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Homological mirror symmetry for hypersurfaces in $(\mathbb{C}^*)^n$

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We prove a homological mirror symmetry result for maximally degenerating families of hypersurfaces in $(\mathbb{C}^*)^n$ (B–model) and their mirror toric Landau–Ginzburg A–models. The main technical ingredient of our construction is a "fiberwise wrapped" version of the Fukaya category of a toric Landau–Ginzburg model. With the definition in hand, we construct a fibered admissible Lagrangian submanifold whose fiberwise wrapped Floer cohomology is isomorphic to the ring of regular functions of the hypersurface. It follows that the derived category of coherent sheaves of the hypersurface quasiembeds into the fiberwise wrapped Fukaya category of the mirror. We also discuss an extension to complete intersections.

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1 Introduction

The range of settings in which mirror symmetry is expected to hold has steadily expanded since the mirror conjectures were first formulated for projective Calabi–Yau varieties, and there are now candidate mirror constructions in a wide range of settings. Outside of the Calabi–Yau setting, the mirrors are in general Landau–Ginzburg models, ie pairs (Y, W) where Y is a quasiprojective variety and $W \in \mathcal{O}(Y)$ is a regular function (the *superpotential*).

We focus on the case of hypersurfaces in $(\mathbb{C}^*)^n$ (or rather hypersurfaces defined over the nonarchimedean Novikov field $\mathbb{K} = \Lambda$, which arise from maximally degenerating families of hypersurfaces near the tropical limit). These have mirror Landau–Ginzburg models which consist of a noncompact toric Calabi–Yau variety Y of dimension $n+1$, equipped with a superpotential W which is a toric monomial vanishing to order 1 on each irreducible toric divisor of Y. The construction is summarized in Section 2, following

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the description given in Abouzaid, Auroux and Katzarkov [3], which arrives at these mirrors from the perspective of SYZ mirror symmetry; see also Hori and Vafa [26], Clarke [13], Chan, Lau and Leung [11] and Gross, Katzarkov and Ruddat [22] for other viewpoints.

To be specific, consider a degenerating family of complex hypersurfaces defined by a Laurent polynomial of the form

$$f = \sum_{\alpha \in P_{\mathbb{Z}}} a_\alpha t^{2\pi \nu(\alpha)} x^\alpha, \tag{1-1}$$

where $P_{\mathbb{Z}}$ is a finite subset of $\mathbb{Z}^n$, the exponents $\nu(\alpha) \in \mathbb{R}$ are assumed to satisfy a convexity condition which ensures that equation (1-1) is a sufficiently generic degeneration, and the coefficients a_α are complex numbers in the simplest situations, but will in general be given by elements of Λ of vanishing valuation; see Section 2. The space Y which we associate to these data is the Kähler toric variety determined by the polytope

$$\Delta_Y = \{(\xi, \eta) \mid \eta \geq \varphi(\xi)\} \subset \mathbb{R}^n \oplus \mathbb{R}, \tag{1-2}$$

where the piecewise linear function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is the tropicalization of f,

$$\varphi(\xi) = \max_{\alpha \in P_{\mathbb{Z}}} \{\langle \alpha, \xi \rangle - \nu(\alpha)\},$$

and the superpotential $W = -z^{(0,\dots,0,1)}$ is (up to sign) the toric monomial associated to the last coordinate of the ambient space $\mathbb{R}^n \oplus \mathbb{R}$ in equation (1-2). The regular fibers of $W : Y \to \mathbb{C}$ are isomorphic to $(\mathbb{C}^*)^n$, while the unique singular fiber $Z = W^{-1}(0) = \bigcup_\alpha Z_\alpha$ is a union of toric varieties (the irreducible toric divisors of Y, which are in one-to-one correspondence with the monomials appearing in f).

In the simplest example, the hypersurface H is the higher-dimensional pair of pants

$$\{(x_1, \dots, x_n) \mid 1 + x_1 + \cdots + x_n = 0\} \subset (\mathbb{C}^*)^n$$

with mirror the Landau–Ginzburg model ($Y = \mathbb{C}^{n+1}$, $W = -z_1 \cdots z_{n+1}$), whose singular fiber is the union of the coordinate hyperplanes in $\mathbb{C}^{n+1}$; however in most cases Y is not affine and depends on the choice of degeneration.

In one direction, homological mirror symmetry predicts that the wrapped Fukaya category of a hypersurface $H \subset (\mathbb{C}^*)^n$ is equivalent to the derived category of singularities of the mirror Landau–Ginzburg model, $D^b_{\mathrm{sg}}(Y, W) = D^b\mathrm{Coh}(Z)/\mathrm{Perf}(Z)$. This was first verified for the wrapped Fukaya categories of open Riemann surfaces in $(\mathbb{C}^*)^2$ and the derived categories of singularities of their mirror Landau–Ginzburg models; see Abouzaid, Auroux, Efimov, Katzarkov and Orlov [2] and Lee [27]. See also Lekili and Polishchuk [28], where the algebraic side is rather the derived category of coherent sheaves of a stacky nodal curve (equivalent to the Landau–Ginzburg model (Y, W) via Orlov's derived Knörrer periodicity). In higher dimensions, the result was first verified for higher-dimensional pairs of pants by Gammage and Nadler [14] and Lekili and Polishchuk [29]; in the first of these, the wrapped Fukaya category is replaced

by the category of wrapped microlocal sheaves, but the two were subsequently shown to be equivalent by Ganatra, Pardon and Shende [17]. Finally, the case of general hypersurfaces in $(\mathbb{C}^*)^n$ was established by Gammage and Shende [15], also using wrapped microlocal sheaves.

Here we consider the other direction of mirror symmetry, comparing coherent sheaves on the family of hypersurfaces H_t defined by f to a suitable version of the Fukaya category of the Landau–Ginzburg model (Y, W), where Y is equipped with a suitable toric Kähler form in the class $[\omega_Y] \in H^2(Y, \mathbb{R})$ determined by the polytope Δ_Y, and also a bulk deformation class (or B–field) $\mathfrak{b} \in H^2(Y, \Lambda_{\geq 0})$ (the subscript ≥ 0 indicates that we only consider elements of nonnegative valuation). This direction has been much less studied; in fact, at the start of our project there wasn't even a candidate definition yet for the appropriate Fukaya category, because the initial formulation required that Y be affine and that W have isolated nondegenerate singularities; see Seidel [35].

1.1 Fiberwise wrapped Fukaya categories

The first step in our approach is to define the *fiberwise wrapped Fukaya category* $\mathcal{W}(Y, W)$ of a toric Landau–Ginzburg model. The objects of $\mathcal{W}(Y, W)$ are properly embedded Lagrangian submanifolds $L \subset Y$ which satisfy two different types of geometric requirements:

(1) In the base direction, we require that L is fibered at infinity, ie that outside of a compact subset of $\mathbb{C}$ the image of L under $W : Y \to \mathbb{C}$ is a union of embedded arcs, which are further required to be disjoint from the negative real axis and along which the distance from the origin increases strictly.

(2) We require L to be fiberwise "flat" at infinity with respect to a weakly plurisubharmonic fiberwise "height" function $h : Y \to [0, \infty)$, ie the restriction of $d^c h$ to L is required to vanish outside of a bounded subset of each fiber of W.

We call such Lagrangians *admissible*; see Definition 3.5. The Lagrangians we consider are also required to be tautologically unobstructed (in the sense of not bounding any holomorphic disc with respect to a prescribed almost complex structure), and are equipped with the grading data and local systems needed to construct Floer complexes.

Morphism spaces in $\mathcal{W}(Y, W)$ are defined as direct limits of Floer complexes for the images of admissible Lagrangians under a suitable geometric flow, which combines

(1) in the base direction, admissible isotopies acting on the complex plane by positive rotations without crossing the negative real axis (as in the more familiar setting of Fukaya categories of Lefschetz fibrations), and

(2) in the fiber direction, the flow of a Hamiltonian $H : Y \to \mathbb{R}$ which preserves the fibers of W and whose restriction to each fiber is a linear-growth wrapping Hamiltonian (hence the name "fiberwise wrapped").

The details of the construction are given in Section 3.

In the toric case, the fiberwise behavior of our admissible Lagrangians is enforced by fixing a collection of monomials $z^{\boldsymbol{v}} \in \mathcal{O}(Y)$ and open subsets $C_{\boldsymbol{v}}$ of Y, and requiring $\arg(z^{\boldsymbol{v}})$ to be locally constant over $L \cap C_{\boldsymbol{v}}$. This amounts to a fiberwise version of the notion of *monomial admissibility* considered in Andrew Hanlon's thesis [24]; in fact, even though we treat the monomial W separately, the condition we impose in the base direction could also be reformulated in the language of monomial admissibility.

Since our Lagrangians are required to be both fibered with respect to $W : Y \to \mathbb{C}$ and fiberwise monomially admissible within the fibers, our setup requires symplectic parallel transport between smooth fibers of W to be compatible with monomial admissibility. This compatibility is easy to achieve for parallel transport along radial lines in the complex plane by using elementary toric geometry (or by directly imposing monomial admissibility in the total space Y). However, the explicit calculation of Floer complexes and differentials at the heart of our verification of homological mirror symmetry requires us to consider Lagrangians that are everywhere fibered over U-shaped arcs in the complex plane. Achieving fiberwise monomial admissibility for such Lagrangians requires some extra care in the choice of the toric Kähler form ω_Y on Y within the given cohomology class; see Section 4 for details.

Remark 1.1 The several years elapsed since our results were first announced have brought forth key advances and new viewpoints on Fukaya categories of Landau–Ginzburg models, which suggest other possible approaches.

For example, partially wrapped Floer theory for Liouville domains with stops (see Sylvan [37]) and sectors (see Ganatra, Pardon and Shende [16]) has led to considerable progress in the exact setting. However, it is not clear that viewing (Y, W) as a nonexact sector would yield any simplification to our setup and main calculation, as the alternative description in terms of wrapped microlocal sheaves used by Nadler [32] in the case of higher-dimensional pairs of pants would not be applicable outside of the exact setting, and direct calculation by counting holomorphic discs would likely be no easier than the approach taken here.

Monomial admissibility, as used by Hanlon to revisit mirror symmetry for toric varieties [24], is much more directly suited to our goals, and in fact we use this viewpoint to constrain the fiberwise behavior of our Lagrangians and to arrive at a maximum principle. Defining $\mathcal{W}(Y, W)$ directly in the language of monomial admissibility (adding W itself to the list of monomials $z^{\boldsymbol{v}}$ whose arguments we constrain at infinity) would be fairly straightforward, but the explicit calculation of Floer cohomology would likely still require the Lagrangian to be everywhere fibered with respect to the projection $W : Y \to \mathbb{C}$ (not just near infinity), making the setup essentially identical to that considered here.

One can alternatively attempt to replace monomial admissibility with a variant of Groman's formulation for Floer theory on open manifolds [20], adapted to the setting of Landau–Ginzburg models. Early drafts of this text pursued a related approach based on geometric estimates on parallel transport and monotonicity type arguments, but the relevant estimates turned out to be quite challenging.

1.2 A Floer cohomology calculation

The main protagonist of our argument is a specific admissible Lagrangian L_0 in the toric Landau–Ginzburg model (Y, W), which is expected to generate the fiberwise wrapped Fukaya category.

Consider a Laurent polynomial $f \in \mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ defining a maximally degenerating family of hypersurfaces H_t as above, and let (Y, W) be the toric Landau–Ginzburg model constructed in Section 2, equipped with the toric Kähler form ω_Y constructed in Section 4 and a bulk deformation[1] $\mathfrak{b} \in H^2(Y, \Lambda_{\geq 0})$. Since the fiber $W^{-1}(-1) \subset Y$ is isomorphic to $(\mathbb{C}^*)^n$, it contains a distinguished Lagrangian $\ell_0 = (\mathbb{R}_+)^n$ along which the toric monomials $z^{\boldsymbol{v}}$ are all real positive. The parallel transport of ℓ_0 over a U-shaped arc in the complex plane connecting -1 to $+\infty$ around the origin yields an admissible Lagrangian submanifold L_0 in (Y, W). Our main result is:

Theorem 1.2 *For a suitable choice of bulk deformation class $\mathfrak{b} \in H^2(Y, \Lambda_{\geq 0})$, the fiberwise wrapped Floer cohomology ring $H\mathcal{W}^*(L_0, L_0)$ is isomorphic to the quotient $\mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]/(f)$ of the ring of Laurent polynomials by the ideal generated by f, the defining equation of the family of hypersurfaces H_t.*

Remark 1.3 We refer the reader to Remark 5.31 for a discussion of the relationship between the bulk class appearing in the statement of the above theorem and the expression of the mirror map in terms of Gromov–Witten theory.

In other terms, $H\mathcal{W}^*(L_0, L_0)$ is isomorphic to the ring of functions of the nonarchimedean hypersurface $\mathcal{H}$ defined by f over $\mathbb{K}$:

$$H\mathcal{W}^*(L_0, L_0) \simeq \mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]/(f) \simeq H^0(\mathcal{H}, \mathcal{O}_{\mathcal{H}}) = \mathrm{Hom}(\mathcal{O}_{\mathcal{H}}, \mathcal{O}_{\mathcal{H}}). \tag{1-3}$$

Since this ring is supported in degree 0, it is intrinsically formal, so we conclude that the Floer algebra $\mathcal{W}^*(L_0, L_0)$ is A_∞ equivalent to the ring of functions on $\mathcal{H}$. On the other hand, since $\mathcal{H}$ is affine, its derived category is generated by the structure sheaf $\mathcal{O}_{\mathcal{H}}$, and by mapping a twisted complex built from copies of $\mathcal{O}_{\mathcal{H}}$ to the corresponding twisted complex built from L_0 inside $\mathcal{W}(Y, W)$, we arrive at:

Corollary 1.4 *The derived category of coherent sheaves of $\mathcal{H}$ admits a fully faithful quasiembedding into $\mathcal{W}(Y, W)$.*

One can then return from the nonarchimedean setting to the complex hypersurfaces H_t by observing that, when f is of the form (1-1) with $a_\alpha \in \mathbb{C}^*$, the outcome of our calculation is manifestly convergent over complex numbers and we can treat t as an actual parameter rather than a formal variable.

[1] In the literature, one usually considers bulk classes of strictly positive valuation; the 0–valuation part of $\mathfrak{b}$ corresponds to (a logarithm of) what is sometimes called a background class, which in our case is valued in $\mathbb{C}^*$, but is usually considered with $\mathbb{Z}_2$ coefficients, and modifies Floer theory by changing the sign contributions of discs.

The calculation of $H\mathcal{W}^*(L_0, L_0)$ involves counts of holomorphic sections of the fibration $W: Y \to \mathbb{C}$ over domains in the complex plane, with boundary in fibered Lagrangians, and the argument is essentially within the realm of the "Seidel TQFT" [35] even though W is not a Lefschetz fibration; see Section 5. Our approach is concrete and explicit, but a more conceptual interpretation can be given in terms of the Orlov cup functor; see below.

Remark 1.5 The object L_0 is expected to generate the category $\mathcal{W}(Y, W)$, which would imply that the embedding of Corollary 1.4 is an equivalence. Stop removal (wrapping past the negative real axis in the base direction) yields an *acceleration* functor from $\mathcal{W}(Y, W)$ to a suitable version of the wrapped Fukaya category of Y, under which L_0 maps to the zero object; see Abouzaid and Seidel [5]. The stop removal results of Sylvan [37] and Ganatra, Pardon and Shende [16] (to the extent that they hold in our setup) should imply that $\mathcal{W}(Y)$ is precisely the quotient of $\mathcal{W}(Y, W)$ by the full subcategory generated by L_0. The generation statement is then equivalent to the vanishing of $\mathcal{W}(Y)$. This argument can be made precise in the case of the pair of pants, where $Y = \mathbb{C}^{n+1}$ is a subcritical Liouville manifold and its wrapped Fukaya category vanishes. However, given that a complete argument in the general case where Y is not exact would involve several pieces of machinery that have not yet been developed in that setting, we do not investigate this question further in this paper.

1.3 A functorial perspective

The fiberwise wrapped Fukaya category is the target of a functor

$$\cup: \mathcal{W}((\mathbb{C}^*)^n) \to \mathcal{W}(Y, W)$$

(the Orlov cup functor), which is given on objects by parallel transport of admissible Lagrangian submanifolds of $W^{-1}(-1) \simeq (\mathbb{C}^*)^n$ along a U-shaped arc in the complex plane, and on morphisms by observing that the portions of the fiberwise wrapped Floer complexes which live in the fiber over -1 are closed under all A_∞–operations. In this language, the computation at the heart of the proof of Theorem 1.2 gives a commutative diagram of functors

$$\begin{tikzcd} \mathrm{Perf}((\mathbb{K}^*)^n) \arrow[r, "i^*"] \arrow[d, "\simeq"'] & \mathrm{Perf}(\mathcal{H}) \arrow[d, hook] \\ \mathcal{W}((\mathbb{C}^*)^n) \arrow[r, "\cup"] & \mathcal{W}(Y, W) \end{tikzcd} \tag{1-4}$$

where the restriction functor i^* and the cup functor $\cup$ intertwine mirror symmetry for the ambient torus $(\mathbb{K}^*)^n$ and the hypersurface $\mathcal{H}$. The core of our argument amounts to a verification of this statement for the structure sheaves on the algebraic side, and the admissible Lagrangians $\ell_0 = (\mathbb{R}_+)^n$ and $L_0 = \cup \ell_0$ on the symplectic side.

To continue further in this direction, the functor $\cup$ has an adjoint functor $\cap: \mathcal{W}(Y, W) \to \mathrm{Perf}\,\mathcal{W}((\mathbb{C}^*)^n)$ ("restriction to the fiber at $+\infty$"), under which a fibered Lagrangian $L = \cup \ell$ maps to a twisted complex built from the fiberwise Lagrangians at the two ends of the U-shaped arc, with a connecting differential s_ℓ^0

which counts holomorphic sections of $W : Y \to \mathbb{C}$ bounded by L over the region enclosed by the U-shaped arc. After choosing a suitable identification of the fiber near $+\infty$ with $(\mathbb{C}^*)^n$, we find that the image of ℓ under the composite functor $\cap\cup$ is isomorphic to a cone

$$\cap \cup \ell \simeq \{\mu^{-1}(\ell) \xrightarrow{s_\ell^0} \ell\},$$

where μ^{-1} is the clockwise monodromy of the fibration W around the origin, acting on the wrapped Fukaya category of the fiber (in our case $\mu^{-1} \simeq \mathrm{id}$), and s_ℓ^0 is a count of sections. This is part of an exact triangle of functors

$$\begin{tikzcd} \mu^{-1} \arrow[rr, "s"] & & \mathrm{id} \arrow[dl] \\ & \cap\cup \arrow[ul, "{[1]}"] & \end{tikzcd}$$

acting on $\mathcal{W}((\mathbb{C}^*)^n)$, originating in Seidel's work [34] on Dehn twists, and which has been the subject of some recent work; see for instance Abouzaid and Ganatra [4], Abouzaid and Smith [6, Appendix A], and Sylvan [38, Theorem 1.3].

Our calculation of the fiberwise wrapped Floer complex of $L_0 = \cup\ell_0$ can then be rewritten as

$$\mathcal{W}_{(Y,W)}(\cup\ell_0, \cup\ell_0) \simeq \mathcal{W}_{(\mathbb{C}^*)^n}(\ell_0, \cap\cup\ell_0) \simeq \mathrm{Cone}\{HW^*(\ell_0, \mu^{-1}(\ell_0)) \xrightarrow{s_{\ell_0}^0} HW^*(\ell_0, \ell_0)\}$$

which, after verifying that the section-counting natural transformation $s_{\ell_0}^0$ amounts to multiplication by the Laurent polynomial f, corresponds on the algebraic side to

$$\mathrm{Hom}_{\mathcal{H}}(\mathcal{O}_{\mathcal{H}}, \mathcal{O}_{\mathcal{H}}) \simeq \mathrm{Hom}_{(\mathbb{K}^*)^n}(\mathcal{O}, i_* i^* \mathcal{O}) \simeq \mathrm{Cone}\{\mathrm{Hom}(\mathcal{O}, \mathcal{O}) \xrightarrow{f} \mathrm{Hom}(\mathcal{O}, \mathcal{O})\}.$$

1.4 Complete intersections and compactifications

Our results admit extensions in at least two directions. The first one, which we briefly discuss in Section 6, concerns complete intersections. The mirror of a codimension k complete intersection in $(\mathbb{C}^*)^n$ (or rather, of a family of complete intersections degenerating to a tropical limit) is a Calabi–Yau toric Kähler manifold Y of complex dimension $n+k$, equipped with a superpotential which is a sum of k toric monomials $W_1, \dots, W_k \in \mathcal{O}(Y)$; taken together these determine a holomorphic map $\boldsymbol{W} : Y \to \mathbb{C}^k$, whose fibers over $(\mathbb{C}^*)^k$ are again isomorphic to $(\mathbb{C}^*)^n$; see Abouzaid, Auroux and Katzarkov [3, Section 11]. We then define a version of the fiberwise wrapped Fukaya category $\mathcal{W}(Y, \boldsymbol{W})$ whose objects are Lagrangian submanifolds which are simultaneously admissible for each of the projections $W_1, \dots, W_k$; the morphisms are direct limits of Floer complexes under a combination of admissible isotopies acting on each factor of $\mathbb{C}^k$ by positive rotations without crossing the negative real axis and wrapping in the fibers of $\boldsymbol{W}$. The key object L_0 under consideration is now obtained by parallel transport of $(\mathbb{R}_+)^n \subset (\mathbb{C}^*)^n$ over a product of U-shaped arcs inside $\mathbb{C}^k$. By an argument similar to our main calculation, its fiberwise wrapped Floer complex is isomorphic to the Koszul resolution of the ring of functions of the complete intersection; the upshot is that the obvious analogues of Theorem 1.2 and Corollary 1.4 continue to hold in this setting. See Section 6 and Theorem 6.7.

Another extension is to hypersurfaces (and complete intersections) in toric varieties. Namely, a Laurent polynomial of the form (1-1) defines not only hypersurfaces in $(\mathbb{C}^*)^n$ or $(\mathbb{K}^*)^n$ but also (partial) compactifications in suitable toric varieties or stacks — for example, the projective toric variety or stack $\overline{V}$ whose moment polytope is the convex hull of $P_{\mathbb{Z}}$. Following [3], the mirror to $\overline{H} \subset \overline{V}$ is the same Calabi–Yau toric variety Y as in our main construction, now equipped with a superpotential $\overline{W}$ which is the sum of the previously encountered monomial $W_0 = -z^{(0,\dots,0,1)}$ and extra terms consisting of one monomial for each of the irreducible toric divisors of $\overline{V}$. The latter turn out to be exactly the collection of monomials $z^{\boldsymbol{v}}$ we consider in Definition 4.14. Consequently, we can define the Fukaya category $\mathcal{F}(Y, \overline{W})$ by considering exactly the same admissible Lagrangian submanifolds of Y as in the construction of $\mathcal{W}(Y, W_0)$: namely, Lagrangians which are fibered at infinity with respect to $W_0 \colon Y \to \mathbb{C}$, and within the fibers of W_0, monomially admissible for the collection of monomials $z^{\boldsymbol{v}}$. However, we now consider colimits under perturbations which only increase the argument of each monomial $z^{\boldsymbol{v}}$ within a small bounded interval, rather than by an unbounded amount of fiberwise wrapping. Starting from monomially admissible Lagrangian sections $\ell, \ell' \subset (\mathbb{C}^*)^n$ such as those considered in [24], which are mirrors to line bundles $\mathcal{L}, \mathcal{L}'$ on the toric variety $\overline{V}$, we now find an isomorphism

$$\mathrm{Hom}_{\mathcal{F}(Y,\overline{W})}(\cup\ell, \cup\ell') \simeq \mathrm{Cone}\{\mathrm{Hom}_{\mathcal{F}((\mathbb{C}^*)^n,\{z^{\boldsymbol{v}}\})}(\ell, \mu^{-1}(\ell')) \xrightarrow{s} \mathrm{Hom}_{\mathcal{F}((\mathbb{C}^*)^n,\{z^{\boldsymbol{v}}\})}(\ell, \ell')\}.$$

After checking that the action of the monodromy μ^{-1} on monomially admissible Lagrangian sections is mirror to the functor $- \otimes \mathcal{O}(-\overline{H})$ and that the natural transformation $s \colon \mu^{-1} \to \mathrm{id}$ still corresponds to multiplication by the defining section f of $\overline{H}$, this corresponds on the algebraic side to the isomorphism

$$\mathrm{Hom}_{\overline{H}}(\mathcal{L}_{|\overline{H}}, \mathcal{L}'_{|\overline{H}}) \simeq \mathrm{Cone}\{\mathrm{Hom}_{\overline{V}}(\mathcal{L}, \mathcal{L}' \otimes \mathcal{O}(-\overline{H})) \xrightarrow{f} \mathrm{Hom}_{\overline{V}}(\mathcal{L}, \mathcal{L}')\}.$$

This in turn implies cohomology-level mirror symmetry statements for restrictions of ample line bundles (compare with Cannizzo [10], which establishes analogous results in a different setting). A more detailed account of this will appear elsewhere (work in progress).

1.5 Related works

In the time elapsed since our results were first announced, at least two papers have appeared establishing conceptually similar homological mirror symmetry results relating coherent sheaves on hypersurfaces to the symplectic geometry of mirror Landau–Ginzburg models.

On one hand, Nadler [32] introduced the category of wrapped microlocal sheaves and gave an explicit computation for the Landau–Ginzburg model $(\mathbb{C}^n, z_1 \cdots z_n)$, which is mirror to the $(n-2)$–dimensional pair of pants. (Wrapped microlocal sheaves were subsequently shown by Ganatra, Pardon and Shende [17] to be equivalent to the Fukaya category of the corresponding Liouville sector.) Nadler's paper showcases the remarkable computational power of microlocal sheaves in the exact setting, and also identifies structural properties which are closely related to those described in Section 1.3 above.

On the other hand, Cannizzo's thesis work [10] considers the case of a genus 2 curve embedded in a principally polarized abelian surface (its Jacobian) and the mirror Landau–Ginzburg model. The approach is fairly similar to ours, but avoids the need to discuss fiberwise admissibility because the mirror is proper (the generic fibers are T^4). However, the monodromy is topologically nontrivial, and involves a twist mirror to the defining section of the genus 2 curve, so that the objects of interest are a sequence of admissible Lagrangians mirror to powers of the canonical bundle of the genus 2 curve (somewhat similarly to the toric variety case outlined above). Another notable difference with our setting is that, despite the nonexact nature of the mirror and the presence of rational curves in the zero fiber, no bulk deformation is required as the instanton corrections only result in a rescaling of the section-counting natural transformation; see [10].

Plan of the paper

The first part of this paper is concerned with the definition of the fiberwise wrapped Fukaya category $\mathcal{W}(Y, W)$. After reviewing the construction of the Landau–Ginzburg model (Y, W) in Section 2, we develop the foundations of fiberwise wrapped Fukaya categories in Section 3, while Section 4 is devoted to the construction of the appropriate toric Kähler form and verification of the required geometric properties. The heart of the paper is then Section 5, which is devoted to the calculation of the fiberwise wrapped Floer cohomology of L_0 and the proof of Theorem 1.2. Finally, in Section 6 we briefly discuss the extension to complete intersections and prove Theorem 6.7.

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2 The mirror Landau–Ginzburg model

2.1 The main construction

Consider a Laurent polynomial with complex coefficients

$$f = \sum_{\alpha \in \mathbb{Z}^n} a_\alpha x^\alpha, \tag{2-1}$$

and denote the corresponding hypersurface by

$$H = f^{-1}(0) \subset (\mathbb{C}^*)^n. \tag{2-2}$$

The construction of a mirror for H depends on a choice of degeneration; we specifically consider a maximal degeneration to a tropical limit, and assume that the corresponding tropical variety is smooth in the sense we explain now.

Let P denote the Newton polytope of f, and $P_{\mathbb{Z}}$ its integral points. For simplicity, we assume that the interior of P is nonempty (ie P has positive volume); otherwise we can always reduce to this case by splitting off some $\mathbb{C}^*$ factors.

A *tropically smooth* maximal degeneration of H is induced by the choice of a strictly convex piecewise linear function

$$\nu\colon P \to \mathbb{R}, \tag{2-3}$$

whose domains of linearity determine a subdivision $\mathcal{P}$ of P into standard integral simplices, ie simplices that are equivalent by an integral affine transformation to the simplex spanned by the origin and the unit coordinate vectors in $\mathbb{Z}^n$; this condition ensures that the mirror toric variety we construct below is smooth. The corresponding degeneration is then

$$f_\nu = \sum_{\alpha \in P_{\mathbb{Z}}} a_\alpha t^{2\pi\nu(\alpha)} x^\alpha. \tag{2-4}$$

We can associate to f_ν either a family of hypersurfaces parametrized by $t \in \mathbb{C}$, or a variety $\mathcal{H}$ over the Novikov field $\mathbb{K} = \Lambda$ of power series in the formal variable t with real exponents. The second point of view is more natural for the purpose of proving the well-definedness and invariance of the Fukaya category, and providing clear formulations of homological mirror symmetry.

Denote by $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ the tropicalization of f_ν, ie the piecewise linear function

$$\varphi(\xi) = \max\{\langle \alpha, \xi\rangle - \nu(\alpha) \mid \alpha \in P_{\mathbb{Z}}\}. \tag{2-5}$$

Let Y be the (noncompact) Kähler toric manifold defined by the moment polytope

$$\Delta_Y = \{(\xi, \eta) \in \mathbb{R}^n \oplus \mathbb{R} \mid \eta \geq \varphi(\xi)\}. \tag{2-6}$$

The polytope Δ_Y determines a Kähler class $[\omega_Y] \in H^2(Y, \mathbb{R})$. In Section 4, we shall specify an explicit Kähler form ω_Y, obtained by Hamiltonian reduction from a vector space, which will be particularly well-adapted to our Floer-theoretic constructions.

Dually, Y can also be described by the fan

$$\Sigma_Y = \mathbb{R}_{\geq 0} \cdot (\mathcal{P} \times \{1\}) \subseteq \mathbb{R}^{n+1} = \mathbb{R}^n \oplus \mathbb{R}, \tag{2-7}$$

whose rays are generated by the integer vectors $(-\alpha, 1)$ for $\alpha \in P_{\mathbb{Z}}$, and which is obtained as the union of the cones on polyhedra appearing in the subdivision $\mathcal{P}$. Since we have assumed that this subdivision

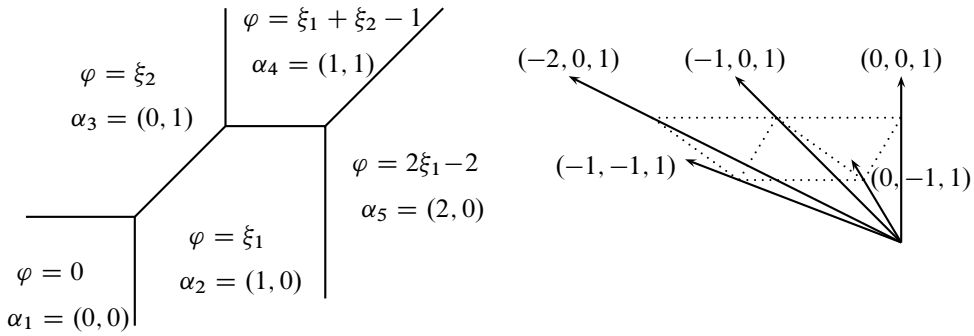

Figure 1: Constructing the mirror: $f(x_1, x_2) = 1 + x_1 + x_2 + t^{2\pi} x_1 x_2 + t^{4\pi} x_1^2$.

is maximal, all such cones are simplicial, and since the simplices are further assumed to be congruent to the standard one, it follows that Y is a smooth toric manifold. It is in fact a smooth toric Calabi–Yau, since the defining equation of its toric anticanonical divisor is a regular function (see below); in particular, its canonical bundle is trivial, ie $c_1(Y) = 0$, which will allow us to introduce $\mathbb{Z}$–gradings in Floer theory (and also simplify our discussion of sphere bubbling).

Denote by $z^{(0,\dots,0,1)} \in \mathcal{O}(Y)$ the toric monomial with weight $(0, \dots, 0, 1)$. Equip Y with the superpotential

$$W = -z^{(0,\dots,0,1)} \colon Y \to \mathbb{C}. \tag{2-8}$$

The toric Landau–Ginzburg model (Y, W) has been constructed as a candidate mirror to H from various perspectives; see in particular [3, Theorem 1.4].

The level set $W^{-1}(0)$ is the union of the toric divisors in Y (each with multiplicity one), while the other level sets of W are smooth and isomorphic to $(\mathbb{C}^*)^n$. (The fact that the toric anticanonical divisor is defined by a regular function, namely W, verifies the above claim that Y is Calabi–Yau).

Example 2.1 As a running example to illustrate the construction, we consider the Laurent polynomial $f(x_1, x_2) = 1 + x_1 + x_2 + t^{2\pi} x_1 x_2 + t^{4\pi} x_1^2$ (which defines a degenerating family of genus 0 curves with 5 punctures in $(\mathbb{C}^*)^2$). The tropicalization of f is given by $\varphi(\xi_1, \xi_2) = \max\{0, \xi_1, \xi_2, \xi_1 + \xi_2 - 1, 2\xi_1 - 2\}$. The domains of linearity of φ, which also correspond to the facets of the polytope Δ_Y "seen from above", are depicted in Figure 1, along with the fan Σ_Y, whose generators $(-\alpha, 1)$ for $\alpha \in P_{\mathbb{Z}}$ give the primitive (inward) normal vectors to the facets of Δ_Y.

2.2 Construction as a Hamiltonian reduction

We have a surjective map

$$\mathbb{Z}^{P_{\mathbb{Z}}} \to \mathbb{Z}^n \oplus \mathbb{Z}, \tag{2-9}$$

which assigns to a lattice point $\alpha \in P$ the pair $(-\alpha, 1)$; the kernel is a lattice, which we denote by M.

We shall consider the reduction of $\mathbb{C}^{P_{\mathbb{Z}}}$ (equipped with a suitable toric Kähler form, described in Section 4.1) by the Hamiltonian action of the torus

$$\mathbb{T}_M = M \otimes (\mathbb{R}/\mathbb{Z}) \subset \mathbb{T}^{P_{\mathbb{Z}}}. \tag{2-10}$$

Fixing a regular value λ for the moment map

$$\mu\colon \mathbb{C}^{P_{\mathbb{Z}}} \to \mathrm{Hom}(M, \mathbb{R}) = M^*_{\mathbb{R}}, \tag{2-11}$$

the quotient $\mu^{-1}(\lambda)/\mathbb{T}_M$ inherits a canonical symplectic form ω_λ. By the Kempf–Ness theorem, this quotient can be naturally identified with the quotient of an open subset of $\mathbb{C}^{P_{\mathbb{Z}}}$ by a complex torus, and the symplectic form ω_λ is Kähler with respect to the induced complex structure. Thus, $\mu^{-1}(\lambda)/\mathbb{T}_M$ is naturally equipped with a toric Kähler form (induced by that constructed in Section 4.1 on $\mathbb{C}^{P_{\mathbb{Z}}}$); see also [23].

We now explain how the choice of level set λ corresponds to the integral affine function in equation (2-3). Dualizing (2-9) we obtain a short exact sequence

$$0 \to \mathbb{R}^{n+1} \xrightarrow{i} \mathbb{R}^{P_{\mathbb{Z}}} \xrightarrow{\pi} M^*_{\mathbb{R}} \to 0, \tag{2-12}$$

where the first map is given explicitly by

$$i(\xi_1, \dots, \xi_n, \eta) = (-\langle \alpha, \xi \rangle + \eta)_{\alpha \in P_{\mathbb{Z}}}. \tag{2-13}$$

Viewing the piecewise linear function $\nu\colon P \to \mathbb{R}$ as an element of $\mathbb{R}^{P_{\mathbb{Z}}}$, we set

$$\lambda = \pi(\nu).$$

Then the image of the moment map for the action of $\mathbb{T}^{n+1} \simeq \mathbb{T}^{P_{\mathbb{Z}}}/\mathbb{T}_M$ on $\mu^{-1}(\lambda)/\mathbb{T}_M$ is the intersection of $\pi^{-1}(\lambda)$ with the nonnegative orthant in $\mathbb{R}^{P_{\mathbb{Z}}}$, ie the set of all $(\xi, \eta) \in \mathbb{R}^{n+1} = \mathbb{R}^n \oplus \mathbb{R}$ such that all the components of $i(\xi, \eta) + \nu$ are nonnegative. Comparing with (2-6), this moment polytope is precisely Δ_Y.

This yields a Hamiltonian quotient description of Y equipped with the toric Kähler form ω_Y. Moreover, the function

$$W_0 = -\textstyle\prod_{\alpha \in P_{\mathbb{Z}}} z_\alpha\colon \mathbb{C}^{P_{\mathbb{Z}}} \to \mathbb{C} \tag{2-14}$$

descends to the toric potential $W\colon Y \to \mathbb{C}$. (Note that both are toric monomials vanishing to order 1 on each toric divisor.) Setting $N = |P_{\mathbb{Z}}|$, we can thus view the Landau–Ginzburg model (Y, W) as a Hamiltonian reduction (by $\mathbb{T}_M$) of the "standard" Landau–Ginzburg model $\big(\mathbb{C}^N, W_0 = -\prod_{i=1}^N z_i\big)$.

Example 2.2 In Example 2.1, the kernel of the map (2-9), ie the space of linear relations among the $(-\alpha_i, 1)$ (the generators of the fan Σ_Y, shown in Figure 1, right) is a rank 2 lattice M, spanned by elements corresponding to the linear relations $\alpha_1 - \alpha_2 - \alpha_3 + \alpha_4 = 0$ and $\alpha_1 - 2\alpha_2 + \alpha_5 = 0$ among the elements of $P_{\mathbb{Z}}$. Thus, we can realize the toric 3–fold Y as the quotient of $\mathbb{C}^5$ by a 2–dimensional torus $\mathbb{T}_M$ whose generators act with weights $(1, -1, -1, 1, 0)$ and $(1, -2, 0, 0, 1)$. The moment map of the $\mathbb{T}_M$–action is obtained from the moment map of the standard action of $\mathbb{T}^5$ by the projection $\pi(\mu_1, \dots, \mu_5) = (\mu_1 - \mu_2 - \mu_3 + \mu_4, \mu_1 - 2\mu_2 + \mu_5)$, where $(\mu_1, \dots, \mu_5)$ take values in the standard

moment polytope of $\mathbb{C}^5$, ie the nonnegative orthant. However, since the toric Kähler form on $\mathbb{C}^5$ we will construct in Section 4.1 differs from the standard one, it will not be the case that $\mu_i = \frac{1}{2}|z_i|^2$.

Setting $\lambda = \pi(\nu) = (1, 2)$, we find that $\mu^{-1}(\lambda) \subset \mathbb{C}^5$ is the set of points whose moment map coordinates for the $\mathbb{T}^5$–action satisfy

$$\mu_1 - \mu_2 - \mu_3 + \mu_4 = 1 \quad \text{and} \quad \mu_1 - 2\mu_2 + \mu_5 = 2. \tag{2-15}$$

The moment polytope of the toric variety $\mu^{-1}(\lambda)/\mathbb{T}_M$ is then the intersection of the nonnegative orthant with the affine subspace determined by (2-15), identified with $\Delta_Y = \{(\xi_1, \xi_2, \eta) \in \mathbb{R}^3 \mid \eta \geq \varphi(\xi_1, \xi_2)\}$ via the affine embedding

$$i(\xi_1, \xi_2, \eta) + \nu = (\eta, \eta - \xi_1, \eta - \xi_2, \eta - \xi_1 - \xi_2 + 1, \eta - 2\xi_1 + 2).$$

Remark 2.3 There is a uniform way of producing all the examples that we consider from a universal construction: $(\mathbb{C}^N, W_0)$ is mirror to an $(N-2)$–dimensional pair of pants, ie the intersection of the hyperplane $X_0 + \cdots + X_{N-1} = 0$ with the open stratum $(\mathbb{K}^*)^{N-1}$ in $\mathbb{P}^{N-1}$. The embedding of $(\mathbb{K}^*)^n$ into the open stratum of $\mathbb{P}(\mathbb{K}^{P_\mathbb{Z}})$ defined by

$$(x_1, \dots, x_n) \mapsto \left(a_\alpha t^{2\pi\nu(\alpha)} x^\alpha\right)_{\alpha \in P_\mathbb{Z}}$$

defines an algebraic subtorus, whose intersection with the pair of pants is the hypersurface $\mathcal{H}$. Thus, the mirror pairs we consider can be viewed as "reductions" of the mirror pair consisting of the $(N-2)$–dimensional pair of pants and the Landau–Ginzburg model $(\mathbb{C}^N, W_0)$: namely, $\mathcal{H}$ is the intersection of the pair of pants with an algebraic subtorus, while its mirror (Y, W) is the quotient of $(\mathbb{C}^N, W_0)$ by the complementary subtorus.

However, the graph of the projection $\mu^{-1}(\lambda) \to Y$, viewed as a Lagrangian correspondence in $\mathbb{C}^N \times Y$, bounds nontrivial families of holomorphic discs; this causes a discrepancy between moduli spaces of discs in Y with boundary on given Lagrangian submanifolds of Y, and moduli spaces of discs in $\mathbb{C}^N$ with boundary on the lifts of these Lagrangians to $\mu^{-1}(\lambda)$. The instanton corrections that arise out of this are responsible for the presence of the bulk deformation class $\mathfrak{b} \in H^2(Y, \Lambda_{\geq 0})$ in the statement of Theorem 1.2, as we shall see in Section 5.

3 The Fukaya category of a Landau–Ginzburg model

3.1 Landau–Ginzburg models

Let (Y, ω) be a symplectic manifold, and

$$W : Y \to \mathbb{C} \tag{3-1}$$

a map which is a symplectic fibration outside a compact subset of $\mathbb{C}$. We shall define a Fukaya category associated to the pair (Y, W), subject to additional auxiliary choices:

(i) A compatible almost complex structure J making W holomorphic outside a compact subset of $\mathbb{C}$.

(ii) A continuous function $h: Y \to [0, \infty)$ which is weakly J–plurisubharmonic.

(iii) A nonnegative *wrapping Hamiltonian*

$$H: Y \to \mathbb{R}. \tag{3-2}$$

(iv) A closed subset $Y^{\text{in}} \subset Y$, whose intersection with every fiber of W is a (compact) sublevel set of h; more precisely, we take Y^{in} to be the set of points where $h \leq r(|W|)$, where $r(|W|)$ is a nondecreasing function of $|W|$, constant over $[0, R_0]$ for some R_0.

We require these data to be compatible as follows:

(1) The restrictions of h and H to every fiber of W are proper.

