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Stable cuspidal curves and the integral Chow ring of $\bar{\mathcal{M}}_{2,1}$

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We introduce the moduli stack $\tilde{\mathcal{M}}_{g,n}$ of n -marked stable at most cuspidal curves of genus g and we use it to determine the integral Chow ring of $\bar{\mathcal{M}}_{2,1}$. Along the way, we also determine the integral Chow ring of $\bar{\mathcal{M}}_{1,2}$.

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Introduction

Rational Chow rings of moduli spaces of curves have been the subject of intensive research in the last forty years, since Mumford's first investigation [1983] of the topic. Nevertheless, rational Chow rings of moduli of *stable* curves of genus larger than 1 have proved to be hard to compute: the complete calculations have been carried out only for \bar{M}_2 [Mumford 1983], $\bar{M}_{2,1}$ and \bar{M}_3 [Faber 1988; 1990].

Integral Chow rings of moduli *stacks* of curves, introduced in [Edidin and Graham 1998], are even harder to study, but they contain way more information: for instance, rational Chow rings of moduli of hyperelliptic curves are trivial, but their integral counterparts are not (see [Vistoli 1998; Edidin and Fulghesu 2009; Di Lorenzo 2021a; Pernice 2022b]); and similarly for $\mathcal{M}_3 \setminus \mathcal{H}_3$, the stack of smooth nonhyperelliptic curves of genus 3 (see [Di Lorenzo et al. 2021]).

For what concerns the moduli stack of stable curves of genus > 1 , the only result available in the integral case is the computation of the Chow ring of $\bar{\mathcal{M}}_2$ by Larson [2021] (Larson's theorem was subsequently reproved using a different approach in [Di Lorenzo and Vistoli 2021]).

The goal of this paper is to compute the integral Chow ring of $\bar{\mathcal{M}}_{2,1}$ using a new geometric approach, involving the stack of what we call *stable A_2 -curves* (these are curves with only nodes or cusps as singularities).

Theorem 6.11 *Suppose that the ground field has characteristic $\neq 2, 3$. Then*

$$\mathrm{CH}^*(\bar{\mathcal{M}}_{2,1}) \simeq \mathbb{Z}[\lambda_1, \psi_1, \vartheta_1, \lambda_2, \vartheta_2]/(\alpha_{2,1}, \alpha_{2,2}, \alpha_{2,3}, \beta_{3,1}, \beta_{3,2}, \beta_{3,3}, \beta_{3,4}),$$

where

- (a) the λ_i are the Chern classes of the Hodge bundle;
- (b) the cycle ψ_1 is the first Chern class of the conormal bundle of the tautological section $\bar{\mathcal{M}}_{2,1} \rightarrow \bar{\mathcal{C}}_{2,1}$, where $\bar{\mathcal{C}}_{2,1} \rightarrow \bar{\mathcal{M}}_{2,1}$ is the universal curve;
- (c) the cycle ϑ_1 is the fundamental class of the locus of marked curves with a separating node;
- (d) the cycle ϑ_2 is the fundamental class of the locus of marked curves with a marked separating node;

and the relations are

$$\begin{aligned} \alpha_{2,1} &= \lambda_2 - \vartheta_2 - \psi_1(\lambda_1 - \psi_1), & \beta_{3,1} &= 20\lambda_1\lambda_2 - 4\lambda_2\vartheta_1, \\ \alpha_{2,2} &= 24\lambda_1^2 - 48\lambda_2, & \beta_{3,2} &= 2\psi_1\vartheta_2, \\ \alpha_{2,3} &= \vartheta_1(\lambda_1 + \vartheta_1), & \beta_{3,3} &= \vartheta_2(\vartheta_1 + \lambda_1 - \psi_1), \\ & & \beta_{3,4} &= 2\psi_1(\lambda_1 + \vartheta_1)(7\psi_1 - \lambda_1) - 24\psi_1^3. \end{aligned}$$

In the above we identify, as usual, $\bar{\mathcal{M}}_{2,1}$ with the universal family over $\bar{\mathcal{M}}_2$ via the stabilization map $\bar{\mathcal{M}}_{2,1} \rightarrow \bar{\mathcal{M}}_2$. Viewed in this way, ψ_1 is the first Chern class of the dualizing sheaf $\omega_{\bar{\mathcal{M}}_{2,1}/\bar{\mathcal{M}}_2}$.

In Corollary 6.12 we reprove Faber's result on the rational Chow ring of $\bar{\mathcal{M}}_{2,1}$ and we further extend it over any base field of characteristic $\neq 2, 3$. Moreover, to show how our strategy works in a simpler case, we also determine the integral Chow ring of $\bar{\mathcal{M}}_{1,2}$.

Theorem 3.6 *Suppose that the ground field has characteristic $\neq 2, 3$. Then*

$$\mathrm{CH}^*(\bar{\mathcal{M}}_{1,2}) \simeq \mathbb{Z}[\lambda_1, \mu_1]/(\mu_1(\lambda_1 + \mu_1), 24\lambda_1^2),$$

where λ_1 is the first Chern class of the Hodge line bundle and $\mu_1 := p_*[\bar{\mathcal{M}}_{1,1}]$ is the fundamental class of the universal section $p: \bar{\mathcal{M}}_{1,1} \rightarrow \bar{\mathcal{M}}_{1,2}$ of $\bar{\mathcal{M}}_{1,2} \rightarrow \bar{\mathcal{M}}_{1,1}$.

Stable A_2 -curves and strategy of proof

The calculations of integral Chow rings of stacks of stable curves that have been carried out so far have been for stacks with a presentation of the form $[X/G]$, where G is an affine algebraic group, and X is an open subscheme of a representation of G ; we will call this a *good presentation*.

In other cases, like $\bar{\mathcal{M}}_{2,1}$, the stack of interest — let us call it \mathcal{X} — is not known to have such a presentation, but contains a closed subscheme $\mathcal{Y} \subseteq \mathcal{X}$ such that both \mathcal{Y} and $\mathcal{X} \setminus \mathcal{Y}$ have a good presentation, and we can compute both $\mathrm{CH}^*(\mathcal{Y})$ and $\mathrm{CH}^*(\mathcal{X})$. One can try to use the localization sequence

$$\mathrm{CH}^{*-c}(\mathcal{Y}) \rightarrow \mathrm{CH}^*(\mathcal{X}) \rightarrow \mathrm{CH}^*(\mathcal{X} \setminus \mathcal{Y}) \rightarrow 0,$$

where c is the codimension of the regular immersion $\mathcal{Y} \hookrightarrow \mathcal{X}$, but this gives no information on the kernel of the pushforward $\mathrm{CH}^{*-c}(\mathcal{Y}) \rightarrow \mathrm{CH}^*(\mathcal{X})$.

