber [29], Jiang and Naber [61] and Cheeger, Jiang and Naber [27].

In the study of limit spaces, there are two different scenarios, depending on whether the sequence of manifolds is *collapsing* or *noncollapsing*. In the first case, the volume of unit balls $B_1(o_\alpha)$ goes to 0 as α tends to infinity, while in the noncollapsed case there is a uniform lower bound on this volume. Since the work of K Fukaya [41] it is known that, in the collapsed case, the Gromov–Hausdorff topology is not sufficient to recover good geometric information on the limit space, such as, for instance, information about the spectrum of its Laplacian. Fukaya then introduced the *measured* Gromov–Hausdorff topology. For that, one rescales the Riemannian measure $\mathrm{d}v_{g_\alpha}$ and considers sequences of manifolds as sequences of metric measure spaces $(M^n_\alpha, \mathsf{d}_{g_\alpha}, \mathrm{d}\mu_\alpha, o_\alpha)$, where $\mathrm{d}\mu_\alpha = v_{g_\alpha}^{-1}(B_1(o_\alpha))\,\mathrm{d}v_{g_\alpha}$. Then again, up to extracting a subsequence, there is convergence to a metric measure space (X, d, μ, o) in the pointed measured Gromov–Hausdorff topology. This allows for finer results on the structure of the limit space.

In the 2000s the work of J Lott and C Villani [68; 94] and K-T Sturm [86; 87] showed that it is possible to define a generalization of a Ricci lower bound in the setting of metric measure spaces. This led to the notions of $\mathrm{CD}(K,\infty)$ and $\mathrm{CD}(K,n)$ metric measure spaces, that are known to include Ricci limit spaces. Later on, L Ambrosio, N Gigli and G Savaré [3] introduced a refinement of the infinite-dimensional $\mathrm{CD}(K,\infty)$ condition, the so-called Riemannian curvature dimension condition $\mathrm{RCD}(K,\infty)$, which is also satisfied by manifolds carrying the lower bound (1) and is preserved under measured Gromov–Hausdorff convergence. The finite-dimensional $\mathrm{RCD}(K,n)$ condition was subsequently introduced and studied by Gigli in [47]. Under this more restrictive condition, it is possible, for instance, to define a Laplacian operator and thus reformulate classical inequalities of Riemannian geometry in the setting of metric measure spaces. Nowadays, the structure of $\mathrm{RCD}(K,n)$ spaces is fairly well understood, and such spaces provide a good conceptual framework comprising Ricci limit spaces; see for instance Mondino and Naber [70], Bruè and Semola [14], De Philippis and Gigli [38].

Ricci limit spaces and beyond The work of Cheeger and Colding relies on several crucial tools and results. The first one is the Bishop–Gromov volume comparison theorem, which provides a monotone quantity given by the *volume ratio*[1]

$$r \mapsto \frac{v_g(B_r(x))}{\mathbb{V}_{n,K}(r)}.$$

Monotone quantities play a crucial role when investigating blow-up phenomena in geometric analysis. In the case of Ricci limit spaces, the monotonicity of the volume ratio is the keystone for understanding the local geometry of limit spaces. Two other very important results are the almost splitting theorem [24] and the theorem now known as "almost volume cones implies almost metric cone" [23, Theorem 3.6]. Together with the monotonicity of the volume ratio, they imply in particular that tangent cones of noncollapsed Ricci limit spaces are genuine metric cones. Many additional technical results are involved in the study of Ricci limit spaces. For example, the existence of good cut-off functions (with suitably bounded gradient and Laplacian) plays an important role in many proofs for exploiting the Bochner formula. The construction of cut-off functions relies on strong analytic properties of manifolds with Ricci curvature bounded from below, such as Laplacian comparisons, Gaussian heat kernel bounds and the Cheng–Yau gradient estimate.

However, there are some very interesting contexts in which a good understanding of the convergence of smooth manifolds is needed, but a uniform lower bound on the Ricci curvature is not satisfied and the previous tools are not available: for example, in the study of the Ricci flow (see Bamler [8; 9], Bamler and Zhang [10] and Simon [82; 83]) and of critical metrics (see Tian and Viaclovsky [89; 90; 91]). It is then important to investigate situations in which weaker assumptions on the curvature are made. The case in which one assumes some L^p bound on the Ricci curvature, for $p > n/2$, has been well studied since the end of the 1990s, and one gets a number of results about the structure of the limit space under an additional smallness assumption; see Petersen and Wei [72; 73] Tian and Zhang [92] and Dai, Wei and Zhang [36]. Very recently, L Chen [30] obtained more results about the structure of the limit space. C Ketterer [64] also opened the way to a new interesting perspective in the study of limits of manifolds with the appropriate L^p bound on the Ricci curvature, by showing, among other things, that tangent cones of the limit space are $\mathrm{RCD}(0,n)$ spaces. One question he raises in this work is whether the same result holds when assuming a Kato condition on the Ricci curvature: we give a positive answer to this question in one of our results (Theorem B).

[1] Here $\mathbb{V}_{n,K}(r)$ is the volume of a geodesic balls with radius r in a simply connected space of constant curvature equal to $K/(n-1)$.

Kato type bound It has been recently remarked (see Carron [17] and C Rose [75]) that a Kato type bound on the Ricci curvature makes it possible to use ideas of Q S Zhang and M Zhu [97] and, as a consequence, to obtain a Li–Yau type bound for solutions of the heat kernel. Many geometric estimates, which are known when the Ricci curvature is bounded from below, follow under such a less restrictive Kato type bound.

Let (M^n, g) be a closed Riemannian manifold. Let $\mathrm{Ric}_-\colon M \to \mathbb{R}_+$ be the smallest function such that

$$\mathrm{Ric} \geq -\mathrm{Ric}_-\, g.$$

Define, for all $\tau > 0$,

$$\mathrm{k}_\tau(M^n, g) = \sup_{x\in M} \int_{[0,\tau]\times M} H(t,x,y)\,\mathrm{Ric}_-(y)\,\mathrm{d}v_g(y)\,\mathrm{d}t,$$

where $H(t,\cdot,\cdot)$ is the Schwartz kernel of the heat operator $e^{-t\Delta}$. If for some $T > 0$ the Kato type bound

$$\mathrm{k}_T(M^n, g) \leq \frac{1}{16n} \tag{2}$$

holds, then one gets geometric and analytic results similar to those implied by the condition (1) with $K = -1/T$. Moreover, as noticed in Carron [17], the set of closed manifolds satisfying (2) is precompact in the Gromov–Hausdorff topology. As a consequence, it is natural to ask under what extent results on the structure of Ricci limit spaces can be obtained under this weaker assumption.

The Kato condition was introduced with the aim of studying Schrödinger operators in the Euclidean space

$$L = \Delta - V,$$

where $V \geq 0$ and our convention is that $\Delta = -\sum_i \partial^2/\partial x_i^2$ on $\mathbb{R}^n$. A nonnegative potential $V\colon \mathbb{R}^n \to \mathbb{R}_+$ is said to be in the Kato class, or to satisfy the Kato condition, if

$$\lim_{T\to 0+} \sup_{x\in\mathbb{R}^n} \iint_{[0,T]\times\mathbb{R}^n} \frac{e^{-\|x-y\|^2/4t}}{(4\pi t)^{n/2}} V(y)\,\mathrm{d}y\,\mathrm{d}t = 0.$$

At the regularity level, this condition only requires that V is the Laplacian of a continuous function. Moreover, if V is in the Kato class, when t tends to 0 one can compare the semigroups $e^{-t\Delta}$ and $e^{-t(\Delta-V)}$ and thus recover good properties of $e^{-t(\Delta-V)}$ for t small enough. We refer to the beautiful survey of B Simon [80] for a extensive overview on the Kato condition in the Euclidean setting, and to the book of B Guneysu [55] for an account of the Kato condition in the Riemannian setting. In our context, the potential V is chosen to be Ric_-.

Now, an assumption in the spirit of the Kato condition in $\mathbb{R}^n$ would require not only that k_T is uniformly bounded along the sequence of manifolds for a fixed $T > 0$, but also some uniform control on the way that k_τ goes to 0 when τ goes to 0. This kind of control is actually required in our analysis in order to be able to compare infinitesimal geometry of limit spaces with Euclidean geometry. In particular, this plays an important role in getting the appropriate monotone quantities that we rely on for studying the geometry of tangent cones.

Main results We begin by illustrating our main results in the noncollapsed case.

Theorem A *Let (X, d, o) be the pointed Gromov–Hausdorff limit of a sequence of closed Riemannian manifolds $\{(M^n_\alpha, \mathsf{d}_{g_\alpha}, o_\alpha)\}_\alpha$ satisfying the uniform Kato bound*

$$\mathrm{k}_\tau(M^n_\alpha, g_\alpha) \le f(\tau) \quad \text{for all } \tau \in (0,1], \tag{3}$$

where $f : [0,1] \to \mathbb{R}_+$ is a nondecreasing function such that

$$\int_0^1 \sqrt{f(\tau)}\,\frac{d\tau}{\tau} < \infty, \tag{4}$$

and the noncollapsing condition

$$v_{g_\alpha}(B_1(o_\alpha)) \ge v > 0. \tag{5}$$

Then the following holds:

(i) ***Volume convergence*** *For any $r > 0$ and $x_\alpha \in M_\alpha$ such that $x_\alpha \to x \in X$, we have*

$$\lim_{\alpha\to\infty} v_{g_\alpha}(B_r(x_\alpha)) = \mathscr{H}^n(B_r(x)),$$

where $\mathscr{H}^n$ is the n–dimensional Hausdorff measure.

(ii) ***Structure of tangent cones*** *For any $x \in X$, tangent cones of X at x are* $\mathrm{RCD}(0, n)$ *metric cones.*

(iii) ***Almost everywhere regularity*** *For $\mathscr{H}^n$–a.e. $x \in X$, $(\mathbb{R}^n, \mathsf{d}_{\mathrm{eucl}})$ is the unique tangent cone of X at x.*

(iv) ***Stratification*** *Let $\mathcal{S}^k$ be the set consisting of the points $x \in X$ such that X does not carry any tangent cone at x that splits isometrically a factor $\mathbb{R}^{k+1}$. Then the Hausdorff dimension of $\mathcal{S}^k$ satisfies*

$$\dim_{\mathscr{H}} \mathcal{S}^k \le k.$$

The first point is a generalization of the volume continuity showed by Colding [31]. The fact that tangent cones are metric cones is the analogue of Cheeger and Colding [24, Theorem 5.2] and the two last points correspond to [24, Theorem 4.7]; see also Cheeger [22, Theorem 10.20]. We also conjecture that, under the same assumptions, we have $\mathcal{S}_{n-1} = \mathcal{S}_{n-2}$, that is, the singular set has codimension at least two. For the case of Ricci limit spaces, it is known that there exists an open subset that is n–rectifiable and bi-Hölder homeomorphic to a manifold; in a subsequent work [18], we show that such a result also holds for noncollapsed Kato limits.

Observe that the uniform Kato bound (3) with a function satisfying (4) is guaranteed as soon as one has an appropriate estimate on the L^p norm of the Ricci curvature. This is due to C Rose and P Stollmann [76]: thanks to their work, it is possible to show that if $p > n/2$ and $\varepsilon(p, n, \kappa)$ is small enough, then the estimate

$$\mathrm{diam}^2(M_\alpha, g_\alpha)\left(\fint_M |\mathrm{Ric}_- - \kappa^2|_+^p \, dv_{g_\alpha}\right)^{1/p} \le \varepsilon(p, n, \kappa),$$

with $x_+ = \max\{x, 0\}$, implies (3) and (4). Similarly, our noncollapsing assumption and the uniform Kato bound is ensured under the assumptions considered by G Tian and Z Zhang [92] in the study of

Kähler–Ricci flow $g(t)$, in which they assumed an a priori bound on the L^p norm of the Ricci curvature for $p > n/2$,

$$\int_M |\operatorname{Ric}_{g(t)}|^p \, dv_{g(t)} \leq \Lambda \quad \text{for all } t \geq 0,$$

and a noncollapsing condition

$$v_{g(t)}(B_r(x)) \geq v r^n \quad \text{for all } t \geq 0,\ x \in M \text{ and } r \in (0,1).$$

In the collapsed case, our results give less information about the structure of the limit space, but apply with a weaker hypothesis.

Theorem B *Let (X, d, μ, o) be the pointed measured Gromov–Hausdorff limit of a sequence*

$$\{(M^n_\alpha, \mathsf{d}_{g_\alpha}, \mu_\alpha, o_\alpha)\}_\alpha$$

satisfying the uniform Kato bound (3) for some nondecreasing positive function $f : [0,1] \to \mathbb{R}_+$ such that

$$\lim_{\tau \to 0} f(\tau) = 0, \tag{6}$$

with the rescaled measure

$$d\mu_\alpha = \frac{dv_{g_\alpha}}{v_{g_\alpha}(B_1(o_\alpha))}. \tag{7}$$

Then we have:

(i) *The Cheeger energy is quadratic and (X, d, μ) is an infinitesimally Hilbertian space in the sense of Gigli [47].*

(ii) *For any $x \in X$, metric measure tangent cones of X at x are* $\mathrm{RCD}(0, n)$.

(iii) *If X is compact, then the spectrum of the Laplace operators of (M^n_α, g_α) converges to the spectrum of the Laplacian associated to (X, d, μ).*

The last point extends Fukaya [41, Theorem 0.4]. The second point generalizes Ketterer [64, Corollary 1.7]: under the same assumptions of [92] that we recalled above, he proved that tangent cones are $\mathrm{RCD}(0,n)$ spaces. Part of his proof relies on an almost splitting theorem of [92]. In our case, we do not use an almost splitting theorem. Nonetheless, we point out that our proof shows that whenever the sequence of manifolds $\{(M_\alpha, g_\alpha)\}$ is such that, for some $\tau > 0$, $\mathrm{k}_\tau(M_\alpha, g_\alpha)$ tends to zero as α goes to infinity, then the limit (X, d, μ) is an $\mathrm{RCD}(0,n)$ space. As a consequence, Gigli's splitting theorem [45; 46] for $\mathrm{RCD}(0,n)$ spaces applies. Moreover, a contradiction argument based on precompactness leads to an almost splitting theorem for manifolds with k_τ small enough. Then we do have an almost splitting theorem in our setting, but in contrast to what happens in the study of Ricci limit and RCD spaces, where such theorem represents a key tool, we obtain it as a consequence of our results rather than relying on it in our proofs.

Outline of proofs We now describe some of the ideas playing a role in our proofs and their organization, starting from Theorem B. The Kato type bound (2) provides very good heat kernel estimates (see for example Proposition 2.3) which imply in particular that a sequence of manifolds satisfying (2), when considered as a sequence of *Dirichlet* spaces, is uniformly doubling and carries a uniform Poincaré inequality. This, together with the results of A Kasue [62] and K Kuwae and T Shioya [67], ensures that the measured Gromov–Hausdorff convergence can be strengthened, in the sense that one additionally obtains Mosco convergence of the Dirichlet energies. More precisely, assume that (X, d, μ, o) is a pointed measured Gromov–Hausdorff limit of a sequence of closed manifolds $\{(M^n_\alpha, \mathsf{d}_{g_\alpha}, \mu_\alpha, o_\alpha)\}_\alpha$, where $\mathrm{d}\mu_\alpha$ is either the Riemannian volume in the noncollapsing case, or its rescaled version (7) in the collapsing one. Up to extraction of a subsequence, it is possible to define a closed, densely defined quadratic form $\mathscr{E}$ on $L^2(X, \mu)$ which is obtained as the Mosco limit of the Dirichlet energies:

$$u \mapsto \int_{M_\alpha} |du|^2_{g_\alpha} \,\mathrm{d}\mu_\alpha.$$

A priori, different subsequences could lead to different quadratic forms. Moreover, the space (X, d, μ, o) carries both the Dirichlet energy $\mathscr{E}$ and the Cheeger energy canonically associated to d and μ. In general, these two energies do not need to coincide — see for instance [2, Theorem 7.1], which gives an example of a limit space such that the distance is a Finsler non-Riemannian metric and thus the Cheeger energy, not being quadratic, cannot coincide with any Dirichlet form.

However, under the Kato bound (3) together with (6), we can use the Li–Yau type inequality in order to get estimates for the solutions of the heat equation on the manifolds (M_α, g_α). We show that such estimates pass to the limit and hold on the Dirichlet limit space $(X, \mathsf{d}, \mu, \mathscr{E})$. As a consequence, we can apply a result due to Ambrosio, Gigli and Savaré [4] and to P Koskela, N Shanmugalingam and Y Zhou [65] and we obtain that the limit Dirichlet energy $\mathscr{E}$ coincides in fact with the Cheeger energy of the metric measure space (X, d, μ). Hence, under conditions (3) and (6), measured Gromov–Hausdorff convergence implies Mosco convergence of the Dirichlet energies to the Cheeger energy.

Our proof also applies when for some $\tau > 0$,

$$\lim_{\alpha\to\infty} \mathrm{k}_\tau(M^n_\alpha, g_\alpha) = 0. \tag{8}$$

Under this condition, we additionally show that the Bakry–Ledoux gradient estimate holds on the limit space and thus (X, d, μ) is an $\mathrm{RCD}(0, n)$ space. Thanks to the rescaling properties of the heat kernel, if (X, d, μ, o) is a limit of manifolds satisfying (3) and (6), then any tangent cone of X is a limit of rescaled manifolds for which (8) holds for all $\tau > 0$. Therefore, this implies Theorem B(ii).

As for the noncollapsed case studied in Theorem A, we prove that the limit measure μ coincides with the n–dimensional Hausdorff measure, so that Gromov–Hausdorff convergence under conditions (4) and (5) implies both measured Gromov–Hausdorff convergence and Mosco convergence of the energies. To prove this, we introduce a new family of monotone quantities that, when the Ricci curvature is nonnegative, interpolates between the Li–Yau inequality and Bishop–Gromov volume comparison theorem. Our quantities are modeled on Huisken's entropy [60] for the mean curvature flow. In order to define them, for a closed manifold (M^n, g) with heat kernel H, we define the function U by setting

$$H(t,x,y) = \frac{\exp\left(-\frac{U(t,x,y)}{4t}\right)}{(4\pi t)^{n/2}}.$$

We then introduce, for any $s, t > 0$, the Gaussian type entropy

$$\theta_x(s,t) = \int_M \frac{\exp\left(-\frac{U(t,x,y)}{4s}\right)}{(4\pi s)^{n/2}} \, \mathrm{d}v_g(y).$$

When the Ricci curvature is nonnegative, we show that for all $x \in M$ the function

$$\lambda \mapsto \theta_x(\lambda s, \lambda t)$$

is monotone nonincreasing for $s \geq t$, and nondecreasing for $s \leq t$. This interpolates between the Bishop–Gromov and Li–Yau inequalities in the following sense. When $t = 0$, we can write

$$\theta_x(s,0) = \int_M \frac{\exp\left(-\frac{\mathrm{d}^2(x,y)}{4s}\right)}{(4\pi s)^{n/2}} \, \mathrm{d}v_g(y) = \frac{1}{2}\int_0^\infty e^{-\rho^2/4}\rho \frac{v_g(B_{\rho\sqrt{s}}(x))}{\mathbb{V}_{n,0}(\rho\sqrt{s})} \, d\rho.$$

Then, Bishop–Gromov volume comparison implies that for any $s \geq 0$ the function $\lambda \mapsto \theta_x(\lambda s, 0)$ is monotone nonincreasing. Moreover, one of the consequences of the Li–Yau inequality is that for all $x \in M$ the map

$$t \mapsto (4\pi t)^{n/2} H(t,x,x)$$

is monotone nondecreasing. When noticing that the semigroup law allows one to write

$$H(2t,x,x) = \int_M H(t,x,y)^2 \, \mathrm{d}v_g(y),$$

a simple computation shows that for any $t > 0$ and $s = t/2$ the function $\lambda \mapsto \theta_x(\lambda t/2, \lambda t)$ is monotone nondecreasing.

Observe that by Varadhan's formula (22) we have

$$\mathsf{d}(x,y)^2 = \lim_{t \to 0} U(t,x,y),$$

so that, when t tends to zero, our quantities θ_x tend to

$$\Theta_x(s) = (4\pi s)^{-n/2} \int_M e^{-d(x,y)^2/4s} \, \mathrm{d}v_g(y).$$

This corresponds to Huisken's entropy and to the $\mathscr{H}_s$ volume considered by W Jiang and A Naber in [61], where it is shown to be monotone nonincreasing if the Ricci curvature is nonnegative. Moreover, in the case of a Ricci limit space (X, d, μ), the limit of Θ_x as s tends to 0 coincides with the *volume density* at x, that is,

$$\vartheta_X(x) = \lim_{r\to 0} \frac{\mu(B(x,r))}{\omega_n r^n},$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. The Bishop–Gromov inequality guarantees that such a limit does exist.

In our setting, with the uniform Kato bounds (6) and (4), the Li–Yau type inequality allows us to show that our quantities θ_x are *almost monotone*, in the sense that there exists a function F of λ, tending to 1 as λ tends to 0, such that the map

$$\lambda \mapsto \theta_x(\lambda s, \lambda t) F(\lambda)$$

has the same monotonicity as θ_x when the Ricci curvature is nonnegative. There is a limitation on the range of parameter where our monotonicity holds: when $t \le s$ we also need $s \preceq t/\sqrt{f(t)}$. As a consequence, the quantity Θ_x is not monotone and we do not get a monotone quantity based on the volume ratio.

Observe that the only bound (3), with a function tending to 0 as t goes to zero, is not enough to obtain the above family of monotone quantities: due to the dependence of the Li–Yau type inequality on k_τ, some kind of integral bound on k_τ is needed. Moreover, for a sequence of smooth manifolds (M^n_α, g_α), the uniform bound (4) implies that function F is the same for all (M_α, g_α), so that we get a corresponding family of monotone quantities on the limit space (X, d, μ).

Thanks to this almost monotonicity, we are able to show that on a tangent cone at $x \in X$ the quantity Θ_x is constant. Then for all $r > 0$, the measure of balls centered at x is equal to $\Theta_x \omega_n r^n$. This, together with the fact that tangent cones are $\mathrm{RCD}(0, n)$ spaces and with the main result of [37], allows us to obtain that tangent cones are metric cones.

We also prove that the almost monotonicity of θ_x implies that the volume density $\vartheta_X(x)$ is well defined on the limit space, despite the lack of monotonicity of the volume ratio. We then show that the volume density is lower semicontinuous under measured Gromov–Hausdorff convergence. As a consequence, we obtain the stratification result from arguments inspired by B White [96] and G De Philippis and N Gigli [38]. In the same proof, we get that μ–almost everywhere tangent cones are Euclidean, with a measure given by $\vartheta_X(x)\mathscr{H}^n$. In order to prove volume convergence, we then show that

$$\vartheta_X(x) = 1 \quad \text{for } \mu\text{–a.e. } x \in X.$$

For this purpose, we prove the existence at almost every point $x \in X$ of harmonic ε–splitting maps $H : B_r(x) \to \mathbb{R}^n$. Splitting maps are "almost coordinates", in the sense that they are $(1+\varepsilon)$–Lipschitz, ∇H is close to the identity and the Hessian is close to zero in L^2. They have been extensively used in the study of Ricci limit spaces and were recently proven to exist on RCD spaces too; see Bruè, Pasqualetto and Semola [13]. In our case, we obtain a very good control of ∇H thanks to the Mosco convergence of Dirichlet energy. This is still not enough to prove, as in Cheeger [22] or Gallot [43], that $\vartheta_X(x) = 1$. But we are also able to obtain the Hessian bound thanks to one of our main technical tools, that is, the existence of good cut-off functions when just the Kato type bound (2) is satisfied.

Outline of the paper Section 1 includes the main preliminary tools that we rely on throughout the paper. After introducing the convergence notions that we need, we focus on Dirichlet spaces. We state a compactness result for PI Dirichlet spaces, originally observed in Kasue [62], for which we give a proof in Appendix D, and we collect the assumptions under which a Dirichlet space satisfies the RCD condition.

In Section 2, we introduce the different Kato type conditions that we consider in the rest of the paper, we state precompactness results, and show that under assumptions (3) and (4), the intrinsic distance associated to the Dirichlet energy coincides with the limit distance. In the noncollapsing case, we recall a useful Ahlfors regularity result due to the first author that also holds in the limit.

Section 3 is devoted to proving some technical tools obtained under assumption (2), in particular the existence of good cut-off functions and the resulting Hessian bound.

In Section 4 we prove Theorem B, first by showing that under assumptions (3) and (6) the Dirichlet energies converge to the Cheeger energy. This immediately implies convergence of the spectrum when X is compact. We then prove that if $\mathrm{k}_\tau(M_\alpha, g_\alpha)$ tends to zero for some fixed $\tau > 0$, the limit space is an $\mathrm{RCD}(0, n)$ space.

In Section 5, we introduce and study the quantity $\theta_x(t, s)$. We show its almost monotonicity and then obtain that, in the noncollapsing case, under assumptions (3) and (4), tangent cones are metric cones and the volume density is well-defined. Section 6 is devoted to proving Theorem A(iv). In particular, we obtain that μ–a.e. tangent cones are unique and coincides with $(\mathbb{R}^n, \mathsf{d}_e, \vartheta_X(x)\mathcal{H}^n, 0)$. In the last section we show that $\vartheta_X(x)$ is equal to one almost everywhere: we prove existence of harmonic splitting maps and as a consequence we get volume convergence.

In Appendix B we show the convergence results that are needed in Section 4, for passing to the limit with the appropriate estimates on manifolds, and in Section 7, to get the existence of ε–splitting harmonic maps with a good bound on the gradient.

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1 Preliminaries

Throughout this paper, n is a positive integer, and A is a countable, infinite, directed set like $\mathbb{N}$, for instance. We choose to denote sequences with countable infinite sets: this means that if $\{u_\alpha\}_{\alpha \in A}$ is a sequence in a topological space $(X, \mathscr{T})$, then $\{u_\alpha\}$ converges to u if and only if for any neighborhood U of u there exists a finite subset $C \subset A$ such that $\alpha \notin C$ implies $u_\alpha \in U$. Similarly, a sequence $\{u_\alpha\}_{\alpha \in A}$ admits a convergent subsequence if and only if there exists an infinite subset $B \subset A$ such that the sequence $\{u_\beta\}_{\beta \in B}$ converges.

All the manifolds we deal with in this paper are smooth and connected, and the Riemannian metrics we consider on these manifolds are all smooth. We often use the notation M^n to specify that a manifold M is n–dimensional. A Riemannian manifold is called closed if it is compact without boundary. Whenever (M, g) is a Riemannian manifold, we write d_g for its Riemannian distance, v_g for its Riemannian volume measure, Δ_g for its Laplacian operator which we choose to define as a nonnegative operator, ie

$$\int_M g(\nabla u, \nabla w)\, \mathrm{d}v_g = \int_M (\Delta_g u) w \, \mathrm{d}v_g$$

for any compactly supported smooth functions $u, w \colon M \to \mathbb{R}$.

We recall that a metric space (X, d) is called proper if all closed balls are compact and that it is called geodesic if for any $x, y \in X$ there exists a rectifiable curve γ joining x to y whose length is equal to $\mathsf{d}(x, y)$, in which case γ is called a geodesic from x to y. We also recall that the diameter of a metric space (X, d) is set as $\operatorname{diam}(X) := \sup\{\mathsf{d}(x, y) : x, y \in X\}$. If $f : X \to \mathbb{R}$ is a locally d–Lipschitz function, we define its local Lipschitz constant $\operatorname{Lip}_{\mathsf{d}} f$ by setting

$$\operatorname{Lip}_{\mathsf{d}} f(x) := \begin{cases} \displaystyle\limsup_{y \to x} \frac{|f(x) - f(y)|}{\mathsf{d}(x, y)} & \text{if } x \in X \text{ is not isolated,} \\ 0 & \text{otherwise.} \end{cases}$$

A metric measure space is a triple (X, d, μ), where (X, d) is a metric space and μ is a Radon measure which is finite and nonzero on balls with positive radius, and we write $B_r(x)$ for the open metric ball centered at $x \in X$ with radius $r > 0$, and $\bar{B}_r(x)$ for the closed metric ball. We may often implicitly consider a Riemannian manifold (M, g) as the metric measure space (M, d_g, v_g), in which case metric balls are geodesic balls.

We use standard notation to denote several classical function spaces: $L^p(X, \mu)$, $L^p_{\mathrm{loc}}(X, \mu)$, $\mathscr{C}(X)$, $\operatorname{Lip}(X)$ or $\operatorname{Lip}(X, \mathsf{d})$, $\mathscr{C}^\infty(M)$ and so on. We use the subscript c to denote the subspace of compactly supported functions of a given function space, like $\mathscr{C}_c(X)$ for compactly supported functions in $\mathscr{C}(X)$, for instance. We write $\mathscr{C}_0(X)$ for the space of continuous functions converging to 0 at infinity, which is the $L^\infty(X, \mu)$–closure of $\mathscr{C}_c(X)$.

We write $\mathbf{1}_A$ for the characteristic function of some set $A \subset X$. By $\operatorname{supp} f$ (resp. $\operatorname{supp} \mu$) we denote the support of a function f (resp. a measure μ). If f is a measurable map from a measured space (X, μ) to a measurable one Y, we write $f_\# \mu$ for the pushforward measure of μ by f. For any Borel set $A \subset X$ with finite measure and any locally integrable function $u \colon A \to \mathbb{R}$, we set

$$u_A := \fint_A u \, \mathrm{d}\mu.$$

For any $s > 0$ we write ω_s for the constant $\pi^{s/2}/\Gamma(s/2+1)$, where Γ is the usual Gamma function; as is well known, if s is an integer k, then ω_k coincides with the Hausdorff measure of the unit Euclidean ball in $\mathbb{R}^k$.

1.1 The doubling condition

Let us begin with recalling the definition of a doubling metric measure space.

Definition 1.1 Given $R \in (0, +\infty]$ and $\kappa \geq 1$, a metric measure space (X, d, μ) is called κ–doubling at scale R if for any ball $B_r(x) \subset X$ with $r \leq R$, we have

$$\mu(B_{2r}(x)) \leq \kappa\mu(B_r(x)).$$

When $R = +\infty$, we simply say that (X, d, μ) is doubling.

Doubling metric measure spaces have the following useful properties.

Proposition 1.2 *Assume that (X, d, μ) is κ–doubling at scale R for some $\kappa \geq 1$ and $R \in (0, +\infty]$, and that (X, d) is geodesic. Then there exist $c, \lambda, \delta > 0$, depending only on κ, such that*

(i) $\mu(B_r(x)) \leq c e^{\lambda \mathsf{d}(x,y)/r} \mu(B_r(y))$ *for any $x, y \in X$ and $0 < r \leq R$,*

(ii) $\mu(B_r(x)) \leq c(r/s)^\lambda \mu(B_s(x))$ *for any $x \in X$ and $0 < s \leq r \leq R$,*

(iii) $\mu(B_S(x)) \leq e^{\lambda S/s} \mu(B_s(x))$ *for any $x \in X$ and $0 < s \leq R \leq S$,*

(iv) $\mu(B_s(x)) e^{-\lambda}(r/s)^\delta \leq \mu(B_r(x))$ *for any $x \in X$ and $0 < s < r < \min\{R, \operatorname{diam}(X, \mathsf{d})/2\}$,*

(v) $\mu(B_r(x) \setminus B_{r-\tau}(x)) \leq c(\tau/r)^\delta \mu(B_r(x))$ *for any $x \in X$, $r > 0$ and $0 < \tau < \min\{r, R\}$.*

We refer to [57, Section 2.3] for the first four properties and to [32, Lemma 3.3] or [88] for the last one.

1.2 Dirichlet spaces

Let us recall now some classical notions from the theory of Dirichlet forms; we refer to [42] for details.

Let H be a Hilbert space of norm $|\cdot|_H$. We recall that a quadratic form $Q\colon H \to [0, +\infty]$ is called closed if its domain $\mathcal{D}(Q)$ equipped with the norm $|\cdot|_Q := (|\cdot|_H^2 + Q(\cdot))^{1/2}$ is a Hilbert space.

Let $(X, \mathcal{T})$ be a locally compact separable topological space equipped with a σ–finite Radon measure μ fully supported in X. A Dirichlet form on $L^2(X, \mu)$ with a dense domain $\mathcal{D}(\mathcal{E}) \subset L^2(X, \mu)$ is a nonnegative definite bilinear map $\mathcal{E}\colon \mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E}) \to \mathbb{R}$ such that $\mathcal{E}(f) := \mathcal{E}(f, f)$ is a closed quadratic form satisfying the Markov property, that is, for any $f \in \mathcal{D}(\mathcal{E})$, the function $f_0^1 = \min(\max(f, 0), 1)$ belongs to $\mathcal{D}(\mathcal{E})$ and $\mathcal{E}(f_0^1) \leq \mathcal{E}(f)$; we denote by $\langle \cdot, \cdot \rangle_{\mathcal{E}}$ the scalar product associated with $|\cdot|_{\mathcal{E}}$. We call such a quadruple $(X, \mathcal{T}, \mu, \mathcal{E})$ a Dirichlet space. When $\mathcal{T}$ is induced by a given distance d on X, we write $(X, \mathsf{d}, \mu, \mathcal{E})$ instead of $(X, \mathcal{T}, \mu, \mathcal{E})$ and call $(X, \mathsf{d}, \mu, \mathcal{E})$ a metric Dirichlet space.

Any Dirichlet form $\mathscr{E}$ is naturally associated with a nonnegative definite self-adjoint operator L with a dense domain $\mathscr{D}(L) \subset L^2(X,\mu)$ defined by

$$\mathscr{D}(L) := \left\{ f \in \mathscr{D}(\mathscr{E}) : \text{there exists } h =: Lf \in L^2(X,\mu) \text{ such that } \mathscr{E}(f,g) = \int_X hg \, \mathrm{d}\mu \text{ for all } g \in \mathscr{D}(\mathscr{E}) \right\}.$$

The spectral theorem implies that L generates an analytic sub-Markovian semigroup $(P_t := e^{-tL})_{t>0}$ acting on $L^2(X,\mu)$, where for any $f \in L^2(X,\mu)$, the map $t \mapsto P_t f$ is characterized as the unique C^1 map $(0,+\infty) \to L^2(X,\mu)$, with values in $\mathscr{D}(L)$, such that

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} P_t f = -L(P_t f) & \text{for all } t > 0, \\ \lim\limits_{t \to 0} \|P_t f - f\|_{L^2(X,\mu)} = 0. \end{cases}$$

Moreover, we get the property that when $0 \le f \le 1$ then $0 \le P_t f \le 1$. Standard functional analytic theory shows that $(P_t)_{t>0}$ extends uniquely for any $p \in [1,+\infty)$ to a strongly continuous semigroup of linear contractions in $L^p(X,\mu)$. Moreover, the spectral theorem yields a functional calculus which justifies the following estimate: for any $t > 0$ and $f \in \mathscr{D}(\mathscr{E})$,

$$\|f - P_t f\|_{L^2(X,\mu)} \le \sqrt{t}\, \mathscr{E}(f). \tag{9}$$

1.2.1 Heat kernel We call a heat kernel of $\mathscr{E}$ any function $H : (0,+\infty) \times X \times X \to \mathbb{R}$ such that for any $t > 0$ the function $H(t,\cdot,\cdot)$ is $(\mu \otimes \mu)$–measurable and

$$P_t f(x) = \int_X H(t,x,y) f(y) \, \mathrm{d}\mu(y) \quad \text{for } \mu\text{–a.e. } x \in X, \tag{10}$$

for all $f \in L^2(X,\mu)$. If $\mathscr{E}$ admits a heat kernel H, then it is nonnegative and symmetric with respect to its second and third variable, and for any $t > 0$ the function $H(t,\cdot,\cdot)$ is uniquely determined up to a $(\mu \otimes \mu)$–negligible subset of $X \times X$. Moreover, the semigroup property of $(P_t)_{t>0}$ results in the so-called Chapman–Kolmogorov property for H,

$$\int_X H(t,x,z) H(s,z,y) \, \mathrm{d}\mu(z) = H(t+s,x,y) \quad \text{for all } x,y \in X \text{ and all } s,t > 0. \tag{11}$$

The space $(X,\mathscr{T},\mu,\mathscr{E})$ — or the heat kernel H — is called stochastically complete whenever for any $x \in X$ and $t > 0$ it holds that

$$\int_X H(t,x,y) \, \mathrm{d}\mu(y) = 1. \tag{12}$$

Strongly local, regular Dirichlet spaces Let us recall now an important definition.

Definition 1.3 A Dirichlet form $\mathscr{E}$ on $L^2(X,\mu)$ is called *strongly local* if $\mathscr{E}(f,g) = 0$ for any $f,g \in \mathscr{D}(\mathscr{E})$ such that f is constant on a neighborhood of $\operatorname{supp} g$, and *regular* if $\mathscr{C}_c(X) \cap \mathscr{D}(\mathscr{E})$ contains a core, that is, a subset which is both dense in $\mathscr{C}_c(X)$ for $\|\cdot\|_\infty$ and in $\mathscr{D}(\mathscr{E})$ for $|\cdot|_{\mathscr{E}}$. If $(X,\mathscr{T},\mu,\mathscr{E})$ is a Dirichlet space where $\mathscr{E}$ is strongly local and regular, we say that $(X,\mathscr{T},\mu,\mathscr{E})$ is a strongly local, regular Dirichlet space.

By a celebrated theorem of A Beurling and J Deny [11], any strongly local, regular Dirichlet form $\mathcal{E}$ on $L^2(X,\mu)$ admits a *carré du champ*, that is, a nonnegative definite symmetric bilinear map

$$\Gamma : \mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E}) \to \mathrm{Rad},$$

where Rad is the set of signed Radon measures on $(X,\mathcal{T})$, such that

$$\mathcal{E}(f,g) = \int_X \mathrm{d}\Gamma(f,g) \quad \text{for all } \ f,g \in \mathcal{D}(\mathcal{E}),$$

where $\int_X \mathrm{d}\Gamma(f,g)$ denotes the total mass of the measure $\Gamma(f,g)$. Moreover, Γ is local, meaning that

$$\int_A \mathrm{d}\Gamma(u,w) = \int_A \mathrm{d}\Gamma(v,w)$$

holds for any open set $A \subset X$ and any $u,v,w \in \mathcal{D}(\mathcal{E})$ such that $u=v$ μ–a.e. on A. Thanks to this latter property, Γ extends to any μ–measurable function f such that for any compact set $K \subset X$ there exists $g \in \mathcal{D}(\mathcal{E})$ such that $f=g$ μ–a.e. on K; we denote by $\mathcal{D}_{\mathrm{loc}}(\mathcal{E})$ the set of such functions. Then Γ satisfies the Leibniz rule and the chain rule: if we set $\Gamma(f) := \Gamma(f,f)$,

$$\Gamma(fg) \le 2(\Gamma(f)+\Gamma(g)) \tag{13}$$

for any $f,g \in \mathcal{D}_{\mathrm{loc}}(\mathcal{E}) \cap L^\infty_{\mathrm{loc}}(X,\mu)$, and

$$\Gamma(\eta \circ h) = (\eta' \circ h)^2 \Gamma(h) \tag{14}$$

for any $h \in \mathcal{D}_{\mathrm{loc}}(\mathcal{E})$ and $\eta \in C^1(\mathbb{R})$ bounded with bounded derivative.

A final consequence of strong locality and regularity is that the operator L canonically associated to $\mathcal{E}$ satisfies the classical chain rule

$$L(\phi \circ f) = (\phi' \circ f) Lf - (\phi'' \circ f)\Gamma(f) \quad \text{for all } f \in \mathbb{G} \text{ and all } \phi \in C^\infty([0,+\infty),\mathbb{R}), \tag{15}$$

where $\mathbb{G}$ is the set of functions $f \in \mathcal{D}(L)$ such that $\Gamma(f)$ is absolutely continuous with respect to μ with density also denoted by $\Gamma(f)$. In particular:

$$Lf^2 = 2fLf - 2\Gamma(f) \quad \text{for all } \ f \in \mathbb{G}. \tag{16}$$

Intrinsic distance The carré du champ operator of a strongly local, regular Dirichlet form $\mathcal{E}$ provides an extended pseudometric structure on X given by the next definition.

Definition 1.4 The *intrinsic* extended pseudodistance $\mathsf{d}_{\mathcal{E}}$ associated with $\mathcal{E}$ is defined by

$$\mathsf{d}_{\mathcal{E}}(x,y) := \sup\{|f(x)-f(y)| : f \in \mathcal{C}(X) \cap \mathcal{D}_{\mathrm{loc}}(\mathcal{E}) \text{ such that } \Gamma(f) \le \mu\} \tag{17}$$

for any $x,y \in X$, where $\Gamma(f) \le \mu$ means that $\Gamma(f)$ is absolutely continuous with respect to μ with density lower than 1 μ–a.e. on X.

A priori, $\mathsf{d}_{\mathcal{E}}(x,y)$ may be infinite, hence we use the word "extended". Of course the case where $\mathsf{d}_{\mathcal{E}}$ does provide a metric structure on X is of special interest. In this regard, if $(X,\mathcal{T},\mu,\mathcal{E})$ is a strongly local, regular Dirichlet space where $\mathsf{d}_{\mathcal{E}}$ is a distance inducing $\mathcal{T}$, we denote it by $(X,\mathsf{d}_{\mathcal{E}},\mu,\mathcal{E})$.

1.3 The Poincaré inequality and PI Dirichlet spaces

Given $R \in (0,+\infty]$, we say that a strongly local, regular Dirichlet space $(X, \mathsf{d}_{\mathcal{E}}, \mu, \mathcal{E})$ satisfies an R–scale-invariant Poincaré inequality if there exists $\gamma > 0$ such that

$$\|u - u_B\|^2_{L^2(B)} \le \gamma r^2 \int_B \mathrm{d}\Gamma(u) \tag{18}$$

for any $u \in \mathcal{D}(\mathcal{E})$ and any ball B with radius $r \in (0, R]$. When $R = +\infty$, we simply say that $(X, \mathsf{d}_{\mathcal{E}}, \mu, \mathcal{E})$ satisfies a Poincaré inequality. The next definition is central in our work.

Definition 1.5 Given $R \in (0,+\infty]$, $\kappa \ge 1$ and $\gamma > 0$, we say that a strongly local, regular Dirichlet space $(X, \mathsf{d}_{\mathcal{E}}, \mu, \mathcal{E})$ is $\mathrm{PI}_{\kappa,\gamma}(R)$ if it satisfies the following conditions:

- $(X, \mathsf{d}_{\mathcal{E}}, \mu)$ is κ–doubling at scale R,
- $(X, \mathsf{d}_{\mathcal{E}}, \mu, \mathcal{E})$ satisfies an R–scale-invariant Poincaré inequality (18) with constant γ.

We may use the terminology $\mathrm{PI}(R)$ if no reference to the doubling or Poincaré constant is required, or even PI if we do not need to mention the scale R.

Geometry and analysis of PI–Dirichlet spaces Assume that $(X, \mathsf{d}_{\mathcal{E}}, \mu, \mathcal{E})$ is a $\mathrm{PI}_{\kappa,\gamma}(R)$–Dirichlet space for some given $R \in (0,+\infty]$, $\kappa \ge 1$ and $\gamma > 0$. According to [85], the strong locality and regularity assumptions on $\mathcal{E}$ imply that the metric space $(X, \mathsf{d}_{\mathcal{E}})$ is geodesic and that it satisfies the Hopf–Rinow theorem: it is proper if and only if it is complete. Moreover, there is a relationship between the local Lipschitz constant and the carré du champ of $\mathsf{d}_{\mathcal{E}}$–Lipschitz functions; see [66, Theorem 2.2] and [65, Lemma 2.4]: if $u \in \mathrm{Lip}(X, \mathsf{d}_{\mathcal{E}})$, then $u \in \mathcal{D}_{\mathrm{loc}}(\mathcal{E})$ and the Radon measure $\Gamma(u)$ is absolutely continuous with respect to μ; moreover, there exists a constant $\eta \in (0,1]$ depending only on κ, γ such that

$$\eta(\mathrm{Lip}_{\mathsf{d}_{\mathcal{E}}} u)^2 \le \frac{\mathrm{d}\Gamma(u)}{\mathrm{d}\mu} \le (\mathrm{Lip}_{\mathsf{d}_{\mathcal{E}}} u)^2 \quad \mu\text{–a.e. on } X. \tag{19}$$

In addition, it follows from [66, Theorem 2.2] that $\mathrm{Lip}_c(X, \mathsf{d}_{\mathcal{E}})$ is dense in $\mathcal{D}(\mathcal{E})$ and that for any $u \in \mathcal{D}(\mathcal{E})$ the Radon measure $\Gamma(u)$ is absolutely continuous with respect to μ with density $\rho_u \in L^2_{\mathrm{loc}}(X,\mu)$ comparable to the approximate Lipschitz constant of u.

For a strongly local, regular Dirichlet space $(X, \mathsf{d}_{\mathcal{E}}, \mu, \mathcal{E})$, to be $\mathrm{PI}_{\kappa,\gamma}(R)$ implies to have a Hölder continuous heat kernel H satisfying Gaussian upper and lower bounds: there exist $C_1, C_2 > 0$ depending only on κ and γ such that

$$\frac{C_2^{-1}}{\mu(B_{\sqrt{t}}(x))} e^{-C_2 \mathsf{d}^2_{\mathcal{E}}(x,y)/t} \le H(t,x,y) \le \frac{C_1}{\mu(B_{\sqrt{t}}(x))} e^{-\mathsf{d}^2_{\mathcal{E}}(x,y)/5t} \tag{20}$$

for all $t \in (0, R^2)$ and $x, y \in X$. This implication is actually an equivalence: see Theorem C.1 in Appendix C, where we provide references. In fact such a Dirichlet space satisfies the Feller property: the heat semigroup extends to a continuous semigroup on $\mathcal{C}_0(X)$.

Moreover, a PI(R)–Dirichlet space is necessarily stochastically complete: this was proved on Riemannian manifolds by A Grigor'yan [50, Theorem 9.1] and extended by K-T Sturm [84, Theorem 4] to Dirichlet spaces.

The above Gaussian upper bound can be improved to get the optimal Gaussian decay rate.

Proposition 1.6 *Let $(X, \mathsf{d}_{\mathscr{E}}, \mu, \mathscr{E})$ be a $\mathrm{PI}_{\kappa,\gamma}(R)$–Dirichlet space. Then there exist $C, \nu > 0$ depending only on κ, γ such that for any $x, y \in X$ and $t \in (0, R^2)$,*

$$H(t,x,y) \le \frac{C}{\mu(B_R(x))} \frac{R^\nu}{t^{\nu/2}} \left(1 + \frac{\mathsf{d}_{\mathscr{E}}^2(x,y)}{t}\right)^{\nu+1} e^{-\mathsf{d}_{\mathscr{E}}^2(x,y)/4t}. \tag{21}$$

Moreover, Varadhan's formula holds: for any $x, y \in X$,

$$\mathsf{d}_{\mathscr{E}}^2(x,y) = -4 \lim_{t \to 0+} t \log H(t,x,y). \tag{22}$$

The Gaussian upper bound can be found in [49, Theorem 5.2] (see also [79; 33] for optimal versions) and Varadhan's formula is due to ter Elst, D Robinson and A Sikora [39] (see also [74] for an earlier result).

1.4 Notions of convergence

We provide now our working definitions of convergence of spaces and of points, functions, bounded operators and Dirichlet forms defined on varying spaces.

1.4.1 Convergence of spaces

Let us start with some classical definitions.

Pointed Gromov–Hausdorff convergence For any $\varepsilon > 0$, an ε–isometry between two metric spaces (X, d) and (X', d') is a map $\Phi \colon X \to X'$ such that $|\mathsf{d}(x_0,x_1) - \mathsf{d}'(\Phi(x_0), \Phi(x_1))| < \varepsilon$ for any $x_0, x_1 \in X$ and $X' = \bigcup_{x \in X} B_\varepsilon(\Phi(x))$. A sequence of pointed metric spaces $\{(X_\alpha, \mathsf{d}_\alpha, o_\alpha)\}_\alpha$ converges in the pointed Gromov–Hausdorff topology (pGH for short) to another pointed metric space (X, d, o) if there exist two sequences $\{R_\alpha\}_\alpha, \{\varepsilon_\alpha\}_\alpha \subset (0, +\infty)$ such that $R_\alpha \uparrow +\infty$, $\varepsilon_\alpha \downarrow 0$, and, for any α, an ε_α–isometry $\Phi_\alpha \colon B_{R_\alpha}(o_\alpha) \to B_{R_\alpha}(o)$ such that $\Phi_\alpha(o_\alpha) = o$. We denote this by

$$(X_\alpha, \mathsf{d}_\alpha, o_\alpha) \xrightarrow{\text{pGH}} (X, \mathsf{d}, o).$$

Pointed measured Gromov–Hausdorff convergence Let us assume that the spaces $\{(X_\alpha, \mathsf{d}_\alpha, o_\alpha)\}_\alpha$ and (X, d, o) are equipped with Radon measures $\{\mu_\alpha\}_\alpha$ and μ, respectively. Then the sequence of pointed metric measure spaces $\{(X_\alpha, \mathsf{d}_\alpha, \mu_\alpha, o_\alpha)\}_\alpha$ converges to (X, d, μ, o) in the pointed measured Gromov–Hausdorff topology (pmGH for short) if there exist two sequences $\{R_\alpha\}_\alpha, \{\varepsilon_\alpha\}_\alpha \subset (0, +\infty)$ such that $R_\alpha \uparrow +\infty$, $\varepsilon_\alpha \downarrow 0$ and, for any α, an ε_α–isometry $\Phi_\alpha \colon B_{R_\alpha}(o_\alpha) \to B_{R_\alpha}(o)$ such that $\Phi_\alpha(o_\alpha) = o$ and

$$(\Phi_\alpha)_\# \mu_\alpha \rightharpoonup \mu,$$

where we recall that $(\Phi_\alpha)_\#\mu_\alpha \rightharpoonup \mu$ means that for any $\varphi \in \mathscr{C}_c(X)$,

$$\lim_\alpha \int_{X_\alpha} \varphi \circ \Phi_\alpha \, \mathrm{d}\mu_\alpha = \int_X \varphi \, \mathrm{d}\mu.$$

We denote this by $(X_\alpha, \mathsf{d}_\alpha, \mu_\alpha, o_\alpha) \xrightarrow{\text{pmGH}} (X, \mathsf{d}, \mu, o)$.