(2) The Hamiltonian flow of H preserves the fibers of W, and outside of Y^{in} it preserves the level sets of h:

$$dW(X_H) = 0 \qquad \text{and} \qquad dh(X_H) = 0 \quad \text{outside } Y^{\text{in}}. \tag{3-3}$$

Also, horizontal parallel transport preserves H everywhere, as well as h outside of Y^{in}. By this we mean that, if $\xi^{\#}$ is the horizontal lift of a vector on $\mathbb{C}$, then

$$dH(\xi^{\#}) = 0 \qquad \text{and} \qquad dh(\xi^{\#}) = 0 \quad \text{outside } Y^{\text{in}}. \tag{3-4}$$

This in turn guarantees that horizontal parallel transport is well-defined (except at critical points) despite the noncompactness of the fibers, since horizontal lifts are contained in the level sets of H which is fiberwise proper.

(3) Outside of Y^{in}, the 1–form $d^c h = -dh \circ J$ vanishes on the symplectic orthogonal to the fibers of W, ie if $\xi^{\#}$ is the horizontal lift of a vector on $\mathbb{C}$, we have

$$d^c h(\xi^{\#}) = 0. \tag{3-5}$$

Moreover, $d^c h$ is preserved by (i) parallel transport and (ii) the Hamiltonian flow X_H, ie the Lie derivative with respect to X_H and to the horizontal pullback of a vector field ξ on $\mathbb{C}$ both vanish:

$$\mathcal{L}_{X_H} d^c h = \mathcal{L}_{\xi^{\#}} d^c h = 0. \tag{3-6}$$

(4) The function h grows along $-JX_H$ outside of Y^{in}, ie

$$0 \leq d^c h(X_H). \tag{3-7}$$

Remark 3.1 Condition (2) essentially states that W, H and h Poisson commute outside of Y^{in}. Moreover, the fact that W is holomorphic outside of a compact subset implies that the horizontal subspace is J–invariant, and hence the vanishings of dh and $d^c h$ on the horizontal distribution are equivalent to each other. On the other hand, the condition $\mathcal{L}_{X_H} d^c h = 0$ is particularly strong, and is analogous to considering only *linear* Hamiltonians in the situation of a manifold with contact boundary.

Remark 3.2 By the Cartan formula, given (3-5) the condition $\mathcal{L}_{\xi^{\#}} d^c h = 0$ is equivalent to requiring that $\iota_{\xi^{\#}} d d^c h = 0$ for every horizontal vector $\xi^{\#}$.

Remark 3.3 In our main examples, the requirements concerning the behavior of h along the horizontal distribution ($dh(\xi^\#) = d^c h(\xi^\#) = 0$ and $\mathcal{L}_{\xi^\#} d^c h = 0$) only hold outside of $Y^{\mathrm{in}} \cup W^{-1}(\Delta')$, where Δ' is a small neighborhood of $\mathrm{crit}(W) = \{0\}$ in the complex plane. We will see that this weakening of the assumptions is not problematic as long as the Lagrangians we consider remain outside of $W^{-1}(\Delta')$ and the isotopies of the complex plane whose lifts we consider are supported outside of Δ'.

Definition 3.4 An *admissible arc* is a properly embedded arc $\gamma\colon [0,\infty) \to \mathbb{C}$ that is disjoint from the critical values of W and from the negative real axis, and along which distance from the origin is strictly increasing outside of the disc of radius R_0.

Definition 3.5 An *admissible Lagrangian* with respect to the above data is a properly embedded Lagrangian $L \subset Y$ such that

(i) the image $W(L) \subset \mathbb{C}$ agrees outside of a compact subset Δ with a finite union of admissible arcs which do not reenter Δ; and

(ii) the restriction of $d^c h$ to L vanishes outside of Y^{in}.

The main examples we consider below are in fact fibered over properly embedded arcs in $\mathbb{C}$ which avoid the critical values of W and are asymptotic to radial straight lines at infinity. In this case we can take Δ to be a single basepoint on the arc.

Given an admissible Lagrangian $L \subset Y$ and an isotopy ρ^t of the complex plane, pointwise preserving $\Delta \cup \mathrm{crit}(W)$ (or the slightly larger set $\Delta \cup \Delta'$) and setwise preserving the negative real axis, there exists a unique Lagrangian isotopy, which we denote by $\rho^t(L)$, with the following properties:

(i) $\rho^t(L) = L$ in $W^{-1}(\Delta)$, and

(ii) outside of $W^{-1}(\Delta)$, $\rho^t(L)$ fibers over the collection of arcs which is the image of $W(L)$ under ρ^t.

We say that the lifted isotopy $\rho^t(L)$ is admissible if the images of the arcs under ρ^t are admissible. The Lagrangian $\rho^t(L)$ can be constructed by intersecting L with $W^{-1}(\Delta)$ and parallel transporting its boundary along the images of the arcs under ρ^t.

Remark 3.6 If the symplectic connection on $W\colon Y \to \mathbb{C}$ has vanishing curvature outside of Δ then ρ^t can be directly constructed as the horizontal lift of the isotopy of the base. However, the geometric models required for our applications do not naturally satisfy this condition.

Lemma 3.7 *The set of admissible Lagrangians is invariant under compositions of*

(i) *Hamiltonian isotopies supported in* Y^{in} *that preserve the fibers of* W *outside of a compact subset*,

(ii) *the Hamiltonian flow of* H, *and*

(iii) *admissible lifted isotopies* $\rho^t(L)$ *as defined above.*

Proof The first statement is obvious from the definition. The Hamiltonian flow of H preserves admissibility because we have required that $dW(X_H) = 0$, so that the projection to the base is preserved, and $\mathcal{L}_{X_H} d^c h = 0$, so that $d^c h$ vanishes on a Lagrangian if and only if it vanishes on its image under the flow. The third statement follows from the observation that parallel transport along an admissible arc preserves Y^{in} and preserves the vanishing of $d^c h$ outside Y^{in} by (3-5) and (3-6). □

We also note that admissible lifted isotopies commute with the Hamiltonian flow of H, so the two operations can be performed in either order.

It will be useful for us to have a more explicit description of lifted admissible isotopies as Hamiltonian flows.

Lemma 3.8 *Given an admissible arc $\gamma : [0, \infty) \to \mathbb{C}$ and a vector field v on the complex plane which vanishes at $\gamma(0)$ and generates an admissible isotopy of arcs $\gamma_t = \rho^t(\gamma)$, we define a Hamiltonian $K_{\gamma,t,v} \in C^\infty(W^{-1}(\gamma_t), \mathbb{R})$ by*

- *$K_{\gamma,t,v} = 0$ everywhere in the fiber $W^{-1}(\gamma_t(0))$,*
- *the derivative of $K_{\gamma,t,v}$ along the horizontal lift of γ_t is*

$$dK_{\gamma,t,v}(\dot{\gamma}_t^{\#}) = \omega(\dot{\gamma}_t^{\#}, v^{\#}), \tag{3-8}$$

where $v^{\#}$ is the horizontal lift of v.

Denote by ψ^t the Hamiltonian flow generated by (an arbitrary extension of) $K_{\gamma,t,v}$.

Then, for any point $p \in W^{-1}(\gamma(0))$, ψ^t maps the horizontal lift of γ through p to the horizontal lift of γ_t through p. In particular, if L is an admissible Lagrangian which fibers over γ, then $\psi^t(L) = \rho^t(L)$.

Moreover, at every point of $W^{-1}(\gamma_t)$ which lies outside of Y^{in}, the Hamiltonian vector field $X_{\gamma,t,v}$ generated by $K_{\gamma,t,v}$ satisfies

$$dh(X_{\gamma,t,v}) = d^c h(X_{\gamma,t,v}) = 0 \quad \text{and} \quad \iota_{X_{\gamma,t,v}} dd^c h = 0. \tag{3-9}$$

Remark 3.9 The ambiguity in extending $K_{\gamma,t,v}$ to a neighborhood of $W^{-1}(\gamma_t)$ affects $X_{\gamma,t,v}$ by a multiple of $\dot{\gamma}_t^{\#}$, which does not affect the conclusions of the lemma, but implies that the isotopy ψ^t that we construct does not in general lift the isotopy ρ_t in the sense that the $W \circ \psi^t = \rho_t$. By appropriately choosing the extension of the Hamiltonian, we may arrange to have such a lift for a fixed point $p \in W^{-1}(\gamma(0))$, but the curvature of the symplectic connection on $W : Y \to \mathbb{C}$ prevents the existence of a lift simultaneously for all p.

We note for future reference that $K_{\gamma,t,v}$ can be extended to a smooth Hamiltonian whose support is contained in a small neighborhood of $W^{-1}(\gamma_t)$ and such that the corresponding vector field satisfies (3-9) everywhere outside of Y^{in}. The simplest way to do this is to foliate a neighborhood of γ in the complex plane by a family of admissible arcs γ^τ, $\tau \in (-\tau_0, \tau_0)$, and consider a Hamiltonian which equals $\chi(\tau) K_{\gamma^\tau,t,v}$ over the preimage of $\rho^t(\gamma^\tau)$, where $\chi(\tau)$ is a cut-off function with compact support.

Proof of Lemma 3.8 Since (3-8) can be rewritten as $\omega(\dot{\gamma}_t^\#, X_{\gamma,t,v}) = \omega(\dot{\gamma}_t^\#, v^\#)$, the vector field $X_{\gamma,t,v} - v^\#$ is symplectically orthogonal to $\dot{\gamma}_t^\#$, hence tangent to $W^{-1}(\gamma_t)$. It follows that the flow ψ^t maps $W^{-1}(\gamma)$ to $W^{-1}(\gamma_t)$.

Since ψ^t is a Hamiltonian diffeomorphism, it maps Lagrangian submanifolds of Y which fiber over γ to Lagrangian submanifolds which fiber over γ_t. Moreover, since v vanishes at $* = \gamma(0)$, the Hamiltonian $K_{\gamma,t,v}$ and its first derivative both vanish along $W^{-1}(*)$, hence $X_{\gamma,t,v} = 0$ everywhere in $W^{-1}(*)$. In particular, given a Lagrangian $\ell \subset W^{-1}(*)$, the flow ψ^t maps the parallel transport of ℓ over γ to the parallel transport of ℓ over γ_t. Now consider two small Lagrangian discs $\ell_1, \ell_2 \subset W^{-1}(*)$ which intersect transversely at a given point p. The parallel transports of ℓ_1 and ℓ_2 over γ intersect cleanly along the horizontal lift of γ through p, and are mapped by ψ^t to the parallel transports of ℓ_1 and ℓ_2 over γ_t, which intersect along the horizontal lift of γ_t through p. Thus, ψ^t maps horizontal lifts of γ to horizontal lifts of γ_t.

In order to prove (3-9), we consider the map $u\colon W^{-1}(*) \times [0,\infty) \times [0,t_0] \to Y$ such that $u(p,s,t)$ is the point of $W^{-1}(\gamma_t(s))$ obtained by parallel transport of p over γ_t. In other words, $u(p,0,0) = p$ and $\partial_s u = \dot{\gamma}_t^\#$.

Since the flow ψ^t maps $u(\{p\} \times [0,\infty) \times \{0\})$ to $u(\{p\} \times [0,\infty) \times \{t\})$ for all p, the vector field $X_{\gamma,t,v}$ lies in the span of $\partial_t u$ and $\partial_s u$. On the other hand, $\partial_s u = \dot{\gamma}_t^\#$ lies in the kernel of $d^c h$ and $dd^c h$ by (3-5) and (3-6).

The 2–form $u^* dd^c h$ vanishes on ∂_s, so it can be written in the form

$$u^* dd^c h = dt \wedge \alpha(s,t) + \beta(s,t),$$

where $\alpha(s,t)$ and $\beta(s,t)$ are forms on $W^{-1}(*)$. Since $\partial_t u = 0$ whenever $s = 0$, we find that $\alpha(0,t) \equiv 0$, and $\beta(0,t) = dd^c h_{|W^{-1}(*)}$ is independent of t. On the other hand, $u^* dd^c h$ is closed, so α and β are necessarily independent of s, ie $\alpha(s,t) \equiv 0$ and $\beta(s,t) \equiv \beta_0 = dd^c h_{|W^{-1}(*)}$. We conclude that the span of ∂_s and ∂_t lies in the kernel of $u^* dd^c h$, and hence $X_{\gamma,t,v}$ lies in the kernel of $dd^c h$.

Similarly, $u^* d^c h$ vanishes on ∂_s, so it can be written in the form

$$u^* d^c h = f(s,t)\, dt + \eta(s,t)$$

for $\eta(s,t)$ a 1–form on $W^{-1}(*)$. Using again the fact that $\partial_t u = 0$ for $s = 0$, we find that $f(0,t) \equiv 0$ and $\eta(0,t)$ is independent of t. Moreover, since $u^* dd^c h$ vanishes on the span of ∂_s and ∂_t, we have $\partial_s f = 0$, so that $f(s,t) \equiv 0$. This in turn implies that $u^* d^c h$ vanishes on the span of ∂_s and ∂_t, and hence $d^c h(X_{\gamma,t,v}) = 0$.

Finally, the vanishing of $dh(X_{\gamma,t,v})$ is a direct consequence of the assumption that horizontal parallel transport preserves the levels of h outside of Y^{in}. □

3.2 Maximum principle and energy estimates

Our construction of the Fukaya category of a Landau–Ginzburg model involves not only structure maps for Lagrangian Floer theory with boundary on admissible Lagrangians, but also natural morphisms and continuation maps associated to certain isotopies of admissible Lagrangians. In this section we establish the results needed to prove compactness of the corresponding moduli spaces.

Let Σ be the complement of finitely many boundary marked points on a compact Riemann surface with boundary, and Λ a moving family of admissible Lagrangian boundary conditions on Σ, ie a smoothly varying family of admissible Lagrangian submanifolds of Y, constant near the ends of each component of $\partial\Sigma$. The manner in which Λ varies along the boundary of Σ can be described by a compactly supported 1–form on $\partial\Sigma$ with values in vector fields.

We assume that Λ varies along each boundary component by a combination of

(i) a multiple of the flow of the wrapping Hamiltonian H, namely $X_H \otimes \eta$ for η a 1–form on $\partial\Sigma$, and

(ii) the lift of an admissible isotopy of the complex plane supported away from $\Delta \cup \Delta'$, where $\Delta' \supset \mathrm{crit}(W)$ (cf Remark 3.3).

We note that Lemma 3.7 asserts the invariance of the class of admissible Lagrangians under this class of isotopies. We shall impose the following (semi)positivity assumption on the isotopy:

- In the fiber direction, we require:

(3-10) The total fiberwise wrapping is nonpositive, ie $\int_{\partial\Sigma} \eta \leq 0$.

- In the base direction, denote by Γ the family of admissible arcs in the complex plane to which Λ projects outside of Δ. We assume:

(3-11) There exists an isotopy ρ^t of the complex plane rel $\Delta \cup \Delta'$, and a function $\tau \in C^\infty(\Sigma, \mathbb{R})$ which is constant near the punctures, such that along each component of $\partial\Sigma$ the arcs $\rho^\tau(\Gamma)$ vary by an admissible isotopy that moves in the clockwise direction outside of a compact set.

For example, if Γ only moves in the clockwise direction outside of a compact set (or does not move at all) then we can take the isotopy ρ^t to be trivial, and $\tau \equiv 0$.

Condition (3-10) implies the existence of a 1–form α on Σ with the following two properties:

$$\alpha \text{ is subclosed, ie } d\alpha \leq 0, \tag{3-12}$$

$$\alpha_{|\partial\Sigma} \geq \eta \text{ pointwise along } \partial\Sigma. \tag{3-13}$$

(As is customary for Floer theory, $\alpha_{|\partial\Sigma}$ and $d\alpha$ should also be compactly supported.) For instance, if η is pointwise nonpositive, then we can take $\alpha \equiv 0$.

We consider maps $u\colon \Sigma \to Y$ with boundary conditions given by Λ (ie $u(z) \in \Lambda_z$ for all $z \in \partial\Sigma$) and subject to a convergence condition (see (3-16) below), satisfying the perturbed pseudoholomorphic curve equation

$$(du - X_H \otimes \alpha + (\xi^\tau)^\# \otimes d\tau)^{0,1} = 0, \tag{3-14}$$

where ξ^t is the vector field on the complex plane which generates the isotopy ρ^t in (3-11), and $(\xi^t)^\#$ is its horizontal lift to Y. The expression (3-14) takes values in the space of $(0,1)$–forms on Σ with values in u^*TY, ie complex antilinear maps from $T\Sigma$ to u^*TY. (We only need to consider smooth maps, but as usual in Lagrangian Floer theory, the functional analysis setup involves an extension to a space of maps of suitable Sobolev regularity; see eg [35, Chapter 8].) We will also consider modifications of this equation by further adding a compactly supported inhomogeneous perturbation term for transversality purposes.

The inhomogeneous term $X_H \otimes \alpha$ in (3-14) is the same type of Hamiltonian perturbation that commonly appears in the construction of continuation maps (and other operations) in (ordinary or wrapped) Lagrangian Floer theory, and the term $(\xi^\tau)^\# \otimes d\tau$ plays a similar role in the horizontal direction. In the presence of moving boundary conditions, one frequently requires that the restriction of the inhomogeneous perturbation term to $\partial\Sigma$ generates the isotopy by which Λ varies; see eg [35, Section 8k]. However, when the variation of Λ is pointwise nonpositive everywhere along $\partial\Sigma$ the maximum principle readily holds without the need for inhomogeneous terms; our setup encompasses both cases.

The vanishing of ξ^t inside Δ' ensures that, even if the compatibility of h with the horizontal distribution is relaxed over $W^{-1}(\Delta')$ as in Remark 3.3, the quantities $dh((\xi^t)^\#)$, $d^c h((\xi^t)^\#)$ and $\iota_{(\xi^t)^\#} dd^c h$ still vanish identically outside of Y^{in}.

We only ever consider finite-energy solutions to (3-14), in the sense that the *geometric energy*

$$E_{\mathrm{geom}}(u) := \int_\Sigma |du - X_H \otimes \alpha + (\xi^\tau)^\# \otimes d\tau|^2 \, d\mathrm{vol}_\Sigma \tag{3-15}$$

is finite. The norm in the above integral is taken with respect to the metric induced by ω and J on Y, and any j–compatible metric on Σ (the integrand is conformally invariant). By the usual decay estimates for solutions of Floer's equation on strips, this is equivalent to the condition that:

(3-16) Near each puncture of Σ, u converges to a generator of the appropriate Floer complex; ie when the perturbation term is compactly supported over Σ, an intersection point between the boundary conditions Λ on either side of the puncture.

Proposition 3.10 *Assuming* (3-11), *solutions to* (3-14) *satisfy the maximum principle with respect to the quantity* $|\rho^\tau \circ W|$ (*outside of a compact subset of* $\mathbb{C}$).

Proof Outside of a compact subset of $\mathbb{C}$, W is J–holomorphic, so $w = W \circ u$ solves the perturbed Cauchy–Riemann equation

$$(dw + \xi^\tau \otimes d\tau)^{0,1} = 0. \tag{3-17}$$

Hence, $\widetilde{w} = \rho^\tau \circ w \colon \Sigma \to \mathbb{C}$ solves an unperturbed Cauchy–Riemann equation with respect to the domain-dependent complex structure $(\rho^\tau)_* j$ on the complex plane:

$$(d\widetilde{w})^{0,1}_{(\rho^\tau)_* j} = 0, \tag{3-18}$$

and the maximum principle holds at interior points. Along $\partial\Sigma$ we use a variant of the maximum principle with Neumann boundary conditions. Namely, pick local coordinates $z = s + it$ which locally identify Σ with the upper half-plane. If $|\widetilde{w}|$ has a local maximum, then necessarily

$$\partial_s |\widetilde{w}| = 0 \quad \text{and} \quad \partial_t |\widetilde{w}| < 0.$$

It follows that $\partial_s \arg(\widetilde{w}) > 0$, since otherwise $\partial_t \widetilde{w}$ would point clockwise from $\partial_s \widetilde{w}$, contradicting (3-18).

On the other hand, recall that the boundary conditions for $\widetilde{w}$ are given by the family of admissible arcs $\rho^\tau(\Gamma)$, along which the distance from the origin is strictly increasing. Thus, at a boundary maximum, $\partial_s \widetilde{w}$ points counterclockwise from the tangent vector to $\rho^\tau(\Gamma)$. This contradicts the assumption (3-11), and we conclude that $|\widetilde{w}|$ has no local maxima. □

Proposition 3.11 *Solutions to* (3-14) *satisfy the maximum principle with respect to h* (*outside of Y^{in}*).

Proof The argument is similar to other instances of the maximum principle in Floer theory: since h is weakly plurisubharmonic, its values along a holomorphic curve satisfy the maximum principle at interior points, and also at the boundary under the assumption that $d^c h$ vanishes there; the conditions (3-5)–(3-7), which govern the behavior of $d^c h$ along the directions of the inhomogeneous terms appearing in (3-14), ensure that the maximum principle continues to hold for solutions of the perturbed Cauchy–Riemann equation, as we now show by an explicit calculation.

We begin by showing that the maximum principle for $h \circ u$ holds at interior points. Let $z = x + iy$ be coordinates near a point in Σ. Since h is weakly plurisubharmonic, we have

$$\begin{aligned} 0 &\le dd^c h\big(\partial_x u - X_H \otimes \alpha(\partial_x) + \partial_x \tau \cdot (\xi^\tau)^\#, J\big(\partial_x u - X_H \otimes \alpha(\partial_x) + \partial_x \tau \cdot (\xi^\tau)^\#\big)\big) \\ &= dd^c h\big(\partial_x u - X_H \otimes \alpha(\partial_x) + \partial_x \tau \cdot (\xi^\tau)^\#, \partial_y u - X_H \otimes \alpha(\partial_y) + \partial_y \tau \cdot (\xi^\tau)^\#\big) \\ &= \big(u^*(dd^c h) - \alpha \wedge u^*(\iota_{X_H} dd^c h) + d\tau \wedge u^*(\iota_{(\xi^\tau)^\#} dd^c h)\big)(\partial_x, \partial_y). \end{aligned} \tag{3-19}$$

By the Cartan formula, we have

$$d(\iota_{X_H} d^c h) = -\iota_{X_H} dd^c h + \mathcal{L}_{X_H} d^c h, \tag{3-20}$$

where the second term vanishes by assumption (3-6), whereas $\iota_{(\xi^\tau)^\#} dd^c h = 0$ by Remark 3.2, so we conclude that

$$0 \le u^*(dd^c h) - d(u^* d^c h(X_H)) \wedge \alpha, \tag{3-21}$$

where the right-hand side is considered as a 2–form on Σ. The Leibniz rule implies that

$$d(u^* d^c h(X_H) \cdot \alpha) = d(u^* d^c h(X_H)) \wedge \alpha + u^* d^c h(X_H) \cdot d\alpha, \tag{3-22}$$

so we derive the inequality

$$0 \le u^*(dd^c h) - d(u^* d^c h(X_H)\cdot\alpha) + u^* d^c h(X_H)\cdot d\alpha. \tag{3-23}$$

The assumptions that $0 \le d^c h(X_H)$ and that α is subclosed imply that

$$0 \le u^*(dd^c h) - d(u^* d^c h(X_H)\cdot\alpha). \tag{3-24}$$

We claim that the right-hand side is the Laplacian of $h\circ u$. Indeed, since $dh(X_H) = dh((\xi^\tau)^\#) = 0$ and $d^c h((\xi^\tau)^\#) = 0$ by assumption, we compute that

$$\begin{aligned} d^c(h\circ u) = -dh\circ du\circ j &= -dh\circ(du\circ j - X_H\otimes\alpha\circ j + (\xi^\tau)^\#\otimes d\tau\circ j) \\ &= -dh\circ(J\circ du - JX_H\otimes\alpha + J(\xi^\tau)^\#\otimes d\tau) \\ &= u^*(d^c h) - u^* d^c h(X_H)\cdot\alpha + u^* d^c h((\xi^\tau)^\#)\cdot d\tau \\ &= u^*(d^c h) - u^* d^c h(X_H)\cdot\alpha. \end{aligned} \tag{3-25}$$

Hence,

$$dd^c(h\circ u) = u^*(dd^c h) - d(u^* d^c h(X_H)\cdot\alpha), \tag{3-26}$$

and comparing with (3-24), we conclude that

$$dd^c(h\circ u) \ge 0. \tag{3-27}$$

Thus, the maximum principle holds at interior points.

Along $\partial\Sigma$ we use the maximum principle with Neumann boundary conditions. For this, we need to check that, in local coordinates $z = s + it$ which locally identify Σ with the upper half-plane, the inequality $d(h\circ u)(\partial_t) \ge 0$ holds, or equivalently, $d^c(h\circ u)(\partial_s) \le 0$. We have computed above that

$$d^c(h\circ u) = u^*(d^c h) - u^* d^c h(X_H)\cdot\alpha, \tag{3-28}$$

and we now need to check that the restriction of this 1–form to $\partial\Sigma$ is everywhere nonpositive.

The vanishing of $d^c h$ on each admissible Lagrangian Λ_s, by Definition 3.5, and on the vector fields which generate lifted admissible isotopies, by Lemma 3.8, imply that the only contribution to $u^*_{|\partial\Sigma}(d^c h)$ comes from the fiberwise wrapping term $X_H\otimes\eta$ in the moving boundary condition, so

$$d^c(h\circ u)_{|\partial\Sigma} = u^* d^c h(X_H)\cdot\eta - u^* d^c h(X_H)\cdot\alpha_{|\partial\Sigma}.$$

The nonpositivity of this quantity is now immediate, since $d^c h(X_H) \ge 0$ and $\alpha_{|\partial\Sigma} \ge \eta$ pointwise by assumption. □

Remark 3.12 In our setting, rather than being smooth, h will be given by the maximum of a finite collection of smooth plurisubharmonic functions $h_{\boldsymbol{v}}$, where for each $\boldsymbol{v}$ the 1–form $d^c h_{\boldsymbol{v}}$ satisfies all the required properties wherever $h_{\boldsymbol{v}}$ achieves the maximum (ie $h_{\boldsymbol{v}} = h$) outside of Y^{in}. The above argument gives the maximum principle for all $h_{\boldsymbol{v}}$ which achieve the maximum, and hence a fortiori for $h = \max\{h_{\boldsymbol{v}}\}$.

The next result asserts the existence of a bound of the geometric energy of solutions to (3-14): such a bound is necessary to appeal to any version of Gromov's compactness theorem, and requires fixing a homotopy class β of maps from $(\Sigma, \partial\Sigma)$ to (Y, Λ) with fixed asymptotic conditions, given by generators of the Floer complexes, at the punctures of Σ. The key point is that Propositions 3.10 and 3.11 provide maximum principles for the solutions of (3-14) in both base and fiber directions, so that solutions which converge to given generators at the punctures of Σ remain within a fixed compact subset of Y. It thus suffices to bound the difference between the topological and geometric energy for solutions to a perturbed Cauchy–Riemann equation with image lying in a bounded region; this goes back all the way to Gromov's original paper [21] which established compactness for perturbed equations, and is standard for Hamiltonian perturbations. We nonetheless provide a detailed proof because of the (nonstandard) appearance of the horizontal lift in our equation.

Proposition 3.13 *There is a constant $E_{\max}(\beta)$ such that all solutions u to* (3-14) *in the homotopy class β satisfy the a priori bound*

$$E_{\mathrm{geom}}(u) \le E_{\max}(\beta) \tag{3-29}$$

Proof Let $z = x + iy$ be coordinates near a point of Σ. Since $du - X_H \otimes \alpha + (\xi^\tau)^\# \, d\tau$ is complex linear with respect to j and J, the integrand in (3-29) is equal to

$$\omega\left(\partial_x u - X_H \otimes \alpha(\partial_x) + \partial_x \tau \, (\xi^\tau)^\#, \partial_y u - X_H \otimes \alpha(\partial_y) + \partial_y \tau \, (\xi^\tau)^\#\right).$$

Since X_H is tangent to the fibers of W and $(\xi^\tau)^\#$ is horizontal, $\omega(X_H, (\xi^\tau)^\#) = 0$, and so

$$E_{\mathrm{geom}}(u) = \int_\Sigma u^*\omega - \alpha \wedge u^*(\iota_{X_H}\omega) + d\tau \wedge u^*(\iota_{(\xi^\tau)^\#}\omega). \tag{3-30}$$

This quantity is not invariant under deformations of the map u relative to the boundary condition Λ. On the other hand, the variation of Λ along the boundary of Σ is described by a vector field valued 1–form on $\partial\Sigma$ of the form

$$X_H \otimes \eta + X_K \otimes \vartheta,$$

where K is a family of Hamiltonians (dependent on the point of $\partial\Sigma$) generating the lifted isotopy, as in Lemma 3.8. Then the variation of $\int_\Sigma u^*\omega$ along a vector field v (tangent to Λ at the boundary) is equal to

$$\int_{\partial\Sigma} \omega(v, \partial_s u)\, ds = \int_{\partial\Sigma} \omega(v, X_H)\, \eta + \omega(v, X_K)\, \vartheta = \int_{\partial\Sigma} dH(v)\, \eta + dK(v)\, \vartheta,$$

so the *topological energy*

$$E_{\mathrm{top}}([u]) = \int_\Sigma u^*\omega - \int_{\partial\Sigma} u^* H \cdot \eta - \int_{\partial\Sigma} u^* K \cdot \vartheta \tag{3-31}$$

depends only on the relative homotopy class $[u]$ of the map u.

Returning to equation (3-30), Stokes' theorem expresses the second term as

$$\int_\Sigma -\alpha \wedge u^*(\iota_{X_H}\omega) = -\int_\Sigma u^* dH \wedge \alpha = -\int_{\partial\Sigma} u^* H \cdot \alpha + \int_\Sigma u^* H \cdot d\alpha.$$

Putting this together with equation (3-31), we conclude that

$$E_{\mathrm{geom}}(u) = E_{\mathrm{top}}([u]) + \int_{\partial\Sigma} u^* H \cdot (\eta - \alpha_{|\partial\Sigma}) + \int_{\Sigma} u^* H \cdot d\alpha + \int_{\partial\Sigma} u^* K \cdot \vartheta + \int_{\Sigma} d\tau \wedge u^*(\iota_{(\xi^\tau)^\#}\omega). \tag{3-32}$$

The first two integrals on the right-hand side of (3-32) are nonpositive, since $H \geq 0$ by assumption and α is required to satisfy (3-12) and (3-13).

The existence of a compact subset $\Omega \subset Y$ which a priori contains the image of u (as a consequence of Propositions 3.10 and 3.11) provides a bound for the last two terms as follows: The third integral can be bounded by $(\sup_\Omega |K|)\|\vartheta\|_{L^1(\partial\Sigma)}$, which depends only on the size of Ω and the geometric bounds on the lifted isotopy of the boundary condition Λ within the compact subset Ω. Finally, the last integral can be rewritten as

$$\int_\Sigma d\tau \wedge \big(\iota_{(\xi^\tau)^\#}\omega \circ du\big) = \int_\Sigma d\tau \wedge \big(\iota_{(\xi^\tau)^\#}\omega \circ (du - X_H \otimes \alpha + (\xi^\tau)^\# \otimes d\tau)\big). \tag{3-33}$$

Since the vector field ξ^τ vanishes at the critical values of W, the norm of its horizontal lift $(\xi^\tau)^\#$ is bounded everywhere in Ω, and we can bound (3-33) by

$$\big(\sup_\Omega |(\xi^\tau)^\#|\big)\|d\tau\|_{L^2(\Sigma)}\big\|du - X_H \otimes \alpha + (\xi^\tau)^\# \otimes d\tau\big\|_{L^2(\Sigma)}.$$

Combining these bounds, we find that

$$E_{\mathrm{geom}}(u) \leq E_{\mathrm{top}}([u]) + \big(\sup_\Omega |K|\big)\|\vartheta\|_{L^1} + \big(\sup_\Omega |(\xi^\tau)^\#|\big)\|d\tau\|_{L^2} E_{\mathrm{geom}}(u)^{1/2}. \tag{3-34}$$

This implies a bound on $E_{\mathrm{geom}}(u)$ in terms of the other quantities appearing in (3-34). □

Remark 3.14 Proposition 3.13 continues to hold if (3-14) is further modified by a compactly supported (hence uniformly bounded) inhomogeneous perturbation term.

Remark 3.15 In the next sections we will define Floer-theoretic operations in terms of certain moduli spaces of solutions to (compactly supported perturbations of) (3-14). In each case we will make specific choices for the parameters α and τ, but we note that, since the set of allowable choices is contractible and hence connected, the operations we define are independent of these up to homotopy, and likewise for other auxiliary data such as compactly supported inhomogeneous perturbation terms or deformations of the almost complex structure.

3.3 Definition of the directed category

We fix a collection $\boldsymbol{L}$ of admissible Lagrangians in Y, for which the subset Δ appearing in Definition 3.5 is always the same, and whose images in $\mathbb{C}$ agree near infinity with a fixed finite collection of radial straight lines. (In our case Δ will be the single point $\{-1\}$). Also fix a subset $\Delta' \supset \mathrm{crit}(W)$; in our case Δ' will be a small disc centered at the origin.

Let ρ be an autonomous flow on $\mathbb{C}$ which fixes $\Delta \cup \Delta'$ and the negative real axis, maps radial lines to radial lines away from a compact set, and moves all radial lines other than the negative real axis in the counterclockwise direction. This isotopy preserves the admissibility of the arcs over which the objects of $\boldsymbol{L}$ fiber outside of Δ. We define

$$L(t) := \phi^t \rho^t(L), \tag{3-35}$$

where ϕ^t is the flow of the wrapping Hamiltonian H, and ρ^t is the lifted admissible isotopy generated by ρ. Since ϕ^t and ρ^t commute, we can think of this as an autonomous flow on Y; in particular, $(L(t))(t') = L(t+t')$.

By construction, the admissible arcs over which $L(t+\lambda)$ and $L'(t)$ fiber outside of Δ are asymptotic to different straight lines for all but finitely many values of λ. We will essentially require that, in the fiberwise direction, these Lagrangians also go to infinity in different directions for generic λ, so that their intersections are contained in a compact subset. More precisely, we assume:

(3-36) There exists an open (or Baire) dense set $U \subset \mathbb{R}$ such that, for all L and L' in $\boldsymbol{L}$ and $\lambda \in U$, $L(\lambda) \cap L'$ is contained in a compact subset of Y. (The same then holds for $L(t+\lambda) \cap L'(t)$ for all $t \in \mathbb{R}$.)

(In our case it will be possible to choose the compact subset in (3-36) to be independent of L, L' and λ, but there is no reason to require this in general.)

In addition, we impose the following conditions on elements $L \in \boldsymbol{L}$:

(3-37) For all $t \in \mathbb{R}$, $L(t)$ does not bound any (unperturbed) holomorphic discs.

(3-38) L is equipped with a spin structure and with a grading (ie after choosing a holomorphic volume form Ω on Y, a lift of the phase map $\arg(\Omega_{|L})$ to $\mathbb{R}$).

Condition (3-37), which may be replaced by unobstructedness, ensures that Floer homology is well-defined; while (3-38) ensures that it is $\mathbb{Z}$–graded and can be constructed over a field of characteristic zero.

We will also on occasion equip Lagrangians in $\boldsymbol{L}$ with local systems; since this will only come up in specific places, we omit local systems from the notation for now.

Lemma 3.16 *There are arbitrarily small values of $\epsilon > 0$ such that, for each pair of Lagrangians $L_0, L_1 \in \boldsymbol{L}$, and for all integers $k_0 \neq k_1$,*

(3-39) *the images of $L_0(\epsilon k_0)$ and $L_1(\epsilon k_1)$ under W are asymptotic to different radial straight lines in $\mathbb{C}$, and $L_0(\epsilon k_0) \cap L_1(\epsilon k_1)$ is compact.*

Proof After removing a finite set of values u from the set U in condition (3-36), we can assume that for $\lambda \in U$ the images of $L(t+\lambda)$ and $L'(t)$ under W are asymptotic to different radial straight lines in $\mathbb{C}$. Now the desired properties hold whenever ϵ lies in the intersection of the sets $k^{-1} \cdot U \subset \mathbb{R}$ for all positive integers k. This is a countable intersection of Baire sets and hence dense as well. □

Choose $0 < \epsilon$ such that condition (3-39) holds for all pairs of objects.

We construct a directed category $\mathbb{O}$ with objects $L^k := L(-\epsilon k)$ for all $k \in \mathbb{Z}$ and $L \in \boldsymbol{L}$, and morphisms

$$\mathbb{O}(L_0^{k_0}, L_1^{k_1}) \equiv \begin{cases} CF^*(L_0(-\epsilon k_0), L_1(-\epsilon k_1)) & \text{if } k_0 < k_1, \\ \mathbb{K} \cdot \mathrm{id} & \text{if } k_0 = k_1 \text{ and } L_0 = L_1, \\ 0 & \text{otherwise.} \end{cases} \tag{3-40}$$

The A_∞ structure is obtained by counting solutions to pseudoholomorphic curve equations (for suitable J, see Remark 3.18) with compactly supported inhomogeneous perturbation terms (when the integers $k_0, k_1, \dots$ form a strictly increasing sequence — in all other cases the structure maps are defined tautologically). The compactly supported perturbations are used to achieve transversality, and are chosen in a consistent manner; see eg [35]. Since we work over the Novikov field, the count of solutions in each homotopy class is weighted by (topological) energy (as well as the bulk deformation class, and holonomies of local systems along the boundary of the disc when applicable).

The key compactness property required for this construction is a direct consequence of the maximum principle:

Lemma 3.17 *Given any sequence of Lagrangians $L_0, \dots, L_r \in \boldsymbol{L}$ and integers $k_0 < k_1 < \dots < k_r$, there exists a bounded subset of Y which contains the images of all J–holomorphic discs with boundary on $L_0(-\epsilon k_0) \cup \dots \cup L_r(-\epsilon k_r)$. The same property also holds in the presence of a compactly supported inhomogeneous perturbation.*

Proof This follows immediately from Propositions 3.10 and 3.11, in the special case where the Lagrangian boundary condition remains constant along each component of $\partial\Sigma$ and there are no perturbation terms. □

Remark 3.18 Disc bubbling is excluded by assumption (3-37), but sphere bubbling can happen in our setting, so the regularity of the moduli spaces we consider is not immediate.

To deal with sphere bubbling, we assume that J is chosen generically within a suitable class of compatible almost-complex structures, so that simple J–holomorphic spheres are regular, and evaluation maps at interior points for somewhere injective J–holomorphic curves are mutually transverse — see [31, Theorem 3.4.1] for the closed case; the argument works similarly for discs. For our main example the standard complex structure is not regular, but all holomorphic spheres lie inside $W^{-1}(0)$, so it is enough to perturb J in a neighborhood of $W^{-1}(0)$, or, in fact, its intersection with the bounded subset provided by Lemma 3.17, so that the conditions we have set in Section 3.1 on the geometry at infinity are not affected.

With this understood, bubbling of simple J–holomorphic spheres is a real codimension 2 phenomenon, and does not affect our ability to count solutions to Floer's equations in zero-dimensional moduli spaces, or to compare counts of solutions by considering one-dimensional moduli spaces. Moreover, since

$c_1(Y) = 0$ we need not worry about multiply covered sphere bubbles either. Indeed, regularity for simple spheres implies that for generic J the union of the images of all pseudoholomorphic spheres in Y has real codimension 4. By transversality of evaluation maps, it is therefore disjoint from the images of holomorphic discs (or solutions to Floer's equation) in (a fixed countable collection of) zero- or one-dimensional moduli spaces.

3.4 Quasiunits and continuation maps

The next ingredient in the construction of the fiberwise wrapped category $\mathcal{W}(Y, W)$ is a distinguished collection of morphisms

$$e_{L^k} \in HF^0(L^k, L^{k+1}) \tag{3-41}$$

for all $L \in \boldsymbol{L}$ and $k \in \mathbb{Z}$, called *quasiunits*.

The quasiunit e_{L^k} is the image of the identity in $H^0(L)$ under a PSS-type homomorphism from $H^*(L)$ to $HF(L^k, L^{k+1})$ which can be constructed exactly as in [8] (see below for the specific case at hand); note however that the reverse map from $HF(L^k, L^{k+1})$ to $H^*(L)$ is not well-defined in our setting, as it involves Floer data for which the analytic estimates of Section 3.2 do not hold. (Nonetheless, given that our Lagrangians do not bound any holomorphic discs, the PSS map often turns out to be an isomorphism for small enough ϵ, under additional geometric assumptions which ensure that L^{k+1} is contained within a Weinstein tubular neighborhood of L^k; this is the case in our main example, by Proposition 5.11.)

Chain-level quasiunits can be constructed by counting solutions to a Cauchy–Riemann equation with moving boundary condition, whose domain Σ is a disc with a single boundary puncture which we consider as an output, and where the boundary condition Λ is given by the isotopy $L^t = L(-\epsilon t)$, $t \in [k, k+1]$, parametrized using some choice of monotonically increasing smooth function from $\partial\Sigma$ to $[k, k+1]$ which is constant near the ends. Since the isotopy along $\partial\Sigma$ moves the complex plane in the clockwise direction and wraps fiberwise in the negative direction only, we can apply the results of Section 3.2, with $\alpha \equiv 0$ and $\tau \equiv 0$, to control the behavior of solutions. We denote again by

$$e_{L^k} \in CF^0(L(-\epsilon k), L(-\epsilon(k+1))) = \mathcal{O}(L^k, L^{k+1}) \tag{3-42}$$

the chain-level quasiunit constructed in this manner. While e_{L^k} depends on auxiliary choices (eg of a function from $\partial\Sigma$ to $[k, k+1]$), the chain-level quasiunits constructed using different choices only differ by an explicit homotopy, and can be used interchangeably.

Let Z denote the collection of all such morphisms. The fiberwise wrapped category $\mathcal{W}(Y, W)$ is the localization of $\mathcal{O}$ with respect to these morphisms:

$$\mathcal{W}(Y, W) := Z^{-1}\mathcal{O}, \tag{3-43}$$

ie the quotient of $\mathcal{O}$ by the cones of the morphisms in Z, in the sense of Lyubashenko and Ovsienko [30]; see also [16, Section 3.1.3], as well as Section 3.5 below. We shall use a concrete model of the morphisms

in $\mathcal{W}(Y, W)$, introduced in the next section, in which they are expressed as homotopy colimits (ie direct limits) of morphism spaces in $\mathbb{O}$. In order to compute these morphism spaces explicitly in terms of Floer theory, we shall introduce *continuation maps*

$$F_{L_0^k, L_1^j} : \mathbb{O}(L_0^k, L_1^j) \to \mathbb{O}(L_0^{k+1}, L_1^{j+1}). \tag{3-44}$$

These are defined by counting solutions to a perturbed Cauchy–Riemann equation, with domain $\Sigma = \mathbb{R} \times [0, 1]$, and where the boundary conditions are given by $\Lambda_{s,0} = L_0^{k+\chi(s)} = L_0(-\epsilon(k + \chi(s)))$ along $\mathbb{R} \times \{0\}$ and $\Lambda_{s,1} = L_1^{j+\chi(s)} = L_1(-\epsilon(j + \chi(s)))$ along $\mathbb{R} \times \{1\}$. Here $\chi : \mathbb{R} \to [0, 1]$ is a monotonically decreasing smooth function, constant near the ends, so that the boundary conditions are (L_0^k, L_1^j) at the input end $s \to +\infty$, and (L_0^{k+1}, L_1^{j+1}) at the output end $s \to -\infty$.