Our approach, first introduced in [Fulghesu 2010] in the context of intersection theory on stacks of curves nodal curves of genus 0, exploits a patching technique (see Lemma 3.1) that is at the heart of the Borel–Atiyah–Segal–Quillen localization theorem, and has been used by many authors working on equivariant cohomology, equivariant Chow rings and equivariant K–theory (see the discussion in the introduction of [Di Lorenzo and Vistoli 2021]). This works when the top Chern class of the normal bundle of \mathcal{Y} in \mathcal{X} is not a zero-divisor in $\mathrm{CH}^*(\mathcal{Y})$. Unfortunately, when \mathcal{Y} is Deligne–Mumford, $\mathrm{CH}^i(\mathcal{Y})$ is torsion for $i > \dim \mathcal{Y}$, which means that the condition can never be satisfied. Thus, to apply this to stacks of stable curves, one needs to enlarge them so that the condition has a chance to be satisfied, compute the Chow ring of the enlarged stack, then use the localization sequence to compute the additional relations coming from the difference between the enlarged stack and the one we are interested in.

In the first section of this paper we introduce the moduli stack $\tilde{\mathcal{M}}_{g,n}$ of *stable A_2 –curves*, obtained by adding to $\bar{\mathcal{M}}_{g,n}$ curves with cuspidal singularities; the stability condition is that the canonical bundle is still required to be ample. This is neither Deligne–Mumford nor separated (thus the stability condition above is only a weak one, and does not ensure GIT stability), but it is still a smooth quotient stack, and hence it has a well-defined integral Chow ring. The same holds for the universal stable A_2 –curve $\tilde{\mathcal{C}}_{g,n} \rightarrow \tilde{\mathcal{M}}_{g,n}$. In particular, as $\bar{\mathcal{M}}_{2,1}$ is contained in $\tilde{\mathcal{C}}_2$ as an open substack, we can break down the computation of $\mathrm{CH}^*(\bar{\mathcal{M}}_{2,1})$ in two main steps:

- (1) The determination of $\mathrm{CH}^*(\tilde{\mathcal{C}}_2)$, which is the content of Proposition 4.11.
- (2) The determination of the image of $\mathrm{CH}^*(\tilde{\mathcal{C}}_2 \setminus \bar{\mathcal{M}}_{2,1}) \rightarrow \mathrm{CH}^*(\tilde{\mathcal{C}}_2)$, ie the cycles coming from the locus of stable A_2 –curves with cuspidal singularities. This is first done abstractly in terms of fundamental classes in Theorem 5.11; afterwards we proceed with the explicit computations.

For the first step, we consider the stratification of $\tilde{\mathcal{C}}_2$ by closed substacks given by

$$\tilde{\Theta}_2 \subset \tilde{\Theta}_1 \subset \tilde{\mathcal{C}}_2,$$

where $\tilde{\Theta}_1$ is the locus of marked curves with a separating node and $\tilde{\Theta}_2$ is the stratum of marked curves with a marked separating node. We compute $\mathrm{CH}^*(\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_1)$, $\mathrm{CH}^*(\tilde{\Theta}_1 \setminus \tilde{\Theta}_2)$ and $\mathrm{CH}^*(\tilde{\Theta}_2)$ separately (each of these stacks has a good presentation); then we use the fact that the top Chern classes of the normal bundles of $\tilde{\Theta}_2$ in $\tilde{\mathcal{C}}_2$, and of $\tilde{\Theta}_1 \setminus \tilde{\Theta}_2$ in $\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_2$, are not zero-divisors, which allows us to use the patching technique to get $\mathrm{CH}^*(\tilde{\mathcal{C}}_2)$.

Relations with previous work

A very natural approach to the problem of computing Chow rings of smooth stacks with a stratification for which we know the Chow rings of the strata is to use higher Chow groups to get a handle on the kernel. For the stack $\bar{\mathcal{M}}_2$ this is carried out in [Larson 2021] (see also [Bae and Schmitt 2022; 2023]). Our approach, explained above, is different, and does not use higher Chow groups at all. It was first introduced in [Fulghesu 2010] in intersection theory, and used to give an alternative proof of Larson’s theorem on $\bar{\mathcal{M}}_2$ in [Di Lorenzo and Vistoli 2021].

The idea of defining alternative compactifications of the moduli spaces of smooth curves by considering curves with more complicated singularities than nodes is already in the literature, starting from [Schubert 1991] (see also [Hassett and Hyeon 2009]), and from a more stack-theoretic standpoint in the work of Smyth [2013], continued by Alper, Fedorchuk, Smyth and van der Wyck [Alper et al. 2016; 2017a; 2017b; 2017c]. Their perspective, however, is very different, as they look for stacks that are proper and Deligne–Mumford, in order to study the birational geometry of moduli spaces of curves.

In [Fulghesu 2010; Bae and Schmitt 2022], the authors study Chow rings of stacks of curves with positive-dimensional automorphism groups; unlike us, they impose no stability condition, and their curves are all nodal.

Future prospects

Our approach should be applicable, maybe after inverting a few primes, to other moduli stacks with a stratification such that the Chow rings of the strata are computable. For the relatively simple cases considered in this paper, it is enough to consider at most cuspidal curves; in other cases one needs more complicated singularities. The second author’s PhD thesis [Pernice 2022a] (see also [Pernice 2023a; 2023b]) includes several results on stacks of stable A_n -curves, as well as a presentation for the Chow ring of $\bar{\mathcal{M}}_3$ with coefficients in $\mathbb{Z}[\frac{1}{6}]$, obtained using A_7 -curves. Of course, the presentation for the rational Chow ring of $\bar{\mathcal{M}}_3$ is an old theorem of Faber [1990]; our technique gives a completely different approach, and a more refined result.

