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Yifei Chen

Constantin Shramov

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Automorphisms of surfaces over fields of positive characteristic

YIFEI CHEN
CONSTANTIN SHRAMOV

We study automorphism and birational automorphism groups of varieties over fields of positive characteristic from the point of view of Jordan and p–Jordan property. In particular, we show that the Cremona group of rank 2 over a field of characteristic $p > 0$ is p–Jordan, and the birational automorphism group of an arbitrary geometrically irreducible algebraic surface is nilpotently p–Jordan of class at most 2. Also, we show that the automorphism group of a smooth geometrically irreducible projective variety of nonnegative Kodaira dimension is Jordan in the usual sense.

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1 Introduction

Birational automorphism groups of algebraic varieties sometimes have extremely complicated structure. Also, their finite subgroups may be hard to classify explicitly. To obtain an approach to the description of finite subgroups, the following definition was introduced by V Popov [60, Definition 2.1].

Definition 1.1 A group Γ is called *Jordan* (alternatively, one says that Γ has the *Jordan property*) if there exists a constant $J = J(\Gamma)$, depending only on Γ, such that any finite subgroup of Γ contains a normal abelian subgroup of index at most J.

A classical theorem of C Jordan [36, Section 40] asserts that the general linear group, and thus also every linear algebraic group, over a field of characteristic zero is Jordan; see also Bieberbach [3], Frobenius [24], Curtis and Reiner [17, Theorem 36.13], Robinson [68, Theorem A], Tao [80, Section 4], Breuillard [12], Breuillard and Green [13, Section 2], Mundet i Riera [58, Section 3], or Serre [71, Theorem 9.9]. Serre proved in [70, Theorem 5.3] that the same property holds for the Cremona group of rank 2, that is, for the group $\mathrm{Bir}(\mathbb{P}^2)$ of birational automorphisms of the projective plane, over fields of characteristic zero. Popov classified in [60, Theorem 2.32] all surfaces over fields of characteristic zero whose birational automorphism group is Jordan. Sh Meng and D-Q Zhang proved in [53, Theorem 1.6] that the Jordan property holds for automorphism groups of projective varieties over fields of characteristic zero.

However, all this fails miserably over fields of positive characteristic. Indeed, let p be a prime number, let $\boldsymbol{F}_{p^k}$ denote the field of p^k elements, and let $\bar{\boldsymbol{F}}_p$ be the algebraic closure of the field $\boldsymbol{F}_p$. Then the group $\mathrm{PGL}_2(\bar{\boldsymbol{F}}_p)$ contains the groups $\mathrm{PSL}_2(\boldsymbol{F}_{p^k})$ for all positive integers k; the latter finite groups are simple apart from a (small) finite number of exceptions; see Wilson [84, Section 3.3.1]. This means that the group $\mathrm{PGL}_2(\bar{\boldsymbol{F}}_p)$ not only fails to be Jordan, but moreover, its finite subgroups cannot contain *any* proper subgroups of bounded index. In particular, no straightforward generalizations of the Jordan property, like the nilpotent Jordan property (see Guld [27]) or the solvable Jordan property (see Prokhorov and Shramov [62, Section 8]), can hold in this case.

The above observation naturally leads to the following definitions.

Definition 1.2 (Hu [32, Definition 1.2]) Let p be a prime number. A group Γ is called *generalized p–Jordan* if there is a constant $J(\Gamma)$, depending only on Γ, such that every finite subgroup G of Γ whose order is not divisible by p contains a normal abelian subgroup of index at most $J(\Gamma)$.

Definition 1.3 (Hu [32, Definition 1.6] and Brauer and Feit [11]) Let p be a prime number. A group Γ is called *p–Jordan* if there exist constants $J(\Gamma)$ and $e(\Gamma)$, depending only on Γ, such that every finite subgroup G of Γ contains a normal abelian subgroup which has order coprime to p and index at most $J(\Gamma)\cdot|G_p|^{e(\Gamma)}$, where G_p is a p–Sylow subgroup of G.

Note that every Jordan group is p–Jordan (see Corollary 2.6), and every p–Jordan group is generalized p–Jordan, but the converse statements do not hold. It was proved in [70, Theorem 5.3] that the Cremona group of rank 2 over a field of characteristic $p > 0$ is generalized p–Jordan. As for the p–Jordan property, the following analog of the theorem of Jordan is known.

Theorem 1.4 (Brauer and Feit [11] and Larsen and Pink [42, Theorem 0.4]) *Let n be a positive integer. Then there exists a constant $J(n)$ such that for every prime p and every field $\Bbbk$ of characteristic p, every finite subgroup G of $\mathrm{GL}_n(\Bbbk)$ contains a normal abelian subgroup of order coprime to p and index at most $J(n) \cdot |G_p|^3$, where G_p is a p–Sylow subgroup of G. In particular, for every field $\Bbbk$ of characteristic $p > 0$, the group $\mathrm{GL}_n(\Bbbk)$ is p–Jordan.*

It immediately follows from Theorem 1.4 that (the group of $\Bbbk$–points of) every linear algebraic group over a field $\Bbbk$ of characteristic $p > 0$ is p–Jordan.

It appears that the p–Jordan property is useful to study automorphism groups and birational automorphism groups of algebraic varieties (by which we mean geometrically reduced separated schemes of finite type) over fields of positive characteristic. The following analog of a theorem of Meng and Zhang [53, Theorem 1.6] was proved by F Hu.

Theorem 1.5 [32, Theorem 1.10] *Let $\Bbbk$ be a field of characteristic $p > 0$, and let X be a projective variety over $\Bbbk$. Then the automorphism group $\mathrm{Aut}(X)$ is p–Jordan.*

The purpose of this paper is to initiate a systematic study of p–Jordan and generalized p–Jordan properties for groups of birational automorphisms of varieties over fields of positive characteristic, and to generalize to this setting the relevant results for surfaces over fields of characteristic zero (as usual, by a surface we mean a variety of dimension 2). Our first goal is to prove an analog of [70, Theorem 5.3].

Theorem 1.6 *There exists a constant J such that for every prime p and every field $\Bbbk$ of characteristic p, every finite subgroup G of the birational automorphism group $\mathrm{Bir}(\mathbb{P}^2)$ contains a normal abelian subgroup of order coprime to p and index at most $J \cdot |G_p|^3$, where G_p is a p–Sylow subgroup of G. In particular, for every field $\Bbbk$ of characteristic $p > 0$, the group $\mathrm{Bir}(\mathbb{P}^2)$ is p–Jordan.*

Moreover, we obtain an analog of [60, Theorem 2.32]; see also Prokhorov and Shramov [66, Theorem 1.7].

Theorem 1.7 *Let $\Bbbk$ be an algebraically closed field of characteristic $p > 0$, and let S be an irreducible algebraic surface over $\Bbbk$. The following assertions hold.*

(i) *If S is birational to a product $E \times \mathbb{P}^1$ for some elliptic curve E, then the group $\mathrm{Bir}(S)$ is not generalized p–Jordan.*

(ii) *If the Kodaira dimension of S is negative but S is not birational to a product $E \times \mathbb{P}^1$ for any elliptic curve E, then the group $\mathrm{Bir}(S)$ is p–Jordan but not Jordan.*

(iii) *If the Kodaira dimension of S is nonnegative, then the group $\mathrm{Bir}(S)$ is Jordan.*

In the proof of Theorem 1.7 we use the following assertion, which makes Theorem 1.5 more precise in one important particular case.

Proposition 1.8 *Let $\Bbbk$ be an arbitrary field, and let X be a smooth geometrically irreducible projective variety of nonnegative Kodaira dimension over $\Bbbk$. Then the group $\mathrm{Aut}(X)$ is Jordan.*

Our next result is an analog of Prokhorov and Shramov [65, Proposition 1.6]. It can be regarded as (a slightly more precise version of) a particular subcase of Theorem 1.7(iii).

Proposition 1.9 *There exists a constant J such that for every field $\Bbbk$ and every geometrically irreducible algebraic surface S of Kodaira dimension 0 over $\Bbbk$, every finite subgroup of $\mathrm{Bir}(S)$ contains a normal abelian subgroup of index at most J.*

Using the terminology of [62, Definition 1.6], one can reformulate Proposition 1.9 by saying that the set of automorphism groups of all geometrically irreducible algebraic surfaces of Kodaira dimension 0 over all fields is uniformly Jordan.

Given a variety X and a point $P \in X$, we denote by $\mathrm{Aut}(X; P)$ the stabilizer of P in the group $\mathrm{Aut}(X)$. The following result is an analog of [65, Proposition 1.3].

Proposition 1.10 *There exists a constant B such that for every field $\Bbbk$, every smooth geometrically irreducible projective surface S of Kodaira dimension 0 over $\Bbbk$, every $\Bbbk$–point $P \in S$, and every finite subgroup $G \subset \mathrm{Aut}(S; P)$ the order of the group G is at most B.*

We point out that the assertion of Proposition 1.10 fails in general for stabilizers of closed points on varieties over algebraically nonclosed fields which are not $\Bbbk$–points; see Example 6.7. However, it holds if one replaces the stabilizers by the inertia groups; see Corollary 11.5. Also, there is the following (partial) generalization of Proposition 1.10, which holds for varieties of arbitrary dimension over arbitrary fields. It is an analog of [65, Theorem 1.5].

Theorem 1.11 *Let $\Bbbk$ be a field, and let X be a smooth geometrically irreducible projective variety of nonnegative Kodaira dimension over $\Bbbk$. Then there exists a constant $B = B(X)$ such that for every closed point $P \in X$ and every finite subgroup $G \subset \mathrm{Aut}(X; P)$ the order of the group G is at most B.*

Since some of the birational automorphism groups are not Jordan, one can weaken it by considering nilpotent subgroups instead of abelian ones in Definition 1.1. Recall that a group G is said to be *nilpotent of class at most c* if its upper central series has length at most c; see [34, Section 1D] for details. In particular, nilpotent groups of class at most 1 are exactly abelian groups, and the only nilpotent group of class at most 0 is the trivial group. This leads to the notion of nilpotently Jordan groups.

Definition 1.12 [27, Definition 1] A group Γ is *nilpotently Jordan of class at most c* if there exists a constant $J(\Gamma)$, depending only on Γ, such that any finite subgroup of Γ contains a normal subgroup N of index at most $J(\Gamma)$, where N is nilpotent of class at most c.

We introduce the analogs of this definition suitable for automorphism groups of varieties over fields of positive characteristic.

Definition 1.13 Let p be a prime number. A group Γ is *generalized nilpotently p–Jordan of class at most c* if there is a constant $J(\Gamma)$, depending only on Γ, such that every finite subgroup of Γ whose order is not divisible by p contains a normal subgroup N of index at most $J(\Gamma)$, where N is nilpotent of class at most c.

Definition 1.14 Let p be a prime number. A group Γ is *nilpotently p–Jordan of class at most c* if there exist constants $J(\Gamma)$ and $e(\Gamma)$, depending only on Γ, such that every finite subgroup G of Γ contains a normal subgroup N of order coprime to p and index at most $J(\Gamma)\cdot|G_p|^{e(\Gamma)}$, where N is nilpotent of class at most c, and G_p denotes a p–Sylow subgroup of G.

Similarly to the situation with the usual Jordan property, every nilpotently Jordan group of class at most c is nilpotently p–Jordan of class at most c (see Corollary 2.6), and every nilpotently p–Jordan group of class at most c is generalized nilpotently p–Jordan of class at most c, while the converse statements do not hold. Note also that a group is generalized nilpotently p–Jordan of class at most 1 (resp. nilpotently p–Jordan group of class at most 1) if and only if it is generalized p–Jordan (resp. p–Jordan).

According to [27, Theorem 2], the birational automorphism group of any (geometrically irreducible) variety over a field of characteristic zero is nilpotently Jordan of class at most 2. We prove an analog of this assertion for surfaces over fields of positive characteristic.

Theorem 1.15 *Let $\Bbbk$ be a field of characteristic $p>0$. Let S be a geometrically irreducible algebraic surface over $\Bbbk$. Then the group $\mathrm{Bir}(S)$ is nilpotently p–Jordan of class at most 2.*

The plan of our paper is as follows. In Section 2 we collect some elementary assertions about groups and lattices used in the rest of the paper. In Section 3 we recall the basic concepts and facts concerning automorphism groups and schemes of projective varieties. In Section 4 we discuss the group of connected components of the automorphism group scheme of a projective variety following [53]. In Section 5 we recall the basics of the Minimal Model Program in dimension 2, including some theorems from its equivariant version. In Section 6 we collect auxiliary facts about automorphism groups of abelian varieties. In Section 7 we make some observations on automorphism groups of varieties of nonnegative Kodaira dimension and prove Proposition 1.8. In Section 8 we discuss automorphism groups of smooth projective curves. In Section 9 we prove Theorem 1.6. In Section 10 we prove Theorem 1.7. In Section 11 we study automorphism groups of surfaces of zero Kodaira dimension and prove Propositions 1.9 and 1.10. In Section 12 we prove Theorem 1.11. In Section 13 we prove Theorem 1.15. In Section 14 we discuss some open questions concerning (birational) automorphism groups of varieties over fields of positive characteristic.

In some cases, the proofs of our main results go along the same lines as the proofs of the corresponding results in characteristic 0. This applies to Propositions 1.9 and 1.10, Theorem 1.11 and, to a certain extent, to Theorem 1.7. The proof of Theorem 1.6 mostly follows the proof of [70, Theorem 5.3] but contains additional arguments needed to treat finite subgroups of $\mathrm{Bir}(\mathbb{P}^2)$ whose orders are divisible by the characteristic of the field. Proposition 1.8 and its proof look new in the context of positive characteristic. Theorem 1.15 (as well as the accompanying Definitions 1.13 and 1.14) and its proof are also entirely new; we point out that the proof does not follow the ideas of [27], but is rather inspired by the approach of Serre [70, Theorem 5.3]. Many statements collected in the preliminary sections of our paper are well known to experts and are widely used at least over fields of characteristic zero, but are not readily available in the literature in the positive characteristic setup. Since one of the goals of our paper is to provide a survey of the methods of studying finite groups of birational automorphisms in arbitrary characteristic, in such cases we take the opportunity to spell out the details of the proofs. This allows us either to emphasize that the proofs do not depend on the characteristic of the base field, see eg Lemmas 3.6 and 4.1 and Corollary 7.4; or to be able to mention the (minor) differences in the proofs appearing in the case of positive characteristic; see eg Theorem 3.7.

Throughout the paper, we use the following standard notation. Given a field $\Bbbk$, we denote by $\overline{\Bbbk}$ its algebraic closure. If $\Bbbk \subset \mathbb{K}$ is a field extension, and X is a scheme defined over $\Bbbk$, we denote by $X_{\mathbb{K}}$ the extension of scalars of X to $\mathbb{K}$. By a $\Bbbk$–point of a scheme X over $\Bbbk$ we mean its closed point of degree 1; the set of all $\Bbbk$–points of X is denoted by $X(\Bbbk)$.

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2 Preliminaries

In this section we collect auxiliary facts about groups and lattices.

Group theory Recall that a subgroup of a group G is called *characteristic* if it is preserved by all automorphisms of G.

Example 2.1 Let G be a finite group that has a normal p–Sylow subgroup G_p. Then G_p is characteristic in G. Indeed, it consists of all elements of G whose order is a power of p.

Example 2.2 Let $G \cong G' \rtimes G''$ be a finite group. Suppose that the orders of G' and G'' are coprime. Then G' is a characteristic subgroup of G. Indeed, it consists of all elements of G whose order divides $|G'|$.

Example 2.3 Let H be an abelian group generated by its elements $h_1, \dots, h_k$, and let r be a positive integer. Then the subgroup H' generated by $h_1^r, \dots, h_k^r$ coincides with the subgroups that consists of r^{th} powers of elements of H. Therefore, H' is a characteristic subgroup of index at most r^k in H.

Remark 2.4 Let G be a group, let H be its normal subgroup, and let F be a normal subgroup of H. Then F is not necessarily normal in G. However, if F is characteristic in H, then it is normal in G.

It appears that the Jordan property implies the p–Jordan property for every p. To see this, we make an auxiliary observation.