Precompactness results Let (X, d, μ, o) be a pointed metric measure space which is doubling at scale $R < +\infty$. Assume that there exists some $\eta \geq 1$ such that

$$\eta^{-1} \leq \mu(B_R(o)) \leq \eta.$$

Then the doubling condition at scale R implies

$$\eta^{-1} \leq \mu(B_{R_+}(o)) \leq e^{\lambda R_+/R}\eta$$

for any $R_+ \geq R$. This remark allows us to apply a well-known precompactness theorem for pointed metric measure spaces (see [94, Theorem 27.32(ii)]), which is a consequence of Gromov's celebrated result (see [54, Proposition 5.2 and Exercise (a) on page 118]).

Proposition 1.7 *For any $R > 0$ and $\kappa, \eta \geq 1$, the space of pointed proper geodesic metric measure spaces (X, d, μ, o) satisfying*

$$(X, \mathsf{d}, \mu) \quad \text{is } \kappa\text{–doubling at scale } R, \tag{23}$$

$$\eta^{-1} \leq \mu(B_R(o)) \leq \eta, \tag{24}$$

is compact in the pointed measured Gromov–Hausdorff topology, ie for every sequence of pointed proper geodesic metric measure spaces $\{(X_\alpha, \mathsf{d}_\alpha, \mu_\alpha, o_\alpha)\}_\alpha$ satisfying (23) and (24), there exist a subsequence $B \subset A$ and a pointed proper geodesic metric measure space (X, d, μ, o) satisfying (23) and (24) such that

$$(X_\beta, \mathsf{d}_\beta, \mu_\beta, o_\beta) \xrightarrow{\text{pmGH}} (X, \mathsf{d}, \mu, o).$$

We point out that Gromov's precompactness theorem is usually stated for complete, locally compact, length metric spaces [54, Proposition 5.2], but the Hopf–Rinow theorem ensures that these assumptions are equivalent to being proper and geodesic.

Note that the doubling condition is stable with respect to multiplication of the measure by a constant factor. Therefore, if $\{(X_\alpha, \mathsf{d}_\alpha, \mu_\alpha, o_\alpha)\}_\alpha$ is a sequence of pointed proper geodesic metric measure spaces satisfying (23) but not (24), we may rescale each measure μ_α into $m_\alpha \mu_\alpha$ for some $m_\alpha > 0$ in such a way that the sequence $\{m_\alpha^{-1}\mu_\alpha(B_R(o_\alpha)) + m_\alpha \mu_\alpha^{-1}(B_R(o_\alpha))\}_\alpha$ is bounded; then $\{(X_\alpha, \mathsf{d}_\alpha, m_\alpha^{-1}\mu_\alpha, o_\alpha)\}_\alpha$ admits a pmGH convergent subsequence. We can choose $m_\alpha = \mu_\alpha(B_R(o_\alpha))$, for instance.

Tangent cones of doubling spaces We recall the classical definition of a tangent cone.

Definition 1.8 Let (X, d, μ) be a metric measure space and $x \in X$. The pointed metric space (Y, d_Y, x) is a *tangent cone* of X at x if there exists a sequence $\{\varepsilon_\alpha\}_{\alpha \in A} \subset (0, +\infty)$ such that $\varepsilon_\alpha \downarrow 0$ and

$$(X, \varepsilon_\alpha^{-1}\mathsf{d}, x) \xrightarrow{\text{pGH}} (Y, \mathsf{d}_Y, x);$$

it can always be equipped with a limit measure μ_Y such that, up to a subsequence,

$$\big(X, \varepsilon_\alpha^{-1}\mathsf{d}, \mu(B^{\mathsf{d}}_{\varepsilon_\alpha}(y))^{-1}\mu, x\big) \xrightarrow{\text{pmGH}} (Y, \mathsf{d}_Y, \mu_Y, x). \tag{25}$$

The pointed metric measure space $(Y, \mathsf{d}_Y, \mu_Y, x)$ is then called a *measured tangent cone* of X at x. If $(Y, \mathsf{d}_Y, \mu_Y, y)$ is a measured tangent cone of X at x and $y \in Y$, we refer to a measured tangent cone $(Z, \mathsf{d}_Z, \mu_Z, y)$ of Y at y as an *iterated measured tangent cone* of X.

Remark 1.9 We often use $(X_x, \mathsf{d}_x, \mu_x, x)$ to denote a measured tangent cone of (X, d, μ) at x.

As is well known, on a geodesic proper metric measure space (X, d, μ) that is κ–doubling at scale R for some $\kappa \geq 1$ and $R \in (0, +\infty)$, the existence of measured tangent cones at any point x is guaranteed and any of these measured tangent cones is κ–doubling. Indeed, for any $\varepsilon > 0$, the rescaled space $(X, \varepsilon^{-1}\mathsf{d}, \mu)$ is κ–doubling at scale R/ε. Hence when $\varepsilon \leq R$, the rescaled space $(X, \varepsilon^{-1}\mathsf{d}, \mu)$ is κ–doubling at scale 1. Hence Proposition 1.7 applies to the rescaled spaces $\big\{\big(X, \varepsilon^{-1}\mathsf{d}, \mu(B_\varepsilon(x))^{-1}\mu, x\big)\big\}_{\varepsilon > 0}$ and yields the existence of measured tangent cones which are κ–doubling at any scale $S \geq 1$.

When for some $m > 0$ the space (X, d, μ) additionally satisfies a (local) m–Ahlfors regularity condition, ie for each $\rho > 0$ there exists $c_\rho > 0$ such that for any $x \in X$, any $r \in (0, 1)$ and $y \in B_\rho(x)$,

$$c_\rho r^m \leq \mu(B_r(y)) \leq r^m / c_\rho,$$

then it is convenient to rescale the measure by ε^{-m} to study measured tangent cones. In this case, the tangent measures are only changed by a positive multiplicative constant.

1.4.2 Convergence of points and functions A natural way to formalize the notions of convergence of points and functions defined on varying spaces is the following. We let $\{(X_\alpha, \mathsf{d}_\alpha, \mu_\alpha, o_\alpha)\}_\alpha$ and (X, d, μ, o) be proper pointed metric measure spaces such that

$$(X_\alpha, \mathsf{d}_\alpha, \mu_\alpha, o_\alpha) \xrightarrow{\text{pmGH}} (X, \mathsf{d}, \mu, o). \tag{26}$$

As the ε_α–isometries between X_α and X are not unique (they can be composed for instance with isometries of X_α or X), we make a specific choice by using the following characterization:

Characterization 1 *The pmGH convergence (26) holds if and only if there exist sequences $\{R_\alpha\}_\alpha$ and $\{\varepsilon_\alpha\}_\alpha \subset (0, +\infty)$ with $R_\alpha \uparrow +\infty$ and $\varepsilon_\alpha \downarrow 0$, and ε_α–isometries $\Phi_\alpha : B_{R_\alpha}(o_\alpha) \to B_{R_\alpha}(o)$ such that*

(i) $\Phi_\alpha(o_\alpha) = o$,

(ii) $(\Phi_\alpha)_\# \mu_\alpha \rightharpoonup \mu$.

Until the end of this section, we work with a fixed family of ε_α–isometries provided by the previous characterization.

Convergence of points Let $x_\alpha \in X_\alpha$ for any α and $x \in X$ be given. We say that the sequence of points $\{x_\alpha\}_\alpha$ converges to x if $\mathsf{d}(\Phi_\alpha(x_\alpha), x) \to 0$. We denote this by $x_\alpha \to x$.

Uniform convergence Let $u_\alpha \in \mathscr{C}(X_\alpha)$ for any α and $u \in \mathscr{C}(X)$ be given. We say that the sequence of functions $\{u_\alpha\}_\alpha$ converges uniformly on compact sets to u if $\|u_\alpha - u \circ \Phi_\alpha\|_{L^\infty(B(o_\alpha,R))} \to 0$ for any $R > 0$. It is easy to show the following useful criterion for uniform convergence on compact sets.

Proposition 1.10 *Let $u_\alpha \in \mathscr{C}(X_\alpha)$ and $u \in \mathscr{C}(X)$ be given. Then $\{u_\alpha\}_\alpha$ converges uniformly on compact sets to u if and only if $u_\alpha(x_\alpha) \to u(x)$ whenever $x_\alpha \to x$.*

In the case that $\varphi_\alpha \in \mathscr{C}_c(X_\alpha)$ for any α and $\varphi \in \mathscr{C}_c(X)$, we write

$$\varphi_\alpha \xrightarrow{\mathscr{C}_c} \varphi$$

if there is an $R > 0$ such that $\operatorname{supp} \varphi_\alpha \subset B_R(o_\alpha)$ for any α large enough and if $\{\varphi_\alpha\}_\alpha$ converges uniformly to φ. When the spaces $\{(X_\alpha, \mathsf{d}_\alpha, \mu_\alpha)\}_\alpha$, (X, d, μ) are all κ–doubling at scale R, then for every $\varphi \in \mathscr{C}_c(X)$ we can build functions $\varphi_\alpha \in \mathscr{C}_c(X_\alpha)$ such that $\varphi_\alpha \xrightarrow{\mathscr{C}_c} \varphi$: see Proposition A.1 in Appendix A.

Weak L^p–convergence Let $p \in (1, +\infty)$. Let $f_\alpha \in L^p(X_\alpha, \mu_\alpha)$ for any α and $f \in L^p(X, \mu)$ be given. We say that the sequence of functions $\{f_\alpha\}_\alpha$ converges weakly in L^p to f, and we write

$$f_\alpha \overset{L^p}{\rightharpoonup} f$$

if $\sup_\alpha \|f_\alpha\|_{L^p} < +\infty$ and

$$\varphi_\alpha \xrightarrow{\mathscr{C}_c} \varphi \implies \int_{X_\alpha} \varphi_\alpha f_\alpha \, \mathrm{d}\mu_\alpha = \int_X \varphi f \, \mathrm{d}\mu.$$

We have the following compactness result:

Proposition 1.11 *If $\sup_\alpha \|f_\alpha\|_{L^p} < +\infty$, then there exists a subsequence $B \subset A$ and $f \in L^p(X, \mu)$ such that $f_\beta \overset{L^p}{\rightharpoonup} f$.*

Strong L^p–convergence and duality Let $p \in (1, +\infty)$. Let $f_\alpha \in L^p(X_\alpha, \mu_\alpha)$ and $f \in L^p(X, \mu)$ be given. We say that the sequence of functions $\{f_\alpha\}_\alpha$ converges strongly in L^p to f, and we write

$$f_\alpha \xrightarrow{L^p} f,$$

if $f_\alpha \overset{L^p}{\rightharpoonup} f$ and $\|f_\alpha\|_{L^p} \to \|f\|_{L^p}$. For every $f \in L^p(X, \mu)$, we can build functions $f_\alpha \in L^p(X_\alpha, \mu_\alpha)$ converging to f strongly in L^p: this follows from approximating f with functions $\{f_i\} \subset \mathscr{C}_c(X)$, approximating each f_i with functions $f_{i,\alpha} \subset \mathscr{C}_c(X_\alpha)$ as mentioned before, and using a diagonal argument.

Moreover there is a duality between weak convergence in L^p and strong convergence in L^q when p and q are conjugate exponents, as detailed in the next proposition.

Proposition 1.12 *Let $p, q \in (1, +\infty)$ be satisfying $1/p + 1/q = 1$. Consider $f_\alpha \in L^p(X_\alpha, \mu_\alpha)$ for any α and $f \in L^p(X, \mu)$. Then*

- *$f_\alpha \xrightarrow{L^p} f$ if and only if $\varphi_\alpha \overset{L^q}{\rightharpoonup} \varphi$ implies $\int_{X_\alpha} \varphi_\alpha f_\alpha \,\mathrm{d}\mu_\alpha = \int_X \varphi f \,\mathrm{d}\mu$,*
- *$f_\alpha \overset{L^p}{\rightharpoonup} f$ if and only if $\varphi_\alpha \xrightarrow{L^q} \varphi$ implies $\int_{X_\alpha} \varphi_\alpha f_\alpha \,\mathrm{d}\mu_\alpha = \int_X \varphi f \,\mathrm{d}\mu$.*

Convergence of bounded operators If $B_\alpha : L^2(X_\alpha, \mu_\alpha) \to L^2(X_\alpha, \mu_\alpha)$ and $B : L^2(X, \mu) \to L^2(X, \mu)$ are bounded linear operators, we say that $\{B_\alpha\}_\alpha$ converges weakly to B if

$$f_\alpha \overset{L^2}{\rightharpoonup} f \implies B_\alpha f_\alpha \overset{L^2}{\rightharpoonup} Bf,$$

and we say that $\{B_\alpha\}_\alpha$ converges strongly to B if

$$f_\alpha \xrightarrow{L^2} f \implies B_\alpha f_\alpha \xrightarrow{L^2} Bf.$$

By duality, $\{B_\alpha\}_\alpha$ converges weakly to B if and only if the sequence of the adjoint operators $\{B_\alpha^*\}_\alpha$ converges strongly to the adjoint operator B^*. In particular, if the operators B_α and B are all self-adjoint, weak and strong convergences are equivalent.

Convergence in energy When each metric measure space is endowed with a Dirichlet form so that $\{(X_\alpha, \mathsf{d}_\alpha, \mu_\alpha, \mathcal{E}_\alpha)\}_\alpha$ and $(X, \mathsf{d}, \mu, \mathcal{E})$ are metric Dirichlet spaces, we can similarly define convergence in energy of functions. Let $f_\alpha \in \mathcal{D}(\mathcal{E}_\alpha)$ and $f \in \mathcal{D}(\mathcal{E})$ be given. We say that the sequence $\{f_\alpha\}_\alpha$ converges weakly in energy to f, and we write

$$f_\alpha \overset{\mathrm{E}}{\rightharpoonup} f,$$

if $f_\alpha \overset{L^2}{\rightharpoonup} f$ and $\sup_\alpha \mathcal{E}_\alpha(f_\alpha) < +\infty$. We say that $\{f_\alpha\}_\alpha$ converges strongly in energy to f, and we write

$$f_\alpha \xrightarrow{\mathrm{E}} f,$$

if $f_\alpha \xrightarrow{L^2} f$ and $\lim_\alpha \mathcal{E}_\alpha(f_\alpha) = \mathcal{E}(f)$.

Using the nonnegative selfadjoint operator L_α (resp. L) associated to $\mathcal{E}_\alpha$ (resp. to $\mathcal{E}$), we have

$$f_\alpha \overset{\mathrm{E}}{\rightharpoonup} f \iff (1 + L_\alpha)^{1/2} f_\alpha \overset{L^2}{\rightharpoonup} (1 + L)^{1/2} f,$$

$$f_\alpha \xrightarrow{\mathrm{E}} f \iff (1 + L_\alpha)^{1/2} f_\alpha \xrightarrow{L^2} (1 + L)^{1/2} f.$$

Remark 1.13 All the above definitions have also a localized version, where each function f_α is defined only on a ball centered at o_α with a fixed radius. For instance, for a given $\rho > 0$, if $f_\alpha \in L^p(B_\rho(o_\alpha))$ and $f \in L^p(B_\rho(o))$, we say that the sequence $\{f_\alpha\}$ converges weakly to f in $L^p(B_\rho)$, and we write

$$f_\alpha \overset{L^p(B_\rho)}{\rightharpoonup} f,$$

provided $\sup_\alpha \int_{B_\rho(o_\alpha)} |f_\alpha|^p \,\mathrm{d}\mu_\alpha < \infty$ and

$$\varphi_\alpha \xrightarrow{\mathcal{E}_c} \varphi \implies \lim_\alpha \int_{X_\alpha} \varphi_\alpha f_\alpha \,\mathrm{d}\mu_\alpha = \int_X \varphi f \,\mathrm{d}\mu.$$

Similarly, we define L^p_{loc}–convergence of functions through pmGH convergence of spaces in the following way: if $f_\alpha \in L^p_{\mathrm{loc}}(X_\alpha, \mu_\alpha)$ for any α and $f \in L^p_{\mathrm{loc}}(X, \mu)$, we say that the sequence $\{f_\alpha\}_\alpha$ converges weakly to f in L^p_{loc}, and we write

$$f_\alpha \xrightarrow{L^p_{\mathrm{loc}}} f$$

if, for any $\rho > 0$,

$$f_\alpha|_{B_\rho(o_\alpha)} \xrightarrow{L^p(B_\rho)} f|_{B_\rho(o)}.$$

1.4.3 Mosco convergence We recall the following notion of convergence that was introduced by U Mosco in [71] for quadratic forms. We formulate it if for Dirichlet forms as this is sufficient for our purposes.

Definition 1.14 Let $\{(X_\alpha, \mathsf{d}_\alpha, \mu_\alpha, \mathcal{E}_\alpha)\}_\alpha$ and $(X, \mathsf{d}, \mu, \mathcal{E})$ be metric Dirichlet spaces. We say that the sequence of Dirichlet forms $\{\mathcal{E}_\alpha\}_\alpha$ converges to $\mathcal{E}$ in the Mosco sense if the two following properties hold:

(i) For any sequence $\{u_\alpha\}_\alpha$ in $\mathcal{D}(\mathcal{E}_\alpha)$ and any $u \in \mathcal{D}(\mathcal{E})$,

$$u_\alpha \xrightarrow{L^2} u \implies \mathcal{E}(u) \leq \liminf_\alpha \mathcal{E}(u_\alpha).$$

(ii) For any $u \in \mathcal{D}(\mathcal{E})$, there exists a sequence $\{u_\alpha\}_\alpha$ in $\mathcal{D}(\mathcal{E}_\alpha)$ such that $u_\alpha \xrightarrow{\mathrm{E}} u$.

Mosco convergence of Dirichlet forms is equivalent to the convergence of many related objects: this follows from [67, Theorem 2.4]. Recall that for a sequence of self-adjoint operators $\{B_\alpha\}_\alpha$ weak and strong convergence are equivalent.

Proposition 1.15 *Let $\{(X_\alpha, \mathsf{d}_\alpha, \mu_\alpha, \mathcal{E}_\alpha)\}_\alpha$ and $(X, \mathsf{d}, \mu, \mathcal{E})$ be metric Dirichlet spaces. For any α let L_α (resp. L) be the nonnegative self-adjoint operator associated with $\mathcal{E}_\alpha$ (resp. $\mathcal{E}$) and let $(P^\alpha_t)_{t>0}$ (resp. $(P_t)_{t>0}$) be the generated semigroup.Then the following statements are equivalent.*

(i) *$\mathcal{E}_\alpha \to \mathcal{E}$ in the Mosco sense.*

(ii) *There exists $t > 0$ such that the sequence of bounded operators $\{P^\alpha_t\}_\alpha$ strongly/weakly converges to P_t.*

(iii) *For all $t > 0$, the sequence of bounded operators $\{P^\alpha_t\}_\alpha$ strongly/weakly converges to P_t.*

(iv) *For any smooth function $\xi\colon [0, +\infty) \to \mathbb{R}$ with $\operatorname{supp} \xi \subset [0, R]$ for some $R > 0$, the sequence of operators $\{\xi(L_\alpha)\}$ strongly converges to $\xi(L)$.*

(v) *For any sequence $\{\xi_\alpha\colon [0, +\infty) \to \mathbb{R}\}$ of continuous functions vanishing at infinity which converges uniformly to a continuous function $\xi\colon [0, +\infty) \to \mathbb{R}$ vanishing at infinity, the sequence of operators $\{\xi_\alpha(L_\alpha)\}$ strongly converges to $\xi(L)$.*

Definition 1.16 Let $\{(X_\alpha, \mathsf{d}_\alpha, \mu_\alpha, \mathcal{E}_\alpha, o_\alpha)\}_\alpha$ and $(X, \mathsf{d}, \mu, \mathcal{E}, o)$ be pointed metric Dirichlet spaces. We say that the sequence $\{(X_\alpha, \mathsf{d}_\alpha, \mu_\alpha, \mathcal{E}_\alpha, o_\alpha)\}_\alpha$ converges to $(X, \mathsf{d}, \mu, \mathcal{E}, o)$ in the *pointed Mosco–Gromov–Hausdorff* sense if

$$(X_\alpha, \mathsf{d}_\alpha, \mu_\alpha, o_\alpha) \xrightarrow{\text{pmGH}} (X, \mathsf{d}, \mu, o) \qquad \text{and} \qquad \mathcal{E}_\alpha \to \mathcal{E} \quad \text{in the Mosco sense.}$$

We note that

$$(X_\alpha, \mathsf{d}_\alpha, \mu_\alpha, \mathcal{E}_\alpha, o_\alpha) \xrightarrow{\text{pMGH}} (X, \mathsf{d}, \mu, \mathcal{E}, o).$$

1.5 A compactness result for Dirichlet spaces

The next theorem is a key tool in our analysis. It was already observed by Kasue [62, Theorem 3.4] and extended by Kuwae and Shioya [67, Theorem 5.2], who proved that the limit space is regular and strongly local. We refer to Appendix D for the proof.

Theorem 1.17 *Let $\kappa, \eta \geq 1$, $\gamma > 0$ and $R \in (0, +\infty]$ be given. Assume that $\{(X_\alpha, \mathsf{d}_{\mathcal{E}_\alpha}, \mu_\alpha, o_\alpha, \mathcal{E}_\alpha)\}_{\alpha \in A}$ is a sequence of complete $\mathrm{PI}_{\kappa,\gamma}(R)$ pointed Dirichlet spaces such that, for all α,*

$$\eta^{-1} \leq \mu_\alpha(B_R(o_\alpha)) \leq \eta.$$

Then there exist a complete pointed metric Dirichlet space $(X, \mathsf{d}, \mu, o, \mathcal{E})$ and a subsequence $B \subset A$ such that $\{(X_\beta, \mathsf{d}_{\mathcal{E}_\beta}, \mu_\beta, o_\beta, \mathcal{E}_\beta)\}_{\beta \in B}$ Mosco–Gromov–Hausdorff converges to $(X, \mathsf{d}, \mu, o, \mathcal{E})$; moreover, $\mathcal{E}$ is regular, strongly local, the intrinsic pseudodistance $\mathsf{d}_\mathcal{E}$ is a distance, and there is a constant $c \in (0, 1]$, depending only on κ and γ, such that

$$c\,\mathsf{d}_\mathcal{E} \leq \mathsf{d} \leq \mathsf{d}_\mathcal{E}.$$

Furthermore, the space $(X, \mathsf{d}_\mathcal{E}, \mu, \mathcal{E})$ is $\mathrm{PI}_{\kappa,\gamma'}(R)$ for some constant $\gamma' \geq 1$. In addition, if H_β (resp. H) is the heat kernel of $(X_\beta, \mathsf{d}_\beta, \mu_\beta, \mathcal{E}_\beta)$ (resp. $(X, \mathsf{d}, \mu, o, \mathcal{E})$) for any β, then for any $t > 0$,

$$H_\beta(t, \cdot, \cdot) \to H(t, \cdot, \cdot) \quad \textit{uniformly on compact sets,} \tag{27}$$

where we make an implicit use of the obvious convergence

$$(X_\beta \times X_\beta, \mathsf{d}_\beta \otimes \mathsf{d}_\beta, (o_\beta, o_\beta)) \xrightarrow{\text{pGH}} (X \times X, \mathsf{d} \times \mathsf{d}, (o, o)).$$

If $(X, \mathsf{d}_\mathcal{E}, \mu, \mathcal{E})$ is a strongly local and regular Dirichlet space satisfying an R–scale-invariant Poincaré inequality for some $R > 0$, one can check that for any $x \in X$ and any $\rho > 0$, the rescaled quadratic form $\mathcal{E}_\rho := \rho^2 \mu(B_\rho(x))^{-1} \mathcal{E}$ is a strongly local and regular Dirichlet form on $\big(X, \mathsf{d}_\mathcal{E}, \mu_\rho := \mu(B_\rho(x))^{-1}\mu\big)$ such that $\mathsf{d}_{\mathcal{E}_\rho} = \rho^{-1}\mathsf{d}_\mathcal{E}$, and the space $(X, \mathsf{d}_{\mathcal{E}_\rho}, \mu_\rho, \mathcal{E}_\rho)$ satisfies a (R/ρ)–scale-invariant Poincaré inequality. This observation coupled with the previous theorem leads to the next result.

Corollary 1.18 *Let $(X, \mathsf{d}_{\mathcal{E}}, \mu, \mathcal{E})$ be a complete $\mathrm{PI}_{\kappa,\gamma}(R)$–Dirichlet space and $x \in X$. If $(X_x, \mathsf{d}_x, \mu_x, x)$ is a measured tangent cone of $(X, \mathsf{d}_{\mathcal{E}}, \mu)$ at x, then it can be endowed with a strongly local and regular Dirichlet form $\mathcal{E}_x$ such that $\mathsf{d}_{\mathcal{E}_x}$ is a distance bi-Lipschitz equivalent to d_x, and the space $(X_x, \mathsf{d}_{\mathcal{E}_x}, \mu_x, \mathcal{E}_x)$ is $\mathrm{PI}_{\kappa,\gamma'}(\infty)$ for some γ'. Moreover, there exists a sequence $\{\rho_\alpha\} \subset (0, +\infty)$ such that $\rho_\alpha \to 0$ and*

$$(X, \mathsf{d}_{\mathcal{E}_{\rho_\alpha}}, \mu_{\rho_\alpha}, x, \mathcal{E}_{\rho_\alpha}) \xrightarrow{\text{pMGH}} (X_x, \mathsf{d}_{\mathcal{E}_x}, \mu_x, x, \mathcal{E}_x),$$

where, for any α,

$$\mu_{\rho_\alpha} := \mu(B_{\rho_\alpha}(x))^{-1}\mu \quad \text{and} \quad \mathcal{E}_{\rho_\alpha} := \rho_\alpha^2 \mu(B_{\rho_\alpha}(x))^{-1}\mathcal{E}.$$

Note that different sequences could lead to different Dirichlet forms on the same measured tangent cone.

1.6 Dirichlet spaces satisfying an RCD condition

Let us conclude these preliminaries with some facts concerning the Riemannian Curvature Dimension condition $\mathrm{RCD}^*(K,n)$, where $K \in \mathbb{R}$ is fixed from now on, in the setting of Dirichlet spaces.

The Cheeger energy [21] of a metric measure space (X, d, μ) is the convex and $L^2(X,\mu)$–lower semicontinuous functional $\mathsf{Ch}\colon L^2(X,\mu) \to [0,+\infty]$ defined by

$$\mathsf{Ch}(f) = \inf_{f_n \to f} \left\{ \liminf_{n\to+\infty} \int_X \mathrm{Lip}_{\mathsf{d}}^2(f_n)\, \mathrm{d}\mu \right\} \tag{28}$$

for any $f \in L^2(X,\mu)$, where the infimum is taken over the set of sequences $\{f_n\}_n \subset L^2(X,\mu) \cap \mathrm{Lip}(X)$ such that $\|f_n - f\|_{L^2(X,\mu)} \to 0$. We set

$$H^{1,2}(X, \mathsf{d}, \mu) := \mathcal{D}(\mathsf{Ch}) = \{\mathsf{Ch} < +\infty\}$$

and call $H^{1,2}(X,\mathsf{d},\mu)$ the Sobolev space of (X,d,μ). A suitable diagonal argument shows that for any $f \in H^{1,2}(X,\mathsf{d},\mu)$ there exists a unique L^2–function $|df|$ called *minimal relaxed slope* of f such that

$$\mathsf{Ch}(f) = \int_X |df|^2\, \mathrm{d}\mu$$

and $|df| = |dg|$ μ–a.e. on $\{f = g\}$ for any $g \in H^{1,2}(X,\mathsf{d},\mu)$. Moreover, this function $|df|$ coincides μ–a.e. with the local Lipschitz function of f in case f is locally Lipschitz.

There is no reason a priori for the Cheeger energy to be a Dirichlet form or even a quadratic form. In this respect, we provide the next definition and the subsequent proposition, which are taken from [47] and [3].

Definition 1.19 A Polish metric measure space (X,d,μ) is called infinitesimally Hilbertian if Ch is a quadratic form.

Proposition 1.20 *Let (X,d,μ) be an infinitesimally Hilbertian space. Then $H^{1,2}(X,\mathsf{d},\mu)$ endowed with the norm $\|\cdot\|_{H^{1,2}} = \sqrt{\|\cdot\|_{L^2} + \mathsf{Ch}(\cdot)}$ is a Hilbert space. Moreover, the Cheeger energy Ch is a strongly local and regular Dirichlet form; its carré du champ operator takes values in the set of absolutely continuous Radon measures, and for any $f_1, f_2 \in H^{1,2}(X,\mathsf{d},\mu)$,*

$$\langle df_1, df_2\rangle := \frac{\mathrm{d}\Gamma(f_1,f_2)}{\mathrm{d}\mu} = \lim_{\epsilon\to 0}\frac{|d(f_1+\epsilon f_2)|^2 - |df_1|^2}{2\epsilon}$$

in $L^1(X,\mu)$. In particular, $\mathrm{d}\Gamma(f) = |df|^2\,\mathrm{d}\mu$ for any $f \in H^{1,2}(X,\mathsf{d},\mu)$.

When (X,d,μ) is infinitesimally Hilbertian, we call Laplacian of (X,d,μ) the nonnegative, self-adjoint operator associated to Ch, and we denote it by Δ. We also write $(e^{-t\Delta})_{t\geq 0}$ for the semigroup generated by Δ.

For the scope of our work, we must know under which conditions the Dirichlet form $\mathscr{E}$ of a Dirichlet space $(X,\mathsf{d},\mu,\mathscr{E})$ coincides with the Cheeger energy of (X,d,μ). The next result brings us such a condition in the context of PI Dirichlet spaces; it follows from [65, Theorem 4.1].

Proposition 1.21 *Let $(X,\mathsf{d}_{\mathscr{E}},\mu,\mathscr{E})$ be a PI–Dirichlet space. Assume that for some $T>0$ there exists a locally bounded function $\kappa\colon [0,T]\to[0,+\infty)$ such that $\liminf_{t\to 0}\kappa(t) = 1$ and*

$$\int_X \varphi\,\mathrm{d}\Gamma(P_t u) \leq \kappa(t)\int_X P_t\varphi\,\mathrm{d}\Gamma(u) \tag{29}$$

for all $u\in\mathscr{D}(\mathscr{E})$, nonnegative $\varphi\in\mathscr{D}(\mathscr{E})\cap\mathscr{C}_c(X)$ and $t\in[0,T]$. Then $\mathsf{Ch}=\mathscr{E}$.

To perform our analysis in Sections 5 and 6, we need some results from the theory of spaces satisfying a Riemannian Curvature Dimension condition $\mathrm{RCD}^*(K,n)$. In our setting, the original formulation of the $\mathrm{RCD}(K,n)$ and $\mathrm{RCD}^*(K,n)$ conditions based on optimal transport [87; 68; 6; 3; 47] is less relevant than the one provided by a suitable combination of [4] and [40, Section 5]. In a general framework which covers our needs, these two articles discuss how to recover a distance d from a measure space (X,μ) equipped with a suitable Dirichlet energy $\mathscr{E}$ in such a way that (X,d,μ) is an $\mathrm{RCD}^*(K,n)$ space. When particularized to our context, these articles provide us with the following definition.

Definition 1.22 A PI–Dirichlet space $(X,\mathsf{d}_{\mathscr{E}},\mu,\mathscr{E})$ is called an $\mathrm{RCD}^*(K,n)$ space if and only if there exists $T>0$ such that for any $t\in(0,T)$ and $f\in\mathscr{D}(\mathscr{E})$,

$$\tfrac{1}{2}(P_t f^2 - (P_t f)^2) \geq I_K(t)|dP_t f|^2 + J_K(t)\frac{(\Delta P_t f)^2}{n} \tag{BL(K,n)}$$

holds in a weak sense, namely against any nonnegative test function $\varphi\in\mathscr{C}_c(X)\cap\mathscr{D}(\mathscr{E})$, where $I_K(t) := (e^{2K}t-1)/2K$ and $J_K(t) := (e^{2Kt}-2Kt-1)/4K^2$ for any $K\neq 0$ and $I_0(t)=t$, $J_0(t)=t^2/2$.

Remark 1.23 Inequality (BL(K,n)) is one of many equivalent forms of an estimate due to Bakry and Ledoux [7] which is equivalent to the well-known Bakry–Émery condition [4, Corollary 2.3].

To conclude this section, let us consider an $\mathrm{RCD}(0,n)$ space (X,d,μ) — we point out that in the case $K=0$, the $\mathrm{RCD}(K,n)$ and $\mathrm{RCD}^*(K,n)$ conditions are equivalent. The Bishop–Gromov theorem for $\mathrm{RCD}(0,n)$ spaces (known even for the broader class of $\mathrm{CD}(0,n)$ spaces, see [94, Theorem 30.11]) ensures that for any $x\in X$ the volume ratio $r\mapsto \mu(B(x,r))/r^n$ is nonincreasing, hence we can define the *volume density* at a point x as follows.

Definition 1.24 Let (X,d,μ) be an $\mathrm{RCD}(0,n)$ space. Then the volume density at $x\in X$ is defined as

$$\vartheta_X(x) := \lim_{r\to 0}\frac{\mu(B(x,r))}{\omega_n r^n}\in(0,+\infty].$$

Note that without any particular assumption $\vartheta_X(x)$ may be infinite. Gigli and De Philippis [38] introduced an important definition.

Definition 1.25 We say that an $\mathrm{RCD}(0,n)$ space (X,d,μ) is weakly noncollapsed if the volume density $\vartheta_X(x)$ is finite for all $x\in X$.

Weakly noncollapsed $\mathrm{RCD}(0,n)$ spaces are important to us because they enjoy the so-called volume-cone-implies-metric-cone property. To state this property, we must recall a couple of definitions.

If (Z,d_Z) is a metric space, then the metric cone over Z is the metric space $(C(Z),\mathsf{d}_{C(Z)})$ defined in the usual way (see eg [15, Section 3.6.2.]), namely

$$C(Z) := [0,+\infty)\times Z/\sim,$$

where $(r,\sigma)\sim(r',\sigma')$ if and only if $r=r'=0$, and

$$\mathsf{d}_{C(Z)}((r,\sigma),(r',\sigma')) := \sqrt{(r-r')^2+4rr'\sin^2\big(\tfrac12\mathsf{d}_Z(\sigma,\sigma')\big)} \tag{30}$$

for any $(r,\sigma),(r',\sigma')\in C(Z)\setminus\{z^*\}$, where z^* is the vertex of $C(Z)$, that is, z^* is the equivalent class of all the points $(r,\sigma)\in[0,+\infty)\times Z$ such that $r=0$.

Definition 1.26 We say that a metric measure space (X,d,μ) is a *α–metric measure cone* with vertex $x\in X$ for some $\alpha\geq 1$ if there exist a metric measure space (Z,d_Z,μ_Z) and an isometry $\varphi\colon X\to C(Z)$ sending x to the vertex of the cone $C(Z)$ and such that

$$\mathrm{d}(\varphi_\#\mu)(r,z)=\mathrm{d}r\otimes r^{\alpha-1}\,\mathrm{d}\mu_Z(z)=:\mu_{C(Z)}. \tag{31}$$

Here is the volume-cone-implies-metric-cone property of weakly noncollapsed $\mathrm{RCD}(0,n)$ spaces (see [38, Theorem 1.1]) we use in a crucial way in our analysis.

Proposition 1.27 *Let (X,d,μ) be a weakly noncollapsed $\mathrm{RCD}(0,n)$ space such that the function $(0,+\infty)\ni r\mapsto\mu(B_r(x))/r^n$ is constant for some $x\in X$. Then (X,d,μ) is an n–metric measure cone with vertex x.*

2 Kato limits

Let (M^n, g) be a closed Riemannian manifold. The $\|\cdot\|_{1,2}$–closure $H^{1,2}(M)$ of the space of smooth functions and the distributional Sobolev space $W^{1,2}(M)$ coincide and do not depend on the metric g (see eg [56, Chapter 2]), hence we indifferently use both symbols in the rest of the article. Moreover, since the space of smooth functions is $\|\cdot\|_{1,2}$–dense in the one of Lipschitz functions, we also have that $H^{1,2}(M, \mathsf{d}_g, v_g) = H^{1,2}(M)$, and the Cheeger energy Ch_g of (M, d_g, v_g) coincides with the usual Dirichlet energy defined by

$$\mathscr{E}(u,w) = \int_M g(\nabla u, \nabla w)\, dv_g \quad \text{and} \quad \mathscr{E}(u) = \int_M |\nabla u|^2\, dv_g$$

for any $u, w \in W^{1,2}(M)$. As is well known, Ch_g is a strongly local and regular Dirichlet form with core $C_c^\infty(M)$ and associated operator the Laplacian Δ_g. Moreover, $\mathsf{d}_{\mathsf{Ch}_g}$ is a distance function that coincides with d_g. We denote by $H : \mathbb{R}_+ \times M \times M \to \mathbb{R}$ the heat kernel of Ch_g, which we call heat kernel of (M, g), and by $(P_t)_{t>0}$ the associated semigroup. For any $x \in M$, we define

$$\rho(x) = \inf_{v \in T_x M,\, g_x(v,v)=1} \mathrm{Ric}_x(v,v) \quad \text{and} \quad \mathrm{Ric}_-(x) = \max\{-\rho(x), 0\}.$$

For all $t > 0$ we introduce the quantity

$$\mathrm{k}_t(M^n, g) = \sup_{x \in M} \int_0^t \int_M H(s,x,y)\, \mathrm{Ric}_-(y)\, dv_g(y)\, ds. \tag{32}$$

This quantity is defined more generally as $\mathrm{k}_t(V)$ for a Borel function V, where Ric_- is replaced by V. Then V is said to be in the *contractive Dynkin class* when $\mathrm{k}_t(V) < 1$ and in the *Kato class* if $\mathrm{k}_t(V)$ tends to 0 as t goes to zero; see for example [55, Chapter VI]. In our case, since the manifold is compact, Ric_- always belongs to the Kato class.

We point out that $\mathrm{k}_t(M^n, g)$ has a useful scaling property,

$$\mathrm{k}_t(M^n, \varepsilon^{-2} g) = \mathrm{k}_{\varepsilon^2 t}(M^n, g) \quad \text{for all } \varepsilon, t > 0. \tag{33}$$

It is an easy consequence of the scaling property of the heat kernel: if H_ε is the heat kernel of $(M^n, \varepsilon^{-2} g)$, then $H_\varepsilon(s,x,y) = \varepsilon^n H(\varepsilon^2 s, x, y)$ for all $s > 0$ and $x, y \in M$.

In the following, we consider the next uniform bounds for sequences of closed smooth manifolds.

Definition 2.1 Let $\{(M^n_\alpha, g_\alpha)\}_{\alpha \in A}$ be a sequence of closed manifolds. We say that $\{(M^n_\alpha, g_\alpha)\}_{\alpha \in A}$ satisfies a

- **uniform Dynkin bound** if there exists $T > 0$ such that

$$\sup_\alpha \mathrm{k}_T(M_\alpha, g_\alpha) \le \frac{1}{16n}, \tag{UD}$$

- **uniform Kato bound** if there exist $T > 0$ and a nondecreasing function $f : (0, T] \to \mathbb{R}_+$ such that $f(t) \to 0$ when $t \to 0$ and for all $t \in (0, T]$,

$$\sup_\alpha \mathrm{k}_t(M_\alpha, g_\alpha) \le f(t), \tag{UK}$$

- **strong uniform Kato bound** if there exist $T, \Lambda > 0$ and a nondecreasing function $f : (0, T] \to \mathbb{R}_+$ such that for all $t \in (0, T]$,

$$\sup_\alpha \mathrm{k}_t(M_\alpha, g_\alpha) \le f(t) \quad \text{and} \quad \int_0^T \frac{\sqrt{f(s)}}{s}\,\mathrm{d}s \le \Lambda. \tag{SUK}$$

We observe that obviously a Kato bound implies a Dynkin bound, and a strong Kato bound implies a Kato bound. Indeed, if f is as in (SUK), then for any $t \in (0, T]$ we have

$$\sqrt{f(t)} \le \Lambda \log\Big(\frac{T}{t}\Big)^{-1}.$$

Therefore f tends to zero when t goes to zero. Without loss of generality, we can always assume that the function f is bounded by $(16n)^{-1}$.

Remark 2.2 The scaling property of $\mathrm{k}_t(M, g)$ ensures that the previous bounds are preserved when rescaling the metrics by a factor ε^{-2} for $\varepsilon \in (0, 1)$. Indeed, for any $t > 0$ and $\varepsilon \in (0, 1)$,

$$\mathrm{k}_t(M^n, \varepsilon^{-2} g) = \mathrm{k}_{\varepsilon^2 t}(M^n, g) < \mathrm{k}_t(M^n, g).$$

2.1 Dynkin limits

In this section, we prove a precompactness result for sequences of manifolds satisfying a uniform Dynkin bound. We start by proving that a uniform Dynkin bound leads to a uniform volume estimate and a uniform Poincaré inequality.

Proposition 2.3 *Let (M^n, g) be a closed Riemannian manifold, and let $T > 0$. Assume*

$$\mathrm{k}_T(M^n, g) \le \frac{1}{16n}$$

and set $\nu := e^2 n$. Then there exist $\theta \ge 1$ and $\gamma > 0$ depending only on n such that

(I) *for any $x \in M$ and $0 < s < r \le \sqrt{T}$,*

$$\frac{v_g(B_r(x))}{v_g(B_s(x))} \le \theta\Big(\frac{r}{s}\Big)^\nu,$$

(II) *for any ball $B \subset M$ with radius $r \le \sqrt{T}$ and any $\varphi \in \mathscr{C}^1(B)$,*

$$\int_B (\varphi - \varphi_B)^2 \,\mathrm{d}v_g \le \gamma r^2 \int_B |d\varphi|^2 \,\mathrm{d}v_g.$$

In particular, $(M^n, \mathsf{d}_g, v_g, \mathsf{Ch}_g)$ is a $\mathrm{PI}_{\kappa,\gamma}(\sqrt{T})$–Dirichlet space for $\kappa = 2^\nu\theta$.

Remark 2.4 The previous proposition is a minor variation of [17, Propositions 3.8 and 3.11], where similar estimates were shown for balls with radii lower than $\operatorname{diam}(M)/2$ but with constants that additionally depended on $\operatorname{diam}(M)$.

Proof **Step 1** Observe that $\nu > 2$. We begin by proving the following Sobolev inequality: there exists $\lambda > 0$ depending only on n such that for any ball $B \subset M$ with radius $r \le \sqrt{T}$ and any $\varphi \in \mathscr{C}_c^1(B)$,

$$\left(\int_B |\varphi|^{2\nu/(\nu-2)}\,\mathrm{d}v_g\right)^{1-2/\nu} \le \lambda \frac{r^2}{(v_g(B))^{2/\nu}}\left[\int_B |d\varphi|^2\,\mathrm{d}v_g + \frac{1}{r^2}\int_B |\varphi|^2\,\mathrm{d}v_g\right]. \tag{34}$$

To this aim, take $r \in (0,\sqrt{T})$, $x \in M$ and $y \in B_r(x)$. From [17, Theorem 3.5], we know that there exists $c_n > 0$ depending only on n such that, for any $s \in (0, r^2/2]$,

$$e^{-s/r^2} H(s,y,y) \le H(s,y,y) \le \frac{c_n}{v_g(B_r(x))}\left(\frac{r^2}{s}\right)^{\nu/2}.$$

Moreover, since the function $s \mapsto H(s,y,y)$ is nonincreasing, for any $s > r^2/2$,

$$e^{-s/r^2} H(s,y,y) \le e^{-s/r^2} H\left(\tfrac{1}{2}r^2, y, y\right) \le e^{-s/r^2}\frac{c_n}{v_g(B_r(x))}2^{\nu/2}.$$

As the function $\xi \mapsto e^{-\xi}\xi^{-\nu/2}$ is bounded from above on $\left[\frac{1}{2}, +\infty\right)$ by some constant $c_n' > 0$ depending only on n, we get

$$e^{-s/r^2} H(s,y,y) \le c_n'\left(\frac{r^2}{s}\right)^{\nu/2}\frac{c_n}{v_g(B_r(x))}2^{\nu/2}.$$

Setting $c_n'' := c_n c_n' 2^{\nu/2}$, we obtain for any $s > 0$,

$$e^{-s/r^2} H(s,y,y) \le c_n''\frac{r^\nu}{v_g(B_r(x))}\frac{1}{s^{\nu/2}}.$$

In particular, the heat kernel of the operator $\Delta + 1/r^2$ acting on $L^2(B,\mu)$ with Dirichlet boundary condition satisfies the same estimate and one deduces the Sobolev inequality (34) from a famous result of Varopoulos [93, Section 7].

Step 2 Let us prove (I). To this aim, we follow the argument of [1; 16]: for given $0 < s < r \le \sqrt{T}$, applying the Sobolev inequality (34) in the case $B = B_r(x)$ and $\varphi = \operatorname{dist}(\cdot, M \setminus B_s(x))$, then we obtain

$$\left(\tfrac{1}{2}s\right)^2\left(v_g(B_{s/2}(x))\right)^{1-2/\nu} \le 2\lambda\frac{r^2}{\left(v_g(B_r(x))\right)^{2/\nu}}v_g(B_s(x)). \tag{35}$$

Set $\mathscr{V}(\tau) := v_g(B_\tau(x))/\tau^\nu$ for any $\tau > 0$ and use elementary manipulations to turn (35) into

$$\mathscr{V}(s/2)^{1-2/\nu} \le \Lambda\mathscr{V}(s) \quad \text{with } \Lambda = \frac{2^{\nu+1}\lambda}{\mathscr{V}(r)^{2/\nu}}. \tag{36}$$

Iterating, we get for any positive integer ℓ,

$$\mathscr{V}(s/2^\ell)^{(1-2/\nu)\ell} \le \Lambda^{\sum_{k=0}^{\ell-1}(1-2/\nu)^k}\mathscr{V}(s).$$

As $\lim_{\ell\to\infty} \mathcal{V}(s/2^\ell)^{(1-2/\nu)^\ell} = 1$ and $\sum_{k=0}^{+\infty}(1-2/\nu)^k = \nu/2$, we obtain

$$1 \leq \Lambda^{\nu/2}\mathcal{V}(s) = \frac{[2^{\nu+1}\lambda]^{\nu/2}}{\mathcal{V}(r)}\mathcal{V}(s),$$

which is (I) with $\theta = [2^{\nu+1}\lambda]^{\nu/2}$.

Let us now prove (II). We recall a classical result (see eg [57, Theorem 2.7]): a metric measure space (X, d, μ) doubling at scale $\sqrt{T}$ equipped with a strongly local and regular Dirichlet form $\mathcal{E}$ with heat kernel H satisfies a Poincaré inequality at scale $\sqrt{T}$ if and only if there exist $c, C, \varepsilon_1, \varepsilon_2 > 0$ such that for all $x \in M$ and $t \in (0, \varepsilon_1\sqrt{T})$,

(i) $H(t,x,x) \leq C\mu(B_{\sqrt{t}}(x))^{-1}$,

(ii) $c\mu(B_{\sqrt{t}}(x))^{-1} \leq \inf\{H(t,x,y) : y \in B_{\varepsilon_2\sqrt{t}}(x)\}$.

In our context, (i) and (ii) hold with $\varepsilon_1 = \varepsilon_2 = 1$ and C, c depending only on n: indeed, (i) is a direct consequence of [17, Theorem 3.5] while (ii) follows from the same argument as in the proof of [17, Proposition 3.11] based on a method from [34]. □

The previous proposition ensures that we can apply Theorem 1.17 to a sequence of manifolds satisfying a uniform Dynkin bound (UD), and obtain the following precompactness result.

Corollary 2.5 *Let $T > 0$ and $\{(M_\alpha, g_\alpha)\}_{\alpha\in A}$ be a sequence of closed Riemannian manifolds satisfying the uniform Dynkin bound (UD). For all $\alpha \in A$, consider $o_\alpha \in M_\alpha$ and set*

$$\mu_{g_\alpha} := \frac{v_{g_\alpha}}{v_g(B_{\sqrt{T}}(o_\alpha))} \quad \text{and} \quad \mathcal{E}_{g_\alpha}(u) := \int_{M_\alpha} |du|^2 \, \mathrm{d}\mu_\alpha$$

for all $u \in \mathcal{C}^1(M_\alpha)$. Then there exist a $\mathrm{PI}_{\kappa,\gamma}(\sqrt{T})$–Dirichlet space $(X, \mathsf{d}, \mu, o, \mathcal{E})$ and a subsequence $B \subset A$ such that $\{(M_\beta, \mathsf{d}_{g_\beta}, \mu_{g_\beta}, o_\beta, \mathcal{E}_\beta)\}_{\beta\in B}$ Mosco–Gromov–Hausdorff converges to $(X, \mathsf{d}, \mu, o, \mathcal{E})$.