The general principle seems to be that the more singularities you allow, the more likely it is that the patching condition be satisfied, making it possible to compute the Chow ring of the larger stack. Then one needs to compute the classes of various loci of unwanted curves, something that can be difficult; allowing very complicated singularities makes it harder. So, it is not clear what the limits of the methods are; still, we think it worth investigating what it can give, for example, for $\bar{\mathcal{M}}_{3,1}$ and $\bar{\mathcal{M}}_4$, whose rational Chow rings are not known.

Outline of the paper

In Section 1 we introduce stable A_2 -curves (Definition 1.1) and the associated stack $\tilde{\mathcal{M}}_{g,n}$. We prove that $\tilde{\mathcal{M}}_{g,n}$ is a quotient stack (Theorem 1.3) and we study the normalization of $\tilde{\mathcal{M}}_{g,n} \rightarrow \bar{\mathcal{M}}_{g,n}$: in particular, we

prove in Theorem 2.9 that it is isomorphic to $\tilde{\mathcal{M}}_{g-1,n+1} \times [\mathbb{A}^1/\mathbb{G}_m]$ via a certain *pinching* morphism, a result that will be essential for our computations.

In Section 3 we show how our strategy works in a simple case, namely for computing $\text{CH}^*(\bar{\mathcal{M}}_{1,2})$ (Theorem 3.6): we first determine $\text{CH}^*(\tilde{\mathcal{C}}_{1,1})$ (Proposition 3.3) and then we excise the cuspidal locus.

Section 4 is devoted to the computation of $\text{CH}^*(\tilde{\mathcal{C}}_2)$ (Proposition 4.11). As explained before, this result is obtained by patching together the explicit presentations of $\text{CH}^*(\tilde{\mathcal{C}} \setminus \tilde{\Theta}_1)$ (Proposition 4.6), $\text{CH}^*(\tilde{\Theta}_1 \setminus \tilde{\Theta}_2)$ (Proposition 4.7) and $\text{CH}^*(\tilde{\Theta}_2)$ (Proposition 4.10).

In Section 5 we give an abstract characterization of $\text{CH}^*(\bar{\mathcal{M}}_{2,1})$. More precisely, we prove in Theorem 5.11 that

$$\text{CH}^*(\bar{\mathcal{M}}_{2,1}) \simeq \text{CH}^*(\tilde{\mathcal{C}}_2)/(J, [\tilde{\mathcal{C}}_2^c], [\tilde{\mathcal{C}}_2^E], c''_* \rho^* T),$$

where J is the ideal generated by the relations coming from $\bar{\mathcal{M}}_2$ and two of the other generators are the fundamental classes of certain loci in $\tilde{\mathcal{C}}_2$.

Finally, in Section 6 we compute the integral Chow ring of $\bar{\mathcal{M}}_{2,1}$ (Theorem 6.11) by determining explicit expressions for $[\tilde{\mathcal{C}}_2^c]$ and $[\tilde{\mathcal{C}}_2^E]$ (Propositions 6.4 and 6.10). Again, a key ingredient for this computation is the patching lemma. In the last part of the section we compare our presentation of $\text{CH}^*(\bar{\mathcal{M}}_{2,1})_{\mathbb{Q}}$ with the one obtained by Faber.

Notation and conventions

For the convenience of the reader, we summarize here the notation for most of the stacks appearing in the paper:

- (1) $\tilde{\mathcal{M}}_{g,n}$, the stack of n -marked stable A_2 -curves of genus g ;
- (2) $\bar{\mathcal{M}}_{g,n}$, the stack of n -marked stable curves of genus g ;
- (3) $\tilde{\mathcal{C}}_{g,n}$, the universal n -marked stable A_2 -curve of genus g ;
- (4) $\tilde{\Delta}_1$, the stack of stable A_2 -curves of genus 2 with a separating node;
- (5) Δ_1 , the stack of stable curves of genus 2 with a separating node;
- (6) $\tilde{\Theta}_1$, the stack of 1-pointed stable A_2 -curves of genus 2 with a separating node;
- (7) Θ_1 , the stack of 1-pointed stable curves of genus 2 with a separating node;
- (8) $\tilde{\Theta}_2$, the stack of 1-pointed stable A_2 -curves of genus 2 with a marked separating node;
- (9) Θ_2 , the stack of 1-pointed stable curves of genus 2 with a marked separating node.

We work over a base commutative ring k such that 2 and 3 are invertible in k . We will almost exclusively use the case in which k is a field, but in the proof of Theorem 2.9 the added generality will be useful, as we will have to assume $k = \mathbb{Z}[\frac{1}{6}]$. From Section 3, k will be a field of characteristic different from 2 and 3.

All schemes, algebraic spaces and morphisms are going to be defined over k . All stacks will be over the étale site (Aff/k) of affine schemes over k (we could take all schemes; it does not really make a difference). For algebraic stacks and algebraic spaces we will follow the conventions of [Knutson 1971; Laumon and Moret-Bailly 2000]; in particular, they will have a separated diagonal of finite type.

A *quotient stack* \mathcal{X} is a stack over k such that there exists an algebraic space $X \rightarrow \text{Spec } k$ with an action of an affine flat group scheme of finite type $G \rightarrow \text{Spec } k$ such that \mathcal{X} is isomorphic to $[X/G]$.

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1 The stack of stable A_2 -curves of fixed genus

Definition 1.1 An A_2 -curve over a scheme S is a proper flat finitely presented morphism of schemes $C \rightarrow S$ whose geometric fibers are reduced connected curves with only nodes or cusps as singularities.

If S is a scheme, an n -marked stable A_2 -curve of genus g over S is an A_2 -curve $C \rightarrow S$ with n sections $s_1, \dots, s_n: S \rightarrow C$ such that, if we denote by Σ_i the image of s_i in C ,

- (1) the Σ_i are disjoint;
- (2) they are contained in the smooth locus of $C \rightarrow S$; and
- (3) if $\omega_{C/S}$ is the relative dualizing sheaf, the invertible sheaf $\omega_{C/S}(\Sigma_1 + \dots + \Sigma_n)$ is relatively ample on S .

In the most general, and correct, definition of an A_2 -curve $C \rightarrow S$, one should not assume that C is a scheme, but an algebraic space; for the purposes of this paper, this will not be needed, but can be done without making any essential changes.