Lemma 2.5 *Let p be a prime number, let J be a positive integer, and let G be a finite group. Let $\widetilde{A}$ be a normal subgroup of index at most J in G. Suppose that $\widetilde{A}$ is abelian (resp. nilpotently Jordan of class at most c). Then G contains a normal subgroup A such that the minimal number of elements that generate A does not exceed that of $\widetilde{A}$, the order of A is coprime to p, the index of A in G is at most $J \cdot |G_p|$, where G_p is a p–Sylow subgroup of G, and A is abelian (resp. nilpotently Jordan of class at most c).*

Proof Let $\widetilde{A}_p$ be the p–Sylow subgroup of $\widetilde{A}$. Then

$$\widetilde{A} \cong \widetilde{A}_p \times A,$$

where the group A is isomorphic to the product of the q–Sylow subgroups of $\widetilde{A}$ for all prime numbers q different from p; see [34, Theorem 1.26]. In particular, the group A is abelian (resp. nilpotent of class at most c), and its order is coprime to p. Also, if $\widetilde{A}$ can be generated by the elements $\tilde{a}_1, \dots, \tilde{a}_r$, then the projections of these elements to A generate A. Note that A is a characteristic subgroup of $\widetilde{A}$ by Example 2.2. Thus, A is normal in G. Finally, one can see that the index of A in G is at most

$$J \cdot |\widetilde{A}_p| \leqslant J \cdot |G_p|. \qquad \square$$

Corollary 2.6 *Let p be a prime number, and let Γ be a group. Suppose that Γ is Jordan (resp. nilpotently Jordan of class at most c). Then Γ is p–Jordan (resp. nilpotently p–Jordan of class at most c).*

Proof Let G be a finite subgroup of Γ. Then G contains a normal subgroup $\widetilde{A}$ such that its index is bounded by some constant $J = J(\Gamma)$ independent of G, and $\widetilde{A}$ is abelian (resp. nilpotent of class at most c). Therefore, the assertion follows from Lemma 2.5. $\square$

The following fact is standard.

Lemma 2.7 *Let G be a finite group, and let G' be its normal subgroup. Let $A' \subset G'$ be a subgroup that is normal in G'. Denote by B the index of G' in G, and by J the index of A' in G'. Then A' contains a subgroup A such that A is normal in G, and the index of A in G is at most BJ^B.*

Proof The group G acts on G' by conjugation, and the conjugation by the elements of G' preserve A'. Let $A'_1 = A', \dots, A'_r$ be the orbit of A' under this action. Then $r \leqslant |G/G'| \leqslant B$, so that the index of the intersection

$$A = \bigcap_{i=1}^{r} A'_i$$

in G' is at most $J^r \leqslant J^B$. Thus, A is a normal subgroup of index at most BJ^B in G. $\square$

Lemma 2.7 can be applied to find normal subgroups of a given group with the properties that are inherited by subgroups, like the properties of being abelian or nilpotent. Moreover, if under the assumptions of Lemma 2.7 the group A' is abelian, then one can find a normal abelian subgroup A in G such that the index of A in G is at most BJ^2; see [34, Theorem 1.41]. However, as is pointed out in [32, Remark 3.1], in the latter case we cannot guarantee that A is contained in A', and thus have no control on divisibility properties of the order of A (which is essential for the notions of a p–Jordan group or a nilpotently p–Jordan group).

One says that a group Γ has *bounded finite subgroups* if there exists a constant $B = B(\Gamma)$ such that every finite subgroup of Γ has order at most B. The next lemma allows one to check Jordan-type properties for certain extensions of groups.

Lemma 2.8 (cf [60, Lemma 2.11]) *Let p be a prime number, and let*

$$1 \to \Gamma' \to \Gamma \to \Gamma''$$

be an exact sequence of groups. Suppose that Γ'' has bounded finite subgroups. Suppose also that the group Γ' is Jordan (resp. p–Jordan, generalized p–Jordan, nilpotently Jordan of class at most c, nilpotently p–Jordan of class at most c, generalized nilpotently p–Jordan of class at most c). Then the group Γ is Jordan (resp. p–Jordan, generalized p–Jordan, nilpotently Jordan of class at most c, nilpotently p–Jordan of class at most c, generalized nilpotently p–Jordan of class at most c).

Proof By assumption, we know that there exists a constant B such that every finite subgroup of Γ'' has order at most B. Let G be a finite subgroup of Γ, and let $G' = G \cap \Gamma'$. Then the index of G' in G is at most B.

Suppose that Γ' is Jordan, or that Γ' is generalized p–Jordan and the order of G is coprime to p. Then G' contains a normal abelian subgroup of index at most J for some constant $J = J(\Gamma')$ that does not depend on G'. Therefore, G contains a normal abelian subgroup of index at most BJ^B by Lemma 2.7.

Similarly, suppose that Γ' is nilpotently Jordan of class at most c, or that Γ' is generalized nilpotently p–Jordan of class at most c, and the order of G is coprime to p. Then G' contains a normal nilpotent subgroup of class at most c that has index at most J for some constant $J = J(\Gamma')$. Therefore, G contains a normal nilpotent subgroup of class at most c that has index at most BJ^B by Lemma 2.7.

Now suppose that Γ' is p–Jordan (resp. nilpotently p–Jordan of class at most c). Let G_p and G'_p be p–Sylow subgroups of G and G'. The group G' contains a normal subgroup A' of order coprime to p and index at most $J \cdot |G'_p|^e$ for some constants $J = J(\Gamma')$ and $e = e(\Gamma')$ such that A' is abelian (resp. nilpotent of class at most c). Applying Lemma 2.7, we find a normal subgroup A in G such that A is abelian (resp. nilpotent of class at most c), its order is coprime to p, and its index in G is at most

$$B \cdot (J \cdot |G'_p|^e)^B = BJ^B \cdot |G'_p|^{Be} \leqslant BJ^B \cdot |G_p|^{Be}. \qquad \square$$

The next results are partial analogs of [62, Lemma 2.8] for p–Jordan groups.

Lemma 2.9 *Let p be a prime number, and let*

$$1 \to F \to H \to \bar{H} \to 1$$

be an exact sequence of finite groups such that $\bar{H}$ is abelian and generated by r elements. Suppose that $|F| \leqslant B \cdot |F_p|^e$ for some positive constants B and e, where F_p is a p–Sylow subgroup of F, and suppose that F is generated by s elements. Then H contains a characteristic abelian subgroup of order coprime to p and index at most $B^{r+s} \cdot |H_p|^{e(r+s)+1}$, where H_p is a p–Sylow subgroup of H.

Proof First, let us bound the index of the center Z of the group H. Let $K \subset H$ be the commutator subgroup. Since $\bar{H}$ is abelian, we see that K is contained in F. For every element $x \in H$, denote by $Z(x)$ the centralizer of x in H, and by $K(x)$ the set of elements of the form $hxh^{-1}x^{-1}$ for various $h \in H$. Then the index of $Z(x)$ does not exceed $|K(x)| \leqslant |K|$.

By assumption, one can choose $r+s$ elements $x_1, \dots, x_{r+s}$ generating H. Thus

$$Z = Z(x_1) \cap \dots \cap Z(x_{r+s}).$$

Hence

$$\begin{aligned}[H:Z] &\leqslant [H:Z(x_1)] \cdots [H:Z(x_{r+s})] \leqslant |K(x_1)| \cdots |K(x_{r+s})| \\ &\leqslant |K|^{r+s} \leqslant |F|^{r+s} \leqslant B^{r+s} \cdot |F_p|^{e(r+s)}.\end{aligned}$$

Now let Z' be the maximal subgroup in Z whose order is coprime to p. Then Z' is a characteristic abelian subgroup of H, and the index of Z' in Z equals the order of the p–Sylow subgroup Z_p of Z. Therefore, we have

$$\begin{aligned}[H:Z'] = [H:Z] \cdot [Z:Z'] &\leqslant (B^{r+s} \cdot |F_p|^{e(r+s)}) \cdot |Z_p| \\ &\leqslant B^{r+s} \cdot |F_p|^{e(r+s)} \cdot |H_p| \leqslant B^{r+s} \cdot |H_p|^{e(r+s)+1}.\end{aligned}$$ □

Corollary 2.10 *Let p be a prime number, and let*

$$1 \to \Gamma' \to \Gamma \to \Gamma''$$

be an exact sequence of groups. Suppose that Γ'' is p–Jordan, and that there exist positive constants r, B, e and s such that

- *every finite abelian subgroup of Γ'' whose order is coprime to p is generated by at most r elements,*
- *for every finite subgroup F of Γ', one has $|F| \leqslant B \cdot |F_p|^e$, where F_p is a p–Sylow subgroup of F, and F is generated by at most s elements.*

Then the group Γ is p–Jordan.

Proof Let G be a finite subgroup of Γ. Then G fits into an exact sequence

$$1 \to F \to G \to \overline{G} \to 1,$$

where $F \subset \Gamma'$ and $\overline{G} \subset \Gamma''$. By assumption, there exist positive constants $\overline{B}$ and $\overline{e}$ that do not depend on $\overline{G}$ such that $\overline{G}$ contains a normal abelian subgroup $\overline{H}$ whose order is coprime to p and whose index is bounded by $\overline{B} \cdot |\overline{G}_p|^{\overline{e}}$, where $\overline{G}_p$ is a p–Sylow subgroup of $\overline{G}$. Moreover, $\overline{H}$ can be generated by r elements. Let H be the preimage of $\overline{H}$ in G. According to Lemma 2.9, there is a characteristic abelian subgroup A in H whose order is coprime to p and whose index is at most $B^{r+s} \cdot |H_p|^{e(r+s)+1}$, where H_p is a p–Sylow subgroup of H. Therefore, A is a normal abelian subgroup of G of index at most

$$(\overline{B} \cdot |\overline{G}_p|^{\overline{e}}) \cdot (B^{r+s} \cdot |H_p|^{e(r+s)+1}) \leqslant \overline{B} B^{r+s} \cdot |G_p|^{\overline{e}+e(r+s)+1},$$

where G_p is a p–Sylow subgroup of G. □

We will use the following general fact.

Lemma 2.11 *Let p be a prime number, and let m be a nonnegative integer. Let F be a group containing a normal subgroup $F' \cong (\mathbb{Z}/p\mathbb{Z})^m$, and let g be an element of F. Then for some positive integer $t \leqslant p^m - 1$, the element g^t commutes with F'.*

Proof Let $L \subset F$ be the subgroup generated by g. The action of L on F' defines a homomorphism

$$\chi \colon L \to \mathrm{Aut}(F') \cong \mathrm{GL}_m(\boldsymbol{F}_p).$$

It is known that the order of any element in $\mathrm{GL}_m(\boldsymbol{F}_p)$ does not exceed $p^m - 1$; see eg [18, Corollary 2]. Therefore, g^t is contained in the kernel of χ for some $t \leqslant p^m - 1$, and the required assertion follows. □

Lemma 2.11 allows us to obtain a version of Lemma 2.9 that is applicable for a certain class of groups without a bound on the number of generators.

Lemma 2.12 *Let p be a prime number, and let*

$$1 \to F \to H \to \overline{H} \to 1$$

be an exact sequence of finite groups such that $\overline{H}$ is abelian and is generated by r elements. Suppose that the p–Sylow subgroup F_p of F is normal in F, and $F_p \cong (\mathbb{Z}/p\mathbb{Z})^m$ for some nonnegative integer m. Furthermore, suppose that $|F| \leqslant B \cdot |F_p|^e$ for some positive constants B and e, and suppose that F is generated by F_p and s additional elements. Then H contains a characteristic abelian subgroup of order coprime to p and index at most $B^{r+s+1} \cdot |H_p|^{e(r+s+1)+r+1}$, where H_p is a p–Sylow subgroup of H.

Proof Let us use the notation of the proof of Lemma 2.9. We are going to estimate the index of the center Z of H, and its maximal subgroup Z' of order coprime to p.

Let $x_1, \dots, x_r$ be the elements of H such that their images in $\overline{H}$ generate $\overline{H}$, and let $y_1, \dots, y_s$ be the elements of F such that F_p and $y_1, \dots, y_s$ generate F. Set

$$R = Z(x_1) \cap \dots \cap Z(x_r) \cap Z(y_1) \cap \dots \cap Z(y_s).$$

Then

$$[H:R] \leqslant |K(x_1)|\cdots|K(x_r)|\cdot|K(y_1)|\cdots|K(y_s)| \leqslant |K|^{r+s} \leqslant |F|^{r+s} \leqslant B^{r+s}\cdot|F_p|^{e(r+s)}.$$

Let $\bar{R}$ be the image of R in $\bar{H}$. Being a subgroup of an abelian group generated by r elements, $\bar{R}$ can be generated by r elements as well. Let $z_1,\dots,z_r$ be elements of R whose images in $\bar{H}$ generate $\bar{R}$. Then $z_1,\dots,z_r$ normalize the group F_p by Example 2.1. According to Lemma 2.11, there exist positive integers $t_i \leqslant p^m-1$, $1\leqslant i\leqslant r$, such that the elements $z_i^{t_i}$ commute with the subgroup F_p. Therefore, the group R' generated by $z_i^{t_i}$, $1\leqslant i\leqslant r$, is contained in the center Z of H. On the other hand, the image $\bar{R}'$ of R' in $\bar{H}$ is a subgroup of index at most $t_1\cdots t_r$ in $\bar{R}$, which implies that

$$[R:R'] \leqslant t_1\cdots t_r\cdot|F| < p^{mr}\cdot B\cdot|F_p|^e = B\cdot|F_p|^{e+r}.$$

Finally, as in the proof of Lemma 2.9, we have

$$\begin{aligned}[H:Z'] = [H:Z]\cdot[Z:Z'] \leqslant [H:R']\cdot[Z:Z'] &= [H:R]\cdot[R:R']\cdot[Z:Z'] \\ &\leqslant (B^{r+s}\cdot|F_p|^{e(r+s)})\cdot(B\cdot|F_p|^{e+r})\cdot|H_p| \\ &\leqslant B^{r+s+1}\cdot|H_p|^{e(r+s+1)+r+1}.\end{aligned}$$

□

Automorphisms of lattices Given a prime number ℓ, we denote by $\mathbb{Z}_\ell$ the ring of ℓ–adic integers. The following assertion is well known, and is proved similarly to the classical theorem of H Minkowski [54, Section 1]; see also Serre [69, Lemma 1] or [71, Theorem 9.9].

Lemma 2.13 *Let n be a positive integer, let ℓ be a prime, and let G be a finite subgroup of $\mathrm{GL}_n(\mathbb{Z}_\ell)$. Then G is isomorphic to a subgroup of $\mathrm{GL}_n(\mathbb{Z}/\ell\mathbb{Z})$ if $\ell\neq 2$, and to a subgroup of $\mathrm{GL}_n(\mathbb{Z}/4\mathbb{Z})$ if $\ell=2$. In particular, the group $\mathrm{GL}_n(\mathbb{Z}_\ell)$ has bounded finite subgroups.*

Proof First assume that $\ell\neq 2$. Let

$$\rho\colon \mathrm{GL}_n(\mathbb{Z}_\ell)\to\mathrm{GL}_n(\mathbb{Z}/\ell\mathbb{Z})$$

be the natural homomorphism. We claim that its kernel does not contain nontrivial elements of finite order. Indeed, denote by $\mathbf{1}$ the identity matrix in $\mathrm{GL}_n(\mathbb{Z}_\ell)$, and suppose that

$$M = \mathbf{1}+\ell M'$$

is an element of $\mathrm{GL}_n(\mathbb{Z}_\ell)$ such that $M^r=\mathbf{1}$ for some positive integer r. Set

$$\log M = \ell M' - \frac{(\ell M')^2}{2} + \cdots + (-1)^{k-1}\frac{(\ell M')^k}{k} + \cdots . \tag{2-1}$$

It is easy to see that the series on the right-hand side of (2-1) converges in $\mathrm{GL}_n(\mathbb{Z}_\ell)$, and so $\log M$ is a well-defined element of $\mathrm{GL}_n(\mathbb{Z}_\ell)$; also, we see that $\log M$ is divisible by ℓ in $\mathrm{GL}_n(\mathbb{Z}_\ell)$. Now let L be an arbitrary element divisible by ℓ in $\mathrm{GL}_n(\mathbb{Z}_\ell)$. Set

$$\exp L = \mathbf{1} + L + \cdots + \frac{L^k}{k!} + \cdots . \tag{2-2}$$

Note that the ℓ–adic valuation of L^k is at least k, while the ℓ–adic valuation of $k!$ equals

$$\left\lfloor \frac{k}{\ell} \right\rfloor + \left\lfloor \frac{k}{\ell^2} \right\rfloor + \cdots \leqslant k\left(\frac{1}{\ell} + \frac{1}{\ell^2} + \cdots\right) = \frac{k}{\ell - 1} \leqslant \frac{k}{2}.$$

Thus the series on the right-hand side of (2-2) converges in $\mathrm{GL}_n(\mathbb{Z}_\ell)$, and so $\exp L$ is a well-defined element of $\mathrm{GL}_n(\mathbb{Z}_\ell)$. We conclude that

$$\exp \log M = M \quad \text{and} \quad r \log M = \log M^r = 0.$$

Hence $\log M = 0$ and $M = \mathbf{1}$.

Now assume that $\ell = 2$. Arguing as above, one shows that the kernel of the natural homomorphism

$$\rho\colon \mathrm{GL}_n(\mathbb{Z}_2) \to \mathrm{GL}_n(\mathbb{Z}/4\mathbb{Z})$$

does not contain nontrivial elements of finite order. □

Lemma 2.13 allows us to deduce more traditional versions of Minkowski's theorem.