Definition 2.6 A metric Dirichlet space $(X, \mathsf{d}, \mu, o, \mathcal{E})$ is called a *Dynkin limit* space if it is obtained as the Mosco–Gromov–Hausdorff limit of a sequence of closed Riemannian manifolds satisfying a uniform Dynkin bound.

Remark 2.7 For any $T > 0$, the class of Dynkin limit spaces obtained as limits of manifolds satisfying (UD) is closed under pointed Mosco–Gromov–Hausdorff convergence: this follows from a direct diagonal argument.

As a consequence, tangent cones of Dynkin limit spaces equipped with their intrinsic distances are Dynkin limit spaces too. Indeed, if $(X, \mathsf{d}, \mu, o, \mathcal{E})$ is the pointed Mosco–Gromov–Hausdorff limit of pointed manifolds $\{(M_\alpha, g_\alpha, o_\alpha)\}_{\alpha\in A}$ satisfying (UD) and $(X_x, \mathsf{d}_x, \mu_x, x, \mathcal{E}_x)$ is a tangent cone at $x \in X$ provided by Corollary 1.18, then this latter can be written as the limit of a sequence of rescaled manifolds

$$\{(M_\beta, \varepsilon_\beta^{-1}\mathsf{d}_{g_\beta}, v_{g_\beta}(B_{\varepsilon_\beta}(x_\beta))^{-1}v_\beta, x_\beta, \varepsilon_\beta^2 v_{g_\beta}(B_{\varepsilon_\beta}(x_\beta))^{-1}\mathcal{E}_\beta)\}_{\beta\in B},$$

where $B \subset A$, $x_\beta \in M_\beta$ and $\varepsilon_\beta > 0$ for any $\beta \in B$, and $x_\beta \to x$, $\varepsilon_\beta \to 0$.

Another consequence is that if $\{z_\alpha\}_{\alpha\in A}$ belongs to a compact set of X and $\{\varepsilon_\alpha\}\subset(0,+\infty)$ satisfies $\varepsilon_\alpha\to 0$, then there exists a subsequence $B\subset A$ such that the sequence

$$\big\{\big(X,\varepsilon_\beta^{-1}\mathsf{d},\mu(B_{\varepsilon_\beta}(z_\beta))^{-1}\mu,z_\beta,\varepsilon_\beta^2\mu(B_{\varepsilon_\beta}(z_\beta))^{-1}\mathscr{E}\big)\big\}_\beta$$

converges to a Dynkin limit space $(Z,\mathsf{d}_Z,\mu_Z,z,\mathscr{E}_Z)$, which may also be written as the limit of a sequence of rescaled manifolds $\big\{\big(M_\beta,\varepsilon_\beta^{-1}\mathsf{d}_{g_\beta},v_{g_\beta}(B_{\varepsilon_\beta}(x_\beta))^{-1}v_{g_\beta},x_\beta,v_{g_\beta}(B_{\varepsilon_\beta}(x_\beta))^{-1}\mathscr{E}_\beta\big)\big\}_\beta$, with $x_\beta\in M_\beta$ for any $\beta\in B$ and $\mathsf{d}(z_\beta,\Phi_\beta(x_\beta))\to 0$, where Φ_β is as in Characterization 1.

2.2 Kato limits

In this section, we consider manifolds with a uniform Kato bound. In this case, some better properties can be proved for the distance in the limit and for tangent cones; see the next remark and Proposition 2.12. Thanks to the previous precompactness result we can give the following definition.

Definition 2.8 A Dirichlet space $(X,\mathsf{d},\mu,o,\mathscr{E})$ is called a *Kato limit* space if it is obtained as a Mosco–Gromov–Hausdorff limit of manifolds with a uniform Kato bound (UK).

Remark 2.9 A Kato limit space is obviously a $\mathrm{PI}_{\kappa,\gamma}(\sqrt{T})$–Dirichlet space for any $T>0$ such that $f(T)\le 1/(16n)$.

Remark 2.10 As in the case of Dynkin limits, tangent cones of Kato limits are Kato limits as well. Furthermore, if $(X,\mathsf{d},\mu,o,\mathscr{E})$ is a Kato limit and $(X_x,\mathsf{d}_x,\mu_x,x,\mathscr{E}_x)$ is a tangent cone at $x\in X$ provided by Corollary 1.18, then this latter is a limit of rescaled manifolds

$$\{(M_\alpha,\varepsilon_\alpha^{-1}\mathsf{d}_{g_\alpha},v_{g_\alpha}(B_{\varepsilon_\alpha}(x_\alpha))^{-1}v_\alpha,x_\alpha,\varepsilon_\alpha^2 v_{g_\alpha}(B_{\varepsilon_\alpha}(x_\alpha))^{-1}\mathscr{E}_\alpha)\}$$

such that for all $t>0$,

$$\mathrm{k}_t(M_\alpha,\varepsilon_\alpha^{-2}g_\alpha)\to 0\quad\text{as } \alpha\to\infty.$$

Indeed, we have $\mathrm{k}_t(M_\alpha,\varepsilon_\alpha^{-2}g_\alpha)=\mathrm{k}_{\varepsilon_\alpha^2 t}(M_\alpha,g_\alpha)\le f(\varepsilon_\alpha^2 t)\to 0$ as $\alpha\to\infty$. This observation also applies to spaces $(Z,\mathsf{d}_Z,\mu_Z,z,\mathscr{E}_Z)$ obtained as limits of rescalings of X centered at varying but convergent points, as considered in Remark 2.7.

As a consequence of Theorem 1.17, any Dynkin limit space $(X,\mathsf{d},\mu,o,\mathscr{E})$ satisfies $\mathsf{d}\le\mathsf{d}_{\mathscr{E}}$. But for Kato limit spaces, this inequality turns out to be an equality. To prove this fact, we need the following Li–Yau inequality, which was proved in [17, Proposition 3.3].

Proposition 2.11 *Let (M^n,g) be a closed Riemannian manifold, and $T>0$. Assume*

$$\mathrm{k}_T(M^n,g)\le\frac{1}{16n}.$$

If u is a positive solution of the heat equation on $[0,T]\times M$, then for any $(t,x)\in[0,T]\times M$,

$$e^{-8\sqrt{n\,\mathrm{k}_t(M,g)}}\frac{|du|^2}{u^2}-\frac{1}{u}\frac{\partial u}{\partial t}\le\frac{n}{2t}e^{8\sqrt{n\,\mathrm{k}_t(M,g)}}. \tag{37}$$

We are now in a position to prove the following.

Proposition 2.12 *Let $(X, \mathsf{d}, \mu, o, \mathcal{E})$ be a Kato limit space. Then $\mathsf{d} = \mathsf{d}_{\mathcal{E}}$.*

Proof We only need to prove $\mathsf{d} \geq \mathsf{d}_{\mathcal{E}}$. Let $\{(M^n_\beta, g_\beta, o_\beta)\}_{\beta \in B}$ be a sequence of closed Riemannian manifolds satisfying a uniform Kato bound and such that the sequence $\{(M_\beta, \mathsf{d}_\beta, \mu_\beta, o_\beta, \mathcal{E}_\beta)\}$ converges in the Mosco–Gromov–Hausdorff sense to $(X, \mathsf{d}, \mu, o, \mathcal{E})$. In particular, there exists $T > 0$ such that $\{(M^n_\beta, g_\beta)\}_{\beta \in B}$ satisfies the uniform Dynkin bound (UD). We claim that for any $x, y \in X$, $t \in (0, T)$ and $\theta \in (0, 1)$,

$$\log\left(\frac{H(\theta t, x, x)}{H(t, x, y)}\right) \leq \tfrac{1}{2} n \log(1/\theta) e^2 + \frac{\mathsf{d}^2(x, y)}{4(1-\theta)t} e^{8\sqrt{n f(t)}}. \tag{38}$$

Let us explain how to conclude from there. Multiply by $4t$ and apply Varadhan's formula (22) as $t \to 0$ to get

$$\mathsf{d}^2_{\mathcal{E}}(x, y) \leq \frac{\mathsf{d}^2(x, y)}{1-\theta}.$$

The desired inequality follows from $\theta \downarrow 0$.

In the following we thus prove (38). Take $\beta \in B$. Let u be a positive solution of the heat equation on $[0, T] \times M_\beta$. Take $x, y \in M_\beta$, $t \in (0, T]$ and $s \in (0, t)$. Let $\gamma \colon [0, t-s] \to M_\beta$ be a minimizing geodesic from y to x. For any $\tau \in [0, t-s]$, set

$$\phi(\tau) := \log u(t-\tau, \gamma(\tau)),$$

and note that $\phi(0) = \log u(t, y)$ and $\phi(t-s) = \log u(s, x)$. Differentiate ϕ at τ and apply the Li–Yau inequality (37) and the simple fact $\mathrm{k}_{t-\tau} \leq \mathrm{k}_t$ to get the first inequality in the following calculation, where u and its derivatives are implicitly evaluated at $(t-\tau, \gamma(\tau))$ and where we omit (M_β, g_β) in the notation $\mathrm{k}_t(M_\beta, g_\beta)$ for the sake of simplicity:

$$\begin{aligned}
\dot\phi(\tau) &= -\frac{1}{u}\frac{\partial u}{\partial t} + \langle \dot\gamma(\tau), d\log u\rangle \\
&\leq \frac{n e^{8\sqrt{n\,\mathrm{k}_{t-\tau}}}}{2(t-\tau)} - e^{-8\sqrt{n\,\mathrm{k}_t}} \frac{|du|^2}{u^2} + \langle \dot\gamma(\tau), d\log u\rangle \\
&= \frac{n e^{8\sqrt{n\,\mathrm{k}_{t-\tau}}}}{2(t-\tau)} - \left| e^{-4\sqrt{n\,\mathrm{k}_t}} d\log u - \frac{e^{4\sqrt{n\,\mathrm{k}_t}}}{2}\dot\gamma(\tau)\right|^2 + \frac{e^{8\sqrt{n\mathrm{k}_t}}}{4}|\dot\gamma(\tau)|^2 \\
&\leq \frac{n e^{8\sqrt{n\,\mathrm{k}_{t-\tau}}}}{2(t-\tau)} + \frac{e^{8\sqrt{n\,\mathrm{k}_t}}}{4}|\dot\gamma(\tau)|^2 \leq \frac{n e^{8\sqrt{n\,\mathrm{k}_{t-\tau}}}}{2(t-\tau)} + \frac{e^{8\sqrt{n\,\mathrm{k}_t}}\mathsf{d}^2_\beta(x, y)}{4(t-s)^2}.
\end{aligned}$$

Hence, when integrating between 0 and $t-s$ and changing variables in the first term, we obtain

$$\log\left(\frac{u(s, x)}{u(t, y)}\right) \leq \tfrac{1}{2} n \int_s^t e^{8\sqrt{n\,\mathrm{k}_\tau}} \frac{d\tau}{\tau} + \frac{e^{8\sqrt{n\,\mathrm{k}_t}}\mathsf{d}^2_\beta(x, y)}{4(t-s)}.$$

Write $s=\theta t$ for some $\theta\in(0,1)$. The uniform Dynkin bound (UD) allows us to bound $e^{8\sqrt{n\,\mathrm{k}_\tau}}$ in the first term of the right-hand side by e^2, while the uniform Kato bound lets us bound $e^{8\sqrt{n\,\mathrm{k}_t}}$ by $e^{8\sqrt{nf(t)}}$ in the second term. Thus

$$\log\left(\frac{u(s,x)}{u(t,y)}\right)\le \tfrac{1}{2}ne^2\log\left(\frac{t}{s}\right)+\frac{\mathsf{d}_\beta^2(x,y)}{4(1-\theta)t}e^{8\sqrt{nf(t)}}.$$

Choose $s=\theta t$ and $u(\tau,z)=H_\beta(\tau,x,z)$ for any $(\tau,z)\in(0,T)\times M$ to get

$$\log\left(\frac{H_\beta(\theta t,x,x)}{H_\beta(t,x,y)}\right)\le \tfrac{1}{2}n\log\left(\frac{1}{\theta}\right)e^2+\frac{\mathsf{d}_\beta^2(x,y)}{4(1-\theta)t}e^{8\sqrt{nf(t)}}.$$

The Mosco–Gromov–Hausdorff convergence $(M_\beta,\mathsf{d}_\beta,\mu_\beta,o_\beta,\mathcal{E}_\beta)\to(X,\mathsf{d},\mu,o,\mathcal{E})$ eventually yields the inequality (38). □

Remark 2.13 The previous proof also applies more generally in the case of a sequence of manifolds $\{(M_\alpha,g_\alpha)\}_{\alpha\in A}$ such that there exists a nondecreasing function $f:(0,T)\to\mathbb{R}_+$, tending to 0 as t goes to 0 and for which

$$\limsup_{\alpha\to\infty}\mathrm{k}_t(M_\alpha,g_\alpha)\le f(t)\quad\text{for all } t\in(0,T].$$

In particular, we have $\mathsf{d}=\mathsf{d}_{\mathcal{E}}$ whenever $(X,\mathsf{d},\mu,o,\mathcal{E})$ is the Mosco–Gromov–Hausdorff limit of a sequence $\{(M_\alpha,g_\alpha)\}_{\alpha\in A}$ such that, for some $T>0$,

$$\lim_{\alpha\to\infty}\mathrm{k}_T(M_\alpha,g_\alpha)=0;$$

in this case f is constantly equal to 0. As a consequence of Remark 2.10, Proposition 2.12 applies to tangent cones (and rescalings centered at convergent points) of Kato limits.

2.3 Ahlfors regularity

We now discuss volume estimates for closed Riemannian manifolds (M^n,g) satisfying the strong Kato condition

(39) $$\mathrm{k}_T(M^n,g)\le\frac{1}{16n}\quad\text{and}\quad\int_0^T\frac{\sqrt{\mathrm{k}_s(M^n,g)}}{s}\,\mathrm{d}s\le\Lambda$$

for some $T,\Lambda>0$. This condition was also considered in [17].

The proof of [17, Proposition 3.13] gives the following volume estimate, which improves the one given in Proposition 2.3.

Proposition 2.14 *Let (M^n,g) be a closed Riemannian manifold satisfying (39) for some $T,\Lambda>0$. Then there exists a constant $C_n>0$ depending only on n such that for all $x\in X$ and $0\le r\le s\le\sqrt{T}$,*

(40) $$v_g(B_r(x))\le C_n^{\Lambda+1}r^n\quad\textit{and}\quad\frac{v_g(B_s(x))}{v_g(B_r(x))}\le C_n^{\Lambda+1}\left(\frac{s}{r}\right)^n.$$

Proof The upper bound is proven in [17, page 3144]. The other estimate is a consequence of the proof of Proposition 2.3 and of the estimate

$$s^{n/2}H(s,x,x) \le C_n^{\Lambda+1} t^{n/2} H(t,x,x),$$

which holds for any $x \in M$ and $0 < s < t \le T$; see again [17, page 3144]. □

Then we get the following uniform local Ahlfors regularity result.

Corollary 2.15 *Let (M^n, g) be a closed Riemannian manifold satisfying (39) for some $T, \Lambda > 0$. There exists a positive constant $C > 0$ depending only on n such that, for all $o, x \in X$ and $0 \le r \le \sqrt{T}$, we have*

$$\frac{v_g(B_{\sqrt{T}}(o))}{T^{n/2}} \le C^{\Lambda+1+\mathsf{d}_g(x,o)/\sqrt{T}} \frac{v_g(B_r(x))}{r^n}. \tag{41}$$

Indeed by using the doubling property [Proposition 1.2(i)] and the volume bound (40) we get

$$\frac{v_g(B_{\sqrt{T}}(o))}{T^{n/2}} \le c e^{\lambda \mathsf{d}_g(o,x)/\sqrt{T}} \frac{v_g(B_{\sqrt{T}}(x))}{T^{n/2}} \le c C_n^{\Lambda+1} e^{\lambda \mathsf{d}_g(o,x)/\sqrt{T}} \frac{v_g(B_r(x))}{r^n}.$$

Thus, choosing $C = \max\{cC_n, C_n, e^{\lambda}\}$, we obtain (41).

Remark 2.16 A metric measure space (X, d, μ) for which there exist $C_1, C_2 > 0$ such that $C_1 \le \mu(B_r(x))/r^n \le C_2$ for any $x \in X$ and $r > 0$ is usually called Ahlfors n–regular. Thus (40) and (41) tell us that, for any $R > 0$, $(B_R(o), \mathsf{d}_g, v_g)$ is Ahlfors n–regular with constants depending on n, Λ, R, T and $v_g(B_{\sqrt{T}}(o))/T^{n/2}$.

2.4 Noncollapsed strong Kato limits

We introduce a last class of limit spaces that we are going to deal with, that is, strong Kato limits with a noncollapsing assumption. In this case, the limit measure carries the local Ahlfors regularity described above. This will be important in proving that tangent cones are metric cones and for our stratification result.

Definition 2.17 A pointed metric Dirichlet space $(X, \mathsf{d}, \mu, o, \mathcal{E})$ is called a *strong Kato limit* if it is obtained as a Mosco–Gromov–Hausdorff limit of pointed manifolds $(M_\alpha, g_\alpha, o_\alpha)$ with a strong uniform Kato bound. It is called a *noncollapsed strong Kato limit* if moreover there exists $v > 0$ such that for all α,

$$v_{g_\alpha}(B_{\sqrt{T}}(o_\alpha)) \ge v T^{n/2}, \tag{NC}$$

where T is given in Definition 2.1.

Remark 2.18 The convergence of the measure ensures that if the manifolds (M_α, g_α) satisfy a strong uniform Kato bound, then inequalities (40) and (41) pass to the limit. In particular, if $(X, \mathsf{d}, \mu, o, \mathscr{E})$ is a noncollapsed strong Kato limit, then there exist constants C and λ depending only on n and Λ such that we have, for all $0 < r \le s \le \sqrt{T}$ and $x \in X$,

$$\mu(B_r(x)) \le C r^n \quad \text{and} \quad \frac{\mu(B_s(x))}{\mu(B_r(x))} \le C\left(\frac{s}{r}\right)^n,$$

as well as the lower bound

$$\mu(B_r(x)) \ge v C^{-1-\Lambda-\mathsf{d}(x,o)/\sqrt{T}} r^n.$$

As a consequence, for any $R > 0$, $(B_R(o), \mathsf{d}, \mu)$ is Ahlfors n–regular, with constants depending on n, Λ, R, T and v.

Remark 2.19 As in the previous cases, tangent cones of strong Kato limits are strong Kato limits. Under the noncollapsing assumption (NC), Remark 2.18 ensures a local Ahlfors n–regularity. Then, as observed in Section 2, we can consider tangent cones as limits of rescaled manifolds $\{(M_\alpha, \varepsilon_\alpha^{-1}\mathsf{d}_{g_\alpha}, \varepsilon_\alpha^{-n} v_{g_\alpha}, x_\alpha)\}$, that is to say we can replace the rescaling factor $v_{g_\alpha}(B_{\sqrt{T}}(x_\alpha))$ of the measures by ε_α^{-n}. This sequence of rescaled manifolds also satisfies the noncollapsing condition.

This also applies to limits $(Z, \mathsf{d}_Z, \mu_Z, z, \mathscr{E}_Z)$ of rescalings of noncollapsed strong Kato manifolds centered at varying but convergent points.

2.5 L^p–Kato condition

The strong Kato condition follows from a uniform estimate on the L^p–Kato constant for $p > 1$. Let us introduce

$$\mathrm{k}_{p,T}(M, g) := \left(\sup_{x \in M} T^{p-1} \int_0^T \int_M H(s, x, y)\mathrm{Ric}_-(y)^p \, \mathrm{d}v_g(y)\, ds\right)^{1/p}. \tag{42}$$

When $p > 1$, and using the Hölder inequality, we obtain

$$\mathrm{k}_t(M, g) \le \left(\frac{t}{T}\right)^{1-1/p} \mathrm{k}_{p,T}(M, g).$$

Hence a sequence $\{(M_\alpha^n, g_\alpha)\}_{\alpha \in A}$ of closed manifolds satisfying

$$\sup_{\alpha \in A} \mathrm{k}_{p,T}(M, g) < \infty$$

satisfies a strong uniform Kato bound.

As noticed in [17, Proposition 3.15], we can estimate the L^2–Kato constant in terms of the $\boldsymbol{Q}$–curvature. Recall that if (M, g) is Riemannian manifold of dimension $n \ge 4$, its $\boldsymbol{Q}$–curvature is defined by

$$\boldsymbol{Q}_g = \frac{1}{2(n-1)}\Delta\, \mathrm{Scal}_g - \frac{2}{(n-2)^2}|\mathrm{Ric}|^2 + c_n\, \mathrm{Scal}_g^2,$$

where

$$c_n = \frac{n^3 - 4n^2 + 16n - 16}{8n(n-1)^2(n-2)^2}.$$

Proposition 2.20 *Let (M^n, g) be a closed Riemannian manifold of dimension $n \geq 4$ such that*

$$-\kappa^4 \leq \boldsymbol{Q}_g \quad \text{and} \quad |\mathrm{Scal}_g| \leq \kappa^2,$$

where $\kappa > 0$. Then

$$\mathrm{k}_{2,T}(M, g) \leq C(n)\kappa\sqrt{T}(1+\kappa\sqrt{T}).$$

3 Analytic properties of manifolds with a Dynkin bound

In this section, we develop some analytic tools in the setting of closed Riemannian manifolds satisfying a Dynkin bound, that is, those manifolds for which there exists $T > 0$ such that

$$\text{(D)} \qquad \mathrm{k}_T(M^n, g) \leq \frac{1}{16n}.$$

3.1 Good cut-off functions

The existence of cut-off functions with suitably bounded gradient and Laplacian is a key technical tool in the theory of Ricci limit spaces [24] and $\mathrm{RCD}^*(K, N)$ spaces [70]. Our next proposition tells that such functions also exist in the context of manifolds with a Dynkin bound.

Proposition 3.1 *Let (M^n, g) be a closed Riemannian manifold satisfying (D) for some $T > 0$. Then for any ball $B_{r+s}(x) \subset M$ there exists a function $\chi \in C^\infty(M)$ such that $0 \leq \chi \leq 1$ and*

(i) *$\chi = 1$ on $B_r(x)$,*
(ii) *$\chi = 0$ on $M \setminus B_{r+s}(x)$, and*
(iii) *there exists a constant $C(n) > 0$ such that*

$$|\nabla\chi| \leq \frac{C(n)}{\min(s, \sqrt{T})} \quad \text{and} \quad |\Delta\chi| \leq \frac{C(n)}{\min(s^2, T)}.$$

We start with the following useful consequence of a Kato bound.

Lemma 3.2 *Let (M^n, g) be a closed Riemannian manifold satisfying (D) for some $T > 0$. Assume that $u : M \to \mathbb{R}$ is a Λ–Lipschitz function. Then for any $t \in (0, T]$ and $x \in M$,*

$$|\nabla e^{-t\Delta}u|(x) \leq 2\Lambda.$$

Proof It follows from [95, Proof of Theorem 1, step (i)] (see also [19, Remark 1.3.2]) that the Dynkin bound (D) implies, for any $t \in (0, T]$,

$$\|e^{-t(\Delta - \mathrm{Ric}_-)}\|_{L^1 \to L^1} \leq \frac{16n}{16n-1} \leq 2.$$

By the self-adjointness of the Schrödinger operator $\Delta - \mathrm{Ric}_-$ this yields

$$\|e^{-t(\Delta - \mathrm{Ric}_-)}\|_{L^\infty \to L^\infty} \leq 2.$$

The domination properties for the Hodge Laplacian $\vec{\Delta} = d^*d + dd^* = \nabla^*\nabla + \mathrm{Ric}$ on one-forms (see [59, first formula on page 32]) and the fact that $d\Delta = \vec{\Delta} d$ yield that for any Lipschitz function u we have

$$|\nabla e^{-t\Delta} u| = |e^{-t\vec{\Delta}} \nabla u| \le e^{-t(\Delta - \mathrm{Ric}_-)} |\nabla u|.$$

Hence we get the result. □

The Li–Yau inequality has the following consequence which will be very useful.

Lemma 3.3 *Let (M^n, g) be a closed Riemannian manifold satisfying (D) for some $T > 0$. The heat kernel of (M^n, g) satisfies, for any $x \in M$ and $t \in (0, T]$,*

$$\int_M |\nabla_z H(t,x,z)| \, \mathrm{d}v_g(z) \le \sqrt{\frac{e^4 n}{2t}} \quad \text{and} \quad \int_M \frac{|\nabla_z H(t,x,z)|^2}{H(t,y,z)} \, \mathrm{d}v_g(z) \le \frac{e^4 n}{2t}.$$

Proof Using Hölder's inequality and the stochastic completeness of H, we get

$$\begin{aligned}\int_M |\nabla_z H(t,x,z)| \, \mathrm{d}v_g(z) &\le \left(\int_M \frac{|\nabla_z H(t,y,z)|^2}{H(t,y,z)} \, \mathrm{d}v_g(z) \right)^{1/2} \left(\int_M H(t,y,z) \, \mathrm{d}v_g(z) \right)^{1/2} \\ &= \left(\int_M \frac{|\nabla_z H(t,y,z)|^2}{H(t,y,z)} \, \mathrm{d}v_g(z) \right)^{1/2}.\end{aligned}$$

Hence the first estimate follows from the second one. By the Li–Yau estimate (37),

$$\frac{|\nabla_z H(t,y,z)|^2}{H(t,y,z)} \le \frac{e^4 n}{2t} H(t,y,z) + e^2 \frac{\partial H(t,y,z)}{\partial t}. \tag{43}$$

Since

$$\int_M \frac{\partial H(t,y,z)}{\partial t} \, \mathrm{d}v_g(z) = \frac{\partial}{\partial t} \underbrace{\int_M H(t,y,z) \, \mathrm{d}v_g(z)}_{=1} = 0,$$

the second estimate follows. □

Proof of Proposition 3.1 Take $x \in M$ and $r, s > 0$. Let ρ_0 be the distance function to x (ie $\rho_0(\cdot) = \mathsf{d}_g(x, \cdot)$) and set

$$\rho_t := e^{-t\Delta} \rho_0$$

for any $t \in (0, T]$. Note that from Lemma 3.2, ρ_t is 2–Lipschitz. Then for any $y \in M$,

$$\begin{aligned}\left| \frac{\partial \rho_t}{\partial t}(y) \right| = |\Delta \rho_t(y)| &= \left| \int_M \langle \nabla_z H(t,y,z), \nabla \rho_0(z) \rangle \, \mathrm{d}v_g(z) \right| \\ &\le \int_M |\nabla_z H(t,y,z)| \underbrace{|\nabla \rho_0(z)|}_{=1 \ v_g\text{–a.e.}} \, \mathrm{d}v_g(z) \le e^2 \sqrt{\frac{n}{2t}},\end{aligned} \tag{44}$$

where we used Lemma 3.3. Then the derivative estimate (44) implies

$$|\rho_t(y) - \rho_0(y)| \le e^2 \sqrt{n/2} \sqrt{t}.$$

Therefore, if $y \in B_r(x)$ then

$$\rho_t(y) \le r + e^2 \sqrt{n/2}\,\sqrt{t},$$

while if $y \in M \setminus B_{r+s}(x)$ then

$$\rho_t(y) \ge r + s - e^2 \sqrt{n/2}\,\sqrt{t}.$$

Let us define

$$t_0 = \left(\frac{s}{4e^2\sqrt{n/2}}\right)^2 \quad \text{and} \quad t = \min(t_0, T).$$

Then

$$\rho_t(y) \le r + \tfrac{1}{4}s \quad \text{if } y \in B_r(x) \qquad \text{and} \qquad \rho_t(y) \ge r + \tfrac{3}{4}s \quad \text{if } y \in M \setminus B_{r+s}(x).$$

Let $u\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a smooth function such that

$$u = \begin{cases} 1 & \text{on } \left[0, \frac{1}{4}\right], \\ 0 & \text{on } \left[\frac{3}{4}, +\infty\right). \end{cases}$$

Set

$$\chi(y) := u\left(\frac{\rho_t(y) - r}{s}\right)$$

for any $y \in M$. Then

$$\chi = \begin{cases} 1 & \text{on } B_r(x), \\ 0 & \text{on } M \setminus B_{r+s}(x). \end{cases}$$

Since, for any $y \in M$,

$$\begin{aligned} d\chi(y) &= \frac{1}{s}u'\left(\frac{\rho_t(y)-r}{s}\right) d\rho_t(y), \\ \Delta\chi(y) &= \frac{1}{s}u'\left(\frac{\rho_t(y)-r}{s}\right)\Delta\rho_t(y) - \frac{1}{s^2}u''\left(\frac{\rho_t(y)-r}{s}\right)|d\rho_t|^2(y), \end{aligned}$$

setting $L := \sup_{\mathbb{R}}(|u'| + |u''|)$, we get

$$\|d\chi\|_\infty \le \frac{2L}{s} \quad \text{and} \quad \|\Delta\chi\|_\infty \le L\left(\frac{e^2\sqrt{n}}{s\sqrt{2t}} + \frac{4}{s^2}\right). \qquad \square$$

Remark 3.4 (complete Kato manifolds) The above proof makes use of an integrated version of the Li–Yau inequality only. In this regard, it would be interesting to study whether the assumption (D) on a complete Riemannian manifold implies

$$\left(\int_M \frac{|\nabla_z H(t,y,z)|^2}{H(t,y,z)}\,\mathrm{d}v_g(z)\right)^{1/2} \le \frac{C(n)}{\sqrt{t}}.$$

This would ensure the existence of good cut-off functions which would in turn provide the Li–Yau inequality, and then make possible the study of limits of complete Riemannian manifolds satisfying the uniform bound (UD).

3.2 Hessian estimates

Good cut-off functions are particularly relevant to deduce the following powerful Hessian estimates, which we will use in Section 7.

Proposition 3.5 *Let (M^n, g) be a closed Riemannian manifold satisfying (D) for some $T > 0$. Then there exists a constant $C(n) > 0$ such that for any ball $B_r(x) \subset M$ and any $u \in \mathscr{C}^\infty(B_r(x))$,*

$$\int_{B_{r/2}(x)} |\nabla du|^2 \, dv_g \le C(n) \int_{B_r(x)} \left[(\Delta u)^2 + \frac{1}{\min(r^2, T)} |du|^2 \right] dv_g. \tag{45}$$

If u is additionally harmonic, then

$$\int_{B_{r/2}(x)} |\nabla du|^2 \, dv_g \le \frac{C(n)}{\min(r^2, T)} \int_{B_r(x)} \left| |du|^2 - \fint_{B_r(x)} |du|^2 \, dv_g \right| dv_g. \tag{46}$$

Proof **Step 1** We are going to obtain a lower bound for the first nonzero eigenvalue λ_0 of $\Delta - 2\operatorname{Ric}_-$. Using [95, Proof of Theorem 1, step (i)] and the self-adjointness of $\Delta - 2\operatorname{Ric}_-$ as in Lemma 3.2, we obtain the estimates

$$\|e^{-t(\Delta - 2\operatorname{Ric}_-)}\|_{L^1 \to L^1} \le \frac{8n}{8n-1} \le 2 \quad \text{and} \quad \|e^{-t(\Delta - 2\operatorname{Ric}_-)}\|_{L^\infty \to L^\infty} \le 2.$$

Then by interpolation we also have, for any $t \in (0, T]$,

$$\|e^{-t(\Delta - 2\operatorname{Ric}_-)}\|_{L^2 \to L^2} \le 2.$$

In particular, this means that the first nonzero eigenvalue of $e^{-T(\Delta - 2\operatorname{Ric}_-)}$ is smaller than 2. But this latter eigenvalue is equal to $e^{-T\lambda_0}$, so that

$$\lambda_0 \ge -\frac{\log 2}{T}. \tag{47}$$

Step 2 All the integrals in this step are taken with respect to v_g, hence we skip the notation dv_g for the sake of brevity. Thanks to (47), we have for any $v \in C^\infty(M)$,

$$\int_M \operatorname{Ric}_- v^2 \le \frac{1}{2} \int_M \left[|dv|^2 + \frac{\log(2)}{T} v^2 \right]. \tag{48}$$

Let $u \in C_c(B_r(x))$ and χ be a cut-off function as built in Proposition 3.1 such that $\chi = 1$ on $B_{r/2}(x)$ and $\chi = 0$ on $M \setminus B_r(x)$. Apply the Bochner formula to u to get

$$|\nabla du|^2 + \tfrac{1}{2}\Delta |du|^2 + \operatorname{Ric}(du, du) = \langle d\Delta u, du \rangle,$$

multiply by χ^2 and integrate over M. This gives

$$\int_M \chi^2 |\nabla du|^2 + \frac{1}{2} \int_M \chi^2 \Delta |du|^2 \le \int_M \operatorname{Ric}_- |du|^2 \chi^2 + \int_M \chi^2 \langle d\Delta u, du \rangle. \tag{49}$$

We control the second term on the right-hand side of (49) as follows, using successively integration by parts, the Cauchy–Schwarz inequality, and the elementary fact $2ab \le a^2+b^2$:

$$\int_M \langle d\Delta u, \chi^2 du\rangle = \int_M \chi^2(\Delta u)^2 - \int_M \Delta u \langle d\chi^2, du\rangle \le \int_M \chi^2(\Delta u)^2 + \int_M 2\chi|\Delta u|\,|d\chi|\,|du| \le 2\int_M \chi^2(\Delta u)^2 + \int_M |d\chi|^2\,|du|^2. \tag{50}$$

Now we control the first term in the right-hand side of (49) as follows. Thanks to (48) we have

$$\int_M \mathrm{Ric}_-\,|du|^2\chi^2 \le \frac{1}{2}\biggl(\int_M |\nabla(\chi|du|)|^2 + \frac{\log 2}{T}\int_M (\chi|du|)^2\biggr).$$

Since

$$\begin{aligned}\int_M |\nabla(\chi|du|)|^2 &= \int_M \langle \chi\nabla|du|, \nabla(\chi|du|)\rangle + \langle |du|\nabla\chi, \nabla(\chi|du|)\rangle \\ &= \int_M \chi^2\bigl|\nabla|du|\bigr|^2 + \chi|du|\langle\nabla\chi, \nabla|du|\rangle + |du|\langle\nabla\chi, \nabla(\chi|du|)\rangle \\ &= \int_M \chi^2\bigl|\nabla|du|\bigr|^2 + \int_M \langle \nabla\chi, \underbrace{\chi|du|\nabla|du| + |du|\nabla(\chi|du|)}_{=\nabla(\chi|du|^2)}\rangle\end{aligned}$$

and $\bigl|\nabla|du|\bigr|^2 \le |\nabla du|^2$, we get

$$\int_M \mathrm{Ric}_-\,|du|^2\chi^2 \le \frac{1}{2}\biggl(\int_M \chi^2|\nabla du|^2 + \int_M (\Delta\chi)\chi|du|^2 + \frac{\log 2}{T}\int_M (\chi|du|)^2\biggr). \tag{51}$$

Combining (49) with (50) and (51), we get

$$\frac{1}{2}\int_M \chi^2|\nabla du|^2 \le 2\int_M \chi^2(\Delta u)^2 + \int_M \biggl[|d\chi|^2 + \frac{\log 2}{2T}\chi^2 + \tfrac{1}{2}|\Delta\chi|\chi - \tfrac{1}{2}(\Delta\chi^2)\biggr]|du|^2,$$

which eventually leads to (45) thanks to the properties of χ.

The second estimate (46) is obtained in a similar way, by replacing $\Delta|du|^2$ with $\Delta(|du|^2-c)$ in (49), where $c = ⨍_{B_r(x)} |du|^2\,dv_g$. □

3.3 Gradient estimates for harmonic functions

We conclude with the following gradient estimates, that will also be useful in Section 7.

Lemma 3.6 *Let (M^n,g) be a closed Riemannian manifold satisfying (D) for some $T>0$, and let $h\colon B_r(x)\to\mathbb{R}$ be a harmonic function. Then for some constant $c_n>0$ depending only on n,*

(i) $\displaystyle \sup_{B_{r/2}(x)} |\nabla h| \le c_n^{1+r/\sqrt{T}}\biggl(⨍_{B_r(x)} |\nabla h|^2\,dv_g\biggr)^{1/2}$,

(ii) $\displaystyle \sup_{B_{r/2}(x)} |\nabla h| \le \frac{c_n^{1+r/\sqrt{T}}}{r}\sup_{B_r(x)} |h|$.

Proof We first prove the result when $r \le \sqrt{T}$. Consider the operator

$$A = \Big(\Delta_g + \frac{1}{T}\Big)^{-1} \operatorname{Ric}_-.$$

Assumption (D) ensures that

$$\|A\|_{L^\infty \to L^\infty} \le \frac{1}{16n}\frac{e}{e-1} \le \frac{1}{8n}.$$

See for example [17, Lemma 2.22].

The same is true when replacing the Laplacian Δ_g on M by the Laplacian Δ_B on the ball $B = B_r(x)$ with the Dirichlet boundary conditions, that is, introducing

$$A_B = \Big(\Delta_B + \frac{1}{T}\Big)^{-1} \operatorname{Ric}_-,$$

we get $\|A_B\|_{L^\infty \to L^\infty} \le 1/(8n)$. As a consequence, if $f = A_B(1)$, then we find a unique $\varphi \in L^\infty(B)$ solving

$$\varphi = A_B \varphi + f.$$

Note that A_B preserves the positivity: $v \ge 0 \implies A_B v \ge 0$. Then it is not difficult to show that φ satisfies the inequality

$$0 \le \varphi \le \frac{1}{8n}\frac{1}{1 - \frac{1}{8n}} \le 1.$$

Moreover, by construction $\varphi \in W^{1,2}(B)$ and is zero along ∂B.

Consider $J = 1 + \varphi$. By definition, J solves the equation

$$\Big(\Delta_B + \frac{1}{T} - \operatorname{Ric}_-\Big) J = \frac{1}{T},$$

and $1 \le J \le 2$. Now consider the Laplacian Δ_{J^2} associated to the quadratic form

$$\mathscr{E}_{J^2}(\Psi) = \int_B |d\Psi|^2 J^2 \, \mathrm{d}v_g$$

on the space $L^2(B, J^2 \, \mathrm{d}v_g)$. Then for any $\Psi \in \mathscr{C}^2(B)$ we have

$$J^{-1}\Big(\Delta + \frac{1}{T} - \operatorname{Ric}_-\Big)(J\Psi) = \Delta\Psi - 2J^{-1}\langle dJ, d\Psi\rangle + \frac{\Psi}{JT} = \Delta_{J^2}\Psi + \frac{\Psi}{JT}.$$

Choosing $\Psi = |dh|/J$, we obtain

$$\Delta_{J^2}\Big(\frac{|dh|}{J}\Big) + \frac{|dh|}{J^2 T} = J^{-1}\Big(\Delta|dh| + \frac{|dh|}{T} - \operatorname{Ric}_- |dh|\Big) \le \frac{|dh|}{JT},$$

where we used that $\Delta|dh| \le \operatorname{Ric}_- |dh|$ because of the Bochner inequality. We then conclude that

$$\Delta_{J^2}\Psi \le \frac{1}{T}\Psi. \tag{52}$$

Now, as we observed in (34), for $\nu = e^2 n$ the following Sobolev inequality holds for all $\varphi \in C_0^\infty(B)$:

$$\left(\int_B |\varphi|^{2\nu/(\nu-2)}\,\mathrm{d}v_g\right)^{1-2/\nu} \le \frac{C(n)r^2}{v_g(B)^{2/\nu}}\left[\int_B |d\varphi|^2\,\mathrm{d}v_g + \frac{1}{r^2}\int_B |\varphi|^2\,\mathrm{d}v_g\right].$$

Using that $1 \le J \le 2$, we also have the analogous Sobolev inequality for the measure $J^2\,\mathrm{d}v_g$:

$$\left(\int_B |\varphi|^{2\nu/(\nu-2)} J^2\,\mathrm{d}v_g\right)^{1-2/\nu} \le \frac{4C(n)r^2}{v_g(B)^{2/\nu}}\left[\int_B |d\varphi|^2 J^2\,\mathrm{d}v_g + \frac{1}{r^2}\int_B |\varphi|^2 J^2\,\mathrm{d}v_g\right].$$

Together with inequality (52) and De Giorgi–Nash–Moser iteration, this leads to

$$\sup_{B_{r/2}(x)} \Psi \le C(n)\sqrt{\frac{1}{v_g(B)}\int_{B_{3r/4}(x)} \Psi^2 J^2\,\mathrm{d}v_g}.$$

Since J is bounded between 1 and 2, we then obtain

$$\sup_{B_{r/2}(x)} |dh| \le C(n)\sqrt{\frac{1}{v_g(B)}\int_{B_{3r/4}(x)} |dh|^2\,\mathrm{d}v_g}, \tag{53}$$

thus the first inequality.

As for the second inequality, take $\xi \in C_c^\infty(M)$ such that $\xi \equiv 1$ on $B_{3r/4}(x)$ and $|d\xi|^2 \le C/r^2$ for some $C > 0$. The integration by parts formula applied to ξh, that is,

$$\int_B |d(\xi h)|^2\,\mathrm{d}v_g = \int_B |d\xi|^2 h^2\,\mathrm{d}v_g + \int_B \xi^2 h \Delta h\,\mathrm{d}v_g,$$

and the fact that h is harmonic imply that

$$\int_B |d(\xi h)|^2\,\mathrm{d}v_g \le \frac{C}{r^2}\int_B |h|^2\,\mathrm{d}v_g \le \frac{C}{r^2} v_g(B)\sup_B |h|^2. \tag{54}$$

The left-hand side of (54) is bounded from below by $\int_{B_{3r/4}(x)} |dh|^2\,\mathrm{d}v_g$, hence the square of the right-hand side of (53) is bounded from above by the right-hand side of (54), so that we get

$$\sup_{B_{r/2}(x)} |dh| \le \frac{C(n)}{r}\sup_{B_r(x)} |h|. \tag{55}$$

Assume now that $r > \sqrt{T}$ and let $h\colon B_r(x) \to \mathbb{R}$ be a harmonic function. For any $y \in B_{r/2}(x)$, we have that

$$B_{\sqrt{T}/2}(y) \subset B_r(x).$$

By using inequality (53) for the restriction of h to $B_{\sqrt{T}/2}(y)$, we conclude that

$$|dh|^2(y) \le \sup_{B_{\sqrt{T}/4}(y)} |dh|^2 \le C \fint_{B_{\sqrt{T}/2}(y)} |dh|^2\,\mathrm{d}v_g.$$

By applying Proposition 1.2(iii) and (i), we get

$$v_g(B_r(x)) \le C_n^{1+r/\sqrt{T}} v_g(B_{1/2\sqrt{T}}(y)).$$

As a consequence, we have

$$|dh|^2(y) \le C\big(v_g(B_{\sqrt{T}/2}(y))\big)^{-1}\int_{B_{\sqrt{T}/2}(y)} |dh|^2\,\mathrm{d}v_g \le C\big(v_g(B_{\sqrt{T}/2}(y))\big)^{-1}\int_{B_r(x)} |dh|^2\,\mathrm{d}v_g$$
$$\le C C_n^{1+r/\sqrt{T}} \fint_{B_r(x)} |dh|^2\,\mathrm{d}v_g.$$

This gives (i) when $r > \sqrt{T}$. The inequality (ii) is proven similarly: for $y \in B_{r/2}(x)$ we use inequality (55) on $B_{\sqrt{T}/2}(y)$ and get

$$|dh|(y) \le \frac{C(n)}{\sqrt{T}} \sup_{B_{\sqrt{T}/2}(y)} |h| \le \frac{C(n)}{\sqrt{T}} \sup_{B_r(x)} |h|.$$

Using the estimate

$$\frac{C(n)}{\sqrt{T}} \le \frac{C(n)}{r} e^{r/\sqrt{T}},$$

we get the desired estimate. □

Remark 3.7 As shown by inequalities (53) and (55), whenever $r \le \sqrt{T}$ we do not need to consider the exponent $r/\sqrt{T}$ in the previous estimates.

4 Curvature-dimension condition for Kato limits

In this section, we prove that the Cheeger energy built from the metric measure structure of a Kato limit space $(X, \mathsf{d}, \mu, o, \mathcal{E})$ always coincides with the limit Dirichlet energy $\mathcal{E}$. Moreover, we show that tangent cones of a Kato limit space are all $\mathrm{RCD}(0, n)$ spaces, and that they are additionally weakly noncollapsed in the case that the space is a noncollapsed strong Kato limit. We obtain these two latter statements by establishing the Bakry–Ledoux gradient estimate $\mathrm{BL}(0, n)$ on a specific class of Kato limit spaces, namely those obtained from a sequence of closed Riemannian manifolds $\{(M^n_\alpha, g_\alpha)\}$ for which there exists $T > 0$ such that $\mathrm{k}_T(M_\alpha, g_\alpha)$ tends to zero as $\alpha \to \infty$.

For these purposes, acting like in [4, Section 2.2], we define the following quantity A on a strongly local, regular Dirichlet space $(X, \mathsf{d}, \mu, \mathcal{E})$. For all $t > 0$ and $u \in \mathcal{D}(\mathcal{E})$, $\varphi \in L^2(X) \cap L^\infty(X)$ with $\varphi \ge 0$, we set

$$A_t(u, \varphi)(s) := \frac{1}{2}\int_X (P_{t-s}u)^2 P_s\varphi\,\mathrm{d}\mu \tag{56}$$

for any $s \in [0, t]$. As shown in [4, Lemma 2.1], the function $s \mapsto A_t(u, \varphi)(s)$ is continuous on $[0, t]$ and continuously differentiable on $(0, t]$ with derivative given by

$$\frac{\mathrm{d}}{\mathrm{d}s} A_t(u, \varphi)(s) = \int_X P_s\varphi\,\mathrm{d}\Gamma(P_{t-s}u) \tag{57}$$

for any $s \in (0, t]$. Whenever φ additionally belongs to $\mathcal{D}(\mathcal{E})$, the map $s \mapsto A_t(u, \varphi)(s)$ is in $C^1([0, t])$ and the previous formula is valid for $s = 0$.

4.1 Differential inequalities

We first need to prove some differential inequalities on closed manifolds (M^n, g) with a smallness condition on $\mathrm{k}_t(M^n, g)$.

Theorem 4.1 *Let (M^n, g) be a closed Riemannian manifold satisfying*

$$\text{(sK)} \qquad \mathrm{k}_t(M^n, g) < \tfrac{1}{8}$$

for some $t > 0$. Then for all $v \in W^{1,2}(M)$ with $\Delta_g v \in L^2(M) \cap L^\infty(M)$ and for any nonnegative $\varphi \in W^{1,2}(M) \cap L^\infty(M)$, we have the inequality

$$\text{(58)} \quad \frac{1}{2}\int_M (P_t\varphi v^2 - \varphi(P_t v)^2)\,\mathrm{d}v_g \geq e^{-12\mathrm{k}_t(M,g)}\biggl(t\int_M \varphi|dP_t v|^2\,\mathrm{d}v_g + \frac{t^2}{n}\int_M \varphi(\Delta_g P_t v)^2\,\mathrm{d}v_g\biggr).$$

In the next subsection, under the assumption $\mathrm{k}_T(M_\alpha, g_\alpha) \to 0$, we aim to pass inequality (58) to a limit space in order to get the Bakry–Ledoux gradient estimate $\mathrm{BL}(0, n)$. To this aim, it is useful to rewrite this inequality in terms of A and its derivative with respect to $s \in (0, t)$. Since on a closed manifold (M^n, g), the derivative of A is simply

$$\frac{\mathrm{d}}{\mathrm{d}s}A_t(u,\varphi)(s) = \int_M |dP_{t-s}u|^2 P_s\varphi\,\mathrm{d}v_g,$$

we can rephrase Theorem 4.1 as follows.

Theorem 4.2 *Let (M^n, g) be a closed Riemannian manifold satisfying (sK) for some $t > 0$. Then for all $v \in W^{1,2}(M)$ with $\Delta_g v \in L^2(M) \cap L^\infty(M)$, for any nonnegative $\varphi \in W^{1,2}(M) \cap L^\infty(M)$ and any $s \in (0, t)$,*

$$\text{(59)} \quad A_t(v,\varphi)(t) - A_t(v,\varphi)(s) \geq e^{-12\mathrm{k}_t(M,g)}\biggl((t-s)\frac{\mathrm{d}}{\mathrm{d}s}A_t(v,\varphi)(s) + \frac{2}{n}(t-s)^2 A_t(\Delta_g v,\varphi)(s)\biggr).$$

Indeed, with A and its derivative, inequality (58) writes as

$$A_t(v,\varphi)(t) - A_t(v,\varphi)(0) \geq e^{-12\mathrm{k}_t(M,g)}\biggl(t\int_M \varphi|dP_t v|^2\,\mathrm{d}v_g + \frac{2}{n}t^2 A_t(\Delta_g v,\varphi)(t)\biggr).$$

For any $s \in [0, t]$ we also have $\mathrm{k}_{t-s}(M, g) \leq \mathrm{k}_t(M, g) < \frac{1}{8}$ so the previous holds with $t - s$ instead of t:

$$\begin{aligned} &A_{t-s}(v, P_s\varphi)(t-s) - A_{t-s}(v, P_s\varphi)(0) \\ &\qquad \geq e^{-12\mathrm{k}_t(M,g)}\biggl[(t-s)\int_M P_s\varphi|dP_{t-s}v|^2\,\mathrm{d}v_g + \frac{2}{n}(t-s)^2 A_{t-s}(\Delta_g v, P_s\varphi)(t-s)\biggr], \end{aligned}$$

and this rewrites easily as inequality (59).