We have the following standard result:

Proposition 1.2 *Let $C \rightarrow S$ a proper flat finitely presented morphism with n -sections $S \rightarrow C$. There exists an open subscheme $S' \subseteq S$ with the property that a morphism $T \rightarrow S$ factors through S' if and only if the projection $T \times_S C \rightarrow T$, with the sections induced by the s_i , is a stable n -pointed A_2 -curve.*

Proof It is well known that a small deformation of a curve with cuspidal or nodal singularities still has cuspidal or nodal singularities. Hence, after restricting to an open subscheme of S , we can assume that $C \rightarrow S$ is an A_2 -curve. By further restricting S we can assume that the sections land in the smooth locus of $C \rightarrow S$ and are disjoint. Then the result follows from openness of ampleness for invertible sheaves. \square

If $S' \rightarrow S$ is a morphism of schemes and $C \rightarrow S$ is an n -marked stable A_2 -curve of genus g , the projection $S' \times_S C \rightarrow S'$ is an n -marked stable A_2 -curve of genus g . Thus, there is an obvious stack $\tilde{\mathcal{M}}_{g,n,k}$ of stable A_2 -curves of genus g over (Aff/k) whose category of sections over an affine scheme S is the groupoid of stable A_2 -curves of genus g over S . Mostly we will omit the indication of the base field, and simply write $\tilde{\mathcal{M}}_{g,n}$.

As is customary, we will denote $\tilde{\mathcal{M}}_{g,0}$ by $\tilde{\mathcal{M}}_g$.

Theorem 1.3 *The stack $\tilde{\mathcal{M}}_{g,n}$ is a smooth algebraic stack of finite type over k . Furthermore, it is a quotient stack; more precisely, there exists a smooth quasiprojective scheme X with an action of GL_N for some positive integer N such that $\tilde{\mathcal{M}}_{g,n} \simeq [X/\text{GL}_N]$.*

If k is a field, then $\tilde{\mathcal{M}}_{g,n}$ is connected.

Proof It follows from Lemma 2.4 that there exists a positive integer m with the property that if Ω is an algebraically closed field with a homomorphism $k \rightarrow \Omega$, and (C, p_1, \dots, p_n) is in $\tilde{\mathcal{M}}_{g,n}(\Omega)$, then $\omega_{C/\Omega}(p_1 + \dots + p_n)^{\otimes m}$ is very ample, and $H^1(C, \omega_{C/\Omega}(p_1 + \dots + p_n)^{\otimes m}) = 0$.

The invertible sheaf $\omega_{C/\Omega}(p_1 + \dots + p_n)^{\otimes m}$ has Hilbert polynomial $P(t) = g + m(2g - 2 + n)t$, and defines an embedding $C \subseteq \mathbb{P}_\Omega^{N-1}$, where $N := m(2g - 2 + n) - g + 1$. If $\pi : C \rightarrow S$ is a stable n -pointed A_2 -curve, and $\Sigma_1, \dots, \Sigma_n \subseteq C$ are the images of the sections, then $\pi_*\omega_{C/S}(\Sigma_1 + \dots + \Sigma_n)^{\otimes m}$ is a locally free sheaf of rank N , and its formation commutes with base change, because of Grothendieck's base-change theorem.

Call X the stack over k whose sections over a scheme S consist of a stable n -pointed A_2 -curve as above, and an isomorphism $\mathcal{O}_S^N \simeq \pi_*\omega_{C/S}(\Sigma_1 + \dots + \Sigma_n)^{\otimes m}$ of sheaves of \mathcal{O}_S -modules. Since $\pi_*\omega_{C/S}(\Sigma_1 + \dots + \Sigma_n)^{\otimes m}$ is very ample, the automorphism group of an object of X is trivial, and X is equivalent to its functor of isomorphism classes.

Call H the Hilbert scheme of subschemes of \mathbb{P}_k^{N-1} with Hilbert polynomial $P(t)$, and $D \rightarrow H$ the universal family. Call F the fiber product of n copies of D over S , and $C \rightarrow F$ the pullback of $D \rightarrow H$ to F ; there are n tautological sections $s_1, \dots, s_n : F \rightarrow C$. Consider the largest open subscheme F' of F such that the restriction C' of C , with the restrictions of the n tautological sections, is a stable n -pointed A_2 -curve, as in Proposition 1.2. Call $Y \subseteq F'$ the open subscheme whose points are those for which the corresponding curve is nondegenerate, $E \rightarrow Y$ the restriction of the universal family, $\Sigma_1, \dots, \Sigma_n \subseteq E$ the tautological sections. Call $\mathcal{O}_E(1)$ the restriction of $\mathcal{O}_{\mathbb{P}_Y^{N-1}}(1)$ via the tautological embedding $E \subseteq \mathbb{P}_Y^{N-1}$; there are two sections of the projection $\text{Pic}_{E/Y}^{m(2g-2+n)} \rightarrow Y$ from the Picard scheme parametrizing invertible sheaves of degree $m(2g - 2 + n)$, one defined by $\mathcal{O}_E(1)$, the other by $\omega_{E/Y}(\Sigma_1 + \dots + \Sigma_n)^{\otimes m}$; let $Z \subseteq Y$ the equalizer of these two sections, which is a locally closed subscheme of Y .

Then Z is a quasiprojective scheme over k representing the functor sending a scheme S into the isomorphism class of tuples consisting of a stable n -pointed A_2 -curve $\pi: C \rightarrow S$, together with an isomorphism of S -schemes

$$\mathbb{P}_S^{N-1} \simeq \mathbb{P}(\pi_*\omega_{C/S}(\Sigma_1 + \cdots + \Sigma_n)).$$

There is an obvious functor $X \rightarrow Z$, associating with an isomorphism $\mathcal{O}_S^N \simeq \pi_*\omega_{C/S}(\Sigma_1 + \cdots + \Sigma_n)^{\otimes m}$ its projectivization. It is immediate to check that $X \rightarrow Z$ is a \mathbb{G}_m -torsor; hence, it is representable and affine, and X is a quasiprojective scheme over $\text{Spec } k$.