Corollary 2.14 *Let Λ be a finitely generated abelian group. Then the group $\mathrm{Aut}(\Lambda)$ has bounded finite subgroups.*

Proof There is an exact sequence of groups

$$1 \to (\Lambda_{\mathrm{tors}})^{\times n} \times \mathrm{Aut}(\Lambda_{\mathrm{tors}}) \to \mathrm{Aut}(\Lambda) \to \mathrm{GL}_n(\mathbb{Z}) \to 1,$$

where Λ_{tors} is the torsion subgroup of Λ, and n is the rank of the free abelian group $\Lambda/\Lambda_{\mathrm{tors}}$. The group $\mathrm{GL}_n(\mathbb{Z})$ is a subgroup of $\mathrm{GL}_n(\mathbb{Z}_\ell)$ for any prime ℓ. Thus, it has bounded finite subgroups by Lemma 2.13. On the other hand, the group $\mathrm{Aut}(\Lambda_{\mathrm{tors}})$ is finite, and the required assertion follows. □

Corollary 2.15 (cf [23, Theorem F]) *For every positive integer n, the group $\mathrm{GL}_n(\mathbb{Q})$ has bounded finite subgroups.*

Proof If G is a finite subgroup of $\mathrm{GL}_n(\mathbb{Q})$, then it acts by automorphisms of some sublattice $\mathbb{Z}^n \subset \mathbb{Q}^n$. This means that G is isomorphic to a subgroup of $\mathrm{GL}_n(\mathbb{Z})$, and thus $\mathrm{GL}_n(\mathbb{Q})$ has bounded finite by Corollary 2.14. □

Projective general linear groups We conclude this section by an easy consequence of Theorem 1.4.

Corollary 2.16 *Let n be a positive integer. Then there exists a constant $J_{\mathrm{PGL}}(n)$ such that every finite subgroup G of $\mathrm{PGL}_n(\Bbbk)$, where $\Bbbk$ is an arbitrary field of characteristic $p > 0$, contains a normal abelian subgroup of order coprime to p and index at most $J_{\mathrm{PGL}}(n) \cdot |G_p|^3$, where G_p is a p–Sylow subgroup of G.*

Proof The adjoint representation embeds the group $\mathrm{PGL}_n(\Bbbk)$ into $\mathrm{GL}_{n^2}(\Bbbk)$, so Theorem 1.4 applies. □

3 Automorphism groups

In this section we recall the basic concepts and facts about varieties and their automorphisms.

General settings The following terminology is standard (although not *universally* accepted); see for example [10, Chapters AG.12 and I.1], and compare [14, Definition 2.1.5].

Definition 3.1 An algebraic group over a field $\Bbbk$ is a geometrically reduced group scheme of finite type over $\Bbbk$.

In other words, an algebraic group is a variety with a structure of a group scheme. Note that any algebraic group is smooth; see for instance [14, Proposition 2.1.12].

If X is an arbitrary algebraic variety over a field $\Bbbk$, then its automorphisms form a group, which we denote by $\mathrm{Aut}(X)$. If $K \supset \Bbbk$ is a field extension, and X_K is the extension of scalars of X to K, then every automorphism of X defines an automorphism of X_K; in other words, one has

$$\mathrm{Aut}(X) \subset \mathrm{Aut}(X_K).$$

If X is irreducible, then one can consider its birational automorphism group $\mathrm{Bir}(X)$; one has a natural embedding $\mathrm{Aut}(X) \subset \mathrm{Bir}(X)$. If X_K is still irreducible, then $\mathrm{Bir}(X) \subset \mathrm{Bir}(X_K)$. In this paper, we will not be interested in any additional structures on the group $\mathrm{Bir}(X)$; the reader can find a discussion of these in some particular cases over fields of characteristic zero in [29] and [30]; see also [7]. However, we will need some structure related to the automorphism group.

Let X be a projective variety over a field $\Bbbk$. Then the automorphism functor of X is represented by the automorphism group scheme $\mathbf{Aut}_X$ which is locally of finite type; see eg [14, Theorem 7.1.1]. The automorphism group $\mathrm{Aut}(X)$ is just the group of $\Bbbk$–points of $\mathbf{Aut}_X$, that is,

$$\mathrm{Aut}(X) = \mathbf{Aut}_X(\Bbbk).$$

Let $\mathbf{Aut}^0_X$ be the neutral component of $\mathbf{Aut}_X$; then $\mathbf{Aut}^0_X$ is a connected group scheme of finite type over $\Bbbk$, but it may be nonreduced, even if $\Bbbk$ is algebraically closed and X is smooth; see eg [14, Example 7.1.5]. Let $\mathbf{Aut}^0_{X,\mathrm{red}}$ be the maximal reduced subscheme of $\mathbf{Aut}^0_X$. If the field $\Bbbk$ is perfect, then $\mathbf{Aut}^0_{X,\mathrm{red}}$ is a group scheme; see [14, Section 2.5]. (Note that over a nonperfect field the maximal reduced subscheme of a group scheme is not necessarily a group scheme itself [14, Example 2.5.3].) Furthermore, in this case the fact that $\mathbf{Aut}^0_{X,\mathrm{red}}$ is reduced implies that it is geometrically reduced, and thus $\mathbf{Aut}^0_{X,\mathrm{red}}$ is an algebraic group; in particular, this means that $\mathbf{Aut}^0_{X,\mathrm{red}}$ is smooth. We set

$$\mathrm{Aut}^0(X) = \mathbf{Aut}^0_{X,\mathrm{red}}(\Bbbk) = \mathbf{Aut}^0_X(\Bbbk).$$

Note that $\mathrm{Aut}^0(X)$ always has a group structure, regardless of whether the field $\Bbbk$ is perfect or not. Anyway, in this paper we will need to deal with the group $\mathrm{Aut}^0(X)$ and the group scheme $\mathbf{Aut}^0_X$ only in the case when the base field is algebraically closed.

The reader is referred to the surveys [14, Section 7.1] and [15, Section 2.1] for more details on automorphism groups and automorphism group schemes.

Resolution of singularities and regularization of birational maps Resolution of singularities is available in arbitrary dimension over fields of characteristic 0, and in small dimensions over fields of positive characteristic. We will need it in the classical case of surfaces.

Theorem 3.2 [46, Theorem; 45, Corollary 27.3] *Let S be a geometrically irreducible algebraic surface over a field $\Bbbk$. Then there exists a minimal resolution of singularities of S. More precisely, there exists a regular surface $\widetilde{S}$ over $\Bbbk$ with a proper birational morphism $\pi\colon \widetilde{S} \to S$ such that any proper birational morphism from a regular projective surface to S factors through π. In particular, if the field $\Bbbk$ is perfect, then the surface $\widetilde{S}$ is smooth.*

Remark 3.3 If in the notation of Theorem 3.2 the surface S is projective, then the surface $\widetilde{S}$ is projective as well. Indeed, $\widetilde{S}$ is complete and regular. Hence $\widetilde{S}_{\overline{\Bbbk}}$ is projective according to [39, Corollary IV.2.4], which implies that $\widetilde{S}$ is projective; see eg [25, Corollaire 9.1.5].

In particular, Theorem 3.2 and Remark 3.3 tell us that every geometrically irreducible algebraic surface has a regular projective birational model.

Corollary 3.4 *Let S be a geometrically irreducible algebraic surface over a field $\Bbbk$. Then there exists a regular projective surface $\widetilde{S}$ over $\Bbbk$ birational to S. In particular, if the field $\Bbbk$ is perfect, then the surface $\widetilde{S}$ is smooth.*

Proof Replace S by its affine open subset, then replace the latter by a projective completion, and take a resolution of singularities. □

The factorization property provided by Theorem 3.2 implies that the minimal resolution of singularities behaves well with respect to the automorphism group.

Corollary 3.5 *Let S be a geometrically irreducible algebraic surface, and let $\pi\colon \widetilde{S} \to S$ be the minimal resolution of singularities. Then there is an action of the group $\mathrm{Aut}(S)$ on $\widetilde{S}$ such that the morphism π is $\mathrm{Aut}(S)$–equivariant.*

The following version of Corollary 3.4 taking into account a birational action of a finite group is classical and widely used, at least in the case of zero characteristic; see eg [20, Lemma 3.5] or [62, Lemma–Definition 3.1]; cf [83; 16]. We recall its proof for the convenience of the reader.

Lemma 3.6 *Let S be a geometrically irreducible algebraic surface over a field $\Bbbk$, and let $G \subset \mathrm{Bir}(S)$ be a finite group. Then there exists a regular projective surface $\widetilde{S}$ with a biregular action of G and a G–equivariant birational map $\widetilde{S} \dashrightarrow S$. In particular, if the field $\Bbbk$ is perfect, then the surface $\widetilde{S}$ is smooth.*

Proof Let $\widehat{V}$ be a normal projective model of the field of invariants $\Bbbk(S)^G$, and let $\widehat{S}$ be the normalization of $\widehat{V}$ in the field $\Bbbk(S)$. Then there is a regular action of G on $\widehat{S}$ and a G–equivariant birational map $\zeta\colon \widehat{S} \dashrightarrow S$.

Let $\pi\colon \widetilde{S} \to \widehat{S}$ be the minimal resolution of singularities provided by Theorem 3.2. Then $\widetilde{S}$ is a regular geometrically irreducible surface; also, $\widetilde{S}$ is projective by Remark 3.3. According to Corollary 3.5, the action of G lifts to $\widetilde{S}$ so that the morphism π is G–equivariant. Therefore, we obtain a G–equivariant birational map $\zeta \circ \pi\colon \widetilde{S} \dashrightarrow S$. $\square$

Stabilizer of a point Let us recall an auxiliary result on fixed points of automorphisms of finite order which is well known and widely used (especially in characteristic 0). We provide its proof to be self-contained. Given an algebraic variety X over a field $\Bbbk$ and a $\Bbbk$–point $P \in X$, denote by $T_P(X)$ the Zariski tangent space to X at P.

Theorem 3.7 *Let X be an irreducible algebraic variety over a field $\Bbbk$ of characteristic p. Let G be a finite group acting on X with a fixed $\Bbbk$–point P. Suppose that $|G|$ is not divisible by p. Then the natural representation*

$$d\colon G \to \mathrm{GL}(T_P(X))$$

is an embedding.

Proof Suppose that d is not an embedding. Replacing G by the kernel of d, we may assume that G acts trivially on $T_P(X)$. So G acts trivially on

$$\mathfrak{m}_P/\mathfrak{m}_P^2 \cong T_P(X)^\vee,$$

where $\mathfrak{m}_P \subset \mathbb{O}_P$ is the maximal ideal in the local ring of the point P on X. The quotient morphism $\mathbb{O}_P \to \mathbb{O}_P/\mathfrak{m}_P \cong \Bbbk$ admits a natural section, which gives a G–invariant decomposition $\mathbb{O}_P \cong \Bbbk \oplus \mathfrak{m}_P$ into a direct sum of vector subspaces.

Consider the G–invariant filtration

$$\mathfrak{m}_P \supset \mathfrak{m}_P^2 \supset \mathfrak{m}_P^3 \supset \cdots.$$

Recall that $\mathfrak{m}_P$ is generated by elements of degree 1, ie generated by a collection of elements whose images form a basis in $\mathfrak{m}_P/\mathfrak{m}_P^2$. Hence G acts trivially on $\mathfrak{m}_P^n/\mathfrak{m}_P^{n+1}$ for every positive integer n. Since the order of G is not divisible by p, every representation of G is completely reducible. Therefore, we have an isomorphism of G–representations

$$\mathfrak{m}_P \cong \bigoplus_{n=1}^{\infty} \mathfrak{m}_P^n/\mathfrak{m}_P^{n+1}.$$

Thus, the action of G on $\mathfrak{m}_P$ and $\mathbb{O}_P$ is trivial.

Let $U \subset X$ be an affine open subset containing the point P. Then $U' = \sigma(U)$ is also an affine open subset of X containing P. Let R and R' be the coordinate rings of U and U', respectively. Then R and R' are subalgebras of $\mathbb{O}_P$, and one has $\sigma^* R' = R$. Since the action of σ on $\mathbb{O}_P$ is trivial, we conclude that $R = R'$, and σ acts trivially on R. This means that U is σ–invariant, and σ acts trivially on U. Finally, since X is irreducible, U is dense in X; hence σ acts trivially on the whole X. $\square$

4 Group of connected components

In this section we recall the following assertion established in [53, Lemma 2.5]; compare with the proof of [32, Theorem 1.9].

Lemma 4.1 *Let X be a (possibly reducible) projective variety. Then the group $\mathrm{Aut}(X)/\mathrm{Aut}^0(X)$ has bounded finite subgroups.*

We provide the proof of Lemma 4.1 for the reader's convenience. The argument below is mostly taken from [53, Remark 2.6]. Let us start with a simple observation.

Lemma 4.2 *Let X be a projective variety over a field $\Bbbk$. Then there is an embedding*

$$\theta\colon \mathrm{Aut}(X)/\mathrm{Aut}^0(X) \hookrightarrow \mathrm{Aut}(X_{\bar{\Bbbk}})/\mathrm{Aut}^0(X_{\bar{\Bbbk}}).$$

Proof Let $\hat{g}$ be an element of the quotient group $\mathrm{Aut}(X)/\mathrm{Aut}^0(X)$, and let g be its preimage in $\mathrm{Aut}(X)$. Thus, g is a $\Bbbk$–point of the group scheme $\mathbf{Aut}_X$. Considering it as a $\bar{\Bbbk}$–point of $\mathbf{Aut}_X$, we obtain the map θ.

To show that θ is injective, suppose that $\hat{g}$ is a nontrivial element, so that g is not contained in $\mathrm{Aut}^0(X)$. In other words, the corresponding $\Bbbk$–point of $\mathbf{Aut}_X$ is not contained in the closed subgroup $\mathbf{Aut}_X^0$, which is also defined over $\Bbbk$. Therefore, g is not contained in the set of $\bar{\Bbbk}$–points of $\mathbf{Aut}_X^0$, which means that it defines a nontrivial element of the quotient group $\mathrm{Aut}(X_{\bar{\Bbbk}})/\mathrm{Aut}^0(X_{\bar{\Bbbk}})$ as well. □

To prove Lemma 4.1, we will use the following standard fact.

Theorem 4.3 [39, Theorem II.2.1] *Let X be a projective variety over an algebraically closed field. Let $\mathscr{F}$ be a coherent sheaf, and let $\mathscr{G}$ be a numerically trivial line bundle on X. Then the Euler characteristic of $\mathscr{F}$ equals the Euler characteristic of $\mathscr{F}\otimes\mathscr{G}$.*

Given a projective variety X over an algebraically closed field $\Bbbk$, let $\mathrm{NS}(X)$ denote Neron–Severi group of line bundles on X modulo algebraic equivalence. Recall from [40, Théorème 5.1] that $\mathrm{NS}(X)$ is a finitely generated abelian group. If L is a line bundle on X, we define the group scheme $\mathbf{Aut}_{X;[L]}$ as the stabilizer in $\mathbf{Aut}_X$ of the class of L in $\mathrm{NS}(X)$. Obviously, one has $\mathbf{Aut}_X^0 \subset \mathbf{Aut}_{X;[L]}$; this means that $\mathbf{Aut}_X^0$ is the neutral component of $\mathbf{Aut}_{X;[L]}$.

Lemma 4.4 *Let X be a projective variety over an algebraically closed field, and let L be an ample line bundle on X. Then $\mathbf{Aut}_{X;[L]}$ is a group scheme of finite type.*

Proof One can identify $\mathbf{Aut}_X$ with an open subscheme of the Hilbert scheme $\mathrm{Hilb}(X\times X)$ by associating with each automorphism f its graph $\Gamma_f \subset X\times X$; see [22, Theorem 5.23], and the exercise after this theorem. Set

$$L_{X\times X} = p_1^*L\otimes p_2^*L,$$

where $p_1, p_2 \colon X \times X \to X$ are the projections to the first and the second factor, respectively. Then $L_{X\times X}$ is an ample line bundle on $X \times X$. Its restriction to Γ_f (identified with X via the projection p_1) is the line bundle

$$L_f \cong L \otimes f^* L.$$

If f preserves the class $[L] \in \mathrm{NS}(X)$, then L_f is algebraically equivalent to L^2; in particular, these line bundles are numerically equivalent. Therefore, by Theorem 4.3 the Hilbert polynomial of Γ_f (with respect to the ample line bundle $L_{X\times X}$ on X) is independent of f provided that $f \in \mathbf{Aut}_{X;[L]}$.

Let P be this polynomial. Then $\mathbf{Aut}_{X;[L]}$ is contained in the Hilbert scheme $\mathrm{Hilb}_P(X \times X)$. Recall that $\mathrm{Hilb}_P(X \times X)$ is projective; see for instance [41, Theorem I.1.4]. Set

$$\mathbf{Aut}_X^P = \mathbf{Aut}_X \cap \mathrm{Hilb}_P(X \times X) \subset \mathrm{Hilb}(X \times X).$$

Then $\mathbf{Aut}_X^P$ is an open subscheme of $\mathrm{Hilb}_P(X \times X)$, and so it is quasiprojective. Furthermore, $\mathbf{Aut}_{X;[L]}$ is a closed subscheme of $\mathbf{Aut}_X^P$. Hence, $\mathbf{Aut}_{X;[L]}$ is quasiprojective as well, thus it is of finite type. □

Denote by $\mathrm{Aut}_{[L]}(X)$ the group of $\mathbb{k}$–points of $\mathbf{Aut}_{X;[L]}$. The following assertion is implied by Lemma 4.4.