The proof of Theorem 4.1 relies on a modified version of the function

$$s \mapsto \frac{\mathrm{d}}{\mathrm{d}s}A_t(u,\varphi)(s).$$

For any closed Riemannian manifold (M^n, g), any for $t > 0$ and for any positive "gauging" function $J\colon [0,t]\times M \to \mathbb{R}$, we define

$$B_J(u,\varphi)(s) := \int_M |dP_{t-s}u|^2 (P_s\varphi) J_{t-s}\,\mathrm{d}v_g$$

for any s, u and φ as above, where $J_{t-s}(\cdot) = J(t-s,\cdot)$. For the sake of simplicity, from now on in this section we write $P_\tau u$, $P_\tau\varphi$ for u_τ, φ_τ for any $\tau > 0$, and we write J instead of J_{t-s}.

Lemma 4.3 *Let (M^n, g) be a closed Riemannian manifold, let $t > 0$ and let $J\colon [0,t]\times M \to (0,+\infty)$ be a smooth function. Then for any $\varepsilon > 0$,*

$$\frac{\mathrm{d}}{\mathrm{d}s} B_J(u,\varphi)(s) \ge \int_M \varphi_s \left(\left(-\Delta_g J - \dot{J} - \frac{2}{\varepsilon}\frac{|dJ|^2}{J} - 2J\,\mathrm{Ric}_-\right)|du_{t-s}|^2 + 2(1-\varepsilon)J\,\frac{(\Delta_g u_{t-s})^2}{n}\right)\mathrm{d}v_g \tag{60}$$

for any $s \in [0,t]$, where $\dot{J}$ is a shorthand for $\partial J/\partial s$.

Proof When deriving $B_J(u,\varphi)$ with respect to s, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s} B_J(u,\varphi)(s) &= \int_M \varphi_s\big(-\Delta_g(J|du_{t-s}|^2) - \dot{J}|du_{t-s}|^2 + 2J\langle d\Delta_g u_{t-s}, du_{t-s}\rangle\big)\,\mathrm{d}v_g \\ &= \int_M \varphi_s\big(-(\Delta_g J)|du_{t-s}|^2 - J\Delta_g|du_{t-s}|^2 + 2\langle dJ, \nabla|du_{t-s}|^2\rangle \\ &\qquad\qquad - \dot{J}|du_{t-s}|^2 + 2J\langle d\Delta_g u_{t-s}, du_{t-s}\rangle\big)\,\mathrm{d}v_g,\end{aligned}$$

where we have used the Leibniz formula

$$\begin{aligned}\Delta_g(J|du_{t-s}|^2) &= (\Delta_g J)|du_{t-s}|^2 + J\Delta_g(|du_{t-s}|^2) - 2\langle\nabla|du_{t-s}|^2, dJ\rangle \\ &= (\Delta_g J)|du_{t-s}|^2 + J\Delta_g(|du_{t-s}|^2) - 4\nabla du_{t-s}(du_{t-s}, dJ).\end{aligned}$$

Then by using the Bochner formula

$$-\Delta_g|df|^2 + 2\langle d\Delta_g f, df\rangle = 2(|\nabla df|^2 + \mathrm{Ric}(df,df))$$

with $f = u_{t-s}$, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}s} B_J(u,\varphi)(s) \\ &\quad= \int_M \varphi_s\big((-\Delta_g J - \dot{J})|du_{t-s}|^2 + 2J(|\nabla du_{t-s}|^2 + \mathrm{Ric}(du_{t-s}, du_{t-s})) + 4\nabla du_{t-s}(dJ, du_{t-s})\big)\,\mathrm{d}v_g.\end{aligned}$$

By using the fact that, for all $x \in M$, $\mathrm{Ric}_x \ge -\mathrm{Ric}_-(x)$ we get the lower bound

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}s} B_J(u,\varphi)(s) \\ &\quad\ge \int_M \varphi_s\left[(-\Delta_g J - \dot{J} - 2J\,\mathrm{Ric}_-)|du_{t-s}|^2 + 2J|\nabla du_{t-s}|^2 + 4\sqrt{J}\nabla du_{t-s}\left(\frac{dJ}{\sqrt{J}}, du_{t-s}\right)\right]\mathrm{d}v_g.\end{aligned}$$

Now, for any $\varepsilon > 0$, we have

$$2\sqrt{J}\nabla du_{t-s}\left(\frac{dJ}{\sqrt{J}}, du_{t-s}\right) \geq -\left(\varepsilon J|\nabla du_{t-s}|^2 + \frac{1}{\varepsilon}\frac{|dJ|^2}{J}|du_{t-s}|^2\right).$$

Therefore we get

$$\frac{\mathrm{d}}{\mathrm{d}s}B_J(u,\varphi)(s) \geq \int_M \varphi_s\left[\left(-\Delta_g J - \dot{J} - \frac{2}{\varepsilon}\frac{|dJ|^2}{J} - 2J\,\mathrm{Ric}_-\right)|du_{t-s}|^2 + 2(1-\varepsilon)J|\nabla du_{t-s}|^2\right]\mathrm{d}v_g.$$

We conclude by using $|\nabla du_{t-s}|^2 \geq (\Delta_g u_{t-s})^2/n$. □

Lemma 4.4 *Let (M^n, g) be a closed Riemannian manifold satisfying (sK) for some $t > 0$. Set $\varepsilon := 1 - e^{-8\,\mathrm{k}_t(M,g)}$. Then there exists a unique solution $J\colon [0,t]\times M \to (0,+\infty)$ to the problem*

$$\text{(EJ)}\qquad \begin{cases} \Delta_g J + \dfrac{\partial J}{\partial s} + \dfrac{2}{\varepsilon}\dfrac{|dJ|^2}{J} + 2J\,\mathrm{Ric}_- = 0, \\ J(0,x) = 1, \end{cases}$$

which satisfies

$$e^{-4\,\mathrm{k}_t(M,g)} \leq J \leq 1.$$

Proof Consider

$$\delta = \frac{2}{\varepsilon} - 1 = \frac{1 + e^{-8\mathrm{k}_t(M,g)}}{1 - e^{-8\mathrm{k}_t(M,g)}} \geq 1.$$

We have

$$\Delta_g(J^{-\delta}) = -\delta J^{-\delta-1}\left(\Delta_g J + (\delta+1)\frac{|dJ|^2}{J}\right) = -\delta J^{-\delta-1}\left(\Delta_g J + \frac{2}{\varepsilon}\frac{|dJ|^2}{J}\right).$$

Define $I := J^{-\delta}$. Then J solves (EJ) if and only if I is a solution of

$$\begin{cases} \Delta_g I + \dfrac{\partial I}{\partial s} - 2\delta I\,\mathrm{Ric}_- = 0, \\ I(0,x) = 1. \end{cases}$$

By Duhamel's formula, this latter equation is equivalent to the integral equation

$$\text{(61)}\qquad I(s,x) = 1 + 2\delta\int_0^s\int_M H(s-\tau,x,y)\,\mathrm{Ric}_-(y)\,I(\tau,y)\,\mathrm{d}v_g(y)\,\mathrm{d}\tau.$$

Consider the map $L^\infty([0,t]\times M) \ni f \mapsto Tf \in L^\infty([0,t]\times M)$ defined by

$$(Tf)(s,x) := 2\delta\int_0^s\int_M H(s-\tau,x,y)\,\mathrm{Ric}_-(y)\,f(\tau,y)\,\mathrm{d}v_g(y)\,\mathrm{d}\tau.$$

By the definition of $\mathrm{k}_t(M,g)$, the operator norm of T satisfies

$$\text{(62)}\qquad \|T\|_{L^\infty\to L^\infty} \leq 2\delta\,\mathrm{k}_t(M,g).$$

Since

$$2x\frac{1+e^{-8x}}{1-e^{-8x}} \le \frac{1+e^{-1}}{4(1-e^{-1})} \le \frac{2}{3}$$

for any $x \in \left(0, \frac{1}{8}\right]$, the definition of δ and (sK) imply that

$$\|T\|_{L^\infty \to L^\infty} < 1.$$

As a consequence, the operator $\mathrm{Id} - T$ is invertible, hence $I = (\mathrm{Id} - T)^{-1}\mathbf{1}$ is the unique solution of the equation (61). Moreover, by (62), I satisfies

$$\|I\|_{L^\infty} \le \frac{1}{1 - 2\delta\,\mathrm{k}_t(M,g)}.$$

Since $J = I^{-1/\delta}$, we get $(1 - 2\delta\,\mathrm{k}_t(M,g))^{1/\delta} \le J \le 1$, and we conclude from the fact that if $x \in \left[0, \frac{2}{3}\right]$ then $e^{-2x} \le (1-x)$. □

Corollary 4.5 *Let (M^n, g) be a closed Riemannian manifold satisfying (sK) for some $t > 0$. Then for all $u \in C^1(M)$ and $\varphi \in C^0(M)$ with $\varphi \ge 0$ and $\tau \in (0, t]$,*

$$\int_M \varphi |dP_\tau u|^2 \,\mathrm{d}v_g \le e^{4\mathrm{k}_t(M,g)} \int_M P_\tau \varphi\, |du|^2 \,\mathrm{d}v_g. \tag{63}$$

Proof We only prove (63) for $\tau = t$ as our proof remains true if t is replaced by any $\tau \in (0, t]$. First observe that inequality (60) and the lower bound for J given by Lemma 4.4 imply

$$\frac{\mathrm{d}}{\mathrm{d}s} B_J(u,\varphi)(s) \ge \frac{2e^{-12\mathrm{k}_t(M,g)}}{n} \int_M \varphi_s (\Delta_g u_{t-s})^2 \,\mathrm{d}v_g. \tag{64}$$

In particular,

$$\frac{\mathrm{d}}{\mathrm{d}s} B_J(u,\varphi)(s) \ge 0.$$

Therefore, when integrating between 0 and t, we get

$$\int_M \varphi (P_t |du|^2 - J\,|dP_t u|^2) \,\mathrm{d}v_g \ge 0,$$

which leads to

$$\int_M P_t \varphi |du|^2 \,\mathrm{d}v_g \ge \int_M J\,|dP_t u|^2 \,\mathrm{d}v_g.$$

Inequality (63) then immediately follows by using the lower bound $J \ge e^{-4\mathrm{k}_t(M,g)}$. □

Remark 4.6 Corollary 4.5 can be rephrased in the following way: if (M^n, g) is a closed Riemannian manifold satisfying (sK), then for any $u \in W^{1,2}(M)$ and any $t > 0$,

$$|dP_t u|^2 \le e^{4\mathrm{k}_t(M,g)} P_t(|du|^2)$$

holds in the weak sense. Of course, the right-hand side can be bounded from above by $e^{1/4n} P_t(|du|^2)$. Thus, as a direct consequence of P_t being nonnegative and sub-Markovian, if u is κ–Lipschitz, then $P_t u$ is $e^{1/8n}\kappa$–Lipschitz.

We are now in a position to prove Theorem 4.1.

Proof of Theorem 4.1 We consider again inequality (64). By definition of A and since $P_{t-s}\Delta_g u = \Delta_g P_{t-s}u$, we can write it as

$$\frac{\mathrm{d}}{\mathrm{d}s}B_J(u,\varphi)(s) \geq e^{-12\mathrm{k}_t(M,g)}\frac{4}{n}A(\Delta_g u,\varphi)(s).$$

Since A is monotone nondecreasing in s, $A(\Delta_g u,\varphi)(s)$ is bounded from below by its value in $s=0$. Then we get

$$\frac{\mathrm{d}}{\mathrm{d}s}B_J(u,\varphi)(s) \geq e^{-12\mathrm{k}_t(M,g)}\frac{2}{n}\int_M \varphi(\Delta_g u_t)^2\,\mathrm{d}v_g.$$

We integrate this latter inequality between 0 and t, so that we get

$$\int_M (\varphi_t|du|^2 - \varphi J_t|du_t|^2)\,\mathrm{d}v_g \geq e^{-12\mathrm{k}_t(M,g)}\frac{2}{n}t\int_M \varphi(\Delta_g u_t)^2\,\mathrm{d}v_g.$$

Using the lower bound of Lemma 4.4 for J, we get

$$\begin{aligned}\int_M \varphi_t|du|^2\,\mathrm{d}v_g &\geq e^{-4\mathrm{k}_t(M,g)}\int_M \varphi|du_t|^2\,\mathrm{d}v_g + e^{-12\mathrm{k}_t(M,g)}\frac{2}{n}t\int_M \varphi(\Delta_g u_t)^2\,\mathrm{d}v_g\\ &\geq e^{-12\mathrm{k}_t(M,g)}\biggl(\int_M \varphi|du_t|^2\,\mathrm{d}v_g + \frac{2}{n}t\int_M \varphi(\Delta_g u_t)^2\,\mathrm{d}v_g\biggr).\end{aligned}$$

We also have $\mathrm{k}_s(M,g) \leq \mathrm{k}_t(M,g) < \frac{1}{8}$ for any $s \in (0,t]$. Hence if $s \in (0,t]$, then

$$\int_M \varphi_s|du|^2\,\mathrm{d}v_g \geq e^{-12\mathrm{k}_t(M,g)}\biggl(\int_M \varphi|du_s|^2\,\mathrm{d}v_g + \frac{2}{n}s\int_M \varphi(\Delta_g u_s)^2\,\mathrm{d}v_g\biggr).$$

Apply this with $t=s$ and $u=P_{t-s}v=v_{t-s}$ to get

(65) $$\int_M \varphi_s|dv_{t-s}|^2\,\mathrm{d}v_g \geq e^{-12\mathrm{k}_t(M,g)}\biggl(\int_M \varphi|dv_t|^2\,\mathrm{d}v_g + \frac{2}{n}s\int_M \varphi(\Delta_g v_t)^2\,\mathrm{d}v_g\biggr).$$

Observe that the left-hand side of the previous inequality can be rewritten as the following derivative with respect to s:

$$\int_M \varphi_s|dv_{t-s}|^2\,\mathrm{d}v_g = \frac{\mathrm{d}}{\mathrm{d}s}\int_M \varphi_s\frac{v_{t-s}^2}{2}\,\mathrm{d}v_g.$$

Taking this into account while integrating (65) between 0 and t yields (58). □

Remark 4.7 Theorem 4.2 and Corollary 4.5 are formulated for Riemannian manifolds, but they also hold for the Dirichlet space obtained when rescaling the Riemannian measure by a constant. More precisely, consider a closed Riemannian manifold (M^n,g) satisfying the condition (sK) and, for any $c>0$, the Dirichlet space $(M,\mathrm{d}_g,\mu,\mathscr{E})$ defined by setting

$$\mu = \frac{v_g}{c} \quad\text{and}\quad \mathscr{E}(u) = \int_M |du|^2\,\mathrm{d}\mu$$

for any $u \in W^{1,2}(M)$. Then the conclusions of Theorem 4.2 and Corollary 4.5 also hold for $(M,\mathrm{d}_g,\mu,\mathscr{E})$.

4.2 Convergence of the energy

Let us prove now that the Cheeger energy of a Kato limit space $(X, \mathsf{d}, \mu, o, \mathcal{E})$ coincides with the limit Dirichlet energy $\mathcal{E}$. Recall that $(X, \mathsf{d}, \mu, o, \mathcal{E})$ is the limit of pointed manifolds $\{(M_\alpha, \mathsf{d}_{g_\alpha}, \mu_\alpha, o_\alpha, \mathcal{E}_\alpha)\}_\alpha$, where the measure and the energy are defined as

$$\mu_\alpha = \frac{v_{g_\alpha}}{v_{g_\alpha}(B_{\sqrt{T}}(o_\alpha))} \quad \text{and} \quad \mathcal{E}_\alpha(u) = \int_{M_\alpha} |du|^2 \, \mathrm{d}\mu_\alpha$$

for any $u \in C^1(M_\alpha)$.

Theorem 4.8 *Let $(X, \mathsf{d}, \mu, o, \mathcal{E})$ be a Kato limit space. Then $\mathcal{E} = \mathsf{Ch}_\mathsf{d}$.*

Remark 4.9 Theorem 4.8 implies that for Kato limit spaces (X, d, μ, o), the pmGH convergence of the approximating sequence of manifolds implies the Mosco convergence of the associated energies. As a consequence, if X is compact then we have convergence of the spectrum of the rescaled Laplacians of the approximating manifolds to the spectrum of the Laplacian associated with the Cheeger energy. This generalizes results of Cheeger and Colding [26, Section 7], where a uniform lower bound on the Ricci curvature is assumed, and of Fukaya [41] under a uniform bound on the sectional curvature.

Proof of Theorem 4.8 Let $\{(M_\alpha, g_\alpha, o_\alpha)\}_\alpha$ be a sequence of pointed Riemannian manifolds satisfying a uniform Kato bound and such that $\{(M_\alpha, \mathsf{d}_{g_\alpha}, \mu_\alpha, o_\alpha, \mathcal{E}_\alpha)\}_{\alpha \in A}$ converges in the Mosco–Gromov–Hausdorff sense to $(X, \mathsf{d}, \mu, o, cE)$. Let $T > 0$ and $f : (0, T] \to [0, +\infty)$ be the nondecreasing function in Definition 2.1. We know from Proposition 2.12 that $\mathsf{d} = \mathsf{d}_\mathcal{E}$. Moreover, by Remark 2.9, we get that $(X, \mathsf{d}_\mathcal{E}, \mu, \mathcal{E})$ is a PI(R)–Dirichlet space. Therefore, thanks to Proposition 1.21, we are left with showing that for any $u \in \mathcal{D}(\mathcal{E})$, any nonnegative $\varphi \in \mathcal{D}(\mathcal{E}) \cap \mathcal{C}_c(X)$ and any $t \in [0, T]$,

$$\int_X \varphi \, \mathrm{d}\Gamma(P_t u) \le e^{4f(t)} \int_X P_t \varphi \, \mathrm{d}\Gamma(u). \tag{66}$$

Let u and φ be as above. Set $L := \|\varphi\|_{L^\infty}$ and let $R > 0$ be such that $\operatorname{supp} \varphi \subset B_R(o)$. Let $\{u_\alpha\}_\alpha, \{\varphi_\alpha\}$ be two sequences, where $u_\alpha \in \mathcal{D}(\mathcal{E}_\alpha)$ and $\varphi_\alpha \in \mathcal{C}_c(X_\alpha) \cap \mathcal{D}(\mathcal{E}_\alpha)$, such that

- $u_\alpha \xrightarrow{\mathrm{E}} u$,
- the sequence $\{\varphi_\alpha\}_\alpha$ converges uniformly to φ, and
- $0 \le \varphi_\alpha \le L$ and $\operatorname{supp} \varphi_\alpha \subset B_{R+1}(o_\alpha)$ for any α.

The Mosco convergence $\mathcal{E}_\alpha \to \mathcal{E}$ guarantees the existence of $\{u_\alpha\}$ while Proposition A.1 ensures the existence of $\{\varphi_\alpha\}$. Let $(P_t^\alpha)_{t \ge 0}$ (resp. $(P_t)_{t \ge 0}$) be the heat semigroup of the Dirichlet space $(M_\alpha, \mathsf{d}_\alpha, \mu_\alpha, \mathcal{E}_\alpha)$ (resp. $(X, \mathsf{d}, \mu, \mathcal{E})$) for any α. Fix $t \in (0, T]$ and $s \in [0, t]$. Set

$$a(s) := A_t(u, \varphi)(s) \quad \text{and} \quad a_\alpha(s) := A_t^\alpha(u_\alpha, \varphi_\alpha)(s) = \frac{1}{2} \int_{M_\alpha} (P_{t-s}^\alpha u_\alpha)^2 P_s^\alpha \varphi_\alpha \, \mathrm{d}\mu_\alpha.$$

We claim that

$$\lim_{\alpha} a_\alpha(s) = a(s). \tag{67}$$

Indeed, by (iii) in Proposition 1.15, the sequence $\{P^\alpha_{t-s}u_\alpha\}_\alpha$ converges strongly in L^2 to $P_{t-s}u$. Moreover, let us prove that $\{P^\alpha_s\varphi_\alpha\}_\alpha$ converges uniformly on compact sets to $P_s\varphi$. Since $P_s\varphi$ is continuous, this follows from showing that for any $x \in X$ and any given sequence $\{x_\alpha\}$ such that $M_\alpha \ni x_\alpha \to x \in X$,

$$\lim_{\alpha} P^\alpha_s\varphi_\alpha(x_\alpha) = P_s\varphi(x). \tag{68}$$

Let H_α (resp. H) be the heat kernel of $(M_\alpha, \mathsf{d}_\alpha, \mu_\alpha, \mathcal{E}_\alpha)$ (resp. $(X, \mathsf{d}, \mu, \mathcal{E})$) for any α. We have

$$P^\alpha_s\varphi_\alpha(x_\alpha) = \int_{M_\alpha} H_\alpha(s, x_\alpha, y)\varphi_\alpha(y)\,\mathrm{d}\mu_\alpha(y)$$

for any α, and similarly

$$P_s\varphi(x) = \int_X H(s,x,y)\varphi(y)\,\mathrm{d}\mu(y).$$

The heat kernel estimate (20) ensures that we can apply Proposition B.3 to the continuous functions $\{H_\alpha(s, x_\alpha, \cdot)\varphi_\alpha(\cdot)\}_\alpha$, $H(s,x,y)\varphi(y)$: this directly establishes (68). We are then in a position to apply Proposition B.1 to make the integrals

$$a_\alpha(s) = \int_X (P^\alpha_{t-s}u_\alpha)^2 P^\alpha_s\varphi_\alpha\,\mathrm{d}\mu_\alpha$$

converge to $a(s)$, as claimed in (67).

From (57), we have

$$a'_\alpha(s) = \int_{M_\alpha} P^\alpha_s\varphi_\alpha |dP^\alpha_{t-s}u_\alpha|^2\,\mathrm{d}\mu_\alpha.$$

Assume from now on that $s \in (0, t/2)$. According to Corollary 4.5 and Remark 4.7, we know that

$$a'_\alpha(s) \le e^{4f(t)} a'_\alpha(t-s).$$

Integrating this inequality between 0 and s and dividing by s yields

$$\frac{a_\alpha(s) - a_\alpha(0)}{s} \le e^{4f(t)}\frac{a_\alpha(t) - a_\alpha(t-s)}{s}.$$

Letting $\alpha \to \infty$, then $s \to 0$, and using expression (57) for $a'(s) = A'_t(u,\varphi)(s)$ imply inequality (66). □

Remarks 4.10 Similarly to what we pointed out in Remark 2.13, it is clear that Theorem 4.8 also holds when we assume the following more general assumption: there exists a nondecreasing function $f : (0,T] \to \mathbb{R}_+$ such that $\lim_0 f = 0$ and

$$\limsup_{\alpha\to\infty} \mathrm{k}_t(M_\alpha, g_\alpha) \le f(t) \quad \text{for all } t \in (0,T].$$

In particular, when $\lim_\alpha \mathrm{k}_T(M_\alpha, g_\alpha) = 0$, we have both $\mathsf{d} = \mathsf{d}_{\mathcal{E}}$ and $\mathcal{E} = \mathsf{Ch}$.

According to Remark 2.10, tangent cones (and rescalings $(Z, \mathsf{d}_Z, \mu_Z, z, \mathcal{E}_Z)$ centered at convergent points) of a Kato limit are obtained as limits of manifolds such that for all $t > 0$, $\mathrm{k}_t(M_\alpha, g_\alpha) \to 0$ as $\alpha \to \infty$. Theorem 4.8 then applies to such spaces.

4.3 The RCD condition for a certain class of Kato limit spaces

In the next key result we prove that any Kato limit space obtained from manifolds $\{(M_\alpha, g_\alpha)\}$ with $\mathrm{k}_T(M_\alpha^n, g_\alpha) \to 0$ for some $T > 0$ satisfies the Riemannian Curvature Dimension condition $\mathrm{RCD}(0, n)$.

Theorem 4.11 *Let $(X, \mathsf{d}, \mu, o, \mathcal{E})$ be a pointed metric Dirichlet space obtained as the pointed Mosco–Gromov–Hausdorff limit of spaces $\{(M_\alpha, \mathsf{d}_{g_\alpha}, \mu_\alpha, o_\alpha, \mathcal{E}_\alpha)\}$, where $\{(M_\alpha^n, g_\alpha, o_\alpha)\}$ is a sequence of closed pointed Riemannian manifolds such that*

$$\text{(H)} \qquad \mathrm{k}_T(M_\alpha^n, g_\alpha) \to 0 \quad \textit{for some } T > 0.$$

Then (X, d, μ) is an $\mathrm{RCD}(0, n)$ space.

Proof Our goal is to establish the Bakry–Ledoux $\mathrm{BL}(0, n)$ estimate: for $t \in (0, T)$, $u \in \mathcal{D}(\mathcal{E})$ and $\varphi \in \mathcal{D}(\mathcal{E}) \cap \mathcal{C}_c(X)$ with $\varphi \geq 0$, we aim at showing that

$$(69) \qquad \frac{1}{2} \int_X \varphi(P_t u^2 - (P_t u)^2) \, \mathrm{d}\mu \geq \int_X \varphi\left(t \frac{\mathrm{d}\Gamma(P_t u)}{\mathrm{d}\mu} + \frac{t^2}{n} (\Delta P_t u)^2\right) \mathrm{d}\mu,$$

starting from inequality (59) in Theorem 4.2. Like in the proof of the previous Theorem 4.8, we let $\{u_\alpha\}_\alpha$ and $\{\varphi_\alpha\}$ be two sequences where $u_\alpha \in \mathcal{D}(\mathcal{E}_\alpha)$ and $\varphi_\alpha \in \mathcal{C}_c(X_\alpha) \cap \mathcal{D}(\mathcal{E}_\alpha)$ for any α, such that

- $u_\alpha \xrightarrow{\mathrm{E}} u$,
- the sequence $\{\varphi_\alpha\}_\alpha$ converges uniformly to φ, and
- for some $L, R > 0$, $0 \leq \varphi_\alpha \leq L$ and $\operatorname{supp} \varphi_\alpha \subset B_{R+1}(o_\alpha)$ for any α.

Take $t \in (0, T]$ and $0 \leq s \leq t$. As in the proof of Theorem 4.8, we set $a_\alpha(s) := A_t^\alpha(u_\alpha, \varphi_\alpha)(s)$ for any α and $a(s) := A_t(u, \varphi)(s)$. We know from there that

$$\lim_\alpha a_\alpha(s) = a(s).$$

For any α, we let $(P_t^\alpha)_{t \geq 0}$ be the heat semigroup associated with $\mathcal{E}_\alpha$ and we set

$$c_\alpha(s) = \int_{M_\alpha} P_s^\alpha \varphi_\alpha (\Delta_\alpha P_{t-s}^\alpha u_\alpha)^2 \, \mathrm{d}\mu_\alpha \quad \text{and} \quad c(s) = \int_X P_s \varphi (\Delta P_{t-s} u)^2 \, \mathrm{d}\mu.$$

Thanks to (v) in Proposition 1.15, we know that $\Delta_\alpha P_t^\alpha$ strongly converges to ΔP_t in the sense of bounded operators. In particular, for any $s \in [0, t)$, we have the strong convergence

$$\Delta_\alpha P_{t-s}^\alpha u_\alpha \xrightarrow{L^2} \Delta P_{t-s} u,$$

hence the same argument to get the convergence $a_\alpha(s) \to a(s)$ gives

$$\lim_\alpha c_\alpha(s) = c(s).$$

Thanks to Theorem 4.2 and Remark 4.7, inequality (59) holds on $(M_\alpha, \mathsf{d}_{g_\alpha}, \mu_\alpha, \mathscr{E}_\alpha)$ for all α, so that for any $0 < s < t < T$ we obtain

$$a_\alpha(0) - a_\alpha(s) \geq e^{-12\mathrm{k}_t(M_\alpha, g_\alpha)}\left((t-s)a'_\alpha(s) + \frac{2(t-s)^2}{n} c_\alpha(s)\right).$$

Consider a nonnegative test function $\xi \in \mathscr{C}_0^\infty((0,t))$. When multiplying the previous inequality by $\xi/(t-s)$ and integrating over $[0,t]$, we obtain

$$\int_0^t \frac{\xi(s)}{(t-s)} \,\mathrm{d}s\, a_\alpha(0) - \int_0^t \frac{\xi(s)}{(t-s)} a_\alpha(s)\,\mathrm{d}s \geq e^{-12\mathrm{k}_t(M_\alpha, g_\alpha)}\left(\int_0^t \xi(s) a'_\alpha(s)\,\mathrm{d}s + \int_0^t \xi(s)\frac{2(t-s)}{n} c_\alpha(s)\,\mathrm{d}s\right).$$

Integrating by parts, we get

$$\int_0^t \frac{\xi(s)}{(t-s)} \,\mathrm{d}s\, a_\alpha(0) - \int_0^t \frac{\xi(s)}{(t-s)} a_\alpha(s)\,\mathrm{d}s \geq e^{-12\mathrm{k}_t(M_\alpha, g_\alpha)}\left(-\int_0^t \xi'(s) a_\alpha(s)\,\mathrm{d}s + \int_0^t \xi(s)\frac{2(t-s)}{n} c_\alpha(s)\,\mathrm{d}s\right).$$

We notice that for any $s \in [0,t]$,

$$a_\alpha(s) \leq L\|u_\alpha\|^2_{L^2} \quad \text{and} \quad c_\alpha(s) \leq \frac{L}{(t-s)^2}\|u_\alpha\|^2_{L^2}.$$

Recall that ξ is compactly supported in the open interval $(0,t)$; hence we can use the dominated convergence theorem to get that

$$\int_0^t \frac{\xi(s)}{(t-s)} \,\mathrm{d}s\, a(0) - \int_0^t \frac{\xi(s)}{(t-s)} a(s)\,\mathrm{d}s \geq -\int_0^t \xi'(s) a(s)\,\mathrm{d}s + \int_0^t \xi(s)\frac{2(t-s)}{n} c(s)\,\mathrm{d}s.$$

The functions a, a' and c are continuous on $[0,t)$. The continuity of a and c follows from [4, Lemma 2.1]. The continuity of a' follows similarly from the same lemma. Indeed, we assumed that φ is continuous with compact support; hence the function $[0,t] \ni s \mapsto P_s\varphi \in \mathscr{C}_0^0(X)$ is continuous. Moreover, the map $[0,t) \ni s \mapsto P_{t-s}u \in \mathscr{D}(\mathscr{E})$ is continuous, so $[0,t) \ni s \mapsto \mathrm{d}\Gamma(P_{t-s}u) \in \mathrm{Rad}(X) = (\mathscr{C}_0^0(X))'$ is also continuous, therefore a' is continuous.

As a consequence, the function a is C^1 on $[0,t)$, thus by integration by parts we obtain

$$\int_0^t \frac{\xi(s)}{(t-s)} \,\mathrm{d}s\, a(0) - \int_0^t \frac{\xi(s)}{(t-s)} a(s)\,\mathrm{d}s \geq \int_0^t \xi(s) a'(s)\,\mathrm{d}s + \int_0^t \xi(s)\frac{2(t-s)}{n} c(s)\,\mathrm{d}s.$$

Moreover, since a and c are continuous on $[0,t)$, for any $s \in [0,t)$ we get

$$a(0) - a(s) \geq (t-s)a'(s) + \frac{2(t-s)^2}{n} c(s),$$

that is to say,

$$\frac{1}{2}\left(\int_X \varphi P_t(u^2)\,\mathrm{d}\mu - \int_X P_s\varphi\,(P_{t-s}u)^2\,\mathrm{d}\mu\right) \geq (t-s)\int_X P_s\varphi\,\mathrm{d}\Gamma(P_{t-s}u) + \frac{(t-s)^2}{n}\int_X P_s\varphi(\Delta P_{t-s}u)^2\,\mathrm{d}\mu.$$

Inequality (69) coincides with this latter inequality when taking $s = 0$. □

Thanks to Remarks 2.13 and 4.10, the two previous theorems allow us to get the following result for tangent cones of Kato limit spaces.

Corollary 4.12 *Let $(X, \mathsf{d}, \mu, o, \mathcal{E})$ be a Kato (resp. noncollapsed strong Kato) limit space. Then for any $x \in X$, any measured tangent cone $(X_x, \mathsf{d}_x, \mu_x, x)$ is a* RCD$(0, n)$ *(resp. weakly noncollapsed* RCD$(0, n)$*) metric measure space.*

The fact that tangent cones of noncollapsed strong Kato limits are weakly noncollapsed follows from the local Ahlfors regularity given in Remark 2.18.

Remark 4.13 Corollary 4.12 applies in particular to iterated tangent cones and to convergent rescalings (not necessarily centered at the same point) of a Kato limit. Indeed, let $(X, \mathsf{d}, \mu, o, \mathcal{E})$ be a Kato limit and assume that for some sequence $\{\varepsilon_\alpha\} \subset (0, +\infty)$ such that $\varepsilon_\alpha \downarrow 0$ and some points $\{x_\alpha\} \subset X$, the sequence of pointed rescalings $\{(X, \varepsilon_\alpha^{-1}\mathsf{d}, \varepsilon_\alpha^{-n}\mu, x_\alpha)\}$ converges to some pointed metric measure space $(Z, \mathsf{d}_Z, \mu_Z, z)$. Thanks to Remark 2.19, we know that (Z, d_Z, μ_Z) has the same properties as a tangent cone. As a consequence, (Z, d_Z, μ_Z) is a weakly noncollapsed RCD$(0, n)$ space.

5 Tangent cones are metric cones

In this section, we prove that the tangent cones of a noncollapsed strong Kato limit are all metric cones. To this aim, we introduce two crucial quantities. Let $(X, \mathsf{d}, \mu, \mathcal{E})$ be a metric Dirichlet space admitting a heat kernel H. We recall that n is a given positive integer. For any $x \in X$ and $s, t > 0$, we define the Θ*–volume* by

$$\Theta_x(s) := (4\pi s)^{-n/2} \int_X e^{-\mathsf{d}^2(x,y)/4s} \, \mathrm{d}\mu(y)$$

and similarly, the θ*–volume* by

$$\theta_x(s, t) := (4\pi s)^{-n/2} \int_X e^{-U(t,x,y)/4s} \, \mathrm{d}\mu(y),$$

where U is defined by

$$H(t, x, y) = (4\pi t)^{-n/2} e^{-U(t,x,y)/4t} \quad \text{for any } y \in X. \tag{70}$$

Remark 5.1 We point out the following properties of θ_x and Θ_x.

(i) $\theta_x(t, t) = \int_X H(t, x, y) \, \mathrm{d}\mu(y)$ is identically 1 if the Dirichlet space is stochastically complete, for instance for PI–Dirichlet spaces.

(ii) The Chapman–Kolmogorov property yields

$$\theta_x\big(\tfrac{1}{4}t, \tfrac{1}{2}t\big) = (4\pi t)^{n/2} H(t, x, x). \tag{71}$$

(iii) We always have the following lower bound: for any $r > 0$,

$$\Theta_x(s) \geq (4\pi s)^{-n/2} e^{-r^2/4s} \mu(B_r(x)), \tag{72}$$

hence if μ is nontrivial, then Θ_x is always positive (but perhaps infinite).

When it is necessary, we specify the space X to which the previous quantities are associated by writing $\Theta_x^X(s)$ and $\theta_x^X(s,t)$.

5.1 On a doubling space

We start by recalling an elementary property of the Θ–volume that relates it to the density of volume at a given point.

Lemma 5.2 *Let (X, d, μ) be κ–doubling at scale R.*

(i) *The function $\mathbb{R}_+^* \times X \ni (s,x) \mapsto \Theta_x(s)$ is continuous.*

(ii) *For all $x \in X$ and $s > 0$ we have that $\Theta_x(s)$ is finite. More precisely, there is a constant A depending only on κ such that for any $x \in X$ and $s \in (0, R^2]$,*

$$\Theta_x(s) \le A \frac{\mu(B_{\sqrt{s}}(x))}{s^{n/2}}. \tag{73}$$

(iii) *If there exists $\vartheta \in (0,+\infty)$ such that*

$$\lim_{r \to 0^+} \frac{\mu(B_r(x))}{\omega_n r^n} = \vartheta,$$

then $\lim_{s\to 0^+} \Theta_x(s) = \vartheta$.

Proof For any $x \in X$, by Cavalieri's formula we can write

$$\Theta_x(s) = \int_0^{+\infty} e^{-r^2/4s} \frac{r}{2s} \mu(B_r(x)) \frac{\mathrm{d}r}{(4\pi s)^{n/2}}.$$

A simple computation using Proposition 1.2(iii) ensures that this integral converges; hence $\Theta_x(s)$ is well-defined for any $s > 0$.

Moreover, Lebesgue's dominated convergence theorem implies that $\Theta_x(s)$ depends continuously on $(s,x) \in \mathbb{R}_+^* \times X$. In addition, when $0 < s \le R^2$, the estimate (73) follows from

$$\begin{aligned}
\Theta_x(s) &\le \frac{\mu(B_{\sqrt{s}}(x))}{(4\pi s)^{n/2}} + \int_{X \setminus B_{\sqrt{s}}(x)} \frac{e^{-\mathrm{d}^2(x,y)/4s}}{(4\pi s)^{n/2}} \, \mathrm{d}\mu(y) \\
&\le \frac{\mu(B_{\sqrt{s}}(x))}{(4\pi s)^{n/2}} + \int_{\sqrt{s}}^{+\infty} e^{-\rho^2/4s} \frac{\rho}{2s} \mu(B_\rho(x)) \frac{\mathrm{d}\rho}{(4\pi s)^{n/2}} \\
&\le \frac{\mu(B_{\sqrt{s}}(x))}{(4\pi s)^{n/2}} \left(1 + \int_{\sqrt{s}}^{+\infty} e^{-\rho^2/4s + \lambda\rho/\sqrt{s}} \frac{\rho}{2s} \, \mathrm{d}\rho\right) \\
&\le \frac{\mu(B_{\sqrt{s}}(x))}{(4\pi s)^{n/2}} \left(1 + \int_1^{+\infty} e^{-\rho^2/4 + \lambda\rho} \frac{\rho}{2} \, \mathrm{d}\rho\right).
\end{aligned}$$

Now assume that

$$\lim_{r \to 0^+} \frac{\mu(B_r(x))}{\omega_n r^n} = \vartheta.$$

We have, similarly,

$$\Theta_x(s) = \int_{B_R(x)} \frac{e^{-\mathsf{d}^2(x,y)/4s}}{(4\pi s)^{n/2}}\,\mathrm{d}\mu(y) + \int_{M\setminus B_R(x)} \frac{e^{-\mathsf{d}^2(x,y)/4s}}{(4\pi s)^{n/2}}\,\mathrm{d}\mu(y).$$

The same estimate yields

$$\begin{aligned}\int_{M\setminus B_R(x)} \frac{e^{-\mathsf{d}^2(x,y)/4s}}{(4\pi s)^{n/2}}\,\mathrm{d}\mu(y) &\le \frac{\mu(B_R(x))}{(4\pi s)^{n/2}} \int_R^\infty e^{-\rho^2/4s+\lambda\rho/R}\,\frac{\rho}{2s}\,\mathrm{d}\rho \\ &= \frac{\mu(B_R(x))}{(4\pi s)^{n/2}} \int_{R/\sqrt{s}}^\infty e^{-\rho^2/4+\lambda\rho\sqrt{s}/R}\,\frac{\rho}{2}\,\mathrm{d}\rho.\end{aligned}$$

Then

$$\lim_{s\to 0+} \int_{M\setminus B_R(x)} \frac{e^{-\mathsf{d}^2(x,y)/4s}}{(4\pi s)^{n/2}}\,\mathrm{d}\mu(y) = 0.$$

Moreover, we have

$$\int_{B_R(x)} \frac{e^{-\mathsf{d}^2(x,y)/4s}}{(4\pi s)^{n/2}}\,\mathrm{d}\mu(y) = \int_0^{R/\sqrt{s}} \frac{e^{-\rho^2/4}\omega_n\rho^{n+1}}{2(4\pi)^{n/2}}\,\frac{\mu(B_{\rho\sqrt{s}}(x))}{\omega_n(\rho\sqrt{s})^n}\,\mathrm{d}\rho.$$

Note that there is some constant C (depending on x) such that for $r \le R$,

$$\mu(B_r(x)) \le C r^n.$$

Hence the dominated convergence theorem and the fact that

$$\int_0^{+\infty} \frac{e^{-\rho^2/4}\omega_n\rho^{n+1}}{2(4\pi)^{n/2}}\,\mathrm{d}\rho = \Theta_0^{\mathbb{R}^n}(1) = 1 \quad \text{imply that} \quad \lim_{s\to 0+} \int_{B_R(x)} \frac{e^{-\mathsf{d}^2(x,y)/4s}}{(4\pi s)^{n/2}}\,\mathrm{d}\mu(y)) = \vartheta. \qquad \square$$

We also have the following assertion about the comparison between the measure of balls and the Θ–volume.

Lemma 5.3 *Let (X,d,μ) be a metric measure space, and $n > 0$.*

(i) *Let $x \in X$ and $c > 0$. Then $\Theta_x(s) = c$ for all $s > 0$ if and only if $\mu(B_r(x)) = c\omega_n r^n$ for all $r > 0$.*

(ii) *If (X,d,μ) is κ–doubling at scale R, then for any $s \in (0, R^2]$,*

$$a\frac{\mu(B_{\sqrt{s}}(x))}{(\sqrt{s})^n} \le \Theta_x(s) \le A\frac{\mu(B_{\sqrt{s}}(x))}{(\sqrt{s})^n}. \tag{74}$$

where the constant $a > 0$ depends only on n and the constant A depends only on κ.

(iii) *Assume that for some $R > 0$ and $x \in X$,*

$$c \le \Theta_x(s) \le C \quad \text{for all } s \in (0, R^2].$$

Then for any $r \in (0, R]$,

$$v r^n \le \mu(B_r(x)) \le V r^n,$$

where the constants v and V depend only on n, c and C.

Proof (i) This follows from Cavalieri's principle and some properties of the Laplace transform, see eg the proof of [20, Lemma 3.2].

(ii) This is a consequence of the two inequalities (72) and (73).

(iii) The upper bound is a consequence of (72). The lower bound follows from an argument used in the proof of [20, Lemma 5.1]. The estimate (72) also implies that, for any $\rho \geq \sqrt{t}$,

$$\mu(B_\rho(x)) \leq (4\pi s)^{n/2} e^{\rho^2/4s}\, C.$$

Hence, for any $\varepsilon \in (0,1)$,

$$\begin{aligned}
\Theta_x(\varepsilon s) &\leq \frac{\mu(B_{\sqrt{s}}(x))}{(4\pi\varepsilon s)^{n/2}} + \int_{X\setminus B_{\sqrt{s}}(x)} \frac{e^{-\mathrm{d}^2(x,y)/4\varepsilon s}}{(4\pi\varepsilon s)^{n/2}}\,\mathrm{d}\mu(y)\\
&\leq \frac{\mu(B_{\sqrt{s}}(x))}{(4\pi\varepsilon s)^{n/2}} + \int_{\sqrt{s}}^{+\infty} e^{-\rho^2/4\varepsilon s}\frac{\rho}{2\varepsilon s}\mu(B_\rho(x))\frac{\mathrm{d}\rho}{(4\pi\varepsilon s)^{n/2}}\\
&\leq \frac{\mu(B_{\sqrt{s}}(x))}{(4\pi\varepsilon s)^{n/2}} + \frac{C}{\varepsilon^{n/2}}\int_{\sqrt{s}}^{+\infty} e^{-(1/\varepsilon-1)\rho^2/4s}\frac{\rho}{2\varepsilon s}\,\mathrm{d}\rho\\
&\leq \frac{\mu(B_{\sqrt{s}}(x))}{(4\pi\varepsilon s)^{n/2}} - \frac{C}{\varepsilon^{n/2}}\left[\frac{e^{-(1/\varepsilon-1)\rho^2/4s}}{1-\varepsilon}\right]_{\sqrt{s}}^{+\infty}\\
&\leq \frac{\mu(B_{\sqrt{s}}(x))}{(4\pi\varepsilon s)^{n/2}} + \frac{Ce^{-(1/\varepsilon-1)/4}}{(1-\varepsilon)\varepsilon^{n/2}}.
\end{aligned}$$

Choosing ε small enough so that

$$\frac{e^{-(1/\varepsilon-1)/4}}{(1-\varepsilon)\varepsilon^{n/2}} \leq \frac{1}{2}\frac{c}{C} \quad \text{yields that} \quad \frac{c}{2}(4\pi\varepsilon s)^{n/2} \leq \mu(B_{\sqrt{s}}(x)).$$ □

The Θ–volume is continuous with respect to pointed measured Gromov–Hausdorff convergence:

Proposition 5.4 *Let $\{(X_\alpha,\mathsf{d}_\alpha,\mu_\alpha,o_\alpha)\}_\alpha$, (X,d,μ,o) be proper geodesic pointed metric measure spaces κ–doubling at scale R such that $(X_\alpha,\mathsf{d}_\alpha,\mu_\alpha,o_\alpha)\to(X,\mathsf{d},\mu,o)$ in the pmGH sense. Let $x_\alpha\in X_\alpha\to x\in X$. Then for any $s>0$,*

$$\lim_\alpha \Theta_{x_\alpha}^{X_\alpha}(s) = \Theta_x^X(s).$$

Proof This is a direct consequence of the Proposition B.3. □

5.2 On a Dirichlet space

The following gives a relationship between the Θ– and θ–volume on a PI–Dirichlet space.

Proposition 5.5 *If $(X,\mathsf{d}_{\mathcal{E}},\mu,\mathcal{E})$ is a PI(R)–Dirichlet space and the Θ–volume is defined with the intrinsic distance, then for any $s>0$,*

$$\Theta_x(s) = \lim_{t\to 0+}\theta_x(s,t).$$

Proof According to Varadhan's formula (22), we get that for any $x, y \in X$,

$$\mathsf{d}_{\mathscr{E}}^2(x,y) = \lim_{t\to 0+} U(t,x,y).$$

The Fatou lemma implies that

$$\Theta_x(s) \le \liminf_{t\to 0} \theta_x(s,t). \tag{75}$$

To get the limsup inequality, we will use the heat kernel upper bound (21)

$$H(t,x,y) \le \frac{C}{\mu(B_R(x))}\frac{R^\nu}{t^{\nu/2}}\left(1+\frac{\mathsf{d}_{\mathscr{E}}^2(x,y)}{t}\right)^{\nu+1} e^{-\mathsf{d}_{\mathscr{E}}^2(x,y)/4t}$$

for any $x, y \in X$ and $t \in (0, R^2)$. Since there exists $C > 0$ such that $\xi \mapsto \xi^{\nu+1}e^{-\xi} \le C$ for any $\xi > 0$, choosing $\xi = \varepsilon(1+\mathsf{d}_{\mathscr{E}}^2(x,y)/t)$ where $\varepsilon \in (0,1)$ leads to

$$H(t,x,y) \le \frac{C}{\mu(B_R(x))}\frac{R^\nu}{t^{\nu/2}}\varepsilon^{-\nu-1}e^{\varepsilon+(4\varepsilon-1)\mathsf{d}_{\mathscr{E}}^2(x,y)/4t},$$

which implies that there is a constant C not depending on $t \in (0, R^2]$ and $y \in X$ such that

$$-\tfrac14 U(t,x,y) \le tC\left(1+\log\left(\frac1t\right)\right) - (1-4\varepsilon)\cdot\tfrac14\mathsf{d}_{\mathscr{E}}^2(x,y) \tag{76}$$

and then, for any $s > 0$,

$$\theta_x(s,t) \le \left(\frac{e}{t}\right)^{Ct/s}\Theta_x\left(\frac{s}{1-4\varepsilon}\right). \tag{77}$$

So for any $\varepsilon \in (0,1)$,

$$\limsup_{t\to 0}\theta_x(s,t) \le \Theta_x\left(\frac{s}{1-4\varepsilon}\right).$$

Then the assertion follows from the continuity of Θ_x upon letting ε tend to 0. ◻

Remark 5.6 It is worth pointing out that if $(X, \mathsf{d}_{\mathscr{E}}, \mu, \mathscr{E})$ is a $\mathrm{PI}_{\kappa,\gamma}(R)$–Dirichlet space then there is a constant α such that for any $t \in (0, R^2)$ and any $x, y \in X$, we get the heat kernel bound

$$\frac{t^{n/2}}{\mu(B_{\sqrt t}(x))}\frac{1}{\alpha t^{n/2}}e^{-\alpha\,\mathsf{d}^2(x,y)/t} \le H(t,x,y) \le \frac{t^{n/2}}{\mu(B_{\sqrt t}(x))}\frac{\alpha}{t^{n/2}}e^{-\mathsf{d}^2(x,y)/\alpha t}.$$

This easily implies that

$$\left(\frac{(4\pi)^{n/2}}{\alpha}\right)^{t/s}\Theta_x\left(\frac{s}{4\alpha}\right) \le \left(\frac{t^{n/2}}{\mu(B_{\sqrt t}(x))}\right)^{-t/s}\theta_x(s,t) \le ((4\pi)^{n/2}\alpha)^{t/s}\Theta_x\left(\tfrac14\alpha s\right).$$

In particular, using Lemma 5.3, there is a positive constant $\eta > 0$ depending only κ, γ and n such that for any $x \in X$ and $t, s > 0$ with $t \le R^2$ and $s \le (16/\alpha^2)R^2$, then

$$\eta\frac{\mu(B_{\sqrt s}(x))}{s^{n/2}}\left(\frac{\mu(B_{\sqrt t}(x))}{t^{n/2}}\right)^{-t/s} \le \theta_x(s,t) \le \eta^{-1}\frac{\mu(B_{\sqrt s}(x))}{s^{n/2}}\left(\frac{\mu(B_{\sqrt t}(x))}{t^{n/2}}\right)^{-t/s}. \tag{78}$$

The heat kernel upper bound implies similarly the continuity of the θ–volume under the pointed Mosco–Gromov–Hausdorff convergence of $\mathrm{PI}(R)$–Dirichlet spaces.