On the other hand, there is an obvious morphism $X \rightarrow \tilde{\mathcal{M}}_{g,n}$ which forgets the isomorphism $\mathcal{O}_S^N \simeq \pi_*\omega_{C/S}(\Sigma_1 + \cdots + \Sigma_n)^{\otimes m}$; this is immediately seen to be a GL_N torsor. We deduce that $\tilde{\mathcal{M}}_{g,n}$ is isomorphic to $[X/\text{GL}_N]$. This shows that $\tilde{\mathcal{M}}_{g,n}$ is a quotient stack, as in the last statement; this implies that $\tilde{\mathcal{M}}_{g,n}$ is an algebraic stack of finite type over k .

The fact that $\tilde{\mathcal{M}}_{g,n}$ is smooth follows from the fact that A_2 -curves are unobstructed.

Finally, to check that $\tilde{\mathcal{M}}_{g,n}$ is connected it is enough to check that the open embedding $\mathcal{M}_{g,n} \subseteq \tilde{\mathcal{M}}_{g,n}$ has a dense image, since $\mathcal{M}_{g,n}$ is well known to be connected. This is equivalent to saying that every stable n -pointed A_2 -curve over an algebraically closed extension Ω of k has a small deformation that is stable and nodal. Let (C, p_1, \dots, p_n) be a stable n -pointed A_2 -curve; the singularities of C are unobstructed, so we can choose a lifting $\bar{C} \rightarrow \text{Spec } \Omega[[t]]$ with smooth generic fiber. The points p_i lift to sections $\text{Spec } \Omega[[t]] \rightarrow \bar{C}$, and then the result follows from Proposition 1.2. \square

Notice that there are many examples of stable A_2 -curves over an algebraically closed field whose group of automorphisms is a positive-dimensional affine group (for example, let C an irreducible rational curve whose only singularity is a single cusp, attached by a smooth point to a smooth curve of genus $g - 1$; then the automorphism group of C is an extension of a finite group by \mathbb{G}_m). This means that $\tilde{\mathcal{M}}_g$ is not Deligne–Mumford, and not separated.

Proposition 1.4 *Let $\pi: C \rightarrow S$ be a stable A_2 -curve of genus g . Then $\pi_*\omega_{C/S}$ is a locally free sheaf of rank g on S , and its formation commutes with base change.*

Proof If C is an A_2 -curve over a field k , the dimension of $H^0(C, \omega_{C/k})$ is g ; so the result follows from Grauert's theorem when S is reduced. But the versal deformation space of an A_2 -curve over a field is smooth, so every A_2 -curve comes, étale-locally, from an A_2 -curve over a reduced scheme. \square

As a consequence we obtain a locally free sheaf $\tilde{\mathbb{H}}_g$ of rank g on $\tilde{\mathcal{M}}_{g,n}$, the *Hodge bundle*.

We will need a classification of stable A_2 -curves of genus 2, and 1-marked stable A_2 -curves of genus 1. The following is straightforward:

Proposition 1.5 *Assume that k is an algebraically closed field. A 1-marked stable A_2 -curve of genus 1 over k is either stable or an irreducible cuspidal rational cubic.*

A stable A_2 -curve of genus 2 over k is of one of the following types:

- (1) a stable curve of genus 2;
- (2) an irreducible curve of geometric genus 1 with a node or a cusp;
- (3) an irreducible rational curve with a node and a cusp, or two cusps;
- (4) the union of two 1-marked stable A_2 -curves of genus 1 meeting transversally at the marked points.

The locus of singular curves in $\tilde{\mathcal{M}}_2$ is a divisor with normal crossing, with two irreducible components $\tilde{\Delta}_0$ and $\tilde{\Delta}_1$. The component $\tilde{\Delta}_0$ is the closure of the locus of irreducible curves; the other component $\tilde{\Delta}_1$ is formed by the curves with a separating node, which are those of type (4).

2 The locus of cusps in the universal curve

The stack $\tilde{\mathcal{M}}_g$ contains a closed substack of codimension 2, the cuspidal locus, whose points correspond to curves with at least a cusp. Its normalization, which we denote by $\tilde{\mathcal{C}}_g^c$, is the stack of curves of genus g with a distinguished cusp. This will play an important role in one of our calculations. Everything that we are going to say in this section generalizes to $\tilde{\mathcal{M}}_{g,n}$, but this would complicate notation, and the added generality would not be useful to us.

2.1 The cuspidal locus in an A_2 -curve

Let $C \rightarrow S$ be an A_2 -curve; we are interested in giving a scheme structure to the locus of cusps $C^c \subseteq C$, which is closed. This is done as follows.

Let $C^{\text{sing}} \subseteq C$ be the singular locus of the map $C \rightarrow S$, with the usual scheme structure given by the first Fitting ideal of the sheaf $\Omega_{C/S}$. Then C^{sing} is finite over S ; it is unramified at the nodal points, and it ramifies at the cusps. We define C^c as the closed subscheme of C defined by the 0th Fitting ideal of $\Omega_{C^{\text{sing}}/S}$.

Proposition 2.1 *The geometric points of the subscheme $C^c \subseteq C$ correspond precisely to the cuspidal points of the geometric fibers of $C \rightarrow S$. The restriction $C^c \rightarrow S$ is finite and unramified.*

Furthermore, the formation of $C^c \subseteq C$ commutes with base change on S . If S is of finite type over k and the family $C \rightarrow S$ is versal at each point of S , then C^c is a smooth scheme over k .

Proof Let us check the first statement. This can be done after restricting to the geometric fibers; so assume that $S = \text{Spec } \ell$, where ℓ is an algebraically closed field. Furthermore, one can pass to an étale cover of C , without assuming that C is proper.

Let $p \in C(\ell)$ be a singular point of C . If p is a node, then it is étale-locally of the form $\text{Spec } \ell[x, y]/(xy)$; a straightforward calculation reveals that the first Fitting ideal of $\Omega_{C/\ell}$ is $(x, y) \subseteq \mathbb{O}_C$. Hence, $\Omega_{C^{\text{sing}}/\ell}$ is zero at p , and p is not in C^c .