Corollary 4.5 *Let X be a projective variety over an algebraically closed field $\mathbb{k}$, and let L be an ample line bundle on X. Then $\mathrm{Aut}_{[L]}(X)/\mathrm{Aut}^0(X)$ is a finite group.*

Proof It follows from Lemma 4.4 that $\mathbf{Aut}_{X;[L]}/\mathbf{Aut}_X^0$ is a finite group scheme. Therefore, its group of $\mathbb{k}$–points

$$(\mathbf{Aut}_{X;[L]}/\mathbf{Aut}_X^0)(\mathbb{k}) \cong \mathrm{Aut}_{[L]}(X)/\mathrm{Aut}^0(X)$$

is finite. □

Proof of Lemma 4.1 By Lemma 4.2, we may assume that X is defined over an algebraically closed field. Choose an ample line bundle L on X. Let G be a finite subgroup of the group $\mathrm{Aut}(X)/\mathrm{Aut}^0(X)$. Since $\mathrm{Aut}^0(X)$ acts trivially on the Neron–Severi group $\mathrm{NS}(X)$, there is a natural action of G on $\mathrm{NS}(X)$. One has an exact sequence of groups

$$1 \to K \to G \to \bar{G} \to 1,$$

where $\bar{G}$ is the image of the representation of G in $\mathrm{NS}(X)$, and K is the kernel of this representation. Since $\mathrm{NS}(X)$ is a finitely generated abelian group, by Corollary 2.14 there is a constant $M = M(X)$ such that every finite subgroup of $\mathrm{Aut}(\mathrm{NS}(X))$ has order at most M; in particular, one has $|\bar{G}| \leqslant M$. On the other hand, the group K preserves the ample divisor class $[L] \in \mathrm{NS}(X)$, and hence

$$K \subset \mathrm{Aut}_{[L]}(X)/\mathrm{Aut}^0(X).$$

By Corollary 4.5 the group $\mathrm{Aut}_{[L]}(X)/\mathrm{Aut}^0(X)$ is finite. Therefore, we have

$$|G| = |\bar{G}| \cdot |K| \leqslant M \cdot |\mathrm{Aut}_{[L]}(X)/\mathrm{Aut}^0(X)|.$$ □

To conclude this section, let us make the following remark. For every projective variety X defined over a field $\Bbbk$, there exists a natural embedding of groups

$$\mathrm{Aut}(X)/\mathrm{Aut}^0(X) \hookrightarrow (\mathbf{Aut}_X/\mathbf{Aut}^0_X)(\Bbbk).$$

However, over an algebraically nonclosed field this embedding may fail to be an isomorphism.

Example 4.6 Let $\Bbbk = \mathbb{Q}(\omega)$, where ω is a nontrivial cubic root of unity. Let E be the curve given in $\mathbb{P}^2$ with homogeneous coordinates x, y and z by equation

$$y^2 z = x^3 + z^3,$$

and choose $(0:1:0)$ to be the marked point on E. Then E is an elliptic curve such that

$$\mathbf{Aut}_E \cong \mathbf{Aut}^0_E \rtimes \mathbb{Z}/6\mathbb{Z},$$

and all six $\bar{\Bbbk}$–points of the group scheme $\mathbf{Aut}_E/\mathbf{Aut}^0_E$ are defined over $\Bbbk$. Note that all the 2–torsion $\bar{\Bbbk}$–points of $\mathbf{Aut}^0_E$ are also defined over $\Bbbk$; let υ be one of the nontrivial 2–torsion $\Bbbk$–points of $\mathbf{Aut}^0_E$. Choose an auxiliary quadratic extension $\mathbb{K}$ of $\Bbbk$, say, $\mathbb{K} = \Bbbk(\sqrt{7})$. Let ζ be the 1–cocycle corresponding to the homomorphism $\mathrm{Gal}(\mathbb{K}/\Bbbk) \to \mathrm{Aut}(E)$ that sends the generator of $\mathrm{Gal}(\mathbb{K}/\Bbbk) \cong \mathbb{Z}/2\mathbb{Z}$ to $\upsilon \in \mathrm{Aut}(E)$. Let E' be the twist of E by ζ. Then E' is a smooth geometrically irreducible projective curve of genus 1 over $\Bbbk$. One can check that $E'(\Bbbk) = \varnothing$. Moreover, the Jacobian $J(E')$ is the twist of the Jacobian $J(E)$ by the same cocycle ζ. However, υ acts trivially on $J(E)$, and thus $J(E') \cong J(E)$. Since E has a $\Bbbk$–point, we also have an isomorphism $J(E) \cong E$.

Now observe that the group scheme $\mathbf{Aut}_{E'}$ acts on the curve $E \cong J(E')$, and $\mathbf{Aut}^0_{E'}$ is contained in the kernel of this action. This gives rise to a homomorphism

$$\mathbf{Aut}_{E'}/\mathbf{Aut}^0_{E'} \to \mathbf{Aut}_E/\mathbf{Aut}^0_E,$$

which becomes an isomorphism after the extension of scalars to $\bar{\Bbbk}$. Thus, this homomorphism is actually an isomorphism. Since $\mathbf{Aut}_E/\mathbf{Aut}^0_E$ has six $\Bbbk$–points, we conclude that $\mathbf{Aut}_{E'}/\mathbf{Aut}^0_{E'}$ has six $\Bbbk$–points as well. On the other hand, the group $\mathrm{Aut}(E')/\mathrm{Aut}^0(E')$ does not contain elements of order 6. Indeed, it follows from the Lefschetz fixed-point formula that any preimage of such an element in $\mathrm{Aut}(E')$ would have a unique fixed point on $E'_{\bar{\Bbbk}} \cong E_{\bar{\Bbbk}}$, and so this point would be defined over $\Bbbk$.

5 Minimal models

In this section, we recall the notion of the Kodaira dimension, and discuss some facts concerning classification of surfaces and the Minimal Model Program in dimension 2.

Kodaira dimension One of the most important birational invariants of projective varieties is the Kodaira dimension. Given a smooth geometrically irreducible projective variety, we will denote by ω_X the canonical sheaf on X, and by K_X the canonical class of X, ie the class of ω_X in $\mathrm{Pic}(X)$.

Definition 5.1 [1, Definition 5.6] Let X be a smooth irreducible projective variety over an algebraically closed field. The Kodaira dimension $\kappa(X)$ of X is defined by

$$\kappa(X) = \begin{cases} \operatorname{tr}\deg\bigl(\bigoplus_{n=0}^{\infty} H^0(X,\omega_X^{\otimes n})\bigr) - 1 & \text{if } \operatorname{tr}\deg\bigl(\bigoplus_{n=0}^{\infty} H^0(X,\omega_X^{\otimes n})\bigr) > 0, \\ -\infty & \text{if } H^0(X,\omega_X^{\otimes n}) = 0 \text{ for all } n \geqslant 1. \end{cases}$$

Remark 5.2 Alternatively, one can define $\kappa(X)$ as the maximal dimension of an image of X with respect to the rational map given by the pluricanonical linear systems $|mK_X|$ for all $m \geqslant 1$; see [33, Section 10.5].

For an arbitrary field $\Bbbk$, we define the Kodaira dimension $\kappa(X)$ of a smooth geometrically irreducible projective variety X as the Kodaira dimension of $X_{\bar{\Bbbk}}$. One has

$$-\infty \leqslant \kappa(X) \leqslant \dim X;$$

see eg [1, Lemma 5.5]. Kodaira dimension is a birational invariant for smooth geometrically irreducible projective varieties; see [33, Section 10.5]. Moreover, the following assertion holds.

Lemma 5.3 [41, Corollary IV.1.11] *Let X be a smooth geometrically irreducible projective variety. Suppose X is birational to $Y \times \mathbb{P}^1$, where Y is a (possibly singular) algebraic variety. Then $\kappa(X) = -\infty$.*

In the case of surfaces (and in other cases when the resolution of singularities is available) one can extend the definition of Kodaira dimension a little further. Namely, for an irreducible algebraic surface S over an algebraically closed field, the Kodaira dimension of S is defined as the Kodaira dimension of (any) smooth projective birational model of S (which exists by Corollary 3.4). Note that due to birational invariance of Kodaira dimension, this definition does not depend on the choice of a birational model. If S is a geometrically irreducible algebraic surface over an arbitrary field $\Bbbk$, we set $\kappa(S) = \kappa(S_{\bar{\Bbbk}})$.

Minimal surfaces Among all smooth projective surfaces, there is a special class of the so-called minimal surfaces. They are the most important ones for studying automorphism groups.

Definition 5.4 [1, Definition 6.1] Let S be a smooth geometrically irreducible projective surface. We say that S is *minimal* if every birational morphism $S \to Y$, where Y is a smooth projective surface, is an isomorphism.

Example 5.5 Let S be a smooth geometrically irreducible projective surface over a field $\Bbbk$. Suppose that the canonical class K_S is numerically trivial. Then S is minimal. Indeed, if S is not minimal, then $S_{\bar{\Bbbk}}$ contains a smooth rational curve with self-intersection -1; see eg [72, Theorem IV.3.4.5] or [1, Section 6]. On the other hand, computing the self-intersection by adjunction formula, we see that such a curve cannot exist on $S_{\bar{\Bbbk}}$.

Minimal surfaces are representatives of the birational equivalent classes of all smooth geometrically irreducible projective surfaces. Indeed, since every birational morphism between smooth projective surfaces over an algebraically closed field is a composition of contractions of smooth rational curves with self-intersection -1 (see [72, Theorem IV.3.4.5]), the number of such consecutive contractions from a given surface S is bounded by the Picard rank of S. Thus, we can replace every smooth geometrically irreducible projective surface S over an arbitrary field by a minimal surface S' such that there exists a birational morphism from S to S'.

Birational automorphism groups of minimal surfaces of nonnegative Kodaira dimension are easy to study due to the following well known result.

Lemma 5.6 [1, Corollary 10.22, Theorem 10.21] *Let S be a minimal surface such that $\kappa(S) \geqslant 0$. Then S is the unique minimal surface in its birational equivalence class, and* $\mathrm{Bir}(S) = \mathrm{Aut}(S)$.

Remark 5.7 In [1], the proof of Lemma 5.6 is given over an algebraically closed field. However, the general case easily follows from this.

Similarly to the case of characteristic zero, over fields of positive characteristic there exists a Kodaira–Enriques classification of minimal surfaces due to E Bombieri and D Mumford [56; 9; 8]; see also [1] and [44]. We recall its part that will be used in this paper. The definitions of the particular classes of surfaces can be found for instance in [44, Sections 6 and 7].

Theorem 5.8 *Let S be a smooth irreducible projective surface over an algebraically closed field. The following assertions hold.*

(i) *If $\kappa(S) = -\infty$, then S is birational either to $\mathbb{P}^2$, or to $C \times \mathbb{P}^1$, where C is a (irreducible smooth projective) curve of positive genus.*

(ii) *If $\kappa(S) = 0$ and S is minimal, then S is either a $K3$ surface, or an Enriques surface, or an abelian surface, or a hyperelliptic surface, or a quasihyperelliptic surface.*

***G*–minimal surfaces** There exists a version of the Minimal Model Program that takes into account an action of a group. Below we recall some of its implications in the case of geometrically rational surfaces.

Definition 5.9 A smooth geometrically irreducible projective surface S with an action of a group G is called *G–minimal* if every G–equivariant birational morphism $S \to T$, where T is a smooth geometrically irreducible projective surface with an action of G, is an isomorphism.

Similarly to the case of the trivial group action, for every smooth geometrically irreducible projective surface with an action of a group G there is a G–equivariant birational morphism to a G–minimal surface. Thus, it is interesting to know the properties of G–minimal surfaces.

Definition 5.10 Let S be a smooth geometrically irreducible projective surface, and let $\phi\colon S \to C$ be a surjective morphism to a smooth curve. One says that ϕ (or S) is a *conic bundle*, if the fiber of ϕ over the scheme-theoretic generic point of C is smooth and geometrically irreducible, and the anticanonical line bundle ω_S^{-1} is very ample over C.

Remark 5.11 Let S be a smooth geometrically irreducible projective surface, and let $\phi\colon S \to C$ be a conic bundle. Denote by S_η the fiber of ϕ over the scheme-theoretic generic point of C. Then S_η is smooth and geometrically irreducible. Moreover, the anticanonical line bundle $\omega_{S_\eta}^{-1}$ is very ample. This implies that S_η is a smooth conic over the field $\Bbbk(C)$ of rational functions on C.

Example 5.12 Let $\Bbbk$ be an algebraically closed field of characteristic 2. Consider the surface $S \cong \mathbb{P}^1 \times \mathbb{P}^1$, and let $\phi'\colon S \to \mathbb{P}^1$ be the projection to the second factor. Then ϕ' is a conic bundle. On the other hand, let $\phi\colon S \to \mathbb{P}^1$ be the composition of ϕ' with an inseparable double cover $\mathbb{P}^1 \to \mathbb{P}^1$. Then ϕ is *not* a conic bundle. Indeed, its scheme-theoretic generic fiber S_η is not smooth over the field $\Bbbk(\mathbb{P}^1)$. Moreover, for every point P of S_η, the scheme S_η is regular at P, but fails to be smooth at this point, because S_η is not geometrically reduced (although it is reduced over $\Bbbk$). Note that ϕ satisfies the second requirement of Definition 5.10, ie the anticanonical sheaf ω_S^{-1} is very ample over $\mathbb{P}^1$; also, each geometric fiber of ϕ is isomorphic to a nonreduced conic in $\mathbb{P}^2$.

Recall that a *del Pezzo surface* is a smooth geometrically irreducible projective surface S with ample anticanonical class. For the following result, we refer the reader to [35, Theorem 1G] or [55, Theorem 2.7]; compare also with [1, Corollary 7.3].

Theorem 5.13 *Let G be a finite group, and let S be a geometrically rational G–minimal surface. Then S is either a del Pezzo surface, or a G–equivariant conic bundle over a smooth curve of genus zero.*

If the base field is perfect, one can use Lemma 3.6 together with Theorem 5.13 to produce nice regularizations of birational actions of finite groups on geometrically rational surfaces (note however that we will use this only over algebraically closed fields in our paper).

Theorem 5.14 *Let S be a geometrically rational algebraic surface over a perfect field. Let G be a finite subgroup of* $\mathrm{Bir}(S)$. *Then there exists a smooth geometrically irreducible projective surface S' with a regular action of G and a G–equivariant map $S \dashrightarrow S'$, such that S' is either a del Pezzo surface, or a G–equivariant conic bundle over a curve of genus zero.*

Proof First, there exists a regular projective surface $\widetilde{S}$ with an action of G and a G–equivariant birational map $\widetilde{S} \dashrightarrow S$; see Lemma 3.6. Since the base field is perfect, $\widetilde{S}$ is actually smooth. Thus there exists a G–minimal surface S' that is G–equivariantly birational to $\widetilde{S}$ (and thus to S). Now the assertion about the geometrically rational case follows from Theorem 5.13. □

The next fact is well known.

Theorem 5.15 *Let S be a del Pezzo surface over a field $\Bbbk$. Then the linear system $|-3K_S|$ defines an embedding $S \hookrightarrow \mathbb{P}^N$, where $N \leqslant 54$.*

Proof The divisor $-3K_S$ is very ample; see [41, Proposition III.3.4.2]. Thus it defines an embedding $S \hookrightarrow \mathbb{P}^N$, where

$$N = h^0(S, \mathcal{O}_S(-3K_S)) - 1.$$

On the other hand, one has $1 \leqslant K_S^2 \leqslant 9$; see eg [48, Theorem IV.2.5] or [41, Exercise III.3.9]. Hence

$$h^0(S, \mathbb{O}_S(-3K_S)) - 1 = 6K_S^2 \leqslant 54$$

by [41, Corollary III.3.2.5]. $\square$

Nonrational ruled surfaces Birational automorphism groups of nonrational surfaces covered by rational curves have simpler structure than those of geometrically rational surfaces.

Definition 5.16 Given a morphism $\phi\colon X \to Y$ between varieties X and Y, and an automorphism (or a birational automorphism) g of X, we will say that g is *fiberwise with respect to* ϕ if it maps every point of X to a point in the same fiber of ϕ (provided that g is well defined at this point). The action of a subgroup Γ of $\mathrm{Aut}(X)$ or $\mathrm{Bir}(X)$ is fiberwise with respect to ϕ if every element of Γ is fiberwise with respect to ϕ.

Lemma 5.17 *Let $\Bbbk$ be a field. Let C be a smooth geometrically irreducible projective curve of positive genus over $\Bbbk$, and set $S = C \times \mathbb{P}^1$. Then there is an exact sequence of groups*

$$1 \to \mathrm{Bir}(S)_\phi \to \mathrm{Bir}(S) \to \Gamma, \tag{5-1}$$

where $\mathrm{Bir}(S)_\phi \subset \mathrm{PGL}_2(\Bbbk(C))$ *and* $\Gamma \subset \mathrm{Aut}(C)$.