Proposition 5.7 *Let $\{(X_\alpha, \mathsf{d}_\alpha, \mu_\alpha, o_\alpha, \mathcal{E}_\alpha)\}_\alpha, (X, \mathsf{d}, \mu, o, \mathcal{E})$ be pointed $\mathrm{PI}_{\kappa,\gamma}(R)$–Dirichlet spaces such that $(X_\alpha, \mathsf{d}_\alpha, \mu_\alpha, o_\alpha, \mathcal{E}_\alpha) \to (X, \mathsf{d}, \mu, o, \mathcal{E})$ in the pointed Mosco–Gromov–Hausdorff sense. Let $x \in X$ and $\{x_\alpha\}$ be such that $x_\alpha \in X_\alpha$ for any α and $x_\alpha \to x$. Then for any $s, t > 0$,*

$$\lim_\alpha \theta_{x_\alpha}^{X_\alpha}(s,t) = \theta_x^X(s,t).$$

Proof Let $s, t > 0$. We know that the sequence $f_\alpha(y) = H_\alpha(t, x_\alpha, y)$ converges uniformly on compact sets to $f(y) = H(t, x, y)$. Hence the same is true for the integrand

$$h_\alpha(y) = (4\pi s)^{-n/2}\big((4\pi t)^{n/2} H_\alpha(t, x_\alpha, y)\big)^{t/s} = (4\pi s)^{-n/2} e^{-U_{X_\alpha}(t,x,y)/4s},$$

which converges uniformly on compact sets to

$$h(y) = (4\pi s)^{-n/2} e^{-U_X(t,x,y)/4s}.$$

The uniform doubling condition and the uniform Poincaré inequality yield that there are positive constants C and ν, depending only on κ and γ, such that

$$H_\alpha(t, x_\alpha, y) \le \frac{C}{\mu(B_{R^2}(x_\alpha))} \max\left\{1, \frac{R^\nu}{t^{\nu/2}}\right\} e^{-\mathsf{d}_\alpha^2(x_\alpha,y)/Ct}$$

for any $y \in X$. But $\lim_\alpha \mu_\alpha(B_{R^2}(x_\alpha)) = \mu(B_{R^2}(x))$; hence we find a constant such that for any α,

$$h_\alpha(y) \le C e^{-\mathsf{d}_\alpha^2(x_\alpha,y)/Cs}.$$

Hence the result follows also from Proposition B.3. □

5.3 A differential inequality

We now study the properties of the θ–volume on manifolds satisfying a Dynkin bound. Whenever this bound is improved to be an upper bound on the integral quantity

$$\int_0^T \frac{\sqrt{\mathrm{k}_s(M,g)}}{s}\, \mathrm{d}s \le \Lambda$$

for some $T, \Lambda > 0$, we obtain a monotone quantity that will be crucial in the remainder of this section.

Proposition 5.8 *Let (M^n, g) be a closed Riemannian manifold satisfying*

$$\mathrm{k}_T(M^n, g) \le \frac{1}{16n}$$

for some $T > 0$. For $\tau \le T$, set $\Gamma_\tau(M^n, g) := e^{8\sqrt{n\,\mathrm{k}_\tau(M^n,g)}} - 1$. Then for any $x \in M$, $t \in (0, \tau)$ and $s \le t/(2\Gamma_\tau(M^n, g))$,

$$t\frac{\partial \theta_x}{\partial t}(s,t) + s\frac{\partial \theta_x}{\partial s}(s,t) + n\Gamma_\tau(M^n, g)\Big(\frac{t}{s} - \frac{s}{t}\Big)\theta_x(s,t) \tag{79}$$

has the same sign as $t - s$.

Remark 5.9 When the Ricci curvature is nonnegative (in which case $\Gamma_\tau(M^n, g) = 0$), our proof shows that the map $\lambda \mapsto \theta_x(\lambda s, \lambda t)$ is monotone nonincreasing for $s \ge t$ and monotone nondecreasing for $s \le t$.

Proof Our proof will use the Li–Yau estimate [17, Proposition 3.3]: if $v\colon [0,\tau]\times M \to \mathbb{R}_+$ is a solution of the heat equation, then

$$e^{-8\sqrt{n\,\mathrm{k}_\tau(M^n,g)}}\,\frac{|dv|^2}{v^2}-\frac{1}{v}\frac{\partial v}{\partial t}\le e^{8\sqrt{n\,\mathrm{k}_\tau(M^n,g)}}\,\frac{n}{2t}. \tag{80}$$

For simplicity, let us write θ, $\partial\theta/\partial t$, $\partial\theta/\partial s$, U, H, k and Γ instead of $\theta_x(s,t)$, $\partial\theta_x(/\partial ts,t)$, $\partial\theta_x(s,t)/\partial s$, $U(t,x,y)$, $H(t,x,y)$, $\mathrm{k}_\tau(M^n,g)$ and $\Gamma_\tau(M^n,g)$, respectively.

A direct computation implies

$$\frac{\partial\theta}{\partial t}=-\frac{1}{4s}\int_M\frac{\partial U}{\partial t}e^{-U/4s}\frac{\mathrm{d}v_g}{(4\pi s)^{n/2}}, \tag{81}$$

$$\frac{\partial\theta}{\partial s}=-\frac{n}{2s}\theta+\frac{1}{4s^2}J,\quad \text{where } J:=\int_M Ue^{-U/4s}\,\frac{\mathrm{d}v_g}{(4\pi s)^{n/2}}. \tag{82}$$

From (70) and the fact that H solves the heat equation, another computation gives

$$-\frac{n}{2t}-\frac{\partial}{\partial t}\Big(\frac{U}{4t}\Big)-\frac{1}{4t}\Delta U-\frac{1}{16t^2}|\nabla U|^2=0, \tag{83}$$

hence

$$-\frac{n}{2}-\frac{1}{4}\frac{\partial U}{\partial t}+\frac{U}{4t}-\frac{1}{4}\Delta U-\frac{1}{16t}|\nabla U|^2=0, \tag{84}$$

where here and in the rest of the proof the Laplacian and gradient are taken with respect to the y variable. Moreover the Li–Yau estimate provides, when $t\le\tau$,

$$e^{-8\sqrt{n\,\mathrm{k}_t}}\,\frac{|\nabla U|^2}{16t^2}+\frac{\partial}{\partial t}\Big(\tfrac{1}{2}n\log t+\frac{U}{4t}\Big)\le e^{8\sqrt{n\,\mathrm{k}_t}}\,\frac{n}{2t}. \tag{85}$$

Adding (83) and (85), together with the fact that $e^{-8\sqrt{n\mathrm{k}_t}}-1=-e^{-8\sqrt{n\,\mathrm{k}_t}}\Gamma\ge-\Gamma$, yields the estimate

$$-\frac{\Delta U}{4}-\Gamma\frac{|\nabla U|^2}{16t}\le\frac{n}{2}+\Gamma\frac{n}{2}. \tag{86}$$

Combine (81) and (84) to get

$$\frac{\partial\theta}{\partial t}=\frac{n}{2s}\theta-\frac{1}{4ts}J+\frac{1}{4s}\int_M\Delta Ue^{-U/4s}\frac{\mathrm{d}v_g}{(4\pi s)^{n/2}}+\int_M\frac{|\nabla U|^2}{16ts}e^{-U/4s}\,\frac{\mathrm{d}v_g}{(4\pi s)^{n/2}}. \tag{87}$$

Integration by parts implies

$$\frac{1}{4s}\int_M\Delta U\,e^{-U/4s}\frac{\mathrm{d}v_g}{(4\pi s)^{n/2}}=-\int_M\frac{|\nabla U|^2}{16s^2}e^{-U/4s}\,\frac{\mathrm{d}v_g}{(4\pi s)^{n/2}},$$

hence from (87) we get the equalities

$$\frac{\partial\theta}{\partial t}=\frac{n}{2s}\theta-\frac{1}{4ts}J+\Big(\frac{1}{s}-\frac{1}{t}\Big)\int_M\tfrac{1}{4}\Delta Ue^{-U/4s}\,\frac{\mathrm{d}v_g}{(4\pi s)^{n/2}}, \tag{88}$$

$$\frac{\partial\theta}{\partial t}=\frac{n}{2s}\theta-\frac{1}{4ts}J-\Big(\frac{1}{s}-\frac{1}{t}\Big)\int_M\frac{|\nabla U|^2}{16s}e^{-U/4s}\,\frac{\mathrm{d}v_g}{(4\pi s)^{n/2}}. \tag{89}$$

We use now inequality (86) in (88). Then if $t \geq s$, we obtain

$$\frac{\partial \theta}{\partial t} - \frac{n}{2s}\theta + \frac{J}{4ts} \geq \left(\frac{1}{s} - \frac{1}{t}\right)\left(-\tfrac{1}{2}n(\Gamma+1)\theta - \Gamma \int_M \frac{|\nabla U|^2}{16t} e^{-U/4s} \frac{\mathrm{d}v_g}{(4\pi s)^{n/2}}\right),$$

and we have the opposite inequality if $t \leq s$. As a consequence, there exists a nonnegative function $\pi\colon \mathbb{R}_+ \times [0,\tau] \times M \to \mathbb{R}_+$ such that

(90) $$\frac{\partial \theta}{\partial t} = \frac{n}{2t}\theta - \frac{1}{4ts}J + \left(\frac{1}{s} - \frac{1}{t}\right)\pi - \frac{n}{2}\left(\frac{1}{s} - \frac{1}{t}\right)\Gamma\theta - \Gamma\left(\frac{1}{s} - \frac{1}{t}\right)\frac{s}{t}\int_M \frac{|\nabla U|^2}{16s} e^{-U/4s} \frac{\mathrm{d}v_g}{(4\pi s)^{n/2}}.$$

We use now (89) to express in a different way the last term of the right-hand side of (90):

(91) $$\frac{\partial \theta}{\partial t} = \frac{n}{2t}\theta - \frac{1}{4ts}J + \left(\frac{1}{s} - \frac{1}{t}\right)\pi - \frac{n}{2}\left(\frac{1}{s} - \frac{1}{t}\right)\Gamma\theta + \Gamma\frac{s}{t}\left(\frac{\partial \theta}{\partial t} - \frac{n}{2s}\theta + \frac{J}{4ts}\right).$$

Using (82) we get

$$-\frac{n}{2s}\theta + \frac{J}{4ts} = \frac{n}{2}\left(\frac{1}{t} - \frac{1}{s}\right)\theta + \frac{s}{t}\frac{\partial \theta}{\partial s},$$

hence

(92) $$\frac{\partial \theta}{\partial t} = -\frac{s}{t}\frac{\partial \theta}{\partial s} + \left(\frac{1}{s} - \frac{1}{t}\right)\pi - \frac{n}{2}\left(\frac{1}{s} - \frac{1}{t}\right)\Gamma\theta + \frac{n}{2}\frac{\Gamma}{t}\left(\frac{s}{t} - 1\right)\theta + \Gamma\frac{s}{t}\left(\frac{\partial \theta}{\partial t} + \frac{s}{t}\frac{\partial \theta}{\partial s}\right).$$

We get the differential equation

(93) $$\left(1 - \Gamma\frac{s}{t}\right)\left(t\frac{\partial \theta}{\partial t} + s\frac{\partial \theta}{\partial s}\right) = \left(\frac{t}{s} - 1\right)\pi - \tfrac{1}{2}n\Gamma\left(\frac{t}{s} - \frac{s}{t}\right)\theta.$$

Since

$$s < \frac{t}{2\Gamma} \quad \text{implies} \quad \tfrac{1}{2} \leq 1 - \Gamma\frac{s}{t},$$

we get that

$$t\frac{\partial \theta}{\partial t} + s\frac{\partial \theta}{\partial s} + n\Gamma\left(\frac{t}{s} - \frac{s}{t}\right)\theta$$

has the same sign as $t - s$. □

The above proposition has the following immediate consequence under a stronger integral bound.

Corollary 5.10 *Let (M^n, g) be a closed manifold satisfying*

$$\int_0^T \frac{\sqrt{\mathrm{k}_s(M^n, g)}}{s}\,\mathrm{d}s \leq \Lambda.$$

For any $\tau \in (0, T]$, we set

$$\Phi(\tau) = \int_0^\tau \frac{\sqrt{\mathrm{k}_s(M^n, g)}}{s}\,\mathrm{d}s.$$

Then there exists a positive constant c_n, depending only on n, such that the following holds. Let $t \in (0, T]$ and $s > 0$. Set $\bar{\lambda} := \bar{\lambda}(s,t) = \min\{e^{-c_n\Lambda s/t}, e^{-4\sqrt{n}\Lambda}\}$. Then the function

$$\lambda \in [0, \bar{\lambda}] \to \theta_x(\lambda s, \lambda t)e^{c_n\Phi(\lambda t)(t/s - s/t)}$$

is monotone:

(i) *it is nondecreasing if $t \geq s$,*

(ii) *it is nonincreasing if $t \leq s$.*

Proof For any $c \in \mathbb{R}$, $\lambda, s > 0$ and $t \in (0, T]$, a direct computation of the derivative

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(\theta_x(\lambda s, \lambda t)e^{c\Phi(\lambda t)(t/s-s/t)})$$

shows that it has the same sign as

$$\lambda t\,\frac{\partial\theta_x(\lambda s,\lambda t)}{\partial t}+\lambda s\,\frac{\partial\theta(\lambda s,\lambda t)}{\partial s}+c\Big(\frac{t}{s}-\frac{s}{t}\Big)\sqrt{\mathrm{k}_{\lambda t}}\,\theta(\lambda s,\lambda t).$$

If $\lambda \le e^{-4\sqrt{n}\Lambda}$, then $\mathrm{k}_{\lambda t} \le (16n)^{-1}$, and since there exists $b_n > 0$ such that $e^{8\sqrt{nr}} - 1 \le b_n\sqrt{r}$ for any $r \in (0, (16n)^{-1}]$, this yields

$$\Gamma_{\lambda t} \le b_n\sqrt{\mathrm{k}_{\lambda t}} \le b_n\frac{1}{\log(1/\lambda)}\int_{\lambda t}^{t}\frac{\sqrt{\mathrm{k}_s}}{s}\,\mathrm{d}s \le b_n\frac{\Lambda}{\log(1/\lambda)}. \tag{94}$$

When choosing $c = c_n := nb_n$ we get that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(\theta_x(\lambda s, \lambda t)e^{c_n\Phi(\lambda t)(t/s-s/t)})$$

has the same sign as

$$\lambda t\,\frac{\partial\theta_x(\lambda s,\lambda t)}{\partial t}+\lambda s\,\frac{\partial\theta(\lambda s,\lambda t)}{\partial s}+n\Gamma_{\bar\lambda t}\Big(\frac{t}{s}-\frac{s}{t}\Big)\theta_x(\lambda s,\lambda t)+\Big(\frac{t}{s}-\frac{s}{t}\Big)(c_n\sqrt{\mathrm{k}_{\lambda t}}-n\Gamma_{\bar\lambda t})\theta_x(\lambda s,\lambda t). \tag{95}$$

In addition, if $\lambda \le e^{-(c_n/n)\Lambda\cdot 2s/t}$ in (94), we get $\Gamma_{\lambda t} \le t/2s$. In particular, we have $s \le t/2\Gamma_{\bar\lambda t}$. Hence by Proposition 5.8, when $\lambda \le \bar\lambda$ the first three summands in the right-hand side of (95) give a term of the same sign as $t-s$. Moreover, with our choice of c_n we have

$$c_n\sqrt{\mathrm{k}_{\lambda t}} - n\Gamma_{\bar\lambda t} > 0,$$

hence the last term of (95) also has the same sign as $t-s$. This concludes the proof. □

5.4 Consequences on noncollapsed strong Kato limits

We are now able to prove that tangent cones of noncollapsed strong Kato limits are metric measure cones.

Throughout this subsection, we fix constants $T, \Lambda, v > 0$ and a nondecreasing function $f : (0, T] \to \mathbb{R}_+$ such that

$$\int_0^T\frac{\sqrt{f(s)}}{s}\,\mathrm{d}s \le \Lambda.$$

Without loss of generality, we assume that $f(T) \le 1/(16n)$ and set

$$\Phi(\tau) := \int_0^\tau\frac{\sqrt{f(t)}}{t}\,\mathrm{d}t \quad \text{for any } \tau \in (0, T].$$

According to Definition 2.17, a noncollapsed strong Kato limit (X, d, μ, o) is obtained as

$$(M_\alpha, \mathsf{d}_\alpha, v_{g_\alpha}, o_\alpha) \xrightarrow{\text{pmGH}} (X, \mathsf{d}, \mu, o),$$

where $\{(M_\alpha, g_\alpha)\}_\alpha$ are closed manifolds satisfying the uniform estimates

$$\sup_\alpha \mathrm{k}_t(M_\alpha, g_\alpha) \le f(t) \quad \text{for all } t \in (0, T] \qquad \text{and} \qquad \inf_\alpha v_{g_\alpha}(B_{\sqrt{T}}(o_\alpha)) \ge vT^{n/2}.$$

Theorem 5.11 *Let (X, d, μ, o) be a noncollapsed strong Kato limit in the sense of Definition 2.17, and $x \in X$. Then the following holds.*

(i) *Any tangent cone of X at x is a metric measure cone.*

(ii) *The volume density*

$$\vartheta_X(x) = \lim_{r \to 0} \frac{\mu(B_r(x))}{\omega_n r^n}$$

is well-defined.

(iii) *We have the following relationship between the behavior of the θ–volume and the volume density:*

$$\lim_{\lambda \to 0} \theta_x(\lambda s, \lambda t) = \vartheta_X^{1-t/s}(x).$$

The proof of this theorem and Remark 5.1(ii) will imply:

Corollary 5.12 *Let (X, d, μ, o) be a noncollapsed strong Kato limit. Then for any $x \in X$ we have*

$$\lim_{t \to 0} (4\pi t)^{n/2} H(t, x, x) = \frac{1}{\vartheta_X(x)}.$$

Moreover, there is a positive constant η and an increasing function $\Phi\colon (0, \eta T] \to \mathbb{R}_+$ which satisfies $\lim_{t \to 0+} \Phi(t) = 0$ and is such that

$$(0, \eta T] \ni t \mapsto \exp(\Phi(t))(4\pi t)^{n/2} H(t, x, x)$$

is nondecreasing.

Proof of Theorem 5.11 Let (X, d, μ, o) be a noncollapsed strong Kato limit obtained as above. According to Theorem 4.8, we know that we have Mosco convergence of the quadratic forms

$$u \mapsto \int_{M_\alpha} |du|^2_{g_\alpha} \, \mathrm{d}v_{g_\alpha}$$

to the Cheeger energy Ch of (X, d, μ). As a consequence, $(X, \mathsf{d}, \mu, \mathsf{Ch})$ is a PI($\sqrt{T}$)–Dirichlet space. Therefore, the Θ– and θ–volume are well-defined on X. Moreover, we know that if $x_\alpha \to x$, then

$$\Theta^X_x(s) = \lim_{\alpha \to +\infty} \Theta^{M_\alpha}_{x_\alpha}(s) \quad \text{and} \quad \theta^X_x(s, t) = \lim_{\alpha \to +\infty} \theta^{M_\alpha}_{x_\alpha}(s, t).$$

According to Corollary 5.10, we also know that for any $t, s > 0$ with $t \le T$ there exist $\varepsilon, \kappa > 0$, both depending on s and t, such that the function

$$(0, \varepsilon] \ni \lambda \mapsto \theta^{M_\alpha}_{x_\alpha}(\lambda s, \lambda t) e^{\kappa \Phi(\lambda t)}$$

is monotone. Hence the same is true for the function

$$(0, \varepsilon] \ni \lambda \mapsto \theta^X_x(\lambda s, \lambda t) e^{\kappa \Phi(\lambda t)}. \tag{96}$$

As observed in Remark 2.18, since (X, d, μ) is a noncollapsed strong Kato limit, the measure μ is locally Ahlfors regular and satisfies, for all $x \in X$ and $0 < r \le s \le \sqrt{T}/2$,

$$\mu(B_s(x)) \le C s^n \quad \text{and} \quad \frac{\mu(B_s(x))}{\mu(B_r(x))} \le C \left(\frac{s}{r}\right)^n.$$

We also have the uniform lower bound

$$v s^n \, C^{-1-\Lambda-\mathsf{d}(o,x)/\sqrt{T}} \le \mu(B_s(x)).$$

Hence by (78), there exist positive constants c and C, depending on $\mathsf{d}(o,x)$ and t/s, such that

$$c \le \theta_x(\lambda s, \lambda t) \le C.$$

As a consequence, the monotone function (96) has a well-defined limit as $\lambda \downarrow 0$, which we denote by $\vartheta_x(s,t)$. Moreover,

$$\vartheta_x(s,t) := \lim_{\lambda\to 0+} \theta_x^X(\lambda s, \lambda t) = \sup_{\lambda\in(0,\varepsilon)} \theta_x^X(\lambda s, \lambda t) e^{\kappa\Phi(\lambda t)} \quad \text{when } t \le s,$$

$$\vartheta_x(s,t) := \lim_{\lambda\to 0+} \theta_x^X(\lambda s, \lambda t) = \inf_{\lambda\in(0,\varepsilon)} \theta_x^X(\lambda s, \lambda t) e^{\kappa\Phi(\lambda t)} \quad \text{when } t \ge s.$$

By construction, the function $(s,t) \mapsto \vartheta_x(s,t)$ is 0–homogeneous:

$$\vartheta_x(\lambda s, \lambda t) = \vartheta_x(s,t) \quad \text{for all } \lambda \in (0,1).$$

Let $x \in X$ and let $(Y, \mathsf{d}_Y, \mu_Y, y)$ be a tangent cone of (X, d, μ) at x. Then there exist a sequence $\{\varepsilon_\beta\}_\beta \in (0,+\infty)$ with $\varepsilon_\beta \downarrow 0$ and a limit measure μ_Y on Y such that

$$(X_\beta = X, \mathsf{d}_\beta := \varepsilon_\beta^{-1}\mathsf{d}, \mu_\beta := \varepsilon_\beta^{-n}\mu, x) \xrightarrow{\text{pmGH}} (Y, \mathsf{d}_Y, \mu_Y, y). \tag{97}$$

Moreover, the sequence $\{(X, \mathsf{d}_\beta := \varepsilon_\beta^{-1}\mathsf{d}, \mu_\beta := \varepsilon_\beta^{-n}\mu, \mathsf{Ch}_\beta = \varepsilon_\beta^{2-n}\mathsf{Ch}, x)\}_\beta$ Mosco–Gromov–Hausdorff converges to $(Y, \mathsf{d}_Y, \mu, \mathsf{Ch}, y)$. Let H_β be the heat kernel of the scaled Dirichlet spaces $(X_\beta, \mathsf{d}_\beta, \mu_\beta, \mathsf{Ch}_\beta)$ and U_β the corresponding function such that

$$H_\beta(t,x,y) = (4\pi t)^{-n/2} e^{-U_\beta(t,x,y)/4t}.$$

The scaling property of the heat kernel leads to

$$U_\beta(t,x,y) = \varepsilon_\beta^{-2} U(\varepsilon_\beta^2 t, x, y),$$

so that we have

$$\theta_x^{X_\beta}(s,t) = \theta_x^X(\varepsilon_\beta^2 s, \varepsilon_\beta^2 t).$$

Moreover, Proposition 5.7 ensures that for any $s,t > 0$ we have

$$\theta_y^Y(s,t) = \lim_{\beta\to+\infty} \theta_x^{X_\beta}(s,t) = \lim_{\beta\to+\infty} \theta_x^X(\varepsilon_\beta^2 s, \varepsilon_\beta^2 t) = \vartheta_x(s,t).$$

Hence the function $(s,t) \to \theta_y^Y(s,t)$ is also 0–homogeneous, but according to Proposition 5.5, we get that

$$\Theta_y^Y(s) = \lim_{t\to 0} \theta_y^Y(s,t).$$

Hence $s \mapsto \Theta_y^Y(s)$ is 0–homogeneous and there is some $c = \Theta_y^Y(1) > 0$ such that for any $s > 0$,

$$\Theta_y^Y(s) = c.$$

As a consequence, Lemma 5.3 implies, for any $r > 0$,

$$\mu_Y(B_r(y)) = c\omega_n r^n.$$

From Corollary 4.12, we know that (Y, d_Y, μ_Y) is a weakly noncollapsed $\mathrm{RCD}(0, n)$ space, so according to Proposition 1.27, we know that (Y, d_Y, μ_Y) is a metric measure cone at y. This shows assertion (i).

We have also shown that for any point $x \in X$ and for any tangent cone $(Y, \mathsf{d}_Y, \mu_Y, y)$ of X at x, the functions Θ_y^Y and θ_y^Y do not depend on the tangent cone Y.

As pointed out earlier in the proof, for any fixed $x \in X$ the function $(0, \sqrt{T}/2] \ni r \mapsto \mu(B_r(x))/\omega_n r^n$ is bounded above and below by positive constants, hence it admits limit points as $r \downarrow 0$. Let ϖ be one of these limit points and $(r_\alpha)_\alpha \subset (0, +\infty)$ a sequence such that $r_\alpha \downarrow 0$ and $\varpi = \lim_\alpha \mu(B_{r_\alpha}(x))/\omega_n r_\alpha^n$. We can assume, up to extracting a subsequence, that the sequence $\{(X, \mathsf{d}_\alpha := r_\alpha^{-1}\mathsf{d}, \mu_\alpha := r_\alpha^{-n}\mu, x)\}$ of rescaled spaces converges for the pointed measured Gromov–Hausdorff topology to some tangent cone $(Y, \mathsf{d}_Y, \mu_Y, y)$. In particular,

$$\varpi = \frac{\mu_Y(B_1(y))}{\omega_n} = \Theta_y^Y(1) = \lim_{t \to 0} \vartheta_x(s, t).$$

Hence all the limit points ϖ are equal, so that the volume density is well-defined.

It remains to show that

$$\lim_{\lambda \to 0} \theta_x(\lambda s, \lambda t) = \vartheta_X^{1-t/s}(x),$$

that is, to verify that

$$\vartheta_x(s, t) = \vartheta_X^{1-t/s}(x).$$

If $(Y, \mathsf{d}_Y, \mu_Y, y)$ is a tangent cone of X at x, then we have shown that

$$\theta_y^Y(s, t) = \vartheta_x(s, t).$$

Since $(Y, \mathsf{d}_Y, \mu_Y, y)$ is a measure metric cone at y, we get that for any $z \in Y$ and $t > 0$,

$$H_Y(t, y, z) = \frac{1}{\vartheta_Y(y)(4\pi t)^{n/2}} e^{-\mathsf{d}_Y^2(y,z)/4t},$$

where we recall that, since Y is a measure metric cone at y and a tangent cone of X at x, for any $r > 0$,

$$\vartheta_Y(y) = \frac{\mu(B_r(y))}{\omega_n r^n} = \vartheta_X(x).$$

Therefore the function U_Y associated to H_Y equals

$$U_Y(t, y, z) = \mathsf{d}_Y^2(y, z) + 4t \log(\vartheta_Y(y)) = \mathsf{d}_Y^2(y, z) + 4t \log(\vartheta_X(x)).$$

When using this equality in the definition of $\theta_y^Y(s, t)$ we obtain

$$\begin{aligned} \theta_y^Y(s, t) &= \int_Y e^{-U_Y(t,y,z)/4s} \frac{\mathrm{d}\mu_Y(z)}{(4\pi s)^{n/2}} = \vartheta_Y(y)^{-t/s} \int_Y e^{-\mathsf{d}_Y^2(y,z)/4s} \frac{\mathrm{d}\mu_Y(z)}{(4\pi s)^{n/2}} \\ &= \vartheta_Y(y)^{1-t/s} \int_Y H_Y(s, y, z)\, \mathrm{d}\mu_Y(z) = \vartheta_Y(y)^{1-t/s} = \vartheta_X(x)^{1-t/s}, \end{aligned}$$

where we have used the stochastic completeness of Y. □

This theorem also has the following useful consequence.

Corollary 5.13 *Let (X, d, μ, o) be a noncollapsed strong Kato limit in the sense of Definition 2.17. Then at every point $x \in X$, the volume density satisfies*

$$\vartheta_X(x) \leq 1.$$

Proof Let (X, d, μ, o) be a noncollapsed strong Kato limit defined as above and recall that we defined for all $t \in (0, T]$,

$$\Phi(t) = \int_0^t \frac{\sqrt{f(s)}}{s}\, \mathrm{d}s.$$

Let $x \in X$. We only need to show that

$$\lim_{t \to 0} \theta_x^X\left(\tfrac{1}{4}t, \tfrac{1}{2}t\right) = \vartheta_X(x)^{-1} \geq 1.$$

Using Corollary 5.12 we know that for some $\eta > 0$, the function

$$(0, \eta T] \ni t \mapsto \exp\left(\frac{\Phi(t)}{\eta}\right)(4\pi t)^{n/2} H_{M_\alpha}(t, x_\alpha, x_\alpha)$$

is nondecreasing. But

$$\lim_{t \to 0+} (4\pi t)^{n/2} H_{M_\alpha}(t, x_\alpha, x_\alpha) = 1,$$

hence for any $t \in [0, \eta T]$,

$$\theta_{x_\alpha}^{M_\alpha}\left(\tfrac{1}{4}t, \tfrac{1}{2}t\right) = (4\pi t)^{n/2} H_{M_\alpha}(t, x_\alpha, x_\alpha) \geq \exp\left(-\frac{\Phi(t)}{\eta}\right).$$

By Proposition 5.7, we also have

$$\theta_x^X\left(\tfrac{1}{4}t, \tfrac{1}{2}t\right) \geq \exp\left(-\frac{\Phi(t)}{\eta}\right),$$

and the result follows from letting t tend to 0. □

Our next result concerns the measure of balls in a noncollapsed strong Kato limit and the comparison between the n–dimensional Hausdorff measure and μ.

Corollary 5.14 *Let (X, d, μ, o) be a noncollapsed strong Kato limit in the sense of Definition 2.17. Then for any $\rho > 0$ and $\varepsilon > 0$, there is some $\delta > 0$ such that for any $x \in B_\rho(o)$ and $r \in (0, \delta]$,*

$$\mu(B_r(x)) \leq \omega_n r^n (1 + \epsilon).$$

As a consequence, $\mu \leq \mathcal{H}^n$.

To prove this corollary we will use, as in [22], the spherical Hausdorff measure defined for any $s > 0$ and any Borel set A in a metric space (Z, d_Z) by

$$\mathcal{H}^s(A) := \lim_{\delta \to 0+} \mathcal{H}^s_\delta(A),$$

where for any $\delta \in (0, +\infty]$,

$$\mathcal{H}^s_\delta(A) := \inf\left\{ \sum_i \omega_s r_i^s : A \subset \bigcup_i B_{r_i}(x_i) \text{ and } r_i < \delta \text{ for all } i \right\}.$$

Following [81, Theorem 3.6] or [69, Theorem 6.6], we have the following result: if $\mathcal{H}^s(A) < \infty$ then

(98) $$\limsup_{r\to 0} \frac{\mathcal{H}^s(B_r(x)\cap A)}{\omega_s r^s} \le 1 \quad \text{for } \mathcal{H}^s\text{–a.e. } x \in A.$$

Proof of Corollary 5.14 If the estimate were not true, then we would find $\rho, \varepsilon > 0$ and sequences $r_\alpha \downarrow 0$, $x_\alpha \in B_\rho(o)$, such that the sequence of rescaled spaces $(X, r_\alpha^{-1}\mathsf{d}, r_\alpha^{-n}\mu, x_\alpha)$ converges to some pointed metric measure space $(Z, \mathsf{d}_Z, \mu_Z, z)$ with

$$\mu_Z(B_1(z)) \ge \omega_n(1+\varepsilon).$$

By Remark 2.19, we know that $(Z, \mathsf{d}_Z, \mu_Z, z)$ is a noncollapsed strong Kato limit as well, then by Corollary 5.13 its volume density is smaller than 1. Moreover, $(Z, \mathsf{d}_Z, \mu_Z, z)$ is obtained as a limit of rescaled manifolds $(M_\alpha, \tilde{g}_\alpha)$ such that for all $t > 0$,

$$\lim_{\alpha\to\infty} \mathrm{k}_t(M_\alpha, \tilde{g}_\alpha) = 0.$$

Then according to Remark 4.13, $(Z, \mathsf{d}_Z, \mu_Z, z)$ is a weakly noncollapsed $\mathrm{RCD}(0,n)$ space. Thus the Bishop–Gromov comparison theorem holds on $(Z, \mathsf{d}_Z, \mu_Z, z)$ and we get

$$\mu_Z(B_1(z)) \le \vartheta_Z(z)\omega_n,$$

hence a contradiction. The comparison with the Hausdorff measure is then straightforward, because if $A \subset B_\rho(o)$ and $\varepsilon > 0$ we find $\delta \in (0,1)$ such that $x \in B_{\rho+1}(o)$ and $r < \delta$ yields $\mu(B_r(x)) \le \omega_n r^n(1+\epsilon)$, so that

$$\mu(A) \le (1+\epsilon)\mathcal{H}^n(A). \qquad \square$$

Remark 5.15 The above volume estimate can in fact be quantified on closed Riemannian manifolds. Let $v, T, \Lambda > 0$ and $f\colon (0,T] \to \mathbb{R}_+$ be a nondecreasing function such that

$$f(T) \le \frac{1}{16n} \quad \text{and} \quad \int_0^T \frac{\sqrt{f(s)}}{s}\,\mathrm{d}s \le \Lambda.$$

Then for any $\rho > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ depending only on n, f, v and T such that if (M^n, g) is a closed Riemannian manifold such that

$$v \le \frac{v_g(B_{\sqrt{T}}(o))}{T^{n/2}} \qquad \text{and} \qquad \mathrm{k}_t(M,g) \le f(t) \quad \text{for all } t \in (0,T],$$

then for any $x \in B_\rho(o)$ and $r < \delta$,

$$v_g(B_r(x)) \le \omega_n r^n(1+\epsilon).$$

6 Stratification

In this section, we prove a stratification theorem for noncollapsed strong Kato limit spaces. To state this result, we first give a useful definition. From now on we equip $\mathbb{R}^k$ with the classical Euclidean distance. We refer to (30) and (31) for the definition of the cone distance and the cone measure, respectively.

Definition 6.1 For $k \in \mathbb{N} \setminus \{0\}$, a pointed metric measure space $(Y, \mathsf{d}_Y, \mu_Y, y)$ is called *metric measure k–symmetric* (mm k–symmetric for short) if there exists a metric measure cone (Z, d_Z, μ_Z) with vertex z such that

$$(Y, \mathsf{d}_Y, \mu_Y, y) = (\mathbb{R}^k \times Z, \mathsf{d}_{\mathbb{R}^k \times Z}, \mathscr{H}^k \otimes \mu_Z, (0_k, z)),$$

where $\mathsf{d}_{\mathbb{R}^k \times Z}$ is the classical Pythagorean product distance, 0_k is the origin of $\mathbb{R}^k$, and the equality sign means that there exists an isometry $\varphi\colon Y \to \mathbb{R}^k \times Z$ such that $\varphi_\# \mu_Y = \mathscr{H}^n \otimes \mu_Z$ and $\varphi(y) = (0_k, z)$.

Let $\dim_{\mathscr{H}} A$ be the Hausdorff dimension of a subset A of a metric space (X, d). Then our stratification theorem writes as follows.

Theorem 6.2 *Let (X, d, μ, o) be a noncollapsed strong Kato limit. We set, for any $x \in X$,*

$$d(x) := \sup\{k \in \mathbb{N} : \text{one tangent cone at } x \text{ is mm } k\text{–symmetric}\} \in \{0, \dots, n\}$$

and $S^k := \{x \in X : d(x) \le k\}$ for any $k \in \{0, \dots, n\}$. Then the sets S^k define a filtration of X

$$S^0 \subset S^1 \subset \cdots \subset S^{n-1} \subset S^n,$$

and the following holds:

(i) *The set S_0 is countable.*

(ii) *For any $k \in \{1, \dots, n\}$ we have*

$$\dim_{\mathscr{H}} S^k \le k.$$

(iii) *For μ–a.e. $x \in X$ the set of tangent cones of (X, d, μ) at x is reduced to $\{(\mathbb{R}^n, \mathsf{d}_e, \vartheta_X(x)\mathscr{H}^n, x)\}$.*

The previous theorem was proven for Ricci limit spaces by Cheeger and Colding [24, Theorem 4.7]; see also [22, Theorem 10.20]. To establish it, they combined the splitting theorem on iterated tangent cones with a general density argument. Such an argument was formalized in the Euclidean setting by White [96]. The key point consists in dealing with an appropriate upper or lower semicontinuous function, hence we begin with showing that the volume density ϑ_X, that is well-defined at any point of a noncollapsed strong Kato limit thanks to Theorem 5.11, is lower semicontinuous.

For the sake of clarity, like in the previous subsection, we fix constants $T, \Lambda, v > 0$ and a nondecreasing function $f\colon (0, T] \to \mathbb{R}_+$ such that $f(T) \le 1/(16n)$ and

$$\int_0^T \sqrt{f(\tau)}\,\frac{\mathrm{d}\tau}{\tau} \le \Lambda.$$

Any noncollapsed strong Kato limit space (X, d, μ, o) considered in this section is the pmGH limit of a sequence of pointed Riemannian manifolds $\{(M_\alpha, g_\alpha, o_\alpha)\}$ satisfying, for all α,

$$\mathrm{k}_t(M_\alpha, g_\alpha) \le f(t) \quad \text{for all } t \in (0, T] \qquad \text{and} \qquad v_{g_\alpha}(B_{\sqrt{T}}(o_\alpha)) \ge vT^{n/2}.$$

Proposition 6.3 *Let (X,d,μ,o) be a noncollapsed strong Kato limit. Then the function ϑ_X is lower semicontinuous. Moreover, if $\{(X_\alpha,\mathsf{d}_\alpha,\mu_\alpha,o_\alpha)\}_\alpha$ is a sequence of noncollapsed strong Kato limits pmGH converging to (X,d,μ,o), then for any $x\in X$ and any sequence $\{x_\alpha\}$ where $x_\alpha\in X_\alpha$ for any α such that $x_\alpha\to x$,*

$$\vartheta_X(x)\le\liminf_{\alpha\to+\infty}\vartheta_{X_\alpha}(x_\alpha).$$

Proof Recall that the infimum of a family of continuous functions is upper semicontinuous. Our result is then a consequence of the fact that ϑ^{-1} is the infimum of a family of continuous functions. Indeed from Corollary 5.12 and Remark 5.1(ii), we know that there exists $\eta>0$ and an increasing function $\Phi\colon(0,\eta T]\to\mathbb{R}_+$ satisfying $\lim_{t\to0+}\Phi(t)=0$ such that the function

$$(0,\eta T]\ni t\mapsto\exp(\Phi(t))(4\pi t)^{n/2}H(t,x,x)$$

is nondecreasing. We also know that

$$\vartheta_X^{-1}(x)=\inf_{t\in(0,\eta T]}\exp(\Phi(t))\,\theta_x^X\left(\tfrac{1}{4}t,\tfrac{1}{2}t\right).$$

The result then follows from Proposition 5.7. □

The next additional result deals with the volume density of weakly noncollapsed $\mathrm{RCD}(0,n)$ measure metric cones and was implicitly present in [38, Lemma 2.9].

Proposition 6.4 *Let (Y,d,μ) be a weakly noncollapsed $\mathrm{RCD}(0,n)$ space which is an n–dimensional metric measure cone with vertex $y\in Y$. Then $\vartheta_Y(y')\ge\vartheta_Y(y)$ for any $y'\in Y$. Moreover, there exists $k\in\mathbb{N}$ such that the level set $\{\vartheta_Y(\cdot)=\vartheta_Y(y)\}$ is isometric to the Euclidean space $\mathbb{R}^k$, and (Y,d,μ,y) is mm k–symmetric but not mm $(k+1)$–symmetric.*

To prove this proposition, we shall make use of a general lemma. It was mentioned in [28].

Lemma 6.5 (cone-splitting principle) *Let (X,d,μ) be a metric measure space. Assume that there exist $o_0\neq o_1$ in X such that for $i\in\{0,1\}$ the space (X,d,μ) is an n–metric measure cone with vertex o_i. Then there exists a metric measure cone (Z,d_Z,μ_Z) with vertex z^* such that*

$$(X,\mathsf{d},\mu,o_0)=(\mathbb{R}\times Z,\mathsf{d}_{\mathbb{R}\times Z},\mathcal{H}^1\otimes\mu_Z,(0,z^*)),$$

and the geodesic connecting o_0 and o_1 coincides with $\mathbb{R}\times\{z^*\}$.

Proof With no loss of generality, assume that $\mathsf{d}(o_0,o_1)=1$. Set $S:=\{x\in X:\mathsf{d}(o_0,x)=1\}$ and equip S with the length distance d_S induced by d. Since (X,d,μ) is a metric cone with vertex o_0, there exists a bijection between $X\setminus\{o_0\}$ and $(0,+\infty)\times S$. We shall often identify an element $x\in X\setminus\{o_0\}$ with its image (r,σ) through this bijection, and $(1,\sigma)\in S$ with σ. Moreover, for any $x_i=(r_i,\sigma_i)\in X\setminus\{o_0\}$, $i\in\{1,2\}$, we can write the distance as

$$\mathsf{d}^2(x_1,x_2)=(r_1-r_2)^2+4r_1r_2\sin^2(\phi/2)=(r_1-r_2)^2+r_1r_2\,\mathsf{d}^2(\sigma_1,\sigma_2),\tag{99}$$

where $\phi = \mathsf{d}_S(\sigma_1,\sigma_2)$. We have a similar formula using that X is a metric cone with vertex o_1. Moreover, the geodesic ray originating from o_0 and passing by o_1 and the geodesic ray originating from o_1 and passing by o_0 produce a minimizing geodesic $\gamma\colon\mathbb{R}\to X$ that is parametrized by arc length and such that $\gamma(i)=o_i$ for $i\in\{0,1\}$. Using (99), it is easy to see that the Busemann function associated to γ satisfies the following: for any $x=(r,\sigma)\in X$,

$$b(x) := \lim_{t\to+\infty} t-\mathsf{d}(x,\gamma(t)) = r\cos(\phi) = \tfrac{1}{2}(\mathsf{d}^2(o_0,x)-\mathsf{d}^2(o_1,x)+1), \tag{100}$$

where $\phi=\mathsf{d}_S(\sigma,o_1)$. We set

$$H := b^{-1}(\{0\}) \quad\text{and}\quad \Sigma := H\cap S.$$

Note that there exists a bijection between H and $C(\Sigma)$, since $\Sigma=\{\sigma\in S:\mathsf{d}_S(o_1,\sigma)=\pi/2\}$. From now on, our goal is to show that X is isometric to $H\times\mathbb{R}$.

Let F_λ be the dilation of center o_1 and dilation factor $\lambda>0$. Notice that $F_\lambda(\gamma(t))=\gamma(\lambda(t-1)+1)$ for any $t\geq 0$. Then for any $x\in X$,

$$b(F_\lambda(x)) = \lim_{t\to+\infty} t-\lambda\mathsf{d}\big(x,F_{\lambda^{-1}}(\gamma(t))\big) = \lambda b(x)-\lambda+1.$$

Let $\sigma\in S\setminus\{o_1\}$ and $\phi=\mathsf{d}_S(\sigma,o_1)$. The previous equality, together with (100), implies that there exists a unique $\lambda=(1-\cos\phi)^{-1}$ such that $F_\lambda(\sigma)\in H$. We identify $\breve{\sigma}:=F_\lambda(\sigma)$ with $(r,\widehat{\sigma})\in(0,+\infty)\times\Sigma$. Notice that $\mathsf{d}(o_1,\sigma)=2\sin(\phi/2)$, so $\mathsf{d}(o_1,\breve{\sigma})=\lambda\mathsf{d}(o_1,\sigma)=\sin(\phi/2)^{-1}$. Moreover, we know that $\mathsf{d}(o_1,\widehat{\sigma})=\sqrt{2}$, thus from (99) we have $\mathsf{d}_S(o_1,\widehat{\sigma})=\pi/2$. By (99) again, $\mathsf{d}^2(o_1,\breve{\sigma})=(r-1)^2+2r$, hence

$$r = \big(\tan(\tfrac{1}{2}\phi)\big)^{-1}.$$

We also define the point $\widetilde{\sigma}$ as follows: let $\mu=(2\sin(\phi/2))^{-1}$ and $\widetilde{\sigma}:=F_\mu(\sigma)$. Since $\mathsf{d}(o_1,\sigma)=2\sin(\phi/2)$, the point $\widetilde{\sigma}:=F_\mu(\sigma)$ satisfies $\mathsf{d}(\widetilde{\sigma},o_1)=1$.

Now consider $\sigma_1,\sigma_2\in S\setminus\{o_1\}$ and the corresponding values ϕ_i,r_i and points $\breve{\sigma}_i,\widehat{\sigma}_i,\widetilde{\sigma}_i$. Since $\mathsf{d}(o_1,\sigma_i)=2\sin(\phi_i/2)$ for $i\in\{1,2\}$, and X is a metric cone with vertex o_1, the analogue of formula (99) shows that

$$\mathsf{d}^2(\sigma_1,\sigma_2) = 4(\sin(\phi_1/2)-\sin(\phi_2/2))^2+4\sin(\phi_1/2)\sin(\phi_2/2)\,\mathsf{d}^2(\widetilde{\sigma}_1,\widetilde{\sigma}_2).$$

Similarly, since $\mathsf{d}(o_1,\breve{\sigma}_i)=(\sin(\phi_i/2))^{-1}$ for $i\in\{1,2\}$, we get

$$\mathsf{d}^2(\breve{\sigma}_1,\breve{\sigma}_2) = \left(\frac{1}{\sin(\phi_1/2)}-\frac{1}{\sin(\phi_2/2)}\right)^2+\frac{1}{\sin(\phi_1/2)\sin(\phi_2/2)}\,\mathsf{d}^2(\widetilde{\sigma}_1,\widetilde{\sigma}_2).$$

Using (99) and the fact that $\mathsf{d}(o_0,\breve{\sigma}_i)=r_i$ for $i\in\{1,2\}$, we also obtain

$$\mathsf{d}^2(\breve{\sigma}_1,\breve{\sigma}_2) = (r_1-r_2)^2+r_1r_2\,\mathsf{d}^2(\widehat{\sigma}_1,\widehat{\sigma}_2),$$

so that, after computation,

$$\begin{aligned}\mathsf{d}^2(\sigma_1,\sigma_2) &= 4\sin^2(\phi_1/2)\sin^2(\phi_2/2)[(r_1-r_2)^2+r_1r_2\,\mathsf{d}^2(\widehat{\sigma}_1,\widehat{\sigma}_2)]\\ &= 4\sin^2((\phi_1-\phi_2)/2)+\sin(\phi_1)\sin(\phi_2)\,\mathsf{d}^2(\widehat{\sigma}_1,\widehat{\sigma}_2).\end{aligned} \tag{101}$$

Now consider $x_i = (\rho_i, \sigma_i) \in X \setminus \{o_0, o_1\}$ and $\overline{x}_i = (\rho_i \sin\phi_i, \widehat{\sigma}_i) \in H$ for $i \in \{1,2\}$. Since (X, d) is a metric cone in o_0,

$$\mathsf{d}^2(x_1,x_2) = (\rho_1-\rho_2)^2 + \rho_1\rho_2\,\mathsf{d}^2(\sigma_1,\sigma_2),$$
$$\mathsf{d}^2(\overline{x}_1,\overline{x}_2) = (\rho_1\sin\phi_1-\rho_2\sin\phi_2)^2 + \rho_1\rho_2\sin\phi_1\sin\phi_2\,\mathsf{d}^2(\widehat{\sigma}_1,\widehat{\sigma}_2),$$

hence (101) yields

$$\mathsf{d}^2(x_1,x_2) - \mathsf{d}^2(\overline{x}_1,\overline{x}_2) = (\rho_1-\rho_2)^2 - (\rho_1\sin\phi_1-\rho_2\sin\phi_2)^2 + 4\rho_1\rho_2\sin^2\bigl(\tfrac{1}{2}(\phi_1-\phi_2)\bigr).$$

After elementary computation and using (100), we get

$$\mathsf{d}^2(x_1,x_2) = \mathsf{d}^2(\overline{x}_1,\overline{x}_2) + (\rho_1\cos\phi_1-\rho_2\cos\phi_2)^2 = \mathsf{d}^2(\overline{x}_1,\overline{x}_2) + (b(x_1)-b(x_2))^2.$$

Extending the map $X \setminus \{o_0, o_1\} \ni x \mapsto \overline{x} \in H$ by setting $\overline{o}_0 = \overline{o}_1 = o_0$, we obtain that $x \mapsto (\overline{x}, b(x))$ is an isometry between (X, d) and the product $H \times \mathbb{R}$ endowed with the Pythagorean product distance. Moreover, the map $x = (r,\sigma) \mapsto F_\lambda((\lambda^{-1}r,\sigma))$ is a measure-preserving isometry of (X, d, μ), and this isometry is the translation by $(1-\lambda)$ along the $\mathbb{R}$ factor in the decomposition $X \simeq \mathbb{R} \times H$. Therefore, the measure μ is invariant by translation, hence $\mu \simeq \mathcal{H}^1 \otimes \mu_H$. Finally, the measure invariance of the isometry $x = (r,\sigma) \mapsto F_\lambda((\lambda^{-1}r,\sigma))$ also ensures that (H, d, μ_H) is a metric measure cone with vertex o_0. □

We are now in a position to prove Proposition 6.4.