We use the setup of Section 3.2, with a fiberwise wrapping perturbation given by $\alpha = -\epsilon\chi'(s)\, ds$ (so that $d\alpha = 0$ and $\alpha_{|\partial\Sigma} = \eta$), and a horizontal perturbation given by the autonomous flow ρ and $\tau = \epsilon\chi(s)$ (so ρ^τ exactly cancels the horizontal isotopy of the boundary condition). Propositions 3.10, 3.11 and 3.13 then imply that the counts of index 0 solutions to (3-14) (weighted by topological energy) can be used to define $F_{L_0^k, L_1^j}$.

Despite the slight differences in technical setup, these continuation maps have all the usual properties of continuation maps associated to symplectic isotopies in Lagrangian Floer theory: they are quasiisomorphisms, and extend to an A_∞–functor $F : \mathbb{O} \to \mathbb{O}$, which acts on objects by $L^k \mapsto L^{k+1}$. Since we shall not need these properties, we omit the proofs.

Lemma 3.19 *The quasiunits are natural with respect to continuation maps, in the sense that both triangles in the diagram*

$$\begin{tikzcd}
\mathbb{O}(L_0^{k+1}, L_1^j) \arrow[r, "\mu^2(e_{L_1^j}, \cdot)"] \arrow[d, "\mu^2(\cdot, e_{L_0^k})"'] & \mathbb{O}(L_0^{k+1}, L_1^{j+1}) \arrow[d, "\mu^2(\cdot, e_{L_0^k})"] \\
\mathbb{O}(L_0^k, L_1^j) \arrow[r, "\mu^2(e_{L_1^j}, \cdot)"] \arrow[ur, "F_{L_0^k, L_1^j}"] & \mathbb{O}(L_0^k, L_1^{j+1})
\end{tikzcd} \tag{3-45}$$

are commutative up to homotopy.

Proof We start with the upper triangle, ie the homotopy between $F(\mu^2(\cdot, e_{L_0^k}))$ and $\mu^2(e_{L_1^j}, \cdot)$. The argument relies on comparing a series of moduli spaces of perturbed holomorphic curves, presented pictorially on Figure 2, where the thick edges correspond to regions where the Lagrangian boundary condition is moving, and the shaded areas correspond to the support of the inhomogeneous perturbation terms in (3-14).

The main protagonists in the homotopy are a family of perturbed holomorphic strips with domain $\Sigma = \mathbb{R} \times [0, 1]$, depicted on the right-hand side of Figure 2. Fix two disjoint compact intervals $I_1, I_2 \subset \mathbb{R}$,

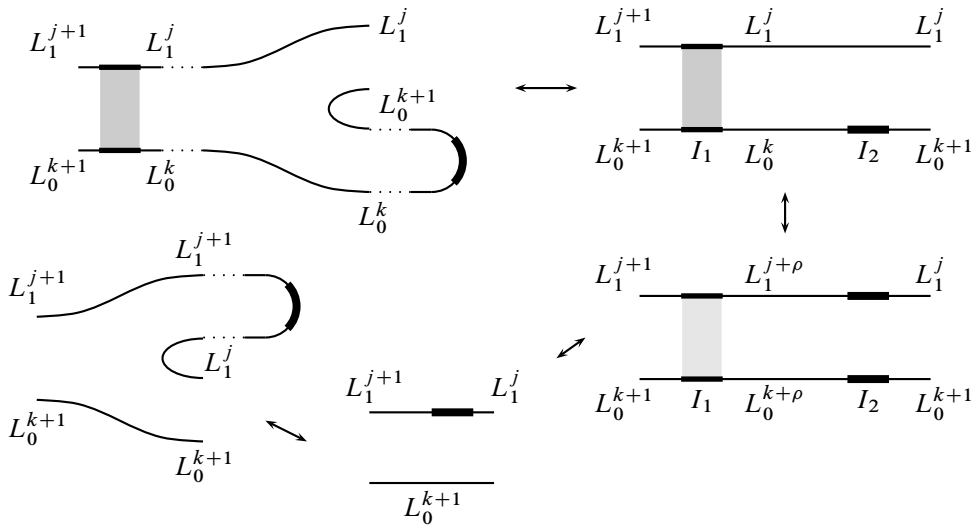

Figure 2: A homotopy between $\mu^2(e_{L_1^j}, \cdot)$ and $F(\mu^2(\cdot, e_{L_0^k}))$.

with I_1 to the left of I_2, as well as two smooth monotonic functions $\chi_1, \chi_2 \colon \mathbb{R} \to [0,1]$ such that χ_1 equals 1 to the left of I_1 and 0 to its right, while χ_2 equals 0 to the left of I_2 and 1 to its right; we arrange that the "profiles" of these functions are identical to those used in the construction of the continuation maps and quasiunits. Also fix a parameter $\rho \in [0,1]$, and define

$$k_\rho(s) = k + \rho + (1-\rho)\chi_1(s) + (1-\rho)\chi_2(s),$$
$$j_\rho(s) = j + \rho + (1-\rho)\chi_1(s) - \rho\chi_2(s).$$

Along $\mathbb{R} \times \{0\}$ we consider the moving boundary condition $\Lambda_{0,s} = L_0^{k_\rho(s)}$, while along $\mathbb{R} \times \{1\}$ we use $\Lambda_{1,s} = L_1^{j_\rho(s)}$. While the boundary condition $\Lambda_{1,s}$ always moves in the negative direction as s decreases (j_ρ is a monotonic function of s), the boundary condition $\Lambda_{0,s}$ moves in the positive direction over I_1. Accordingly, we set $\alpha = -\epsilon(1-\rho)\chi_1'(s)\,ds$, and $\tau = \epsilon(1-\rho)\chi_1(s)$, for the perturbation terms in (3-14).

By Section 3.2 the solutions to (3-14) with these boundary conditions and perturbations satisfy maximum principles and energy estimates, so we can define operations

$$\Phi_{I_1,I_2,\rho} \colon \mathbb{O}(L_0^{k+1}, L_1^j) \to \mathbb{O}(L_0^{k+1}, L_1^{j+1})$$

by counting rigid (index 0) solutions. These operations are chain maps, since the ends of the moduli spaces of index 1 solutions for fixed I_1, I_2 and ρ are in bijection with the broken trajectories which contribute to $\partial \circ \Phi_{I_1,I_2,\rho}$ and $\Phi_{I_1,I_2,\rho} \circ \partial$; and they are all homotopic to each other, with explicit homotopies given by counts of index -1 solutions that may arise as the parameters I_1, I_2 and ρ vary, as can be seen by considering the ends of moduli spaces of index 0 solutions for a one-parameter family of choices of I_1, I_2 and ρ. (These are standard arguments in Lagrangian Floer theory, so we omit the details; see eg [9, Section 2], [35, Section 17] or [8] for similar proofs.)

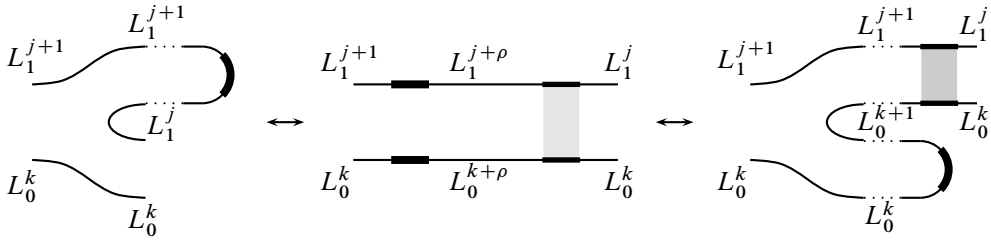

Figure 3: A homotopy between $\mu^2(e_{L_1^j}, \cdot)$ and $\mu^2(F(\cdot), e_{L_0^k})$.

For $\rho = 0$, the boundary conditions and perturbations near I_1 are identical to those used to define the continuation map, while along $I_2 \times \{0\}$ the boundary condition $\Lambda_{0,s}$ varies from L_0^k to L_0^{k+1} (top right diagram in Figure 2). Moving I_1 towards $-\infty$ and shrinking I_2 to a point then causes the solutions to converge to limit configurations consisting of (typically) three components (upper left diagram in Figure 2). The "main" component is an unperturbed holomorphic disc with two inputs, corresponding to the Floer product μ^2, while at $s = -\infty$ we have a strip with moving boundary conditions and inhomogeneous perturbations, corresponding to the continuation map F, and the rescaling limit near $I_2 \times \{0\}$ gives a half-plane with a moving boundary condition, which corresponds to the quasiunit. Thus, the operations $\Phi_{I_1,I_2,\rho}$ are homotopic to $F(\mu^2(\cdot, e_{L_0^k}))$.

On the other hand, for $\rho = 1$, there are no perturbations near I_1, and along $I_2 \times \{1\}$ the boundary condition $\Lambda_{1,s}$ varies from L_1^j to L_1^{j+1}. Shrinking I_2 to a point then causes a holomorphic half-plane with moving boundary condition to break off (lower left diagram in Figure 2), showing that $\Phi_{I_1,I_2,\rho}$ is also homotopic to $\mu^2(e_{L_1^j}, \cdot)$.

The commutativity up to homotopy of the lower triangle in (3-45) is proved in exactly the same manner, by considering a family of perturbed holomorphic strips depicted in Figure 3. The construction is identical, except that the roles of the two regions I_1 and I_2 are now reversed. By considering the limit configurations as ρ becomes 0 or 1 while the leftmost interval is degenerated to a point and the rightmost interval escapes towards $s = +\infty$, this yields a homotopy between $\mu^2(e_{L_1^j}, \cdot)$ (Figure 3, left) and $\mu^2(F(\cdot), e_{L_0^k})$ (Figure 3, right). □

Remark 3.20 Lemma 3.19 can be strengthened to show that the quasiunits form the leading-order term of a natural transformation e from the identity to the A_∞–functor F. The next (order 1) term in the natural transformation is precisely the homotopy between $\mu^2(F(\cdot), e_{L_0^k})$ and $\mu^2(e_{L_1^j}, \cdot)$ that arises in the proof of Lemma 3.19, ie it can be defined by counting index -1 solutions that come up in the family of perturbed Cauchy–Riemann equations depicted in Figure 3. The construction of the higher-order terms of the natural transformation is technically more involved, and we do not discuss it here since we will not be needing it.

Since the localization at all quasiunits amounts to making the natural transformation e invertible up to homotopy, the localized category $\mathcal{W}(Y, W)$ is also sometimes denoted by $\mathcal{O}[e^{-1}]$; this notation is also

suggestive of the fact that the localization effectively enlarges morphism spaces by inverting all quasiunits (up to homotopy).

Remark 3.21 It is natural to ask to what extent the category $\mathcal{W}(Y, W)$ depends on the choice of the collection of Lagrangians $\boldsymbol{L}$ and on the parameter ϵ (the time-step with respect to which we consider quasiunits). Here we do not address the first question, which relates to the existence of generation criteria for $\mathcal{W}(Y, W)$; we simply assume that we have a collection $\boldsymbol{L}$ satisfying the required hypotheses, and if this collection is too small the category we construct might only be a subcategory of the one we would obtain from a larger collection of objects.

On the other hand, the algebraic properties of quasiunits imply that the choice of the parameter ϵ does not affect the outcome of our construction. The key observation is that we can define quasiunits $e_{L(t')\to L(t)} \in HF^0(L(t'), L(t))$ for all $t' > t$ such that $L(t') \cap L(t)$ is contained in a compact subset of Y, and an argument similar to the proof of Lemma 3.19 shows that, for $t'' > t' > t$, these satisfy

$$e_{L(t')\to L(t)} \cdot e_{L(t'')\to L(t')} = e_{L(t'')\to L(t)}$$

(in cohomology, or up to homotopy). Assume that $L(t')$ and $L(t)$ are both objects of $\mathbb{O}$ for some $t' > t$, and let n be such that $n\epsilon > t'-t$. Since $e_{L(t+n\epsilon)\to L(t)}$ is the product of the quasiunits $e_{L(t+k\epsilon)\to L(t+(k-1)\epsilon)}$ for $1 \le k \le n$, it is a quasiisomorphism in the localized category, hence admits a quasiinverse $f_{L(t)\to L(t+n\epsilon)}$; similarly for $e_{L(t')\to L(t'-n\epsilon)}$, whose quasiinverse we denote by $f_{L(t'-n\epsilon)\to L(t')}$. Then in $H^0\mathcal{W}$ we have

$$\begin{aligned} e_{L(t')\to L(t)} \cdot (e_{L(t+n\epsilon)\to L(t')} \cdot f_{L(t)\to L(t+n\epsilon)}) &= \mathrm{id}_{L(t)}, \\ (f_{L(t'-n\epsilon)\to L(t')} \cdot e_{L(t)\to L(t'-n\epsilon)}) \cdot e_{L(t')\to L(t)} &= \mathrm{id}_{L(t')}, \end{aligned}$$

giving left and right inverses for $e_{L(t')\to L(t)}$ up to homotopy and proving that it is a quasiisomorphism. Hence, localizing with respect to quasiunits for a fixed step size ϵ actually inverts *all* quasiunits; and $L(t)$ and $L(t')$ are quasiisomorphic in the localized category whenever they belong to the set of objects. This implies that up to quasiequivalence the category we construct does not depend on the choice of ϵ.

3.5 Fiberwise wrapped category via colimits

Our goal in this section is to construct the fiberwise wrapped Fukaya category as a subcategory of the category of modules over $\mathbb{O}$. This approach is adapted from unpublished work [5] of the first author with Seidel, where the starting point is the more abstract formalism of localization of categories, and the point of view which we take here is used as a computational tool.

The basic idea is that we seek an A_∞–category where morphism spaces between Lagrangians are taken after passing to a limit with respect to positive wrapping. We implement this by assigning to each Lagrangian L an object of the category of modules over $\mathbb{O}$ given as a homotopy colimit (or direct limit)

$$\mathcal{Y}_{L^\infty} \equiv \operatorname*{hocolim}_{k\to+\infty} \mathcal{Y}_{L^k}, \tag{3-46}$$

where $\mathcal{Y}_{L^k}$ is the Yoneda module

$$X \mapsto \mathcal{O}(X, L^k), \tag{3-47}$$

and the connecting maps $\mathcal{Y}_{L^k} \to \mathcal{Y}_{L^{k+1}}$ are given by composition with the quasiunits e_{L^k}. We take as model for the homotopy colimit the mapping telescope

$$\operatorname{Cone}\Big(\bigoplus_{k=0}^{\infty} \mathcal{Y}_{L^k} \to \bigoplus_{k=0}^{\infty} \mathcal{Y}_{L^k}\Big), \tag{3-48}$$

where the arrow is the direct sum of the differences $\mathrm{id} - e_{L^k}$.

We write $\mathcal{W}$ for the full subcategory of modules over $\mathcal{O}$ with these objects, ie objects are admissible Lagrangians in $\boldsymbol{L}$, and morphisms between L_0 and L_1 given by

$$\mathcal{W}(L_0, L_1) \equiv \operatorname{Hom}_{\mathcal{O}}(\mathcal{Y}_{L_0^\infty}, \mathcal{Y}_{L_1^\infty}). \tag{3-49}$$

The first computation we need is:

Lemma 3.22 *There is a natural quasiisomorphism*

$$\operatorname*{holim}_{k\to\infty} \operatorname*{hocolim}_{j\to\infty} \mathcal{O}(L_0^k, L_1^j) \to \mathcal{W}(L_0, L_1). \tag{3-50}$$

Proof The cone of the complex

$$\operatorname{Hom}_{\mathcal{O}}\Big(\bigoplus_{k=0}^{\infty} \mathcal{Y}_{L_0^k}, \mathcal{Y}_{L_1^\infty}\Big) \to \operatorname{Hom}_{\mathcal{O}}\Big(\bigoplus_{k=0}^{\infty} \mathcal{Y}_{L_0^k}, \mathcal{Y}_{L_1^\infty}\Big) \tag{3-51}$$

maps quasiisomorphically to the space of morphisms from $\mathcal{Y}_{L_0^\infty}$ to $\mathcal{Y}_{L_1^\infty}$, and is isomorphic to the cone of the map

$$\prod_{k=0}^{\infty} \operatorname{Hom}_{\mathcal{O}}(\mathcal{Y}_{L_0^k}, \mathcal{Y}_{L_1^\infty}) \to \prod_{k=0}^{\infty} \operatorname{Hom}_{\mathcal{O}}(\mathcal{Y}_{L_0^k}, \mathcal{Y}_{L_1^\infty}), \tag{3-52}$$

which is a model for

$$\operatorname*{holim}_{k} \operatorname{Hom}_{\mathcal{O}}(\mathcal{Y}_{L_0^k}, \mathcal{Y}_{L_1^\infty}). \tag{3-53}$$

On the other hand, the Yoneda map induces a quasiisomorphism

$$\operatorname*{hocolim}_{j\to\infty} \mathcal{O}(L_0^k, L_1^j) \to \operatorname*{hocolim}_{j\to\infty} \operatorname{Hom}_{\mathcal{O}}(\mathcal{Y}_{L_0^k}, \mathcal{Y}_{L_1^j}) \cong \operatorname{Hom}_{\mathcal{O}}(\mathcal{Y}_{L_0^k}, \mathcal{Y}_{L_1^\infty}). \tag{3-54}$$

The desired map follows by composition. □

The next result reduces the computation of morphisms in $\mathcal{W}$ to a direct limit.

Lemma 3.23 *For all L_0, L_1 and k, the map*

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{Y}_{L_0^{k+1}}, \mathcal{Y}_{L_1^\infty}) \to \mathrm{Hom}_{\mathcal{O}}(\mathcal{Y}_{L_0^k}, \mathcal{Y}_{L_1^\infty}) \tag{3-55}$$

induced by multiplication by quasiunits is a quasiisomorphism.

Proof The Yoneda lemma reduces the problem to the statement that the map

$$\operatorname*{hocolim}_{j\to\infty} \mathcal{O}(L_0^{k+1}, L_1^j) \to \operatorname*{hocolim}_{j\to\infty} \mathcal{O}(L_0^k, L_1^j) \tag{3-56}$$

induces an isomorphism on cohomology. Since direct limits commute with passing to cohomology, it suffices to show that the map of cohomology groups

$$\operatorname*{colim}_{j\to\infty} HF^*(L_0^{k+1}, L_1^j) \to \operatorname*{colim}_{j\to\infty} HF^*(L_0^k, L_1^j) \tag{3-57}$$

is an isomorphism, where we use the fact that the morphisms in $\mathcal{O}$ are given by Floer cochains whenever j is sufficiently large. We claim that the continuation maps from equation (3-44) provide an inverse. Indeed, by taking the cohomology of diagram (3-45) we obtain a commutative diagram

$$\begin{tikzcd}
HF^*(L_0^{k+1}, L_1^j) \arrow[r, "\mu^2(e_{L_1^j},\cdot)"] \arrow[d, "\mu^2(\cdot,e_{L_0^k})"'] & HF^*(L_0^{k+1}, L_1^{j+1}) \arrow[d, "\mu^2(\cdot,e_{L_0^k})"] \\
HF^*(L_0^k, L_1^j) \arrow[r, "\mu^2(e_{L_1^j},\cdot)"] \arrow[ur, "F_{L_0^k,L_1^j}"] & HF^*(L_0^k, L_1^{j+1})
\end{tikzcd} \tag{3-58}$$

in which the horizontal maps are those used to define the direct limits, while the vertical maps assemble into the map (3-57).

To show that (3-57) is injective, note that every element of the left-hand side is represented by an element of $HF^*(L_0^{k+1}, L_1^j)$ for some j. The above diagram implies that the image of this element in $HF^*(L_0^{k+1}, L_1^{j+1})$ agrees with the image under our proposed inverse (the continuation map $F_{L_0^k,L_1^j}$) of its image under the map of direct limits (3-57). By definition of the direct limit, this implies that the continuation map is a left inverse to (3-57), and injectivity follows.

Considering the composition in the other order yields surjectivity: every element of the right-hand side of (3-57) is represented by an element of $HF^*(L_0^k, L_1^j)$ for some j, whose image in $HF^*(L_0^k, L_1^{j+1})$ is also the image under (3-57) of its image under the continuation map, so the continuation map is a right inverse. □

Corollary 3.24 *For each pair L_0^k and L_1 of objects of $\mathcal{O}$, there is a natural isomorphism*

$$\operatorname*{colim}_{j\to\infty} HF^*(L_0^k, L_1^j) \to H\mathcal{W}(L_0, L_1). \tag{3-59}$$

Proof The above lemma implies that bonding maps in the inverse system appearing in equation (3-50) are quasiisomorphisms. In particular, the Mittag-Leffler condition is satisfied,[2] and for each integer k the projection map

$$\operatorname*{holim}_{k\to\infty} \operatorname*{hocolim}_{j\to\infty} \mathcal{O}(L_0^k, L_1^j) \to \operatorname*{hocolim}_{j\to\infty} \mathcal{O}(L_0^k, L_1^j) \tag{3-60}$$

induces an isomorphism on cohomology. Inverting this map, and composing with the one induced by equation (3-50) on cohomology yields the desired isomorphism. □

Remark 3.25 The most straightforward way to compare our construction with the approach of [5] is to consider the localization functor from $\mathcal{O}$–modules to $\mathcal{O}[e^{-1}]$–modules. By the universal property of localization, the images of the Yoneda objects L^k are equivalent, hence the image of the colimit $\mathcal{Y}_{L^\infty}$ under localization is equivalent to these Yoneda modules. Lemma 3.23 can be restated as the fact that the modules $\mathcal{Y}_{L^\infty}$ lie in the e–local subcategory of $\mathcal{O}$–modules, which is quasiisomorphic to the category of $\mathcal{O}[e^{-1}]$–modules. We therefore conclude that the category generated by the modules $\mathcal{Y}_{L^\infty}$ is equivalent to the localization of $\mathcal{O}$, which is the point of view taken by [5].

4 Kähler forms and admissibility

In this section, we study the geometry of parallel transport in toric Landau–Ginzburg models, and construct suitable Kähler forms for which *fiberwise monomial admissibility* is preserved by parallel transport; we then show that the technical assumptions we have made in the previous section follow from this property.

Definition 4.1 A *fiberwise monomial subdivision* for the toric Landau–Ginzburg model $W\colon Y \to \mathbb{C}$ consists of a finite collection of toric monomials $z^{\boldsymbol{v}}$, $\boldsymbol{v} \in \mathcal{V} \subset \mathbb{Z}^{n+1}$, weights $d(\boldsymbol{v}) \in \mathbb{Z}_{>0}$, open subsets $C_{\boldsymbol{v}} \subset Y$, and a closed subset $\Omega \subset Y$, such that

(1) $z^{\boldsymbol{v}} \in \mathcal{O}(Y)$ for all $\boldsymbol{v} \in \mathcal{V}$, and $z \mapsto (z^{\boldsymbol{v}})_{\boldsymbol{v}\in\mathcal{V}}$ defines a proper map $Y \to \mathbb{C}^{|\mathcal{V}|}$,

(2) the restriction of W to Ω is a proper map,

(3) $\Omega \cup \bigcup_{\boldsymbol{v}\in\mathcal{V}} C_{\boldsymbol{v}} = Y$, and

(4) for $z \in Y \setminus \Omega$, if $|z^{\boldsymbol{v}_0}|^{1/d(\boldsymbol{v}_0)} = \max\{|z^{\boldsymbol{v}}|^{1/d(\boldsymbol{v})} \mid \boldsymbol{v} \in \mathcal{V}\}$ then $z \in C_{\boldsymbol{v}_0}$.

Definition 4.2 Given a fiberwise monomial subdivision, a Lagrangian submanifold $\ell \subset W^{-1}(c) \cong (\mathbb{C}^*)^n$ is *monomially admissible* with phase angles $\{\varphi_{\boldsymbol{v}}, \boldsymbol{v} \in \mathcal{V}\}$ if, outside of the compact subset $W^{-1}(c) \cap \Omega$, $\arg(z^{\boldsymbol{v}}) = \varphi_{\boldsymbol{v}}$ at every point of $\ell \cap C_{\boldsymbol{v}}$.

A Lagrangian submanifold $L \subset Y$ is *fiberwise monomially admissible* with phase angles $\{\varphi_{\boldsymbol{v}}\}$ if, outside of Ω, $\arg(z^{\boldsymbol{v}}) = \varphi_{\boldsymbol{v}}$ at every point of $L \cap C_{\boldsymbol{v}}$.

[2] An inverse system $A_1 \leftarrow A_2 \leftarrow A_3 \leftarrow \cdots$ is said to satisfy the Mittag-Leffler condition if for each k, there exists $j > k$ such that, for all $i > j$, the maps $A_i \to A_k$ and $A_j \to A_k$ have the same image; this condition implies vanishing of the first derived functor of the inverse limit, and that inverse limits are well-behaved with respect to cohomology; see eg [39, Definition 3.5.6].

Example 4.3 We can define a fiberwise monomial subdivision for the toric Landau–Ginzburg model $\left(\mathbb{C}^N, W_0 = -\prod z_j\right)$ as follows (the construction below will be a slight modification of this example). Take the collection of monomials to be the coordinate functions z_j, $1 \le j \le N$ (ie the exponent vectors $\boldsymbol{v}_j$ are the standard basis of $\mathbb{Z}^N$); take $d(\boldsymbol{v}_j) = 1$ for all j, and let $C_{\boldsymbol{v}_j}$ be the set of points of $\mathbb{C}^N$ where $|z_j| > K\,|W_0|^{1/N}$ for some constant $K > 1$, and $\Omega = \mathbb{C}^N \setminus \bigcup C_{\boldsymbol{v}_j} = \{z \in \mathbb{C}^N \mid \max(|z_j|) \le K\,|W_0|^{1/N}\}$. Condition (2) holds since the coordinates of points of Ω are bounded by $K\,|W_0|^{1/N}$, and condition (4) holds since if $|z_j| = \max(|z_1|, \dots, |z_N|) > K\,|W_0|^{1/N}$ then $z \in C_{\boldsymbol{v}_j}$. A Lagrangian submanifold $L \subset \mathbb{C}^N$ is then fiberwise monomially admissible with phase angles $\varphi_1, \dots, \varphi_N$ if, at every point of L where $|z_j| > K|W_0|^{1/N}$, one has $\arg(z_j) = \varphi_j$. For instance, the real positive locus $(\mathbb{R}_+)^N$ satisfies this condition with all phase angles equal to zero. We shall see below how to build more interesting examples under the assumption that the toric Kähler form on $\mathbb{C}^N$ is chosen suitably; see Section 4.1.

The notions of monomial subdivision and monomial admissibility for Lagrangians in $(\mathbb{C}^*)^n$ already appear in Andrew Hanlon's thesis [24]. One technical difference is that we consider a fiberwise version of monomial admissibility and its compatibility with parallel transport between the fibers of W. The more important difference is philosophical: we use monomial admissibility merely as a technical tool to ensure the flatness condition of Definition 3.5(ii), rather than as a geometric way of restricting the fiberwise wrapping by introducing additional stops (though we will do so in a sequel (work in progress) for mirrors of hypersurfaces in toric varieties).

4.1 A toric Kähler form on $\mathbb{C}^N$

We first consider the case of $\mathbb{C}^N$ equipped with a complete toric Kähler form $\omega = dd^c\Phi$ (for Φ a $\mathbb{T}^N$–invariant Kähler potential) and the superpotential $W_0 = -\prod z_j$. Writing $z_j = \exp(\rho_j + i\theta_j)$, we have

$$\omega = dd^c\Phi = \sum_{i,j} \frac{\partial^2\Phi}{\partial\rho_i\,\partial\rho_j} d\rho_i \wedge d\theta_j .$$

In particular, ω is a Kähler form if and only if the potential is a strictly convex function of the ρ–coordinates, ie the Hessian matrix $\Psi = (\partial^2\Phi/\partial\rho_i\,\partial\rho_j)_{ij}$ is positive definite. The moment map $\mu = (\mu_1, \dots, \mu_N) \colon \mathbb{C}^N \to \mathbb{R}^N$ is given by the partial derivatives of Φ:

$$\mu_j = \partial\Phi/\partial\rho_j .$$

The horizontal distribution, ie the symplectic orthogonal to the level sets of W_0, is spanned (over $\mathbb{C}$) by the Hamiltonian vector field generated by $\log|W_0| = \sum \rho_j$. We can express $d\log|W_0|$ as a linear combination of the differentials of the moment maps,

$$d\log|W_0| = \sum_j d\rho_j = \sum_j \lambda_j\, d\mu_j, \quad \text{where } (\lambda_1, \dots, \lambda_N) = \Psi^{-1}(1, \dots, 1). \tag{4-1}$$

Angular parallel transport (ie along circles centered at the origin in the base of the fibration given by W_0) is then given by rotating each coordinate at a rate proportional to λ_i, so that the horizontal lifts of the angular and radial vector fields are given by

$$(\partial_\theta)^\# = \frac{\sum \lambda_j \partial_{\theta_j}}{\sum \lambda_j} \quad \text{and} \quad (r\partial_r)^\# = -i(\partial_\theta)^\# = \frac{\sum \lambda_j \partial_{\rho_j}}{\sum \lambda_j}. \tag{4-2}$$

One checks that the quantities $\mu_j - \mu_i$ are conserved by parallel transport, as expected (since parallel transport is equivariant with respect to the standard Hamiltonian $\mathbb{T}^{N-1}$–action on the fibers of W_0).

Example 4.4 For the standard Kähler form on $\mathbb{C}^N$, with potential $\Phi = \frac{1}{4}\sum |z_j|^2 = \frac{1}{4}\sum e^{2\rho_j}$, the moment map is given by $\mu_j = \frac{1}{2}|z_j|^2$, and Ψ is diagonal with entries $|z_j|^2$, so that $\lambda_j = |z_j|^{-2}$, and $(\partial_\theta)^\# = \big(1/\sum |z_j|^{-2}\big)\sum |z_j|^{-2}\partial_{\theta_j}$. Thus, when $|z_j| \to \infty$ for $|W_0|$ fixed, the rate of change of $\arg(z_j)$ under angular parallel transport tends to zero. This in turn implies that a weaker form of asymptotic admissibility (only requiring arguments of monomials to converge to prescribed limit values at infinity) is preserved under parallel transport, and it should be possible to carry out the whole construction using the standard Kähler form. However, the stronger admissibility requirement that we impose is necessary for the maximum principle of Proposition 3.11; thus we will need to ensure that $\arg(z_j)$ remains strictly constant (rather than approximately constant) under parallel transport, and this in turn motivates the introduction of a different Kähler form.

Our choice of Kähler form involves smooth approximations of the maximum function:

Definition 4.5 Given a constant $\delta > 0$, denote by $M : \mathbb{R}^2 \to \mathbb{R}$ a smooth convex function such tha:

(1) $M(u, v) = \max(u, v)$ whenever $|u - v| \geq \delta$,

(2) $M(u + a, v + a) = M(u, v) + a$ for all $u, v, a \in \mathbb{R}$, and

(3) $M(u, v) = M(v, u)$.

These conditions imply that M is monotonically increasing with either variable, and

$$\max(u, v) \leq M(u, v) \leq \max(u, v) + \delta, \quad 0 \leq \frac{\partial M}{\partial u} \leq 1, \quad 0 \leq \frac{\partial M}{\partial v} \leq 1.$$

We then define $\widehat{M} : \mathbb{R}^2_{\geq 0} \to \mathbb{R}_{\geq 0}$ by

$$\begin{aligned} \widehat{M}(U, V) &= \exp M(\log U, \log V) \quad \text{for } U, V > 0, \\ \widehat{M}(U, 0) &= \widehat{M}(0, U) = U, \end{aligned}$$

and note that $\widehat{M}$ is continuous, smooth everywhere except at the origin, and $\widehat{M}(U, V) = \max(U, V)$ whenever $U/V \notin (e^{-\delta}, e^{\delta})$.

In fact, the second condition above implies that M is determined by a smoothing, near the origin, of the absolute value function on $\mathbb{R}$.

Definition 4.6 Choosing some small $\varepsilon > 0$, we equip $\mathbb{C}^N$ with $\omega = dd^c\Phi$, where

$$\Phi = \sum_{i=1}^{N} \widehat{M}\Big(\varepsilon, \prod_{j=1,\, j\neq i}^{N} \widehat{M}(|z_i|^2, |z_j|^2)\Big)|z_i|^2. \tag{4-3}$$

Remark 4.7 The only purpose of taking $\widehat{M}(\varepsilon, \dots)$ is that otherwise ω would be degenerate (and nonsmooth) along the coordinate axes. In fact,

$$\prod_{j\neq i} \widehat{M}(|z_i|^2, |z_j|^2) \geq \prod_{j\neq i} \max(|z_i|^2, |z_j|^2) \geq \frac{|W_0|^2}{\min\{|z_1|^2, \dots, |z_N|^2\}} \geq |W_0|^{2(N-1)/N},$$

so we have the simpler expression

$$\Phi = \sum_{i=1}^{N}\Big(\prod_{j\neq i} \widehat{M}(|z_i|^2, |z_j|^2)\Big)|z_i|^2 \quad \text{whenever } |W_0|^2 \geq (\varepsilon e^{\delta})^{N/(N-1)}. \tag{4-4}$$

Since we will only consider Lagrangian submanifolds which stay away from the preimage of a small disc under W_0, choosing ε and δ sufficiently small we can always work with the simpler formula (4-4) to study the geometry of admissible Lagrangians.

Lemma 4.8 *This ω is a toric Kähler form on $\mathbb{C}^N$.*

Proof Φ is obviously $\mathbb{T}^N$–invariant, and we will momentarily check that outside of the coordinate axes it is strictly convex as a function of the variables $\rho_j = \log|z_j|$. Meanwhile, smoothness and nondegeneracy of the Hessian near $z_i = 0$ follow from the observation that the coefficient $\widehat{M}(\varepsilon, \dots)$ in the ith term of (4-3) is bounded below by $\varepsilon > 0$.

To prove the strict convexity of Φ outside of the coordinate axes, we observe that each term in the sum (4-3) is *log-convex* as a function of $\rho_j = \log|z_j|$, ie its logarithm is convex. Indeed, using the convexity of M and the fact that the composition of a convex monotonically increasing function with a convex function is itself convex, we find that

$$\varphi_i(\rho_1, \dots, \rho_N) := M\Big(\log\varepsilon, \sum_{j\neq i} M(2\rho_i, 2\rho_j)\Big) + 2\rho_i$$

is a convex function. Since the exponential function is strictly increasing and strictly convex, we conclude that

$$\Phi_i(\rho_1, \dots, \rho_N) = e^{\varphi_i(\rho_1,\dots,\rho_N)} = \widehat{M}\Big(\varepsilon, \prod_{j\neq i} \widehat{M}(|z_i|^2, |z_j|^2)\Big)|z_i|^2$$

is a convex function, and that its Hessian is nondegenerate on all tangent vectors which are transverse to the level sets of φ_i, ie $d^2\Phi_i(v, v) > 0$ whenever $d\varphi_i(v) \neq 0$.

Thus, in order to conclude that $\Phi = \sum \Phi_i$ is strictly convex, it suffices to show that $d\varphi_1, \dots, d\varphi_N$ are everywhere linearly independent. Equivalently, we need to show that the matrix A with entries

$a_{ij} = \partial\varphi_i/\partial\rho_j$ is invertible. We only do this in the region where $\varphi_i = \sum_{j\neq i} M(2\rho_i, 2\rho_j) + 2\rho_i$ for simplicity; in light of Remark 4.7 this is the only case of genuine interest to us.

Let $\widehat{A} = A + A^T$, with entries $\widehat{a}_{ij} = a_{ij} + a_{ji} = \partial\varphi_i/\partial\rho_j + \partial\varphi_j/\partial\rho_i$. For $i \neq j$, it follows from property (2) of Definition 4.5 that

$$\widehat{a}_{ij} = \frac{\partial}{\partial\rho_i} M(2\rho_i, 2\rho_j) + \frac{\partial}{\partial\rho_j} M(2\rho_i, 2\rho_j) = 2.$$

Meanwhile, $\widehat{a}_{ii} = 2\,\partial\varphi_i/\partial\rho_i \geq 4$. Thus, given any nonzero vector $v \in \mathbb{R}^N$,

$$\langle v, \widehat{A}v\rangle = \sum_{i,j=1}^{N} \widehat{a}_{ij} v_i v_j = 2\left(\sum_i v_i\right)^2 + \sum_i (\widehat{a}_{ii} - 2)\, v_i^2 > 0,$$

and it follows that $\langle v, Av\rangle = \frac{1}{2}\langle v, \widehat{A}v\rangle$ is positive as well, which implies that A is invertible, and hence Φ is strictly convex. □

The key feature of the Kähler form ω which makes it possible for fibered Lagrangians to be fiberwise monomially admissible is that all "large" coordinates are preserved under parallel transport. We first make the notion of "large" coordinate more precise.

Definition 4.9 A partition $\{1, \dots, N\} = K \sqcup J$ into two nonempty subsets is called a δ–gap at a point $(z_1, \dots, z_N) \in \mathbb{C}^N$ if $\inf\{|z_i|^2 \mid i \in J\} \geq e^\delta \sup\{|z_i|^2 \mid i \in K\}$. We say that z_j lies above a δ–gap if there exists a δ–gap $\{1, \dots, N\} = K \sqcup J$ with $j \in J$.

Lemma 4.10 *If* $|z_\ell| \geq e^{\frac{1}{4}(N-1)\delta}|W_0|^{1/N}$, *or if* $|z_\ell| \geq e^{\frac{1}{2}(N-1)\delta} \min|z_i|$, *then* z_ℓ *lies above a* δ*–gap.*

Proof Assume z_ℓ does not lie above any δ–gap. Then listing all $|z_i|^2$ in decreasing order, the entry just after $|z_\ell|^2$ (if there is one) is bounded below by $e^{-\delta}|z_\ell|^2$, the next one is bounded below by $e^{-2\delta}|z_\ell|^2$, and so on, whereas the entries preceding $|z_\ell|^2$ are bounded below by $|z_\ell|^2$. Thus, we conclude that $\min|z_i|^2 > e^{-(N-1)\delta}|z_\ell|^2$, and $|W_0|^2 = \prod_{i=1}^N |z_i|^2 > e^{-\frac{1}{2}N(N-1)\delta}|z_\ell|^{2N}$. Taking the square root and the $2N^{\text{th}}$ root of both sides of these respective inequalities, we obtain a contradiction. □

Lemma 4.11 *Assume that* $|W_0|^2 \geq (\varepsilon e^\delta)^{N/(N-1)}$, *and that* z_ℓ *lies above a* δ*–gap. Then the coordinate* z_ℓ *is* (*locally*) *invariant under parallel transport.*

Before giving the proof, we provide some intuition by briefly considering the case $N = 2$: when $|z_2|^2 \geq e^\delta|z_1|^2$, our Kähler potential is $\Phi = |z_1|^2|z_2|^2 + |z_2|^4$, and ω is locally a product Kähler form when expressed in the coordinates (W_0, z_2), which readily implies that parallel transport for W_0 preserves z_2. Alternatively, the first component of the moment map is $\mu_1 = \partial\Phi/\partial\rho_1 = 2|W_0|^2$, as is also the case more generally whenever z_1 is the smallest coordinate and separated from $z_2, \dots, z_N$ by a δ–gap. Since $d\log|W_0|$ is proportional to $d\mu_1$, comparing (4-1) and (4-2) we conclude that only z_1 varies

along the horizontal distribution, while $z_2, \dots, z_N$ are preserved. (However, as parallel transport towards $|W_0| \to \infty$ proceeds by varying z_1 while $z_2, \dots, z_N$ remain fixed, eventually $|z_1|$ becomes large enough to "close" the δ–gap and the statement no longer holds.) The argument in the general case is less explicit but similarly involves the vanishing of certain coefficients in (4-1).

Proof Let $\{1, \dots, N\} = K \sqcup J$ be a δ–gap with $\ell \in J$. Recall that the Kähler potential is given by (4-4), ie $\Phi = \sum_{i=1}^N e^{\varphi_i}$, where $\varphi_i = \sum_{j \neq i} M(2\rho_i, 2\rho_j) + 2\rho_i$. Property (1) of Definition 4.5 implies that, for $i \in J$ and $k \in K$, $\partial\varphi_i/\partial\rho_k \equiv 0$, whereas for $i \in K$ and $j \in J$, $\partial\varphi_i/\partial\rho_j \equiv 2$. Thus, for $k \in K$ and $j \in J$,

$$\Psi_{jk} = \frac{\partial^2 \Phi}{\partial\rho_j \partial\rho_k} = \sum_{i=1}^N \left(\frac{\partial^2 \varphi_i}{\partial\rho_j \partial\rho_k} + \frac{\partial\varphi_i}{\partial\rho_j} \frac{\partial\varphi_i}{\partial\rho_k} \right) e^{\varphi_i} = \sum_{i \in K} 2 \frac{\partial\varphi_i}{\partial\rho_k} e^{\varphi_i}, \tag{4-5}$$

which is *independent of* j. We denote this quantity by c_k.

Next, property (2) of Definition 4.5 implies that for all i we have $\sum_{m=1}^N \partial\varphi_i/\partial\rho_m = 2N$, and for $i \in K$ we have $\sum_{m \in K} \partial\varphi_i/\partial\rho_m = 2N - 2|J| = 2|K|$. Thus,

$$\sum_{m \in K} \frac{\partial\Phi}{\partial\rho_m} = \sum_{i,m \in K} \frac{\partial\varphi_i}{\partial\rho_m} e^{\varphi_i} = 2|K| \sum_{i \in K} e^{\varphi_i}.$$

Differentiating with respect to ρ_k for $k \in K$, we find that

$$\frac{1}{|K|} \sum_{m \in K} \Psi_{mk} = \sum_{i \in K} 2 \frac{\partial\varphi_i}{\partial\rho_k} e^{\varphi_i} = c_k. \tag{4-6}$$

The nondegeneracy of ω implies that the symmetric matrix Ψ is positive definite, and in particular its restriction $\Psi_{|K}$ to the coordinates labeled by elements of K is also nondegenerate. For $k \in K$, denote by λ_k the components of $(\Psi_{|K})^{-1}(1, \dots, 1)$, ie by definition $\sum_{k \in K} \Psi_{ik}\lambda_k = 1$ for all $i \in K$. Averaging over $i \in K$ and using (4-6), we also have $\sum_{k \in K} c_k \lambda_k = 1$. Thus, using (4-5) we find that

$$\sum_{k \in K} \Psi_{ik}\lambda_k = 1 \quad \text{for all } i = 1, \dots, N.$$

Setting $\lambda_j = 0$ for $j \in J$, we conclude that $\Psi^{-1}(1, \dots, 1) = (\lambda_1, \dots, \lambda_N)$, ie $(\lambda_1, \dots, \lambda_N)$ are the coefficients which appear in (4-1) and (4-2).