Proof Consider the projection $\phi\colon S \to C$. If $g \in \mathrm{Bir}(S)$, and $F \cong \mathbb{P}^1$ is a general fiber of ϕ, then the map

$$\phi \circ g\colon F \to C$$

is either surjective, or maps F to a point. The former option is impossible, because a rational curve cannot dominate a curve of positive genus; see for instance [31, Corollary IV.2.4] and [31, Proposition IV.2.5]. This means that an image of F under g is again a fiber of ϕ, so that ϕ is equivariant with respect to the whole group $\mathrm{Bir}(S)$.

Hence there is an exact sequence (5-1), where the action of the group $\mathrm{Bir}(S)_\phi$ is fiberwise with respect to ϕ, and Γ is a subgroup of $\mathrm{Bir}(C) = \mathrm{Aut}(C)$. Thus, the group $\mathrm{Bir}(S)_\phi$ is a subgroup of $\mathrm{Bir}(S_\eta)$, where S_η is the scheme-theoretic generic fiber of the map ϕ. Since S_η is isomorphic to the projective line over the field $\Bbbk(C)$, we have

$$\mathrm{Bir}(S_\eta) = \mathrm{Aut}(S_\eta) \cong \mathrm{PGL}_2(\Bbbk(C)).$$

$\square$

6 Abelian varieties

In this section we collect auxiliary facts about automorphism groups of abelian varieties. The latter groups are usually infinite, but are rather easy to understand. Here we regard an abelian variety just as a variety, and do not take the group structure into account. Also, if the base field is not algebraically closed, it is not necessary for us to assume that an abelian variety has a point.

Theorem 6.1 *Let $\Bbbk$ be an algebraically closed field of characteristic $p \geqslant 0$, and let $\ell \neq p$ be a prime number. Let X be an n–dimensional abelian variety over $\Bbbk$. Then*

$$\operatorname{Aut}(X) \cong X(\Bbbk) \rtimes \operatorname{Aut}(X; P), \tag{6-1}$$

where $X(\Bbbk)$ is the group of $\Bbbk$–points of X, and $\operatorname{Aut}(X; P)$ is the stabilizer of a $\Bbbk$–point $P \in X$ in $\operatorname{Aut}(X)$. Furthermore, $\operatorname{Aut}(X; P)$ is isomorphic to a subgroup of $\mathrm{GL}_{2n}(\mathbb{Z}_\ell)$.

Proof The isomorphism (6-1) is obvious. Note that $\operatorname{Aut}(X; P)$ can be identified with the group of automorphisms that preserve the group structure of X.

Let $T_\ell(X) \cong \mathbb{Z}_\ell^{2n}$ be the Tate module of X; see [57, Section 18] for the definition. Let $\operatorname{Hom}(X, X)$ be the $\mathbb{Z}$–algebra of endomorphisms of the abelian variety X. According to [57, Section 19, Theorem 3], there is an injective homomorphism

$$\operatorname{Hom}(X, X) \hookrightarrow \operatorname{Hom}_{\mathbb{Z}_\ell}(T_\ell(X), T_\ell(X)).$$

The group $\operatorname{Aut}(X; P)$ is the multiplicative group of invertible elements in $\operatorname{Hom}(X, X)$. Thus, it embeds into the group of invertible elements in $\operatorname{Hom}_{\mathbb{Z}_\ell}(T_\ell(X), T_\ell(X))$, which is

$$\operatorname{Aut}_{\mathbb{Z}_\ell}(T_\ell(X), T_\ell(X)) \cong \mathrm{GL}_{2n}(\mathbb{Z}_\ell). \qquad \square$$

Remark 6.2 In the notation of Theorem 6.1, if the characteristic of the field $\Bbbk$ is zero, one can check that $\operatorname{Aut}(X; P)$ is a subgroup of $\mathrm{GL}_{2n}(\mathbb{Z})$. It is well known that this also holds if X is a complex torus of dimension n; see eg [66, Theorem 8.4]. However, one can produce an example of an elliptic curve X over a field of characteristic 2 such that the stabilizer of a point on X is a group of order 24; see [75, Exercise A.1(b)]. Such a group cannot be embedded into $\mathrm{GL}_2(\mathbb{Z})$; see [79, Section 1].

The next two results are consequences of Theorem 6.1. To formulate them, we set

$$J_A(n) = |\mathrm{GL}_{2n}(\mathbb{Z}/4\mathbb{Z})| = (2^{4n} - 2^{2n}) \cdot (2^{4n} - 2^{2n+1}) \cdots (2^{4n} - 2^{4n-1}). \tag{6-2}$$

Note that

$$J_A(n) > (3^{2n} - 1) \cdot (3^{2n} - 3) \cdots (3^{2n} - 3^{2n-1}) = |\mathrm{GL}_{2n}(\mathbb{Z}/3\mathbb{Z})|$$

for every positive integer n.

Corollary 6.3 *Let n be a positive integer. For every field $\Bbbk$, every abelian variety X of dimension n over $\Bbbk$, every finite subgroup $G \subset \operatorname{Aut}(X)$ contains a normal abelian subgroup of index at most $J_A(n)$ that can be generated by at most $2n$ elements. In particular, for every abelian variety X the group $\operatorname{Aut}(X)$ is Jordan.*

Proof Let X be an abelian variety over a field $\Bbbk$ of characteristic $p \geqslant 0$, and let G be a finite subgroup of $\operatorname{Aut}(X)$. It is enough to consider the case when $\Bbbk$ is algebraically closed. Set $\ell = 2$ if $p \neq 2$, and $\ell = 3$ if $p = 2$. Note that the intersection G_X of G with the subgroup $X(\Bbbk) \subset \operatorname{Aut}(X)$ of $\Bbbk$–points of X is abelian and normal in G. Moreover, G_X can be generated by at most $2n$ elements; see eg [57, Section 15]. On the other hand, by Theorem 6.1 the quotient G/G_X is isomorphic to a subgroup of $\mathrm{GL}_{2n}(\mathbb{Z}_\ell)$. Thus, by Lemma 2.13 the order of G/G_X does not exceed

$$\max\{|\mathrm{GL}_{2n}(\mathbb{Z}/3\mathbb{Z})|, |\mathrm{GL}_{2n}(\mathbb{Z}/4\mathbb{Z})|\} = J_A(n). \qquad \square$$

Applying Corollary 6.3 together with Lemma 2.5, we obtain:

Corollary 6.4 *Let p be a prime number, and let n be a positive integer. For every field $\Bbbk$, every abelian variety X of dimension n over $\Bbbk$, every finite subgroup $G \subset \mathrm{Aut}(X)$ contains a normal abelian subgroup A such that A can be generated by at most $2n$ elements, the order of A is coprime to p, and the index of A in G is at most $J_A(n) \cdot |G_p|$, where G_p is a p–Sylow subgroup of G.*

Arguing as in the proof of Corollary 6.3, we can obtain restrictions on stabilizers of points on abelian varieties.

Corollary 6.5 *Let n be a positive integer. For every algebraically closed field $\Bbbk$, every abelian variety X of dimension n over $\Bbbk$, every $\Bbbk$–point P on X, and every finite subgroup G of the stabilizer $\mathrm{Aut}(X;P)$ of P, the order of G is at most $J_A(n)$.*

Proof Let $\Bbbk$ be an algebraically closed field of characteristic $p \geqslant 0$, and let X be an n–dimensional abelian variety over $\Bbbk$. Set $\ell = 2$ if $p \neq 2$, and $\ell = 3$ if $p = 2$. According to Theorem 6.1, the group $\mathrm{Aut}(X;P)$ is isomorphic to a subgroup of $\mathrm{GL}_{2n}(\mathbb{Z}_\ell)$. Therefore, the assertion follows from Lemma 2.13. $\square$

Remark 6.6 In the notation of Corollaries 6.3, 6.4 and 6.5, one can replace the constant $J_A(1)$ by 24; see [75, Theorem III.10.1] and [75, Proposition A.1.2(c)] — actually, in this case the whole group $\mathrm{Aut}(X;P)$ is finite. This bound is stronger than the bound given by Corollaries 6.3, 6.4 and 6.5.

The assertion of Corollary 6.5 does not hold for stabilizers of closed points over algebraically nonclosed fields, as shown by the following example.

Example 6.7 Let $\Bbbk$ be an algebraically closed field of characteristic $p \geqslant 0$, let A be a positive-dimensional abelian variety over $\Bbbk$ (with a chosen group structure), and let m be a positive integer not divisible by p. Choose a primitive m^{th} root of unity $\zeta \in \Bbbk$, and a point $c \in A(\Bbbk)$ whose order m in the group $A(\Bbbk)$ equals m. Consider the action of the group $\Gamma \cong \mathbb{Z}/m\mathbb{Z}$ on $\mathbb{A}^1 \times A$ such that the generator of Γ acts by the transformation

$$(t,a) \mapsto (\zeta t, a + c),$$

where t is a coordinate on $\mathbb{A}^1$, and $a \in A$. Set $\mathscr{X} = (\mathbb{A}^1 \times A)/\Gamma$. Then there is a morphism

$$\phi \colon \mathscr{X} \to \mathbb{A}^1 \cong \operatorname{Spec} \Bbbk[t^m]$$

that fits into the commutative diagram

$$\begin{tikzcd} \mathbb{A}^1 \times A \arrow[r] \arrow[d] & \mathscr{X} \arrow[d, "\phi"] \\ \mathbb{A}^1 \arrow[r] & \mathbb{A}^1 \end{tikzcd}$$

Let X be the scheme-theoretic generic fiber of ϕ. Then X is an abelian variety over the field $\mathbb{K} = \Bbbk(t^m)$. Let Q be a $\Bbbk$–point of A, and let P be the closed point of X corresponding to the image of the section $\mathbb{A}^1 \times \{Q\}$ of the projection $\mathbb{A}^1 \times A \to \mathbb{A}^1$; thus, P is not a $\mathbb{K}$–point of X if $m > 1$. The translation by c defines a faithful action of the group $\mathbb{Z}/m\mathbb{Z}$ on X, and this action preserves the point P. Taking m arbitrarily large, we obtain a series of varieties and their closed points preserved by automorphisms of arbitrarily large finite orders.

7 Varieties of nonnegative Kodaira dimension

In this section we make some general observations on automorphism groups of varieties of nonnegative Kodaira dimension and prove Proposition 1.8. The following assertions (and the arguments that prove them) are well known to experts. We provide their proofs to be self-contained.

Theorem 7.1 (cf [43, Section 2.2]) *Let X be a smooth geometrically irreducible projective variety such that $\kappa(X) = \dim X$. Then for some positive integer n, the rational map defined by the linear system $|nK_X|$ is birational onto its image.*

Proof Since $\kappa(X) = \dim X$, there exists a positive integer m such that $\dim \Phi_m(X) = \dim X$, where $\Phi_m \colon X \dashrightarrow \mathbb{P}^N$ is the rational map given by the linear system $|mK_X|$. By elimination of indeterminacy (see for instance [31, Example II.7.17.3]), there is a commutative diagram

$$\begin{tikzcd} & \widetilde{X} \arrow[dl, "\pi"'] \arrow[dr, "\widetilde{\Phi}_m"] & \\ X \arrow[rr, dashed, "\Phi_m"] & & \mathbb{P}^N \end{tikzcd}$$

Here π is the blow-up of the indeterminacy locus of Φ_m. Let $E \subset \widetilde{X}$ be the exceptional divisor of the morphism π. Then E is an effective Cartier divisor, and

$$\pi^*(mK_X) \sim \widetilde{\Phi}_m^* H + E,$$

where H is a hyperplane in $\mathbb{P}^N$.

Let $W = \widetilde{\Phi}_m(\widetilde{X})$. Since $\widetilde{\Phi}_m$ is surjective, we conclude that $\widetilde{\Phi}_m^*$ induces an injective map

$$H^0(W, \mathcal{O}_W(H|_W)) \hookrightarrow H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(\widetilde{\Phi}_m^* H)).$$

Therefore, one has

$$h^0(X, \mathcal{O}_X(mK_X)) = h^0\big(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(\pi^*(mK_X))\big) \geq h^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(\widetilde{\Phi}_m^* H)) \geq h^0(W, \mathcal{O}_W(H|_W)).$$

Write $r = \dim W = \dim X = \dim \widetilde{X}$. Since $h^0(W, \mathbb{O}_W(m'H|_W))$ is a polynomial in m' for $m' \gg 0$, there exists a positive constant C such that

$$h^0(W, \mathbb{O}_W(m'H|_W)) \geqslant C(m')^r.$$

Hence for $m' \gg 0$ we have

$$h^0(X, \mathbb{O}_X(m'mK_X)) \geqslant C(m')^r. \tag{7-1}$$

Let A be a very ample divisor on X. Consider the exact sequence

$$0 \to \mathbb{O}_X(-A) \to \mathbb{O}_X \to \mathbb{O}_A \to 0.$$

It gives the exact sequence

$$0 \to \mathbb{O}_X(m'mK_X - A) \to \mathbb{O}_X(m'mK_X) \to \mathbb{O}_A(m'mK_X|_A) \to 0.$$

Since $\dim A = r-1$, we know that $h^0(X, \mathbb{O}_A(m'mK_X|_A))$ grows as a polynomial of degree at most $r-1$ in m'. Therefore, from (7-1) we get

$$h^0(X, \mathbb{O}_X(m'mK_X - A)) \neq 0 \quad \text{for } m' \gg 0.$$

Let $F \in |m'mK_X - A|$ be an effective divisor, so that $m'mK_X \sim A + F$. Over the open subset $U = X \setminus F \subset X$, the linear system $|A|$ can be viewed as a linear subsystem of $|m'mK_X|$, where the embedding is given by $L \mapsto L + F$ for $L \in |A|$. By assumption, A is very ample, so the rational map $\Phi_{m'm}$ induced by $|m'mK_X|$ is an embedding on U. Therefore, the map $\Phi_{m'm}$ is birational. □

Lemma 7.2 *Let Γ be a nontrivial connected linear algebraic group over an algebraically closed field. Then Γ contains a subgroup isomorphic either to $\mathbb{G}_{\mathrm{m}}$ or to $\mathbb{G}_{\mathrm{a}}$.*

Proof Let R be the radical of Γ, ie a maximal closed, connected, normal, solvable subgroup of Γ. First, suppose that R is nontrivial. If R is not a torus, then it contains a subgroup isomorphic to $\mathbb{G}_{\mathrm{a}}$ by [76, Lemma 6.3.4]. If R is a torus, then it contains a subgroup isomorphic to $\mathbb{G}_{\mathrm{m}}$. Thus, we may assume that R is trivial, so that Γ is semisimple. In this case Γ is generated by its maximal torus T and a certain collection $\{U_\alpha\}$ of subgroups isomorphic to $\mathbb{G}_{\mathrm{a}}$; see [76, Proposition 8.1.1]. Since Γ is nontrivial, we conclude that either the torus T is positive-dimensional, or the collection $\{U_\alpha\}$ is nonempty. Thus, one finds a subgroup of Γ isomorphic to $\mathbb{G}_{\mathrm{m}}$ or to $\mathbb{G}_{\mathrm{a}}$, respectively. □

Proposition 7.3 (cf [81, Theorem 14.1] and [14, Proposition 7.1.4]) *Suppose that X is a smooth geometrically irreducible projective variety of nonnegative Kodaira dimension over a field $\Bbbk$. Then the group scheme $\mathbf{Aut}_X$ does not contain nontrivial connected linear algebraic subgroups.*

Proof Since a connected algebraic group is geometrically connected, we may assume that the field $\Bbbk$ is algebraically closed. Suppose that $\mathbf{Aut}_X$ contains a nontrivial connected linear algebraic group. Then it contains a linear algebraic group $\mathbb{G}$ isomorphic either to $\mathbb{G}_\mathrm{m}$ or to $\mathbb{G}_\mathrm{a}$; see Lemma 7.2. Hence X is birational to $Y \times \mathbb{P}^1$, where Y is the geometric quotient of some open subset of X by the action of $\mathbb{G}$; see [61] or [73, Lemma A.5]. By Lemma 5.3 one has

$$\kappa(X) = -\infty,$$

which contradicts our assumptions. □

The following corollary of Theorem 7.1 and Proposition 7.3 will be used only in dimensions 1 and 2 in this paper. However, we state it for arbitrary dimension because its proof is essentially the same.

Corollary 7.4 (see [51] and [49, Corollaire 4]; cf [81, Corollary 14.3]) *Let X be a smooth geometrically irreducible projective variety over a field $\Bbbk$ such that $\kappa(X) = \dim X$. Then $\mathbf{Aut}_X$ is a finite group scheme, and $\mathrm{Aut}(X)$ is a finite group.*

Proof We may assume that the field $\Bbbk$ is algebraically closed. For some $m > 0$, the rational map

$$\phi_m \colon X \dashrightarrow Z \subset \mathbb{P}^n$$

defined by the linear system $|mK_X|$ is birational; see Theorem 7.1. This realizes the group scheme $\mathbf{Aut}_X$ as the stabilizer of Z in the linear algebraic group PGL_{n+1}. Therefore, $\mathbf{Aut}_X$ is a group scheme of finite type, and the reduced neutral component $\mathbf{Aut}^0_{X,\mathrm{red}}$ of $\mathbf{Aut}_X$ is a linear algebraic group. Moreover, $\mathbf{Aut}^0_{X,\mathrm{red}}$ is trivial by Proposition 7.3. This implies that $\mathbf{Aut}_X$ is a finite group scheme, and in particular $\mathrm{Aut}(X)$ is a finite group. □

We will need the following general fact concerning algebraic groups.