Proof By the Bishop–Gromov theorem for $\mathrm{RCD}(0,n)$ spaces, the volume ratio is nonincreasing, hence we know that for any $y' \in Y$ and $r > 0$,

$$\vartheta_Y(y') \geq \frac{\mu(B_r(y'))}{\omega_n r^n} \geq \inf_{s>0} \frac{\mu(B_s(y'))}{\omega_n s^n} = \lim_{s\to+\infty} \frac{\mu(B_s(y'))}{\omega_n s^n}.$$

The Bishop–Gromov theorem classically implies that the asymptotic volume ratio

$$\lim_{s\to+\infty} \frac{\mu(B_s(y'))}{\omega_n s^n}$$

does not depend on $y' \in Y$. Thus

$$\vartheta_Y(y') \geq \lim_{s\to+\infty} \frac{\mu(B_s(y))}{\omega_n s^n}.$$

Since (Y, d, μ) is an n–metric measure cone with vertex y, the function $s \mapsto \mu(B_s(y))/\omega_n s^n$ is constantly equal to $\vartheta_Y(y)$. As a consequence,

$$\vartheta_Y(y') \geq \lim_{s\to+\infty} \frac{\mu(B_s(y))}{\omega_n s^n} = \vartheta_Y(y).$$

Since for any $r > 0$,

$$\vartheta_Y(y') \geq \frac{\mu(B_r(y'))}{\omega_n r^n} \geq \vartheta_Y(y),$$

then $\vartheta_Y(y') = \vartheta_Y(y)$ if and only if the function $r \mapsto \mu(B_r(y'))/\omega_n r^n$ is constantly equal to $\vartheta_Y(y)$, which occurs if and only if Y is a n–metric measure cone at y' thanks to Proposition 1.27. By Lemma 6.5, if $y' \neq y$, this implies that Y is mm 1–symmetric along the geodesic connecting y and y'. □

Recall that, thanks to Corollary 4.12 and Theorem 5.11, any tangent cone of a noncollapsed strong Kato limit space is a weakly noncollapsed $\mathrm{RCD}(0,n)$ n–metric measure cone. Then in our setting we directly obtain the following reformulation of Proposition 6.4.

Corollary 6.6 *Let (X,d,μ) be a noncollapsed strong Kato limit and $x \in X$. Let $(X_x,\mathsf{d}_x,\mu_x,x)$ be a tangent cone. Then $\vartheta_{X_x}(z) \geq \vartheta_{X_x}(x)$ for any $z \in X_x$, and there exists $k \in \mathbb{N}$ such that the level set $\{\vartheta_{X_x}(\cdot) = \vartheta_{X_x}(x)\}$ is isometric to the Euclidean space $\mathbb{R}^k$ and $(X_x,\mathsf{d}_x,\mu_x,x)$ is mm k–symmetric but not mm $(k+1)$–symmetric.*

Before proving Theorem 6.2, we recall the definition of $\mathscr{H}^s_\infty$ from the previous section and provide some of its classical properties. For any $s \in \mathbb{R}_+$ and any subset E of a metric space (X,d),

$$\mathscr{H}^s_\infty(E) := \inf\left\{\sum_i \omega_s r_i^s : E \subseteq \bigcup_i B_{r_i}(x_i)\right\}.$$

Lemma 6.7 *The function $\mathscr{H}^s_\infty$ satisfies the following properties.*

(i) $\dim_{\mathscr{H}}(E) = \sup\{s>0 : \mathscr{H}^s_\infty(E) > 0\} = \inf\{s>0 : \mathscr{H}^s_\infty(E) = 0\}$.

(ii) *If $\mathscr{H}^s_\infty(E) > 0$, then for $\mathscr{H}^s$–a.e. $x \in E$,*
$$\limsup_{r\to 0+} \frac{\mathscr{H}^s_\infty(E\cap B_r(x))}{\omega_s r^s} \geq 2^{-s}.$$

(iii) *If E is a countable union of sets $\{E_j\}$, then $\mathscr{H}^s_\infty(E) > 0$ if and only if there exists j such that $\mathscr{H}^s_\infty(E_j) > 0$.*

(iv) *$\mathscr{H}^s_\infty$ is upper semicontinuous with respect to the Gromov–Hausdorff convergence of compact metric sets.*

We are now in a position to prove the existence of a well-defined stratification for noncollapsed strong Kato limits.

Proof of Theorem 6.2 The proof is divided into three cases: first the case $k=0$, then the case $k \in \{1,\dots,n-1\}$, and eventually the case $k=n$.

Case I ($k=0$) Our argument to prove the assertion about S_0 is inspired by [96, Proposition 3.3]. It suffices to prove the inclusion

$$S_0 \subseteq \mathscr{M} := \left\{x \in X : \vartheta_X(x) < \liminf_{y\to x} \vartheta_X(y)\right\}.$$

Indeed, $\mathscr{M}$ is countable, as it can be written as the union over $\ell \in \mathbb{N}\setminus\{0\}$ of the sets

$$\mathscr{M}_\ell = \left\{x \in X : \vartheta_X(x) + \frac{1}{\ell} < \vartheta_X(y) \text{ for all } y \in B_{1/\ell}(x)\setminus\{x\}\right\},$$

which are all discrete and countable, since whenever two disjoint points x,y are in $\mathscr{M}_\ell$, they satisfy $\mathsf{d}(x,y) \geq 1/\ell$.

To show the inclusion $S_0 \subseteq \mathcal{M}$, let us take $x \notin \mathcal{M}$. Then there exists a sequence $\{y_\ell\}_\ell \subset X$ such that $0 < r_\ell := \mathsf{d}(x, y_\ell) < 1/\ell$ and $\vartheta_X(x) + 1/\ell \geq \vartheta_X(y_\ell)$ for any $\ell \in \mathbb{N} \setminus \{0\}$. In particular, $y_\ell \to x$ and

$$\lim_\ell \vartheta_X(y_\ell) = \vartheta_X(x).$$

Consider the sequence of rescalings $\{(X, r_\ell^{-1}\mathsf{d}, r_\ell^{-n}\mu, x)\}_\ell$. Since $r_\ell \downarrow 0$, there exists a subsequence $\{(X, r_{\ell'}^{-1}\mathsf{d}, r_{\ell'}^{-n}\mu, x)\}_{\ell'}$ which pmGH converges to a tangent cone $(X_x, \mathsf{d}_x, \mu_x, x)$. Moreover, the points $\{y_{\ell'}\}_{\ell'}$ converge to some $y \in X_x$ such that $\mathsf{d}_x(x, y) = 1$. The lower semicontinuity of ϑ_X through pmGH convergence, together with the choice of $\{y_\ell\}_\ell$, ensures that

$$\vartheta_{X_x}(y) \leq \liminf_{\ell'} \vartheta_X(y_{\ell'}) = \vartheta_X(x) = \vartheta_{X_x}(x).$$

Thanks to Corollary 6.6, this implies that $\vartheta_{X_x}(y) = \vartheta_{X_x}(x)$ and $(X_x, \mathsf{d}_x, \mu_x, x)$ is mm 1–symmetric. Hence $d(x) \geq 1$ and $x \notin S_0$.

Case II $(k \in \{1, \dots, n-1\})$ From (i) in Lemma 6.7, we only need to prove that for any $s > 0$, $\mathcal{H}^s_\infty(S^k) > 0$ implies $s \leq k$. Thus we assume

$$\mathcal{H}^s_\infty(S^k) > 0. \tag{102}$$

Step 1 Let us write S^k as a countable union of closed sets. From Remark 2.18, we know that there exist $C, \lambda > 0$ such that for all $x \in X$ and all $0 < s < r \leq R$,

$$vC^{-1-\Lambda}e^{-C\mathsf{d}(x,o)/R}r^n \leq \mu(B_r(x)) \leq Cr^n \quad \text{and} \quad \frac{\mu(B_r(x))}{\mu(B_s(x))} \leq C\left(\frac{r}{s}\right)^n. \tag{103}$$

Arguing as in [22, Proof of Theorem 10.20], we write S^k as the countable union over $j \in \mathbb{N} \setminus \{0\}$ of the closed sets

$$S^{k,j} := \{x \in X : \mathrm{D}(B_r(x), B_r^Z(z)) \geq r/j \text{ for all } r \in (0, 1/j) \text{ and all } (Z, \mathsf{d}_Z, \mu_Z, z) \in \mathrm{Adm}_{k+1}\},$$

where Adm_{k+1} is the set of mm $(k+1)$–symmetric spaces $(Z, \mathsf{d}_Z, \mu_Z, z)$ that satisfy (103), and $\mathrm{D}(B_r(x), B_r^Z(z))$ is the sum of the L^2–transportation distance [86, page 69] between the normalized metric measure spaces $(B_r(x), \mathsf{d}, \mu(B_r(x))^{-1}\mu \llcorner B_r(x))$ and $(B_r^Z(z), \mathsf{d}_Z, \mu_Z(B_r^Z(z))^{-1}\mu_Z \llcorner B_r^Z(z))$, and $|\mu(B_r(x)) - \mu_Z(B_r^Z(z))|$. Moreover, for any j,

$$S^{k,j} = \bigcup_{N \in \mathbb{N}} S^{k,j} \cap \overline{B_N(o)},$$

so (iii) in Lemma 6.7 ensures from (102) that there exist $j, N \in \mathbb{N} \setminus \{0\}$ such that

$$\mathcal{H}^s_\infty(S^{k,j} \cap \overline{B_N(o)}) > 0.$$

Step 2 Let us write $S^{k,j} \cap \overline{B_N(o)}$ as a countable union of closed sets. Take $\varepsilon > 0$. For any $x \in X$, since $\vartheta_X(x) < +\infty$, there exists $\eta(x, \varepsilon) > 0$ such that for all $r \in (0, \eta(x, \varepsilon)]$,

$$\left|\frac{\mu(B_r(x))}{\omega_n r^n} - \vartheta_X(x)\right| \leq \varepsilon,$$

and we can define

$$\delta(x,\varepsilon) := \sup\left\{ r>0 : \left| \frac{\mu(B_\sigma(x))}{\omega_n \sigma^n} - \frac{\mu(B_\rho(x))}{\omega_n \rho^n} \right| \le \varepsilon \text{ for all } \sigma,\rho \in (0,r] \right\} > 0.$$

Then for all $c>0$, the set $A_{\varepsilon,c} \subset X$ defined by

$$A_{\varepsilon,c} := \{x \in X : \delta(x,\varepsilon) \ge c\} = \bigcap_{0<\sigma\le\rho\le c} \left\{ x\in X : \left| \frac{\mu(B_\sigma(x))}{\omega_n \sigma^n} - \frac{\mu(B_\rho(x))}{\omega_n \rho^n} \right| \le \varepsilon \right\}$$

is closed. Hence for any $p,q \in \mathbb{Q}$ with $q<p<q+\varepsilon$, the set

$$S_{\varepsilon,p,q} := A_{\varepsilon,2(p-q)} \cap \left\{ x \in S^{k,j} \cap \overline{B_N(o)} : q \le \frac{\mu(B_{p-q}(x))}{\omega_n (p-q)^n} \le p \right\}$$

is compact since it is a closed subset of the compact set $\overline{B_N(o)}$. Observe that for any $x \in S_{\varepsilon,p,q}$ and $\rho \in (0, 2(p-q)]$, we have

$$q-\varepsilon \le \frac{\mu(B_\rho(x))}{\omega_n \rho^n} \le p+\varepsilon, \tag{104}$$

and then $q-\varepsilon \le \vartheta_X(x) \le p+\varepsilon$ as $\rho \downarrow 0$. Finally, note that

$$S^{k,j} \cap \overline{B_N(o)} = \bigcup_{p,q\in\mathbb{Q},\, q<p<q+\varepsilon} S_{\varepsilon,p,q}.$$

Step 3 Now let us consider the sequence $\{\varepsilon_\ell := 2^{-\ell}\}_{\ell\in\mathbb{N}\setminus\{0\}}$. By (iii) in Lemma 6.7, for any ℓ there exist $p_\ell, q_\ell \in \mathbb{Q}$ with $q_\ell < p_\ell < q_\ell + \varepsilon_\ell$ such that $\mathscr{H}^s_\infty(S_{\varepsilon_\ell,p_\ell,q_\ell}) > 0$, hence by (ii) in Lemma 6.7 there exist $x_\ell \in S_{\varepsilon_\ell,p_\ell,q_\ell}$ and $r_\ell > 0$ with $r_\ell \le 2^{-\ell+1}(p_\ell - q_\ell)$ such that

$$\frac{\mathscr{H}^s_\infty(S_{\varepsilon_\ell,p_\ell,q_\ell} \cap B_{r_\ell}(x_\ell))}{\omega_s r_\ell^s} \ge 4^{-s}. \tag{105}$$

As the pointed metric measure spaces $\{(X, r_\ell^{-1}\mathsf{d}, r_\ell^{-n}\mu, x_\ell)\}_\ell$ all satisfy the volume estimates (103), up to extracting a subsequence we can assume that they pmGH–converge as $\ell\to+\infty$ to a pointed metric measure space (Z,d_Z,μ_Z,z). Since the sets $\{S_{\varepsilon_\ell,p_\ell,q_\ell}\}_\ell$ are compact, up to extracting another subsequence we can assume that the compact sets $\{S_{\varepsilon_\ell,p_\ell,q_\ell} \cap \overline{B_{r_\ell}(x_\ell)}\}_\ell$ GH–converge to some compact set $K \subset \overline{B_1^Z(z)}$ containing z. Because of the upper semicontinuity of $\mathscr{H}^s_\infty$ with respect to GH–convergence (ie (iv) in Lemma 6.7) and because of (105), we have $\mathscr{H}^s_\infty(K) \ge \omega_s 4^{-s}$. In particular,

$$\dim_{\mathscr{H}} K \ge s.$$

Finally, up to extracting a further subsequence, we can assume that the bounded sequence of rational numbers $\{q_\ell\}$ tends to some number $Q>0$ as $\ell\to+\infty$.

Step 4 Now we consider $y\in K$ and we let $y_\ell \in S_{\varepsilon_\ell,p_\ell,q_\ell}$ be such that $y_\ell \to y$. Take $r>0$ and consider ℓ sufficiently large so that $r<2^\ell$. We set $\rho_\ell := r r_\ell$. Recalling that $r_\ell \le 2^{-\ell+1}(p_\ell-q_\ell)$, we get that $\rho_\ell \in (0, 2(p_\ell - q_\ell)]$. Then the triangle inequality and (104) lead to

$$\left| \frac{\mu(B_{\rho_\ell}(y_\ell))}{\omega_n \rho_\ell^n} - Q \right| \le 2\varepsilon_\ell + |q_\ell - Q|,$$

which rewrites as

$$\left|\frac{\mu(B_{rr_\ell}(y_\ell))}{\omega_n r_\ell^n} - Qr^n\right| \le r^n(2\varepsilon_\ell + |q_\ell - Q|). \tag{106}$$

Since

$$\lim_{\ell\to+\infty} \frac{\mu(B_{r_\ell r}(y_\ell))}{r_\ell^n} = \mu_Z(B_r(y)),$$

inequality (106) yields $\mu_Z(B_r(y)) = \omega_n Q r^n$ as $\ell \to +\infty$.

Because of Remark 4.13, we know that (Z, d_Z, μ_Z) is a weakly noncollapsed $\mathrm{RCD}(0,n)$ metric measure space. In particular, its volume density ϑ_Z is well defined at all points, and thanks to Proposition 1.27 the previous computation shows that for any $y \in K$, (Z, d_Z, μ_Z) is a metric measure cone at y and $\vartheta_Z(y) = Q$. By Proposition 6.4, this means that there exists an integer $k' \ge \dim_{\mathscr{H}} K$ such that $(Z, \mathsf{d}_Z, \mu_Z, z)$ is metric measure k'–symmetric. In particular,

$$k' \ge s.$$

Step 5 To conclude, let us show that $k \ge k'$. Since (X, d, μ, o) satisfies the volume estimates (103), so do the rescalings $\{(X, r_\ell^{-1}\mathsf{d}, r_\ell^{-n}\mu, x_\ell)\}_{\ell\in\mathbb{N}\setminus\{0\}}$. As $(Z, \mathsf{d}_Z, \mu_Z, z)$ is the pmGH limit of these rescalings, this implies that $(Z, \mathsf{d}_Z, \mu_Z, z)$ belongs to $\mathrm{Adm}_{k'}$. Since for ℓ large enough we have $r_\ell < 1/j$ and

$$\mathrm{D}(B_{r_\ell}(x_\ell), B^Z_{r_\ell}(z)) < \frac{r_\ell}{j},$$

this means that x_ℓ does not belong to $S^{k'-1,j}$. But $x_\ell \in S^{k,j}$ and by definition, we have that if $\bar{k} \ge k$ then $S^{k,j} \subset S^{\bar{k},j}$, hence $k'-1 < k$.

Case III $(k = n)$ Lebesgue differentiation theorem holds on locally doubling spaces [58, Section 3.4] so μ–a.e. $x \in X$ is a Lebesgue point of the locally integrable function ϑ_X. Thus it is enough to show that whenever $x \in X$ is a Lebesgue point of ϑ_X, that is,

$$\lim_{r\to 0} \fint_{B_r(x)} \vartheta_X \,\mathrm{d}\mu = \vartheta_X(x),$$

then any tangent cone at x is equal to $(\mathbb{R}^n, \mathsf{d}_e, \vartheta_X(x)\mathscr{H}^n, 0_n)$.

Let x be a Lebesgue point of ϑ_X, $(X_x, \mathsf{d}_x, \mu_x, x)$ be a tangent cone and $\{r_\alpha\}_\alpha \subset (0, +\infty)$ be such that $r_\alpha \downarrow 0$ and $(X, \mathsf{d}_\alpha := r_\alpha^{-1}\mathsf{d}, \mu_\alpha := r_\alpha^{-n}\mu, x) \to (X_x, \mathsf{d}_x, \mu_x, x)$ in the pmGH sense. Let us denote by X_α the rescaled spaces $(X, \mathsf{d}_\alpha, \mu_\alpha, x)$.

According to Corollary 5.12, we know that if we set

$$\beta_X(z,t) := \frac{1}{\exp(\Phi(t))(4\pi t)^{n/2} H_X(t,z,z)}$$

for any $z \in X$ and any t small enough, then

$$\vartheta_X(z) = \lim_{t\to 0} \beta_X(z,t)$$

and $t \mapsto \beta_X(z,t)$ is nonincreasing. The same is true if we define β_{X_α} (resp. β_{X_x}) in a similar way on the rescaled space X_α for any α (resp. on the tangent cone $(X_x, \mathsf{d}_x, \mu_x, x)$). Moreover we have $\beta_{X_\alpha}(\cdot, t) \to \beta_{X_x}(\cdot, t)$ uniformly on compact sets for any t small enough, implying

$$\begin{aligned}\fint_{B_1^{\mathsf{d}_x}(x)} \beta_{X_x}(z,t)\,\mathrm{d}\mu_x(z) &= \lim_\alpha \fint_{B_1^{\mathsf{d}_\alpha}(x)} \beta_{X_\alpha}(z,t)\,\mathrm{d}\mu_\alpha(z)\\ &= \lim_\alpha \fint_{B_{r_\alpha}^{\mathsf{d}}(x)} \beta_X(z, r_\alpha^2 t)\,\mathrm{d}\mu_(z)\\ &\le \lim_\alpha \fint_{B_{r_\alpha}^{\mathsf{d}}(x)} \vartheta_X(z)\,\mathrm{d}\mu_(z) = \vartheta_X(x).\end{aligned}$$

By monotone convergence, letting $t \downarrow 0$ gives

$$\fint_{B_1^{\mathsf{d}_x}(x)} \vartheta_{X_x}(z)\,\mathrm{d}\mu_x(z) \le \vartheta_X(x).$$

By the first statement in Proposition 6.4,

$$\fint_{B_1^{\mathsf{d}_x}(x)} \vartheta_{X_x}(z)\,\mathrm{d}\mu_x(z) \ge \vartheta_{X_x}(x) = \vartheta_X(x),$$

hence ϑ_{X_x} is constantly equal to $\vartheta_X(x)$ on $B_1^{\mathsf{d}_x}(x)$. The second statement of Proposition 6.4 implies that X_x is isometric to $\mathbb{R}^n$ equipped with the Euclidean distance and μ_{X_x} is given by $c\mathcal{H}^n$ for some $c \in (0,1]$. But since for all $r > 0$ we have $\mu_{X_x}(B_r(x)) = \vartheta_X(x)\omega_n r^n$, with $\mathcal{H}^n(B_r(x)) = \omega_n r^n$ in $\mathbb{R}^n$, we get $c = \vartheta_X(x)$. □

7 Volume continuity

This section is devoted to proving the following analogue of volume continuity for Ricci limit spaces as in [31, Theorem 0.1] and [24, Theorem 5.9].

Theorem 7.1 *Let $(X, \mathsf{d}, \mu, o, \mathcal{E})$ be a noncollapsed strong Kato limit. Then μ coincides with the n–dimensional Hausdorff measure $\mathcal{H}^n$.*

The proof of the previous is a direct consequence of the next key result, of [69, Theorem 6.9] and of the fact that we already know $\mu \le \mathcal{H}^n$. Recall that at any point x of a noncollapsed strong Kato limit, the volume density is well-defined:

$$\vartheta_X(x) = \lim_{r\to 0} \frac{\mu(B_r(x))}{\omega_n r^n}.$$

Theorem 7.2 *Let $(X, \mathsf{d}, \mu, o, \mathcal{E})$ be a noncollapsed strong Kato limit and $x \in X$ be such that the set of tangent cones at x is reduced to $\{(\mathbb{R}^n, \mathsf{d}_e, \vartheta_X(x)\mathcal{H}^n, 0^n)\}$. Then $\vartheta_X(x) = 1$.*

As a consequence, we also obtain the following corollary, which generalizes [22, Theorem 9.31] for manifolds with Ricci curvature bounded below.

Corollary 7.3 *Let $n \geq 1$, let $T, v, \Lambda > 0$ and let $f\colon (0,T] \to \mathbb{R}$ be a function such that*

$$\int_0^T \frac{\sqrt{f(s)}}{s}\,\mathrm{d}s \leq \Lambda.$$

Then for all $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, n, \Lambda, v, T, f)$ such that the following holds. Assume that for all $t \in (0,T]$,

$$\mathrm{k}_t(M^n, g) \leq f(t) \quad \text{and} \quad \frac{v_g(B_{\sqrt{T}}(x))}{T^{n/2}} \geq v.$$

If, for some $r \in (0, \delta\sqrt{T}]$,

$$\mathrm{d}_{\mathrm{GH}}(B_r(x), \mathbb{B}(r)) \leq \delta r,$$

then

$$1 - \varepsilon \leq \frac{v_g(B_r(x))}{\omega_n r^n} \leq 1 + \varepsilon.$$

Proof Assume by contradiction that there exists ε_0 such that for all δ the conclusion of the corollary is false. Then we can consider a sequence δ_i tending to zero and manifolds (M_i^n, g_i) satisfying the assumptions above, for which there exist $r_i \in (0, \delta_i\sqrt{T}]$ and $x_i \in M_i$ such that

$$\mathrm{d}_{\mathrm{GH}}(B_{r_i}(x_i), \mathbb{B}(r_i)) \leq \delta_i r_i, \tag{107}$$

$$\left| \frac{v_{g_i}(B_{r_i}(x_i))}{\omega_n r_i^n} - 1 \right| \geq \varepsilon_0. \tag{108}$$

The rescaled sequence $\{(M_i^n, r_i^{-2} g_i, r_i^{-n} v_{g_i}, x_i)\}$ is a noncollapsing sequence satisfying the strong Kato bound (SUK), with $\mathrm{k}_t(M_i, r_i^{-2} g_i)$ tending to zero. As a consequence, up to a subsequence, it converges to a pointed metric measure space (X, d, μ, x). Because of (107) and (108), the unit ball $B_1(x)$ is isometric to the unit Euclidean ball $\mathbb{B}(1)$ and satisfies

$$\left| \frac{\mu(B_1(x))}{\omega_n} - 1 \right| \geq \varepsilon_0. \tag{109}$$

But according to Theorem 7.1, $\mu = \mathscr{H}^n$ hence in particular μ coincides with the Lebesgue measure on $B_1(x)$, contradicting (109). □

In the remainder of this section, we prove Theorem 7.2. In order to do this, we start by proving the existence of GH–isometries with the appropriate regularity properties.

7.1 Existence of splitting maps

One of the most powerful tools in the study of Ricci limit spaces and RCD spaces is given by ε–splitting maps; see for example Definition 4.10 in [27]. We are going to show that whenever a point x in a noncollapsed strong Kato limit admits a Euclidean tangent cone, we can construct an ε–splitting map from a ball around x to a Euclidean ball. To this aim, we need an approximation result for harmonic functions defined on PI Mosco–Gromov–Hausdorff limits, which is proven in Appendix A, together with the gradient and Hessian estimates shown in Section 3.

In the following, we denote a Euclidean ball of radius r centered at 0^n as $\mathbb{B}(r)$, and we write $\mathbb{B}(p,r) = p + \mathbb{B}(r)$ for any $p \in \mathbb{R}^k$.

Theorem 7.4 *Suppose that (X, d, μ, o) is a noncollapsed strong Kato limit obtained from a sequence $\{(M_\alpha, g_\alpha, o_\alpha)\}_\alpha$, and $x \in X$ is a point admitting $(\mathbb{R}^n, \mathsf{d}_e, \vartheta_X(x)\mathcal{H}^n, 0^n)$ as a tangent cone. Then there exist sequences $\{r_\alpha\}, \{\varepsilon_\alpha\} \subset (0,\infty)$ that tend to zero, points x_α in M_α and maps*

$$H_\alpha = (h_{1,\alpha}, \dots, h_{n,\alpha})\colon B_{r_\alpha}(x_\alpha) \to \mathbb{B}(r_\alpha)$$

such that $h_{i,\alpha}$ is harmonic on $B_{r_\alpha}(x_\alpha)$ for all $i = 1, \dots, n$. Moreover, the following holds:

(i) *H_α is an $\varepsilon_\alpha r_\alpha$–GH isometry between $B_{r_\alpha}(x_\alpha)$ and $\mathbb{B}(r_\alpha)$.*

(ii) *H_α is $(1+\varepsilon_\alpha)$–Lipschitz.*

(iii) $\displaystyle \fint_{B_{r_\alpha}(x_\alpha)} |{}^t dH_\alpha \circ dH_\alpha - \mathrm{Id}_n| \, \mathrm{d}v_{g_\alpha} \le \varepsilon_\alpha.$

(iv) $\displaystyle r_\alpha^2 \fint_{B_{r_\alpha}(x_\alpha)} |\nabla dH_\alpha|^2 \, \mathrm{d}v_{g_\alpha} \le \varepsilon_\alpha.$

(v) $\displaystyle \lim_\alpha \frac{v_{g_\alpha}(B_{t r_\alpha}(x_\alpha))}{\omega_n (t r_\alpha)^n} = \vartheta_X(x)$ *for all $t > 0$.*

Before proving the previous theorem, we show an improvement of the Lipschitz constant of Lipschitz harmonic functions whose gradient is suitably close to 1. The argument we use is originally due to Cheeger and Naber [29, Lemma 3.34], and it relies on the existence of good cut-off functions, the Bochner formula and some appropriate estimates for the heat kernel.

Proposition 7.5 *Let (M^n, g) be a closed Riemannian manifold and $u\colon B_r(x) \to \mathbb{R}$ a κ–Lipschitz harmonic function for some $\kappa \ge 1$. Assume that there exists $\delta > 0$ such that*

$$\mathrm{k}_{r^2}(M^n, g) \le \delta \le \frac{1}{16n} \quad \text{and} \quad \fint_{B_r(x)} \big||du|^2 - 1\big| \, \mathrm{d}v_g \le \delta^2.$$

Then $|du| \le 1 + C(n,\kappa)\delta$ on $B_{r/2}(x)$.

Proof Let $\chi \in C_c^\infty(M)$ be a cut-off function as constructed in Proposition 3.1 such that:

(i) $\chi = 1$ on $B_{3r/4}(x)$,

(ii) $\chi = 0$ on $M \setminus B_r(x)$,

(iii) $|d\chi| \le C(n)/r$ and $|\Delta_g \chi| \le C(n)/r^2$ on $B_r(x) \setminus B_{3r/4}(x)$.

Apply the Bochner formula on $B_r(x)$ to the κ–Lipschitz harmonic function u in order to get

$$|\nabla du|^2 + \tfrac{1}{2}\Delta_g(|du|^2 - 1) = -\mathrm{Ric}(\nabla u, \nabla u).$$

Since $|\nabla du|^2 \ge 0$ and $-\mathrm{Ric}(\nabla u, \nabla u) \le \mathrm{Ric}_- \,\kappa^2$, this leads to

$$\tfrac{1}{2}\Delta_g(|du|^2 - 1) \le \mathrm{Ric}_- \,\kappa^2.$$

Take $y \in B_{r/2}(x)$ and multiply the previous inequality evaluated at some $z \in B_r(x)$ by $2H(t,y,z)\chi(z)$, where $t \in [0, r^2]$, then integrate with respect to z and t:

$$\iint_{[0,r^2]\times B_r(x)} H(t,y,z)\chi(z)\Delta_g(|du|^2(z)-1)\,\mathrm{d}v_g(z)\,\mathrm{d}t \le 2\iint_{[0,r^2]\times B_r(x)} H(t,y,z)\chi(z)\operatorname{Ric}_-(z)\kappa^2\,\mathrm{d}v_g(z)\,\mathrm{d}t.$$

As is immediately seen, the previous right-hand side is not greater than $2\,\mathrm{k}_{r^2}(M^n,g)\kappa^2$, which is not greater than $2\delta\kappa^2$. Thus,

$$\iint_{[0,r^2]\times B_r(x)} H(t,y,z)\chi(z)\Delta_g(|du|^2(z)-1)\,\mathrm{d}v_g(z)\,\mathrm{d}t \le 2\delta\kappa^2. \tag{110}$$

Use integration by parts to rewrite the left-hand side, with simplified notation, as follows:

$$\begin{aligned}\int H\,\chi\,\Delta(|du|^2-1) &= \int \Delta(H\,\chi)\,(|du|^2-1)\\ &= \int(\Delta H)\,\chi\,(|du|^2-1) - 2\int\langle\nabla H,\nabla\chi\rangle(|du|^2-1) + \int H\,(\Delta\chi)\,(|du|^2-1).\end{aligned}$$

Now

$$\begin{aligned}&\iint_{[0,r^2]\times B_r(x)} \Delta_z(H(t,y,z))\chi(z)(|du|^2(z)-1)\,\mathrm{d}v_g(z)\,\mathrm{d}t\\ &\qquad= -\int_{B_r(x)}\left(\int_0^{r^2}\frac{\partial H(t,y,z)}{\partial t}\,\mathrm{d}t\right)\chi(z)(|du|^2(z)-1)\,\mathrm{d}v_g(z)\\ &\qquad= -\int_{B_r(x)} H(r^2,y,z)\chi(z)(|du|^2(z)-1)\,\mathrm{d}v_g(z) + \underbrace{\chi(y)}_{=1}(|du|^2(y)-1).\end{aligned}$$

Combining these three last estimates with the properties of the cut-off function χ, we get

$$|du|^2(y)-1 \le 2\kappa^2\delta + \mathrm{I}(y) + \mathrm{II}(y) + \mathrm{III}(y),$$

where

$$\begin{aligned}\mathrm{I}(y) &= \int_M H(r^2,y,z)\chi(z)\big||du|^2(z)-1\big|\,\mathrm{d}v_g(z),\\ \mathrm{II}(y) &= \frac{C(n)}{r}\iint_{[0,r^2]\times[B_r(x)\setminus B_{3r/4}(x)]} |\nabla_z H(t,y,z)|\,\big||du|^2(z)-1\big|\,\mathrm{d}v_g(z)\,\mathrm{d}t,\\ \mathrm{III}(y) &= \frac{C(n)}{r^2}\iint_{[0,r^2]\times[B_r(x)\setminus B_{3r/4}(x)]} H(t,y,z)\big||du|^2(z)-1\big|\,\mathrm{d}v_g(z)\,\mathrm{d}t,\end{aligned}$$

and we are going to establish the estimates

$$\mathrm{I}(y) \le C(n)\delta^2,\quad \mathrm{II}(y) \le C(n)\delta\sqrt{1+\kappa^2},\quad \mathrm{III}(y)\le C(n)\delta^2,$$

which are enough to complete the proof.

We recall the upper bound for the heat kernel, with $\nu = e^2 n$ and $t \le r^2, z \in M$:

$$H(t, y, z) \le \frac{C(n)}{v_g(B_r(y))} \frac{r^\nu}{t^{\nu/2}} e^{-\mathrm{d}_g^2(y,z)/5t}.$$

Moreover, since $y \in B_{r/2}(x)$, the doubling condition implies

$$H(t, y, z) \le \frac{C(n)}{v_g(B_r(x))} \frac{r^\nu}{t^{\nu/2}} e^{-\mathrm{d}_g^2(y,z)/5t}. \tag{111}$$

Therefore, for any $z \in B_r(x) \setminus B_{3r/4}(x)$, we obtain

$$H(r^2, y, z) \le \frac{C(n)}{v_g(B_r(x))}.$$

Using this inequality and the assumption on $|du|$ leads to the estimate

$$\mathrm{I}(y) \le \frac{C(n)}{v_g(B_r(x))} \int_{B_r(x)} \big||du|^2(z) - 1\big| \, \mathrm{d}v_g(z) \le C(n)\delta^2.$$

We now obtain the estimate for $\mathrm{III}(y)$. Consider $z \in B_r(x) \setminus B_{3r/4}(x)$, as above. Inequality (111) and the fact that

$$\int_0^{r^2} \frac{r^\nu}{t^{\nu/2}} e^{-r^2/80t} \, \mathrm{d}t = r^2 \int_0^1 \frac{1}{t^{\nu/2}} e^{-1/80t} \, \mathrm{d}t$$

imply that

$$\int_0^{r^2} H(t, y, z) \, \mathrm{d}t \le \frac{C(n)}{v_g(B_r(x))} \int_0^{r^2} \frac{r^\nu}{t^{\nu/2}} e^{-r^2/80t} \, \mathrm{d}t = \frac{C(n) r^2}{v_g(B_r(x))},$$

and as a consequence

$$\mathrm{III}(y) \le C(n) \fint_{B_r(x)} \big||du|^2(z) - 1\big| \, \mathrm{d}v_g(z) \le C(n)\delta^2.$$

As for $\mathrm{II}(y)$, we use the Cauchy–Schwarz inequality twice, first in $\mathrm{d}v_g$ and then dt, together with the result of Lemma 3.3,

$$\int_M \frac{|\nabla_z H(t, y, z)|^2}{H(t, y, z)} \, \mathrm{d}v_g(z) \le \frac{C(n)}{t},$$

in order to obtain

$$\begin{aligned}
\mathrm{II}(y) &= \frac{C(n)}{r} \int_0^{r^2} \int_{B_r(x) \setminus B_{3r/4}(x)} \frac{|\nabla_z H(t, y, z)|}{\sqrt{H(t, y, z)}} \sqrt{H(t, y, z)} \, \big||du|^2(z) - 1\big| \, \mathrm{d}v_g(z) \, \mathrm{d}t \\
&\le \frac{C(n)}{r} \int_0^{r^2} \frac{1}{\sqrt{t}} \bigg(\int_{B_r(x) \setminus B_{3r/4}(x)} H(t, y, z) \, \big||du|^2(z) - 1\big|^2 \, \mathrm{d}v_g(z) \bigg)^{1/2} \mathrm{d}t \\
&\le C(n) \bigg(\int_{[0, r^2] \times (B_r(x) \setminus B_{3r/4}(x))} \frac{H(t, y, z)}{t} \big||du|^2(z) - 1\big|^2 \, \mathrm{d}v_g(z) \, \mathrm{d}t \bigg)^{1/2}.
\end{aligned}$$

Using

$$\int_0^{r^2} \frac{r^\nu}{t^{1+\nu/2}} e^{-r^2/80t} \, \mathrm{d}t = \int_0^1 \frac{1}{t^{1+\nu/2}} e^{-1/80t} \, \mathrm{d}t,$$

we have

$$\int_0^{r^2} \frac{H(t,z,y)}{t}\,\mathrm{d}t \leq \int_0^{r^2} \frac{C(n)}{v_g(B_r(x))}\frac{r^\nu}{t^{1+\nu/2}} e^{-r^2/80t}\,\mathrm{d}t \leq \frac{C(n)}{v_g(B_r(x))}.$$

Then we obtain

$$\mathrm{II}(y) \leq C(n)(1+\kappa^2)^{1/2}\left(\fint_{B_r(x)} \big||du|^2(z)-1\big|\,\mathrm{d}v_g(z)\right)^{1/2} \leq C(n)(1+\kappa^2)^{1/2}\delta.$$

This allows us to obtain the desired bound on $|du|$ over $B_{r/2}(x)$. □

We are now in a position to prove Theorem 7.4.

Proof of Theorem 7.4 Let $x \in X$ and assume that $(\mathbb{R}^n, \mathrm{d}_e, \vartheta_X(x)\mathcal{H}^n, 0^n)$ is a tangent cone at x. Then by definition of tangent cones of strong Kato limit spaces, there exist sequences $(r_\alpha)_\alpha \subset (0,+\infty)$, $r_\alpha \downarrow 0$ and $x_\alpha \in M_\alpha$ such that

$$(M_\alpha, r_\alpha^{-1}\mathrm{d}_{g_\alpha}, r_\alpha^{-n}\,\mathrm{d}v_{g_\alpha}, x_\alpha) \longrightarrow (\mathbb{R}^n, \mathrm{d}_e, \vartheta_X(x)\mathcal{H}^n, 0^n),$$

and property (v) holds. Write $\tilde{g}_\alpha = r_\alpha^{-2} g_\alpha$ and $\mu_\alpha = r_\alpha^{-n}\,\mathrm{d}v_{g_\alpha}$. Balls with respect to $\tilde{g}_\alpha$ are denoted by $\widetilde{B}_s(y)$. It is enough to prove the existence of a map $H_\alpha\colon \widetilde{B}_1(x_\alpha) \to \mathbb{B}(1)$ satisfying properties (i) to (iv) with respect to the rescaled metric $\tilde{g}_\alpha$ and with r_α replaced by 1. Then the map on $B_{r_\alpha}(x_\alpha)$ is simply obtained by rescaling H_α by a factor r_α. As a consequence, in the rest of the proof, we only work with the rescaled manifolds $(M_\alpha, \tilde{g}_\alpha)$.

Consider the coordinate maps $x_i\colon \mathbb{R}^n \to \mathbb{R}$ for all $i = 1, \dots, n$. Then x_i are harmonic and we can apply Proposition E.10: for all α, there exist harmonic functions $h_{i,\alpha}\colon \widetilde{B}_1(x_\alpha) \to \mathbb{B}(1)$ such that

(i) $h_{i,\alpha} \to x_i|_{\mathbb{B}(1)}$ uniformly, and

(ii) for all $s \leq 1$,

$$\lim_{\alpha\to\infty} \int_{\widetilde{B}_s(x_\alpha)} |dh_{i,\alpha}|^2_{\tilde{g}_\alpha}\,\mathrm{d}\mu_\alpha = \int_{\mathbb{B}(s)} |dx_i|^2 \vartheta_X(x)\,\mathrm{d}\mathcal{H}^n = \omega_n s^n\,\vartheta_X(x).$$

Define

$$H_\alpha = (h_{1,\alpha}, \dots, h_{n,\alpha})\colon \widetilde{B}_1(x_\alpha) \to \mathbb{B}(1).$$

Since H_α converges uniformly to the identity $\mathrm{Id}_n = (x_1, \dots, x_n)$, it is not difficult to show that H_α is an ε_α–GH isometry between $\widetilde{B}_1(x_\alpha)$ and $\mathbb{B}(1)$, where $(\varepsilon_\alpha)_\alpha \subset (0,+\infty)$ is a sequence tending to zero. In the rest of the proof, we will take the freedom of modifying this sequence tending to zero while keeping its notation.

Since $\mu_\alpha(\widetilde{B}_1(x_\alpha))$ tends to $\vartheta_X(x)\omega_n$, the second property implies that

$$\lim_{\alpha\to\infty} \fint_{\widetilde{B}_1(x_\alpha)} |dh_{i,\alpha}|^2_{\tilde{g}_\alpha}\,\mathrm{d}\mu_\alpha = 1.$$

Using the first estimate in Lemma 3.6 we then deduce that there exists $C(n) > 0$ such that

$$\sup_{\widetilde{B}_{3/4}(x_\alpha)} |dh_{i,\alpha}|_{\widetilde{g}_\alpha} \leq C(n),$$

that is, $h_{i,\alpha}$ is $C(n)$–Lipschitz on $\widetilde{B}_{1/2}(x_\alpha)$. We can then apply Proposition 3.5 and get some uniform estimates

$$\fint_{\widetilde{B}_{1/2}(x_\alpha)} |\nabla dh_{i,\alpha}|^2_{\widetilde{g}_\alpha} \, \mathrm{d}\mu_\alpha \leq C(n).$$

Then, Proposition E.6 in Appendix E ensures that $|dh_{i,\alpha}|_{\widetilde{g}_\alpha}$ tends to 1 in L^2. By additionally using the fact that $(M_\alpha, \widetilde{g}_\alpha)$ is doubling, we have

$$\lim_{\alpha\to\infty} \fint_{\widetilde{B}_{1/2}(x_\alpha)} \bigl||dh_{i,\alpha}|_{\widetilde{g}_\alpha} - 1\bigr|^2 \, \mathrm{d}\mu_\alpha = 0.$$

This, together with $h_{i,\alpha}$ being Lipschitz, implies

$$\lim_{\alpha\to\infty} \fint_{\widetilde{B}_{1/2}(x_\alpha)} \bigl||dh_{i,\alpha}|^2_{\widetilde{g}_\alpha} - 1\bigr| \, \mathrm{d}\mu_\alpha = 0.$$

Recall that, as observed in Remark 2.10, a sequence of rescaled manifolds converging to a tangent cone of a strong Kato limit is such that for all $t > 0$

$$\lim_{\alpha\to\infty} \mathrm{k}_t(M_\alpha, \widetilde{g}_\alpha) = 0.$$

Modifying the sequence $(\varepsilon_\alpha)_\alpha$ if necessary, we have

$$\mathrm{k}_{1/4}(M_\alpha, \widetilde{g}_\alpha) < \varepsilon_\alpha \quad \text{and} \quad \fint_{\widetilde{B}_{1/2}(x_\alpha)} \bigl||dh_{i,\alpha}|^2_{\widetilde{g}_\alpha} - 1\bigr| \, \mathrm{d}\mu_\alpha < \varepsilon_\alpha^2.$$

This means that the assumptions of Proposition 7.5 are satisfied, therefore for α large enough $h_{i,\alpha}$ is $(1+\varepsilon_\alpha)$–Lipschitz on the ball $\widetilde{B}_{1/4}(x_\alpha)$.

Applying Proposition 3.5 and using again the doubling property, we also obtain

$$\fint_{\widetilde{B}_{1/4}(x_\alpha)} |\nabla dh_{i,\alpha}|^2_{\widetilde{g}_\alpha} \, \mathrm{d}\mu_\alpha \leq C_n \fint_{\widetilde{B}_{1/2}(x_\alpha)} \left| |dh_{i,\alpha}|^2_{\widetilde{g}_\alpha} - \fint_{\widetilde{B}_{1/2}(x_\alpha)} |dh_{i,\alpha}|^2_{\widetilde{g}_\alpha} \, \mathrm{d}\mu_\alpha \right| \mathrm{d}\mu_\alpha < \varepsilon_\alpha.$$

Here we have used that $\mathrm{k}_{r_\alpha^{-2}T}(M_\alpha, \widetilde{g}_\alpha) \leq 1/16n$. Then for large enough α, we have $\min\{\frac{1}{2}, r_\alpha^{-2}T\} = \frac{1}{2}$. As a consequence, up to replacing r_α by $r_\alpha/4$, $h_{i,\alpha}$ is $(1+\varepsilon_\alpha)$–Lipschitz on $\widetilde{B}_1(x_\alpha)$ and satisfies

$$\fint_{\widetilde{B}_1(x_\alpha)} \bigl||dh_{i,\alpha}|^2_{\widetilde{g}_\alpha} - 1\bigr| \, \mathrm{d}\mu_\alpha < \varepsilon_\alpha \quad \text{and} \quad \fint_{\widetilde{B}_1(x_\alpha)} |\nabla dh_{i,\alpha}|^2_{\widetilde{g}_\alpha} \, \mathrm{d}\mu_\alpha < \varepsilon_\alpha.$$

In order to obtain properties (iii) and (iv) for H_α, one can consider the function $x_i + x_j$. Since we know that $h_{i,\alpha} + h_{j,\alpha}$ converges uniformly to $x_i + x_j$, by arguing as above we get

$$\lim_{\alpha\to\infty} \fint_{\widetilde{B}_1(x_\alpha)} |\langle dh_{i,\alpha}, dh_{j,\alpha}\rangle_{\widetilde{g}_\alpha} - \delta_{ij}| \, \mathrm{d}\mu_\alpha = 0.$$

Then the same argument that we used for $h_{i,\alpha}$ finally leads to properties (iii) and (iv) for H_α. □

Remark 7.6 The same argument as above shows that if (X, d, μ) is a noncollapsed strong Kato limit and $x \in X$ admits an mm k–symmetric tangent cone, then there exist harmonic ε–splitting maps from a ball around x to a Euclidean ball of the same radius in $\mathbb{R}^k$.

7.2 Proof of Theorem 7.2

Our proof is inspired by the argument illustrated in [22, Theorem 9.31] and [43, Theorem 1.6]. Both proofs are based on degree theory.

Proof Let $x \in X$ admit a Euclidean tangent cone $(\mathbb{R}^n, \mathsf{d}_e, \vartheta_X(x)\mathcal{H}^n, 0^n)$. Consider sequences r_α, ε_α and $H_\alpha \colon B_{r_\alpha}(x_\alpha) \to \mathbb{R}^n$ as in Theorem 7.4. Let $\rho_\alpha \colon B_{r_\alpha}(x_\alpha) \to \mathbb{R}$ be defined as $\rho_\alpha(\cdot) = \|H_\alpha(\cdot)\|^2$. Fix $\tau_\alpha \in \left(\frac{1}{4}r_\alpha - 2\varepsilon_\alpha r_\alpha, \frac{1}{4}r_\alpha - \varepsilon_\alpha r_\alpha\right)$. Since ρ_α is smooth, by Sard's theorem τ_α can be chosen so that τ_α^2 is a regular value of ρ_α. Now define the compact set

$$\Omega_\alpha = \{y \in B_{r_\alpha}(x_\alpha) : \|H_\alpha(y)\| \le \tau_\alpha\}.$$

Since H_α is an $(\varepsilon_\alpha r_\alpha)$–GH isometry, we have

$$H_\alpha(B_{r_\alpha/4 - 3r_\alpha\varepsilon_\alpha}(x_\alpha)) \subset \mathbb{B}\left(\tfrac{1}{4}r_\alpha - 2\varepsilon_\alpha r_\alpha\right).$$

Also, any $y \in B_{r_\alpha}(x_\alpha)$ such that $\|H_\alpha(y)\| \le r_\alpha/4 - \varepsilon_\alpha r_\alpha$ belongs to $B_{r_\alpha/4}(x_\alpha)$. Then with our choice of interval and τ_α we have

$$B_{r_\alpha/4 - 3\varepsilon_\alpha r_\alpha}(x_\alpha) \subset \Omega_\alpha \subset B_{r_\alpha/4}(x_\alpha). \tag{112}$$

We claim that if there exists $B \subset A$ finite and such that $H_\alpha : \Omega_\alpha \to \mathbb{B}(\tau_\alpha)$ is surjective for any $\alpha \notin B$, then $\vartheta_X(x) = 1$. Indeed, if this latter statement is established, then the estimates on dH_α and on the Lipschitz constant of H_α imply

$$\mathcal{H}^n(\mathbb{B}(\tau_\alpha)) \le (1 + \varepsilon_\alpha)^n v_{g_\alpha}(\Omega_\alpha),$$

which, together with the inclusion above leads to

$$\omega_n\left((1 - 12\varepsilon_\alpha) \cdot \tfrac{1}{4}r_\alpha\right)^n \le (1 + \varepsilon_\alpha)^n v_{g_\alpha}(B_{r_\alpha/4}(x_\alpha)).$$

Together with (v) in Theorem 7.4, this shows that $\vartheta_X(x) \ge 1$. But since ϑ_X is lower semicontinuous, we already know that $\vartheta_X(x) \le 1$, then $\vartheta_X(x) = 1$.