For $j \in J$, the vanishing of λ_j implies that the jth components of $(\partial_\theta)^\#$ and $(r\partial_r)^\#$ are zero, and thus, parallel transport preserves z_j. This is true, in particular, for $j = \ell$. □

We conclude this section with some estimates for the moment map, which will be used to establish the analogue of Lemma 4.11 for Kähler forms obtained from ω by symplectic reduction. Since the formula for the moment map is obviously equivariant under permutation of the variables, it suffices to consider the case where $|z_1| \leq |z_2| \leq \cdots \leq |z_N|$.

Lemma 4.12 *Assume that* $|W_0|^2 \geq (\varepsilon e^{\delta})^{N/(N-1)}$, *and that* $|z_1| \leq |z_2| \leq \cdots \leq |z_N|$. *Then*:

(1) $0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_N$,

(2) $2 \leq \dfrac{\mu_j}{|z_j|^{2j} \prod_{i=j+1}^N |z_i|^2} \leq 4Ne^{2N\delta}$ *for all* $1 \leq j \leq N$,

(3) $(2N)^{-1/(2N)} e^{-\delta} \left(\dfrac{\mu_j}{\mu_k}\right)^{1/(2N)} \leq \dfrac{|z_j|}{|z_k|} \leq (2N)^{1/2} e^{N\delta} \left(\dfrac{\mu_j}{\mu_k}\right)^{1/2}$ *for all* $1 \leq k < j \leq N$.

Proof Recall that, by (4-4), $\Phi = \sum e^{\varphi_i}$ with $\varphi_i = \sum_{j \neq i} M(2\rho_i, 2\rho_j) + 2\rho_i$. Thus,

$$\mu_j = \frac{\partial \Phi}{\partial \rho_j} = \sum_{i=1}^N \frac{\partial \varphi_i}{\partial \rho_j} e^{\varphi_i} = \sum_{i \neq j} \frac{\partial M(2\rho_i, 2\rho_j)}{\partial \rho_j} (e^{\varphi_i} + e^{\varphi_j}) + 2\, e^{\varphi_j}. \tag{4-7}$$

We first establish the inequality (1). For $j < k$, we have $|z_j| \leq |z_k|$ by assumption, and using the monotonicity of M we immediately deduce that $\varphi_j \leq \varphi_k$. Moreover, for $i \notin \{j, k\}$, the convexity of M implies that $0 \leq \partial M(2\rho_i, 2\rho_j)/\partial \rho_j \leq \partial M(2\rho_i, 2\rho_k)/\partial \rho_k$, and hence

$$\frac{\partial M(2\rho_i, 2\rho_j)}{\partial \rho_j} (e^{\varphi_i} + e^{\varphi_j}) \leq \frac{\partial M(2\rho_i, 2\rho_k)}{\partial \rho_k} (e^{\varphi_i} + e^{\varphi_k}).$$

Properties (2) and (3) of Definition 4.5 and the convexity of M imply that $\partial M(2\rho_j, 2\rho_k)/\partial \rho_j \leq 1 \leq \partial M(2\rho_j, 2\rho_k)/\partial \rho_k$, so

$$\frac{\partial M(2\rho_j, 2\rho_k)}{\partial \rho_j} (e^{\varphi_j} + e^{\varphi_k}) + 2e^{\varphi_j} \leq \frac{\partial M(2\rho_j, 2\rho_k)}{\partial \rho_k} (e^{\varphi_j} + e^{\varphi_k}) + 2e^{\varphi_k}.$$

Combining these inequalities we conclude that $\mu_j \leq \mu_k$, which proves (1).

To establish (2), we first observe that $|z_{i_2}|^2 \leq \widehat{M}(|z_{i_1}|^2, |z_{i_2}|^2) \leq e^{\delta} |z_{i_2}|^2$ for $i_1 < i_2$. Therefore,

$$|z_j|^{2j} \prod_{i>j} |z_i|^2 \leq e^{\varphi_j} = \left(\prod_{i \neq j} \widehat{M}(|z_i|^2, |z_j|^2) \right) |z_j|^2 \leq e^{N\delta} |z_j|^{2j} \prod_{i>j} |z_i|^2.$$

Since $\mu_j \geq 2\, e^{\varphi_j}$ by (4-7), the lower bound on e^{φ_j} immediately yields the lower bound in (2). Meanwhile, to obtain an upper bound on μ_j we observe that in the sum (4-7) the terms corresponding to i such that $|z_i|^2 \geq e^{\delta} |z_j|^2$ vanish identically, since for such i we have $M(2\rho_i, 2\rho_j) \equiv 2\rho_i$. Otherwise, the inequality $2\rho_i \leq 2\rho_j + \delta$ implies that $\varphi_i \leq \varphi_j + N\delta$. Meanwhile, $\partial M(2\rho_i, 2\rho_j)/\partial \rho_j \leq 2$. Thus,

$$\mu_j \leq \sum_{i \neq j,\, 2\rho_i \leq 2\rho_j + \delta} 2(e^{\varphi_i} + e^{\varphi_j}) + 2e^{\varphi_j} \leq (2N + 2(N-1)e^{N\delta}) e^{\varphi_j} \leq 4Ne^{2N\delta} |z_j|^{2j} \prod_{i>j} |z_i|^2.$$

This establishes the upper bound in (2). Finally, (3) is a direct consequence of (2) using the observation that

$$\left(\frac{|z_j|}{|z_k|}\right)^{2k} \leq \frac{|z_j|^{2j} \prod_{i=j+1}^N |z_i|^2}{|z_k|^{2k} \prod_{i=k+1}^N |z_i|^2} = \left(\frac{|z_j|}{|z_k|}\right)^{2k} \prod_{i=k+1}^{j-1} \frac{|z_j|^2}{|z_i|^2} \leq \left(\frac{|z_j|}{|z_k|}\right)^{2j-2},$$

which in turn implies that

$$\left(\frac{|z_j|^{2j} \prod_{i=j+1}^N |z_i|^2}{|z_k|^{2k} \prod_{i=k+1}^N |z_i|^2}\right)^{1/(2N)} \leq \frac{|z_j|}{|z_k|} \leq \left(\frac{|z_j|^{2j} \prod_{i=j+1}^N |z_i|^2}{|z_k|^{2k} \prod_{i=k+1}^N |z_i|^2}\right)^{1/2}. \qquad \square$$

4.2 Symplectic reduction and monomial admissibility

Recall that the toric variety Y described in Section 2 is the symplectic reduction of $\mathbb{C}^{P_{\mathbb{Z}}}$ by a subtorus $\mathbb{T}_M \subset \mathbb{T}^{P_{\mathbb{Z}}}$, ie $Y = \mu^{-1}(\lambda)/\mathbb{T}_M$, and the superpotential $W_0 \in \mathcal{O}(\mathbb{C}^{P_{\mathbb{Z}}})$ descends to $W \in \mathcal{O}(Y)$. We equip $\mathbb{C}^{P_{\mathbb{Z}}}$ with the toric Kähler form constructed in the previous section, and the reduced space Y with the induced toric Kähler form.

Our goal in this section is to show that symplectic reduction preserves the compatibility of parallel transport with fiberwise monomial admissibility, ie to establish an analogue of Lemma 4.11 for symplectic parallel transport between the fibers of $W : Y \to \mathbb{C}$. Our starting point is the observation that "parallel transport commutes with reduction":

Lemma 4.13 *The horizontal vector fields $(\partial_\theta)^\#$ and $(r\partial_r)^\#$ described by* (4-2), *which span the symplectic orthogonal to the fibers of $W_0 : \mathbb{C}^{P_{\mathbb{Z}}} \to \mathbb{C}$, are $\mathbb{T}_M$–equivariant and tangent to $\mu^{-1}(\lambda)$. Their images under the projection from $\mu^{-1}(\lambda)$ to $\mu^{-1}(\lambda)/\mathbb{T}_M = Y$ span the symplectic orthogonal to the fibers of $W : Y \to \mathbb{C}$ with respect to the reduced Kähler form, and in fact they are the horizontal lifts to Y of ∂_θ and $r\partial_r$.* □

To take advantage of this property, we need a criterion to determine when a $\mathbb{T}_M$–invariant monomial on $\mathbb{C}^{P_{\mathbb{Z}}}$ involves only coordinates to which Lemma 4.11 applies.

Recall that the moment polytope Δ_Y of Y, given by (2-6), arises as the intersection of an affine linear subspace of $\mathbb{R}^{P_{\mathbb{Z}}}$ (expressing the condition $\mu = \lambda$) with the nonnegative orthant (the moment polytope for $\mathbb{C}^{P_{\mathbb{Z}}}$). Embedding Δ_Y into $\mathbb{R}^{P_{\mathbb{Z}}}$ in this way, the coordinate hyperplanes correspond to the facets of Δ_Y, and the ambient coordinates (ie the components of the moment map for $\mathbb{C}^{P_{\mathbb{Z}}}$) are given by the affine distances to the facets of Δ_Y. Thus, in terms of the description (2-6), the point $(\xi, \eta) \in \Delta_Y \subset \mathbb{R}^n \oplus \mathbb{R}$ corresponds to a $\mathbb{T}^{n+1}$–orbit in Y whose preimage in $\mu^{-1}(\lambda) \subset \mathbb{C}^{P_{\mathbb{Z}}}$ is the $\mathbb{T}^{P_{\mathbb{Z}}}$–orbit whose moment map coordinates $(\mu_\alpha)_{\alpha \in P_{\mathbb{Z}}}$ are given by

$$\mu_\alpha = \eta - \langle \alpha, \xi \rangle + \nu(\alpha) \quad \text{for all } \alpha \in P_{\mathbb{Z}}. \tag{4-8}$$

Given a vector $\boldsymbol{v} = (\vec{v}, v^0) \in \mathbb{Z}^n \oplus \mathbb{Z}$, the toric monomial $z^{\boldsymbol{v}}$ defines a regular function on Y if and only if it pairs nonnegatively with the inward normal vector to each facet of Δ_Y, ie

$$v^\alpha := (-\alpha, 1) \cdot \boldsymbol{v} = v^0 - \alpha \cdot \vec{v} \geq 0 \quad \text{for all } \alpha \in P_{\mathbb{Z}}. \tag{4-9}$$

The monomial $z^{\boldsymbol{v}}$ vanishes to order v^α along the toric divisor of Y corresponding to $\alpha \in P_{\mathbb{Z}}$. Moreover, the monomial $\prod_{\alpha \in P_{\mathbb{Z}}} z_\alpha^{v^\alpha} \in \mathcal{O}(\mathbb{C}^{P_{\mathbb{Z}}})$ is invariant under the $\mathbb{T}_M$–action and descends to $z^{\boldsymbol{v}} \in \mathcal{O}(Y)$ under reduction. With a slight abuse of notation, we will therefore write

$$z^{\boldsymbol{v}} = \prod_{\alpha \in P_{\mathbb{Z}}} z_\alpha^{v^\alpha}. \tag{4-10}$$

The vectors $\boldsymbol{v}$ satisfying (4-9) are the integer points of a polyhedral convex cone, whose extremal rays are in one-to-one correspondence with the facets of P.

Definition 4.14 Given a facet of the polytope P with primitive outward normal vector $\vec{v}$, contained in the affine hyperplane $\langle \vec{v}, \cdot \rangle = v^0$, the corresponding extremal vector is $\boldsymbol{v} = (\vec{v}, v^0)$; we denote the set of these vectors by $\mathcal{V}$.

The elements of $\mathcal{V}$ can be characterized equivalently as the primitive inward normal vectors to the n–dimensional cones which lie on the boundary of the fan Σ_Y, or as the primitive tangent vectors to the unbounded edges of the moment polytope Δ_Y.

For $\boldsymbol{v} \in \mathcal{V}$ we denote by $A_{\boldsymbol{v}}$ the set of all $\alpha \in P_{\mathbb{Z}}$ which lie on the corresponding facet of P, ie those α for which $\alpha \cdot \vec{v} = v^0$, or equivalently, the quantity v^α defined by (4-9) vanishes. These correspond exactly to the facets of Δ_Y to which $\boldsymbol{v}$ is parallel.

Given a small positive constant $\gamma > 0$ and $\boldsymbol{v} = (\vec{v}, v^0) \in \mathcal{V}$, we define

$$S_{\boldsymbol{v},\gamma} = \{\xi \in \mathbb{R}^n \mid \langle \alpha, \xi \rangle - \nu(\alpha) < \varphi(\xi) - \gamma \|\xi\| \text{ for all } \alpha \in P_{\mathbb{Z}} \setminus A_{\boldsymbol{v}}\}, \tag{4-11}$$

where $\|\cdot\|$ is an arbitrary norm (eg the Euclidean norm) on $\mathbb{R}^n$. In other terms, recalling that $\varphi(\xi) = \max\{\langle \alpha, \xi \rangle - \nu(\alpha) \mid \alpha \in P_{\mathbb{Z}}\}$, $S_{\boldsymbol{v},\gamma}$ is the set of points where the maximum is achieved by some $\alpha \in A_{\boldsymbol{v}}$, and no $\alpha \notin A_{\boldsymbol{v}}$ comes close to the maximum. We also define $C_{\boldsymbol{v},\gamma} \subset Y$ to be the inverse image of $S_{\boldsymbol{v},\gamma} \times \mathbb{R}$ under the moment map.

Denote by Δ_α the polyhedral subset of $\mathbb{R}^n$ where α achieves the maximum in φ (which is also the projection to $\mathbb{R}^n$ of the corresponding facet of Δ_Y). Then $S_{\boldsymbol{v},\gamma}$ is a retract of $\bigcup_{\alpha \in A_{\boldsymbol{v}}} \Delta_\alpha$, obtained by removing those points which are too close (within distance of the order of $\gamma\|\xi\|$) to some other Δ_α. Thus, for sufficiently small γ, the subsets $S_{\boldsymbol{v},\gamma}$ for $\boldsymbol{v} \in \mathcal{V}$ cover the complement of a compact subset of $\mathbb{R}^n$.

Example 4.15 Consider $f(x_1, x_2) = 1 + x_1 + x_2 + t^{2\pi} x_1 x_2 + t^{4\pi} x_1^2$ (as in Example 2.1) and its tropicalization $\varphi(\xi_1, \xi_2) = \max\{0, \xi_1, \xi_2, \xi_1 + \xi_2 - 1, 2\xi_1 - 2\}$. The convex hull P of $P_{\mathbb{Z}} = \{(0,0), (1,0), (0,1), (1,1), (2,0)\}$ is a trapezoid with primitive outward normal vectors $\vec{v}_1 = (0,-1)$, $\vec{v}_2 = (1,1)$, $\vec{v}_3 = (0,1)$ and $\vec{v}_4 = (-1,0)$, and we find that $\mathcal{V}$ consists of the four elements $\boldsymbol{v}_1 = (0,-1,0)$, $\boldsymbol{v}_2 = (1,1,2)$, $\boldsymbol{v}_3 = (0,1,1)$ and $\boldsymbol{v}_4 = (-1,0,0)$, which are indeed the tangent vectors to the unbounded edges of the moment polytope $\Delta_Y = \{(\xi_1, \xi_2, \eta) \mid \eta \geq \varphi(\xi_1, \xi_2)\}$, shown "from above" in Figure 4.

For $\boldsymbol{v}_1 = (0,-1,0)$, the elements of $P_{\mathbb{Z}}$ which lie on the facet of P with outward normal vector $\vec{v}_1 = (0,-1)$ are $\alpha_1 = (0,0)$, $\alpha_2 = (1,0)$ and $\alpha_5 = (2,0)$, whereas $P_{\mathbb{Z}} \setminus A_{\boldsymbol{v}_1}$ consists of $\alpha_3 = (0,1)$ and $\alpha_4 = (1,1)$, so

$$S_{\boldsymbol{v}_1,\gamma} = \{\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \mid \xi_2 < \varphi(\xi) - \gamma\|\xi\| \text{ and } \xi_1 + \xi_2 - 1 < \varphi(\xi) - \gamma\|\xi\|\}$$

is the set of points where the two terms ξ_2 and $\xi_1 + \xi_2 - 1$ are sufficiently far from achieving the maximum in $\varphi(\xi_1, \xi_2)$; see Figure 4. This is a retract of the region $\Delta_{\alpha_1} \cup \Delta_{\alpha_2} \cup \Delta_{\alpha_5}$ where the maximum is achieved by one of the three other terms. Similarly for the other regions $S_{\boldsymbol{v}_i,\gamma}$. □

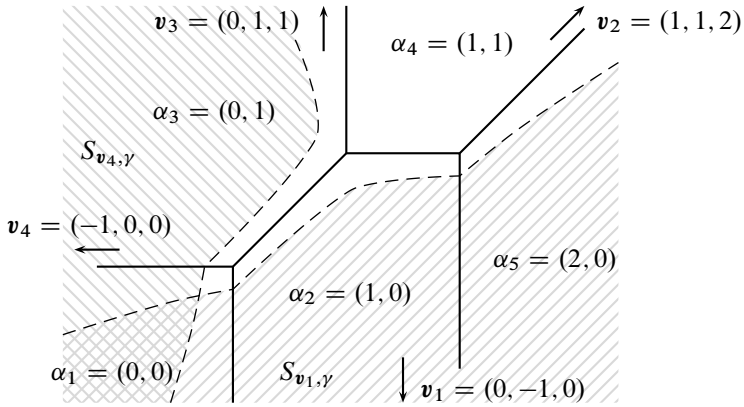

Figure 4: The extremal vectors $\boldsymbol{v} \in \mathcal{V}$ and the regions $S_{\boldsymbol{v},\gamma}$, for $f(x_1,x_2) = 1 + x_1 + x_2 + t^{2\pi}x_1x_2 + t^{4\pi}x_1^2$; cf Example 4.15.

For $c \in \mathbb{C}^*$, the fiber $W^{-1}(c)$ of $W\colon Y \to \mathbb{C}$ is $\mathbb{T}^n$–invariant, and its image under the moment map is the graph $\{(\xi,\eta) \in \mathbb{R}^n \oplus \mathbb{R} \mid \eta = f_c(\xi)\}$ of a piecewise smooth function $f_c\colon \mathbb{R}^n \to \mathbb{R}$ (with $f_c(\xi) > \varphi(\xi)$ everywhere). We now show that, outside of a bounded subset of $W^{-1}(c)$ (whose size depends on c), the monomial $z^{\boldsymbol{v}}$ is locally preserved by parallel transport at all points of $C_{\boldsymbol{v},\gamma}$.

Proposition 4.16 *Let $z \in W^{-1}(c) \cap C_{\boldsymbol{v},\gamma} \subset Y$ for some $\boldsymbol{v} \in \mathcal{V}$ and $\gamma > 0$, with moment map coordinates (ξ,η) for $\xi \in S_{\boldsymbol{v},\gamma}$. Assume that $|c|^2 \geq (\varepsilon e^{\delta})^{N/(N-1)}$ and $\|\xi\| \geq R = R(c,\gamma) := 8N^2 e^{N(N+3)\delta}\gamma^{-1}|c|^2$, where $N = |P_{\mathbb{Z}}|$, and ε and δ are the same constants as in Section 4.1. Then the monomial $z^{\boldsymbol{v}} \in \mathcal{O}(Y)$ is locally invariant under parallel transport.*

Example 4.17 Continuing with Example 4.15, consider the case of $\boldsymbol{v}_1 = (0,-1,0)$, for which the quantities v^α defined by (4-9) are $0,0,1,1,0$ for $\alpha_1,\dots,\alpha_5$, respectively. Thus, $z^{\boldsymbol{v}_1} \in \mathcal{O}(Y)$ arises by toric reduction from the monomial $z_3z_4 \in \mathcal{O}(\mathbb{C}^{P_{\mathbb{Z}}})$, which is indeed invariant under the action of the two-dimensional torus $\mathbb{T}_M$ described in Example 2.2. By Lemma 4.11, the monomial z_3z_4 is invariant under parallel transport for $W_0\colon \mathbb{C}^{P_{\mathbb{Z}}} \to \mathbb{C}$ wherever z_3 and z_4 lie above a δ–gap. The main ingredient of the proof is therefore to prove that such a gap exists whenever $\xi \in S_{\boldsymbol{v}_1,\gamma}$ and $\|\xi\|$ is sufficiently large; the key point being that, by (4-8), when $\xi \in S_{\boldsymbol{v}_1,\gamma}$ the moment map coordinates μ_{α_3} and μ_{α_4} are bounded below by $\min(\mu_{\alpha_i}) + \gamma\|\xi\|$.

Proof Denote by $(z_\alpha)_{\alpha\in P_{\mathbb{Z}}}$ the coordinates of a lift of $z \in Y$ to $\mu^{-1}(\lambda) \subset \mathbb{C}^{P_{\mathbb{Z}}}$, and let $\alpha_0 \in P_{\mathbb{Z}}$ be such that $\xi \in \Delta_{\alpha_0}$. Then by (4-8) the smallest moment map coordinate is $\min(\mu_\alpha) = \mu_{\alpha_0} = \eta - \varphi(\xi) = f_c(\xi) - \varphi(\xi)$. On the other hand, Lemma 4.12(2) gives a bound on the ratio between μ_{α_0} and $|W_0(z)|^2 = |c|^2$. We conclude that

$$2|c|^2 \leq \mu_{\alpha_0} = f_c(\xi) - \varphi(\xi) \leq 4Ne^{2N\delta}|c|^2. \tag{4-12}$$

On the other hand, since $\xi \in S_{\boldsymbol{v},\gamma}$ and $\|\xi\| \geq R$, for all $\alpha \notin A_{\boldsymbol{v}}$ we have

$$\mu_\alpha = f_c(\xi) - \langle \alpha, \xi \rangle + \nu(\alpha) \geq \mu_{\alpha_0} + \gamma \|\xi\| \geq \gamma R = 8N^2 e^{N(N+3)\delta} |c|^2 \geq 2N e^{N(N+1)\delta}\, \mu_{\alpha_0}.$$

Hence, by Lemma 4.12(3),

$$\frac{|z_\alpha|}{|z_{\alpha_0}|} \geq (2N)^{-1/2N} e^{-\delta} \left(\frac{\mu_\alpha}{\mu_{\alpha_0}} \right)^{1/2N} \geq e^{(N-1)\delta/2}.$$

By Lemma 4.10, we conclude that z_α lies above a δ–gap for all $\alpha \notin A_{\boldsymbol{v}}$. Hence, by Lemma 4.11 the coordinates z_α for $\alpha \notin A_{\boldsymbol{v}}$ are locally invariant under parallel transport in $\mathbb{C}^{P_\mathbb{Z}}$. Using the fact that the exponents v^α in (4-10) vanish for all $\alpha \in A_{\boldsymbol{v}}$ (by definition of $A_{\boldsymbol{v}}$) and the compatibility of parallel transport with reduction (Lemma 4.13), we conclude that $z^{\boldsymbol{v}}$ is locally preserved under parallel transport in Y. □

Finally, we show that, at every point where $\|\xi\|$ is sufficiently large, Proposition 4.16 applies to the largest (in a suitably renormalized sense) among the monomials $z^{\boldsymbol{v}}$ where $\boldsymbol{v} \in \mathcal{V}$. More precisely, for $\boldsymbol{v} \in \mathcal{V}$ and v^α as in (4-9), we set

$$d(\boldsymbol{v}) = \sum_{\alpha \in P_\mathbb{Z}} v^\alpha. \tag{4-13}$$

Proposition 4.18 *There exist positive constants γ_0 and K_0, depending only on the polytope Δ_Y (and on N, ε and δ) with the following property. Let $z \in W^{-1}(c) \subset Y$ be a point with moment map coordinates (ξ, η), where $|c|^2 \geq (\varepsilon e^\delta)^{N/(N-1)}$ and $\|\xi\| \geq K_0 |c|^2$. Let $\boldsymbol{v}_0 \in \mathcal{V}$ be such that*

$$|z^{\boldsymbol{v}_0}|^{1/d(\boldsymbol{v}_0)} = \max\{|z^{\boldsymbol{v}}|^{1/d(\boldsymbol{v})} \mid \boldsymbol{v} \in \mathcal{V}\}.$$

Then $\xi \in S_{\boldsymbol{v}_0,\gamma_0}$ and $z \in C_{\boldsymbol{v}_0,\gamma_0}$.

Proof Let $(z_\alpha)_{\alpha \in P_\mathbb{Z}}$ be a lift of $z \in Y$ to $\mu^{-1}(\lambda) \subset \mathbb{C}^{P_\mathbb{Z}}$. Recall from (4-12) that the smallest moment map coordinate μ_{α_0} corresponds to $\alpha_0 \in P_\mathbb{Z}$ such that $\xi \in \Delta_{\alpha_0}$, and μ_{α_0} is bounded by $4Ne^{2N\delta}|c|^2$. On the other hand, let $\alpha_1 \in P_\mathbb{Z}$ be such that $|z_{\alpha_1}| = \max\{|z_\alpha| \mid \alpha \in P_\mathbb{Z}\}$, or equivalently, $\mu_{\alpha_1} = \max\{\mu_\alpha \mid \alpha \in P_\mathbb{Z}\}$. By (4-8), $\mu_{\alpha_1} - \mu_{\alpha_0} = \langle \alpha_0 - \alpha_1, \xi \rangle + \nu(\alpha_1) - \nu(\alpha_0)$, so there exist positive constants c_1, c_2 depending only on Δ_Y such that

$$\mu_{\alpha_1} \leq \mu_{\alpha_0} + c_1 \|\xi\| + c_2. \tag{4-14}$$

On the other hand, since P is assumed to have nonempty interior, the quantity $\max\{\langle \alpha - \alpha', \xi \rangle \mid \alpha, \alpha' \in P_\mathbb{Z}\}$ is bounded below by a positive constant times $\|\xi\|$, and there are positive constants c_1', c_2' depending only on Δ_Y such that

$$\mu_{\alpha_1} \geq \mu_{\alpha_0} + c_1' \|\xi\| - c_2'. \tag{4-15}$$

Assume that $\xi \in S_{\boldsymbol{v},\gamma}$ for some $\boldsymbol{v} \in \mathcal{V}$ and $\gamma > 0$. Then for all $\alpha \notin A_{\boldsymbol{v}}$ we have

$$\mu_\alpha = f_c(\xi) - \langle \alpha, \xi \rangle + \nu(\alpha) \geq \mu_{\alpha_0} + \gamma \|\xi\|. \tag{4-16}$$

Thus, assuming some lower bound on $\|\xi\|$ (eg $\|\xi\| \geq 1$), the upper bound on μ_{α_1} implies the existence of a constant $c_3 > 0$ (still depending only on Δ_Y) such that, for all $\alpha \notin A_{\boldsymbol{v}}$, $\mu_\alpha \geq c_3 \gamma \mu_{\alpha_1}$. Using Lemma 4.12(3), this in turn yields the inequality

$$|z_\alpha| \geq (2N)^{-1/2} e^{-N\delta} c_3^{1/2} \gamma^{1/2} |z_{\alpha_1}| \quad \text{for all } \alpha \notin A_{\boldsymbol{v}}. \tag{4-17}$$

Taking a weighted geometric mean (and recalling that $v^\alpha = 0$ for $\alpha \in A_{\boldsymbol{v}}$), we get

$$|z^{\boldsymbol{v}}|^{1/d(\boldsymbol{v})} \geq (2N)^{-1/2} e^{-N\delta} c_3^{1/2} \gamma^{1/2} |z_{\alpha_1}|. \tag{4-18}$$

Conversely, if $\xi \notin S_{\boldsymbol{v},\gamma}$, then $\langle \alpha, \xi \rangle - \nu(\alpha) \geq \varphi(\xi) - \gamma \|\xi\|$ for some $\alpha \notin A_{\boldsymbol{v}}$, hence

$$\mu_\alpha = f_c(\xi) - \langle \alpha, \xi \rangle + \nu(\alpha) \leq \mu_{\alpha_0} + \gamma \|\xi\|.$$

When $\|\xi\|$ is sufficiently large, namely $\|\xi\| \geq \max(2c_2'/c_1', 4Ne^{2N\delta}\gamma^{-1}|c|^2)$, we have $\mu_\alpha \leq 2\gamma\|\xi\|$ and $\mu_{\alpha_1} \geq \frac{1}{2} c_1' \|\xi\|$. Therefore, $\mu_\alpha \leq c_3' \gamma \mu_{\alpha_1}$, where $c_3' = 4(c_1')^{-1}$. Using Lemma 4.12(3), this in turn yields the inequality

$$|z_\alpha| \leq (2N)^{1/2N} e^{\delta} (c_3')^{1/2N} \gamma^{1/2N} |z_{\alpha_1}|. \tag{4-19}$$

Since $\alpha \notin A_{\boldsymbol{v}}$, by definition the exponent v^α of z_α in the expression for $z^{\boldsymbol{v}}$ is at least 1. Since the other coordinates which appear in the expression for $z^{\boldsymbol{v}}$ are all bounded by $|z_{\alpha_1}|$, we obtain

$$|z^{\boldsymbol{v}}|^{1/d(\boldsymbol{v})} \leq e^{\delta/d(\boldsymbol{v})} (2N\, c_3' \gamma)^{1/(2Nd(\boldsymbol{v}))} |z_{\alpha_1}|. \tag{4-20}$$

With the necessary estimates in hand, we now proceed with the proof. First, there exists $\gamma_1 > 0$ depending only on Δ_Y such that the subsets $S_{\boldsymbol{v},\gamma_1}$ cover all but a bounded subset of $\mathbb{R}^n$, ie for some constant $K_1 > 0$ (depending only on Δ_Y), every point with $\|\xi\| \geq K_1$ belongs to some $S_{\boldsymbol{v},\gamma_1}$. Thus, whenever $\|\xi\| \geq K_1$, the estimate (4-18) implies that

$$\max\{|z^{\boldsymbol{v}}|^{1/d(\boldsymbol{v})} \mid \boldsymbol{v} \in \mathcal{V}\} \geq (2N)^{-1/2} e^{-N\delta} c_3^{1/2} \gamma_1^{1/2} \max\{|z_\alpha| \mid \alpha \in P_{\mathbb{Z}}\}. \tag{4-21}$$

Let $\mathcal{D} = \{d(\boldsymbol{v}) \mid \boldsymbol{v} \in \mathcal{V}\}$ — a finite set of positive integers. We now choose γ_0 so that

$$e^{\delta/d} (2Nc_3'\gamma_0)^{1/(2Nd)} < (2N)^{-1/2} e^{-N\delta} c_3^{1/2} \gamma_1^{1/2} \quad \text{for all } d \in \mathcal{D}, \tag{4-22}$$

and choose K_0 so that

$$K_0 \geq 4Ne^{2N\delta}\gamma_0^{-1} \quad \text{and} \quad K_0(\varepsilon e^\delta)^{N/(N-1)} \geq \max(K_1, 2c_2'/c_1').$$

Assume $|c|^2 \geq (\varepsilon e^\delta)^{N/(N-1)}$ and $\|\xi\| \geq K_0|c|^2$, and let $\boldsymbol{v}_0$ be such that

$$|z^{\boldsymbol{v}_0}|^{1/d(\boldsymbol{v}_0)} = \max\{|z^{\boldsymbol{v}}|^{1/d(\boldsymbol{v})} \mid \boldsymbol{v} \in \mathcal{V}\}.$$

If $\xi \notin S_{\boldsymbol{v}_0,\gamma_0}$, then (4-20) and (4-22) give

$$\begin{aligned} |z^{\boldsymbol{v}_0}|^{1/d(\boldsymbol{v}_0)} &\leq e^{\delta/d(\boldsymbol{v}_0)} (2N\, c_3'\gamma_0)^{1/(2Nd(\boldsymbol{v}_0))} \max\{|z_\alpha| \mid \alpha \in P_{\mathbb{Z}}\} \\ &< (2N)^{-1/2} e^{-N\delta} c_3^{1/2} \gamma_1^{1/2} \max\{|z_\alpha| \mid \alpha \in P_{\mathbb{Z}}\}, \end{aligned}$$

which contradicts (4-21). Thus $\xi \in S_{\boldsymbol{v}_0,\gamma_0}$, or equivalently, $z \in C_{\boldsymbol{v}_0,\gamma_0}$. □

Propositions 4.16 and 4.18 imply the following.

Corollary 4.19 *The extremal monomials* $z^{\boldsymbol{v}}$ *for* $\boldsymbol{v} \in \mathcal{V}$ *introduced in Definition 4.14, the weights* $d(\boldsymbol{v})$ *defined in* (4-13), *the open subsets* $C_{\boldsymbol{v}} = C_{\boldsymbol{v},\gamma_0}$, *and the closed subset*

$$\Omega = \{z \in Y \mid \|\xi\| \le K_0' \max(1, |W|^2)\},$$

where $K_0' = \max(8N^2 e^{N(N+3)\delta}\gamma_0^{-1}, K_0)$, *define a fiberwise monomial subdivision on the toric Landau–Ginzburg model* (Y, W, ω).

Moreover, with respect to this subdivision, fiberwise monomial admissibility (*with fixed phase angles*) *is preserved by parallel transport between the fibers of* W *over any path* $\gamma\colon [0,1] \to \mathbb{C}$ *such that* $|\gamma(0)|^2 \ge (\varepsilon e^{\delta})^{N/(N-1)}$ *and* $|\gamma(t)|$ *is nondecreasing.*

Proof The fact that the collection of extremal monomials $(z^{\boldsymbol{v}})_{\boldsymbol{v}\in\mathcal{V}}$ defines a proper map is a classical fact of toric geometry, but can also be seen directly from the lower bound (4-21). Properties (2) and (3) of Definition 4.1 are clear from the construction, and property (4) follows from Proposition 4.18.

When $|W|^2 \ge (\varepsilon e^{\delta})^{N/(N-1)}$, Proposition 4.16 implies that $z^{\boldsymbol{v}}$ is invariant under parallel transport at every point of $C_{\boldsymbol{v}}$ which lies outside of Ω. Thus, the property that $\arg(z^{\boldsymbol{v}}) = \varphi_{\boldsymbol{v}}$ is preserved under parallel transport. The reason we require $|\gamma(t)|$ to be nondecreasing with respect to t is to ensure that $C_{\boldsymbol{v}} \setminus (C_{\boldsymbol{v}} \cap \Omega)$ is preserved under parallel transport (using the fact that ξ is preserved under parallel transport). □

4.3 The wrapping Hamiltonian

We now define a Hamiltonian $H : Y \to \mathbb{R}$ whose flow preserves both the fibers of W and monomial admissibility within them. This Hamiltonian is constructed by reduction from the case of $\mathbb{C}^N$. The construction involves a smooth approximation of the minimum function, conceptually similar to Definition 4.5 but with N variables.

Definition 4.20 Given a constant $\delta' > 0$, denote by $m\colon \mathbb{R}^N \to \mathbb{R}$ a smooth concave function with the following properties:

(1) Letting $I = \{i \mid u_i < \min(u_1, \dots, u_N) + \delta'\}$, locally $m(u_1, \dots, u_N)$ depends only on $(u_i)_{i\in I}$, and if $I = \{i_0\}$, then $m(u_1, \dots, u_N) = u_{i_0} = \min(u_1, \dots, u_N)$.

(2) $m(u_1 + a, \dots, u_N + a) = m(u_1, \dots, u_N) + a$ for all $a \in \mathbb{R}$.

(3) m is symmetric, ie $m(u_{\sigma(1)}, \dots, u_{\sigma(N)}) = m(u_1, \dots, u_N)$ for all $\sigma \in \mathfrak{S}_N$.

These conditions imply that m is monotonically increasing in all variables, and

$$\min(u_1, \dots, u_N) - \delta' \le m(u_1, \dots, u_N) \le \min(u_1, \dots, u_N).$$

For instance, for $\delta' \geq N\delta$ we can take

$$m(u_1,\dots,u_N) = -\frac{1}{N!}\sum_{\sigma\in\mathfrak{S}_N} M\big(-u_{\sigma(1)}, M\big(\dots, M(-u_{\sigma(N-1)}, -u_{\sigma(N)})\cdots\big)\big).$$

Denoting $\mu_1,\dots,\mu_N$ the moment map coordinates for the chosen toric Kähler form on $\mathbb{C}^N$, the Hamiltonian we consider is

$$H_0 = \sum_{i=1}^{N}\mu_i - Nm(\mu_1,\dots,\mu_N). \tag{4-23}$$

Setting $N = |P_{\mathbb{Z}}|$ and viewing Y as a symplectic reduction of $\mathbb{C}^{P_{\mathbb{Z}}}$, recall that the moment map coordinates $\mu_1,\dots,\mu_N$ descend to functions $(\mu_\alpha)_{\alpha\in P_{\mathbb{Z}}}$ on the moment polytope Δ_Y, given by (4-8). We then define the Hamiltonian H on Y via reduction:

Definition 4.21 Given a point of Y with moment map coordinates $(\xi,\eta)\in\Delta_Y$, set $\mu_\alpha = \eta - \langle\alpha,\xi\rangle + \nu(\alpha)$ for all $\alpha\in P_{\mathbb{Z}}$ as before. Then we define $H: Y\to\mathbb{R}$ by

$$H = \sum_{\alpha\in P_{\mathbb{Z}}}\mu_\alpha - |P_{\mathbb{Z}}|\, m(\{\mu_\alpha\}_{\alpha\in P_{\mathbb{Z}}}). \tag{4-24}$$

Proposition 4.22 *H only depends on the moment map coordinates $(\xi_1,\dots,\xi_n)$, and as a function of these variables it is proper, convex, and grows linearly at infinity. In particular, the flow of H preserves the fibers of W, and the restriction of H to every fiber of W is proper.*

Proof Clearly H is a function of the moment map coordinates $(\xi_1\dots,\xi_n,\eta)$. Since $\partial\mu_\alpha/\partial\eta = 1$ for all $\alpha\in P_{\mathbb{Z}}$, property (2) of Definition 4.20 implies that $\partial H/\partial\eta = 0$, ie H only depends on $(\xi_1,\dots,\xi_n)$. This in turns implies that X_H is everywhere in the linear span of the generators of the $\mathbb{T}^n$–action and preserves the fibers of W.

Since the coordinates μ_α are affine linear functions of $(\xi_1,\dots,\xi_n,\eta)$, the convexity of H as a function of these variables (and hence of $(\xi_1,\dots,\xi_n)$) follows from the concavity of m. Meanwhile, the properness of H follows from our assumption that P has nonempty interior, which yields the lower bound (4-15) on $\max\{\mu_\alpha\} - \min\{\mu_\alpha\}$; the linear growth is manifest. □

Proposition 4.23 *The flow of H preserves monomial admissibility with respect to the fiberwise monomial subdivision of Corollary 4.19. More precisely, if $\ell\subset W^{-1}(c)$ is monomially admissible with phase angles $\{\varphi_{\boldsymbol{v}} \mid \boldsymbol{v}\in\mathcal{V}\}$, then its image under the time t flow is monomially admissible at infinity with phase angles $\{\varphi_{\boldsymbol{v}} + t\, d(\boldsymbol{v}) \mid \boldsymbol{v}\in\mathcal{V}\}$, where $d(\boldsymbol{v})\in\mathbb{Z}_+$ is given by* (4-13).

Proof On $\mathbb{C}^N$ the Hamiltonian H_0 defined by (4-23) is a function of the moment map coordinates $\mu_1,\dots,\mu_N$. Letting $I = \{i \mid \mu_i < \min(\mu_1,\dots,\mu_N) + \delta'\}$ as in Definition 4.20(1), we observe that $\partial H_0/\partial\mu_i \equiv 1$ for all $i\notin I$. Thus, the flow of H_0 rotates the coordinates z_i uniformly at unit speed for all $i\notin I$. Moreover, this flow is $\mathbb{T}_M$–equivariant, preserves $\mu^{-1}(\lambda)\subset\mathbb{C}^N$, and descends to $Y = \mu^{-1}(\lambda)/\mathbb{T}_M$ as the Hamiltonian flow generated by H.

Using the same notations as in the previous section, fix $\boldsymbol{v} \in \mathcal{V}$, and consider a point of $C_{\boldsymbol{v}} = C_{\boldsymbol{v},\gamma_0} \subset Y$ with moment map coordinates (ξ, η) such that $\|\xi\| \geq \gamma_0^{-1}\delta'$. (Choosing δ' sufficiently small in Definition 4.20, we can ensure that every point outside of Ω satisfies this inequality.) Write $\mu_{\alpha_0} = \min\{\mu_\alpha\}$. By (4-16), for $\alpha \notin A_{\boldsymbol{v}}$ we have $\mu_\alpha \geq \mu_{\alpha_0} + \gamma_0\|\xi\| \geq \mu_{\alpha_0} + \delta'$. Thus, $m(\{\mu_\alpha\}_{\alpha \in P_{\mathbb{Z}}})$ only depends on $(\mu_\alpha)_{\alpha \in A_{\boldsymbol{v}}}$, and the flow generated by H rotates all the other coordinates $(z_\alpha, \alpha \notin A_{\boldsymbol{v}})$ at unit speed. Recalling that $z^{\boldsymbol{v}} = \prod_\alpha z_\alpha^{v^\alpha}$ with $v^\alpha = 0$ whenever $\alpha \in A_{\boldsymbol{v}}$, we conclude that the flow of X_H rotates $z^{\boldsymbol{v}}$ uniformly at a rate of $\sum_\alpha v^\alpha = d(\boldsymbol{v})$ at every point of $C_{\boldsymbol{v}}$ which lies outside of Ω. The result follows. □

Remark 4.24 Essentially any Hamiltonian satisfying the conditions of Propositions 4.22 and 4.23 (possibly with different values of the phase shifts $d(\boldsymbol{v})$, as long as these remain positive) would be equally suitable for our purposes; see eg Hanlon's work [24] for another construction. The Hamiltonian of Definition 4.21 is particularly natural from the perspective of symplectic reduction from $\mathbb{C}^N$ to Y, but the category $\mathcal{W}(Y, W)$ is, up to equivalence, independent of the choice, as will be clear from the arguments in Section 5.

5 Computation of fiberwise wrapped Floer cohomology

5.1 Geometric setup

We now fix the geometric data needed for our construction of the admissible Lagrangian $L_0 \in \mathcal{W}(Y, W)$, besides the Kähler forms and wrapping Hamiltonians defined in Section 4, and check that the various conditions imposed in Section 3 are satisfied.

Let $(Y, W = -z^{(0,\dots,0,1)})$ be the Landau–Ginzburg model constructed in Section 2, equipped with the toric Kähler form ω which is the result of symplectic reduction by $\mathbb{T}_M$ of the Kähler form on $\mathbb{C}^{P_{\mathbb{Z}}}$ introduced in Definition 4.6. Let $\mathcal{V} \subset \mathbb{Z}^{n+1}$ be the set of extremal vectors of Definition 4.14, $d(\boldsymbol{v})$ the positive integers given by (4-13), and the subsets $C_{\boldsymbol{v}}$ and Ω of Y as in Corollary 4.19. We consider the height function

$$h = \max\{h_{\boldsymbol{v}} \mid \boldsymbol{v} \in \mathcal{V}\} \colon Y \to [0, \infty), \quad \text{where } h_{\boldsymbol{v}} = |z^{\boldsymbol{v}}|^{1/d(\boldsymbol{v})}, \tag{5-1}$$

and the wrapping Hamiltonian H introduced in Definition 4.21.