Theorem 7.5 [14, Lemma 4.3.1] *Let Γ be a connected algebraic group. Suppose that Γ does not contain nontrivial connected linear algebraic subgroups. Then Γ is an abelian variety.*

Corollary 7.6 *Let X be a smooth geometrically irreducible projective variety over a perfect field. Suppose that $\kappa(X) \geqslant 0$. Then $\mathbf{Aut}^0_{X,\mathrm{red}}$ is an abelian variety.*

Proof The connected algebraic group $\mathbf{Aut}^0_{X,\mathrm{red}}$ does not contain nontrivial connected linear algebraic subgroups by Proposition 7.3. Therefore, the required assertion follows from Theorem 7.5. □

Now we are ready to prove Proposition 1.8.

Proof of Proposition 1.8 We may assume that X is defined over an algebraically closed field. Then the group $\mathrm{Aut}^0(X)$ is the group of points of some abelian variety by Corollary 7.6. In particular, it is abelian, On the other hand, the quotient group $\mathrm{Aut}(X)/\mathrm{Aut}^0(X)$ has bounded finite subgroups by Lemma 4.1. Hence the group $\mathrm{Aut}(X)$ is Jordan by Lemma 2.8. □

8 Jordan property for curves

In this section we describe the Jordan properties for automorphism groups of smooth projective curves.

Lemma 8.1 *Let $\Bbbk$ be a field of characteristic $p > 0$, and let C be a smooth geometrically irreducible projective curve of genus g over $\Bbbk$. The following assertions hold.*

(i) *If $g = 0$, then the group $\mathrm{Aut}(C)$ is p–Jordan.*

(ii) *If $g = 1$, then the group $\mathrm{Aut}(C)$ is Jordan.*

(iii) *If the $g \geqslant 2$, then the group $\mathrm{Aut}(C)$ is finite.*

Proof We may assume that $\Bbbk$ is algebraically closed. Now assertion (i) follows from Theorem 1.5. Assertion (ii) is given by Corollary 6.3. Assertion (iii) is given by Corollary 7.4. □

Similarly to the Hurwitz bound over fields of characteristic zero, there exists a bound on the order of the automorphism group of a curve of genus $g \geqslant 2$ over a field of positive characteristic that depends only on g, but not on the characteristic of the field; see [78]. In the case of elliptic curves, the constant arising in Lemma 8.1(ii) is always bounded by 24; see Remark 6.6. For a more explicit version of Lemma 8.1(i), one can use a classification of finite groups acting on $\mathbb{P}^1$.

Theorem 8.2 [19, Theorem 2.1] *Let $\Bbbk$ be a field of characteristic $p > 0$, let $G \subset \mathrm{PGL}_2(\Bbbk)$ be a finite group, and let G_p be a p–Sylow subgroup of G. Then G is one of the following groups:*

(i) *A dihedral group of order $2n$, where $n > 1$ is coprime to p.*

(ii) *One of the groups $\mathfrak{A}_4$, $\mathfrak{S}_4$ or $\mathfrak{A}_5$.*

(iii) *The group $\mathrm{PSL}_2(\boldsymbol{F}_{p^k})$ for some $k \geqslant 1$.*

(iv) *The group $\mathrm{PGL}_2(\boldsymbol{F}_{p^k})$ for some $k \geqslant 1$.*

(v) *A group of the form $G_p \rtimes \mathbb{Z}/n\mathbb{Z}$, where $n \geqslant 1$ is coprime to p, and G_p is a p–subgroup of the additive group of $\Bbbk$.*

Remark 8.3 Let $\Bbbk$ be a field of characteristic $p > 0$, and let $g \in \mathrm{PGL}_2(\Bbbk)$ be a nontrivial element of finite order n coprime to p. Then g is semisimple, and thus is contained in some algebraic torus $T \subset \mathrm{PGL}_2(\Bbbk)$. Moreover, if $n \geqslant 3$, then the centralizer of g coincides with T. In particular, every finite group that commutes with g is cyclic. Moreover, the order of such a group is coprime to p, because an algebraic torus over $\Bbbk$ does not contain elements of order p. This provides additional restrictions for the possible structure of the groups of type (v) in Theorem 8.2.

Remark 8.4 Any group of one of types (i)–(iv) in the notation of Theorem 8.2 can be generated by two elements. Indeed, this clearly holds for a dihedral group. Also, it is well known that the symmetric group on n elements is generated by a cycle of length 2 and a cycle of length n. Furthermore, any nonabelian finite simple group is generated by two elements; see eg [38]. In particular, this holds for an alternating group $\mathfrak{A}_5$. A similar assertion for the group $\mathfrak{A}_4$ can be checked directly. The groups $\mathrm{PSL}_2(\boldsymbol{F}_{p^k})$ are simple except for $\mathrm{PSL}_2(\boldsymbol{F}_2) \cong \mathfrak{S}_3$ and $\mathrm{PSL}_2(\boldsymbol{F}_3) \cong \mathfrak{A}_4$; see [84, Section 3.3.1]. Thus, any group of this type can be generated by two elements. Finally, for the group $\mathrm{PGL}_2(\boldsymbol{F}_{p^k})$, the required assertion follows from [82].

Now we can give an explicit bound for the index of a normal abelian subgroup in a finite group acting on $\mathbb{P}^1$.

Lemma 8.5 *For every prime p and every field $\Bbbk$ of characteristic p, every finite subgroup*

$$G \subset \mathrm{Aut}(\mathbb{P}^1) \cong \mathrm{PGL}_2(\Bbbk)$$

contains a characteristic cyclic subgroup of order coprime to p and index at most $60|G_p|^3$, where G_p is a p–Sylow subgroup of G.

Proof Let $\Bbbk$ be a field of characteristic $p > 0$, let $G \subset \mathrm{PGL}_2(\Bbbk)$ be a finite group, and let G_p be a p–Sylow subgroup of G. Then G is a group of one of types (i)–(v) in the notation of Theorem 8.2.

If G is a dihedral group of type (i), then the commutator subgroup of G is a characteristic cyclic subgroup of index at most 4 in G, and its order is coprime to p. If G is one of the groups $\mathfrak{A}_4$, $\mathfrak{S}_4$ or $\mathfrak{A}_5$, then the trivial subgroup has index at most 60 in G. If $G \cong \mathrm{PSL}_2(\boldsymbol{F}_{p^k})$ and $p \neq 2$, then the trivial subgroup of G has index

$$|G| = \tfrac{1}{2}|\mathrm{SL}_2(\boldsymbol{F}_{p^k})| = \tfrac{1}{2}p^k(p^{2k}-1) < p^{3k} = |G_p|^3.$$

If $G \cong \mathrm{PSL}_2(\boldsymbol{F}_{2^k})$, then the trivial subgroup of G has index

$$|G| = |\mathrm{SL}_2(\boldsymbol{F}_{2^k})| = 2^k(2^{2k}-1) < 2^{3k} = |G_2|^3.$$

If $G \cong \mathrm{PGL}_2(\boldsymbol{F}_{p^k})$, then the trivial subgroup of G has index

$$|G| = p^k(p^{2k}-1) < p^{3k} = |G_p|^3.$$

Now suppose that G is of type (v). Then G contains a cyclic group $L \cong \mathbb{Z}/n\mathbb{Z}$ such that $G \cong G_p \rtimes L$, and $G_p \cong (\mathbb{Z}/p\mathbb{Z})^m$ for some positive integer m. Let L' be the centralizer of G_p in L. By Lemma 2.11, the index of L' in L is at most p^m-1. Thus, L' is a cyclic subgroup of G of index at most

$$|G_p|(p^m-1) < |G_p|^2,$$

and the order of L' is coprime to p.

We claim that L' is a characteristic subgroup of G. Indeed, G_p is characteristic in G by Example 2.2. Thus its centralizer $C(G_p)$ in G is also a characteristic subgroup of G. On the other hand, one has

$$C(G_p) \cong G_p \times L'.$$

Therefore, L' is characteristic in $C(G_p)$ (cf Example 2.2), and hence also characteristic in G. $\square$

9 Cremona group

In this section we prove Theorem 1.6. Our proof is (nearly) identical to that presented in [70].

We start by recalling the assertion proved in [70, Lemma 5.2]. We provide its detailed proof for the convenience of the reader.

Lemma 9.1 *Let n be a positive integer, and let $\Bbbk$ be a field of characteristic p such that p does not divide n and $\Bbbk$ contains a primitive n^{th} root of 1. Let S be a smooth geometrically irreducible projective surface over $\Bbbk$, and let $H \subset \operatorname{Aut}(S)$ be a finite group. Suppose that there exists an H–equivariant conic bundle $\phi\colon S \to C$. Denote by F the subgroup of H that consists of automorphisms that are fiberwise with respect to ϕ; see Definition 5.16. Let $R \subset F$ be a cyclic group of order n, and let $\alpha \in H$ be an element normalizing R. Then α^2 commutes with R.*

Proof Let S_η be the fiber of ϕ over the scheme-theoretic generic point η of C. Then S_η is a smooth conic over the field $\mathbb{K} = \Bbbk(C)$. The group F can be considered as a subgroup of $\operatorname{Aut}(S_\eta) \subset \operatorname{PGL}_2(\overline{\mathbb{K}})$. Thus, the cyclic group R has exactly two fixed $\overline{\mathbb{K}}$–points on the conic $S_{\eta,\overline{\mathbb{K}}} \cong \mathbb{P}^1_{\overline{\mathbb{K}}}$. Denote them by P_+ and P_-. Recall that the action of R in the Zariski tangent spaces $T_{P_\pm}(S_{\eta,\overline{\mathbb{K}}})$ is faithful by Theorem 3.7. Thus, there is a primitive n^{th} root ζ of 1 such that a generator g of R acts on $T_{P_\pm}(S_{\eta,\overline{\mathbb{K}}}) \cong \overline{\mathbb{K}}$ by $\zeta^{\pm 1}$. For any positive integer r, the element g^r acts on $T_{P_\pm}(S_{\eta,\overline{\mathbb{K}}})$ by $\zeta^{\pm r}$; thus, an element of R is uniquely defined by its action in any of these two Zariski tangent spaces.

One can consider α as an automorphism of the scheme $S_{\eta,\overline{\mathbb{K}}}$ (of nonfinite type) over the field $\Bbbk$. Write $\alpha g \alpha^{-1} = g^r$. Since the fixed points of g^r on $S_{\eta,\overline{\mathbb{K}}}$ are P_+ and P_-, we see that $\alpha(P_+) \in \{P_+, P_-\}$. Hence $\alpha^2(P_+) = P_+$, and

$$g^{r^2} = \alpha^2 g \alpha^{-2}$$

acts on $T_{P_+}(S_{\eta,\overline{\mathbb{K}}})$ by $\zeta^{r^2} = \bar{\alpha}^2(\zeta)$, where $\bar{\alpha}$ is the automorphism of the field $\mathbb{K}$ over $\Bbbk$ induced by α. However, ζ is contained in $\Bbbk$, so $\bar{\alpha}(\zeta) = \zeta$. Therefore, we see that r^2 is congruent to 1 modulo n, which means that α^2 commutes with g. □

The next lemma allows one to deal with finite groups acting on surfaces with conic bundle structure. It also provides an additional observation concerning the number of generators of abelian subgroups in such groups.

Lemma 9.2 *Let $\Bbbk$ be a field of characteristic $p > 0$, and let S be a smooth geometrically rational projective surface over $\Bbbk$. Let $G \subset \operatorname{Aut}(S)$ be a finite group, and let $\phi\colon S \to C$ be a G–equivariant conic bundle. Then G contains a normal abelian subgroup generated by at most two elements that has order coprime to p and index at most $7200|G_p|^3$, where G_p is a p–Sylow subgroup of G.*

Proof We can assume that the field $\Bbbk$ is algebraically closed; in particular, one has $C \cong \mathbb{P}^1$.

There is an exact sequence of groups

$$1 \to F \to G \to \overline{G} \to 1,$$

where the action of F is fiberwise with respect to ϕ, while $\bar{G}$ acts faithfully on $\mathbb{P}^1$. By Lemma 8.5 there exists a normal cyclic subgroup $\bar{H}$ in $\bar{G}$ of order coprime to p whose index in $\bar{G}$ does not exceed $60|\bar{G}_p|^3$, where $\bar{G}_p$ is a p–Sylow subgroup of $\bar{G}$. Let H be the preimage of $\bar{H}$ in G, so that there is an exact sequence of groups

$$1 \to F \to H \to \bar{H} \to 1.$$

In particular, H is a normal subgroup of G.

By Lemma 8.5 there exists a characteristic cyclic subgroup R of order coprime to p and index at most $60|F_p|^3$ in F, where F_p is a p–Sylow subgroup of F. The group H acts on F by conjugation, and this action preserves the characteristic subgroup R.

Pick an element α of H such that its image $\bar{\alpha}$ in $\bar{H}$ generates $\bar{H}$. Since $|\bar{H}|$ is coprime to p, the element $\bar{\alpha}^p$ generates $\bar{H}$ as well. Hence, replacing α by its appropriate power, we may assume that the order of α is coprime to p. Since α normalizes the subgroup R, and the (algebraically closed) field $\Bbbk$ contains a primitive root of 1 or degree $|R|$, by Lemma 9.1 the element α^2 commutes with R. Let $\tilde{A}$ be the subgroup of H generated by R and α^2. Then $\tilde{A}$ is abelian, and its order is coprime to p. Note that the subgroup $\bar{H}_{\bar{\alpha}^2} \subset \bar{H}$ generated by $\bar{\alpha}^2$ either coincides with $\bar{H}$, or has index 2 in $\bar{H}$.

The subgroup R is characteristic in F, and hence normal in G. Also, the subgroup $\bar{H}_{\bar{\alpha}^2}$ is characteristic in $\bar{H}$, because the cyclic group $\bar{H}$ contains at most one subgroup of given order; cf Example 2.3. Hence $\bar{H}_{\bar{\alpha}^2}$ is normal in $\bar{G}$. Still we cannot conclude from this that $\tilde{A}$ is normal in G. However, let A be the intersection of all subgroups of G conjugate to $\tilde{A}$. Then A is an abelian group of order coprime to p, and it is normal in G. Since $\tilde{A}$ is generated by two elements, the same holds for A. Also, we know that A contains the subgroup R, because R is normal in G. Similarly, the image of A in $\bar{H}$ coincides with $\bar{H}_{\bar{\alpha}^2}$, because $\bar{H}_{\bar{\alpha}^2}$ is normal in $\bar{G}$. Therefore, the index of A in G is

$$\frac{|G|}{|A|} = \frac{|\bar{G}|}{|\bar{H}_{\bar{\alpha}^2}|} \cdot \frac{|F|}{|A \cap F|} \leqslant 2\frac{|\bar{G}|}{|\bar{H}|} \cdot \frac{|F|}{|R|} \leqslant 7200\,|\bar{G}_p|^3 \cdot |F_p|^3 = 7200\,|G_p|^3. \qquad \square$$

Finally, we are ready to prove our main result.

Proof of Theorem 1.6 Let $\Bbbk$ be a field of characteristic $p > 0$. We may assume that $\Bbbk$ is algebraically closed. Let G be a finite subgroup of $\mathrm{Bir}(\mathbb{P}^2)$. By Theorem 5.14 there exists a surface S with a faithful regular action of G, such that S is either a del Pezzo surface or a G–equivariant conic bundle.

Suppose that S is a del Pezzo surface. By Theorem 5.15 the linear system $|-3K_S|$ defines an embedding $\iota\colon S \hookrightarrow \mathbb{P}^N$, where $N \leqslant 54$. Since the linear system $|-3K_S|$ is G–invariant, we see that the map ι is G–equivariant, and thus G can be realized as a subgroup of $\mathrm{PGL}_{55}(\Bbbk)$. By Corollary 2.16 there exists a constant J_{dP}, independent of p, $\Bbbk$ and G, such that G contains a normal abelian subgroup of order coprime to p and index at most $J_{\mathrm{dP}} \cdot |G_p|^3$, where G_p is a p–Sylow subgroup of G.

Now we are left with the case when S is a G–equivariant conic bundle, which is covered by Lemma 9.2. $\square$

10 Jordan property for surfaces

In this section we prove Theorem 1.7. The proof of the following assertion is identical to that in [86].

Lemma 10.1 *Let $\Bbbk$ be an algebraically closed field of characteristic $p > 0$. Let A be an abelian variety of positive dimension n over $\Bbbk$, and let $S = A \times \mathbb{P}^1$. Then the group $\mathrm{Bir}(S)$ is not generalized p–Jordan.*

Proof Choose a prime number $\ell \neq p$. Fix a $\Bbbk$–point $O \in A$ to define a group structure on A. For every $\Bbbk$–point $x \in A$, let $t_x\colon A \to A$ be the translation by x.