If p is a cusp, then C is étale-locally of the form $\text{Spec } \ell[x, y]/(y^2 - x^3)$. Then the first Fitting ideal of $\Omega_{C/\ell}$ is $(2y, 3x^3) = (y, x^2)$ (recall that we are assuming $\text{char } k \neq 2, 3$), so étale locally $\mathbb{O}_{C^{\text{sing}}}$ is $\text{Spec } \ell[x, y]/(y, x^2)$. Hence, étale-locally, the 0th Fitting ideal of $\Omega_{C^{\text{sing}}/\ell}$ is $(x) \subseteq \mathbb{O}_{C^{\text{sing}}}$, and $\mathbb{O}_{C^c} = \text{Spec } \ell$. This proves the first statement.

Formation of sheaves of differentials, and of Fitting ideals, commutes with base change on S ; hence, formation of C^c also commutes with base change. Therefore, to prove the remaining statements we can assume that $S = \text{Spec } R$ is affine and of finite type over k .

The projection $C^c \rightarrow S$ is proper and representable; hence, to check that it is finite, it is enough to show that it has finite fibers, which follows from the first statement.

For the rest of the statements, take a geometric point $p: \text{Spec } \ell \rightarrow C^c$. We can base change from k to ℓ , and assume that p is a rational point, and k is algebraically closed; call q the image of p in S . Notice that the definition of C^c does not require $C \rightarrow S$ to be proper; furthermore, if $U \rightarrow C$ is an étale map, then U^c is the inverse image of C^c in U .

By [Talpo and Vistoli 2013, Example 6.46], after passing to an étale neighborhood of the image of q in S , there exist two elements a and b of R , vanishing at q , and an étale neighborhood $p \rightarrow U \rightarrow C$ of p such that, if we denote by $V \subseteq \mathbb{A}_{\mathbb{R}}^2$ the subscheme defined by the equation $y^2 = x^3 + ax + b$, there is an étale map $U \rightarrow V$ of S -schemes sending $p \in U$ into the point over q defined by $x = y = 0$. Then U^c is the inverse image of V^c ; a straightforward calculation shows that V^c is the subscheme defined by $x = y = a = b$, which proves that C^c is unramified over S . Furthermore, if $C \rightarrow S$ is versal then S is smooth, and a and b are part of a system of parameters around q , which shows that C^c is smooth, as claimed. \square

2.2 The pinching construction

Suppose that we have an A_2 -curve D over an algebraically closed field, with a cuspidal point $q \in C(k)$. Call C the normalization of D at the point q ; then the inverse image of q consists of a single point p . Then it is standard that the pair (D, q) can be recovered from (C, p) : as a topological space, $D = C$, while the structure sheaf \mathbb{O}_D is the subsheaf of \mathbb{O}_C consisting of functions whose Taylor expansions at p have first-order term equal to 0. This construction, which we call *pinching*, works in families, and plays a fundamental role in this paper; a depiction of it can be found in Figure 1.

Let $\pi: C \rightarrow S$ be an A_2 -curve with a section $\sigma: S \rightarrow C$ landing into the smooth locus of $C \rightarrow S$. We will assume that C is a scheme (the pinching construction will also work for algebraic spaces, using the small étale site, but we will not need this). We define $\hat{C} \rightarrow S$ as follows. Denote by $\Sigma \subseteq C$ the image of σ ; call $I_{\Sigma} \subseteq \mathbb{O}_C$ the sheaf of ideals of $\Sigma \subseteq C$. Sending a section f of \mathbb{O}_C into $f - \pi^* \sigma^* f$ gives an \mathbb{O}_S -linear splitting $\mathbb{O}_C \rightarrow I_{\Sigma}$ of the embedding $I_{\Sigma} \subseteq \mathbb{O}_C$.

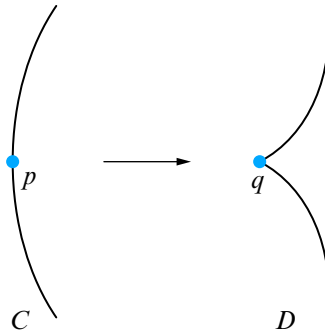


Figure 1: The pinching construction.

As a topological space, we set $\hat{C} = C$; the structure sheaf $\mathcal{O}_{\hat{C}} \subseteq \mathcal{O}_C$ is defined as the subring of sections f of \mathcal{O}_C with the property that the class $[f - \pi^* \sigma^* f] \in I_{\Sigma}/I_{\Sigma}^2$ is zero. Then $\mathcal{O}_{\hat{C}}$ is a sheaf of rings with local stalks. Furthermore, $\pi^{\#}: \mathcal{O}_S \rightarrow \mathcal{O}_C$ has image contained in $\mathcal{O}_{\hat{C}}$, so it defines a morphism of locally ringed spaces $\hat{C} \rightarrow S$.

Proposition 2.2 *The morphism $\hat{C} \rightarrow S$ is an A_2 -curve. The morphism $C \rightarrow \hat{C}$ induced by the embedding $\mathcal{O}_{\hat{C}} \subseteq \mathcal{O}_C$ is an isomorphism outside of the image of the section σ . Furthermore, the composite $\Sigma \subseteq C \xrightarrow{\pi} \hat{C}$ gives a closed and open embedding of Σ into the cuspidal locus \hat{C}^c .*

Proof There is an exact sequence

$$0 \rightarrow \mathcal{O}_{\hat{C}} \rightarrow \mathcal{O}_C \rightarrow I_{\Sigma}/I_{\Sigma}^2 \rightarrow 0$$

of sheaves of \mathcal{O}_S -modules; since I_{Σ}/I_{Σ}^2 is flat over S , it is clear that formation of \hat{C} commutes with base change on S , and that \hat{C} is flat over \mathcal{O}_S .

Let us check that \hat{C} is finitely presented over S . Since formation of \hat{C} commutes with base change on S , and C is finitely presented, we can assume that S is noetherian. But the extension $\mathcal{O}_{\hat{C}} \subseteq \mathcal{O}_C$ is finite, while \mathcal{O}_C is of finite type over \mathcal{O}_S ; from this it follows that $\mathcal{O}_{\hat{C}}$ is also of finite type over \mathcal{O}_S .

Since $C \rightarrow \hat{C}$ is proper and surjective, it follows that $C \rightarrow S$ is also proper.

The fact that \hat{C} is an A_2 -curve can now be checked when S is the spectrum of an algebraically closed field, and it is straightforward.

The second statement is evident from the construction.