We fix a properly embedded U-shaped arc $\gamma_0 \colon \mathbb{R} \to \mathbb{C}$ such that $\gamma_0(0) = -1$; $|\gamma_0(s)|$ passes through a minimum at $s = 0$ and increases monotonically as a function of $|s|$; $\arg \gamma_0(s)$ increases monotonically as a function of s; $\arg \gamma_0(s) = \theta_0$ for $s \ll 0$ and $\arg \gamma_0(s) = 2\pi - \theta_0$ for $s \gg 0$, for some positive angle $0 < \theta_0 < \pi/2$. (Thus, γ_0 intersects the negative real axis transversely at -1, remains at distance at least 1 from the origin, and outside of a compact subset it coincides with the rays $e^{\pm i\theta_0}\mathbb{R}_+$.)

Given a monomially admissible Lagrangian submanifold $\ell \subset W^{-1}(-1) \cong (\mathbb{C}^*)^n$ (in the sense of Definition 4.2), with all phase angles equal to zero, we denote by $L = \cup \ell$ the fibered Lagrangian submanifold of Y obtained from ℓ by parallel transport in the fibers of W over the arc γ_0. It follows from

Corollary 4.19 that L is fiberwise monomially admissible, with all phase angles equal to zero. We will in particular consider the case where $\ell = \ell_0$ is the real positive locus of $W^{-1}(-1)$, ie the set of points where all toric monomials are real positive and $z^{(0,\dots,0,1)} = 1$, and denote its parallel transport by $L_0 = \cup \ell_0$.

As in Section 3.3, we choose an autonomous flow ρ^t in the complex plane which fixes the negative real axis pointwise as well as a small neighborhood of the origin, specifically the disc Δ' of radius $(\varepsilon e^{\delta})^{N/(2N-2)}$ (in particular ρ^t fixes the points -1 and 0), maps radial lines to radial lines outside of a compact subset, and moves all radial lines other than the negative real axis in the counterclockwise direction. We will furthermore assume that the flow rotates the tangent vector to γ_0 at -1 (the imaginary axis) counterclockwise, so that

(1) for $t \neq 0$, $\gamma_t = \rho^t(\gamma_0)$ intersects γ_0 transversely at -1,

(2) $\gamma_0 \cap \gamma_t = \{-1\}$ for $|t| \in (0, t_0)$, where t_0 is the value of t for which ρ^t pushes the ray $e^{-i\theta_0}\mathbb{R}_+$ past $e^{i\theta_0}\mathbb{R}_+$, and

(3) for $|t| > t_0$, γ_0 and γ_t intersect transversely in exactly two points (-1 and one other intersection).

(These requirements on $\gamma_0 \cap \gamma_t$ are natural and easy to achieve given the other requirements on ρ^t; see Figure 5.)

Since the arcs γ_t are strictly radial outside of a bounded subset, their mutual intersections, and the bounded polygonal regions they delimit in the complex plane are all contained within a bounded subset, say the disc of radius R_0. For $R \in \mathbb{R}_{\geq 0}$, let $r(R)$ be the maximum of h on the compact subset $\Omega \cap \{|W| \leq \max(R, R_0)\}$ of Y. Then r is a nondecreasing function, constant over $[0, R_0]$, and we take the closed subset $Y^{\mathrm{in}} \subset Y$ appearing in Section 3.1 to be the set of points of Y where $h \leq r(|W|)$. This ensures that Y^{in} contains Ω.

Finally, we take the almost-complex structure J to be the standard complex structure of Y outside of the bounded subset

$$Y^{\mathrm{in}} \cap \{|W| < \epsilon\} \tag{5-2}$$

for some $\epsilon \ll 1$ (smaller than the radius of Δ'), and a generic small perturbation of the standard complex structure inside that subset. This ensures that simple J–holomorphic spheres which intersect this subset are regular, and evaluation maps for rigid somewhere injective discs and spheres are mutually transverse, as explained in Remark 3.18.

Proposition 5.1 *The above geometric data on Y satisfy the requirements listed in Section 3.1, and the Lagrangian submanifolds $L_0(t) = \phi^t \rho^t(L_0)$ are admissible in the sense of Definition 3.5.*

Proof We start with the geometric conditions in Section 3.1. First, the properness of $h = \max\{|z^{\boldsymbol{v}}|^{1/d(\boldsymbol{v})}\}$ follows from that of the map $(z^{\boldsymbol{v}})_{\boldsymbol{v}\in\mathcal{V}} \colon Y \to \mathbb{C}^{|\mathcal{V}|}$ (item (1) in Definition 4.1). Next, we have already seen in Proposition 4.22 that H is proper on every fiber of W, and its Hamiltonian flow preserves the fibers of W, ie $dW(X_H) = 0$. Thus, H Poisson commutes with the real and imaginary parts of W, whose

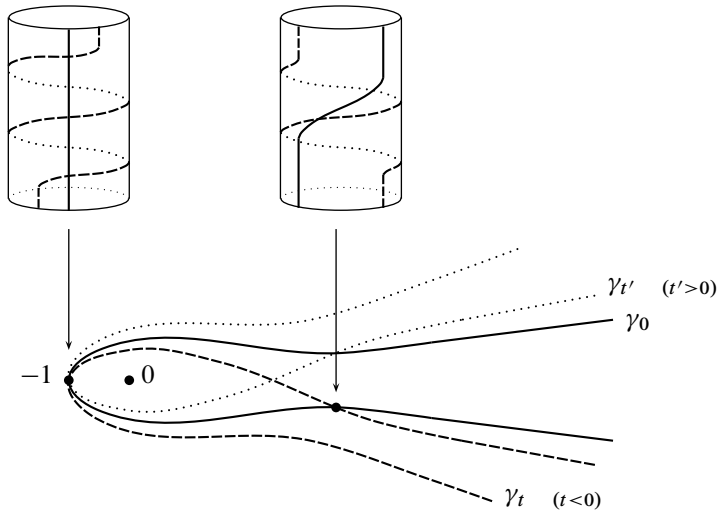

Figure 5: The Lagrangians L_0 and $L_0(t) = \phi^t \rho^t(L_0)$ for $t < 0$.

Hamiltonian vector fields span the horizontal distribution; it follows that dH vanishes on horizontal vector fields. Moreover, since H is a function of the moment map coordinates only, X_H is in the span of the vector fields generating the toric action, hence its flow preserves the norms of all toric monomials, and so $dh(X_H) = 0$.

Next we consider the behavior of h along the horizontal distribution — or more precisely, by Remark 3.12, the behavior of the term(s) $h_{\boldsymbol{v}}$ that achieve the maximum in $h = \max\{h_{\boldsymbol{v}}\}$. By Proposition 4.16, for each $\boldsymbol{v} \in \mathcal{V}$, and at every point of $C_{\boldsymbol{v}}$ which lies outside of $Y^{\text{in}} \cup W^{-1}(\Delta')$, the monomial $z^{\boldsymbol{v}}$ is invariant under parallel transport. Therefore, $dh_{\boldsymbol{v}} = (1/d(\boldsymbol{v}))h_{\boldsymbol{v}}\, d\log|z^{\boldsymbol{v}}|$ and $d^c h_{\boldsymbol{v}} = (1/d(\boldsymbol{v}))h_v\, d\arg(z^{\boldsymbol{v}})$ both vanish on horizontal vectors, and their Lie derivatives along horizontal vector fields also vanish. It then follows from Proposition 4.18 that, everywhere outside of $Y^{\text{in}} \cup W^{-1}(\Delta')$, these properties hold for any $h_{\boldsymbol{v}}$ that achieves the maximum in $h = \max\{h_{\boldsymbol{v}}\}$.

Finally, Proposition 4.23 implies that the flow of X_H rotates $z^{\boldsymbol{v}}$ uniformly at a rate of $d(\boldsymbol{v})$ at every point of $C_{\boldsymbol{v}}$ which lies outside of Y^{in}. Therefore,

$$dh_{\boldsymbol{v}}(X_H) = 0, \quad d^c h_{\boldsymbol{v}}(X_H) = \frac{1}{d(\boldsymbol{v})} h_v\, d\arg(z^{\boldsymbol{v}})(X_H) = h_{\boldsymbol{v}} \geq 0, \quad \mathcal{L}_{X_H}(d^c h_{\boldsymbol{v}}) = 0.$$

As before, these properties hold everywhere in $Y \setminus Y^{\text{in}}$ for any $h_{\boldsymbol{v}}$ that achieves the maximum in h. This completes the verification of the requirements listed in Section 3.1.

Next we prove the admissibility of L_0 in the sense of Definition 3.5. The construction of the U-shaped arc γ_0 ensures that its two halves connecting -1 to infinity are admissible arcs in the sense of Definition 3.4. The monomial admissibility of $\ell_0 = (\mathbb{R}_+)^n \subset W^{-1}(-1)$ and the compatibility of parallel transport with fiberwise monomial admissibility (Corollary 4.19) imply that L_0 is fiberwise monomially admissible; therefore, $\arg(z^{\boldsymbol{v}})$ vanishes identically on the portion of L_0 which lies in $C_{\boldsymbol{v}} \setminus (C_{\boldsymbol{v}} \cap Y^{\text{in}})$, which in

turn implies the vanishing of $d^c h_{\boldsymbol{v}} = (1/d(\boldsymbol{v}))h_{\boldsymbol{v}} d\arg(z^{\boldsymbol{v}})$. It follows that the restriction of $d^c h$ to L_0 vanishes outside of Y^{in} (wherever h is differentiable, and otherwise in the sense of Remark 3.12).

Since $L_0(t) = \phi^t \rho^t(L_0)$ is obtained from the admissible Lagrangian L_0 by the admissible lifted isotopy ρ^t and the flow of the wrapping Hamiltonian H, it is also admissible by Lemma 3.7. (Alternatively, $\ell_0(t) = \phi^t(\ell_0) \subset W^{-1}(-1)$ is monomially admissible by Proposition 4.23, and the two portions of the arc $\gamma_t = \rho^t(\gamma_0)$ connecting -1 to infinity are admissible in the sense of Definition 3.4; since $L_0(t)$ is obtained by parallel transport of $\ell_0(t)$ over γ_t, its admissibility follows from the same argument as above.) □

5.2 The Floer complex $CF^*(L_0(t'), L_0(t))$

Recall that $L_0(t)$ is fibered over γ_t, and fiberwise monomially admissible with phase angles $\varphi_{\boldsymbol{v}} = d(\boldsymbol{v})t$ (by Proposition 4.23). Thus, the asymptotic directions of the noncompact ends of $L_0(t)$ and $L_0(t')$ are disjoint whenever $t' - t \in U = \mathbb{R} \setminus (\{\pm t_0\} \cup (2\pi/d_0)\mathbb{Z})$, where we denote by d_0 the least common multiple of the positive integers $d_{\boldsymbol{v}}$, $\boldsymbol{v} \in \mathcal{V}$. Since the arcs γ_t are strictly radial outside of the disc of radius R_0, and monomial admissibility precludes the existence of intersections outside of Y^{in} when the phase angles are distinct, for $t' - t \in U$ all the intersections of $L_0(t)$ and $L_0(t')$ lie within the compact subset $Y^{\mathrm{in}} \cap \{|W| \le R_0\}$.

The intersections of $L_0(t)$ and $L_0(t')$ are concentrated in the fibers of W above the intersection points of γ_t and $\gamma_{t'}$; we will now see that Lagrangian Floer theory for these submanifolds can be expressed in terms of the fiberwise Floer complexes in those fibers and counts of holomorphic sections of $W : Y \to \mathbb{C}$ over regions of the complex plane delimited by the arcs γ_t and $\gamma_{t'}$.

Because our construction of the wrapping Hamiltonian does not guarantee that $L_0(t')$ and $L_0(t)$ intersect transversely, we will allow ourselves to modify our Lagrangians by small Hamiltonian isotopies supported inside Y^{in} (and preserving the fibers of W, so that admissibility is not affected) in order to achieve transversality of intersections; we will see in the next sections that our main calculation reduces to a cohomology-level argument, so we do not specify the exact choice of perturbation involved in the definition of the Floer complex.

For $t' - t \in \mathbb{R}_+ \cap U$, we denote by $C_0(t', t)$ the portion of the Floer complex $CF^*(L_0(t'), L_0(t))$ generated by intersection points which lie in the fiber $W^{-1}(-1)$, ie the Floer complex of the monomially admissible Lagrangian submanifolds $\ell_0(t') = \phi^{t'}(\ell_0)$ and $\ell_0(t) = \phi^t(\ell_0)$ inside $W^{-1}(-1) \simeq (\mathbb{C}^*)^n$ (possibly after a small compactly supported perturbation to achieve transversality). We similarly denote by $C_1(t', t)$ the portion of the Floer complex generated by intersection points which lie in the fiber above the other intersection point $c_{t',t}$ of $\gamma_{t'}$ and γ_t for $t' - t > t_0$; this amounts to the Floer complex of the monomially admissible Lagrangian submanifolds $\ell_-(t')$ and $\ell_+(t)$ of $W^{-1}(c_{t',t})$ obtained by parallel transport of $\ell_0(t')$ and $\ell_0(t)$ along the portions of $\gamma_{t'}$ and γ_t which run from -1 to $c_{t',t}$ (clockwise on γ_t, and counterclockwise on $\gamma_{t'}$). For $t' - t < t_0$ we set $C_1(t', t) = 0$.

The choice of a grading (for instance the usual one) on $\ell_0 = (\mathbb{R}_+)^n \subset (\mathbb{C}^*)^n$ and on the arc γ_0 in the complex plane induces a grading on the admissible Lagrangian L_0, and also, by following the various isotopies, on the monomially admissible Lagrangians $\ell_0(t)$ and their images under parallel transport, as well as $L_0(t)$. We view $C_0(t', t)$ and $C_1(t', t)$ as the Floer complexes of the monomially admissible Lagrangian submanifolds $\ell_0(t'), \ell_0(t)$ and $\ell_-(t'), \ell_+(t)$ of $W^{-1}(-1)$ and $W^{-1}(c_{t',t})$, respectively, with the grading induced by that of ℓ_0; in the case of $C_0(t', t)$ this coincides with the grading of $CF^*(L_0(t'), L_0(t))$, but in the case of $C_1(t', t)$ the grading in $CF^*(L_0(t'), L_0(t))$ is one less than the fiberwise degree, due to the phase angles of the arcs $\gamma_t, \gamma_{t'}$ at $c_{t',t}$ differing by an amount in the interval $(\pi, 2\pi)$ for $t' - t > t_0$; see Figure 5. Thus,

$$\begin{aligned} CF^*(L_0(t'), L_0(t)) &= C_0(t', t) \oplus C_1(t', t)[1] \\ &= \begin{cases} CF^*(\ell_0(t'), \ell_0(t)) \oplus CF^*(\ell_-(t'), \ell_+(t))[1] & \text{if } t' - t > t_0, \\ CF^*(\ell_0(t'), \ell_0(t)) & \text{if } 0 < t' - t < t_0. \end{cases} \end{aligned} \tag{5-3}$$

Since the almost-complex structure J coincides with the standard one outside of the subset $Y^{\text{in}} \cap \{|W| < \epsilon\}$ introduced in (5-2), J–holomorphic curves satisfy the open mapping principle with respect to the projection $W : Y \to \mathbb{C}$ and intersect positively with the fibers of W outside of the disc of radius ϵ. (However, this fails near the origin.) This implies immediately that J–holomorphic discs with boundary on a union of fibered Lagrangian submanifolds (disjoint from the region where $|W| < \epsilon$) are either contained in the fibers of W, or behave (away from the zero fiber) like sections or multisections of $W : Y \to \mathbb{C}$ over regions of the complex plane delimited by the arcs over which the Lagrangians fiber. By abuse of terminology, we call such J–holomorphic discs "sections" when their intersection number with the fibers is one, even though they need not be genuine sections over the disc of radius ϵ.

The fibers of W outside of the origin are isomorphic to $(\mathbb{C}^*)^n$, and the monomially admissible Lagrangians $\ell_0(t)$ and their images under parallel transport do not bound any holomorphic discs inside the fibers of W (eg because they are contractible and hence exact). It follows that $L_0(t)$ does not bound any J–holomorphic discs.

Moreover, our choice of J ensures that we can also avoid sphere bubbling by arguing as in Remark 3.18. Since the intersections of $L_0(t')$ and $L_0(t)$ lie within the region of Y where $|W| \le R_0$ and $h \le r(R_0)$, the maximum principles for W and h (Propositions 3.10 and 3.11) imply that the J–holomorphic discs contributing to the Floer differential (and later on, to continuation maps or product operations) also remain within $Y^{\text{in}} \cap \{|W| \le R_0\}$. Since the fibers of W away from the origin are aspherical, the only possible sphere bubbles are configurations contained in the region where $|W| \le \epsilon$, at least one component of which must pass within $Y^{\text{in}} \cap \{|W| < \epsilon\}$. The choice of a generic perturbation of the standard complex structure within this subset ensures that the underlying simple spheres are disjoint from all J–holomorphic discs in the 0– or 1–dimensional moduli spaces we consider, and hence that no sphere bubbles can form.

We can now state and prove the main result of this section, which describes the structure of the Floer differential on $CF^*(L_0(t'), L_0(t))$.

Proposition 5.2 *For $0 < t' - t < t_0$, the Floer complex $CF^*(L_0(t'), L_0(t))$ in Y is isomorphic to the Floer complex $CF^*(\ell_0(t'), \ell_0(t))$ in $W^{-1}(-1) \simeq (\mathbb{C}^*)^n$.*

For $t' - t > t_0$, $CF^*(L_0(t'), L_0(t))$ is isomorphic to the mapping cone

$$CF^*(\ell_0(t'), \ell_0(t)) \oplus CF^*(\ell_-(t'), \ell_+(t))[1], \quad \partial = \begin{pmatrix} \partial_0 & s \\ 0 & \partial_1 \end{pmatrix}, \tag{5-4}$$

where the diagonal entries are the Floer differentials on the fiberwise Floer complexes, and the off-diagonal term

$$s = s^0_{\ell_0,t',t} \colon CF^*(\ell_-(t'), \ell_+(t)) \to CF^*(\ell_0(t'), \ell_0(t)) \tag{5-5}$$

is a chain map defined by a (weighted) count of J–holomorphic sections of $W : Y \to \mathbb{C}$ over the bounded region of the complex plane delimited by γ_t and $\gamma_{t'}$ (see Figure 5).

Proof The open mapping principle implies that the J–holomorphic discs that contribute to the Floer differential on $CF^*(L_0(t'), L_0(t))$ are either contained within the fibers of W, or (for $t' - t > t_0$) sections of W over the bounded region of the complex plane delimited by γ_t and $\gamma_{t'}$. The contributions of discs contained within $W^{-1}(-1)$ and $W^{-1}(c_{t',t})$ correspond exactly to the Floer differentials on the fiberwise Floer complexes $C_0(t', t) = CF^*(\ell_0(t'), \ell_0(t))$ and $C_1(t', t) = CF^*(\ell_-(t'), \ell_+(t))$, while the sections contribute the off-diagonal term s. The fact that s is a chain map follows directly from the vanishing of the square of the Floer differential. □

It follows that the Floer cohomology group $HF^*(L_0(t'), L_0(t))$ is isomorphic to $HF^*(\ell_0(t'), \ell_0(t))$ for $0 < t' - t < t_0$, while for $t' - t > t_0$ it is determined by the map induced by s on cohomology, which we again denote by

$$s = s^0_{\ell_0,t',t} \colon HF^*(\ell_-(t'), \ell_+(t)) \to HF^*(\ell_0(t'), \ell_0(t)). \tag{5-6}$$

Even though the Floer complexes and the chain map (5-5) depend on the choices made in the construction, the maps constructed from different choices are related by homotopies, so that the cohomology-level map (5-6) is independent of choices.

Indeed, deformations of Floer data among the set of choices which satisfy our technical requirements (eg compactly supported fiberwise Hamiltonian isotopies, modifications of J near $W^{-1}(0)$, or even admissible isotopies of the arcs γ_t and $\gamma_{t'}$ which preserve transversality at all times) induce continuation quasiisomorphisms on the Floer complexes (5-4). In every instance, by considering the projection $W : Y \to \mathbb{C}$ one shows that continuation trajectories, just like contributions to the Floer differential, can map generators in $W^{-1}(c_{t',t})$ to generators in $W^{-1}(-1)$ but not vice-versa. Thus, our continuation homomorphisms are upper-triangular with respect to the decomposition (5-4) and induce quasiisomorphisms on the summands $C_0 = CF^*(\ell_0(t'), \ell_0(t))$ and $C_1 = CF^*(\ell_-(t'), \ell_+(t))$. Denoting by C_0 and C_1 the

two summands in (5-4) with respect to one set of choices, and C_0' and C_1' the two summands for the other set of choices, we obtain a diagram

$$\begin{tikzcd} C_1 \arrow[r, "s"] \arrow[d, "f_1"'] \arrow[dr, "h"] & C_0 \arrow[d, "f_0"] \\ C_1' \arrow[r, "s'"] & C_0' \end{tikzcd} \tag{5-7}$$

where f_0, f_1 and h are the components of the continuation homomorphism, and $f_0\colon (C_0, \partial_0) \to (C_0', \partial_0')$ and $f_1\colon (C_1, \partial_1) \to (C_1', \partial_1')$ are quasiisomorphisms.

The fact that the continuation homomorphism is a chain map implies that

$$f_0 \circ s + h \circ \partial_1 = s' \circ f_1 + \partial_0' \circ h.$$

Therefore $f_0 \circ s$ and $s' \circ f_1$ are homotopic, and so the cohomology level maps induced by s and s' coincide under the isomorphisms $H^*(C_1, \partial_1) \simeq H^*(C_1', \partial_1')$ and $H^*(C_0, \partial_0) \simeq H^*(C_0', \partial_0')$ induced by f_1 and f_0. In this sense, the map (5-6) is independent of the choices made in the construction and invariant under admissible isotopies.

To put this in proper context, the map s is part of the "Seidel TQFT" (cf [35]) associated to the symplectic fibration $W\colon Y \to \mathbb{C}$. As a general principle, counts of J–holomorphic sections over given domains in the complex plane with boundary on given fibered Lagrangian submanifolds give rise to maps between the respective fiberwise Floer complexes which are independent of choices up to homotopy and satisfy algebraic relations that can be understood in terms of gluing axioms (we shall not elaborate on the latter point here; see [35] for details).

5.3 Floer cohomology for monomially admissible Lagrangians in $(\mathbb{C}^*)^n$

To proceed further, we need to discuss Floer theory for monomially admissible Lagrangian submanifolds in the fibers of W, which we identify with $(\mathbb{C}^*)^n$ by considering the toric monomials $z_1, \dots, z_n$ on the open stratum of Y whose weights correspond to first n basis vectors. (So, for each $\boldsymbol{v} = (\vec{v}, v^0) = (v_1, \dots, v_n, v^0) \in \mathcal{V}$, the monomial $z^{\boldsymbol{v}}$ restricts to $W^{-1}(c)$ as $(-c)^{v^0} z_1^{v_1} \dots z_n^{v_n}$.) The material in this section closely parallels Hanlon's work [24, Section 3.4].

The moment map for the standard $\mathbb{T}^n$–action on $W^{-1}(c) \simeq (\mathbb{C}^*)^n$ is given by the first n coordinates $(\xi_1, \dots, \xi_n)$ of the moment map of Y, and for each $\boldsymbol{v} \in \mathcal{V}$ the intersection of $C_{\boldsymbol{v}}$ with $W^{-1}(c)$ is the inverse image under the moment map of the subset $S_{\boldsymbol{v}} = S_{\boldsymbol{v},\gamma} \subset \mathbb{R}^n$ defined by (4-11) (for an appropriate value of the constant $\gamma > 0$, matching that used for $C_{\boldsymbol{v}}$ at the beginning of Section 5.1).

We consider Lagrangian submanifolds of $(\mathbb{C}^*)^n$ which are sections over the moment map projection; any such Lagrangian is the graph of the differential of a function $K = K(\xi)\colon \mathbb{R}^n \to \mathbb{R}$, ie the arguments $\arg(z_j) = \theta_j$ are determined as functions of the moment map variables $(\xi_1, \dots, \xi_n)$ by $\theta_j = \partial K / \partial \xi_j$. (For a given Lagrangian, K is unique up to an affine function whose gradient is 2π times an integer

vector.) The monomial admissibility condition can then be expressed in terms of the gradient of K: the graph $\ell = \Gamma_{dK} \subset W^{-1}(c) \simeq (\mathbb{C}^*)^n$ is monomially admissible with phase angles $\{\varphi_{\boldsymbol{v}}\}$ if and only if, outside of a compact subset,

$$\langle \nabla K(\xi), \vec{v}\rangle \equiv \varphi_{\boldsymbol{v}} - v^0 \arg(-c) \bmod 2\pi\mathbb{Z} \quad \text{for all } \xi \in S_{\boldsymbol{v}} \text{ and all } \boldsymbol{v} = (\vec{v}, v^0) \in \mathcal{V}. \tag{5-8}$$

Definition 5.3 The *slope* of the monomially admissible Lagrangian section $\ell = \Gamma_{dK}$ is the tuple $\sigma(K) = (\sigma_{\boldsymbol{v}}(K))_{\boldsymbol{v}\in\mathcal{V}} \in \mathbb{R}^{|\mathcal{V}|}$, where $\sigma_{\boldsymbol{v}}(K) = \langle \nabla K(\xi), \vec{v}\rangle_{|S_{\boldsymbol{v}}}$.

When K is a convex function, we associate to its slope $\sigma = \sigma(K)$ the polytope

$$P(\sigma) = \{u \in \mathbb{R}^n \mid \langle u, \vec{v}\rangle \le \sigma_{\boldsymbol{v}} \text{ for all } \boldsymbol{v} = (\vec{v}, v^0) \in \mathcal{V}\}. \tag{5-9}$$

Recall that the vectors $\vec{v}$ appearing in (5-9) are the primitive normal vectors to the facets of the Newton polytope P associated to the Laurent polynomial f; see Definition 4.14. Given any vertex $\alpha \in P_{\mathbb{Z}}$ of P, the subsets $S_{\boldsymbol{v}}$ associated to the various facets of P which meet at α have a nonempty and unbounded intersection U_α (comprising most of the region of $\mathbb{R}^n$ where α achieves the maximum in the tropicalization of f; see Figure 4). Over U_α, the value of ∇K is fully constrained by the slope $\sigma = \sigma(K)$, since $\langle \nabla K, \vec{v}\rangle = \sigma_{\boldsymbol{v}}$ whenever $\vec{v}$ is the normal vector to a facet of P containing α. This corresponds to the equality case in the inequalities (5-9) for a maximal collection of linearly independent $\vec{v}$, ie a vertex of the polytope $P(\sigma)$. From this and standard facts about convex functions we deduce:

Lemma 5.4 *If K is convex with slope σ, then $P(\sigma)$ is a convex polytope with the same normal vectors and normal fan as P, and the range of values taken by the gradient ∇K is exactly $P(\sigma)$.*

Example 5.5 The monomially admissible section $\ell_0(t) = \phi^t(\ell_0) \subset W^{-1}(-1)$ is the graph of $d(tH)$, so by Proposition 4.23 and (5-8) its slope is

$$\sigma_0(t) := \sigma(tH) = (\sigma_{\boldsymbol{v}}(tH))_{\boldsymbol{v}\in\mathcal{V}} = (t\, d(\boldsymbol{v}))_{\boldsymbol{v}\in\mathcal{V}}. \tag{5-10}$$

Moreover, for $t' - t > t_0$, parallel transport of $\ell_0(t)$ and $\ell_0(t')$ from $W^{-1}(-1)$ to $W^{-1}(c_{t',t})$ along the relevant portions of γ_t and $\gamma_{t'}$ preserves the phase angles $\varphi_{\boldsymbol{v}} = t\, d(\boldsymbol{v})$, so by (5-8) the monomially admissible Lagrangian sections $\ell_-(t')$ and $\ell_+(t)$ in $W^{-1}(c_{t',t})$ have slopes

$$\sigma_-(t') = (t' d(\boldsymbol{v}) - v^0\,(\arg(c_{t',t}) + \pi))_{\boldsymbol{v}\in\mathcal{V}}, \tag{5-11}$$

$$\sigma_+(t) = (t\, d(\boldsymbol{v}) - v^0\,(\arg(c_{t',t}) - \pi))_{\boldsymbol{v}\in\mathcal{V}}, \tag{5-12}$$

where we take $\arg(c_{t',t}) \in (-\pi, \pi)$; the values of $\arg(-c_{t',t})$ in these two formulas differ by 2π because we consider parallel transport from -1 to $c_{t',t}$ clockwise around the origin for $\ell_+(t)$ and counterclockwise for $\ell_-(t')$.

Let ℓ and ℓ' be two monomially admissible Lagrangian sections, expressed as the graphs of dK and dK'. If the slopes of K and K' differ by amounts that aren't multiples of 2π, then the intersections of ℓ and ℓ' remain within a compact subset of $(\mathbb{C}^*)^n$, and their Floer cohomology is well-defined. We claim that $HF^*(\ell', \ell)$ only depends on the slopes. (As we shall see in the argument below, this is an instance of the invariance of Floer cohomology under Hamiltonian isotopies which preserve admissibility and disjointness at infinity, and follows from the existence of well-defined continuation maps; see [16, Lemma 3.21] for the analogous result in the setting of Liouville sectors.)

Proposition 5.6 *Let $\ell = \Gamma_{dK}$ and $\ell' = \Gamma_{dK'}$ be two monomially admissible Lagrangian sections, with slopes $\sigma(K) = \sigma$ and $\sigma(K') = \sigma'$, and assume that $\sigma_{\boldsymbol{v}} - \sigma'_{\boldsymbol{v}} \notin 2\pi\mathbb{Z}$ $\forall \boldsymbol{v} \in \mathcal{V}$. Then $HF^*(\ell', \ell)$ only depends on the slopes σ and σ' of K and K'. Moreover, if $K' - K$ is convex, then the Floer cohomology is concentrated in degree zero, and*

$$HF^0(\ell', \ell) \cong \bigoplus_{p \in P(\sigma' - \sigma) \cap (2\pi\mathbb{Z})^n} \mathbb{K} \cdot \vartheta_p. \tag{5-13}$$

Proof First we prove invariance. Given any two Hamiltonians K_0, K_1 with $\sigma(K_0) = \sigma(K_1) = \sigma$, the convex combinations $K_s = (1-s)K_0 + sK_1$ also have slope σ, and the graphs $\ell_s = \Gamma_{dK_s}$ are monomially admissible Lagrangian sections. Similarly, given K'_0, K'_1 with $\sigma(K'_0) = \sigma(K'_1) = \sigma'$, we set $K'_t = (1-s)K'_0 + sK'_1$ and $\ell'_s = \Gamma_{dK'_s}$. We then define continuation maps

$$\Phi_{01} \colon CF^*(\ell'_0, \ell_0) \to CF^*(\ell'_1, \ell_1) \quad \text{and} \quad \Phi_{10} \colon CF^*(\ell'_1, \ell_1) \to CF^*(\ell'_0, \ell_0)$$

by counting index zero J–holomorphic strips $u \colon \mathbb{R} \times [0,1] \to (\mathbb{C}^*)^n$ with moving boundary conditions given by ℓ'_s (for s a suitable function of the real coordinate) along $\mathbb{R} \times 0$ and ℓ_s along $\mathbb{R} \times 1$.

Since the slopes of $K_1 - K_0$ and $K'_1 - K'_0$ are all zero, the Hamiltonian vector fields $X = X_{K_1 - K_0}$ and $X' = X_{K'_1 - K'_0}$ (which generate the isotopies of the moving boundary conditions ℓ_s and ℓ'_s) satisfy $d^c h(X) = d^c h(X') = 0$ outside of a compact subset (recalling that $h = \max\{h_{\boldsymbol{v}}\} = \max\{|z^{\boldsymbol{v}}|^{1/d(\boldsymbol{v})}\}$). More precisely, the vanishing of $\langle \nabla(K_1 - K_0), \vec{v} \rangle$ and $\langle \nabla(K'_1 - K'_0), \vec{v} \rangle$ implies the invariance of the monomial $z^{\boldsymbol{v}}$ under the flows of X and X' at all points of $C_{\boldsymbol{v}} \cap W^{-1}(c)$ which lie outside of Y^{in}, and hence the vanishing of $d^c h_{\boldsymbol{v}}(X) = (1/d(\boldsymbol{v}))h_v d \arg(z^{\boldsymbol{v}})(X)$ and $d^c h_{\boldsymbol{v}}(X')$.

This in turn implies that J–holomorphic strips with moving boundary conditions ℓ_s and ℓ'_s satisfy the maximum principle with respect to the proper function h outside of a compact subset of $(\mathbb{C}^*)^n$, and hence that the continuation maps Φ_{01} and Φ_{10} are well-defined. The argument is similar to the last part of the proof of Proposition 3.11: the vanishing of $d^c h$ on the tangent spaces to the monomially admissible Lagrangians ℓ_s and ℓ'_s and on the vector fields X and X' along which these boundary conditions move implies that the restriction of $d^c(h \circ u)$ to the boundary of the strip $\mathbb{R} \times [0,1]$ vanishes identically (outside of $u^{-1}(Y^{\text{in}})$), and the result then follows from the maximum principle with Neumann boundary conditions.

The usual argument for Floer continuation maps then shows that Φ_{01} and Φ_{10} are chain maps, and that $\Phi_{01} \circ \Phi_{10}$ and $\Phi_{10} \circ \Phi_{01}$ are homotopic to identity; it follows that $HF^*(\ell'_0, \ell_0) \simeq HF^*(\ell'_1, \ell_1)$.

We now turn to the second part of the statement. Assume that $K'-K$ is convex, and observe that the generators of $CF^*(\ell',\ell)$ correspond to points where dK' and dK differ by an integer multiple of 2π, ie $\nabla(K'-K)\in(2\pi\mathbb{Z})^n$. By Lemma 5.4, the set of possible values of $\nabla(K'-K)$ is $P(\sigma'-\sigma)$. For each $p\in P(\sigma'-\sigma)\cap(2\pi\mathbb{Z})^n$, the function $K'(\xi)-K(\xi)-\langle p,\xi\rangle$ is convex; up to a small perturbation (preserving convexity) we can assume that its critical points are nondegenerate. Convexity then ensures that the critical point (guaranteed to exist by Lemma 5.4) is unique and a minimum, so that it contributes a single generator to $CF^0(\ell',\ell)$, which (up to a suitable rescaling, see below) we denote by ϑ_p. Taking the direct sum over all p, we find that the Floer complex $CF^*(\ell',\ell)=\operatorname{span}\{\vartheta_p \mid p\in P(\sigma'-\sigma)\cap(2\pi\mathbb{Z})^n\}$ is concentrated in degree zero, which in turn implies the vanishing of the Floer differential, and (5-13) follows. □

As a general convention, we rescale all generators of the Floer complexes for monomially admissible Lagrangian sections by their action (suitably defined, see below), using the exactness of these Lagrangians to eliminate geometrically irrelevant powers of the Novikov variable and ensure that continuation isomorphisms map generators to generators. In the setting of Proposition 5.6, given $\ell=\Gamma_{dK}$ and $\ell'=\Gamma_{dK'}$ with $K'-K$ convex and $p\in P(\sigma'-\sigma)\cap(2\pi\mathbb{Z})^n$, and denoting by ξ_p the critical point of $K'-K-\langle p,\cdot\rangle$, we define the *action* of this intersection point to be the associated critical value of $K'-K-\langle p,\cdot\rangle$, and the generator we denote by ϑ_p is actually $t^{K'(\xi_p)-K(\xi_p)-\langle p,\xi_p\rangle}$ times the standard generator associated to the intersection point. (Of note: our basis depends not only on the Lagrangians ℓ and ℓ' but also on the normalizations of K and K'; different choices yield differently scaled bases, which can be related explicitly by isomorphisms mapping each generator to a power of t times a generator.)

Proposition 5.7 *Let $\ell=\Gamma_{dK}$, $\ell'=\Gamma_{dK'}$ and $\ell''=\Gamma_{dK''}$ be three monomially admissible Lagrangian sections such that none of the pairwise differences of their slopes σ, σ' and σ'' is a multiple of 2π. Assume moreover that $K''-K'$ and $K'-K$ are convex. Then for any $p\in P(\sigma'-\sigma)\cap(2\pi\mathbb{Z})^n$ and $p'\in P(\sigma''-\sigma')\cap(2\pi\mathbb{Z})^n$, the Floer product of $\vartheta_p\in HF^0(\ell',\ell)$ and $\vartheta_{p'}\in HF^0(\ell'',\ell')$ is given by*

$$\vartheta_p\cdot\vartheta_{p'}=\vartheta_{p+p'}\in HF^0(\ell'',\ell). \tag{5-14}$$

Proof We lift ℓ, ℓ' and ℓ'' to the universal cover $T^*\mathbb{R}^n$ of $(\mathbb{C}^*)^n$ by considering the graphs $\widetilde{\ell}$, $\widetilde{\ell}'$ and $\widetilde{\ell}''$ of $d(K+\langle p,\cdot\rangle)$, dK' and $d(K''-\langle p',\cdot\rangle)$, respectively. By construction, the generator ϑ_p lifts to an intersection point of $\widetilde{\ell}$ and $\widetilde{\ell}'$, and similarly $\vartheta_{p'}$ lifts to an intersection of $\widetilde{\ell}'$ with $\widetilde{\ell}''$. Thus, any holomorphic disc in $(\mathbb{C}^*)^n$ contributing to the Floer product of ϑ_p and $\vartheta_{p'}$ lifts to a disc in the universal cover with boundary on $\widetilde{\ell}$, $\widetilde{\ell}'$ and $\widetilde{\ell}''$. It follows that the output of the disc corresponds to an intersection of $\widetilde{\ell}$ with $\widetilde{\ell}''$, ie a critical point of $K''-K-\langle p+p',\cdot\rangle$; hence $\vartheta_p\cdot\vartheta_{p'}$ is a multiple of $\vartheta_{p+p'}$.

The critical points of the convex functions $K'-K-\langle p,\cdot\rangle$, $K''-K'-\langle p',\cdot\rangle$ and $K''-K-\langle p+p',\cdot\rangle$ will be denoted by ξ_p, $\xi_{p'}$ and $\xi_{p+p'}\in\mathbb{R}^n$, respectively. By Stokes' theorem, the symplectic area of any holomorphic triangle contributing to the coefficient of $\vartheta_{p+p'}$ in $\vartheta_p\cdot\vartheta_{p'}$ is equal to the difference of the

actions of the input and output generators, ie

$$(K''(\xi_{p+p'}) - K(\xi_{p+p'}) - \langle p+p', \xi_{p+p'}\rangle) - (K'(\xi_p) - K(\xi_p) - \langle p, \xi_p\rangle) \\ - (K''(\xi_{p'}) - K'(\xi_{p'}) - \langle p', \xi_{p'}\rangle). \tag{5-15}$$

Thus, since our chosen bases of the Floer complexes are already rescaled by action, the powers of t cancel out and each holomorphic disc contributes ± 1.

It remains to show that the overall count of discs is $+1$. Since our calculation is at the level of Floer cohomology, the count we consider is homotopy invariant and we can deform the Lagrangian submanifolds $\widetilde{\ell}$, $\widetilde{\ell}'$ and $\widetilde{\ell}''$ to simplify the problem. We use the same trick as [24, Proposition 3.22], and replace K and K'' by modified functions $\widehat{K}$ and $\widehat{K}''$ such that

$$(K' - \widehat{K})(\xi) = (K' - K)(\xi + \xi_p) \quad \text{and} \quad (\widehat{K}'' - K')(\xi) = (K'' - K)(\xi + \xi_{p'}).$$

This modification ensures that $K' - \widehat{K}$ and $\widehat{K}'' - K'$ remain convex and have the same slopes at infinity as $K' - K$ and $K'' - K$, but the critical points of $K' - \widehat{K} - \langle p, \cdot\rangle$ and $\widehat{K}'' - K' - \langle p', \cdot\rangle$ now lie at the origin; considering their sum, the critical point of $\widehat{K}'' - \widehat{K} - \langle p+p', \cdot\rangle$ also lies at the origin. (A note of caution: modifying $K' - K$ and $K'' - K'$ by translations in the ξ–coordinate in this manner doesn't quite preserve monomial admissibility, as the control over $\arg(z^{\boldsymbol{v}})$ is now achieved over a slightly smaller subset of $(\mathbb{C}^*)^n$; since the collection of these modified subsets still covers the complement of a compact subset, this does not affect in any significant manner the maximum principle arguments we use to control holomorphic curves.) Thus we have reduced the problem to the case where $\widetilde{\ell}$, $\widetilde{\ell}'$ and $\widetilde{\ell}''$ all intersect (transversely) in a single point (near which they are the graphs of the differentials of functions whose differences have nondegenerate minima). The formula (5-15) now shows that any holomorphic disc contributing to the Floer product must have area zero, ie the only contribution is from the constant map. By linearization and reduction to a product setting, the constant disc is easily checked to be regular and contribute $+1$ to the count (using the preferred trivializations of the orientation lines at even-degree generators and the sign conventions from [35, Section 13c]). □

Next, we consider continuation elements (quasiunits) for the action of the wrapping Hamiltonian H on monomially admissible Lagrangian sections in $W^{-1}(-1) \simeq (\mathbb{C}^*)^n$. Recall that H is proper and convex by Proposition 4.22; to simplify normalizations, we assume that its minimum value is zero (otherwise the formula below should be corrected by a factor of $t^{\tau \min H}$).

Proposition 5.8 *Let $\ell = \Gamma_{dK}$ be a monomially admissible Lagrangian section in $(\mathbb{C}^*)^n$, and denote by $\ell' = \phi^\tau(\ell) = \Gamma_{d(K+\tau H)}$ its image under the time τ flow of the wrapping Hamiltonian H for $\tau > 0$, chosen so that $\tau d(\boldsymbol{v}) \notin 2\pi\mathbb{Z}$ for all $\boldsymbol{v} \in \mathcal{V}$. Then the quasiunit $e = e_{\ell',\ell} \in HF^0(\ell', \ell)$ is the generator $e = \vartheta_0$ corresponding to the minimum of H.*

Proof As in Section 3.4 (now working in $(\mathbb{C}^*)^n$ rather than in Y), the quasiunit $e_{\ell',\ell}$ is defined by counting solutions to a Cauchy–Riemann equation whose domain Σ is a disc with a single output boundary

puncture, with moving boundary condition along $\partial\Sigma$ given by the images by ℓ under the flow generated by H. Such a disc lifts to the universal cover $T^*\mathbb{R}^n$ as a disc whose output marked point maps to an intersection of the graphs of dK and $d(K+\tau H)$; it follows that e is a multiple of ϑ_0.

The count of solutions to the Cauchy–Riemann equation is homotopy invariant, so we modify the setting slightly from Section 3.4 in order to make it apparent that the only contribution is from the constant solution at the point of ℓ where H reaches its minimum. Denote by η the 1–form on $\partial\Sigma$ (vanishing near the puncture) such that the variation of the boundary condition along $\partial\Sigma$ is induced by the flow of $X_H \otimes \eta$. Then we consider the perturbed Cauchy–Riemann equation

$$(du - X_H \otimes \alpha)^{0,1} = 0, \tag{5-16}$$

where α is a subclosed 1–form on Σ ($d\alpha \le 0$) which vanishes in the output striplike end and satisfies $\alpha_{|\partial\Sigma} = \eta$.