Let $\mathscr{L}_0$ be an ample line bundle on A, and set $\mathscr{L} = \mathscr{L}_0^{\otimes \ell}$. Let $K(\mathscr{L})$ denote the group of $\Bbbk$–points $x \in A$ such that $t_x^*\mathscr{L} \cong \mathscr{L}$, and let $K(\mathscr{L})_\ell$ be its subgroup that consists of all elements whose order is a power of ℓ. According to [57, Section 23, Theorem 3], the group $K(\mathscr{L})$ contains all the ℓ–torsion $\Bbbk$–points of A. Since $\ell \neq p$, the group of ℓ–torsion $\Bbbk$–points of A is isomorphic to $(\mathbb{Z}/\ell\mathbb{Z})^{2n}$; see eg [57, Section 6]. Thus, the group $K(\mathscr{L})_\ell$ is nontrivial. Furthermore, since $\mathscr{L}$ is ample, the groups $K(\mathscr{L})$ and $K(\mathscr{L})_\ell$ are finite by [57, Section 23, Theorem 4].

Let $\mathrm{Tot}(\mathscr{L})$ be the total space of $\mathscr{L}$. For every $x \in K(\mathscr{L})$, there exists a (nonunique) fiberwise linear isomorphism $\tilde{t}_x^*\colon \mathrm{Tot}(\mathscr{L}) \to \mathrm{Tot}(\mathscr{L})$ that fits into a commutative diagram

$$\begin{tikzcd}
\mathrm{Tot}(\mathscr{L}) \arrow[r, "\tilde{t}_x^*"] \arrow[d] & \mathrm{Tot}(\mathscr{L}) \arrow[d] \\
A \arrow[r, "t_x"] & A
\end{tikzcd}$$

Moreover, there exists an exact sequence of groups

$$1 \to \Bbbk^* \to \mathscr{G}(\mathscr{L}) \to K(\mathscr{L}) \to 1,$$

where $\mathscr{G}(\mathscr{L})$ is a central extension of $K(\mathscr{L})$ acting by automorphisms of $\mathrm{Tot}(\mathscr{L})$ of the form $\tilde{t}_x^*$. The preimage of $K(\mathscr{L})_\ell$ in $\mathscr{G}(\mathscr{L})$ contains a finite group $\mathscr{H}_\ell$ that is a central extension of $K(\mathscr{L})_\ell$ by a cyclic group $\mathfrak{Z}$ of order ℓ^k for some nonnegative integer k.

For every two elements $x, y \in K(\mathscr{L})_\ell$, let (x, y) be their commutator pairing; in other words, write

$$\tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1} = (x, y)z,$$

where $\tilde{x}$ and $\tilde{y}$ are (arbitrary) preimages of x and y in $\mathscr{H}_\ell$, and z is a generator of the cyclic group $\mathfrak{Z} \subset \mathscr{H}_\ell$. According to [57, Section 23, Theorem 4], the commutator pairing is nontrivial on $K(\mathscr{L})_\ell$. Hence $\mathscr{H}_\ell$ is a nonabelian group whose order is a power of the prime number ℓ. Thus every abelian subgroup of $\mathscr{H}_\ell$ has index at least ℓ in $\mathscr{H}_\ell$.

It remains to notice that the $\mathbb{A}^1$–bundle $\mathrm{Tot}(\mathscr{L}) \to A$ is birational to $S = A \times \mathbb{P}^1$. Therefore, the group $\mathrm{Bir}(S)$ contains a group $\mathscr{H}_\ell$ for every prime $\ell \neq p$, and the required assertion follows. □

Lemma 10.2 *Let $\Bbbk$ be an algebraically closed field of characteristic $p > 0$. Let C be a smooth geometrically irreducible projective curve of genus at least 2 over $\Bbbk$, and set $S = C \times \mathbb{P}^1$. Then the group $\operatorname{Bir}(S)$ is p–Jordan.*

Proof According to Lemma 5.17, the group $\operatorname{Bir}(S)$ fits into the exact sequence

$$1 \to \operatorname{Bir}(S)_\phi \to \operatorname{Bir}(S) \to \Gamma,$$

where $\operatorname{Bir}(S)_\phi \subset \mathrm{PGL}_2(\Bbbk(C))$ and $\Gamma \subset \operatorname{Aut}(C)$. Since the genus of C is at least 2, the group Γ is finite by Lemma 8.1(iii). On the other hand, the group $\mathrm{PGL}_2(\Bbbk(C))$, and thus also the group $\operatorname{Bir}(S)_\phi$, is p–Jordan by Corollary 2.16. Therefore, the required assertion follows from Lemma 2.8. □

Now we can prove the main result of this section.

Proof of Theorem 1.7 Assertion (i) is given by Lemma 10.1. In the rest of the proof we may assume that S is smooth and projective by Corollary 3.4. If $\kappa(S) = -\infty$, then S is birational to a product $C \times \mathbb{P}^1$ for some irreducible smooth projective curve C; see Theorem 5.8(i). Therefore, if C is not an elliptic curve, then it is either rational, or has genus at least 2, and so the group $\operatorname{Bir}(S)$ is p–Jordan by Theorem 1.6 and Lemma 10.2. On the other hand, in each of these cases $\operatorname{Bir}(S)$ is not Jordan because it contains a subgroup isomorphic to $\operatorname{Aut}(\mathbb{P}^1) \cong \mathrm{PGL}_2(\Bbbk)$ which is not Jordan. This proves assertion (ii). Thus, we may assume that the Kodaira dimension of S is nonnegative, and replace S by its minimal model. Then $\operatorname{Bir}(S) = \operatorname{Aut}(S)$ by Lemma 5.6, so that assertion (iii) follows from Proposition 1.8. □

Remark 10.3 There are alternative ways that do not use Proposition 1.8 to prove Theorem 1.7(iii) for many birational classes of surfaces appearing in the Kodaira–Enriques classification. For $\kappa(S) = 0$ the assertion follows from Proposition 1.9 (which we will prove later in Section 11 independently of Theorem 1.7). For elliptic surfaces with $\kappa(S) = 1$, the assertion is given by [26, Corollary 1.6]. It was communicated to us by Yi Gu that a result similar to [26, Corollary 1.6] can be established also for quasielliptic surfaces with $\kappa(S) = 1$. Finally, for $\kappa = 2$ the assertion follows from Corollary 7.4.

11 Surfaces of zero Kodaira dimension

In this section we study automorphism groups of surfaces of zero Kodaira dimension and prove Propositions 1.9 and 1.10. We start with the case of $K3$ and Enriques surfaces.

Lemma 11.1 *There exists a constant B_{K3} such that for every algebraically closed field $\Bbbk$, every surface S over $\Bbbk$ such that S is either a $K3$ surface or an Enriques surface, and for every finite subgroup $G \subset \operatorname{Aut}(S)$, the order of G is at most B_{K3}.*

Proof Let S be either a $K3$ surface or an Enriques surface over a field $\Bbbk$ of characteristic p. By [65, Corollary 2.8] we may assume that $p > 0$. Set $\ell = 2$ if $p \neq 2$, and $\ell = 3$ if $p = 2$. Set

$$H(S) = H^2_{\text{ét}}(S, \mathbb{Z}_\ell)/T,$$

where T is the torsion subgroup of $H^2_{\text{ét}}(S,\mathbb{Z}_\ell)$; note that T is trivial in the $K3$ case. Then $H(S)\cong\mathbb{Z}_\ell^b$, where $b=22$ if S is a $K3$ surface (see [44, Section 7.2]), and $b=10$ if S is an Enriques surface (see [44, Section 7.3]).

Consider the representation

$$\rho\colon \mathrm{Aut}(S)\to \mathrm{GL}\big(\mathrm{H}(S)\big).$$

By Lemma 2.13, the order of every finite subgroup in the image of ρ is bounded by the constant $J_A(11)$; see (6-2). On the other hand, the kernel of the homomorphism ρ is trivial in the case when S is a $K3$ surface; see [37, Theorem 1.4]; cf [59, Corollary 2.5]. This kernel has order at most 4 if S is an Enriques surface; see [21, Theorem]. □

Next, we deal with hyperelliptic and quasihyperelliptic surfaces. Recall that a minimal surface S over an algebraically closed field $\Bbbk$ (of arbitrary characteristic) is called *hyperelliptic*, if $\kappa(S)=0$, and the fibers of the Albanese map of S are smooth elliptic curves. Similarly, S is called *quasihyperelliptic* if $\kappa(S)=0$ and the fibers of the Albanese map of S are singular rational curves. The latter case is possible only if the characteristic of $\Bbbk$ equals 2 or 3.

Lemma 11.2 *There exists a constant B_{hyp} with the following property. Let $\Bbbk$ be an algebraically closed field, let S be a hyperelliptic or a quasihyperelliptic surface over $\Bbbk$, and let G be a (possibly infinite) subgroup of $\mathrm{Aut}(S)$. Then the following assertions hold.*

(i) *The group G contains a normal abelian subgroup of index at most B_{hyp}.*

(ii) *If G fixes a $\Bbbk$–point on S, then $|G|\leqslant B_{\text{hyp}}$.*

Proof Let S be a hyperelliptic or a quasihyperelliptic surface over an algebraically closed field $\Bbbk$. The Albanese morphism $\phi\colon S\to C$ maps S to an elliptic curve C; see [1, Theorem 8.6]. Furthermore, there exists an elliptic curve E, a curve F that is either an elliptic curve or a cuspidal rational curve, and a finite group scheme Γ acting faithfully on E by translations and acting faithfully on F, such that S is included in the commutative diagram

$$\begin{tikzcd} E\times F \arrow[r] \arrow[d] & S \arrow[d, "\phi"] \\ E \arrow[r] & C \end{tikzcd}$$

We refer the reader to [9, Section 3] and [8, Section 2] for details. In particular, this construction implies that the group $C(\Bbbk)$ of $\Bbbk$–points of the elliptic curve $C\cong E/\Gamma$ is a normal subgroup of $\mathrm{Aut}(S)$, and its action on S agrees with the action of $C(\Bbbk)$ on C by translations.

By the classification of automorphism groups of hyperelliptic and quasihyperelliptic surfaces provided in [50], there is a constant B_{hyp} independent of $\Bbbk$ and S such that the index of $C(\Bbbk)$ in $\mathrm{Aut}(S)$ does not exceed B_{hyp}. This means that every subgroup G of $\mathrm{Aut}(S)$ has a normal abelian subgroup of index at most B_{hyp}, which gives assertion (i).

Now suppose that a subgroup $G \subset \mathrm{Aut}(S)$ fixes a $\Bbbk$–point on S. Since ϕ is equivariant with respect to $\mathrm{Aut}(S)$, we conclude that G acts on C with a fixed $\Bbbk$–point. Therefore, the intersection of G with $C(\Bbbk)$ is trivial. Hence $|G| \leqslant B_{\mathrm{hyp}}$, which proves assertion (ii). □

Proof of Proposition 1.9 Let S be a geometrically irreducible algebraic surface of Kodaira dimension 0 over a field $\Bbbk$ of characteristic p. We may assume that $\Bbbk$ is algebraically closed. By Corollary 3.4 we can also assume that S is smooth and projective. Furthermore, by [65, Proposition 1.6] it is enough to consider the case when $p > 0$.

We can replace S by its minimal model, so that $\mathrm{Bir}(S) = \mathrm{Aut}(S)$ by Lemma 5.6. By Theorem 5.8(ii), we need to provide bounds for the indices of normal abelian subgroups of $\mathrm{Aut}(S)$ in the cases when S is a $K3$ surface, an Enriques surface, an abelian surface, a hyperelliptic surface, or a quasihyperelliptic surface. In the first two cases this is done by Lemma 11.1. In the third case this is done by Corollary 6.3. In the last two cases this is done by Lemma 11.2(i). □

Proof of Proposition 1.10 Let S be a smooth irreducible projective surface of Kodaira dimension 0 over a field $\Bbbk$ of characteristic p, and let P be a $\Bbbk$–point on S. Replacing the surface S by the surface $S_{\bar{\Bbbk}}$, and the $\Bbbk$–point P by the $\bar{\Bbbk}$–point $P_{\bar{\Bbbk}}$ of $S_{\bar{\Bbbk}}$, we may assume that the field $\Bbbk$ is algebraically closed. By [65, Proposition 1.3] it is enough to deal with the case when $p > 0$.

Consider the minimal model S' of the surface S. Then $\mathrm{Aut}(S) \subset \mathrm{Bir}(S')$; hence by Lemma 5.6 there is an embedding $\mathrm{Aut}(S) \subset \mathrm{Aut}(S')$. Consider the birational morphism $\pi : S \to S'$. Since the minimal model S' is unique by Lemma 5.6, the morphism π is equivariant with respect to the group $\mathrm{Aut}(S)$. Therefore, the image $\pi(P)$ of the point P is invariant under the group $\mathrm{Aut}(S; P)$. Thus we can assume from the very beginning that the surface S is minimal. By Theorem 5.8(ii), we need to provide bounds for the order of $\mathrm{Aut}(S; P)$ in the cases when S is a $K3$ surface, an Enriques surface, an abelian surface, a hyperelliptic surface, or a quasihyperelliptic surface. In the first two cases this is done by Lemma 11.1. In the third case this is done by Corollary 6.5. In the last two cases this is done by Lemma 11.2(ii). □

There is a partial analog of Proposition 1.10 that is valid over arbitrary fields, suggested to us by Yi Gu.

Definition 11.3 Let X be an algebraic variety over a field $\Bbbk$, and let P be a closed point of X. The *inertia group* $\mathrm{Ine}(X; P)$ of P is the kernel of the action of the stabilizer $\mathrm{Aut}(X; P)$ on the residue field $\Bbbk(P)$.

Lemma 11.4 *Let X be an algebraic variety over a field $\Bbbk$, and let P be a closed point of X. Let P_1 be one of the $\bar{\Bbbk}$–points of $P_{\bar{\Bbbk}}$. Then $\mathrm{Ine}(X; P) \subset \mathrm{Aut}(X_{\bar{\Bbbk}}; P_1)$.*

Proof The action of $\mathrm{Ine}(X; P) \subset \mathrm{Aut}(X_{\bar{\Bbbk}})$ on

$$P_{\bar{\Bbbk}} \cong \mathrm{Spec}(\Bbbk(P) \otimes_{\Bbbk} \bar{\Bbbk})$$

is trivial. □

Corollary 11.5 *There exists a constant B such that for every field $\Bbbk$, every smooth geometrically irreducible projective surface S of Kodaira dimension 0 over $\Bbbk$, every closed point $P \in S$, and every finite subgroup $G \subset \mathrm{Ine}(S; P)$ the order of the group G is at most B.*

Proof Apply Proposition 1.10 together with Lemma 11.4. □

12 Fixed points in arbitrary dimension

In this section we prove Theorem 1.11. We start by recalling the rigidity theorem for projective varieties; see [72, Section III.4.3] or [14, Lemma 3.3.3].

Theorem 12.1 *Let U, V and W be varieties over an algebraically closed field $\Bbbk$. Suppose that U is irreducible, and V is irreducible and projective. Let*

$$f: U \times V \to W$$

be a morphism. Suppose that for some $\Bbbk$–point $u_0 \in U$ the subvariety

$$\{u_0\} \times V \subset U \times V$$

is mapped to a point by f. Then for every $\Bbbk$–point $u \in U$, the subvariety $\{u\} \times V$ is also mapped to a point.

Corollary 12.2 *Let X be a geometrically irreducible projective variety over a field $\Bbbk$, and let $\mathscr{A} \subset \mathbf{Aut}_X$ be a positive-dimensional abelian variety. Then $\mathscr{A}$ has no fixed (closed) points on X.*

Proof Suppose that $\mathscr{A}$ acts on X with a fixed closed point P. Since $\mathscr{A}$ is (geometrically) connected, and $P_{\bar{\Bbbk}}$ is a finite union of $\bar{\Bbbk}$–points, one can see that $\mathscr{A}_{\bar{\Bbbk}}$ acts on $X_{\bar{\Bbbk}}$ with a fixed point as well. Hence, we may assume that the field $\Bbbk$ is algebraically closed.

The action of $\mathscr{A}$ on X is given by a morphism

$$\Psi: \mathscr{A} \times X \to X.$$

The image $\Psi(\mathscr{A} \times \{P\})$ is a point. On the other hand, since the action of $\mathscr{A}$ on X is nontrivial, for a general $\Bbbk$–point $Q \in X$ the image $\Psi(\mathscr{A} \times \{Q\})$ is not a point. This is impossible by Theorem 12.1 because $\mathscr{A}$ is projective. □

We will need the following simple but convenient fact.