The last statement is proved with a straightforward local calculation. □

The pinching construction is functorial, in the sense that an isomorphism $C \simeq C'$ of pointed A_2 -curves over S induces an isomorphism $\hat{C} \simeq \hat{C}'$ of A_2 -curves over S .

Suppose that C is a geometrically connected A_2 -curve over $\text{Spec } k$, with $p \in C(k)$ a smooth rational point. Let $U = \text{Spec } R$ be a miniversal deformation space for the pair (C, p) as a 1-pointed A_2 -curve. Here R is a complete local k -algebra; since A_2 -curves are unobstructed, R is a power series algebra $k[[x_1, \dots, x_r]]$. Denote by $C_U \rightarrow U$ the corresponding 1-pointed A_2 -curve; this C_U is a priori a formal scheme; but the closed fiber C is projective, and an ample line bundle on C extends to C_U , because C is 1-dimensional, so it follows from Grothendieck's existence theorem that $C_U \rightarrow U$ is a projective scheme.

Analogously, let $V = \text{Spec } S$ be a miniversal deformation space for the pinched A_2 -curve $\hat{C} \rightarrow \text{Spec } k$; like in the previous case, V is the spectrum of a power series algebra over k . Call $D_V \rightarrow V$ the universal A_2 -curve. Its closed fiber \hat{C} has a distinguished cusp \hat{p} , the image of $p \in C(k)$; call $\Delta \subseteq D_V^c$ the connected component of D_V^c containing \hat{p} ; by Proposition 2.1, the $\Delta \rightarrow V$ is an embedding, and Δ is once again the spectrum of a power series algebra.

By pinching the tautological section $U \rightarrow C_U$, we get a family of A_2 -curves $\hat{C}_U \rightarrow U$ with a distinguished section $U \rightarrow \hat{C}_U^c$; this induces a (nonunique) morphism $U \rightarrow V$, which factors through Δ .

Lemma 2.3 *The morphism $U \rightarrow \Delta$ described above is an isomorphism.*

Proof Call $\phi: U \rightarrow \Delta$ the morphism above. To produce a morphism in the other direction $\psi: \Delta \rightarrow U$, consider the pullback $D_\Delta := \Delta \times_V D_V$, with its tautological section $\Delta \rightarrow D_\Delta^c$. A straightforward local calculation reveals that the normalization \bar{D}_Δ along $\Delta \subseteq D_\Delta$ is an A_2 -curve, the reduced inverse image of $\Delta \subseteq D_\Delta$ in \bar{D}_Δ maps isomorphically onto Δ , thus giving a section $\Delta \rightarrow \bar{D}_\Delta$. Hence, we get a pointed A_2 -curve \bar{D}_Δ , whose closed fiber is C ; so, there exists a morphism $\psi: \Delta \rightarrow U$, with an isomorphism $\bar{D}_\Delta \simeq \Delta \times_U C_U$ such that the section $\Delta \rightarrow \bar{D}_\Delta$ corresponds to the section $\Delta \times_U C_U$ coming from the tautological section $U \rightarrow C_U$.

Consider the composite $\phi\psi: U \rightarrow U$. The pullback of C_U along $\phi\psi$ is the normalization of \hat{C}_U along the canonical section $U \rightarrow \hat{C}_U^c$; hence, it is isomorphic to C_U . This implies that this pullback is C_U , which, in turn, together with the fact that $C_U \rightarrow U$ is miniversal, implies that $\psi\phi$ is an isomorphism. An analogous argument shows that the pullback of D_Δ along $\phi\psi$ is isomorphic to D_Δ , so that $\phi\psi$ is also an isomorphism. It follows that ϕ is an isomorphism, as claimed. \square

As an application of the pinching construction, let us prove the following lemma, which is needed in the proof of Theorem 1.3:

Lemma 2.4 *There exists a scheme of finite type U over k and a morphism $U \rightarrow \tilde{\mathcal{M}}_{g,n}$ which is surjective on geometric points.*

Proof Let Ω be an algebraically closed extension of k . Let (C, p_1, \dots, p_n) be a stable n -pointed A_2 -curve over Ω , and call q_1, \dots, q_s the cuspidal points of C . Denote by \bar{C} the normalization of C at the q_i ,

\bar{p}_i the point of \bar{C} lying over p_i , and \bar{q}_i the point of \bar{C} lying over q_i . Then $(\bar{C}, \bar{p}_1, \dots, \bar{p}_n, \bar{q}_1, \dots, \bar{q}_s)$ is an $(n+s)$ -pointed nodal curve of genus $g - s$. Hence, $s \leq g$; it is enough to produce for each s with $1 \leq s \leq g$ a scheme of finite type and a morphism $U_s \rightarrow \tilde{\mathcal{M}}_{g,n}$ whose image contains all n -pointed stable A_2 -curves with exactly s cuspidal points.

Notice that the n -pointed curve $(\bar{C}, \bar{p}_1, \dots, \bar{p}_n, \bar{q}_1, \dots, \bar{q}_s)$ above is not necessarily stable, because the pullback of $\omega_{C/\Omega}(p_1 + \dots + p_n)$ is

$$\omega_{\bar{C}/\Omega}(\bar{p}_1 + \dots + \bar{p}_n + 2\bar{q}_1 + \dots + 2\bar{q}_s);$$

but this instability can only occur when C has a rational component whose only singularity is a single cusp, and intersects the rest of the curve in only one point. For each $i = 1, \dots, s$ let r_i be a smooth point on the component of \bar{C} containing \bar{q}_i , but different from \bar{q}_i ; then the $(n+2s)$ -pointed curve

$$(\bar{C}, \bar{p}_1 + \dots + \bar{p}_n + 2\bar{q}_1 + \dots + 2\bar{q}_s, r_1, \dots, r_s)$$

is stable.

Now, starting from a stable $(n+2s)$ -curve of genus $g - s$ $(D, t_1, \dots, t_{n+2s})$ over a scheme S , let us construct an n -pointed A_2 -curve $(\tilde{D}, \tilde{t}_1, \dots, \tilde{t}_n)$ of genus g , over the same S , as follows:

- (1) The curve \tilde{D} is obtained from D by pinching down the images of t_{n+1}, \dots, t_{n+s} .
- (2) For each $i = 1, \dots, n$, the section $\tilde{t}_i: S \rightarrow \tilde{D}$ is the composite of $t_i: S \rightarrow D$ with the projection $D \rightarrow \tilde{D}$.
- (3) The markings $t_{n+s+1}, \dots, t_{n+2s}$ are forgotten.