As in [1, Appendix B], the geometric energy

$$E_{\mathrm{geo}}(u) = \int_\Sigma \|du - X_H \otimes \alpha\|^2 = \int_\Sigma u^*\omega - u^*(dH) \wedge \alpha$$

of a solution to (5-16) and the topological energy

$$E_{\mathrm{top}}(u) = \int_\Sigma u^*\omega - d(u^*(H)\alpha) = E_{\mathrm{geo}}(u) - \int_\Sigma u^*(H)\, d\alpha$$

satisfy $0 \le E_{\mathrm{geo}}(u) \le E_{\mathrm{top}}(u)$ (since $H \ge 0$ and $d\alpha \le 0$). Denoting by s a coordinate along $\partial\Sigma$ and by $t(s)$ the function such that the boundary condition at s is given by $\phi^{t(s)}(\ell) = \Gamma_{K+t(s)H}$ (so $t(s)$ decreases from τ to zero along the boundary, and its differential coincides with η), Stokes' theorem gives

$$E_{\mathrm{top}}(u) = \int_{\partial\Sigma} -\big(u^*(dK) + t(s)u^*(dH)\big) - u^*(H)\,\eta = \int_{\partial\Sigma} -d(u^*K + t(s)u^*H) = \tau\, H_{\mathrm{out}},$$

where H_{out} is the value of H at the output marked point, ie zero. Thus any solution has vanishing geometric and topological energies, ie it is a constant map at the point where H reaches its minimum. Moreover, the constant map is regular (using the fact that its index equals the degree of the output generator, ie zero, and the linearized Cauchy–Riemann operator is injective since essentially the same argument as above shows that the energy of any element of the kernel must be zero); thus the count of solutions is ± 1. Since the sign is independent of ℓ and τ, it follows from the multiplicativity of quasiunits ($e_{\ell'',\ell} = e_{\ell',\ell} \cdot e_{\ell'',\ell'}$, see eg [24, Proposition 3.15]) that the sign is $+1$, and thus $e = \vartheta_0$. □

Finally, we consider the Floer theory of admissible sections with Lagrangian tori, which will allow us in the next part to reduce Floer-theoretic computations involving noncompact Lagrangians to computations involving only tori. Given $x = (x_1, \dots, x_n) \in (\mathbb{K}^*)^n$, we denote by $\mathfrak{t}_x$ the Lagrangian torus $\{\xi\} \times T^n$ consisting of those points of $(\mathbb{C}^*)^n$ whose moment map coordinates satisfy $\xi_i = -(1/2\pi)\,\mathrm{val}(x_i)$ for all $i = 1, \dots, n$, equipped with a rank-one unitary local system over $\mathbb{K}$ whose holonomy y_i around the ith S^1 factor satisfies $x_i = t^{-2\pi\xi_i} y_i^{-1}$. Given a Lagrangian section $\ell = \Gamma_{dK}$, the Floer complex

$CF^*(\ell, \mathfrak{t}_x)$ has rank one, and we denote by ε_x a suitably rescaled generator: namely, we define ε_x to be $t^{K(\xi)}$ times the element of the local system at the intersection point $(\xi, dK(\xi))$ obtained by parallel transport of a fixed element at $(\xi, 0)$ from the origin to $dK(\xi)$ along $\mathfrak{t}_x$.

Proposition 5.9 *Let $\ell = \Gamma_{dK}$ and $\ell' = \Gamma_{dK'}$ be two monomially admissible Lagrangian sections whose slopes σ and σ' do not differ by a multiple of 2π and are such that $K' - K$ is convex, and let $\mathfrak{t}_x$ be the Lagrangian torus with local system associated to the point $x \in (\mathbb{K}^*)^n$ as above. For $p \in P(\sigma' - \sigma) \cap (2\pi\mathbb{Z})^n$, the Floer product of the generators $\vartheta_p \in HF^0(\ell', \ell)$ and $\varepsilon_x \in HF^0(\ell, \mathfrak{t}_x)$ is given by*

$$\varepsilon_x \cdot \vartheta_p = x^{\overline{p}}\, \varepsilon'_x, \tag{5-17}$$

where $\overline{p} = p/2\pi \in \mathbb{Z}^n$, $x^{\overline{p}} = \prod x_i^{\overline{p}_i} \in \mathbb{K}^*$, and ε'_x is the generator of $HF^0(\ell', \mathfrak{t}_x)$ rescaled in the same manner as ε_x.

Proof The argument is similar to the proof of Proposition 5.7. We lift ℓ and ℓ' to $T^*\mathbb{R}^n$ by considering the graphs $\widetilde{\ell}$ and $\widetilde{\ell}'$ of $d(K + \langle p, \cdot \rangle)$ and dK', which intersect at a lift of ϑ_p, and lift $\mathfrak{t}_x$ to the cotangent fiber at $\xi = -(1/2\pi)\,\mathrm{val}(x)$. Any holomorphic disc contributing to the Floer product of ϑ_p and ε_x lifts to $T^*\mathbb{R}^n$, and its symplectic area can be calculated by integrating $d(K' - K - \langle p, \cdot \rangle)$ from ξ_p to ξ, where ξ_p is the critical point of $K' - K - \langle p, \cdot \rangle$, which gives

$$(K'(\xi) - K(\xi) - \langle p, \xi \rangle) - (K'(\xi_p) - K(\xi_p) - \langle p, \xi_p \rangle).$$

The contribution to the Floer product also involves a holonomy factor, given by the ratio between the parallel transport of ε_x along $\mathfrak{t}_x$ from $(\xi, dK(\xi) + p)$ to $(\xi, dK'(\xi))$ and ε'_x. Given the above choices of normalizations of the generators ϑ_p, ε_x and ε'_x, we find that the contribution of each holomorphic disc to the coefficient of ε'_x in the product of ε_x and ϑ_p is, up to sign, $t^{-\langle p, \xi \rangle}$ times the holonomy of $\mathfrak{t}_x$ along a closed loop whose lift to the universal cover runs from $(\xi, dK(\xi) + p)$ to $(\xi, dK(\xi))$. This loop represents the homotopy class $-\overline{p} \in \mathbb{Z}^n \simeq \pi_1(T^n)$; hence, the holonomy can be expressed as $y^{-\overline{p}}$, and one ends up with

$$t^{-\langle p, \xi \rangle}\, y^{-\overline{p}} = x^{\overline{p}}.$$

It only remains to show that the signed count of holomorphic discs contributing to the Floer product of ε_x and ϑ_p is $+1$. Since this count is invariant under deformations, it does not depend on the value of ξ (the position of the cotangent fiber), and it suffices to determine it for a particular value of ξ. We take $\xi = \xi_p$, when all three intersection points coincide and the only contribution is from the constant map, which is regular and contributes $+1$. □

5.4 Floer products on $HF^*(L_0(t'), L_0(t))$

We now return to our main topic, namely the calculation of the Floer cohomology $HF^*(L_0(t'), L_0(t))$ for $t' > t$ and its product operations. As seen in Example 5.5, the slopes of the monomially admissible Lagrangian sections $\ell_0(t'), \ell_0(t) \subset W^{-1}(-1)$ and $\ell_-(t'), \ell_+(t) \subset W^{-1}(c_{t',t})$ (for $t' - t > t_0$) are given by (5-10)–(5-12).

Definition 5.10 For $\tau > 0$, we define

$$\sigma_0(\tau) = (\tau\, d(\boldsymbol{v}))_{\boldsymbol{v}\in\mathcal{V}} \quad \text{and} \quad \sigma_1(\tau) = (\tau\, d(\boldsymbol{v}) - 2\pi v^0)_{\boldsymbol{v}\in\mathcal{V}} \tag{5-18}$$

and denote by $P_0(\tau)$ and $P_1(\tau)$ the corresponding polytopes defined by (5-9).

Since H is convex by Proposition 4.22, the results of Section 5.3 apply to the pair $(\ell_0(t'), \ell_0(t))$ whenever $t'-t>0$. However, because the clockwise monodromy of $W : Y \to \mathbb{C}$ does not act by a convex Hamiltonian, there is no similar guarantee for the pair $(\ell_-(t'), \ell_+(t))$; nonetheless, $\sigma_1(\tau)$ is the slope of a convex Hamiltonian for $\tau = t' - t$ sufficiently large (larger than some constant $t_1 \geq t_0$), so Propositions 5.6–5.9 apply to the Floer cohomology $HF^*(\ell_-(t'), \ell_+(t))$ whenever $t' - t > t_1$.

Proposition 5.11 *For $\tau = t' - t \in (0, t_0) \cap U$, the Floer complex $CF^*(L_0(t'), L_0(t))$ is concentrated in degree zero, the Floer differential vanishes, and*

$$HF^0(L_0(t'), L_0(t)) \simeq HF^0(\ell_0(t'), \ell_0(t)) \simeq \bigoplus_{p\in P_0(t'-t)\cap(2\pi\mathbb{Z})^n} \mathbb{K}\cdot\vartheta_p^{t'\to t}, \tag{5-19}$$

where the generators $\vartheta_p^{t'\to t}$ correspond to the intersections of $\ell_0(t')$ and $\ell_0(t)$ inside $W^{-1}(-1)$, rescaled by action as explained in Section 5.3.

For $\tau = t' - t \in (t_1, \infty) \cap U$, the Floer cohomology $HF^*(L_0(t'), L_0(t))$ is isomorphic to the cohomology of the complex

$$\begin{aligned}\big\{HF^0(\ell_-(t'), \ell_+(t)) \xrightarrow{s} HF^0(\ell_0(t'), \ell_0(t))\big\} \\ \simeq \Big\{\bigoplus_{p\in P_1(t'-t)\cap(2\pi\mathbb{Z})^n} \mathbb{K}\cdot\zeta_p^{t'\to t} \xrightarrow{s} \bigoplus_{p\in P_0(t'-t)\cap(2\pi\mathbb{Z})^n} \mathbb{K}\cdot\vartheta_p^{t'\to t}\Big\},\end{aligned} \tag{5-20}$$

where the generators $\zeta_p^{t'\to t}$ (in degree -1) and $\vartheta_p^{t'\to t}$ (in degree zero) correspond to intersections of $\ell_-(t')$ and $\ell_+(t)$ inside $W^{-1}(c_{t',t})$ and to intersections of $\ell_0(t')$ and $\ell_0(t)$ inside $W^{-1}(-1)$, rescaled by action within the fibers of W; and $s = s^0_{\ell_0,t',t}$ is defined by a weighted count of J–holomorphic sections of $W : Y \to \mathbb{C}$ over the bounded region of the complex plane delimited by γ_t and $\gamma_{t'}$.

Proof This follows immediately from Propositions 5.2 and 5.6. □

Remark 5.12 There are two ways to understand the complex (5-20) and its relation to the Floer complex $CF^*(L_0(t'), L_0(t))$ for $t' - t > t_1$.

(1) Perturbing $L_0(t')$ or $L_0(t)$ by an admissible Hamiltonian isotopy (preserving the fibers of W, and preserving fiberwise monomial admissibility) if necessary, we can assume that (suitably perturbed versions of) the monomially admissible Lagrangian sections $\ell_-(t')$ and $\ell_+(t)$ differ by a convex Hamiltonian. After such a perturbation, both of the Floer complexes $CF^*(\ell_-(t'), \ell_+(t))$ and $CF^*(\ell_0(t'), \ell_0(t))$ are concentrated in degree 0 and their differentials vanish, so that $CF^*(L_0(t'), L_0(t))$ is given by (5-20).

(2) Alternatively, consider the filtration $0 \subset CF^*(\ell_0(t'), \ell_0(t)) \subset CF^*(L_0(t'), L_0(t))$, which is compatible with the Floer differential and products, as any holomorphic disc contributes in a manner that decreases the filtration index by its intersection number with the fibers of W near the origin.[3] This filtration gives rise to a spectral sequence computing $HF^*(L_0(t'), L_0(t))$, in which the second page (after taking the cohomology of the portion of the differential which preserves the filtration index, ie the contributions of holomorphic discs contained in the fibers of W) is precisely (5-20).

Definition 5.13 We call the complex (5-20) (or (5-19) for $t' - t \in (0, t_0)$) the *vertical Floer complex* of $L_0(t')$ and $L_0(t)$, and denote it by $CF^*_{\mathrm{vert}}(L_0(t'), L_0(t))$.

The vertical Floer complex carries a Floer product operation

$$CF^*_{\mathrm{vert}}(L_0(t'), L_0(t)) \otimes CF^*_{\mathrm{vert}}(L_0(t''), L_0(t')) \to CF^*_{\mathrm{vert}}(L_0(t''), L_0(t)) \tag{5-21}$$

for $t'' > t' > t$; this can be understood either as the chain-level product μ^2 after suitable fiberwise perturbations, or as an induced product on the second page of the spectral sequence computing the Floer cohomology (using the fact that the product operation is compatible with the filtration). It follows from the algebraic properties of the Floer product that this operation is associative and satisfies the Leibniz rule with respect to the section-counting differential s.

Proposition 5.14 *Assume that $t'' > t' > t$, and that $t' - t, t'' - t, t'' - t' \in U$, and label the generators as in Proposition 5.11. Then the Floer product* (5-21) *is given by*:

- *For $p \in P_0(t' - t) \cap (2\pi\mathbb{Z})^n$ and $p' \in P_0(t'' - t') \cap (2\pi\mathbb{Z})^n$,*
$$\vartheta_p^{t' \to t} \cdot \vartheta_{p'}^{t'' \to t'} = \vartheta_{p+p'}^{t'' \to t} \in HF^0(\ell_0(t''), \ell_0(t)).$$
- *When $t' - t > t_1$, for $p \in P_1(t' - t) \cap (2\pi\mathbb{Z})^n$ and $p' \in P_0(t'' - t') \cap (2\pi\mathbb{Z})^n$,*
$$\zeta_p^{t' \to t} \cdot \vartheta_{p'}^{t'' \to t'} = C_{t'' \to t', t}\, \zeta_{p+p'}^{t'' \to t} \in HF^0(\ell_-(t''), \ell_+(t)),$$
where $C_{t'' \to t', t}$ is a nonzero constant (independent of p and p').
- *When $t'' - t' > t_1$, for $p \in P_0(t' - t) \cap (2\pi\mathbb{Z})^n$ and $p' \in P_1(t'' - t') \cap (2\pi\mathbb{Z})^n$,*
$$\vartheta_p^{t' \to t} \cdot \zeta_{p'}^{t'' \to t'} = C_{t'', t' \to t}\, \zeta_{p+p'}^{t'' \to t} \in HF^0(\ell_-(t''), \ell_+(t)),$$
where $C_{t'', t' \to t}$ is a nonzero constant (independent of p and p').
- *When $t' - t > t_1$ and $t'' - t' > t_1$, for all p and p', we have $\zeta_p^{t' \to t} \cdot \zeta_{p'}^{t'' \to t'} = 0$.*

[3] Reinterpreting Floer generators as Hamiltonian chords on L_0, their filtration index is their intersection number with the preimage under W of the real positive axis, making this an instance of the filtration associated to a stop (and its removal) in partially wrapped Floer theory [37].

Proof Since the projection $W: Y \to \mathbb{C}$ is holomorphic away from a neighborhood of the zero fiber, it follows from the open mapping principle and from degree constraints that all the holomorphic discs contributing to the Floer product are either contained in the fiber $W^{-1}(-1)$ or sections over a triangular region of the complex plane delimited by the arcs $\gamma_{t''}$, $\gamma_{t'}$ and γ_t; see Figure 5.

When both inputs lie in $W^{-1}(-1)$, the output must also lie in $W^{-1}(-1)$ for degree reasons, and the only contributions come from discs contained inside $W^{-1}(-1)$. Given the relative positions of the tangent lines to $\gamma_{t''}$, $\gamma_{t'}$ and γ_t at -1, the base of the fibration $W: Y \to \mathbb{C}$ doesn't contribute anything to the index of the Cauchy–Riemann operator, so the product operation agrees with the product on the Floer complexes of the monomially admissible sections $\ell_0(t'')$, $\ell_0(t')$ and $\ell_0(t)$ within $W^{-1}(-1) \simeq (\mathbb{C}^*)^n$. Hence, using the same normalization of the generators as in Section 5.3, it follows from Proposition 5.7 that $\vartheta_p^{t'\to t} \cdot \vartheta_{p'}^{t''\to t'} = \vartheta_{p+p'}^{t''\to t}$.

Next we consider the case where one input lies in $W^{-1}(c_{t',t})$ (with $t'-t > t_1$) and the other one is in $W^{-1}(-1)$. The output then necessarily lies in $W^{-1}(c_{t'',t})$ for degree reasons, and the contributions to the Floer product come from holomorphic sections over the triangle $\mathcal{T}_{t''\to t',t}$ delimited by $\gamma_{t''}$, $\gamma_{t'}$ and γ_t with vertices at -1, $c_{t',t}$ and $c_{t'',t}$. Since we are considering cohomology-level operations on the fiberwise Floer complexes, the count we consider is homotopy invariant under deformations; it is in fact one of the operations of the cohomology-level "Seidel TQFT" [35] associated to the fibration $W: Y \to \mathbb{C}$ (in a fairly simple case, since the region over which we count sections does not contain the critical value 0). Thus, we can simplify the counting problem either by trivializing the fibration and deforming the symplectic and complex structures to product ones over $\mathcal{T}_{t''\to t',t}$, or more simply, by deforming the arc γ_t (without crossing the origin) by a compactly supported isotopy in order to bring the intersection points $c_{t',t}$ and $c_{t'',t}$ to -1 and shrink the triangular region $\mathcal{T}_{t''\to t',t}$ to a single point. After this deformation, we are once again reduced to a calculation of the Floer product for the admissible Lagrangian sections within a fiber of W, as the horizontal direction does not contribute to the index of the Cauchy–Riemann operator. Since the slopes of the relevant admissible Lagrangian sections differ by $\sigma_1(t'-t)$ at one input and by $\sigma_0(t''-t')$ at the other, it follows again from Proposition 5.7 that, for all $p \in P_1(t'-t) \cap (2\pi\mathbb{Z})^n$ and $p' \in P_0(t''-t') \cap (2\pi\mathbb{Z})^n$, the product of $\zeta_p^{t'\to t}$ and $\vartheta_{p'}^{t''\to t'}$ is equal to $\zeta_{p+p'}^{t''\to t}$ up to a scaling factor (some power of the Novikov parameter) coming from the amount of symplectic area swept in the deformation to a single fiber.

Next we show that, when all the generators are normalized by action within the fibers of W, the coefficient of $\zeta_{p+p'}^{t''\to t}$ in the product of $\zeta_p^{t'\to t}$ and $\vartheta_{p'}^{t''\to t'}$ depends only on t'', t', t but not on p and p'. Let $K_c: \mathbb{R}^n \to \mathbb{R}$ (resp. K'_c, K''_c) be such that the intersection of $L_0(t)$ (resp. $L_0(t')$, $L_0(t'')$) with $W^{-1}(c)$ is the graph of dK_c (resp. dK'_c, dK''_c) for each $c \in \gamma_t$ (resp. $\gamma_{t'}$, $\gamma_{t''}$). Normalizing K_c, K'_c, K''_c suitably, we can ensure that they vanish at $\xi = 0$, and that a holomorphic section u of $W: Y \to \mathbb{C}$ over $\mathcal{T}_{t''\to t',t}$ which contributes to the product of $\zeta_p^{t'\to t}$ and $\vartheta_{p'}^{t''\to t'}$ lifts to the universal cover of $W^{-1}(\mathcal{T}_{t''\to t',t})$ as a section with boundary values on the graphs of $dK_c + p$, dK'_c and $dK''_c - p'$ for each $c \in \partial\mathcal{T}_{t''\to t',t}$. With this

understood, the holomorphic section u represents the same relative homology class as the chain obtained by adding together:

(1) The "zero section" of W over $\mathcal{T}_{t''\to t',t}$, consisting of the points with moment map coordinates $\xi = 0$ and angular coordinates $\theta_i = \arg(z_i) = 0$ in each fiber.

(2) Over each edge of $\mathcal{T}_{t''\to t',t}$, a path in each fiber $W^{-1}(c)$, $c \in \partial\mathcal{T}_{t''\to t',t}$, connecting the zero section to the boundary value $u(c)$ of the holomorphic section u by running first along $\xi = 0$ from the origin to $dK_c(0) + p$, $dK'_c(0)$ or $dK''_c(0) - p'$, and then along the graph of $dK_c + p$, dK'_c or $dK''_c - p'$ from $\xi = 0$ in a straight line to the ξ–coordinate of $u(c)$.

(3) Over each vertex of $\mathcal{T}_{t''\to t',t}$, a chain in $W^{-1}(c)$, where $c \in \{-1, c_{t',t}, c_{t'',t}\}$, which lies over a straight line path from $\xi = 0$ to the ξ–coordinate of $u(c)$, and for each ξ–value runs in a straight line from $dK'_c(\xi)$ to $dK''_c(\xi) - p'$ for $c = -1$, from $dK_c(\xi) + p$ to $dK'_c(\xi)$ for $c = c_{t',t}$, and from $dK_c(\xi) + p$ to $dK''_c(\xi) - p'$ for $c = c_{t'',t}$.

Denote by $A_{t''\to t',t}$ the symplectic area of the first part of our chain (the "zero section"), which manifestly does not depend on p and p'. The second portion of our chain (over the edges of $\mathcal{T}_{t''\to t',t}$) runs partly along the Lagrangians obtained by parallel transport of the torus $\{\xi = 0\}$ over $\gamma_t, \gamma_{t'}, \gamma_{t''}$, and partly along the Lagrangians $L_0(t), L_0(t'), L_0(t'')$, so its symplectic area vanishes. Finally, the third piece (over the vertices) contributes at each vertex an area equal to the fiberwise action of the corresponding Floer generator, given that we have normalized the Hamiltonians K_c, K'_c, K''_c so that they vanish at $\xi = 0$. For instance, the portion which lies in $W^{-1}(c_{t',t})$, over the path from 0 to $\xi = \xi_p$ and between the graphs of $dK_c + p$ and dK'_c, has symplectic area given by the integral of $dK'_c - dK_c - p$ from zero to ξ_p, ie $(K'_c(\xi_p) - K_c(\xi_p) - \langle p, \xi_p\rangle) - (K'_c(0) - K_c(0))$, which coincides with the fiberwise action for the generator $\zeta_p^{t'\to t}$ within $W^{-1}(c_{t',t})$ since the last term vanishes. Similarly at the two other vertices. Because a rescaling by action is built into the definition of our Floer generators, this implies that the coefficient of $\zeta_{p+p'}^{t''\to t}$ in the product $\zeta_p^{t'\to t} \cdot \vartheta_{p'}^{t''\to t'}$ is $C_{t''\to t',t} = t^{A_{t''\to t',t}}$.

The case of the product $\vartheta_p^{t'\to t} \cdot \zeta_{p'}^{t''\to t'}$ is handled by exactly the same argument, deforming the problem from a count of sections over a triangular region of the complex plane to a fiberwise Floer product and appealing to Proposition 5.7. Finally, the product of two degree -1 generators vanishes for degree reasons. □

For $x = (x_1, \dots, x_n) \in (\mathbb{K}^*)^n$ and $t \in \mathbb{R}$, we denote by $T_x(t)$ the admissible Lagrangian with local system obtained by parallel transport over the arc γ_t of the Lagrangian torus with local system $\mathfrak{t}_x$ introduced in Section 5.3. To be more specific, we fix a T^n–equivariant structure on the local system of $\mathfrak{t}_x$, ie a family of isomorphisms between the local system and its pullbacks under rotations by elements of T^n. (This can be done for instance by thinking of the local system as a trivial complex line bundle equipped with a translation-invariant connection.) With this understood, $\mathfrak{t}_x$ is invariant under both parallel transport between the fibers of W and the action of the wrapping Hamiltonian, and the restriction of $T_x(t)$ to the fiber of W over any point of γ_t can be identified (as a Lagrangian submanifold with local system) with $\mathfrak{t}_x$.

For $t'-t>t_0$, $L_0(t')$ and $T_x(t)$ intersect transversely once in $W^{-1}(-1)$ and once in $W^{-1}(c_{t',t})$; we denote by $\varepsilon_x^{t'\to t}\in HF^0(\ell_0(t'),\mathfrak{t}_x)$ and $\eta_x^{t'\to t}\in HF^0(\ell_-(t'),\mathfrak{t}_x)$ the corresponding Floer generators, rescaled by action as in Section 5.3. We now consider the Floer product

$$CF^*_{\mathrm{vert}}(L_0(t'),T_x(t))\otimes CF^*_{\mathrm{vert}}(L_0(t''),L_0(t'))\to CF^*_{\mathrm{vert}}(L_0(t''),T_x(t)). \tag{5-22}$$

Proposition 5.15 *For $t'-t>t_0$, $CF^*_{\mathrm{vert}}(L_0(t'),T_x(t))=CF^*(L_0(t'),T_x(t))$ is given by*

$$\{\mathbb{K}\cdot\eta_x^{t'\to t}\xrightarrow{s_x}\mathbb{K}\cdot\varepsilon_x^{t'\to t}\}, \tag{5-23}$$

where the generators $\eta_x^{t'\to t}$ (in degree -1) and $\varepsilon_x^{t'\to t}$ (in degree zero) correspond to intersections of $\ell_-(t')$ and $\ell_0(t')$ with $\mathfrak{t}_x$ inside $W^{-1}(c_{t',t})$ and $W^{-1}(-1)$, respectively, rescaled by action, and s_x is defined by a weighted count of J–holomorphic sections of $W:Y\to\mathbb{C}$ over the bounded region of the complex plane delimited by γ_t and $\gamma_{t'}$.

Moreover, given $t''>t'>t$ with $t'-t>t_0$, the Floer product (5-22) is given by:

- *For $p=2\pi\bar p\in P_0(t''-t')\cap(2\pi\mathbb{Z})^n$,*
$$\begin{aligned}\varepsilon_x^{t'\to t}\cdot\vartheta_p^{t''\to t'}&=x^{\bar p}\varepsilon_x^{t''\to t}&&\in HF^0(\ell_0(t''),\mathfrak{t}_x),\\ \eta_x^{t'\to t}\cdot\vartheta_p^{t''\to t'}&=C_{\xi;t''\to t',t}\,x^{\bar p}\eta_x^{t''\to t}&&\in HF^0(\ell_-(t''),\mathfrak{t}_x).\end{aligned}$$
- *If moreover $t''-t'>t_1$, then for $p=2\pi\bar p\in P_1(t''-t')\cap(2\pi\mathbb{Z})^n$,*
$$\begin{aligned}\varepsilon_x^{t'\to t}\cdot\zeta_p^{t''\to t'}&=C_{\xi;t'',t'\to t}\,x^{\bar p}\,\eta_x^{t''\to t}\in HF^0(\ell_-(t''),\mathfrak{t}_x),\\ \eta_x^{t'\to t}\cdot\zeta_p^{t''\to t'}&=0.\end{aligned}$$

Here $C_{\xi;t''\to t',t}$ and $C_{\xi;t'',t'\to t}$ are nonzero constants which depend on t'', t' and t, and possibly on $\xi=-(1/2\pi)\,\mathrm{val}(x)$ but not on p.

Proof The proof is identical to that of Proposition 5.14, except after reduction to a Floer product within the fiber of W we now appeal to Proposition 5.9. The other difference with our previous argument is that the scaling constant $C_{\xi;t''\to t',t}$ is now determined by the symplectic area of a reference section of W over $\mathcal{T}_{t''\to t',t}$ whose edge along γ_t lies at the ξ–value of $\mathfrak{t}_x$, ie $\xi=-(1/2\pi)\,\mathrm{val}(x)$, rather than at $\xi=0$, hence it generally depends on ξ; similarly for $C_{\xi;t'',t'\to t}$. □

Our next result concerns the quasiunits induced by continuation:

Proposition 5.16 *For $t'>t$, the quasiunit $e^{t'\to t}\in HF^0(L_0(t'),L_0(t))$ is given by $e^{t'\to t}=\vartheta_0^{t'\to t}$.*

Proof It suffices to prove the result for $t'-t\in(0,t_0)$, as the general case follows using the multiplicative property of quasiunits ($e^{t''\to t}=e^{t'\to t}\cdot e^{t''\to t'}$ for $t''>t'>t$) and Proposition 5.14.

Recall that the quasiunit is defined by counting solutions to a Cauchy–Riemann equation whose domain Σ is a disc with a single output boundary puncture, with moving boundary condition given by the Lagrangians $L_0(\tau)$ for τ varying between t and t'. Along $\partial\Sigma$, the boundary condition is obtained from the flow of

$X_K \otimes \eta$ for some 1–form η on $\partial\Sigma$ and some Hamiltonian K, namely the sum of a Hamiltonian generating the admissible lifted isotopy ρ^τ, cf Lemma 3.8, which we assume to be supported over a neighborhood V of $\bigcup_{\tau\in[t,t']} \gamma_\tau$, and the wrapping Hamiltonian H. The restriction of K to $L_0(\tau)$ is proper and achieves its minimum at the point of $W^{-1}(-1)$ where H has its minimum; we normalize K so that this minimum value is zero.

As in the proof of Proposition 5.8, we consider solutions to the perturbed Cauchy–Riemann equation $(du - X_K \otimes \alpha)^{0,1} = 0$, and α is a subclosed 1–form on Σ whose restriction to $\partial\Sigma$ agrees with η. Solutions to this equation satisfy the open mapping principle with respect to the projection $W : Y \to \mathbb{C}$ everywhere outside of V (where X_K is not purely vertical) and a neighborhood of the origin (where W isn't necessarily J–holomorphic); this implies that solutions remain within $W^{-1}(V)$, where the Kähler form is exact and the same energy argument as in the proof of Proposition 5.8 shows that the only solution is the constant map at the point of $W^{-1}(-1)$ where H reaches its minimum. It follows that $e^{t'\to t} = \vartheta_0^{t'\to t}$. □

5.5 The Floer differential

Propositions 5.11–5.16 give all the information needed to determine the fiberwise wrapped Floer cohomology $H\mathcal{W}^*(L_0, L_0)$ and its ring structure, except for one key piece of data: the differential of the complex (5-20), ie the section-counting map $s = s^0_{\ell_0,t',t} : HF^0(\ell_-(t'), \ell_+(t)) \to HF^0(\ell_0(t'), \ell_0(t))$. We will first show that this map is given by multiplication with a Laurent polynomial, then show that this polynomial also controls the section-counting map for the parallel transport of the tori $\mathfrak{t}_x$.

Fix $t_+ > t_-$ with $t_+ - t_- > t_1$, and for $p = 2\pi\,\overline{p} \in P_0(t_+ - t_-) \cap (2\pi\mathbb{Z})^n$, denote by $c_{\overline{p}} \in \mathbb{K}$ the coefficients such that

$$s^0_{\ell_0,t_+,t_-}(\zeta_0^{t_+\to t_-}) = \sum_p c_{\overline{p}}\, \vartheta_p^{t_+\to t_-}. \tag{5-24}$$

Lemma 5.17 *For all $t' > t$ such that $t' - t > t_1$ and all $p' \in P_1(t' - t) \cap (2\pi\mathbb{Z})^n$,*

$$s^0_{\ell_0,t',t}(\zeta_{p'}^{t'\to t}) = C(t', t) \sum_p c_{\overline{p}}\, \vartheta_{p+p'}^{t'\to t}, \tag{5-25}$$

where $C(t', t)$ is a nonzero constant depending only on t and t'. Moreover, if $c_{\overline{p}} \neq 0$ then $\overline{p} \in P_{\mathbb{Z}}$.

Proof The compatibility of the Floer product with the differential (ie the Leibniz rule), together with the product formulas of Proposition 5.14, implies that

$$s^0_{\ell_0,t',t}(\zeta_{p_1}^{t'\to t}) \cdot \vartheta_{p_2}^{t''\to t'} = C_{t''\to t',t}\, s^0_{\ell_0,t'',t}(\zeta_{p_1+p_2}^{t''\to t}) \tag{5-26}$$

for all $(t'' > t' > t)$ with $t' - t > t_1$, $p_1 \in P_1(t' - t) \cap (2\pi\mathbb{Z})^n$ and $p_2 \in P_0(t'' - t') \cap (2\pi\mathbb{Z})^n$; and

$$\vartheta_{p_1}^{t'\to t} \cdot s^0_{\ell_0,t'',t'}(\zeta_{p_2}^{t''\to t'}) = C_{t'',t'\to t}\, s^0_{\ell_0,t'',t}(\zeta_{p_1+p_2}^{t''\to t}) \tag{5-27}$$

for all $(t'' > t' > t)$ with $t'' - t' > t_1$, $p_1 \in P_0(t' - t) \cap (2\pi\mathbb{Z})^n$ and $p_2 \in P_1(t'' - t') \cap (2\pi\mathbb{Z})^n$.

We now deduce the lemma from these two identities. First, choose $t'' > \max(t', t_+)$ such that $P_1(t'-t) \subset P_0(t''-t_+)$. It follows from (5-26) for $(t'' > t_+ > t_-)$, $p_1 = 0$ and $p_2 = p'$ that

$$s^0_{\ell_0,t'',t_-}(\zeta^{t''\to t_-}_{p'}) = C^{-1}_{t''\to t_+,t_-}\, s^0_{\ell_0,t_+,t_-}(\zeta^{t_+\to t_-}_0)\cdot \vartheta^{t''\to t_+}_{p'} = C^{-1}_{t''\to t_+,t_-}\sum_p c_{\overline{p}}\,\vartheta^{t''\to t_-}_{p+p'}$$

for all $p' \in P_1(t'-t)\cap(2\pi\mathbb{Z})^n \subset P_0(t''-t_+)\cap(2\pi\mathbb{Z})^n$.

Next, considering (5-27) for either $(t'' > t_- > t)$ or $(t'' > t > t_-)$, with $p_1 = 0$ and $p_2 = p'$ again, yields

$$s^0_{\ell_0,t'',t}(\zeta^{t''\to t}_{p'}) = C(t'',t)\sum_p c_{\overline{p}}\,\vartheta^{t''\to t}_{p+p'}$$

for all $p' \in P_1(t'-t)\cap(2\pi\mathbb{Z})^n$, where $C(t'',t)$ is equal to $C^{-1}_{t''\to t_+,t_-}C^{-1}_{t'',t_-\to t}$ if $t < t_-$, or is equal to $C^{-1}_{t''\to t_+,t_-}C_{t'',t\to t_-}$ if $t > t_-$. This is precisely (5-25), except with t'' everywhere instead of t'. Finally, we use (5-26), now for $(t'' > t' > t)$, $p_1 = p'$ and $p_2 = 0$, to conclude that

$$s^0_{\ell_0,t',t}(\zeta^{t'\to t}_{p'}) = C_{t''\to t',t}C(t'',t)\sum_p c_{\overline{p}}\,\vartheta^{t'\to t}_{p+p'},$$

which is the desired result.

Moreover, the final step of the calculation implies that $p + p' \in P_0(t'-t)\cap(2\pi\mathbb{Z})^n$ for all $p = 2\pi\overline{p}$ such that $c_{\overline{p}} \neq 0$ and for all $p' \in P_1(t'-t)\cap(2\pi\mathbb{Z})^n$. Recall that $P_0(t'-t)$ is defined by the inequalities

$$\langle \vec{v}, \cdot\rangle \le (t'-t)\, d(\boldsymbol{v}) \tag{5-28}$$

for all $\boldsymbol{v} = (\vec{v}, v^0) \in \mathcal{V}$, while $P_1(t'-t)$ is defined by

$$\langle \vec{v}, \cdot\rangle \le (t'-t)\, d(\boldsymbol{v}) - 2\pi v^0 \tag{5-29}$$

for all $\boldsymbol{v} \in \mathcal{V}$, and P is defined by the inequalities $\langle \vec{v}, \cdot\rangle \le v^0$ for all $\boldsymbol{v} \in \mathcal{V}$; cf Definition 4.14. For every $\boldsymbol{v} \in \mathcal{V}$, we can choose t and t' such that $P_1(t'-t)\cap(2\pi\mathbb{Z})^n$ contains some p' which realizes the equality in (5-29). Thus, since $p + p'$ satisfies (5-28) whenever $c_{\overline{p}} \neq 0$, it follows that $\langle p, \vec{v}\rangle \le 2\pi v^0$, ie $\langle \overline{p}, \vec{v}\rangle \le v^0$, whenever $c_{\overline{p}} \neq 0$. Since this holds for all $\boldsymbol{v} \in \mathcal{V}$, it follows that $\overline{p} \in P \cap \mathbb{Z}^n = P_{\mathbb{Z}}$. □

Lemma 5.17 implies that the coefficients $c_{\overline{p}} \in \mathbb{K}$ ($\overline{p} \in P_{\mathbb{Z}}$) suffice to determine the fiberwise wrapped Floer cohomology of L_0. More explicitly:

Proposition 5.18 *Let $g(x) = \sum_p c_{\overline{p}} x^{\overline{p}} \in \mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, and assume that g is not identically zero. Then $H\mathcal{W}^*(L_0, L_0)$ is isomorphic to the quotient $\mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]/(g)$ of the ring of Laurent polynomials by the ideal generated by g.*

Proof By Corollary 3.24, we can calculate $H\mathcal{W}^*(L_0, L_0)$ as a colimit of Floer cohomology groups $HF^*(L_0(t'), L_0(t))$ for $t'-t \to \infty$. For $t'-t > t_1$, we use Proposition 5.11 and Lemma 5.17 to identify $CF^*_{\mathrm{vert}}(L_0(t'), L_0(t))$ with a subcomplex of the chain complex

$$\mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \xrightarrow{g} \mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}], \tag{5-30}$$

where in degree 0 we identify $\vartheta_p^{t'\to t}$ with the monomial $x^{\bar{p}}$ for all $p \in P_0(t'-t) \cap (2\pi\mathbb{Z})^n$, and in degree -1 we identify $\zeta_p^{t'\to t}$ with $C(t',t)\,x^{\bar{p}}$ for all $p \in P_1(t'-t) \cap (2\pi\mathbb{Z})^n$, and the subcomplex corresponds to those Laurent polynomials whose Newton polytopes are contained inside $(1/2\pi)P_0(t'-t)$ and $(1/2\pi)P_1(t'-t)$, respectively.

It follows from Proposition 5.14 that, with these identifications, the product operations on these Floer complexes are given by multiplication of Laurent polynomials; and Proposition 5.16 implies that the continuation maps as $t'-t$ increases to infinity are given by inclusion. Thus, the naive limit of the complexes (5-20) as $t'-t \to \infty$ is given by (5-30).

Since by assumption g is not zero, multiplication by g is injective, and the cohomology of (5-20) is concentrated in degree zero; specifically, $HF^0(L_0(t'), L_0(t))$ is the quotient of the space of Laurent polynomials whose Newton polytope is contained in $(1/2\pi)P_0(t'-t)$ by the subspace of those which are g times a Laurent polynomial with Newton polytope contained in $(1/2\pi)P_1(t'-t)$. Taking the colimit under inclusion maps as $t'-t \to \infty$, we conclude that $H\mathcal{W}^*(L_0, L_0)$ is also concentrated in degree zero, and we have an isomorphism of $\mathbb{K}$–vector spaces

$$H\mathcal{W}^0(L_0, L_0) \simeq \mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]/(g).$$

This isomorphism is compatible with the ring structure, since by Proposition 5.14 the Floer product operation corresponds to multiplication of Laurent polynomials. □

Given Proposition 5.18, the proof of Theorem 1.2 reduces to the determination of the Laurent polynomial g. More precisely, we need to show that, after equipping Y with a suitable bulk deformation class, g can be assumed to coincide with the Laurent polynomial f defining the hypersurface H up to an overall scaling factor. To this end, we first reinterpret g as a count of holomorphic sections with boundary on the objects $T_x(t)$ obtained by parallel transport of product tori with rank one local systems. Recalling the calculation of the vertical Floer complex $CF^*_{\mathrm{vert}}(L_0(t'), T_x(t))$ from Proposition 5.15, we have:

Proposition 5.19 *For $t'-t > t_1$, and for $x \in (\mathbb{K}^*)^n$, the differential on the complex $CF^*_{\mathrm{vert}}(L_0(t'), T_x(t))$ is given by*

$$s_x(\eta_x^{t'\to t}) = C_\xi(t',t)\, g(x)\, \varepsilon_x^{t'\to t},$$

where $C_\xi(t',t)$ is a nonzero constant depending only on t, t' and $\xi = -(1/2\pi)\,\mathrm{val}(x)$.

Proof For $t'' > t' + t_1$, the compatibility of the Floer product (5-22) with the differentials on the vertical Floer complexes implies that

$$s_x(\eta_x^{t'\to t}) \cdot \zeta_0^{t''\to t'} - \eta_x^{t'\to t} \cdot s^0_{\ell_0,t'',t}(\zeta_0^{t''\to t'}) = 0.$$

Using Lemma 5.17 and Proposition 5.15, this yields

$$s_x(\eta_x^{t'\to t}) \cdot \zeta_0^{t''\to t'} = C(t'',t') \sum_p c_{\bar{p}}\, \eta_x^{t'\to t} \cdot \vartheta_p^{t''\to t'} = C(t'',t')\, C_{\xi;t''\to t',t}\, g(x)\, \eta_x^{t''\to t}.$$

Since $s_x(\eta_x^{t'\to t})$ is a multiple of $\varepsilon_x^{t'\to t}$, comparing with the formula for $\varepsilon_x^{t'\to t}\cdot\zeta_0^{t''\to t}$ given by Proposition 5.15 we conclude that

$$s_x(\eta_x^{t'\to t}) = C^{-1}_{\xi;t'',t'\to t}\, C(t'',t')\, C_{\xi;t''\to t',t}\, g(x)\, \varepsilon_x^{t'\to t}.$$

The result follows, setting $C^{-1}_{t'',t'\to t}\, C(t'',t')\, C_{t''\to t',t} = C_\xi(t',t)$. □

Remark 5.20 Another way to prove Proposition 5.19, still using the Leibniz rule, Lemma 5.17 and Proposition 5.15, is to argue that, for $t''>t'>t$ with $t''-t'>t_1$,

$$\begin{aligned} s_x(\eta_x^{t''\to t}) &= C^{-1}_{\xi;t'',t'\to t}\, s_x(\varepsilon_x^{t'\to t}\cdot\zeta_0^{t''\to t'}) \\ &= C^{-1}_{\xi;t'',t'\to t}\, \varepsilon_x^{t'\to t}\cdot s^0_{\ell_0,t'',t'}(\zeta_0^{t''\to t'}) \\ &= C^{-1}_{\xi;t'',t'\to t} C(t'',t')\sum_p c_{\bar p}\, \varepsilon_x^{t'\to t}\cdot\vartheta_p^{t''\to t'} \\ &= C_\xi(t'',t)\, g(x)\, \varepsilon_x^{t''\to t}. \end{aligned}$$

Next we consider the Floer complex $CF^*_{\mathrm{vert}}(T_x(t'),T_x(t))$ for $t'-t>t_0$. The Lagrangian submanifolds (with local systems) $T_x(t')$ and $T_x(t)$ obtained by parallel transport of $\mathfrak{t}_x$ over the arcs $\gamma_{t'}$ and γ_t intersect cleanly along tori within the fibers $W^{-1}(-1)$ and $W^{-1}(c_{t',t})$, rather than transversely, so the definition of their Floer complex requires a bit of care. One approach is to use a small Hamiltonian perturbation to achieve transversality within the fibers of W; another approach that is better suited to computations is to use a "Morse–Bott" model. Namely, we choose a Morse function on the n–torus, and consider holomorphic discs with boundary in $T_x(t')\cup T_x(t)$ together with Morse flow lines (within a component of $T_x(t')\cap T_x(t)$) from the boundary marked points of the disc to critical points of the Morse function; see for example [36, Section 4], with the difference that we only use Morse theory within the fibers of W, while in the base direction we have usual striplike ends. Equivalently, instead of involving Morse flow lines, one could simply require the boundary marked points of the holomorphic discs to lie on the stable or unstable manifolds of the Morse critical points.