In the rest of the proof we show by contradiction that there exists $B \subset A$ finite and such that for all $\alpha \notin B$, $H_\alpha \colon \Omega_\alpha \to \mathbb{B}(\tau_\alpha)$ is surjective.

We first assume that M_α is oriented. Assume that H_α is not surjective. We let Θ_α be the unit volume form of (M_α, g_α), that is to say,

$$\Theta_\alpha \in \mathscr{C}^\infty(\Lambda^n T^* M_\alpha), \qquad |\Theta_\alpha|_{g_\alpha}(y) = 1 \quad \text{for all } y \in M_\alpha.$$

Since Ω_α is compact, the set $\mathbb{B}(\tau_\alpha) \setminus H_\alpha(\Omega_\alpha)$ is open and there exists an open ball $\mathbb{B}(p_\alpha, \eta_\alpha) \subset \mathbb{B}(\tau_\alpha) \setminus H_\alpha(\Omega_\alpha)$. Observe that there is a diffeomorphism $\Phi_\alpha\colon [0,1]\times\mathbb{S}^{n-1} \to \mathbb{B}(\tau_\alpha)\setminus\mathbb{B}(p_\alpha,\eta_\alpha)$ such that $\Phi_\alpha(\{0\}\times\mathbb{S}^{n-1}) = \partial\mathbb{B}(\tau_\alpha)$. Any differential n–form on $[0,1]\times\mathbb{S}^{n-1}$ can be written as

$$f\,dt \wedge \Omega_{\mathbb{S}^{n-1}} = d(F\,\Omega_{\mathbb{S}^{n-1}}),$$

where $F(t,\sigma) = \int_0^t f(s,\sigma)\,ds$ and $\Omega_{\mathbb{S}^{n-1}}$ is the unit volume form of $\mathbb{S}^{n-1}$. Since the pullback of $F\,\Omega_{\mathbb{S}^{n-1}}$ on $\{0\}\times\mathbb{S}^{n-1}$ is zero, we obtain an $(n-1)$–form γ_α on the set $\mathbb{B}(\tau_\alpha)\setminus\mathbb{B}(p_\alpha,\eta_\alpha)$ such that

$$\mathrm{d}\gamma_\alpha = dx_1\wedge\cdots\wedge dx_n \quad\text{and}\quad \iota_\alpha^*\gamma_\alpha = 0,$$

where $\iota_\alpha\colon \partial\mathbb{B}(\tau_\alpha)\to\mathbb{B}(\tau_\alpha)$ is the inclusion map. We then define $f_\alpha\colon B_{r_\alpha}(x_\alpha)\to\mathbb{R}$ by

$$H_\alpha^*(\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_n) = \mathrm{d}h_{1,\alpha}\wedge\cdots\wedge\mathrm{d}h_{n,\alpha} = f_\alpha\Theta_\alpha,$$

so that for $y\in B_{r_\alpha}(x_\alpha)$,

$$f_\alpha(y) = \det(\mathrm{d}_y H_\alpha(e_1),\dots,\mathrm{d}_y H_\alpha(e_n)),$$

where $(e_1,\dots,e_n)$ is a direct orthonormal basis of $(T_yM_\alpha, g_\alpha(y))$. We also have that for any $y\in B_{r_\alpha}(x_\alpha)$,

$$f_\alpha^2(y) = \det({}^t\mathrm{d}_y H_\alpha\circ\mathrm{d}_y H_\alpha).$$

We also have the estimate

$$|f_\alpha| \le (1+\varepsilon_\alpha)^n, \tag{113}$$

and with the formula

$$\nabla_X f_\alpha\Theta_\alpha = \sum_j \mathrm{d}h_{1,\alpha}\wedge\cdots\wedge\nabla_X\mathrm{d}h_{j,\alpha}\wedge\cdots\wedge\mathrm{d}h_{n,\alpha}$$

we easily obtain

$$|\nabla f_\alpha| \le n(1+\varepsilon_\alpha)^{n-1}|\nabla dH_\alpha|. \tag{114}$$

Now observe that $H_\alpha(\partial\Omega_\alpha) = \partial\mathbb{B}_{\tau_\alpha}$: if $x\in\partial\Omega_\alpha$, then $\|H_\alpha(x)\| = \tau_\alpha$ and $H_\alpha(x)\in\partial\mathbb{B}(\tau_\alpha)$ because τ_α is a regular value of $\rho_\alpha = \|H_\alpha\|^2$. Then Stokes' theorem implies

$$\int_{\Omega_\alpha} H_\alpha^*(\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_n) = \int_{\Omega_\alpha}\mathrm{d}H_\alpha^*(\gamma_\alpha) = \int_{\partial\Omega_\alpha}(\iota_\alpha\circ H_\alpha)^*\gamma_\alpha = 0.$$

Hence

$$\int_{\Omega_\alpha} f_\alpha\,\mathrm{d}v_{g_\alpha} = 0. \tag{115}$$

Denote by B_α the ball $B_{r_\alpha/2}(x_\alpha)$. We introduce

$$m_\alpha := \fint_{B_\alpha} f_\alpha\,\mathrm{d}v_{g_\alpha}.$$

We can write

$$|m_\alpha| = \left| m_\alpha - \fint_{\Omega_\alpha} f_\alpha \right| = \left| \fint_{\Omega_\alpha \times B_\alpha} (f_\alpha(x) - f_\alpha(y)) \,\mathrm{d}v_{g_\alpha}(x)\,\mathrm{d}v_{g_\alpha}(y) \right|$$
$$\leq \frac{v_{g_\alpha}(B_\alpha)}{v_{g_\alpha}(\Omega_\alpha)} \fint_{B_\alpha \times B_\alpha} |f_\alpha(x) - f_\alpha(y)| \,\mathrm{d}v_{g_\alpha}(x)\,\mathrm{d}v_{g_\alpha}(y)$$
$$\leq \frac{v_{g_\alpha}(B_\alpha)}{v_{g_\alpha}(\Omega_\alpha)} \left(\fint_{B_\alpha \times B_\alpha} |f_\alpha(x) - f_\alpha(y)|^2 \,\mathrm{d}v_{g_\alpha}(x)\,\mathrm{d}v_{g_\alpha}(y) \right)^{1/2}.$$

Thanks to the first inclusion in (112), for α large enough $B_{r_\alpha/8}(x_\alpha) \subset \Omega_\alpha$. Then, since (M_α, g_α) is doubling, we know that the ratio $v_{g_\alpha}(B_\alpha)/v_{g_\alpha}(\Omega_\alpha)$ satisfies a uniform upper bound. Moreover

$$\fint_{B_\alpha \times B_\alpha} |f_\alpha(x) - f_\alpha(y)|^2 \,\mathrm{d}v_{g_\alpha}(x)\,\mathrm{d}v_{g_\alpha}(y) \leq 2 \fint_{B_\alpha} |f_\alpha - m_\alpha|^2 \,\mathrm{d}v_{g_\alpha}.$$

Using the Poincaré inequality, estimate (114), the doubling condition and estimate (iv) from Theorem 7.4, we obtain that

$$\fint_{B_\alpha} |f_\alpha - m_\alpha|^2 \,\mathrm{d}v_{g_\alpha} \leq C r_\alpha^2 \fint_{B_\alpha} |\nabla dH_\alpha|^2 \,\mathrm{d}v_{g_\alpha} \leq C r_\alpha^2 \fint_{2B_\alpha} |\nabla dH_\alpha|^2 \,\mathrm{d}v_{g_\alpha} \leq C \varepsilon_\alpha. \tag{116}$$

We can now conclude that

$$|m_\alpha| \leq C\sqrt{\varepsilon_\alpha} \quad \text{and} \quad \fint_{B_\alpha} |f_\alpha| \,\mathrm{d}v_{g_\alpha} \leq |m_\alpha| + \fint_{B_\alpha} |f_\alpha - m_\alpha| \,\mathrm{d}v_{g_\alpha} \leq C\sqrt{\varepsilon_\alpha}. \tag{117}$$

Observe that if S is a $n \times n$ symmetric matrix with $\|S\| \leq 2$, then

$$\det S = \det(\mathrm{Id}_n + (S - \mathrm{Id}_n)) \geq 1 - C_n \|S - \mathrm{Id}_n\|.$$

Hence,

$$(1+\varepsilon_\alpha)^n \fint_{B_\alpha} |f_\alpha| \,\mathrm{d}v_{g_\alpha} \geq \fint_{B_\alpha} |f_\alpha|^2 \,\mathrm{d}v_{g_\alpha}$$
$$= \fint_{B_\alpha} \det({}^t\mathrm{d}_y H_\alpha \circ \mathrm{d}_y H_\alpha) \,\mathrm{d}v_{g_\alpha}$$
$$\geq 1 - C_n \fint_{B_\alpha} \|{}^t\mathrm{d}_y H_\alpha \circ \mathrm{d}_y H_\alpha - \mathrm{Id}_n\| \,\mathrm{d}v_{g_\alpha}.$$

Then using the doubling condition and estimate (iii) in Theorem 7.4, we conclude that

$$\fint_{B_\alpha} |f_\alpha| \,\mathrm{d}v_{g_\alpha} \geq (1+\epsilon_\alpha)^{-n} (1 - C\epsilon_\alpha),$$

and for α large enough we obtain a contradiction with (117).

If M_α is not oriented, consider the two-fold orientation covering $\hat{\pi}_\alpha : \widehat{M}_\alpha \to M_\alpha$ and choose $\hat{x} \in \hat{\pi}_\alpha^{-1}(x_\alpha)$. We observe that $\widehat{M}_\alpha$ endowed with the pullback metric $\hat{g}_\alpha = \hat{\pi}_\alpha^* g_\alpha$ satisfies for all $s > 0$,

$$\mathrm{k}_s(\widehat{M}_\alpha, \hat{g}_\alpha) = \mathrm{k}_s(M_\alpha, g_\alpha).$$

Then $(\widehat{M}_\alpha, \hat{g}_\alpha)$ is PI at the same scale as (M_α, g_α). Moreover, the map

$$\widehat{H}_\alpha = H_\alpha \circ \hat{\pi}_\alpha : B(\hat{x}, r_\alpha) \to \mathbb{B}(\tau_\alpha)$$

is $(1+\varepsilon_\alpha)$–Lipschitz and satisfies properties (iii) and (iv) of Theorem 7.4, Then we can apply the same argument as above and show that $\widehat{H}_\alpha : \hat{\pi}_\alpha^{-1}(\Omega_\alpha) \to \mathbb{B}(\tau_\alpha)$ is surjective. It finally follows that $H_\alpha : \Omega_\alpha \to \mathbb{B}(\tau_\alpha)$ is also surjective. □

Appendix

In this appendix, we provide a proof of Theorem 1.17 and of several other useful convergence results.

For the next two sections, we put ourselves in the following setting: we let $\{(X_\alpha, \mathsf{d}_\alpha, \mu_\alpha, o_\alpha)\}_\alpha, (X, \mathsf{d}, \mu, o)$ be proper geodesic pointed metric measure spaces such that

- $(X_\alpha, \mathsf{d}_\alpha, \mu_\alpha, o_\alpha) \xrightarrow{\text{pmGH}} (X, \mathsf{d}, \mu, o)$, and we use the sequences $\{R_\alpha\}$, $\{\varepsilon_\alpha\}$ and $\{\Phi_\alpha\}$ given by Characterization 1, and
- there exists $\kappa \geq 1$ and $R > 0$ such that the spaces $\{(X_\alpha, \mathsf{d}_\alpha, \mu_\alpha)\}_\alpha$ are all κ–doubling at scale R, hence so is (X, d, μ).

Appendix A Approximation of functions

The following result is known by experts. It says that the space $\mathscr{C}_c(X)$ is somehow the limit of the spaces $\{\mathscr{C}_c(X_\alpha)\}$, in the sense that any $\varphi \in \mathscr{C}_c(X)$ can be nicely approximated by functions $\varphi_\alpha \in \mathscr{C}_c(X_\alpha)$.

Proposition A.1 *For any $r > 0$ and any α large enough, we can build a linear map*

$$\mathscr{A}_\alpha : \mathscr{C}_c(B_r(o)) \to \mathscr{C}_c(X_\alpha)$$

such that the following holds for any $\varphi \in \mathscr{C}_c(B_r(o))$:

(i) *If $0 \leq \varphi \leq L$, then $0 \leq \mathscr{A}_\alpha \varphi \leq L$ for any α.*

(ii) *The convergence $\mathscr{A}_\alpha \varphi \xrightarrow{\mathscr{C}_c} \varphi$ holds.*

(iii) *The functions $\{\mathscr{A}_\alpha \varphi\}_\alpha$ are uniformly equicontinuous.*

(iv) *There exists a constant $\bar{C} > 0$ depending only on κ such that if φ is Λ–Lipschitz, then $\mathscr{A}_\alpha \varphi$ is $\bar{C}\Lambda$–Lipschitz.*

Proof Let $r > 0$. With no loss of generality, we assume that $\sup_\alpha \varepsilon_\alpha \leq r/8$ and that $2r \leq R_\alpha$. Let $\varphi \in \mathscr{C}_c(B_r(o))$.

Step 1 (construction of $\mathcal{A}_\alpha\varphi$) Let α be arbitrary. Let $\mathcal{D}_\alpha \subset X_\alpha$ be a maximal $2\varepsilon_\alpha$–separated set of points, ie a maximal set such that $X_\alpha = \bigcup_{p\in\mathcal{D}_\alpha} B_{2\varepsilon_\alpha}(p)$ and for any $p,q\in\mathcal{D}_\alpha$,

$$p\neq q \implies B_{\varepsilon_\alpha}(p)\cap B_{\varepsilon_\alpha}(q)=\varnothing.$$

For any $x\in X_\alpha$, we set $\mathcal{V}(x) := \mathcal{D}_\alpha\cap B_{4\varepsilon_\alpha}(x)$ and we point out that

$$\#\mathcal{V}(x)\leq \kappa^4. \tag{118}$$

Indeed we have $\bigcup_{p\in\mathcal{V}(x)} B_{\varepsilon_\alpha}(p)\subset B_{5\varepsilon_\alpha}(x)$ and $\mu_\alpha(B_{5\varepsilon_\alpha}(x))\leq\mu_\alpha(B_{9\varepsilon_\alpha}(p))\leq\kappa^4\mu_\alpha(B_{\varepsilon_\alpha}(p))$ for any $p\in\mathcal{V}(x)$.

Let us consider the 1–Lipschitz function $\chi\colon[0,+\infty)\to[0,1]$ defined by

$$\chi(t)=\begin{cases}1 & \text{if } t\leq 1,\\ 2-t & \text{if } t\in[1,2],\\ 0 & \text{if } t\in[2,+\infty).\end{cases} \tag{119}$$

For any $p\in\mathcal{D}_\alpha$, we define functions $\hat{\xi}^\alpha_p,\sigma^\alpha,\xi^\alpha_p\colon X_\alpha\to\mathbb{R}$ by

$$\hat{\xi}^\alpha_p(x)=\chi\left(\frac{\mathsf{d}_\alpha(p,x)}{2\varepsilon_\alpha}\right),\quad \sigma^\alpha(x)=\sum_{p\in\mathcal{D}_\alpha}\hat{\xi}^\alpha_p(x),\quad \xi^\alpha_p(x)=\frac{\hat{\xi}^\alpha_p(x)}{\sigma^\alpha(x)},$$

for $x\in X_\alpha$. By construction, $1\leq\sigma^\alpha\leq\kappa^4$ (the upper bound actually follows from (118)), the function ξ^α_p is $(1+\kappa^4)(2\varepsilon_\alpha)^{-1}$–Lipschitz, and

$$\sum_{p\in\mathcal{D}_\alpha}\xi^\alpha_p=1.$$

Then we define

$$\varphi_\alpha=\mathcal{A}_\alpha\varphi:=\sum_{p\in\mathcal{D}_\alpha}\varphi(\Phi_\alpha(p))\,\xi^\alpha_p.$$

As a linear combination of compactly supported Lipschitz functions, $\varphi_\alpha\in\mathrm{Lip}_c(X_\alpha)$. Moreover, $\operatorname{supp}\varphi_\alpha\subset B_{r+5\varepsilon_\alpha}(o_\alpha)$. Linearity of the map $\mathcal{A}_\alpha$ is clear from the construction. Finally, property (i) is trivially respected.

Step 2 (convergence $\varphi_\alpha\xrightarrow{\mathscr{C}_c}\varphi$) Let us show that if ω_φ is the modulus of continuity of φ, defined by $\omega_\varphi(\delta):=\sup_{\mathsf{d}(x,y)\leq\delta}|\varphi(x)-\varphi(y)|$ for any $\delta>0$, then

$$\|\varphi\circ\Phi_\alpha-\varphi_\alpha\|_{L^\infty(X_\alpha,\mu_\alpha)}\leq\kappa^4\omega_\varphi(5\varepsilon_\alpha)\quad\text{for any }\alpha. \tag{120}$$

Take $x\in X_\alpha$. Then

$$\begin{aligned}|\varphi\circ\Phi_\alpha(x)-\varphi_\alpha(x)| = \left|\sum_{p\in\mathcal{D}_\alpha}\big(\varphi(\Phi_\alpha(x))-\varphi(\Phi_\alpha(p))\big)\xi^\alpha_p(x)\right| &\leq \sum_{p\in\mathcal{V}(x)}\omega_\varphi\big(\mathsf{d}(\Phi_\alpha(x),\Phi_\alpha(p))\big)\xi^\alpha_p(x)\\ &\leq \omega_\varphi(5\varepsilon_\alpha).\end{aligned}$$

Hence (120) is proved, and consequently $\varphi_\alpha\xrightarrow{\mathscr{C}_c}\varphi$.

Step 3 (equicontinuity and Lipschitz estimate) Take $x, y \in X_\alpha$. Then

$$\begin{aligned}\varphi_\alpha(x)-\varphi_\alpha(y) &= \sum_{p\in\mathcal{D}_\alpha}[\varphi(\Phi_\alpha(p))-\varphi(\Phi_\alpha(x))](\xi_p^\alpha(x)-\xi_p^\alpha(y)) \\ &= \sum_{p\in\mathcal{V}(x)\cup\mathcal{V}(y)}[\varphi(\Phi_\alpha(p))-\varphi(\Phi_\alpha(x))](\xi_p^\alpha(x)-\xi_p^\alpha(y)).\end{aligned}$$

Observe that when $p \in \mathcal{V}(x)\cup\mathcal{V}(y)$, then $\mathsf{d}_\alpha(x,p) \le 4\varepsilon_\alpha + \mathsf{d}_\alpha(x,y)$, hence we have

$$|\varphi(\Phi_\alpha(p))-\varphi(\Phi_\alpha(x))| \le \omega_\varphi(\mathsf{d}_\alpha(x,p)+\varepsilon_\alpha) \le \omega_\varphi(5\varepsilon_\alpha+\mathsf{d}_\alpha(x,y)).$$

Using

$$\sum_{p\in\mathcal{D}_\alpha}|\xi_p^\alpha(x)-\xi_p^\alpha(y)| \le \min\left\{2, \frac{1+\kappa^4}{2\varepsilon_\alpha}\mathsf{d}_\alpha(x,y)\right\},$$

we obtain the estimate

$$|\varphi_\alpha(x)-\varphi_\alpha(y)| \le \begin{cases}\dfrac{(1+\kappa^4)\omega_\varphi(6\varepsilon_\alpha)}{2\varepsilon_\alpha}\mathsf{d}_\alpha(x,y) & \text{if } \mathsf{d}_\alpha(x,y)\le\varepsilon_\alpha, \\ 2\omega_\varphi(6\mathsf{d}_\alpha(x,y)) & \text{if } \mathsf{d}_\alpha(x,y)\ge\varepsilon_\alpha.\end{cases}$$

In particular, if φ is Λ–Lipschitz, then φ_α is $12(1+\kappa^4)\Lambda$ -Lipschitz. This estimate also implies the equicontinuity of the sequence: if $\delta \in (0,1)$ and $\mathsf{d}_\alpha(x,y) \le \delta$, then

$$|\varphi_\alpha(x)-\varphi_\alpha(y)| \le \omega_\varphi(6\sqrt{\delta}) + 2(1+\kappa^4)\|\varphi\|_{L^\infty}\sqrt{\delta}.$$ □

Appendix B Convergence of integrals

In this section, we prove two results about convergence of integrals under pmGH convergence that are used repeatedly in this article.

We recall our setting: $\{(X_\alpha, \mathsf{d}_\alpha, \mu_\alpha, o_\alpha)\}_\alpha$, (X, d, μ, o) are κ–doubling at scale R proper geodesic pointed metric measure spaces such that $(X_\alpha, \mathsf{d}_\alpha, \mu_\alpha, o_\alpha) \to (X, \mathsf{d}, \mu, o)$ in the pmGH sense, and we use the notation of Characterization 1.

It is easy to prove the first convergence result.

Proposition B.1 *Let $u \in C(X)$, $v \in L^2(X,\mu)$ and $u_\alpha \in C(X_\alpha)$, $v_\alpha \in L^2(X_\alpha,\mu_\alpha)$ be such that*

- $\sup_\alpha \|u_\alpha\|_{L^\infty} < \infty$,
- *$u_\alpha \to u$ uniformly on compact subsets, and*
- *$v_\alpha \to v$ strongly in L^2.*

Then

$$\lim_\alpha \int_{X_\alpha} u_\alpha v_\alpha^2 \, \mathrm{d}\mu_\alpha = \int_{X_\alpha} u v^2 \, \mathrm{d}\mu.$$

Proof The result follows from establishing the weak convergence $u_\alpha v_\alpha \xrightarrow{L^2} uv$. Notice first that the hypotheses of the proposition imply

$$\sup_\alpha \|u_\alpha v_\alpha\|_{L^2} < +\infty.$$

Moreover, when $\varphi_\alpha \xrightarrow{\mathscr{C}_c} \varphi$, then obviously $\varphi_\alpha u_\alpha \xrightarrow{\mathscr{C}_c} \varphi u$, and as $v_\alpha \xrightarrow{L^2} v$ we get

$$\lim_{\alpha\to+\infty} \int_{X_\alpha} \varphi_\alpha u_\alpha v_\alpha \,\mathrm{d}\mu_\alpha = \int_X \varphi u v \,\mathrm{d}\mu. \qquad \square$$

We make now a few useful remarks. The first point is that for any $r > 0$,

$$x_\alpha \in X_\alpha \to x \in X \implies \lim_{\alpha\to+\infty} \mu_\alpha(B_r(x_\alpha)) = \mu(B_r(x)). \tag{121}$$

In full generality, this convergence result holds when $\mu(\partial B_r(x)) = 0$ [12, Theorem 2.1], and this condition is guaranteed by the doubling condition Proposition 1.2(v). We also have that for any $r > 0$,

$$\varphi_\alpha \xrightarrow{\mathscr{C}_c} \varphi, \quad x_\alpha \in X_\alpha \to x \in X \quad \implies \quad \lim_{\alpha\to+\infty} \int_{B_r(x_\alpha)} \varphi_\alpha \,\mathrm{d}\mu_\alpha = \int_{B_r(x)} \varphi \,\mathrm{d}\mu. \tag{122}$$

Even better, the convergence result takes place as soon as $\varphi_\alpha \in \mathscr{C}(X_\alpha)$ converges uniformly on compact set to $\varphi \in \mathscr{C}(X)$.

The above convergence results (121) and (122) imply, by definition, that when $r > 0$, $p > 1$ and $x_\alpha \in X_\alpha \to x \in X$, then

$$\mathbf{1}_{B_r(x_\alpha)} \xrightarrow{L^p} \mathbf{1}_{B_r(x)}. \tag{123}$$

This implies the following criterion for L^p weak convergence.

Lemma B.2 *For $p \in (1, \infty)$, let $u_\alpha \in L^p(X_\alpha, \mu_\alpha)$ and $u \in L^p(X, \mu)$ be given. Then $u_\alpha \xrightarrow{L^p} u$ if and only if*

$$\begin{cases} \sup_\alpha \|u_\alpha\|_{L^p} < \infty, \\ x_\alpha \in X_\alpha \to x \in X,\ r > 0 \implies \lim_\alpha ⨍_{B_r(x_\alpha)} u_\alpha \,\mathrm{d}\mu_\alpha = ⨍_{B_r(x)} u \,\mathrm{d}\mu. \end{cases} \tag{124}$$

Proof The direct implication follows from (123). For the converse one, consider $B \subset A$ such that $\{u_\beta\}_{\beta\in B}$ converges weakly in L^p to some v. Then by (123), for any $r > 0$ and $x_\beta \to x$ we have $⨍_{B_r(x_\beta)} u_\beta \,\mathrm{d}\mu_\beta \to ⨍_{B_r(x)} v \,\mathrm{d}\mu$. The assumption gives $⨍_{B_r(x_\beta)} u_\beta \,\mathrm{d}\mu_\beta \to ⨍_{B_r(x)} u \,\mathrm{d}\mu$, hence we get

$$⨍_{B_r(x)} u \,\mathrm{d}\mu = ⨍_{B_r(x)} v \,\mathrm{d}\mu.$$

By the Lebesgue differentiation theorem (true on any doubling space), this implies $u = v$ μ–a.e. $\square$

Our second convergence result is the following.

Proposition B.3 *Let $u \in \mathscr{C}(X)$ and $u_\alpha \in \mathscr{C}(X_\alpha)$ be such that*

- *$u_\alpha \to u$ uniformly on compact subsets, and*
- *there exist $C, \gamma > 0$ such that for any α and μ_α–a.e. $x \in X_\alpha$,*

$$|u_\alpha(x)| \le C\, e^{-\gamma \mathsf{d}_\alpha^2(o_\alpha, x)}. \tag{125}$$

Then the functions u_α and u are L^p–integrable for any $p \ge 1$ and

(i) $\displaystyle\int_{X_\alpha} u_\alpha \,\mathrm{d}\mu_\alpha \to \int_X u \,\mathrm{d}\mu$,

(ii) *$u_\alpha \to u$ strongly in L^p when $p > 1$.*

For the proof of this proposition, we use the following lemma, which is a consequence of the ideas of the proof of (73).

Lemma B.4 *For any $c > 0$ there exists $A > 0$ depending only on c, κ and R such that for any $o \in X$,*

$$\int_X e^{-c\mathsf{d}^2(o,x)} \,\mathrm{d}\mu(x) \le A \mu(B_R(o)). \tag{126}$$

Moreover, there exists $\beta\colon (0, +\infty) \to (0, +\infty)$ depending only on c, κ and R such that $\beta(\rho) \to 0$ when $\rho \to +\infty$ and for any $\rho > 0$,

$$\int_{X \setminus B_\rho(o)} e^{-c\mathsf{d}^2(o,x)} \,\mathrm{d}\mu(x) \le \beta(\rho)\mu(B_R(o)). \tag{127}$$

Proof We have

$$\int_X e^{-c\mathsf{d}^2(o,x)} \,\mathrm{d}\mu(x) \le \mu(B_R(o)) + \int_{X \setminus B_R(o)} e^{-c\mathsf{d}^2(o,x)} \,\mathrm{d}\mu(x).$$

By Cavalieri's formula and Proposition 1.2(ii) we get that for any $\rho \ge R$,

$$\int_{X \setminus B_\rho(o)} e^{-c\mathsf{d}^2(o,x)} \,\mathrm{d}\mu(x) = \int_\rho^{+\infty} 2cre^{-cr^2} \mu(B_r(o)) \,\mathrm{d}r \le \mu(B_R(o)) \int_\rho^{+\infty} 2cre^{-cr^2 + \lambda r/R} \,\mathrm{d}r. \qquad \square$$

We can now prove Proposition B.3.

Proof of Proposition B.3 As a consequence of the previous lemma, for any $p > 1$, we get

$$\int_{X_\alpha \setminus B_\rho(o_\alpha)} |u_\alpha|^p \,\mathrm{d}\mu_\alpha \le \beta(\rho)\mu_\alpha(B_R(o_\alpha)), \tag{128}$$

where β depends only on p, γ and κ. The discussion above implies that

$$u_\alpha \xrightarrow{L^p_{\mathrm{loc}}} u.$$

With the estimate (128), we get that the sequence $\{\|u_\alpha\|_{L^p}\}$ is bounded, hence $u_\alpha \xrightarrow{L^p} u$. But the estimate (128) implies the validity of the interchange of limits

$$\lim_{\rho \to +\infty} \lim_\alpha \int_{B_\rho(o_\alpha)} |u_\alpha|^p \,\mathrm{d}\mu_\alpha = \lim_\alpha \lim_{\rho \to +\infty} \int_{B_\rho(o_\alpha)} |u_\alpha|^p \,\mathrm{d}\mu_\alpha,$$

that is to say,

$$\lim_{\alpha} \int_{X_\alpha} |u_\alpha|^p \, \mathrm{d}\mu_\alpha = \int_X |u|^p \, \mathrm{d}\mu.$$

Thus $u_\alpha \xrightarrow{L^p} u$. The statement Proposition B.3(i) is proven in the same way. □

Remark B.5 When the functions u_α are only assumed to be measurable, the conclusion Proposition B.3(i) holds assuming $u_\alpha \xrightarrow{L^2_{\mathrm{loc}}} u$ in place of the uniform convergence on compact sets. Indeed this hypothesis implies that for any $R > 0$,

$$\lim_{\alpha} \int_{B_R(o_\alpha)} u_\alpha \, \mathrm{d}\mu_\alpha = \int_{B_R(o)} u \, \mathrm{d}\mu,$$

and the proof of Proposition B.3 can be applied. But we won't need this refinement here.

Appendix C Heat kernel characterization of PI–Dirichlet spaces

In this section, we provide a set of conditions on the heat kernel of a metric Dirichlet space $(X, \mathsf{d}, \mu, \mathscr{E})$ for it to be regular, strongly local and with $\mathsf{d}_{\mathscr{E}}$ being a distance bi-Lipschitz equivalent to d. We use this result in the next section to prove Theorem 1.17. We let $R > 0$ be fixed throughout this section.

C.1 Heat kernel bound

We need an important statement about regular, strongly local Dirichlet spaces. It is the combination of several well-known theorems [48; 77; 85]. If $(X, \mathsf{d}, \mu, \mathscr{E})$ is a metric measure space equipped with a Dirichlet form with associated operator L, a local solution of the heat equation is any function u satisfying $(\partial_t + L)u = 0$ in the sense of [84]; see also [20, Definition 2.3].

Theorem C.1 *Let $(X, \mathscr{T}, \mu, \mathscr{E})$ be a regular, strongly local Dirichlet space and let d be a distance compatible with $\mathscr{T}$. Then the following are equivalent.*

(c1) *$(X, \mathsf{d}, \mu, \mathscr{E})$ is a $\mathrm{PI}_{\kappa,\gamma}(R)$–Dirichlet space.*

(c2) *$\mathscr{E}$ admits a heat kernel H satisfying Gaussian bounds: there exists $\beta > 0$ such that*

$$\frac{\beta^{-1}}{\mu(B_{\sqrt{t}}(x))} e^{-\beta \mathsf{d}^2(x,y)/t} \le H(t,x,y) \le \frac{\beta}{\mu(B_{\sqrt{t}}(x))} e^{-\mathsf{d}^2(x,y)/\beta t} \tag{129}$$

for all $x, y \in X$ and $t \in (0, R^2]$.

(c3) *The local solutions of the heat equation satisfy a uniform Hölder regularity estimate: there exist constants $\alpha \in (0,1]$ and $A > 0$ such that if B is a ball of radius $r \le R$ and $u\colon (0, r^2) \times 2B \to (0, \infty)$ is a local solution of the heat equation, then for any $s, t \in \left(\tfrac14 r^2, \tfrac34 r^2\right)$ and $x, y \in B$,*

$$|u(s,x) - u(t,y)| \le \frac{A}{r^\alpha} \left(\sqrt{|t-s|} + \mathsf{d}(x,y)\right)^\alpha \sup_{(0,r^2)\times B} |u|. \tag{130}$$

As a corollary, the heat kernel of a regular, strongly local PI–Dirichlet space satisfies the following properties.

Proposition C.2 *Let $(X, \mathsf{d}_{\mathcal{E}}, \mu, \mathcal{E})$ be a regular, strongly local, $\mathrm{PI}_{\kappa,\gamma}(R)$–Dirichlet space for some $\kappa \geq 1$ and $\gamma > 0$. Let d be a distance on X bi-Lipschitz equivalent to $\mathsf{d}_{\mathcal{E}}$. Then $\mathcal{E}$ admits a heat kernel H such that the following hold.*

(a) *H is stochastically complete (12).*

(b) *There exists $\beta \geq 1$ such that the Gaussian two-sided bounds*

$$\frac{\beta^{-1}}{\mu(B_{\sqrt{t}}(x))} e^{-\beta \mathsf{d}^2(x,y)/t} \leq H(t,x,y) \leq \frac{\beta}{\mu(B_{\sqrt{t}}(x))} e^{-\mathsf{d}^2(x,y)/\beta t} \tag{131}$$

hold for all $x, y \in X$ and $t \in (0, R^2]$.

(c) *There exist $\alpha \in (0,1)$ and $A > 0$ such that for any $x, y, z \in X$ and $s, t \in (0, R)$ such that $|t-s| \leq t/4$ and $\mathsf{d}(y,z) \leq \sqrt{t}$,*

$$|H(s,x,z) - H(t,x,y)| \leq A\left(\frac{\sqrt{|t-s|} + \mathsf{d}(y,z)}{\sqrt{t}}\right)^{\alpha} H(t,x,y). \tag{132}$$

The next theorem, which is the converse of Proposition C.2, is our key statement to establish Theorem 1.17. We dedicate the rest of this section to proving it.

Theorem C.3 *Let $(X, \mathsf{d}, \mu, \mathcal{E})$ be a metric Dirichlet space such that (X, d) is geodesic and for which a heat kernel H exists and satisfies* (a)–(c) *of Proposition C.2. Then $(X, \mathsf{d}_{\mathcal{E}}, \mu, \mathcal{E})$ is a $\mathrm{PI}_{\kappa,\gamma}(R)$–Dirichlet space for some $\kappa \geq 1$ and $\gamma > 0$, depending only on the constants from* (b) *and* (c)*, and the distance d is bi-Lipschitz equivalent to the intrinsic distance $\mathsf{d}_{\mathcal{E}}$.*

C.2 Domain characterization

In order to prove Theorem C.3, we start by showing the following crucial proposition. It is a generalization of a similar result of A Grigor'yan, J Hu and K-S Lau [52, Theorem 4.2] — see also [51, Corollary 4.2] and references therein — where the measure is additionally assumed to be uniformly Ahlfors regular.

Proposition C.4 *Under the assumptions of Theorem C.3, the domain of $\mathcal{E}$ coincides with the Besov space $B_{2,\infty}(X)$, consisting of the functions $u \in L^2(X,\mu)$ such that*

$$N(u)^2 := \limsup_{r \to 0^+} \frac{1}{r^2} \int_X \fint_{B_r(x)} (u(x) - u(y))^2 \, \mathrm{d}\mu(y) \, \mathrm{d}\mu(x) < \infty.$$

Moreover, there is a constant C depending only on β such that for any $u \in \mathcal{D}(\mathcal{E})$,

$$\frac{1}{C} N(u)^2 \leq \mathcal{E}(u) \leq C N(u)^2.$$

Proof For any function $u \in L^2(X,\mu)$, define the decreasing function $t \mapsto \mathscr{E}_t(u)$ where for any $t>0$,

$$\mathscr{E}_t(u) := \frac{1}{t}\langle u - e^{-tL}u, u\rangle = \int_{X\times X} H(t,x,y)(u(x)-u(y))^2 \frac{\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)}{2t}.$$

We first observe that, because of assumption (a), a function u belongs to $\mathscr{D}(\mathscr{E})$ if and only if $\sup_t \mathscr{E}_t(u) < \infty$. Moreover, if $u \in \mathscr{D}(\mathscr{E})$, then $\mathscr{E}(u) = \lim_{t\to 0^+} \mathscr{E}_t(u)$. This is explained in [51, Section 2.2], for instance.

Step 1 We begin with showing the easiest inclusion, namely $\mathscr{D}(\mathscr{E}) \subset B_{2,\infty}(X)$. Take $u \in \mathscr{D}(\mathscr{E})$. For $t>0$, set

$$I(t) := \int_{\{(x,y)\in X\times X : \mathrm{d}(x,y)\le\sqrt{t}\}} H(t,x,y)(u(x)-u(y))^2 \frac{\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)}{2t}.$$

Observe that $I(t) \le \mathscr{E}_t(u) \le \mathscr{E}(u)$. The lower bound for the heat kernel given by assumption (b) implies

$$\begin{aligned} I(t) &\ge \int_{\{(x,y)\in X\times X : \mathrm{d}(x,y)\le\sqrt{t}\}} \frac{\beta^{-1}}{\mu(B_{\sqrt{t}}(x))} e^{-\beta \mathrm{d}^2(x,y)/t}(u(x)-u(y))^2 \frac{\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)}{2t} \\ &\ge \frac{\beta^{-1}e^{-\beta}}{2t}\int_X \fint_{B_{\sqrt{t}}(x)} (u(x)-u(y))^2\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y), \end{aligned}$$

hence letting t tend to 0 shows that $N(u)^2 \le 2\beta e^{\beta}\mathscr{E}(u)$.

Step 2 In order to prove the converse inclusion, we need some volume estimates. Our assumptions imply that the measure μ is doubling at scale R. Indeed for all $x \in X$ and $r \le R$, thanks to assumptions (a) and (b) we have

$$\beta^{-1}e^{-4\beta}\frac{\mu(B_{2r}(x))}{\mu(B_r(x))} \le \int_{B_{2r}(x)} H(r^2,x,y)\,\mathrm{d}\mu(y) \le \int_X H(r^2,x,y)\,\mathrm{d}\mu(y) = 1,$$

and therefore

$$\mu(B_{2r}(x)) \le \beta e^{4\beta}\mu(B_r(x)).$$

Because of the doubling condition at scale R, we obtain for $s \le r \le 2R$ and $x \in X$,

$$\mu(B_r(x)) \le C\left(\frac{r}{s}\right)^{\nu}\mu(B_s(x)),$$

where ν and C depend only on β; see Proposition 1.2. The Gaussian estimate of the heat kernel (131) implies that if $0 < t \le \tau \le R^2$, then

$$H(t,x,y) \le H(\tau,x,y)\,C\left(\frac{\tau}{t}\right)^{\nu/2} e^{-\mathrm{d}^2(x,y)(1/\beta t - \beta/\tau)} \tag{133}$$

We introduce $\Omega_r = \{(x,y)\in X\times X : \mathrm{d}(x,y)\ge r\}$ and

$$I_\lambda(t) := \iint_{(X\times X)\setminus\Omega_{\lambda\sqrt{t}}} H(t,x,y)(u(x)-u(y))^2\frac{\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)}{2t}.$$

The same argument as in Step 1 implies that for $\lambda \ge 1$ and $t>0$ such that $\lambda\sqrt{t} < R$,

$$I_\lambda(t) \le \frac{C\lambda^{\nu}}{2t}\int_X \fint_{B_{\lambda\sqrt{t}}(x)} (u(x)-u(y))^2\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x). \tag{134}$$

Using the estimate (133) with $\tau = \lambda t$ and assuming $\lambda \geq 1$ and $\lambda^2 t \leq R^2$, we estimate

$$\mathscr{E}_t(u) - I_\lambda(t) \leq C\lambda^{\nu/2} e^{-\lambda^2 t(1/\beta t - \beta/\lambda t)} \iint_{\Omega_{\lambda\sqrt{t}}} H(\lambda t, x, y)(u(x)-u(y))^2 \frac{\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)}{2t}$$
$$\leq C\lambda^{\nu/2+1} e^{-\lambda^2(1/\beta - \beta/\lambda)} \mathscr{E}_{\lambda t}(u) \leq C\lambda^{\nu/2+1} e^{-\lambda^2(1/\beta-\beta/\lambda)} \mathscr{E}_t(u),$$

where we have used that $t \mapsto \mathscr{E}_t(u)$ is nonincreasing. If we choose $\lambda = \lambda(\beta)$ sufficiently large that $C\lambda^{\nu/2+1} e^{-\lambda^2(1/\beta-\beta/\lambda)} \leq \frac{1}{2}$, then we get

$$\mathscr{E}_t(u) \leq \frac{2C\lambda^\nu}{2t} \int_X \fint_{B_{\lambda\sqrt{t}}(x)} (u(x)-u(y))^2 \,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x).$$

Hence the result. □

Remark C.5 The above reasoning implies that if $U : X \times X \to \mathbb{R}_+$ is a nonnegative integrable function such that the limit

$$\lim_{t\to 0+} \int_{X\times X} H(t,x,y) C(x,y) \frac{\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)}{2t}$$

exists and is finite, then

$$\lim_{t\to 0+} \int_{X\times X} H(t,x,y) C(x,y) \frac{\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)}{2t} \leq C(\beta) \limsup_{r\to 0^+} \int_X \fint_{B_r(x)} C(x,y) \frac{\mathrm{d}\mu(y)\,\mathrm{d}\mu(x)}{r^2}.$$

We are now in position to prove Theorem C.3.

Proof of Theorem C.3 • **Regularity** We start by showing that $(X, \mathsf{d}, \mu, \mathscr{E})$ is regular, that is, we prove that the space $\mathscr{C}_c(X) \cap \mathscr{D}(\mathscr{E})$ is dense in $(\mathscr{C}_c(X), \|\cdot\|_\infty)$ and in $(\mathscr{D}(\mathscr{E}), |\cdot|_{\mathscr{D}(\mathscr{E})})$.

Observe that $\mathrm{Lip}_c(X)$ is contained in $\mathscr{D}(\mathscr{E})$. Indeed, for any $u \in \mathrm{Lip}_c(X)$ there exists Λ such that for all $x, y \in X$,

$$|u(x) - u(y)| \leq \Lambda \mathsf{d}(x,y),$$

and there exists $\rho > 0$ such that the support of u is included in the ball $B_\rho(o)$. Therefore for any $r > 0$ and $x \in X$ we have

$$e_r(x) := \fint_{B_r(x)} (u(x)-u(y))^2 \,\mathrm{d}\mu(y) \leq \Lambda^2 r^2,$$

and moreover, $e_r(x) = 0$ if $x \notin B_{\rho+1}(o)$. As a consequence, for any $u \in \mathrm{Lip}_c(X)$, there exists ρ such that

$$N(u)^2 \leq \mu(B_{\rho+1}(o))\Lambda^2,$$

thus $\mathrm{Lip}_c(X) \subset B_{2,\infty}(X)$, and by the previous theorem $\mathrm{Lip}_c(X) \subset \mathscr{D}(\mathscr{E})$. Since $\mathrm{Lip}_c(X)$ is dense in $(\mathscr{C}_c(X), \|\cdot\|_\infty)$, this implies that $\mathscr{C}_c(X) \cap \mathscr{D}(\mathscr{E})$ is also dense in $(\mathscr{C}_c(X), \|\cdot\|_\infty)$.

In order to prove that $\mathscr{C}_c(X) \cap \mathscr{D}(\mathscr{E})$ is dense in $\mathscr{D}(\varepsilon)$, we follow the same argument as in the proof of [20, Proposition 3.7], ie we show that if $t \in (0, R^2)$ and $u \in L^2(X,\mu)$, then $f = e^{-tL}u$ belongs to $\mathscr{C}_0(X)$. To see that f tends to zero at infinity, notice that the upper bound for the heat kernel implies

$$|f(x)| \leq \beta \frac{1}{\mu(B_{\sqrt{t}}(x))} e^{-\mathsf{d}^2(x,\operatorname{supp} u)/\beta t} \int_X |u|\,\mathrm{d}\mu$$

for any $x \in X$; from Proposition 1.2(i), if $o \in \operatorname{supp} u$ we obtain

$$|f(x)| \leq \frac{C}{\mu(B_{\sqrt{t}}(o))} e^{-\mathsf{d}^2(x,\operatorname{supp} u)/\beta t + \lambda \mathsf{d}(o,x)/\sqrt{t}} \int_X |u| \, \mathrm{d}\mu,$$

therefore f is bounded and tends to zero at infinity.

As for the continuity of f, assumption (c) ensures that for any $x, x' \in X$ such that $\mathsf{d}(x,x') \leq \sqrt{t}$ we have

$$|f(x) - f(x')| \leq A \left(\frac{\mathsf{d}(x,x')}{\sqrt{t}} \right)^{\alpha} f(x).$$

Since f is bounded, this shows that f is continuous.

- **Strong locality** We aim to prove that if $u, v \in \mathcal{D}(\mathcal{E})$ have compact supports and if u is constant in a neighborhood of $\operatorname{supp}(v)$, then $\mathcal{E}(u,v) = 0$. Assume that both u and v are supported in $B_\rho(o)$ and denote by K the support of v. There exist $\eta > 0$ and $c \in \mathbb{R}$ such that if $\mathsf{d}(x,K) \leq \eta$, then $u(x) = c$. Let us introduce $K^r = \bigcup_{x \in K} B_r(x)$. Then for any $r \leq \eta$, u is constantly equal to c on K^r.

As in the proof of the previous theorem, we can define

$$\mathcal{E}_t(u,v) := \int_{X \times X} H(t,x,y)(u(x)-u(y))(v(x)-v(y)) \frac{\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)}{2t},$$

and we have $\mathcal{E}(u,v) = \lim_{t \to 0^+} \mathcal{E}_t(u,v)$. From Remark C.5, there exists a constant $C > 0$ such that

$$|\mathcal{E}(u,v)| \leq C \limsup_{r \to 0^+} \frac{1}{r^2} \int_X \left(\fint_{B_r(x)} |u(x)-u(y)|\,|v(x)-v(y)| \, \mathrm{d}\mu(y) \right) \mathrm{d}\mu(x).$$

Now observe that for any $r > 0$, if $x \notin K^r$ and $y \in B_r(x)$, the triangle inequality ensures that $\mathsf{d}(y,K) > 0$, thus both $v(x)$ and $v(y)$ are equal to zero. We are then left with considering

$$\limsup_{r \to 0^+} \frac{1}{r^2} \int_{K^r} \left(\fint_{B_r(x)} |u(x)-u(y)|\,|v(x)-v(y)| \, \mathrm{d}\mu(y) \right) \mathrm{d}\mu(x).$$

But the same argument implies that when $r \leq \eta/2$, $x \in K^r$ and $y \in B_r(x)$, then $y \in K^{2r}$; as a consequence both x and y belong to K^η, so we have $u(x) = u(y) = c$. Finally, for $r \leq \eta/2$ we get

$$\int_X \left(\fint_{B_r(x)} |u(x)-u(y)|\,|v(x)-v(y)| \, \mathrm{d}\mu(y) \right) \mathrm{d}\mu(x) = 0.$$

This ensures that $\mathcal{E}(u,v) = 0$ and thus $(X, \mathsf{d}, \mu, \mathcal{E})$ is strongly local.