Lemma 12.3 *Let Z and X be projective schemes over a field $\Bbbk$, and let $\pi: Z \to X$ be a morphism such that every fiber of π is finite. Then there is a constant $C = C(Z, X, \pi)$ such that the number of $\Bbbk$–points in every fiber of π is at most C.*

Proof The number of irreducible components of Z is finite, so to bound the number of $\Bbbk$–points in the fibers we may replace Z by its irreducible component. Furthermore, we can assume that Z and X are reduced (so that they are projective varieties). Since Z is projective, the morphism π is projective; see eg [47, Corollary 3.3.32(e)]. Thus we conclude that π is a finite morphism; see for instance [47, Corollary 4.4.7]. By generic flatness (see for instance [77, Tag 052B]), there exists a dense open subset $X^0 \subset X$ such that the restriction π^0 of π to $Z^0 = \pi^{-1}(X^0)$ is flat. Thus the number of $\Bbbk$–points in the fibers of π^0 is bounded by the degree of π^0; see [31, Corollary III.9.10]. Replacing Z and X by $Z \setminus Z^0$ and $X \setminus X^0$ and proceeding by Noetherian induction, we obtain the assertion of the lemma. □

Proof of Theorem 1.11 The proof is identical to that of [65, Theorem 1.5]. The reduced neutral component $\mathbf{Aut}^0_{X,\mathrm{red}}$ of the group scheme $\mathbf{Aut}_X$ is an abelian variety by Corollary 7.6. Let $\mathbf{Aut}^0_{X;P}$ be the stabilizer of P in $\mathbf{Aut}^0_{X,\mathrm{red}}$. Then $\mathbf{Aut}^0_{X;P}$ is a group scheme of finite type over $\Bbbk$ (note however that it may be nonreduced and not connected). Denote by $\mathrm{Aut}^0(X;P)$ the group of $\Bbbk$–points of $\mathbf{Aut}^0_{X;P}$, so that $\mathrm{Aut}^0(X;P)$ is the stabilizer of P in $\mathrm{Aut}^0(X)$.

We claim that the group scheme $\mathbf{Aut}^0_{X;P}$ is finite. Indeed, suppose that $\mathbf{Aut}^0_{X;P}$ is infinite. Then it contains some positive-dimensional abelian variety $\mathscr{A}$. Thus $\mathscr{A}$ acts on X with the fixed point P. This is impossible by Corollary 12.2.

Now we know that the group scheme $\mathbf{Aut}^0_{X;P}$ is finite. We claim that the order of the group $\mathrm{Aut}^0(X;P)$ is bounded by some constant $C(X)$ that does not depend on P. Consider the incidence relation

$$Z = \{(\sigma, Q) \mid \sigma(Q) = Q\} \subset \mathbf{Aut}^0_{X,\mathrm{red}} \times X,$$

and denote by $\pi : Z \to X$ the projection to the second factor. Then Z is a projective scheme, and a fiber of π over a point Q is exactly the group scheme $\mathbf{Aut}^0_{X;Q}$. Thus, the fibers of π are finite. Therefore, according to Lemma 12.3, the number of $\Bbbk$–points in the fibers of π, ie the order of the groups $\mathrm{Aut}^0(X;Q)$, is bounded by a constant $C(X)$.

Finally, we recall from Lemma 4.1 that the quotient $\mathrm{Aut}(X)/\mathrm{Aut}^0(X)$ has bounded finite subgroups. Hence the orders of the finite subgroups of the group

$$\mathrm{Aut}(X;P)/\mathrm{Aut}^0(X;P) \subset \mathrm{Aut}(X)/\mathrm{Aut}^0(X)$$

are bounded by some constant $B(X)$ that does not depend on P. This implies the required assertion about the group $\mathrm{Aut}(X;P)$. □

13 Nilpotent groups

In this section we study finite nilpotent subgroups of birational automorphism groups and prove Theorem 1.15. Let us start with the case of ruled surfaces over elliptic curves.

Lemma 13.1 *Let $\Bbbk$ be a field of characteristic $p > 0$, and let S be a geometrically irreducible algebraic surface over $\Bbbk$ birational to $E \times \mathbb{P}^1$, where E is an elliptic curve. Let $G \subset \mathrm{Bir}(S)$ be a finite subgroup, and let G_p denote a p–Sylow subgroup of G. Then G contains either a normal abelian subgroup of order coprime to p and index at most $2^{15} \cdot 3^5 \cdot 5^4 \cdot |G_p|^{15}$, or a normal nilpotent subgroup of class at most 2, order coprime to p, and index at most $2^9 \cdot 3^2 \cdot 5 \cdot |G_p|^3$.*

Proof We may assume that $\Bbbk$ is algebraically closed. According to Lemma 5.17, the group $\mathrm{Bir}(S)$ fits into the exact sequence

$$1 \to \mathrm{Bir}(S)_\phi \to \mathrm{Bir}(S) \to \Gamma,$$

where $\mathrm{Bir}(S)_\phi \subset \mathrm{PGL}_2(\Bbbk(E))$ and $\Gamma \subset \mathrm{Aut}(E)$. Thus, we obtain an exact sequence of finite groups

$$1 \to F \to G \to \bar{G} \to 1,$$

where F is a subgroup of $\mathrm{PGL}_2(\Bbbk(E))$, and $\bar{G}$ acts faithfully on E. By Corollary 6.4 and Remark 6.6 there exists a normal subgroup $\bar{H}$ in $\bar{G}$ such that $\bar{H}$ is generated by at most two elements, the order of $\bar{H}$ is coprime to p, and the index of $\bar{H}$ in $\bar{G}$ does not exceed $24|\bar{G}_p|$, where $\bar{G}_p$ is a p–Sylow subgroup of $\bar{G}$. Let H be the preimage of $\bar{H}$ in G, so that there is an exact sequence of groups

$$1 \to F \to H \to \bar{H} \to 1.$$

In particular, H is a normal subgroup of G. By Lemma 8.5 there exists a characteristic cyclic subgroup R of order n coprime to p and index at most $60|F_p|^3$ in F, where F_p is a p–Sylow subgroup of F.

Suppose that $n \leqslant 2$. Then $|F| \leqslant 120|F_p|^3$. If F is a group of one of types (i)–(iv) in the notation of Theorem 8.2, then it is generated by at most two elements by Remark 8.4. Hence H contains a characteristic abelian subgroup A of order coprime to p and index at most $120^4 \cdot |H_p|^{13}$ by Lemma 2.9. Thus A is normal in G, and its index in G is at most

$$24 \cdot 120^4 \cdot |\bar{G}_p| \cdot |H_p|^{13} \leqslant 2^{15} \cdot 3^5 \cdot 5^4 \cdot |G_p|^{13} \leqslant 2^{15} \cdot 3^5 \cdot 5^4 \cdot |G_p|^{15}.$$

If F is a group of type (v) in the notation of Theorem 8.2, then H contains a characteristic abelian subgroup A of order coprime to p and index at most $120^4 \cdot |H_p|^{15}$ by Lemma 2.12. Thus A is normal in G, and its index in G is at most

$$24 \cdot 120^4 \cdot |\bar{G}_p| \cdot |H_p|^{15} \leqslant 2^{15} \cdot 3^5 \cdot 5^4 \cdot |G_p|^{15}.$$

Therefore, we can assume that $n \geqslant 3$. The group H acts on F by conjugation, and this action preserves the characteristic subgroup R. Pick two elements α_1 and α_2 of H such that their images $\bar{\alpha}_1$ and $\bar{\alpha}_2$ in $\bar{H}$ generate $\bar{H}$. Since each of the elements α_i normalizes the subgroup R, and the (algebraically closed) field $\Bbbk$ contains a primitive root of 1 or degree $n = |R|$, by Lemma 9.1 the elements α_i^2 commute with R.

Let A be the subgroup of H generated by α_1^2 and α_2^2. Then $A_F = A \cap F$ is contained in the centralizer of R. Since $n \geqslant 3$, the group A_F is cyclic and has order coprime to p by Remark 8.3. In particular, this means that the order of A is coprime to p. Furthermore, since A_F is normalized by α_i^2, we can use Lemma 9.1 once again and conclude that A_F commutes with α_1^4 and α_2^4.

Denote by A' the subgroup of H generated by α_1^4 and α_2^4, and set $A'_F = A' \cap F$. Then $A'_F \subset A_F$. Hence A'_F is a cyclic central subgroup of A'. Thus A' is a central extension of an abelian group, which implies that it is a nilpotent group of class at most 2. Let $\widetilde{N}$ be the subgroup of H generated by A' and R. Since R is abelian and commutes with A', we see that $\widetilde{N}$ is a nilpotent group of class at most 2. Moreover, its order is coprime to p. We want to replace $\widetilde{N}$ by a subgroup with similar properties that is normal in G.

The subgroup R is characteristic in F, and hence normal in G. Also, the subgroup $\bar{H}_{\bar{\alpha}_1^4,\bar{\alpha}_2^4}$ generated by $\bar{\alpha}_1^4$ and $\bar{\alpha}_1^4$ is characteristic in $\bar{H}$, and its index in $\bar{H}$ is at most 16; see Example 2.3. Thus $\bar{H}_{\bar{\alpha}_1^4,\bar{\alpha}_2^4}$ is normal in $\bar{G}$. Let N be the intersection of all subgroups of G conjugate to $\widetilde{N}$. Then N is a nilpotent group of class at most 2; moreover, its order is coprime to p, and it is normal in G. Also, we know that N contains the subgroup R, because R is normal in G. Similarly, the image of N in $\bar{H}$ coincides with $\bar{H}_{\bar{\alpha}_1^4,\bar{\alpha}_2^4}$, because $\bar{H}_{\bar{\alpha}_1^4,\bar{\alpha}_2^4}$ is normal in $\bar{G}$. Therefore, the index of N in G is

$$\begin{aligned}\frac{|G|}{|N|} = \frac{|\bar{G}|}{|\bar{H}_{\bar{\alpha}_1^4,\bar{\alpha}_2^4}|}\cdot\frac{|F|}{|N\cap F|} \leqslant 16\frac{|\bar{G}|}{|\bar{H}|}\cdot\frac{|F|}{|R|} \leqslant 16\cdot(24\cdot|\bar{G}_p|^3)\cdot(60\cdot|F_p|^3) &= 2^9\cdot 3^2\cdot 5\cdot|F_p|^3\cdot|\bar{G}_p|^3 \\ &= 2^9\cdot 3^2\cdot 5\cdot|G_p|^3.\end{aligned}$$

□

Proof of Theorem 1.15 We may assume that the field $\Bbbk$ is algebraically closed. If S is not birational to the product $E\times\mathbb{P}^1$, where E is an elliptic curve, then the group $\mathrm{Bir}(S)$ is p–Jordan by Theorem 1.7; in particular, this means that $\mathrm{Bir}(S)$ is nilpotently p–Jordan of class at most 2. On the other hand, if S is birational to such a product, then $\mathrm{Bir}(S)$ is nilpotently p–Jordan of class at most 2 by Lemma 13.1. □

14 Conclusion

In this section we discuss some open questions concerning birational automorphism groups of varieties over fields of positive characteristic.

Cremona groups of higher rank In [70, 6.1] Serre asked whether the groups $\mathrm{Bir}(\mathbb{P}^n)$ over a field of characteristic p are generalized p–Jordan for arbitrary n. Fei Hu [32, Question 1.11] strengthened this question by asking whether these groups are actually p–Jordan. The answer to both questions is not known, but Theorem 1.6 gives a hope that it should be positive. Over fields of characteristic zero, we know from [63] that all the groups $\mathrm{Bir}(\mathbb{P}^n)$ — and more generally, all birational automorphism groups of rationally connected varieties — are Jordan. However, the proof of this fact given in [63] relies on several theorems that are not known to hold in positive characteristic. Most importantly (except for resolution of singularities, which people are used to routinely assume), it uses boundedness of terminal Fano varieties, which is a particular case of the result of C Birkar [5]. Except for this, the proof uses the Minimal Model Program, which is available only up to dimension 3 over fields of characteristic $p>5$ (see [28; 4; 6]), and some properties of minimal centers of log canonical singularities. It may be interesting to try to generalize this proof to the case of positive characteristic *assuming* resolution of singularities, boundedness of Fanos, and the Minimal Model Program. Even in this setup we expect it to be a complicated problem.

Nonuniruled varieties The case of nonuniruled varieties looks much more accessible. Based on [62, Theorem 1.8(ii)], we ask the following.

Question 14.1 *Let X be a nonuniruled geometrically irreducible algebraic variety over a field of characteristic $p > 0$. Is it true that the group* $\mathrm{Bir}(X)$ *is p–Jordan*?

Note that the approach to birational automorphism groups of nonuniruled varieties over fields of characteristic zero used in [62] does not require boundedness of Fanos, and also does not use much of the Minimal Model Program (that is, does not require termination of flips). Therefore, we expect that the answer to Question 14.1 may be obtained using the method of [62].

Nilpotent groups Based on [27] and Theorem 1.15, one can ask:

Question 14.2 *Let X be a geometrically irreducible algebraic variety over a field of characteristic $p > 0$. Is it true that the group* $\mathrm{Bir}(X)$ *is nilpotently p–Jordan*?

The approach to nilpotent Jordan property used in [27] is based on many features specific to characteristic zero. In particular, it uses the results of [63], which in turn require boundedness of Fanos etc.

Complete varieties One can wonder if the analog of Theorem 1.5 holds for automorphism groups of complete algebraic varieties.

Question 14.3 *Let X be a complete algebraic variety over a field of characteristic $p > 0$. Is it true that the group* $\mathrm{Aut}(X)$ *is p–Jordan*?

The difficulty here is that an analog of Lemma 4.1 is not known in this case, even in characteristic 0. However, according to [52, Corollary 1.2] the answer to an analog of Question 14.3 is positive over fields of characteristic 0; compare with [67] for an alternative proof in the three-dimensional case.

Quasiprojective varieties Similarly to [60, Question 2.30], one can ask the following.

Question 14.4 *Let X be a quasiprojective variety over a field of positive characteristic p. Is it true that the group* $\mathrm{Aut}(X)$ *is p–Jordan*?

Note that the answer to a similar question is known to be positive for quasiprojective surfaces over fields of zero characteristic; see [2].

Explicit estimates It would be interesting to find the precise value of the constant J appearing in Theorem 1.6, similarly to what was done in [64, Proposition 1.2.3] and [85]. As in the case of zero characteristic, this will amount to accurate analysis of finite groups acting on del Pezzo surfaces. Note that some automorphism groups of del Pezzo surfaces over fields of small positive characteristic do not appear in large characteristics and characteristic zero; see for instance [19, Lemma 5.1]. Thus it may happen that the resulting constants are different for small and large values of the characteristic.

Exponents It would be interesting to find the exact values of the constants $e(\Gamma)$ of Definition 1.3 for various groups Γ that enjoy the p–Jordan property. This applies to algebraic groups and automorphism groups of projective varieties; see [32, Theorems 1.7 and 1.10] and cf [32, Remark 1.8] for a (possibly nonsharp) upper bound for $e(\Gamma)$ in these cases. Also, this applies to our Theorem 1.7(ii). Similarly, we do not know the values of the constants $e(\Gamma)$ of Definition 1.14 for birational automorphism groups of surfaces; see Theorem 1.15. Provided that one understands an analogous bound for Theorem 1.7, computing or bounding these values will boil down to analyzing the proof of Lemma 13.1. Some of the progress in this direction may be made by optimizing the bounds provided by Lemmas 2.9 and 2.12.

Generalized p–Jordan property We are not aware of examples of algebraic varieties over a field of positive characteristic p whose birational automorphism group is not p–Jordan, but is generalized p–Jordan. Theorem 1.7 shows that there are no such varieties in dimension 2. It would be interesting to find out if examples of this kind exist in higher dimensions.

Multiplicative bounds In certain cases it is useful to consider multiplicative bounds for indices of normal abelian subgroups in finite subgroups of a given group (ie the least common multiples of such indices); see Serre [69; 70], and Shramov and Vologodsky [74, Theorem 1.2(ii)]. We point out that in the context of p–Jordan groups such bounds fail to exist already in the most simple situations, even if we consider such numbers up to the powers of p. For instance, if $\Bbbk$ is an algebraically closed field of positive characteristic p, the p–Jordan group $\mathrm{PGL}_2(\Bbbk)$ contains a subgroup $\mathrm{PSL}_2(\boldsymbol{F}_{p^k})$ for every positive integer k. The latter group is simple if $p^k > 3$, so that the largest normal abelian subgroup therein is the trivial group, whose index equals $|\mathrm{PSL}_2(\boldsymbol{F}_{p^k})| = n_k p^k$. Here $n_k = \frac{1}{2}(p^{2k}-1)$ if $p \geqslant 3$, and $n_k = 2^{2k}-1$ if $p = 2$; thus, n_k is coprime to p. We see that the numbers n_k, $k \geqslant 1$, are unbounded, and so they do not have a finite common multiple. That being said, it would be interesting to find examples of p–Jordan groups of geometric origin where the multiplicative bounds for the arising constants do exist.

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YC: *Hua Loo-Keng Key Laboratory of Mathematics, Academy of Mathematics and Systems Science*
Chinese Academy of Sciences
Beijing, China

CS: *Steklov Mathematical Institute of Russian Academy of Sciences*
Moscow, Russia

Laboratory of Algebraic Geometry, National Research University Higher School of Economics
Moscow, Russia

`yifeichen@amss.ac.cn`, `costya.shramov@gmail.com`

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