Regardless of the chosen approach, the Floer complex is built from two copies of the fiberwise Floer complex $CF^*(\mathfrak{t}_x,\mathfrak{t}_x)$, corresponding to generators and Floer trajectories which lie entirely within each of the two fibers $W^{-1}(-1)$ and $W^{-1}(c_{t',t})$, together with a connecting differential which counts J–holomorphic sections of $W:Y\to\mathbb{C}$ over the region delimited by γ_t and $\gamma_{t'}$ (with the usual caveat regarding our use of the word "section", since J differs from the standard complex structure near $W^{-1}(0)$), with boundary on $\mathfrak{t}_x$, and satisfying incidence conditions at -1 and at $c_{t',t}$.

As before, we denote by $CF^*_{\mathrm{vert}}(T_x(t'),T_x(t))$ the "vertical Floer complex" obtained by taking the cohomology with respect to the contributions to the Floer differential which lie entirely within a fiber of W. Since $\mathfrak{t}_x\subset(\mathbb{C}^*)^n$ does not bound any holomorphic discs, the Floer differential on $CF^*(\mathfrak{t}_x,\mathfrak{t}_x)$ only involves a classical part, and reduces to the usual cohomology of T^n (with coefficients in endomorphisms of the local system, which are canonically isomorphic to the ground field $\mathbb{K}$). We claim:

Proposition 5.21 *For $t'-t>t_0$, and for $x\in(\mathbb{K}^*)^n$, the vertical Floer complex $CF^*_{\mathrm{vert}}(T_x(t'),T_x(t))$ is given by*

$$\{H^*(T^n,\mathbb{K})\xrightarrow{s_x}H^*(T^n,\mathbb{K})\}, \tag{5-31}$$

where the connecting differential s_x, which is defined by a weighted count of J–holomorphic sections of $W:Y\to\mathbb{C}$ over the region delimited by γ_t and $\gamma_{t'}$, with incidence conditions on cycles in $\mathfrak{t}_x$ at -1 and $c_{t',t}$, is given by multiplication by $C'_\xi(t',t)g(x)\in\mathbb{K}$ for some nonzero constant $C'_\xi(t',t)$ depending only on t, t' and $\xi=-(1/2\pi)\,\mathrm{val}(x)$.

The first part of the statement is clear from the above description of the Floer complex $CF^*(T_x(t'),T_x(t))$; the remaining part, namely showing that the differential s_x is given by multiplication by $g(x)$, relies on an algebraic argument similar to the proof of Proposition 5.19 using the Leibniz rule. Thus, we first need to establish a couple of lemmas (analogous to Propositions 5.14 and 5.15), before providing the proof.

We denote by $\delta_x^{t'\to t}$ and $1_x^{t'\to t}$ respectively the elements which correspond to $1\in H^0(T^n,\mathbb{K})$ in the left and right summands of (5-31); given $\alpha\in H^*(T^n,\mathbb{K})$, the corresponding elements of the left and right summands of (5-31) are denoted by $\alpha\,\delta_x^{t'\to t}$ and $\alpha\,1_x^{t'\to t}$. With this notation, we have:

Lemma 5.22 *Assuming $t'-t>t_0$ and $t''-t'>t_0$, the Floer product*

$$CF^*_{\mathrm{vert}}(T_x(t'),T_x(t))\otimes CF^*_{\mathrm{vert}}(T_x(t''),T_x(t'))\to CF^*_{\mathrm{vert}}(T_x(t''),T_x(t))$$

is as follows: for all $\alpha,\alpha'\in H^*(T^n,\mathbb{K})$,

$$\begin{aligned}
(\alpha\,1_x^{t'\to t})\cdot(\alpha'\,1_x^{t''\to t'})&=(\alpha\smile\alpha')\,1_x^{t''\to t},\\
(\alpha\,\delta_x^{t'\to t})\cdot(\alpha'\,1_x^{t''\to t'})&=C'_{\xi;t''\to t',t}\,(\alpha\smile\alpha')\,\delta_x^{t''\to t},\\
(\alpha\,1_x^{t'\to t})\cdot(\alpha'\,\delta_x^{t''\to t'})&=C'_{\xi;t'',t'\to t}\,(\alpha\smile\alpha')\,\delta_x^{t''\to t},\\
(\alpha\,\delta_x^{t'\to t})\cdot(\alpha'\,\delta_x^{t''\to t'})&=0,
\end{aligned}$$

where $C'_{\xi;t''\to t',t},C'_{\xi;t'',t'\to t}\in\mathbb{K}^*$ depend only on t, t', t'' and $\xi=-(1/2\pi)\,\mathrm{val}(x)$.

Proof The proof is essentially the same as for Proposition 5.14: by considering the projection under $W:Y\to\mathbb{C}$, we find that the only holomorphic discs contributing to the Floer product are either contained in $W^{-1}(-1)$, or sections over one of the two triangular regions delimited by $\gamma_{t''}$, $\gamma_{t'}$ and γ_t; in the latter case, we use a deformation argument to shrink the triangular region to a single point and reduce to a count within the fiber of W. Either way, things reduce to the Floer product on $HF^*(\mathfrak{t}_x,\mathfrak{t}_x)\simeq H^*(T^n,\mathbb{K})$, which coincides with the ordinary cup product since there are no nonconstant holomorphic discs in $(\mathbb{C}^*)^n$ with boundary on $\mathfrak{t}_x$. As in the proof of Proposition 5.14, the constant factors $C'_{\xi;t''\to t',t}$ and $C'_{\xi;t'',t'\to t}$ account for the symplectic area of a reference section (now chosen to lie at the same ξ–value as $\mathfrak{t}_x$, ie $\xi=-(1/2\pi)\,\mathrm{val}(x)$) over the appropriate triangular region of the complex plane, which turns out to coincide with the amount of area swept in the deformation used to reduce to a single fiber. □

Lemma 5.23 *Assume $t'-t>t_0$ and $t''-t'>t_0$. The Floer product*

$$CF^*_{\mathrm{vert}}(T_x(t'),T_x(t))\otimes CF^*_{\mathrm{vert}}(L_0(t''),T_x(t'))\to CF^*_{\mathrm{vert}}(L_0(t''),T_x(t))$$

vanishes identically on elements of the form $(\alpha\, 1_x^{t'\to t})$ or $(\alpha\,\delta_x^{t'\to t})$ whenever α is a cohomology class of positive degree, whereas

$$\begin{aligned} 1_x^{t'\to t}\cdot\eta_x^{t''\to t'} &= C''_{\xi;t'',t'\to t}\,\eta_x^{t''\to t}, \quad 1_x^{t'\to t}\cdot\varepsilon_x^{t''\to t'}=\varepsilon_x^{t''\to t},\\ \delta_x^{t'\to t}\cdot\varepsilon_x^{t''\to t'} &= C''_{\xi;t''\to t',t}\,\eta_x^{t''\to t}, \quad \delta_x^{t'\to t}\cdot\eta_x^{t''\to t'}=0, \end{aligned}$$

where $C''_{\xi;t''\to t',t}, C''_{\xi;t'',t'\to t}\in\mathbb{K}^*$ depend only on t,t',t'' and $\xi=-(1/2\pi)\,\mathrm{val}(x)$.

Proof The argument is again similar, reducing to the calculation of Floer products within the fiber $W^{-1}(-1)\simeq(\mathbb{C}^*)^n$, specifically the product

$$HF^*(\mathfrak{t}_x,\mathfrak{t}_x)\otimes HF^*(\ell_0(t''),\mathfrak{t}_x)\to HF^*(\ell_0(t''),\mathfrak{t}_x).$$

The vanishing for elements of $HF^*(\mathfrak{t}_x,\mathfrak{t}_x)\simeq H^*(T^n,\mathbb{K})$ of positive degree then follows from the fact that $HF^*(\ell_0(t''),\mathfrak{t}_x)$ has rank one and is concentrated in a single degree; whereas $1\in H^0(T^n,\mathbb{K})\simeq HF^0(\mathfrak{t}_x,\mathfrak{t}_x)$ acts by identity by cohomological unitality. □

Proof of Proposition 5.21 Given t and t' with $t'-t>t_0$, choose t'' so that $t''>t'+t_1$. The compatibility of Floer products and differentials on vertical Floer complexes (the Leibniz rule) implies that

$$s_x(\delta_x^{t'\to t})\cdot\eta_x^{t''\to t'}-\delta_x^{t'\to t}\cdot s_x(\eta_x^{t''\to t'})=0,$$

which using Proposition 5.19 and Lemma 5.23 yields

$$s_x(\delta_x^{t'\to t})\cdot\eta_x^{t''\to t'}=C_\xi(t'',t')\,g(x)\,\delta_x^{t'\to t}\cdot\varepsilon_x^{t''\to t'}=C''_{\xi;t''\to t',t}\,C_\xi(t'',t')\,g(x)\,\eta_x^{t''\to t}.$$

Using again Lemma 5.23 (and degree constraints), it follows that

$$s_x(\delta_x^{t'\to t})=(C''_{\xi;t'',t'\to t})^{-1}\,C''_{\xi;t''\to t',t}\,C_\xi(t'',t')\,g(x)\,1_x^{t'\to t}.$$

Setting $C'_\xi(t',t)=(C''_{\xi;t'',t'\to t})^{-1}\,C''_{\xi;t''\to t',t}\,C_\xi(t'',t')$, we rewrite this as

$$s_x(\delta_x^{t'\to t})=C'_\xi(t',t)\,g(x)\,1_x^{t'\to t}$$

whenever $t'-t>t_0$, which is the desired result for the generators of $H^0(T^n,\mathbb{K})$.

To extend the result to higher-degree cohomology classes, we use the product formulas of Lemma 5.22: given $t'>t+t_0$, and choosing $t''>t'+t_0$, the Leibniz rule implies that

$$\begin{aligned} s_x(\alpha\,\delta_x^{t'\to t})\cdot\delta_x^{t''\to t'}=(\alpha\,\delta_x^{t'\to t})\cdot s_x(\delta_x^{t''\to t'}) &= C'_\xi(t'',t')\,g(x)\,(\alpha\,\delta_x^{t'\to t})\cdot 1_x^{t''\to t'}\\ &= C'_{\xi;t''\to t',t}\,C'_\xi(t'',t')\,g(x)\,(\alpha\,\delta_x^{t''\to t}), \end{aligned}$$

and hence

$$s_x(\alpha\,\delta_x^{t'\to t}) = C'^{-1}_{\xi;t'',t'\to t}\, C'_{\xi;t''\to t',t}\, C'_\xi(t'',t')\, g(x)\,(\alpha\, 1_x^{t'\to t}) = C'_\xi(t',t)\, g(x)\,(\alpha\, 1_x^{t'\to t}),$$

where the identity $C'_\xi(t',t) = C'^{-1}_{\xi;t'',t'\to t}\, C'_{\xi;t''\to t',t}\, C'_\xi(t'',t')$ follows from considering the special case $\alpha = 1$. □

Given Propositions 5.18 and 5.21, the remaining step in the proof of Theorem 1.2 is a direct calculation of the differential in (5-31), with the aim of showing that the Laurent polynomials f and g agree up to a constant scaling factor.

5.6 Holomorphic sections of W with boundary on product tori

We now turn to the problem of explicitly determining the differential on the complex (5-31), ie counting J–holomorphic sections of $W: Y \to \mathbb{C}$ over the region delimited by $\gamma_{t'}$ and $\gamma_{t''}$, with boundary in the product torus $\mathfrak{t}_x$ in each fiber. (In this section we use t' and t'' instead of t and t' to avoid notation conflicts with the Novikov parameter).

By Proposition 5.21, the differential s_x is given by multiplication by some element of $\mathbb{K}$; thus it is enough to determine the image of the generator of $H^0(T^n,\mathbb{K})$ (or equivalently, that of $H^n(T^n,\mathbb{K})$); this amounts to counting J–holomorphic sections whose boundary passes through some prescribed input point in $W^{-1}(-1)$ (or output point in $W^{-1}(c_{t'',t'})$ if we consider H^n rather than H^0; or in fact a point anywhere on the Lagrangian boundary condition, as the end result does not depend on this choice).

While our definitions involve a perturbation of the standard complex structure J_0 near $W^{-1}(0)$ in order to achieve regularity of moduli spaces, actually counting discs in practice requires one to consider the limit as J converges to the (nonregular) standard complex structure J_0. Under this limit, the J–holomorphic discs contributing to the differential s_x converge either to holomorphic discs (holomorphic sections of $W: Y \to \mathbb{C}$), or to stable configurations consisting of a holomorphic disc (a section of W) together with one or more rational curves contained inside the singular fiber $W^{-1}(0)$. (This is a standard instance of Gromov compactness for a C^∞–convergent sequence of almost-complex structures, cf [31, Theorem 5.3.1] for the closed case; as usual when considering sections, it follows from positivity of intersection of the nonvertical components with the fibers of W that any bubbles arising in the limit must be contained in a fiber of W, hence in $W^{-1}(0)$.) Thus, the first step is to understand moduli spaces of holomorphic sections of W bounded by $T_x(t'') \cup T_x(t')$.

Proposition 5.24 *For $t''-t' > t_0$ and $x \in (\mathbb{K}^*)^n$, the homotopy classes of holomorphic sections of $W: Y \to \mathbb{C}$ with boundary on $T_x(t'') \cup T_x(t')$ are in one-to-one correspondence with the elements of $P_\mathbb{Z}$. For each such class, the moduli space of sections consists of a single orbit under the action of T^n, and the count of sections through any given point of $\mathfrak{t}_x \subset W^{-1}(-1)$ is equal to one.*

Proof Denote by S the region of the complex plane delimited by $\gamma_{t''}$ and $\gamma_{t'}$. Since S contains the origin, a holomorphic section of $W: Y \to \mathbb{C}$ over S has intersection number one with $Z = W^{-1}(0) = \bigcup_\alpha Z_\alpha$,

which is the union of the irreducible toric divisors of Y. Hence it must intersect exactly one of these, say Z_α for some $\alpha \in P_{\mathbb{Z}}$, and be disjoint from $Z_{\alpha'}$ for all $\alpha' \neq \alpha$. For fixed α, we are thus reduced to studying holomorphic discs contained in $Y_\alpha = Y \setminus \bigcup_{\alpha' \neq \alpha} Z_{\alpha'}$, the partial compactification of the open stratum of the toric variety Y obtained by adding the open stratum of Z_α.

Y_α is biholomorphic to $\mathbb{C} \times (\mathbb{C}^*)^n$, and we choose such an identification where the first coordinate is given by $W = -z^{(0,\dots,0,1)}$, and the remaining coordinates $(z_1, \dots, z_n) \in (\mathbb{C}^*)^n$ are given by toric monomials, in such a way that product tori in the fibers $W^{-1}(c)$, $c \in \partial S$ correspond to standard product tori in $\{c\} \times (\mathbb{C}^*)^n$.

We parametrize holomorphic sections of $W_{|Y_\alpha} : Y_\alpha \to \mathbb{C}$ over S by the first coordinate (ie W), so that the domain is S, and we are reduced to finding holomorphic maps $S \to (\mathbb{C}^*)^n$, $w \mapsto (z_1(w), \dots, z_n(w))$, which satisfy the appropriate boundary conditions over ∂S. Specifically, our boundary condition is given by product tori in $(\mathbb{C}^*)^n$, ie the value of $|z_i|$ is prescribed at every point of the boundary. We claim that solutions, if they exist, are unique up to the action of T^n on $(\mathbb{C}^*)^n$ by rotations. Indeed, if $z_i, \widetilde{z}_i : S \to \mathbb{C}^*$ are both holomorphic and $|z_i(w)| = |\widetilde{z}_i(w)|$ for all $w \in \partial S$, then the ratio $\widetilde{z}_i(w)/z_i(w)$ defines a holomorphic map from S to $\mathbb{C}^*$, taking values in the unit circle along ∂S; the open mapping principle thus implies that it is constant, ie there exists $e^{i\theta} \in S^1$ such that $\widetilde{z}_i(w) = e^{i\theta} z_i(w)$ for all $w \in S$. Thus the moduli space of sections in the given class consists of at most one T^n–orbit.

One approach to proving existence is to use complex analysis. For each $i \in \{1, \dots, n\}$, the boundary condition prescribes the value of $\log |z_i| = \mathrm{Re}(\log z_i)$ at every point of ∂S. Using the Riemann mapping theorem to identify S with the unit disc, it is a classical result of Schwarz that, up to a pure imaginary additive constant, there exists a unique analytic function $\log z_i : \mathrm{int}(S) \to \mathbb{C}$ (given by the Schwarz integral formula) whose real part has a continuous extension and takes the prescribed values at the boundary of S; see eg [7, Sections 4.6.3–4.6.4]. Because the given real boundary condition along the unit circle is Hölder continuous (even after pullback from S to the disc, see eg [33, Chapter 3]), the imaginary part $\mathrm{Im}(\log z_i)$ (the harmonic conjugate of $\mathrm{Re}(\log z_i)$) also has a (Hölder) continuous extension to the boundary, given by the Hilbert transform of the real part [19, Theorem III.1.3]. Exponentiating, we arrive at the desired mapping $z_i : S \to \mathbb{C}^*$, and conclude that, up to the action of T^n by rotation of the coordinates of $(\mathbb{C}^*)^n$, there is a unique continuous map $w \mapsto (z_1(w), \dots, z_n(w))$ from S to $(\mathbb{C}^*)^n$ which is holomorphic over the interior of S and satisfies the given boundary conditions.

An alternative approach to existence is to use the invariance of the count of holomorphic sections of W upon deforming the given boundary condition to a product one, given by the same torus (in terms of the coordinates z_i) in all the fibers of W over ∂S; ie we modify the problem so that the prescribed value of $|z_i|$ is the same at every point of ∂S, rather than possibly varying from one point to another. (This can viewed either as deforming the totally real boundary condition being imposed on the sections of W, or as keeping the same Lagrangian boundary condition but modifying the coordinates and the complex structure on Y_α by rescaling each of $z_1, \dots, z_n$ by an amount which varies smoothly over S.) After this

deformation, one is led to look for holomorphic maps from S to $(\mathbb{C}^*)^n$ such that $|z_i|$ is equal to a fixed constant at every point of ∂S: in other terms, holomorphic discs (parametrized by S) in $(\mathbb{C}^*)^n$ with boundary on a fixed product torus. By the maximum principle, the only solutions are constant maps, and these are regular. Thus, in the deformed setting, the moduli space of sections consists of precisely one T^n–orbit, and the count of holomorphic sections through a given point is equal to one. Because of the homotopy invariance of Floer-theoretic section-counting invariants under deformations, it follows that the moduli space of sections for our initial problem is also nonempty, consisting of a single T^n–orbit, and the count of sections through a given point is equal to one. □

Remark 5.25 The argument can be simplified if we assume that $\xi = -(1/2\pi)\,\mathrm{val}(x)$ lies in the intersection of n of the subsets $S_{\boldsymbol{v},\gamma}$, $\boldsymbol{v} \in \mathcal{V}$, defined by (4-11); since nonempty such intersections always exist, and our comparison of f and g only requires us to determine the differential s_x for x of arbitrary fixed valuation, this simpler setting would in fact suffice for our purposes. When ξ lies in the intersection of n of the $S_{\boldsymbol{v},\gamma}$, by Proposition 4.16 we can choose the toric monomials $z_1, \dots, z_n$ in the above argument in such a way that they are all invariant under parallel transport along ∂S at all points of $\mathfrak{t}_x$. This implies that the radii $|z_i|$ of the boundary tori remain constant all along ∂S, ie the boundary condition consists of the same product torus in $(\mathbb{C}^*)^n$ over each point of ∂S; we can then directly classify the holomorphic sections without appealing to complex analysis or to a deformation argument.

Each of the families of holomorphic sections identified in Proposition 5.24, representing a relative homology class $[D_\alpha] \in H_2(Y, T_x(t'') \cup T_x(t'))$, contributes to the Floer differential on $CF^*_{\mathrm{vert}}(T_x(t''), T_x(t'))$ with a weight

$$\mathrm{weight}([D_\alpha]) = t^{\int_{[D_\alpha]}\omega}\,\mathrm{hol}([\partial D_\alpha])\exp\biggl(\int_{[D_\alpha]}\mathfrak{b}\biggr) \in \Lambda_{\geq 0}. \tag{5-32}$$

In this formula, $\mathrm{hol}([\partial D_\alpha])$ denotes the holonomy of the local system along the boundary of D_α, which requires some clarification. Since the local systems on $T_x(t')$ and $T_x(t'')$ are isomorphic over $T_x(t') \cap T_x(t'')$ (canonically over $W^{-1}(-1)$, and in a preferred manner up to a constant factor over $W^{-1}(c_{t'',t'})$ using the T^n–equivariant structure of $\mathfrak{t}_x$), they can be glued into a local system on the portion of $T_x(t') \cup T_x(t'')$ which fibers over ∂S. Noting that this subset of $T_x(t') \cup T_x(t'')$ can be deformed isotopically to a product torus in Y, we choose the gluing at $W^{-1}(c_{t'',t'})$ in such a way that the holonomy of the local system along a loop which deforms to an orbit of the last S^1–factor of the toric action (with moment map η) is equal to identity. (Meanwhile, the holonomies along the first n circle factors, within the fibers of W, coincide with those of $\mathfrak{t}_x$.) With this choice in hand, we define $\mathrm{hol}([\partial D_\alpha])$ to be the holonomy of the local system on $T_x(t') \cup T_x(t'')$ along the boundary of D_α. Also, we denote by $\mathfrak{b}$ a representative of the bulk deformation class which is supported near $W^{-1}(0)$ (so its pairing with $[D_\alpha]$ is well-defined). Specifically, we choose the bulk deformation to be of the form

$$\mathfrak{b} = \sum_{\alpha \in P_{\mathbb{Z}}} \mathfrak{b}_\alpha\, \delta_{Z_\alpha}, \tag{5-33}$$

where the constants $\mathfrak{b}_\alpha \in \Lambda_{\geq 0}$ are coefficients to be determined later, and δ_{Z_α} is a representative of the cohomology class Poincaré dual to the divisor Z_α, supported in a small neighborhood of Z_α. Since $[D_\alpha]$ has intersection number one with Z_α and zero with the other components of $W^{-1}(0)$, we find that $\exp\bigl(\int_{[D_\alpha]} \mathfrak{b}\bigr) = \exp(\mathfrak{b}_\alpha)$.

Proposition 5.26 *For all $t'' > t' + t_0$, $\alpha \in P_\mathbb{Z}$ and $x \in (\mathbb{K}^*)^n$, there exists a nonzero constant $K_\xi(t'', t')$ depending only on t', t'' and $\xi = -(1/2\pi)\,\mathrm{val}(x)$ such that the weight of a holomorphic section of $W : Y \to \mathbb{C}$ bounded by $T_x(t'') \cup T_x(t')$ and representing the class $[D_\alpha]$ is given by*

$$\mathrm{weight}([D_\alpha]) = K_\xi(t'', t')\, t^{2\pi\nu(\alpha)} x^\alpha \exp(\mathfrak{b}_\alpha). \tag{5-34}$$

Proof The portion of $T_x(t'') \cup T_x(t')$ which fibers over ∂S can be deformed by an isotopy into a product torus in Y (by deforming S to a disc), so $H_2(Y, T_x(t'') \cup T_x(t')) \simeq H_2(Y, T^{n+1}) \simeq \mathbb{Z}^{P_\mathbb{Z}}$ (where the latter isomorphism follows from standard facts in toric geometry). Concretely, this means that relative homology classes are uniquely determined by their algebraic intersection numbers with each of the toric divisors Z_α.

Let $\alpha_1, \alpha_2 \in P_\mathbb{Z}$ be two lattice points which are connected by an edge in the subdivision $\mathcal{P}$ of P determined by the tropicalization of the Laurent polynomial f (see Section 2), ie such that the toric divisors $Z_{\alpha_1}, Z_{\alpha_2} \subset Y$ intersect along an $(n-1)$–dimensional toric stratum $Z_{\alpha_1\alpha_2}$. In terms of the moment polytope Δ_Y, $Z_{\alpha_1\alpha_2}$ corresponds to the codimension 2 stratum of points (ξ, η) where α_1 and α_2 both achieve the maximum in the piecewise linear polynomial φ, and

$$\eta = \varphi(\xi) = \langle \alpha_1, \xi \rangle - \nu(\alpha_1) = \langle \alpha_2, \xi \rangle - \nu(\alpha_2). \tag{5-35}$$

The stabilizer of the T^{n+1}–action on Y along $Z_{\alpha_1\alpha_2}$ is the subtorus spanned by the weights $(-\alpha_1, 1)$ and $(-\alpha_2, 1)$ — the generators of the two rays of the fan Σ_Y which span the cone corresponding to $Z_{\alpha_1\alpha_2}$, or equivalently, the normal vectors to the face (5-35) of Δ_Y. Thus, we can define a 2–chain $D_{\alpha_1\alpha_2}$ in Y, with boundary in $T_x(t'') \cup T_x(t')$, by considering a path in the complex plane which connects some $w_1 \in \partial S$ to the origin, and in every fiber of W over this path, a suitably chosen orbit of the S^1–action with weight $(\alpha_1 - \alpha_2, 0)$. We take these S^1–orbits to lie at moment map values which start at $\xi_1 = -(1/2\pi)\,\mathrm{val}(x)$ over $w_1 \in \partial S$ (so that the boundary of our 2–chain lies in $T_x(t'') \cup T_x(t')$), and end at a point (ξ_0, η_0) which satisfies (5-35) over the origin (whence the S^1–orbit collapses to a point by our above observation on the stabilizer along $Z_{\alpha_1\alpha_2}$).

By comparing intersection numbers with the toric divisors of Y, we find that, for a suitable choice of orientation, $[D_{\alpha_1\alpha_2}] = [D_{\alpha_2}] - [D_{\alpha_1}]$. Thus, since the weight formula (5-32) is manifestly multiplicative, we conclude that

$$\mathrm{weight}([D_{\alpha_2}]) = \mathrm{weight}([D_{\alpha_1\alpha_2}]) \cdot \mathrm{weight}([D_{\alpha_1}]). \tag{5-36}$$

On the other hand, the weight of $D_{\alpha_1\alpha_2}$ can be calculated explicitly. Parametrizing this disc by a map $u\colon D^2 \to Y$ and using polar coordinates ρ (along the path in the moment polytope Δ_Y) and θ (along the S^1–orbits), and observing that $\omega(\,\cdot\,, \partial_\theta u) = d(\langle \alpha_1 - \alpha_2, \xi\rangle)$ by definition of the moment map, we have

$$\int_{D_{\alpha_1\alpha_2}} \omega = \iint_{D^2} \omega(\partial_\rho u, \partial_\theta u)\, d\rho\, d\theta = 2\pi \int_0^1 \partial_\rho(\langle \alpha_1 - \alpha_2, \xi(\rho)\rangle)\, d\rho = 2\pi\, \langle \alpha_1 - \alpha_2, \xi_1 - \xi_0\rangle.$$

Since ξ_0 satisfies (5-35), $\langle \alpha_1 - \alpha_2, \xi_0\rangle = \nu(\alpha_1) - \nu(\alpha_2)$, so

$$\int_{D_{\alpha_1\alpha_2}} \omega = \langle \alpha_2 - \alpha_1, \mathrm{val}(x)\rangle + 2\pi\nu(\alpha_2) - 2\pi\nu(\alpha_1).$$

Denoting by $y = (y_1, \dots, y_n)$ the holonomies of the local system of $\mathfrak{t}_x$ along the various circle factors, the holonomy along the boundary of $D_{\alpha_1\alpha_2}$ is given by $y^{\alpha_1-\alpha_2}$. Recalling that $x_i = t^{\mathrm{val}(x_i)} y_i^{-1}$, we conclude that the weight of $D_{\alpha_1\alpha_2}$ is

$$\begin{aligned} \mathrm{weight}([D_{\alpha_1\alpha_2}]) &= y^{\alpha_1-\alpha_2}\, t^{\langle \alpha_2-\alpha_1, \mathrm{val}(x)\rangle + 2\pi\nu(\alpha_2) - 2\pi\nu(\alpha_1)} \exp(\mathfrak{b}_{\alpha_2} - \mathfrak{b}_{\alpha_1}) \\ &= x^{\alpha_2-\alpha_1}\, t^{2\pi\nu(\alpha_2) - 2\pi\nu(\alpha_1)} \exp(\mathfrak{b}_{\alpha_2} - \mathfrak{b}_{\alpha_1}). \end{aligned} \tag{5-37}$$

In light of (5-36), and using connectedness of the 1–skeleton of the subdivision $\mathcal{P}$ (ie any two elements of $P_{\mathbb{Z}}$ can be connected via a sequence of elements of $P_{\mathbb{Z}}$ such that the above calculation can be applied to consecutive terms in the sequence), this implies that for fixed t', t'', x, the weight of D_α is proportional to

$$x^\alpha t^{2\pi\nu(\alpha)} \exp(\mathfrak{b}_\alpha). \tag{5-38}$$

This is basically the desired formula (5-34), except we have not yet shown that the scaling constant depends only on the valuation of x (and t', t'') rather than on x itself.

To show the constant only depends on ξ (and t', t''), we observe that for fixed $\xi = -(1/2\pi)\,\mathrm{val}(x)$, the only role played by x is in determining the holonomy of the local system. Recalling that $T_x(t'') \cup T_x(t')$ (after restriction to ∂S) is isotopic to a product torus $\mathfrak{t}_x \times S^1 \simeq T^{n+1}$ in Y, and noting that the boundary of D_α represents the class $(-\alpha, 1)$ in $\pi_1(T_x(t'') \cup T_x(t)) \simeq \pi_1(T^{n+1}) \simeq \mathbb{Z}^{n+1}$, we find that $\mathrm{hol}([\partial D_\alpha]) = y^{-\alpha}$, so that the dependence of the weight of D_α on x is indeed as in (5-38), and the scaling factor $K_\xi(t'', t')$ does not depend on the holonomy, ie it depends only on $\xi = -(1/2\pi)\,\mathrm{val}(x)$ and not on x itself. □

We now return to the problem of counting J–holomorphic sections of $W\colon Y \to \mathbb{C}$ with boundary on $T_x(t') \cup T_x(t'')$. As previously noted, when J converges to the standard complex structure J_0, the J–holomorphic discs contributing to the differential (5-31) limit to stable curves consisting of a holomorphic disc, representing one of the classes $[D_\alpha]$ for some $\alpha \in P_{\mathbb{Z}}$ (by Proposition 5.24), and a (possibly empty) configuration of rational curves contained in $Z = W^{-1}(0)$, representing some homology class $\beta \in H_2(Y)$ (with $[\omega]\cdot\beta > 0$ whenever $\beta \neq 0$).

Definition 5.27 For fixed t', t'' and ξ, and for each $\alpha \in P_{\mathbb{Z}}$ and $\beta \in H_2(Y)$, we denote by $n_{\alpha,\beta}$ the (signed) count of J–holomorphic sections of W (for generic J close to J_0) whose relative homology class in $H_2(Y, T_x(t') \cup T_x(t''))$ is equal to $[D_\alpha] + \beta$, passing through a generic point of $\mathfrak{t}_x \subset W^{-1}(-1)$.

By considering the limit as $J \to J_0$ and using the classification of holomorphic discs in Proposition 5.24, we see that every J–holomorphic section under consideration is in one of these homology classes, $n_{\alpha,0} = 1$ for all $\alpha \in P_{\mathbb{Z}}$, and $n_{\alpha,\beta} = 0$ for all $\beta \neq 0$ such that $[\omega] \cdot \beta \leq 0$.

Remark 5.28 The invariance of counts of holomorphic sections under deformations of the Lagrangian boundary condition implies that $n_{\alpha,\beta}$ is independent of t', t'' (as long as $t'' - t' > t_0$) and ξ; hence the notation. However, our argument does not depend on it, so we will not elaborate further.

Since the weight of a section in the class $[D_\alpha] + \beta$ is given by

$$\mathrm{weight}([D_\alpha] + \beta) = \mathrm{weight}([D_\alpha])\, t^{[\omega]\cdot\beta}\ \exp([\mathfrak{b}] \cdot \beta),$$

we arrive at:

Proposition 5.29 *The Laurent polynomial g of Propositions 5.18–5.21 satisfies*

$$C'_\xi(t'', t')g(x) = K_\xi(t'', t') \sum_{\alpha \in P_{\mathbb{Z}}} t^{2\pi\nu(\alpha)} x^\alpha \exp(\mathfrak{b}_\alpha)\Bigg(1 + \sum_{\substack{\beta \in H_2(Y) \\ [\omega]\cdot\beta > 0}} n_{\alpha,\beta}\, t^{[\omega]\cdot\beta} \exp([\mathfrak{b}]\cdot\beta)\Bigg). \tag{5-39}$$

Proof This follows directly from a comparison of the weighted counts of sections which determine the differential on (5-31) (the coefficient of $1_x^{t''\to t'}$ in $s_x(\delta_x^{t''\to t'})$) as given by Proposition 5.21 and by direct calculation of $\sum_{\alpha,\beta} n_{\alpha,\beta}\, \mathrm{weight}([D_\alpha] + \beta)$. □

Corollary 5.30 *There exists a constant $C \in \mathbb{K}^*$ such that*

$$g(x) = C \sum_{\alpha \in P_{\mathbb{Z}}} t^{2\pi\nu(\alpha)} x^\alpha \exp(\mathfrak{b}_\alpha)\Bigg(1 + \sum_{\substack{\beta \in H_2(Y) \\ [\omega]\cdot\beta > 0}} n_{\alpha,\beta}\, t^{[\omega]\cdot\beta} \exp([\mathfrak{b}] \cdot \beta)\Bigg). \tag{5-40}$$

Proof The key point is that, for any $\xi \in \mathbb{R}^n$, the coefficients of a Laurent polynomial in $\mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ are determined by its evaluation at points $x \in (\mathbb{K}^*)^n$ with fixed valuation $\mathrm{val}(x) = -2\pi\xi$. Thus, comparing the left- and right-hand sides of (5-39) for fixed ξ, t'' and t', we find that $g(x)$ and the Laurent polynomial appearing in the right-hand side coincide up to a constant factor. Incidentally, this also implies that the ratio $C'_\xi(t'', t')/K_\xi(t'', t')$ is a genuine constant independent of t', t'' and ξ, and that the power series appearing as coefficients in the right-hand side are independent of t', t'' and ξ; in general this is slightly weaker than asserting that the $n_{\alpha,\beta}$ themselves are independent of these choices. □

Remark 5.31 The power series in the right-hand side of (5-40) are also exactly those which appear in expressions for the instanton-corrected superpotential for product tori in the toric Calabi–Yau variety Y (cf eg [3; 11]), and more explicitly in terms of Gromov–Witten invariants in [12], where these quantities are also interpreted as correction terms in the mirror map for the toric variety Y. Indeed, deforming (a subset of) $T_x(t') \cup T_x(t'')$ to a product torus in Y, it is apparent that the enumerative geometry problems we consider here and those discussed in [3; 11; 12] are equivalent.

Example 5.32 Let $f(x) = t^{2\pi}x^{-1} + 1 + x$, so $H = f^{-1}(0)$ consists of two points. Then $P_{\mathbb{Z}} = \{-1, 0, 1\}$, $\varphi(\xi) = \max(-\xi - 1, 0, \xi)$, and Y is isomorphic to the total space of $\mathcal{O}(-2) \to \mathbb{CP}^1$. In this example, the term in (5-40) corresponding to $\alpha = 0$ (ie discs in Y which intersect the zero section $\mathbb{CP}^1$) includes a nontrivial contribution from $\beta = [\mathbb{CP}^1]$, with $n_{\alpha=0,[\mathbb{CP}^1]} = 1$, whereas all the other $n_{\alpha,\beta}$ are zero; cf eg [11, Example 5.3.1]. Hence, $g(x)$ is proportional to

$$e^{\mathfrak{b}_{-1}} t^{2\pi} x^{-1} + e^{\mathfrak{b}_0}(1 + t^{2\pi} e^{[\mathfrak{b}]\cdot[\mathbb{CP}^1]}) + e^{\mathfrak{b}_1} x,$$

which matches $f(x)$ when $\mathfrak{b}_1 = \mathfrak{b}_{-1} = 0$ and $e^{\mathfrak{b}_0}(1 + t^{2\pi} e^{-2\mathfrak{b}_0}) = 1$. See also [11, Section 5.3] for examples where infinitely many $n_{\alpha,\beta}$ are nonzero. On the other hand, the coefficients $n_{\alpha,\beta}$ all vanish when every rational curve in Y is contained in a toric stratum of complex codimension at least two.

Finally, we observe that, as in the above example, it is always possible by a suitable choice of the bulk deformation class $[\mathfrak{b}] \in H^2(Y, \Lambda_{\geq 0})$ to ensure that the right-hand side of (5-40) matches the Laurent polynomial f used to define the hypersurface H.

Proposition 5.33 *Given any collection of unitary (ie valuation zero) elements $a_\alpha \in \mathbb{K}^*$ for all $\alpha \in P_{\mathbb{Z}}$, there exists a unique collection of unitary elements $e^{\mathfrak{b}_\alpha} \in \mathbb{K}^*$, with $\alpha \in P_{\mathbb{Z}}$, such that*

$$e^{\mathfrak{b}_\alpha}\Bigg(1 + \sum_{\substack{\beta \in H_2(Y) \\ [\omega]\cdot\beta > 0}} n_{\alpha,\beta}\, t^{[\omega]\cdot\beta} e^{[\mathfrak{b}]\cdot\beta}\Bigg) = a_\alpha \quad \text{for all } \alpha \in P_{\mathbb{Z}}. \tag{5-41}$$

Proof We can solve for $e^{\mathfrak{b}_\alpha}$ order by order. Namely, the series $\sum_\beta n_{\alpha,\beta} t^{[\omega]\cdot\beta} e^{[\mathfrak{b}]\cdot\beta}$ consist of terms whose valuations are positive and bounded below by some constant $\lambda > 0$ (by Gromov compactness, the symplectic areas of the rational curves which can appear in these expressions form a discrete set). Thus, (5-41) implies that $e^{\mathfrak{b}_\alpha} = a_\alpha \bmod t^\lambda$. Moreover, once $e^{\mathfrak{b}_\alpha}$ is determined mod $t^{N\lambda}$ for some $N \in \mathbb{N}$ and for all $\alpha \in P_{\mathbb{Z}}$, the power series appearing in the left-hand side of (5-41) are determined mod $t^{(N+1)\lambda}$, and thus (5-41) determines $e^{\mathfrak{b}_\alpha} \bmod t^{(N+1)\lambda}$ for all $\alpha \in P_{\mathbb{Z}}$. □

Proof of Theorem 1.2 We equip Y with the bulk deformation class $\mathfrak{b} = \sum \mathfrak{b}_\alpha \delta_{Z_\alpha}$, where the coefficients $\mathfrak{b}_\alpha$ are determined by Proposition 5.33 so that the expression (5-40) agrees with the Laurent polynomial f in (1-1) up to scaling by a nonzero constant. The result then follows from Proposition 5.18 and Corollary 5.30. □

6 Complete intersections

6.1 Geometric setup

In this section we describe the geometric setup for extending Theorem 1.2 to complete intersections in $(\mathbb{C}^*)^n$. Consider k Laurent polynomials

$$f_i = \sum_{\alpha \in P_{i,\mathbb{Z}}} a_{i,\alpha}\, t^{2\pi \nu_i(\alpha)} x^\alpha \in \mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \quad \text{for } 1 \leq i \leq k, \tag{6-1}$$

where the finite subsets $P_{i,\mathbb{Z}} \subset \mathbb{Z}^n$, the exponents $\nu_i(\alpha) \in \mathbb{R}$, and the coefficients $a_{i,\alpha}$ ensure that the hypersurfaces $H_i = f_i^{-1}(0)$ satisfy the same "tropical smoothness" conditions as in Section 2, and where additionally we assume that the tropical hypersurfaces associated to the tropicalizations

$$\varphi_i(\xi) = \max\{\langle \alpha, \xi\rangle - \nu_i(\alpha) \mid \alpha \in P_{i,\mathbb{Z}}\} \tag{6-2}$$

are in generic position relative to each other, ie all intersections between strata are transverse. Following [3, Section 11], we define Y to be the Kähler toric $(n+k)$–fold defined by the moment polytope

$$\Delta_Y = \{(\xi, \eta_1, \dots, \eta_k) \in \mathbb{R}^n \oplus \mathbb{R}^k \mid \eta_i \geq \varphi_i(\xi) \text{ for all } i = 1, \dots, k\}. \tag{6-3}$$

Dually, Y is also described by a fan $\Sigma_Y \subseteq \mathbb{R}^n \oplus \mathbb{R}^k$ whose rays are generated by the integer vectors $(-\alpha, e_i)$ for all $1 \leq i \leq k$ and $\alpha \in P_{i,\mathbb{Z}}$, where $e_1, \dots, e_k$ is the standard basis of $\mathbb{Z}^k$.

For $1 \leq i \leq k$, we define $W_i : Y \to \mathbb{C}$ to be the negative of the toric monomial with weight $(0, e_i) = (0, \dots, 0, 1, 0, \dots, 0)$, where the 1 is in the $(n+i)^{\text{th}}$ position. (Thus, the zero set of W_i is the union of the toric divisors of Y corresponding to the rays of Σ_Y generated by $(-\alpha, e_i)$, $\alpha \in P_{i,\mathbb{Z}}$, or equivalently, to the facets of Δ_Y on which $\eta_i = \varphi_i(\xi)$.) The candidate mirror to the complete intersection $\boldsymbol{H} = H_1 \cap \dots \cap H_k$ is then the Landau–Ginzburg model $(Y, W_1 + \dots + W_k)$; however, our version of the (fiberwise wrapped) Fukaya category of this Landau–Ginzburg model will involve Lagrangian submanifolds which are simultaneously admissible for each of the projections $W_1, \dots, W_k$. Accordingly, we view our k monomials as the components of a (toric) map

$$\boldsymbol{W} = (W_1, \dots, W_k) : Y \to \mathbb{C}^k.$$

We call $(Y, \boldsymbol{W})$ the *toric Landau–Ginzburg mirror* to the complete intersection $\boldsymbol{H}$ determined by the Laurent polynomials $(f_1, \dots, f_k)$.