- **Equivalence between the distance and the intrinsic distance** Let us begin with proving the existence of $C > 0$ such that

$$\mathsf{d}_{\mathcal{E}} \geq C\,\mathsf{d}.$$

Again from Remark C.5, there exists a constant C such that for any $u \in \mathcal{D}(\mathcal{E})$ and $\phi \in \mathcal{C}_c(X) \cap \mathcal{D}(\mathcal{E})$ with $\phi \geq 0$ then

$$\int_X \phi \, \mathrm{d}\Gamma(u) \leq C \limsup_{r \to 0^+} \frac{1}{r^2} \int_X \phi(x) \left(\fint_{B_r(x)} (u(x)-u(y))^2 \, \mathrm{d}\mu(y) \right) \mathrm{d}\mu(x).$$

If $u \in \mathrm{Lip}_c(X)$ then

$$\int_X \phi \,\mathrm{d}\Gamma(u) \le C\,\mathrm{Lip}(u)^2 \int_X \phi \,\mathrm{d}\mu.$$

Hence

$$\mathrm{d}\Gamma(u) \le C\,\mathrm{Lip}(u)^2 \,\mathrm{d}\mu.$$

Take $x, y \in X$ and set $r := \mathsf{d}(x,y)$ and

$$u_x(z) := \chi\left(\frac{\mathsf{d}(x,z)}{2r}\right)\mathsf{d}(x,z)$$

for any $z \in X$, where χ is defined as in (119). Then $u_x \in \mathrm{Lip}_c(X)$ and $u_x(y) - u_x(x) = \mathsf{d}(x,y)$. Moreover, $\mathrm{Lip}(u_x) \le 3$. Thus, testing $u_x/(3\sqrt{C})$ in the definition of $\mathsf{d}_{\mathscr{E}}(x,y)$, we get

$$\mathsf{d}_{\mathscr{E}}(x,y) \ge (3\sqrt{C})^{-1}\mathsf{d}(x,y).$$

Now let us prove

$$\mathsf{d}_{\mathscr{E}} \le \sqrt{\beta}\,\mathsf{d}.$$

We act as in the proof of [20, Proposition 3.9]. We consider a bounded function $v \in \mathscr{D}_{\mathrm{loc}}(\mathscr{E}) \cap C(X)$ such that $\Gamma(v) \le \mu$. For any $a \ge 0$, $t \in (0, R^2)$ and $x \in X$ we set $\xi_a(x,t) := a v(x) - a^2 t/2$. Take $x, y \in X$ and assume with no loss of generality that $v(y) - v(x) > 0$. From [20, Claim 3.10] applied to $f = \mathbf{1}_{B_{\sqrt{t}}(y)}$, one gets

$$\int_{B_{\sqrt{t}}(x)} \left(\int_{B_{\sqrt{t}}(y)} H(t,z_1,z_2)\,\mathrm{d}\mu(z_2)\right)^2 e^{\xi_a(z_1,t)}\,\mathrm{d}\mu(z_1) \le \int_{B_{\sqrt{t}}(y)} e^{av}\,\mathrm{d}\mu,$$

which leads to

$$\mu(B_{\sqrt{t}}(x))\mu(B_{\sqrt{t}}(y))\exp\left(a\delta_t(x,y) - \tfrac{1}{2}a^2 t\right) \inf_{B_{\sqrt{t}}(x)\times B_{\sqrt{t}}(y)} H(t,\cdot,\cdot)^2 \le 1, \tag{135}$$

where we define

$$\delta_t(x,y) := \inf_{B_{\sqrt{t}}(x)} v - \sup_{B_{\sqrt{t}}(y)} v.$$

Observe that

$$\sup_{B_{\sqrt{t}}(x)\times B_{\sqrt{t}}(y)} \mathsf{d}(\cdot,\cdot) \le \mathsf{d}(x,y) + 2\sqrt{t},$$

so that the Gaussian lower bound in (131) yields, for any $(z_1,z_2) \in B_{\sqrt{t}}(x) \times B_{\sqrt{t}}(y)$,

$$H(t,z_1,z_2) \ge \frac{\beta^{-1}}{\mu(B_{\sqrt{t}}(z_1))}\exp\left(-\beta\frac{(\mathsf{d}(x,y)+2\sqrt{t})^2}{4t}\right).$$

The doubling condition implies $\mu(B_{\sqrt{t}}(z_1)) \le \mu(B_{2\sqrt{t}}(x)) \le \kappa\mu(B_{\sqrt{t}}(x))$, and we get

$$\left(\inf_{B_{\sqrt{t}}(x)\times B_{\sqrt{t}}(y)} H(t,\cdot,\cdot)^2\right) \ge \frac{(\beta\kappa)^{-2}}{\mu(B_{\sqrt{t}}(x))^2}\exp\left(-\beta\frac{(\mathsf{d}(x,y)+2\sqrt{t})^2}{2t}\right).$$

The continuity of v yields $\lim_{t\to 0+}\delta_t(x,y)=v(y)-v(x)$, hence we can take t small enough to ensure $\delta_t(x,y)>0$ and choose $a=\delta_t(x,y)/t$. Then (135) implies

$$\frac{\mu(B_{\sqrt{t}}(y))}{\mu(B_{\sqrt{t}}(x))}\exp\biggl(-\beta\frac{(\mathsf{d}(x,y)+2\sqrt{t})^2}{2t}+\frac{\delta_t^2(x,y)}{2t}\biggr)\le(\beta\kappa)^2.$$

Thanks to Proposition 1.2(i), this leads to

$$\exp\biggl(-\lambda\frac{\mathsf{d}(x,y)}{\sqrt{t}}-\beta\frac{(\mathsf{d}(x,y)+2\sqrt{t})^2}{2t}+\frac{\delta_t^2(x,y)}{2t}\biggr)\le C(\beta\kappa)^2.$$

Letting $t\to 0$, we get

$$(v(y)-v(x))^2\le\beta\mathsf{d}(x,y)^2.$$

Since v is arbitrary, we finally obtain

$$\mathsf{d}_{\mathcal{E}}(x,y)\le\sqrt{\beta}\,\mathsf{d}(x,y). \qquad \square$$

Appendix D Proof of Theorem 1.17

Proof We assume that the spaces $\{(X_\alpha,\mathsf{d}_\alpha=\mathsf{d}_{\mathcal{E}_\alpha},\mu_\alpha,o_\alpha,\mathcal{E}_\alpha)\}_{\alpha\in A}$ are $\mathrm{PI}_{\kappa,\gamma}(R)$–Dirichlet spaces and that for any α,

$$\eta^{-1}\le\mu_\alpha(B_R(o_\alpha))\le\eta. \tag{136}$$

The existence of (X,d,μ,o) and a subsequence $B\subset A$ such that

$$(X_\beta,\mathsf{d}_\beta,\mu_\beta,o_\beta)\xrightarrow{\text{pmGH}}(X,\mathsf{d},\mu,o)$$

follow from Proposition 1.7. Moreover (X,d) is complete and geodesic, and (X,d,μ) is κ–doubling at scale R.

Furthermore, Proposition C.2 ensures that any $\mathcal{E}_\beta$ admits a stochastically complete heat kernel H_β satisfying the Gaussian bounds (131) and the estimate (132) with constants β, α and A depending only on κ and γ. Let $t\in(0,R^2)$ and $\rho>0$. By Proposition 1.2(i), we get that for any $x,y\in B_\rho(o_\beta)$,

$$H_\beta(t,x,y)\le\frac{Ce^{\lambda\rho/\sqrt{t}}}{\mu_\beta(B_{\sqrt{t}}(o_\beta))},$$

from which the doubling condition and the noncollapsing condition (136) yield the uniform estimate

$$H_\beta(t,x,y)\le C\biggl(\frac{R}{\sqrt{t}}\biggr)^{\nu}\exp\biggl(\lambda\frac{\rho}{\sqrt{t}}\biggr)\eta, \tag{137}$$

where ν,C,λ depend only on κ,γ. Hence for any $t\in(0,R^2)$ and $\rho>0$, there is a constant Λ depending only on $t,\rho,\kappa,\gamma,R,\eta$ such that for $x,x',y,y'\in B_\rho(o_\beta)$, we get the Hölder estimate

$$|H_\beta(t,x,y)-H_\beta(t,x',y')|\le\Lambda\min\{t^{\alpha/2},[\mathsf{d}(x,x')+\mathsf{d}(y,y')]^{\alpha}\}. \tag{138}$$

Thanks to this local Hölder continuity estimate and the uniform estimate (137), the Arzelà–Ascoli theorem with respect to pGH convergence (see eg [94, Proposition 27.20]) implies that, up to extracting another subsequence, the functions $H_\beta(t,\cdot,\cdot)$ converge uniformly on compact sets to some function $H(t,\cdot,\cdot) \in \mathscr{C}(X\times X)$, where $t>0$ is fixed from now on. A priori this subsequence may depend on t, but for the moment $t\in(0,R^2)$ is fixed.

Let $S\colon L^2_c(X,\mu)\to L^\infty_{\mathrm{loc}}(X,\mu)$ be the integral operator on X defined by setting

$$Su(x) := \int_X u(y)H(t,x,y)\,\mathrm{d}\mu(y)$$

for any $u\in L^2_c(X,\mu)$ and $x\in X$.

We claim that S has a bounded linear extension $S\colon L^2(X,\mu)\to L^2(X,\mu)$. Firstly, thanks to the uniform convergence on compact sets $H_\beta(t,\cdot,\cdot)\to H(t,\cdot,\cdot)$, the symmetry with respect to the two space variables of H_β transfers to H. Moreover, PI–Dirichlet spaces are stochastically complete, hence for any $x\in X_\beta$,

$$\int_{X_\beta} H_\beta(t,x,y)\,\mathrm{d}\mu_\beta(y) = 1.$$

Using the uniform Gaussian estimate (131) and Proposition B.3, we have similarly

$$\int_X H(t,x,y)\,\mathrm{d}\mu(y) = 1$$

for any $x\in X$. The Schur test implies that for any $p\in[0,+\infty]$, S extends to a bounded operator $S\colon L^p(X,\mu)\to L^p(X,\mu)$ with operator norm satisfying

$$\|S\|_{L^p\to L^p}\le 1. \tag{139}$$

The symmetry with respect to the two space variables of H implies that

$$S\colon L^2(X,\mu)\to L^2(X,\mu)$$

is self-adjoint. Hence there exists a nonnegative self-adjoint operator L with dense domain $\mathscr{D}(L)\subset L^2(X,\mu)$ such that

$$S = e^{-tL}.$$

Moreover, we have $f\ge 0 \implies Sf\ge 0$.

Let us show now the strong convergence of bounded operators

$$e^{-tL_\beta}\to e^{-tL}. \tag{140}$$

The operators are all self-adjoint, hence it is enough to show the weak convergence of bounded operators, and this amounts to showing that if $u_\beta \xrightarrow{L^2} u$, then $e^{-tL_\beta}u_\beta \xrightarrow{L^2} e^{-tL}u$. Note that $\sup_\beta \|u_\beta\|_{L^2}<+\infty$. Since the operators e^{-tL_β} have all an operator norm less than 1, then

$$\sup_\beta \|e^{-tL_\beta}u_\beta\|_{L^2}<+\infty.$$

Now take $x_\beta\to x$. The uniform Gaussian estimate (131) and Proposition B.3 ensure that the functions $f_\beta = H_\beta(t,x_\beta,\cdot)$ converge strongly in L^2 to the function $f=H(t,x,\cdot)$. Then

$$\langle f_\beta,u_\beta\rangle_{L^2(X_\beta,\mu_\beta)}\to\langle f,u\rangle_{L^2(X,\mu)},$$

that is to say

$$e^{-tL_\beta}u_\beta(x_\beta) = \int_{X_\beta} H_\beta(t, x_\beta, y)u_\beta(y)\,\mathrm{d}\mu_\beta(y) \to \int_X H(t, x, y)u(y)\,\mathrm{d}\mu(y) = e^{-tL}u(x).$$

The same argument can be used with

$$f_{\beta,r}(y) = \int_{B_r(x_\beta)} H_\beta(t, z, y)\,\mathrm{d}\mu_\beta(z) \quad \text{and} \quad f_r(y) = \int_{B_r(x)} H(t, z, y)\,\mathrm{d}\mu(z)$$

for any $r > 0$, hence Lemma B.2 eventually implies

$$e^{-tL_\beta}u_\beta \xrightarrow{L^2} e^{-tL}u.$$

Using now [67, Theorem 2.1], we get the strong convergence of bounded operators

$$e^{-\tau L_\beta} \to e^{-\tau L}$$

for any $\tau > 0$. But the above argumentation shows that for any $\tau \in (0, R^2)$, the function $H_\beta(\tau, \cdot, \cdot)$ has a unique limit (for the uniform convergence on compact set of $X \times X$) and this limit is the Schwartz kernel of the operator $e^{-\tau L}$. Moreover

$$0 \le f \le 1 \implies 0 \le e^{-\tau L} f \le 1,$$

hence the quadratic form

$$\mathscr{E}(u) := \int_X (Lu)u\,\mathrm{d}\mu$$

is a Dirichlet form $\mathscr{E}$ on (X, d, μ).

From Proposition 1.15, the strong convergence (140) implies the Mosco convergence $\mathscr{E}_\beta \to \mathscr{E}$. As a consequence, the functions $H_\beta\colon (0, R^2] \times X_\beta \times X_\beta \to (0, +\infty)$ uniformly converge on compact sets to the heat kernel $\widetilde{H}$ of $\mathscr{E}$ restricted to $(0, R^2] \times X \times X$. Then the dominated convergence theorem ensures that $\widetilde{H}$ satisfies the assumptions (a)–(c) in Theorem C.3, thus $(X, \mathsf{d}, \mu, \mathscr{E})$ is regular, strongly local, and with intrinsic distance $\mathsf{d}_{\mathscr{E}}$ bi-Lipschitz equivalent to d so that $(X, \mathsf{d}, \mu, \mathscr{E})$ is a $\mathrm{PI}_{\kappa,\gamma'}(R)$–Dirichlet space.

It remains to show that $\mathsf{d} \le \mathsf{d}_{\mathscr{E}}$. According to (21), the heat kernel of $\mathscr{E}_\beta$ satisfies the uniform upper Gaussian estimate

$$H_\beta(t, x, y) \le \frac{C}{\mu_\beta(B_R(x))} \frac{R^\nu}{t^{\nu/2}} \left(1 + \frac{\mathsf{d}^2_{\mathscr{E}_\beta}(x, y)}{t}\right)^{\nu+1} \exp\left(-\frac{\mathsf{d}^2_{\mathscr{E}_\beta}(x, y)}{4t}\right),$$

which is valid for any $x, y \in X_\beta$ and $t \in (0, R^2)$. By uniform convergence on compact sets $H_\beta \to H$ and $\mathsf{d}_{\mathscr{E}_\beta} \to \mathsf{d}$, we get the same estimate for the heat kernel of $\mathscr{E}$,

$$H(t, x, y) \le \frac{C}{\mu(B_R(x))} \frac{R^\nu}{t^{\nu/2}} \left(1 + \frac{\mathsf{d}^2(x, y)}{t}\right)^{\nu+1} \exp\left(-\frac{\mathsf{d}^2(x, y)}{4t}\right).$$

From there it is easy to conclude that $\mathsf{d} \le \mathsf{d}_{\mathscr{E}}$ thanks to Varadhan's formula (22). □

Appendix E Further convergence results

In this last section we assume that $(X, \mathsf{d}, \mu, \mathscr{E}, o)$ is a $\mathrm{PI}_{\kappa,\gamma}(R)$–Dirichlet space that is a pointed Mosco–Gromov–Hausdorff limit of a sequence $\{(X_\alpha, \mathsf{d}_\alpha, \mu_\alpha, \mathscr{E}_\alpha, o_\alpha)\}_\alpha$ of $\mathrm{PI}_{\kappa,\gamma}(R)$–Dirichlet spaces, and we use the notation $\{\varepsilon_\alpha\}$, $\{R_\alpha\}$, $\{\Phi_\alpha\}$ of Characterization 1.

We begin with a technical result.

Proposition E.1 *Let $\{u_\alpha\}$ be such that $u_\alpha \in L^2(B_r(o_\alpha), \mu_\alpha)$ for any α, for some $r > 0$. Assume that*

(i) *there exists $u \in L^2(B_r(o), \mu)$ such that $u_\alpha \xrightarrow{L^2} u$, and*

(ii) $\sup_\alpha \int_{B_r(o_\alpha)} \mathrm{d}\Gamma_\alpha(u_\alpha) < +\infty$.

Then

$$\lim_\alpha \int_{B_r(o_\alpha)} u_\alpha^2 \,\mathrm{d}\mu_\alpha = \int_{B_r(o)} u^2 \,\mathrm{d}\mu. \tag{141}$$

Proof We first prove that for any $s < r$,

$$\lim_\alpha \int_{B_s(o_\alpha)} u_\alpha^2 \,\mathrm{d}\mu_\alpha = \int_{B_s(o)} u^2 \,\mathrm{d}\mu. \tag{142}$$

For $\varepsilon < (r-s)/4$, we introduce

$$u_{\alpha,\varepsilon}(x) = \fint_{B_\varepsilon(x)} u_\alpha \,\mathrm{d}\mu_\alpha.$$

The Poincaré inequality implies the pseudo-Poincaré inequality [35, Lemme in page 301]

$$\|u_\alpha - u_{\alpha,\varepsilon}\|_{L^2(B_s(o_\alpha))} \le C\varepsilon \quad \text{and} \quad \|u - u_\varepsilon\|_{L^2(B_s(o))} \le C\varepsilon,$$

where C depends only on $\sup_\alpha \int_{B_r(o_\alpha)} \mathrm{d}\Gamma_\alpha(u_\alpha)$, the doubling constant and the Poincaré constant. The Hölder inequality and the doubling property of Proposition 1.2(v) imply that for fixed $\varepsilon > 0$, the sequence $\{u_{\alpha,\varepsilon}\}_\alpha$ is equicontinuous on $B_{(s+r)/2}(o_\alpha)$, hence $u_{\alpha,\varepsilon} \to u_\varepsilon$ uniformly in $B_s(o_\alpha)$. Hence we get the strong convergence in $L^2(B_s(o))$ and the convergence (142).

Since the spaces are uniformly PI, they satisfy a same local Sobolev inequality [85, Theorem 2.6] meaning that there exist $C > 0$ and $\nu > 2$ independent on α such that

$$\left(\int_{B_r(o_\alpha)} |\psi|^{2\nu/(\nu-2)} \,\mathrm{d}\mu_\alpha\right)^{1-2/\nu} \le C\left(\int_{B_r(o_\alpha)} d\Gamma_\alpha(\psi) + \int_{B_r(o_\alpha)} |\psi|^2 \,\mathrm{d}\mu_\alpha\right) \quad \text{for any } \psi \in \mathscr{D}(\mathscr{E}_\alpha).$$

In particular, we get the a priori bound

$$\sup_\alpha \|u_\alpha\|_{L^{2\nu/(\nu-2)}(B_r(o_\alpha))} \le C.$$

With Hölder's inequality and the doubling property of Proposition 1.2(v), this yields

$$\begin{aligned}\left|\int_{B_s(o_\alpha)} u_\alpha^2 \,\mathrm{d}\mu_\alpha - \int_{B_r(o_\alpha)} u_\alpha^2 \,\mathrm{d}\mu_\alpha\right| &= \left|\int_{B_r(o_\alpha)} u_\alpha^2(\mathbf{1}_{B_s(o_\alpha)} - 1)\,\mathrm{d}\mu_\alpha\right| \\ &\le \left(\int_{B_r(o_\alpha)} |u_\alpha|^{2\nu/(\nu-2)}\,\mathrm{d}\mu_\alpha\right)^{1-2/\nu} \mu(B_r(o_\alpha)\setminus B_s(o_\alpha))^{2/\nu} \\ &\le C(r-s)^{2\delta/\nu},\end{aligned}$$

which justifies the interchange

$$\lim_{s\to r}\lim_{\alpha}\int_{B_s(o_\alpha)} u_\alpha^2\,\mathrm{d}\mu_\alpha = \lim_{\alpha}\lim_{s\to r}\int_{B_s(o_\alpha)} u_\alpha^2\,\mathrm{d}\mu_\alpha. \qquad \square$$

E.1 Convergence of the core $\mathscr{C}_c \cap \mathscr{D}(\mathscr{E})$

The next result indicates that in a certain sense the space $\mathscr{C}_c(X)\cap\mathscr{D}(\mathscr{E})$ is the limit of the spaces $\mathscr{C}_c(X_\alpha)\cap\mathscr{D}(\mathscr{E}_\alpha)$.

Proposition E.2 *Let $\varphi\in\mathscr{C}_c(X)\cap\mathscr{D}(\mathscr{E})$. Then there exists $\{\varphi_\alpha\}$, with $\varphi_\alpha\in\mathscr{C}_c(X_\alpha)\cap\mathscr{D}(\mathscr{E}_\alpha)$ for any α, such that*

$$\varphi_\alpha \xrightarrow{\mathscr{C}_c} \varphi \quad\text{and}\quad \varphi_\alpha \xrightarrow{\mathrm{E}} \varphi.$$

Moreover, if φ is nonnegative then each φ_α can be chosen to be also nonnegative.

Proof **Step 1** We construct $\psi_\alpha\in\mathscr{C}_0(X_\alpha)\cap\mathscr{D}(\mathscr{E}_\alpha)$ such that $\psi_\alpha\to\varphi$ uniformly on compact sets and such that $\psi_\alpha\xrightarrow{\mathrm{E}}\varphi$.

Proposition A.1 allows us to build $f_\alpha\in\mathscr{C}_c(X_\alpha)$ such that $f_\alpha\xrightarrow{\mathscr{C}_c}\varphi$. Moreover, we know that the sequence $\{f_\alpha\}$ is uniformly equicontinuous: there is $\omega\colon\mathbb{R}_+\to\mathbb{R}_+$ nondecreasing, bounded and satisfying $\omega(\delta)\to 0$ when $\delta\to 0$, such that

$$|f_\alpha(x)-f_\alpha(y)|\le\omega(\mathrm{d}_\alpha(x,y))$$

for any α and any $x,y\in X_\alpha$. In addition, if it turns out that φ is nonnegative, so is f_α. As $f_\alpha\xrightarrow{\mathscr{C}_c}\varphi$, we also have $f_\alpha\xrightarrow{L^2}\varphi$ and $P^\alpha_\varepsilon f_\alpha\xrightarrow{\mathrm{E}}P_\varepsilon\varphi$ for any $\varepsilon>0$. With φ being in $\mathscr{D}(\mathscr{E})$ we get that

$$|P_\varepsilon\varphi-\varphi|_{\mathscr{D}(\mathscr{E})}\to 0.$$

Let us show now that if $\varepsilon_\alpha\downarrow 0$, then we have $P^\alpha_{\varepsilon_\alpha}f_\alpha\to\varphi$ uniformly. It is sufficient to demonstrate that

$$\lim_{\varepsilon\to 0}\sup_\alpha\|P^\alpha_\varepsilon f_\alpha - f_\alpha\|_{L^\infty}=0.$$

Using the stochastic completeness, we know that for any $x\in X_\alpha$,

$$P^\alpha_\varepsilon f_\alpha(x)-f_\alpha(x)=\int_{X_\alpha}H_\alpha(\varepsilon,x,y)(f_\alpha(y)-f_\alpha(x))\,\mathrm{d}\mu_\alpha(y).$$

Hence for any $\kappa > 0$, we have

$$\begin{aligned}|P^\alpha_\varepsilon f_\alpha(x) - f_\alpha(x)| &\le \int_{X_\alpha} H_\alpha(\varepsilon, x, y)\omega(\mathrm{d}_\alpha(x, y))\,\mathrm{d}\mu_\alpha(y)\\ &\le \omega(\kappa\sqrt{\varepsilon}) + \|\omega\|_{L^\infty} \int_{X_\alpha\setminus B_{\kappa\sqrt{\varepsilon}}(x)} H_\alpha(\varepsilon, x, y)\,\mathrm{d}\mu_\alpha(y)\\ &\le \omega(\kappa\sqrt{\varepsilon}) + \frac{\|\omega\|_{L^\infty}\gamma}{\mu(B_{\sqrt{\varepsilon}}(x))} \int_{X_\alpha\setminus B_{\kappa\sqrt{\varepsilon}}(x)} e^{-\mathrm{d}^2_\alpha(x,y)/\gamma\varepsilon}\,\mathrm{d}\mu_\alpha(y)\\ &\le \omega(\kappa\sqrt{\varepsilon}) + \frac{\|\omega\|_{L^\infty}\gamma}{\mu(B_{\sqrt{\varepsilon}}(x))} \int_{\kappa\sqrt{\varepsilon}}^\infty e^{-r^2/\gamma\varepsilon}\frac{2r}{\gamma\varepsilon}\mu(B_r(x))\,\mathrm{d}r.\end{aligned}$$

We use the doubling condition of Proposition 1.2(iii) to deduce that if $r \ge \kappa\sqrt{\varepsilon} > R > \sqrt{\varepsilon}$, then

$$\mu(B_r(x)) \le e^{\lambda r/\sqrt{\varepsilon}}\mu(B_{\sqrt{\varepsilon}}(x)).$$

Hence if $\kappa\sqrt{\varepsilon} > R > \sqrt{\varepsilon}$, then

$$\begin{aligned}|P^\alpha_\varepsilon f_\alpha(x) - f_\alpha(x)| &\le \omega(\kappa\sqrt{\varepsilon}) + \|\omega\|_{L^\infty} \int_{\kappa\sqrt{\varepsilon}}^\infty e^{-r^2/\gamma\varepsilon+\lambda r/\sqrt{\varepsilon}}\frac{2r}{\varepsilon}\,\mathrm{d}r\\ &\le \omega(\kappa\sqrt{\varepsilon}) + \|\omega\|_{L^\infty} \int_{\kappa}^\infty e^{-r^2/\gamma+\lambda r}\,2r\,\mathrm{d}r\\ &\le \omega(\kappa\sqrt{\varepsilon}) + C(\lambda,\gamma)e^{-\kappa^2/2\gamma}\|\omega\|_{L^\infty}.\end{aligned}$$

We then choose $\kappa = \varepsilon^{-1/4}$ and we get, for ε small enough,

$$\|P^\alpha_\varepsilon f_\alpha - f_\alpha\|_{L^\infty} \le \omega(\varepsilon^{1/4}) + C(\lambda,\gamma)e^{-1/2\gamma\sqrt{\varepsilon}}\|\omega\|_{L^\infty}.$$

Remark E.3 The same estimate leads to the following decay estimate for $P^\alpha_\varepsilon f_\alpha$. Assume that $R > 0$ and L are such that $\operatorname{supp} f_\alpha \subset B_R(o_\alpha)$ and that $\|f_\alpha\|_{L^\infty} \le L$. Then for $x \in X_\alpha \setminus B_{2R}(o_\alpha)$,

$$|P^\alpha_\varepsilon f_\alpha(x)| \le C L\, e^{-\mathrm{d}^2_\alpha(o_\alpha,x)/4\gamma\varepsilon+\lambda R/\sqrt{\varepsilon}}.$$

To build ψ_α we use Mosco's argument for the proof of the implication (ii) $\Longrightarrow$ (i) in Proposition 1.15. We find a decreasing sequence $\eta_\ell \downarrow 0$ and an increasing sequence $\alpha_\ell \uparrow +\infty$ such that

$$\begin{aligned}0 < \varepsilon \le \eta_\ell &\implies \left|\|P_\varepsilon\varphi\|^2_{L^2} - \|\varphi\|^2_{L^2}\right| + |\mathscr{E}(P_\varepsilon\varphi) - \mathscr{E}(\varphi)| \le 2^{-\ell},\\ \alpha \ge \alpha_\ell &\implies \left|\|P^\alpha_{\eta_\ell} f_\alpha\|^2_{L^2} - \|P_{\eta_\ell}\varphi\|^2_{L^2}\right| + |\mathscr{E}_\alpha(P^\alpha_{\eta_\ell} f_\alpha) - \mathscr{E}(P_{\eta_\ell}\varphi)| \le 2^{-\ell}.\end{aligned}$$

Then if $\alpha \in [\alpha_\ell, \alpha_{\ell+1})$, we define $\varepsilon_\alpha = \eta_\ell$ and $\delta_\alpha = 2^{1-\ell}$ and we let

$$\psi_\alpha = P^\alpha_{\varepsilon_\alpha} f_\alpha.$$

Then we have $\lim_\alpha \delta_\alpha = 0$ and $\psi_\alpha \to \varphi$ uniformly and

$$|\|\psi_\alpha\|^2_{L^2} - \|\varphi\|^2_{L^2}| + |\mathscr{E}_\alpha(\psi_\alpha) - \mathscr{E}(\varphi)| \le \delta_\alpha.$$

We necessarily have $\psi_\alpha \xrightarrow{L^2} \varphi$ and the above estimate implies the strong convergence $\psi_\alpha \xrightarrow{\mathrm{E}} \varphi$.

Step 2 We modify each ψ_α with appropriate cut-off functions. Let $R>0$ be such that $\operatorname{supp} f_\alpha \subset B_R(o_\alpha)$ for any α and $\operatorname{supp}\varphi \subset B_R(o)$, and let L be such that

$$\sup_\alpha \|f_\alpha\|_{L^\infty} \leq L.$$

We let $\chi_\alpha\colon X_\alpha \to \mathbb{R}$ be defined by

$$\chi_\alpha(x) = \chi\left(\frac{\mathrm{d}_\alpha(o_\alpha, x)}{2R}\right),$$

where χ is defined by (119), and we set

$$\varphi_\alpha := \chi_\alpha\, \psi_\alpha.$$

It is easy to check that $\varphi_\alpha \xrightarrow{\mathscr{C}_c} \varphi$. In order to verify that $\varphi_\alpha \xrightarrow{\mathrm{E}} \varphi$, we need to check that

$$\lim_{\alpha\to\infty} \mathscr{E}_\alpha((1-\chi_\alpha)\psi_\alpha) = 0.$$

The chain rule implies that

$$\mathscr{E}_\alpha((1-\chi_\alpha)\psi_\alpha) = \mathscr{E}_\alpha(\psi_\alpha, (1-\chi_\alpha)^2\psi_\alpha) + \int_{X_\alpha} \psi_\alpha^2 \,\mathrm{d}\Gamma_\alpha(\chi_\alpha). \tag{143}$$

We have $\psi_\alpha \xrightarrow{\mathrm{E}} \varphi$ and $(1-\chi_\alpha)^2\psi_\alpha \xrightarrow{\mathrm{E}} 0$, hence the first term on the right-hand side of (143) tends to 0 when $\alpha\to\infty$. The functions χ_α are uniformly $(1/2r)$–Lipschitz hence by (19) there is a constant C independent of α such that

$$\int_{X_\alpha} \psi_\alpha^2 \,\mathrm{d}\Gamma_\alpha(\chi_\alpha) \leq C\mu_\alpha(B_{4R}(o_\alpha)) \sup_{B_{4R}(o_\alpha)\setminus B_{2R}(o_\alpha)} |\psi_\alpha|^2.$$

Using Remark E.3, we can conclude that

$$\lim_{\alpha\to\infty} \int_{X_\alpha} \psi_\alpha^2 \,\mathrm{d}\Gamma_\alpha(\chi_\alpha) = 0.$$ □

E.2 Energy convergence and convergence of the carré du champ

We can now easily deduce the following convergence result for the *carré du champ* under convergence in energy.

Proposition E.4 *Assume that $\varphi \in \mathscr{C}(X)\cap\mathscr{D}(\mathscr{E})$, $u \in \mathscr{D}_{\mathrm{loc}}(\mathscr{E})$, and $\varphi_\alpha \in \mathscr{C}(X_\alpha)\cap\mathscr{D}(\mathscr{E}_\alpha)$, $u_\alpha \in \mathscr{D}_{\mathrm{loc}}(\mathscr{E}_\alpha)$ for any α, are such that $\varphi_\alpha \xrightarrow{\mathscr{C}_c} \varphi$, $\varphi_\alpha \xrightarrow{\mathrm{E}} \varphi$, $u_\alpha \xrightarrow{\mathrm{E}_{\mathrm{loc}}} u$ and $\sup_\alpha \|u_\alpha\|_{L^\infty} < \infty$. Then*

$$\lim_{\alpha\to\infty} \int_{X_\alpha} \varphi_\alpha \,\mathrm{d}\Gamma(u_\alpha) = \int_X \varphi \,\mathrm{d}\Gamma(u). \tag{144}$$

Moreover, if for each $\rho>0$ there is some $p>1$ such that

$$\sup_\alpha \int_{B_\rho(o_\alpha)} \left|\frac{\mathrm{d}\Gamma(u_\alpha)}{\mathrm{d}\mu_\alpha}\right|^p \mathrm{d}\mu_\alpha < \infty,$$

then for each $\rho>0$,

$$\lim_{\alpha\to\infty} \int_{B_\rho(o_\alpha)} \mathrm{d}\Gamma(u_\alpha) = \int_{B_\rho(o)} \mathrm{d}\Gamma(u). \tag{145}$$

Proof We give a proof only in the case $u_\alpha \xrightarrow{\mathrm{E}} u$, the demonstration in the stated case being identical up to a few immediate but cumbersome justifications.

To prove (144) we use the definition of the carré du champ,

$$\int_{X_\alpha} \varphi_\alpha \,\mathrm{d}\Gamma(u_\alpha) = \mathscr{E}_\alpha(\varphi_\alpha u_\alpha, u_\alpha) - \tfrac{1}{2}\mathscr{E}_\alpha(\varphi_\alpha, u_\alpha^2), \tag{146}$$

together with the following observation: the chain rule implies that the sequences $\{\varphi_\alpha u_\alpha\}_\alpha$ and $\{u_\alpha^2\}_\alpha$ are bounded in energy and thus have weak limit in $\mathscr{E}$. However, when $\psi_\alpha \xrightarrow{\mathrm{E}} \psi$, we get $\psi_\alpha\varphi_\alpha \xrightarrow{\mathrm{E}} \psi\varphi$ and then $\int_{X_\alpha} \psi_\alpha\varphi_\alpha u_\alpha \,\mathrm{d}\mu_\alpha = \int_X \psi\varphi u \,\mathrm{d}\mu$, so that

$$\varphi_\alpha u_\alpha \xrightarrow{\mathrm{E}} \varphi u.$$

Moreover, when $\psi_\alpha \xrightarrow{\mathrm{E}} \psi$ we get $\psi_\alpha u_\alpha \xrightarrow{L^2} \psi u$ and then $\int_{X_\alpha} \psi_\alpha u_\alpha^2 \,\mathrm{d}\mu_\alpha = \int_X \psi u^2 \,\mathrm{d}\mu$, so that

$$u_\alpha^2 \xrightarrow{\mathrm{E}} u^2.$$

Thus (146) converges to $\int_X \varphi \,\mathrm{d}\Gamma(u)$.

To prove (145), take $\varepsilon > 0$. Acting as in the proof of Proposition E.1, with Hölder's inequality and the doubling condition of Proposition 1.2(v) we can find $\tau \in (0,\rho)$ such that for any α,

$$\left| \int_{B_{\rho-\tau}(o_\alpha)} \mathrm{d}\Gamma(u_\alpha) - \int_{B_\rho(o_\alpha)} \mathrm{d}\Gamma(u_\alpha) \right| \le \tfrac{1}{3}\varepsilon.$$

Moreover, by regularity of the Radon measure $\Gamma(u)$, we can assume that

$$\left| \int_{B_{\rho-\tau}(o)} \mathrm{d}\Gamma(u) - \int_{B_\rho(o)} \mathrm{d}\Gamma(u) \right| \le \tfrac{1}{3}\varepsilon.$$

We set

$$\varphi(x) := \begin{cases} 1 & \text{if } x \in B_{\rho-\tau}(o), \\ (2\rho - \tau - 2\mathrm{d}(o,x))/\tau & \text{if } x \in B_{\rho-\tau/2}(o) \setminus B_{\rho-\tau}(o), \\ 0 & \text{if } x \notin B_{\rho-\tau/2}(o). \end{cases}$$

We have

$$\left| \int_{B_\rho(o)} \mathrm{d}\Gamma(u) - \int_X \varphi \,\mathrm{d}\Gamma(u) \right| \le \tfrac{1}{3}\varepsilon. \tag{147}$$

Thanks to Proposition E.2, we can choose $\varphi_\alpha \in \mathscr{C}_c(X_\alpha) \cap \mathscr{D}(\mathscr{E}_\alpha)$ nonnegative for any α such that $\varphi_\alpha \xrightarrow{\mathscr{C}_c} \varphi$ and $\varphi_\alpha \xrightarrow{\mathrm{E}} \varphi$. Then there is some sequence $\delta_\alpha \downarrow 0$ such that

- $|\varphi_\alpha - 1| \le \delta_\alpha$ on $B_{\rho-\tau}(o_\alpha)$,
- $\varphi_\alpha \le 1 + \delta_\alpha$,
- $\varphi_\alpha \le \delta_\alpha$ outside $B_\rho(o_\alpha)$.

We easily get

$$\left| \int_{B_\rho(o_\alpha)} \mathrm{d}\Gamma(u_\alpha) - \int_{X_\alpha} \varphi_\alpha \,\mathrm{d}\Gamma(u_\alpha) \right| \le \delta_\alpha \mathscr{E}_\alpha(u_\alpha) + (1+\delta_\alpha)\cdot \tfrac{1}{3}\varepsilon. \tag{148}$$

Using (147) and (148), we find $\underline{\alpha}$ such that

$$\alpha \geq \underline{\alpha} \implies \left| \int_{B_\rho(o_\alpha)} \mathrm{d}\Gamma(u_\alpha) - \int_{B_\rho(o)} \mathrm{d}\Gamma(u) \right| < \varepsilon. \qquad \square$$

The above argument also implies the lower semicontinuity of the Carré du champ under weak convergence in energy.

Remark E.5 The above result can be localized: if we assume that functions $u_\alpha \in \mathcal{D}(B_\rho(o_\alpha), \mathcal{E}_\alpha)$ satisfy

- $u_\alpha \xrightarrow{\mathrm{E}} u$,
- $\sup_\alpha \|u_\alpha\|_{L^\infty} < \infty$,
- $\sup_\alpha \int_{B_\rho(o_\alpha)} |\mathrm{d}\Gamma(u_\alpha)/\mathrm{d}\mu_\alpha|^p \,\mathrm{d}\mu_\alpha < \infty$ for some $p > 1$,

then

$$\lim_{\alpha\to\infty} \int_{B_\rho(o_\alpha)} \mathrm{d}\Gamma(u_\alpha) = \int_{B_\rho(o)} \mathrm{d}\Gamma(u).$$

Using Proposition E.1, we also get the following strong convergence result for the energy measure density.

Proposition E.6 *Assume that functions $u_\alpha \in \mathcal{D}_{\mathrm{loc}}(B_\rho(o_\alpha), \mathcal{E}_\alpha)$ satisfy*

- $u_\alpha \xrightarrow{\mathrm{E}} u$,
- $\sup_\alpha \|u_\alpha\|_{L^\infty} < \infty$,
- $\sup_\alpha \int_{B_\rho(o_\alpha)} \mathrm{d}\Gamma(e_\alpha) < \infty$, *where* $e_\alpha := |\mathrm{d}\Gamma(u_\alpha)/\mathrm{d}\mu_\alpha|^{1/2}$ *for any* α,

then

$$e_\alpha \xrightarrow{L^2} \left|\frac{\mathrm{d}\Gamma(u)}{\mathrm{d}\mu}\right|^{1/2}.$$

Proof Proposition E.1 implies that, up to extracting a subsequence, we can assume that $e_\alpha \xrightarrow{L^2} f$. We want to show that $f = |\mathrm{d}\Gamma(u)/\mathrm{d}\mu|^{1/2}$ μ–a.e. on $B_\rho(o)$. Following the proof of Proposition E.1 (using the Sobolev inequality), we have some $p > 1$ such that $\sup_\alpha \|e_\alpha\|_{L^{2p}(B_\rho(o_\alpha))} < \infty$, hence Remark E.5 implies that if $x_\alpha \in B_\rho(o_\alpha) \to x \in B_\rho(o)$, for any $r > 0$ such that $r + \mathsf{d}(o,x) < \rho$ we have

$$\lim_{\alpha\to\infty} \int_{B_r(x_\alpha)} \mathrm{d}\Gamma(u_\alpha) = \int_{B_r(x)} \mathrm{d}\Gamma(u).$$

But the strong L^2–convergence also yields that

$$\lim_{\alpha\to\infty} \int_{B_r(x_\alpha)} \mathrm{d}\Gamma(u_\alpha) = \lim_{\alpha\to\infty} \int_{B_r(x_\alpha)} e_\alpha^2 \,\mathrm{d}\mu_\alpha = \int_{B_r(x)} f^2 \,\mathrm{d}\mu.$$

Hence for any $x \in B_\rho(o)$ and $r > 0$ such that $r + \mathsf{d}(o,x) < \rho$,

$$\int_{B_r(x)} f^2 \,\mathrm{d}\mu = \int_{B_r(x)} \mathrm{d}\Gamma(u).$$

By the Lebesgue differentiation theorem (which holds true on any doubling space), this implies that $f^2 = \mathrm{d}\Gamma(u)/\mathrm{d}\mu$ μ–a.e. $\square$

E.3 Convergence of harmonic functions

Proposition E.7 *Let $\{u_\alpha\}$ be such that $u_\alpha \in \mathcal{D}(B_\rho(o_\alpha), \mathcal{E}_\alpha) \cap L^\infty(B_\rho(o_\alpha))$ for any α and*

$$\sup_\alpha (\|u_\alpha\|_{L^\infty} + \|L_\alpha u_\alpha\|_{L^2}) < \infty.$$

Then there exist a subsequence $B \subset A$ and $u \in \mathcal{D}_{\mathrm{loc}}(B_\rho(o), \mathcal{E})$ such that $Lu \in L^2(B_\rho(o))$ and

$$u_\beta \xrightarrow{L^2} u \quad \text{and} \quad L_\beta u_\beta \xrightarrow{L^2_{\mathrm{loc}}} Lu.$$

Moreover, if $\varphi_\beta \in \mathscr{C}_c(B_\rho(o_\beta)) \cap \mathcal{D}(\mathcal{E}_\beta)$ and $\varphi \in \mathscr{C}_c(B_\rho(o)) \cap \mathcal{D}(\mathcal{E})$ are such that $\varphi_\beta \xrightarrow{\mathscr{C}_c} \varphi$ and $\varphi_\beta \xrightarrow{\mathrm{E}} \varphi$, then

$$\lim_{\beta\to\infty} \int_{X_\beta} \varphi_\beta \, \mathrm{d}\Gamma(u_\beta) = \int_X \varphi \, \mathrm{d}\Gamma(u).$$

Remark E.8 From the proof of Proposition E.2, we notice that for any $\varphi \in \mathscr{C}_c(B_\rho(o)) \cap \mathcal{D}(\mathcal{E})$, we can find $\varphi_\alpha \in \mathscr{C}_c(B_\rho(o_\alpha)) \cap \mathcal{D}(\mathcal{E}_\alpha)$ for any α such that $\varphi_\alpha \xrightarrow{\mathscr{C}_c} \varphi$ and $\varphi_\alpha \xrightarrow{\mathrm{E}} \varphi$.

Proof We can find a subsequence $B \subset A$, $u \in L^2(B_\rho(o)) \cap L^\infty(B_\rho(o))$ and $f \in L^2(B_\rho(o))$ such that

$$u_\beta \xrightarrow{L^2 \cap L^4} u \quad \text{and} \quad f_\beta := L_\beta u_\beta \rightharpoonup f.$$

For any $r < \rho$, we consider the function

$$\chi_\beta(x) = \chi\left(2\frac{\mathrm{d}_\beta(o_\beta, x) - \frac{1}{2}(3r-\rho)}{\rho - r}\right),$$

where χ is defined by (119), which has the following properties:

- $\chi_\beta = 1$ on $B_r(o_\beta)$,
- $\chi_\beta = 0$ outside $B_{(\rho+r)/2}(o_\beta)$,
- χ_β is $2/(\rho - r)$–Lipschitz.

We have the estimate

$$\int_{B_r(o_\beta)} \mathrm{d}\Gamma(u_\beta) \le \int_{B_\rho(o_\beta)} \mathrm{d}\Gamma(\chi_\beta u_\beta),$$

but

$$\begin{aligned}\int_{B_\rho(o_\beta)} \mathrm{d}\Gamma(\chi_\beta u_\beta) &= \int_{B_\rho(o_\beta)} u_\beta^2 \, \mathrm{d}\Gamma(\chi_\beta) + \mathcal{E}_\beta(\chi_\beta^2 u_\beta, u_\beta) \\ &= \int_{B_\rho(o_\beta)} u_\beta^2 \, \mathrm{d}\Gamma(\chi_\beta) + \int_{B_\rho(o_\beta)} u_\beta \chi_\beta^2 f_\beta \, \mathrm{d}\mu_\beta.\end{aligned}$$

Using the comparison (19), we get

$$\int_{B_r(o_\beta)} \mathrm{d}\Gamma(u_\beta) \le \frac{4}{(\rho - r)^2} \|u_\beta\|_{L^\infty}^2 \mu_\beta(B_\rho(o_\beta)) + \|u_\beta\|_{L^2} \|f_\beta\|_{L^2}.$$

Hence u_β is locally bounded in $\mathcal{D}_{\text{loc}}(B_\rho(o), \mathcal{E}_\beta)$, $u \in \mathcal{D}_{\text{loc}}(B_\rho(o), \mathcal{E})$, $u_\beta \xrightarrow{\text{E}_{\text{loc}}} u$ and $u_\beta^2 \xrightarrow{\text{E}_{\text{loc}}} u^2$. The formula

$$\int_{B_\rho(o_\beta)} \varphi_\beta \, \mathrm{d}\Gamma(u_\beta) = \int_{B_\rho(o_\beta)} \varphi_\beta u_\beta f_\beta \, \mathrm{d}\mu_\beta - \tfrac{1}{2}\mathcal{E}_\beta(\varphi_\beta, u_\beta^2)$$

and the fact that from Proposition E.1, we have $u_\beta \xrightarrow{L^2_{\text{loc}}} u$, imply the claimed convergence result. □

In the case where we have a sequence of harmonic functions, this result can be slightly improved.

Proposition E.9 *Let $\{h_\alpha\}$ be such that $h_\alpha \colon B_\rho(o_\alpha) \to \mathbb{R}$ is L_α–harmonic for any α and*

$$\sup_\alpha \|h_\alpha\|_{L^\infty} < \infty.$$

Then there exist a subsequence $B \subset A$ and a harmonic function $h \colon B_\rho(o) \to \mathbb{R}$ such that

(i) $h_\beta \xrightarrow{L^2} h$,

(ii) $h_\beta|_{B_r(o_\beta)} \to h|_{B_r(o)}$ *uniformly for each $r < \rho$, and*

(iii) $\lim_{\beta\to\infty} \int_{B_r(o_\beta)} \mathrm{d}\Gamma(h_\beta) = \int_{B_r(o)} \mathrm{d}\Gamma(h)$ *for each $r < \rho$.*

Proof Proposition E.7 implies the existence of a subsequence $B \subset A$ and of a harmonic function $h \colon B_\rho(o) \to \mathbb{R}$ such that we have the strong convergence in L^2. The uniform convergence follows from the fact that each $(X_\beta, \mathrm{d}_\beta, \mu_\beta, \mathcal{E}_\beta)$ satisfies the uniform parabolic/elliptic Harnack inequality and hence uniform local Hölder estimate for harmonic functions; see [78, Lemma 2.3.2] or [53, Lemma 5.2]. In our case, there exist $\theta \in (0,1)$ and $C > 0$ such that for any α and $x, y \in B_r(o_\alpha)$,

$$|h_\alpha(x) - h_\alpha(y)| \le C \left(\frac{\mathrm{d}_\alpha(x,y)}{\rho - r} \right)^\theta \|h_\alpha\|_{L^\infty}.$$

The last point is a consequence of a uniform reverse Hölder inequality for the energy density of harmonic functions: there exist $p > 1$ and $C > 0$ such that if $B \subset X_\alpha$ is a ball of radius $r(B) \le R$ and $f : B \to \mathbb{R}$ is harmonic, then

$$\left(\fint_{B/2} \left| \frac{\mathrm{d}\Gamma(f)}{\mathrm{d}\mu_\alpha} \right|^p \mathrm{d}\mu_\alpha \right)^{1/p} \le C \fint_B \mathrm{d}\Gamma(f).$$

This is explained in [5, Section 2.1]; it relies on a self-improvement of the L^2–Poincaré inequality to an $L^{2-\varepsilon}$–Poincaré inequality [63] and on the Gehring lemma [44, Chapter V]. □

E.4 Approximation of harmonic functions

Let us conclude with an approximation result for harmonic functions.

Proposition E.10 *Let $h \colon B_\rho(o) \to \mathbb{R}$ be a harmonic function and let $r < \rho$. Then there exists $\{h_\alpha\}$, with $h_\alpha \colon B_r(o_\alpha) \to \mathbb{R}$ harmonic for any α, such that*

(i) $h_\alpha \to h|_{B_r(o)}$ *uniformly on compact sets,*

(ii) $\int_{B_s(o_\alpha)} \mathrm{d}\Gamma(h_\alpha) \to \int_{B_s(o)} \mathrm{d}\Gamma(h)$ *for any $s \le r$.*

Proof Set $\delta := (\rho - r)/4$ and

$$\xi(x) := \chi\left(1 + \frac{\mathsf{d}(o,x) - (r+2\delta)}{\delta}\right)$$

for any $x \in X$, where χ is as in (119). Set

$$\varphi := \xi h \in \mathscr{C}_c(X) \cap \mathscr{D}(\mathscr{E}).$$

Let $\{\varphi_\alpha\}$ be given by Proposition E.2, ie $\varphi_\alpha \in \mathscr{C}_c(X_\alpha) \cap \mathscr{D}(\mathscr{E}_\alpha)$ for any α, and $\varphi_\alpha \xrightarrow{\mathscr{C}_c} \varphi$ and $\varphi_\alpha \xrightarrow{\mathrm{E}} \varphi$. For any α, let h_α be the harmonic replacement of φ_α on $B_{r+\delta}(o_\alpha)$ that is to say $h_\alpha \in \mathscr{D}(\mathscr{E}_\alpha)$ is the unique solution of

$$\begin{cases} L_\alpha h_\alpha = 0 & \text{on } B_{r+\delta}(o_\alpha), \\ h_\alpha = \varphi_\alpha & \text{outside } B_{r+\delta}(o_\alpha), \end{cases}$$

which is characterized by

$$\mathscr{E}_\alpha(h_\alpha) = \inf\{\mathscr{E}_\alpha(f) : f \in \mathscr{D}(\mathscr{E}_\alpha) \text{ and } f = \varphi_\alpha \text{ outside } B_{r+\delta}(o_\alpha)\}.$$

In particular, $\mathscr{E}_\alpha(h_\alpha) \le \mathscr{E}_\alpha(\varphi_\alpha)$ for any α, hence we can find a subsequence $B \subset A$ and $f \in \mathscr{D}(\mathscr{E})$ such that $h_\beta \xrightarrow{\mathrm{E}} f$. The lower semicontinuity of the energy implies

$$\mathscr{E}(f) \le \mathscr{E}(\varphi),$$

and we have $f = \varphi$ on $X \setminus B_{r+\delta}(o)$. However, φ is its own harmonic replacement on $B_{r+\delta}(o)$, hence the variational characterization of the harmonic replacement implies $f = \varphi$. Thus $h_\beta \xrightarrow{\mathrm{E}} \varphi$ and the result is then a consequence of Proposition E.9. □

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