In the course of our argument, we will also consider mirrors of partial intersections determined by a subset of the Laurent polynomials $f_1, \dots, f_k$. Given any subset $I \subset \{1, \dots, k\}$, let $W_I = (W_i)_{i \in I} : Y \to \mathbb{C}^{|I|}$ denote the projection of $\boldsymbol{W}$ onto the subset of coordinates associated to I. We also write $\bar{I} = \{1, \dots, k\} - I$ for the complement of I.

Proposition 6.1 *Given any $c_{\bar{I}} \in (\mathbb{C}^*)^{k-|I|}$, the submanifold $Y_I = W_{\bar{I}}^{-1}(c_{\bar{I}}) \subset Y$ equipped with the restriction of W_I is isomorphic (as a toric Kähler manifold together with an $|I|$–tuple of monomials) to the toric Landau–Ginzburg mirror of the complete intersection determined by $(f_i)_{i \in I}$.*

For $I = \varnothing$, this says that the fiber of $\boldsymbol{W}$ over a point of $(\mathbb{C}^*)^k$ is isomorphic to $(\mathbb{C}^*)^n$.

Proof Algebraically, $W_{\bar{I}} : Y \to \mathbb{C}^{k-|I|}$ is a dominant toric morphism, induced by the morphism of fans from Σ_Y to the fan of $\mathbb{C}^{k-|I|}$ induced by the linear map from $\mathbb{R}^n \oplus \mathbb{R}^k$ to $\mathbb{R}^{k-|I|}$ given by projection to the $(n+i)^{\text{th}}$ coordinates for all $i \in \bar{I}$; we call these the components *indexed by* $\bar{I}$. Thus, the fibers of $W_{\bar{I}}$ over the points of the open dense orbit $(\mathbb{C}^*)^{k-|I|}$ are all isomorphic, and described by the fiber of the morphism

of fans over the trivial cone $\{0\}$, ie the intersection of Σ_Y with the subspace $\mathbb{R}^n \oplus \mathbb{R}^I \subset \mathbb{R}^n \oplus \mathbb{R}^k$; or, dually, the projection of Δ_Y from $\mathbb{R}^n \oplus \mathbb{R}^k$ onto $\mathbb{R}^n \oplus \mathbb{R}^I$ given by forgetting the components η_i for $i \in \bar{I}$. This agrees exactly with the toric variety Y_I obtained by applying our construction to the complete intersection determined by the Laurent polynomials f_i for $i \in I$. Moreover, it is clear that the monomials W_i for $i \in I$ restrict from Y to Y_I in the expected manner (the toric weights match after forgetting the components indexed by the elements of $\bar{I}$).

Symplectically, we observe that the moment map $\mu_I : Y \to \mathbb{R}^n \times \mathbb{R}^I$ for the action of $\mathbb{T}^n \times \mathbb{T}^I$ (the subtorus which preserves the fibers of $W_{\bar{I}}$) is obtained from the moment map μ of the $\mathbb{T}^{n+k}$–action on Y by forgetting the components indexed by the elements of $\bar{I}$. The image of μ_I is therefore

$$\Delta_{Y|I} = \{(\xi, (\eta_i)_{i\in I}) \mid \eta_i \geq \varphi_i(\xi) \text{ for all } i \in I\} \subset \mathbb{R}^n \oplus \mathbb{R}^I.$$

Moreover, $W_{\bar{I}}$ maps every stratum of Y on which $(\mathbb{C}^*)^{\bar{I}}$ acts freely (ie the strata where $\eta_i > \varphi_i(\xi)$ for all $i \in \bar{I}$) onto the open stratum $(\mathbb{C}^*)^{k-|I|}$; this implies that every such stratum intersects $W_{\bar{I}}^{-1}(c_{\bar{I}})$. In particular, $W_{\bar{I}}^{-1}(c_{\bar{I}})$ contains points in strata which map to the vertices of $\Delta_{Y|I}$ under μ_I, as well as strata which map to its unbounded edges. By convexity of the moment map image (and given that there are no other toric fixed points, hence no additional vertices), this implies that the restriction of μ_I to $W_{\bar{I}}^{-1}(c_{\bar{I}})$ is surjective onto $\Delta_{Y|I}$. Thus the Kähler form on the generic fiber of $W_{\bar{I}}$ has moment polytope equal to $\Delta_{Y|I}$, as expected. □

Example 6.2 One case where the geometry of $(Y, \boldsymbol{W})$ is particularly simple is when $\boldsymbol{H}$ is a product of hypersurfaces in $(\mathbb{C}^*)^{n_i}$ for $i = 1, \dots, k$, ie each Laurent polynomial f_i involves a different subset of the coordinates $x_1, \dots, x_n$, where $n = \sum n_i$. In this case, Y ends up being the product of the mirrors we associate to each hypersurface $f_i^{-1}(0) \subset (\mathbb{C}^*)^{n_i}$, with $W_1, \dots, W_k$ the (pullbacks of the) respective superpotentials. In general Y is not a product, but the above considerations nonetheless make it possible to argue in terms of subsets of the collection $\{f_1, \dots, f_k\}$.

We can also describe the toric Kähler manifold Y in terms of toric reduction, as we have done in Section 2.2 for the case of hypersurfaces. We start from the product $\prod_{i=1}^k \mathbb{C}^{P_{i,\mathbb{Z}}}$, equipped with the product of the toric Kähler forms described in Section 4.1. Denote by M the kernel of the surjective map

$$\prod_{i=1}^{k} \mathbb{Z}^{P_{i,\mathbb{Z}}} \to \mathbb{Z}^n \oplus \mathbb{Z}^k \tag{6-4}$$

which maps the generator corresponding to $\alpha \in P_{i,\mathbb{Z}}$ to the element $(-\alpha, e_i)$ of $\mathbb{Z}^n \oplus \mathbb{Z}^k$, and by $\mathbb{T}_M = M \otimes (\mathbb{R}/\mathbb{Z})$ the corresponding subtorus of $\prod \mathbb{T}^{P_{i,\mathbb{Z}}}$. Dualizing (6-4) we have a short exact sequence

$$0 \to \mathbb{R}^{n+k} \xrightarrow{\iota} \prod \mathbb{R}^{P_{i,\mathbb{Z}}} \xrightarrow{\pi} M_{\mathbb{R}}^* \to 0,$$

where the first map is given by

$$\iota(\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_k) = (-\langle \alpha, \xi \rangle + \eta_i)_{\alpha \in P_{i,\mathbb{Z}},\, 1 \leq i \leq k}.$$

Viewing the exponents $\nu_i(\alpha)$ in (6-1) as an element $(\nu_1,\dots,\nu_k)$ of $\prod \mathbb{R}^{P_{i,\mathbb{Z}}}$, we consider the reduction of $\prod \mathbb{C}^{P_{i,\mathbb{Z}}}$ by $\mathbb{T}_M$ at the level $\lambda=\pi(\nu_1,\dots,\nu_k)$, and observe that

$$\mu^{-1}(\lambda)/\mathbb{T}_M \simeq Y,$$

since the moment polytope for the action of $\mathbb{T}^{n+k} \simeq \left(\prod \mathbb{T}^{P_{i,\mathbb{Z}}}\right)/\mathbb{T}_M$ on the reduced space is the intersection of $\pi^{-1}(\lambda)=\mathrm{Im}(\iota)+(\nu_1,\dots,\nu_k)$ with the nonnegative orthant in $\prod \mathbb{R}^{P_{i,\mathbb{Z}}}$, which is naturally identified with Δ_Y.

The toric Kähler manifold Y, its Kähler form ω_Y, and $\boldsymbol{W}=(W_1,\dots,W_k)$ are thus obtained by Hamiltonian reduction from the product of the spaces $\mathbb{C}^{P_{i,\mathbb{Z}}}$ for $i=1,\dots,k$, each equipped with the toric Kähler form of Section 4.1 and the functions $W_{0,i}=-\prod_{\alpha\in P_{i,\mathbb{Z}}} z_{i,\alpha}\colon \mathbb{C}^{P_{i,\mathbb{Z}}}\to\mathbb{C}$. (More precisely: the pullback of $W_{0,i}$ to $\prod \mathbb{C}^{P_{i,\mathbb{Z}}}$ is $\mathbb{T}_M$–invariant and descends to $W_i\colon Y\to\mathbb{C}$.)

This description of $(Y,\boldsymbol{W})$ as a reduction of the product of k "standard" Landau–Ginzburg models $(\mathbb{C}^{P_{i,\mathbb{Z}}},W_{0,i})$ corresponds to viewing $\boldsymbol{H}$ as the intersection of an n–dimensional algebraic subtorus of the open stratum of $\prod_{i=1}^k \mathbb{P}(\mathbb{K}^{P_{i,\mathbb{Z}}})$ with a product of $(|P_{i,\mathbb{Z}}|-2)$–dimensional pairs of pants, as in Remark 2.3.

6.2 The fiberwise wrapped Fukaya category of $(Y,\boldsymbol{W})$

The construction of the partially wrapped Fukaya category $\mathcal{W}(Y,\boldsymbol{W})$ parallels that introduced in Section 3, except we now consider properly embedded Lagrangian submanifolds of Y whose image under *each* of the projections $W_i\colon Y\to\mathbb{C}$ agrees outside of a compact subset with a finite union of admissible arcs in the complex planes; in fact, we shall only consider Lagrangians which fiber over product of U-shaped arcs (the same arcs γ_t as in our main construction) with respect to $\boldsymbol{W}\colon Y\to\mathbb{C}^k$.

As before, we control the behavior of holomorphic curves by equipping Y with a compatible almost-complex structure J making each of $W_1,\dots,W_k$ holomorphic outside of a neighborhood of the zero fiber (as before, J will be taken to agree with the standard complex structure of Y except for a small perturbation near $\bigcup_i W_i^{-1}(0)$), and by choosing a continuous weakly plurisubharmonic function $h\colon Y\to[0,\infty)$ which is proper on the fibers of $\boldsymbol{W}$; in addition, we fix a nonnegative wrapping Hamiltonian $H\colon Y\to\mathbb{R}$. The functions H and h are required to satisfy the same conditions as in Section 3 with respect to *each* of $W_1,\dots,W_k$, ie with respect to the whole horizontal distribution given by the symplectic orthogonals to the fibers of $\boldsymbol{W}\colon Y\to\mathbb{C}^k$, thus ensuring that the maximum principle estimates of Section 3 (with respect to h and to the various $|W_i|$) continue to hold. Specific choices of h and H satisfying these requirements are given below.

6.2.1 Parallel transport preserves fiberwise monomial admissibility The function h is again defined as the maximum of the (rescaled) norms of certain monomials $z^{\boldsymbol{v}}\in\mathcal{O}(Y)$ for $\boldsymbol{v}$ in a set of "extremal" vectors $\mathcal{V}$ (primitive integer vectors parallel to the unbounded edges of Δ_Y),

$$h=\max\{|z^{\boldsymbol{v}}|^{1/\delta(\boldsymbol{v})}\mid \boldsymbol{v}\in\mathcal{V}\}, \tag{6-5}$$

where $\delta(\boldsymbol{v})$ is defined below in (6-8). As in the case of hypersurfaces, the key point which ensures that h has all the required properties is that, at every point outside of a bounded subset of each fiber of $\boldsymbol{W}$, any monomial $z^{\boldsymbol{v}}$ which achieves the maximum in (6-5) is invariant under parallel transport between the fibers of $\boldsymbol{W}$ (Propositions 6.3 and 6.4 below). This property, which amounts to a compatibility of fiberwise monomial admissibility with parallel transport, is proved similarly to the arguments in Section 4.2.

Given a vector $\boldsymbol{v} = (\vec{v}, v^{1,0}, \dots, v^{k,0}) \in \mathbb{Z}^n \oplus \mathbb{Z}^k$, the toric monomial $z^{\boldsymbol{v}}$ defines a regular function on Y if and only if

$$v^{i,\alpha} := (-\alpha, e_i) \cdot \boldsymbol{v} = v^{i,0} - \alpha \cdot \vec{v} \geq 0 \quad \text{for all } 1 \leq i \leq k \text{ and } \alpha \in P_{i,\mathbb{Z}}. \tag{6-6}$$

In fact $z^{\boldsymbol{v}}$ vanishes to order $v^{i,\alpha}$ along the toric divisor of Y which corresponds to the ray $(-\alpha, e_i)$ of the fan Σ_Y. Next we observe that the monomial

$$\prod_{i=1}^{k} \prod_{\alpha \in P_{i,\mathbb{Z}}} z_{i,\alpha}^{v^{i,\alpha}} \in \mathcal{O}\big(\textstyle\prod \mathbb{C}^{P_{i,\mathbb{Z}}}\big)$$

is invariant under the action of $\mathbb{T}_M$ and descends to $z^{\boldsymbol{v}} \in \mathcal{O}(Y)$ under reduction.

For $\boldsymbol{v} \in \mathbb{Z}^{n+k}$ satisfying (6-6), $i \in \{1, \dots, k\}$, and $\gamma > 0$ small, we define a subset $S_{\boldsymbol{v},i,\gamma}$ of $\mathbb{R}^n$ as in (4-11), namely we set

$$S_{\boldsymbol{v},i,\gamma} = \{\xi \in \mathbb{R}^n \mid \langle \alpha, \xi \rangle - \nu_i(\alpha) < \varphi_i(\xi) - \gamma \|\xi\| \text{ for all } \alpha \in P_{i,\mathbb{Z}} \text{ such that } v^{i,\alpha} > 0\}. \tag{6-7}$$

The exact same argument as in the proof of Proposition 4.16 then shows:

Proposition 6.3 *Given $\boldsymbol{v} \in \mathbb{Z}^{n+k}$ satisfying* (6-6) *and $i \in \{1, \dots, k\}$, the monomial $z^{\boldsymbol{v}} \in \mathcal{O}(Y)$ is locally invariant under parallel transport between the fibers of the map $W_i : Y \to \mathbb{C}$ at every point $z \in Y$ whose moment map coordinates (ξ, η) satisfy $\xi \in S_{\boldsymbol{v},i,\gamma}$ as well as lower bounds on $|W_i(z)|$ and on $\|\xi\|$ as in Proposition 4.16.* □

The first consequence, setting $\boldsymbol{v} = (0, e_j)$ and observing that $S_{(0,e_j),i,\gamma} = \mathbb{R}^n$ for all $i \neq j$, is that $W_j = -z^{(0,e_j)}$ is invariant under parallel transport in the direction of W_i for all $i \neq j$. (Inspection of the argument shows that in this case no restriction on $|W_i(z)|$ or on $\|\xi\|$ is needed: the point is that the lift of W_j to $\prod \mathbb{C}^{P_{i,\mathbb{Z}}}$ only involves the variables $z_{j,\alpha}$, all of which are preserved under parallel transport for the ith component.) This ensures that the parallel transports along the different factors in the base of the fibration $\boldsymbol{W} : Y \to \mathbb{C}^k$ commute with each other, and that the parallel transport of a Lagrangian in a fiber of $\boldsymbol{W}$ over a product of arcs in $\mathbb{C}^k$ is well-defined.

Next, to each $\vec{v} \in \mathbb{Z}^n$, we associate an element of $\mathbb{Z}^{n+k}$ as follows: set $v^{i,0} = \max\{\alpha \cdot \vec{v} \mid \alpha \in P_{i,\mathbb{Z}}\}$, and $\boldsymbol{v} = (\vec{v}, v^{1,0}, \dots, v^{k,0})$. Denote by $A_{\vec{v},i}$ the set of $\alpha \in P_{i,\mathbb{Z}}$ which achieve the maximum in the definition of $v^{i,0}$, or equivalently, those α for which $v^{i,\alpha}$ as defined by (6-6) is zero. Denoting by $\Delta_{\alpha,i}$ the polyhedral subset of $\mathbb{R}^n$ where α achieves the maximum in φ_i, we observe that $S_{\boldsymbol{v},i,\gamma}$ is nonempty (for sufficiently small γ) and is a retract of $\bigcup_{\alpha \in A_{\vec{v},i}} \Delta_{\alpha,i}$ obtained by removing those points which are

too close to some other $\Delta_{\alpha',i}$, for some $\alpha' \notin A_{\vec{v},i}$. We also note that the $\Delta_{\alpha,i}$ appearing in this union are those which are unbounded in the direction of $\vec{v}$. Given this, we define $\mathcal{V}$ to be the set of all $\boldsymbol{v}$ obtained by this process from some $\vec{v} \in \mathbb{Z}^n$ which is the primitive outward normal vector to any facet of the Newton polytope P_i of any of the Laurent polynomials f_i, $1 \le i \le k$. Equivalently and much more concisely, the elements of $\mathcal{V}$ are the primitive tangent vectors to the unbounded edges of Δ_Y.

For $\boldsymbol{v} \in \mathcal{V}$ and $v^{i,\alpha}$ as in (6-6), we set

$$\delta(\boldsymbol{v}) = \sum_{i=1}^{k} \frac{d_i(\boldsymbol{v})}{2N_i}, \quad \text{where } d_i(\boldsymbol{v}) = \sum_{\alpha \in P_{i,\mathbb{Z}}} v^{i,\alpha} \text{ and } N_i = |P_{i,\mathbb{Z}}|. \tag{6-8}$$

For sufficiently small $\gamma > 0$, and for all $\boldsymbol{v} \in \mathcal{V}$, $S_{\boldsymbol{v},\gamma} := \bigcap_{i=1}^{k} S_{\boldsymbol{v},i,\gamma}$ is nonempty (it is a retract of the union of those regions of $\mathbb{R}^n$ delimited by the union of the tropical hypersurfaces of $\varphi_1, \dots, \varphi_k$ which are unbounded in the direction of $\vec{v}$), and the union $\bigcup_{\boldsymbol{v} \in \mathcal{V}} S_{\boldsymbol{v},\gamma}$ covers the complement of a compact subset in $\mathbb{R}^n$. We have the following analogue of Proposition 4.18:

Proposition 6.4 *There exist positive constants γ_0 and K_0 such that, at every point $z \in Y$ at which $|W_i(z)|^2 \ge (\varepsilon e^{\delta})^{N_i/(N_i-1)}$ for all i, and whose moment map coordinates (ξ, η) satisfy $\|\xi\| \ge K_0 |\boldsymbol{W}(z)|^2$, if $\boldsymbol{v}_0 \in \mathcal{V}$ achieves the maximum in* (6-5), *then $\xi \in S_{\boldsymbol{v}_0,\gamma_0}$.*

Proof Consider a point $z \in Y$ and its lift $(z_{i,\alpha}) \in \mu^{-1}(\lambda) \subset \prod \mathbb{C}^{P_{i,\mathbb{Z}}}$. For each i, let $\alpha_{i,0}, \alpha_{i,1} \in P_{i,\mathbb{Z}}$ correspond respectively to the smallest and largest $|z_{i,\alpha}|$ (or equivalently, moment map coordinate $\mu_{i,\alpha}$) of all $\alpha \in P_{i,\mathbb{Z}}$. By Lemma 4.12(2), up to bounded constant factors, it holds that $\mu_{i,\alpha_{i,0}} \sim |W_i(z)|^2$, while $\mu_{i,\alpha_{i,1}} \sim |z_{i,\alpha_{i,1}}|^{2N_i}$. Bounding $\mu_{i,\alpha_{i,1}} - \mu_{i,\alpha_{i,0}}$ in terms of $\|\xi\|$ as in the proof of Proposition 4.18, we find that $\mu_{i,\alpha_{i,1}} \sim \|\xi\|$ and hence $|z_{i,\alpha_{i,1}}| \sim \|\xi\|^{1/(2N_i)}$ up to a bounded factor whenever $\|\xi\| \gg |W_i(z)|^2$.

We now proceed as in the proof of Proposition 4.18: if $\xi \in S_{\boldsymbol{v},\gamma}$ then $|z_{i,\alpha}|$ satisfies a lower bound (4-17) by a constant multiple of $|z_{i,\alpha_{i,1}}| \sim \|\xi\|^{1/(2N_i)}$ for all $\alpha \in P_{i,\mathbb{Z}} - A_{\vec{v},i}$; here the constant depends on γ. Hence, $|z^{\boldsymbol{v}}|$ has a lower bound by a constant multiple of

$$\|\xi\|^{\sum v^{i,\alpha}/2N_i} = \|\xi\|^{\delta(\boldsymbol{v})},$$

where the constant again depends on γ. Applying this for some fixed $\gamma = \gamma_1 > 0$ such that $\bigcup_{\boldsymbol{v} \in \mathcal{V}} S_{\boldsymbol{v},\gamma}$ covers the complement of a compact subset in $\mathbb{R}^n$, we find that $h(z) = \max\{|z^{\boldsymbol{v}}|^{1/\delta(\boldsymbol{v})} \mid \boldsymbol{v} \in \mathcal{V}\}$ is bounded from below by a constant $c(\gamma_1)$ times $\|\xi\|$ (still assuming that $\|\xi\| \gg |\boldsymbol{W}|^2$).

Conversely, if $\xi \notin S_{\boldsymbol{v},\gamma}$ for $\gamma > 0$ (now chosen much smaller than γ_1), then there exists some i and $\alpha \in P_{i,\mathbb{Z}} - A_{\vec{v},i}$ such that $|z_{i,\alpha}|$ satisfies the upper bound (4-19), which implies that $|z^{\boldsymbol{v}}|$ is bounded by a constant times $\gamma^{1/2N_i}$ times $\|\xi\|^{\delta(\boldsymbol{v})}$. Choosing $\gamma = \gamma_0$ sufficiently small, so that $\gamma_0^{1/(2N_i\delta(\boldsymbol{v}))}$ is much smaller than $c(\gamma_1)$, this implies that $|z^{\boldsymbol{v}}|^{1/\delta(v)}$ cannot achieve the maximum in (6-5). □

Propositions 6.3 and 6.4 imply that $h = \max\{|z^{\boldsymbol{v}}|^{1/\delta(v)} \mid \boldsymbol{v} \in \mathcal{V}\}$ is invariant under parallel transport between the fibers of $\boldsymbol{W}$ outside of a compact subset of each fiber. This in turn implies, first, that perturbed holomorphic curves satisfy maximum principles with respect to $|\boldsymbol{W}|$ and h as in Propositions 3.10 and 3.11, and second, that we can construct admissible Lagrangian submanifolds of Y by parallel transport of (monomially admissible) Lagrangian submanifolds of the fiber of $\boldsymbol{W}$ (ie $(\mathbb{C}^*)^{\boldsymbol{n}}$) over products of admissible arcs.

6.2.2 The wrapping Hamiltonian We define the wrapping Hamiltonian $H : Y \to \mathbb{R}$ as in Section 4.3: the moment map coordinates of $\prod \mathbb{C}^{P_{i,\mathbb{Z}}}$ descend to real-valued functions $\mu_{i,\alpha}$ on Y, for $i = 1, \dots, k$ and $\alpha \in P_{i,\mathbb{Z}}$, given by

$$\mu_{i,\alpha} = \eta_i - \langle \alpha, \xi \rangle + \nu_i(\alpha).$$

We then define $H : Y \to \mathbb{R}$ by

$$H = \sum_{i=1}^{k} \Bigg(\sum_{\alpha \in P_{i,\mathbb{Z}}} \mu_{i,\alpha} - |P_{i,\mathbb{Z}}|\, m(\{\mu_{i,\alpha}\}_{\alpha \in P_{i,\mathbb{Z}}}) \Bigg), \tag{6-9}$$

where m is a smooth approximation of the minimum function as in Definition 4.20. Propositions 4.22 and 4.23 carry over with essentially the same proofs. To summarize:

Proposition 6.5 *The wrapping Hamiltonian H only depends on $(\xi_1, \dots, \xi_n)$, and as a function of these coordinates it is proper and convex. The flow generated by H preserves the fibers of $\boldsymbol{W}$, and within each fiber it preserves monomial admissibility with respect to the collection of monomials $z^{\boldsymbol{v}}$, for $\boldsymbol{v} \in \mathcal{V}$: if $\ell \subset \boldsymbol{W}^{-1}(c)$ is monomially admissible with phase angles $\arg(z^{\boldsymbol{v}}) = \varphi_{\boldsymbol{v}}$, for $\boldsymbol{v} \in \mathcal{V}$, then its image under the time t flow is monomially admissible at infinity with phase angles $\varphi_{\boldsymbol{v}} + t\, d(\boldsymbol{v})$, where $d(\boldsymbol{v}) = \sum_{i=1}^{k} d_i(\boldsymbol{v}) = \sum_{i,\alpha} v^{i,\alpha}$.* □

6.2.3 The fiberwise wrapped category As in Section 3 we first associate to $(Y, \boldsymbol{W})$ a directed category whose objects are a given collection of admissible Lagrangian submanifolds of Y, whose images under each of the projections $W_1, \dots, W_k$ agree near infinity with some fixed collection of radial straight lines in the complex plane, and their images under an autonomous flow $L(t) = \phi^t \rho^t(L)$, where ρ^t is the lifted admissible isotopy generated by applying the same autonomous flow ρ as in Section 3.3 to each factor of $\mathbb{C}^k$, and ϕ^t is the flow generated by the wrapping Hamiltonian H. This geometric setup gives rise to quasiunits and continuation maps with the exact same properties as in Section 3.4, and we again define $\mathcal{W}(Y, \boldsymbol{W})$ to be the localization of the directed category with respect to the quasiunits.

Remark 6.6 Our construction of $\mathcal{W}(Y, \boldsymbol{W})$ is rather ad hoc, but it can be recast in the language of monomial admissibility on Y, using the collection of toric monomials $\{z^{\boldsymbol{v}} \mid \boldsymbol{v} \in \mathcal{V}\} \cup \{W_1, \dots, W_k\}$. Indeed, our conditions on objects of $\mathcal{W}(Y, \boldsymbol{W})$ require each of these monomials to have locally constant argument (equal to a prescribed phase angle, or a pair of possible phase angles in the case of W_i) over each end of the Lagrangian within a suitable subset of Y; and the flow we consider has the effect of

increasing the phase angles within the interval $(-\pi, \pi)$ for each W_i, and in an unbounded manner for $z^{\boldsymbol{v}}$, ie we have removed the "stops" that monomial admissibility would normally place at each $\arg(z^{\boldsymbol{v}}) = \pi$.

Even though the appropriate notions have yet to be developed outside of the Liouville setting, one also expects that monomial admissibility can be recast in the language of stops in the sense of [18] (see [25] for an instance of this), or even better, wrapped Floer theory on a (nonexact) sector with sectorial corners, in the spirit of [18, Section 12]. A rough candidate for the appropriate sector with corners is the subset of Y consisting of those points where $\mathrm{Re}(W_i) \geq -R$ for all $i = 1, \dots, k$, for some $R \gg 0$; however, making the collection of hypersurfaces $\{\mathrm{Re}(W_i) = -R\}$ for $i = 1, \dots, k$ sectorial requires a modification of the Kähler form on Y.

6.3 The main theorem

As in Section 5.1, fix a properly embedded U-shaped admissible arc γ_0 in the complex plane which crosses the real axis at -1, and consider the admissible Lagrangian submanifold $L_0 \subset Y$ obtained by parallel transport over $\gamma_0 \times \cdots \times \gamma_0 \subset \mathbb{C}^k$ of the real positive locus $\ell_0 \cong (\mathbb{R}_+)^n$ in $\boldsymbol{W}^{-1}(-1, \dots, -1) \cong (\mathbb{C}^*)^n$.

Theorem 6.7 *For a suitable choice of bulk deformation class $\mathfrak{b} \in H^2(Y, \Lambda_{\geq 0})$, the fiberwise wrapped Floer cohomology ring $H\mathcal{W}^*(L_0, L_0)$ is isomorphic to the quotient $\mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]/(f_1, \dots, f_k)$, ie the ring of functions of the complete intersection $\boldsymbol{H}$. Hence, the derived category of coherent sheaves of $\boldsymbol{H}$ admits a fully faithful quasiembedding into $\mathcal{W}(Y, \boldsymbol{W})$.*

As in Section 5, the main step to calculate the fiberwise wrapped Floer cohomology $H\mathcal{W}^*(L_0, L_0)$ is to determine the Floer complex of $L_0(t')$ and $L_0(t) = \phi^t \rho^t(L_0)$ for $t' - t$ sufficiently positive. We start by observing that $L_0(t)$ is obtained from $\ell_0(t) = \phi^t(\ell_0)$ by parallel transport over $\gamma_t \times \cdots \times \gamma_t$, where $\gamma_t = \rho^t(\gamma_0)$ as in Section 5. Thus, for $t' - t > t_0$, the intersections of $L_0(t')$ and $L_0(t)$ lie in the fibers of $\boldsymbol{W}$ above the 2^k points $(c_1, \dots, c_k) \in \mathbb{C}^k$, where each c_i belongs to $\gamma_t \cap \gamma_{t'} = \{-1, c_{t',t}\}$.

For $I \subset \{1, \dots, k\}$ we denote by $c_I \in \mathbb{C}^k$ the point with coordinates $c_i = -1$ if $i \notin I$ and $c_i = c_{t',t}$ if $i \in I$. We then find that, for $t' - t > t_0$,

$$CF^*(L_0(t'), L_0(t)) = \bigoplus_{I \subset \{1,\dots,k\}} C_I(t', t)[|I|], \tag{6-10}$$

where $C_I(t', t) = CF^*(\ell_{I,-}(t'), \ell_{I,+}(t))$ is the Floer complex of the fiberwise Lagrangians obtained by intersecting $L_0(t')$ and $L_0(t)$ with $\boldsymbol{W}^{-1}(c_I)$, and the grading shift by $|I|$ comes from considering the grading contributions of the phase angles of the arcs γ_t and $\gamma_{t'}$ in the various factors of $\mathbb{C}^k$. Moreover, by considering intersection numbers of holomorphic discs with fibers of $\boldsymbol{W}$ (outside a small neighborhood of the coordinate planes), we find that the Floer differential maps each summand $C_I(t', t)$ of (6-10) to the span of the $C_{I'}(t', t)$ for $I' \subseteq I$.

Thus, the complex (6-10) carries a natural filtration (by $|I|$); we can proceed as in Section 5 and calculate $HF^*(L_0(t'), L_0(t))$ as the cohomology of a "vertical Floer complex" built from the fiberwise Floer cohomology groups

$$H^*(C_I(t',t)) = HF^*(\ell_{I,-}(t'), \ell_{I,+}(t)),$$

together with the maps from $H^*(C_I(t',t))$ to $H^*(C_{I'}(t',t))$ for $I' \subsetneq I$ induced by the relevant portions of the Floer differential on (6-10), ie discs which are not contained within the fibers of $\boldsymbol{W}$.

Observing that for each $\boldsymbol{v} = (\vec{v}, v^{1,0}, \dots, v^{k,0})$ the restriction of the monomial $z^{\boldsymbol{v}}$ to $\boldsymbol{W}^{-1}(c_1, \dots, c_k) \simeq (\mathbb{C}^*)^n$ is given by $\prod_{i=1}^k (-c_i)^{v^{i,0}} z_1^{v_1} \cdots z_n^{v_n}$, the same calculation as in Example 5.5 shows that the monomially admissible Lagrangian sections $\ell_{I,-}(t')$ and $\ell_{I,+}(t)$ in $\boldsymbol{W}^{-1}(c_I)$ have slopes

$$\sigma_{I,-}(t') = \Bigg(t' d(\boldsymbol{v}) - \Bigg(\sum_{i \in I} v^{i,0} \Bigg) \big(\arg(c_{t',t}) + \pi \big) \Bigg)_{\boldsymbol{v} \in \mathcal{V}},$$
$$\sigma_{I,+}(t) = \Bigg(t\, d(\boldsymbol{v}) - \Bigg(\sum_{i \in I} v^{i,0} \Bigg) \big(\arg(c_{t',t}) - \pi \big) \Bigg)_{\boldsymbol{v} \in \mathcal{V}}.$$

Because H is convex, for $t' - t$ sufficiently large (larger than some constant $t_1 \geq t_0$)

$$\sigma_I(t'-t) = \sigma_{I,-}(t') - \sigma_{I,+}(t) = \Bigg((t'-t)\, d(\boldsymbol{v}) - 2\pi \sum_{i \in I} v^{i,0} \Bigg)_{\boldsymbol{v} \in \mathcal{V}} \tag{6-11}$$

is the slope of a convex Hamiltonian for all $I \subset \{1, \dots, k\}$, so that the results of Section 5.3 apply to the Floer cohomology groups $HF^*(\ell_{I,-}(t'), \ell_{I,+}(t))$. In particular, these cohomology groups are concentrated in degree zero; since the differential on the vertical Floer complex has degree one, the only nonzero connecting maps are those which take $H^0(C_I(t',t))$ to $H^0(C_{I'}(t',t))$ for $I' \subset I$, $|I'| = |I| - 1$. Writing $I = I' \cup \{i\}$, we denote by $s_{I,i}$ the relevant portion of the differential.

Next, we recall that for $t' - t > t_1$ and $I \subset \{1, \dots, k\}$, $HF^0(\ell_{I,-}(t'), \ell_{I,+}(t))$ has a basis consisting of action-rescaled Floer generators $\zeta_{I,p}^{t' \to t}$, whose elements are indexed by the points of $P_I(t'-t) \cap (2\pi\mathbb{Z})^n$, where $P_I(t'-t)$ is the polytope associated to the slope $\sigma_I(t'-t)$ by (5-9). For $I = \varnothing$ we also use the notation $\vartheta_p^{t' \to t} = \zeta_{\varnothing,p}^{t' \to t}$. Hence:

Proposition 6.8 *For $t' - t > t_1$, the Floer cohomology $HF^*(L_0(t'), L_0(t))$ is isomorphic to the cohomology of the vertical Floer complex*

$$CF^*_{\mathrm{vert}}(L_0(t'), L_0(t)) = \bigoplus_{I \subset \{1,\dots,k\}} HF^{*+|I|}(\ell_{I,-}(t'), \ell_{I,+}(t)) \simeq \bigoplus_{\substack{I \subset \{1,\dots,k\} \\ p \in P_I(t'-t) \cap (2\pi\mathbb{Z})^n}} \mathbb{K} \cdot \zeta_{I,p}^{t' \to t}, \tag{6-12}$$

where the generators $\zeta_{I,p}^{t' \to t}$ (in degree $-|I|$) correspond to intersections in $\boldsymbol{W}^{-1}(c_I)$, rescaled by action within the fiber; together with a differential which is a sum of maps

$$s_{I,i} : HF^0(\ell_{I,-}(t'), \ell_{I,+}(t)) \to HF^0(\ell_{I',-}(t'), \ell_{I',+}(t))$$

for all $I = I' \sqcup \{i\} \subset \{1, \dots, k\}$.

Since the projections $W_1, \dots, W_k : Y \to \mathbb{C}$ are holomorphic outside of a small neighborhood of the origin, the open mapping principle implies that any J–holomorphic disc which contributes to the portion of the Floer differential mapping $C_I(t', t)$ to $C_{I'}(t', t)$ ($I' \subset I$) is contained within a single fiber of W_i (over either -1 or $c_{t',t}$) whenever $i \in I'$ or $i \notin I$, while for $i \in I - I'$ it is a section (except possibly near the origin) of $W_i : Y \to \mathbb{C}$ over the bounded region delimited by $\gamma_{t'}$ and γ_t.

Thus, in the case at hand, the contributions to the differentials $s_{I,i}$ correspond to holomorphic discs which are contained in a level set of $W_{\bar{\imath}} = (W_j)_{j \neq i} : Y \to \mathbb{C}^{k-1}$. By Proposition 6.1, this fiber Y_i, equipped with the restriction of W_i, is isomorphic to the mirror of the hypersurface $H_i = f_i^{-1}(0)$ considered in our main argument. Moreover, the restrictions of $L_0(t')$ and $L_0(t)$ to (Y_i, W_i) are exactly the same sort of fibered admissible Lagrangians we have considered in Section 5 — even though for $I' \neq \varnothing$ the relevant fiberwise monomially admissible Lagrangian sections differ from those previously considered by the monodromy of $W_{i'}$ around the origin for $i' \in I'$, as is manifest from the expression (6-11) for the slopes $\sigma_I(t' - t)$ and $\sigma_{I'}(t' - t)$. Despite this minor difference, the core calculation of Section 5 applies to this setting, and implies:

Proposition 6.9 *For all $I = I' \sqcup \{i\} \subset \{1, \dots, k\}$, the differential*

$$s_{I,i} : HF^0(\ell_{I,-}(t'), \ell_{I,+}(t)) \to HF^0(\ell_{I',-}(t'), \ell_{I',+}(t))$$

is, up to a nonzero multiplicative constant $C_{I,i}(t', t) \in \mathbb{K}^*$, given by multiplication by a Laurent polynomial

$$g_i(x) = \sum_{\bar{p} \in P_{i,\mathbb{Z}}} c_{i,\bar{p}} x^{\bar{p}} \in \mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

with the same Newton polytope as f_i. Namely, for $p' \in P_I(t' - t) \cap (2\pi\mathbb{Z})^n$,

$$s_{I,i}(\zeta_{I,p'}^{t' \to t}) = C_{I,i}(t', t) \sum_{\bar{p} \in P_{i,\mathbb{Z}}} c_{i,\bar{p}}\, \zeta_{I',p'+2\pi\bar{p}}^{t' \to t}.$$

Moreover, equipping Y with a bulk deformation class $\mathfrak{b} = \sum_i \sum_{\alpha \in P_{i,\mathbb{Z}}} \mathfrak{b}_{i,\alpha} \delta_{Z_{i,\alpha}}$, where the $\delta_{Z_{i,\alpha}}$ are Poincaré dual to the irreducible toric divisors of Y and the coefficients $\mathfrak{b}_{i,\alpha} \in \Lambda_{\geq 0}$ are determined as in Proposition 5.33, ensures that $g_i = f_i$ for all i.

Thus, denoting by $\mathbb{K}[(x_i^{\pm 1})]_P$ the subspace of $\mathbb{K}[(x_i^{\pm 1})]$ consisting of Laurent polynomials whose Newton polytope is contained in $(1/2\pi)P$, we have:

Proposition 6.10 *For a suitable choice of bulk deformation class $\mathfrak{b} \in H^2(Y, \Lambda_{\geq 0})$, the Floer cohomology group $HF^*(L_0(t'), L_0(t))$ for $t' - t > t_1$ is concentrated in degree zero and isomorphic as a vector space, via $\vartheta_p^{t' \to t} \mapsto x^{\bar{p}}$, to the quotient*

$$\mathbb{K}[(x_i^{\pm 1})]_{P_0(t'-t)} / \big(f_1 \mathbb{K}[(x_i^{\pm 1})]_{P_{\{1\}}(t'-t)} + \cdots + f_k \mathbb{K}[(x_i^{\pm 1})]_{P_{\{k\}}(t'-t)} \big). \tag{6-13}$$

The Floer product

$$CF^*_{\mathrm{vert}}(L_0(t''), L_0(t')) \otimes CF^*_{\mathrm{vert}}(L_0(t'), L_0(t)) \to CF^*_{\mathrm{vert}}(L_0(t''), L_0(t)) \tag{6-14}$$

can be determined as in Section 5, by observing that any contributing J–holomorphic disc projects under $W_i : Y \to \mathbb{C}$ to either a single point or a triangular region of the complex plane delimited by $\gamma_{t''}$, $\gamma_{t'}$ and γ_t (not enclosing the origin), and reducing to a calculation within the fiber of $\boldsymbol{W}$. This yields an analogue of Proposition 5.14:

Proposition 6.11 *For $t'' - t' > t_1$ and $t' - t > t_1$, the product* (6-14) *is given by*

$$\zeta^{t' \to t}_{I,p} \cdot \zeta^{t'' \to t'}_{J,p'} = \begin{cases} C_{I,J,t'',t',t}\, \zeta^{t'' \to t}_{I \sqcup J,\, p+p'} & \text{if } I \cap J = \varnothing, \\ 0 & \text{if } I \cap J \neq \varnothing, \end{cases} \tag{6-15}$$

for all $I, J \subset \{1, \dots, k\}$, $p \in P_I(t' - t) \cap (2\pi\mathbb{Z})^n$ and $p' \in P_J(t'' - t') \cap (2\pi\mathbb{Z})^n$, where $C_{I,J,t'',t',t}$ is a nonzero constant. In particular, for $I = J = \varnothing$ we have

$$\vartheta^{t' \to t}_p \cdot \vartheta^{t'' \to t'}_{p'} = \vartheta^{t'' \to t}_{p+p'}. \tag{6-16}$$

It follows from (6-16) that the cohomology-level product structure corresponds to multiplication of Laurent polynomials on the quotient spaces (6-13).

Finally, the quasiunit $e^{t' \to t} \in HF^0(L_0(t'), L_0(t))$ is again given by $e^{t' \to t} = \vartheta^{t' \to t}_0$, by the same argument as in Proposition 5.16. Thus, computing $H\mathcal{W}(L_0, L_0)$ as a colimit of the Floer cohomology groups $HF^*(L_0(t'), L_0(t))$ as $t' - t \to \infty$ amounts to taking the colimit of (6-13) under the naive inclusion maps, and we arrive at

$$H\mathcal{W}(L_0, L_0) \simeq \mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]/(f_1, \dots, f_k),$$

which completes the proof of Theorem 6.7.

Remark 6.12 It is not a coincidence that the structure of the vertical Floer complex (6-12) matches that of the Koszul complex which resolves $i_* \mathbb{O}_{\boldsymbol{H}}$. This can be understood using the same perspective as in Section 1.3, given the interpretation of the Landau–Ginzburg models (Y_I, W_I) for $I \subset \{1, \dots, k\}$ provided by Proposition 6.1 and observing that for $I = I' \sqcup \{i\}$ the categories $\mathcal{W}(Y_I, W_I)$ and $\mathcal{W}(Y_{I'}, W_{I'})$ are related to each other by cap and cup functors which correspond under mirror symmetry to the inclusion and restriction functors between the derived categories of $H_I = \bigcap_{i \in I} f_i^{-1}(0)$ and $H_{I'}$.

Remark 6.13 The object L_0 is expected to generate $\mathcal{W}(Y, \boldsymbol{W})$, which would imply that the embedding of Theorem 6.7 is an equivalence. The argument should proceed by induction on k, using stop removal. Namely, for $I = I' \sqcup \{i\}$ it should follow from a suitable stop removal result (for the stop at $W_i \to -\infty$) that $\mathcal{W}(Y_I, W_{I'})$ is the quotient of $\mathcal{W}(Y_I, W_I)$ by the image of the cup functor from $\mathcal{W}(Y_{I'}, W_{I'})$. On the other hand, the category $\mathcal{W}(Y_I, W_{I'})$ is expected to be trivial for I' a strict subset of I; at least, SYZ mirror symmetry suggests that $(Y_I, W_{I'})$ admits a B–side Landau–Ginzburg mirror whose superpotential

has no critical points [3], which implies the triviality of its derived category of singularities. Thus, one expects that $\mathcal{W}(Y_I, W_I)$ is generated by the image under the cup functor of a generator of $\mathcal{W}(Y_{I'}, W_{I'})$; the result then follows by induction on k.

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