The curve $(\tilde{D}, \tilde{t}_1, \dots, \tilde{t}_n)$ is not necessarily stable; however, by Proposition 1.2, there exists an open substack $\mathcal{U}_s \subseteq \bar{\mathcal{M}}_{g-s, n+2s}$ whose objects are curves

$$(D, t_1, \dots, t_{n+2s})$$

whose associated curve $(\tilde{D}, \tilde{t}_1, \dots, \tilde{t}_n)$ is stable. By sending $(D, t_1, \dots, t_{n+2s})$ into $(\tilde{D}, \tilde{t}_1, \dots, \tilde{t}_n)$, we get a morphism $\mathcal{U}_s \rightarrow \tilde{\mathcal{M}}_{g,n}$. For each Ω , the stable n -pointed curves with exactly s cuspidal points are precisely those coming from \mathcal{U}_s ; since \mathcal{U}_s is of finite type, we have a smooth surjective morphism $U_s \rightarrow \mathcal{U}_s$, in which U_s is a scheme of finite type over k . The composite $U_s \rightarrow \mathcal{U}_s \rightarrow \tilde{\mathcal{M}}_{g,n}$ is the desired morphism. □

2.3 The description of the cuspidal locus

Consider an A_2 -curve $C \rightarrow S$. The fact that formation of C^c commutes with base changes allows us define C^c even when S is an Artin stack. In particular, let $\tilde{\mathcal{C}}_g \rightarrow \tilde{\mathcal{M}}_g$ be the universal stable A_2 -curve of genus g ; define $\tilde{\mathcal{C}}_g^c$ to be the cuspidal locus inside $\tilde{\mathcal{C}}_g$. Then $\tilde{\mathcal{C}}_g^c$ is the stack of stable A_2 -curves of genus g with a marked cusp. More precisely, one can describe $\tilde{\mathcal{C}}_g^c$ as the stack of stable A_2 -curves $C \rightarrow S$ of genus g , with a section $S \rightarrow C^c$.

By Proposition 2.1, the projection $\tilde{\mathcal{C}}_g^c \rightarrow \tilde{\mathcal{M}}_g$ is finite and unramified; its image is precisely the locus of curves with at least one cusp. Furthermore, $\tilde{\mathcal{C}}_g^c$ is a smooth stack; hence, it is in fact the normalization of the locus of curves in $\tilde{\mathcal{M}}_g$ with at least one cusp. We aim to give a description of $\tilde{\mathcal{C}}_g^c$. This is a little subtle, as there are two distinct possibilities for a stable A_2 -curve, revealed by the following:

Proposition 2.5 *Assume that k is algebraically closed. Let C be a stable A_2 -curve of genus g over k , and let $p \in C(k)$ be a cuspidal point. Call \bar{C} the normalization of C at p , and \bar{p} the point of \bar{C} lying over p . Furthermore, call D the component of C containing p and \bar{D} its inverse image in \bar{C} . Then there are two possibilities:*

- (1) (\bar{C}, \bar{p}) is a 1-marked stable A_2 -curve of genus $g - 1$, or
- (2) \bar{D} is smooth of genus 0, and meets the rest of \bar{C} transversally in one smooth point.

In case (1), the morphism $\bar{C} \rightarrow C$ induces an isomorphism of group schemes $\underline{\text{Aut}}_k(\bar{C}, \bar{p}) \simeq \underline{\text{Aut}}_k(C, p)$.

In case (2), call C_1 the union of the irreducible components of C different from D , and q the point of intersection of D with C_1 . Then $\underline{\text{Aut}}_k(D, p, q) = \mathbb{G}_m$; there is an isomorphism of group schemes $\underline{\text{Aut}}_k(C_1, q) \times \mathbb{G}_m \simeq \underline{\text{Aut}}_k(C, p)$, in which $\underline{\text{Aut}}_k(C_1, q)$ acts trivially on D , and \mathbb{G}_m acts trivially on C_1 .

Proof The degree of the restriction $\omega_{\bar{C}}(\bar{p})|_{\bar{D}}$ equals the degree of the restriction $\omega_C|_D$ minus one. This means that there are two possibilities: either $\omega_{\bar{C}}(\bar{p})|_{\bar{D}}$ has positive degree, which gives case (1), or it has degree 0, which gives case (2).

The statement about automorphism group schemes is straightforward. □

This shows that there are two possibilities for the pair (C, p) . It can be obtained from a stable 1-marked A_2 -curve (\bar{C}, \bar{p}) by either pinching down \bar{p} , or attaching to \bar{p} a rational curve with a cusp.

Let us give a stack-theoretic version of this, by showing that $\tilde{\mathcal{C}}_g^c$ is isomorphic to $\tilde{\mathcal{M}}_{g-1,1} \times [\mathbb{A}^1/\mathbb{G}_m]$. The stack $[\mathbb{A}^1/\mathbb{G}_m]$ has a closed substack $\mathcal{B}\mathbb{G}_m = [0/\mathbb{G}_m]$, whose complement is $[(\mathbb{A}^1 \setminus \{0\})/\mathbb{G}_m] = \text{Spec } k$. Thus a geometric point of $\tilde{\mathcal{M}}_{g-1,1} \times [\mathbb{A}^1/\mathbb{G}_m]$ is of one of two types (see Figures 2 and 3):

- (1) It can be in the open substack $\tilde{\mathcal{M}}_{g-1,1} = \tilde{\mathcal{M}}_{g-1,1} \times \text{Spec } k$; in this case it corresponds to a stable 1-marked A_2 -curve (\bar{C}, \bar{p}) of genus $g - 1$. In this case the associated A_2 -curve is of type (1), and is obtained by pinching \bar{C} at \bar{p} .
- (2) Alternatively, it can be in the closed substack $\tilde{\mathcal{M}}_{g-1,1} \times \mathcal{B}\mathbb{G}_m \subseteq \tilde{\mathcal{M}}_{g-1,1} \times [\mathbb{A}^1/\mathbb{G}_m]$; then it corresponds, up to isomorphism, to a stable 1-marked A_2 -curve (\bar{C}, \bar{p}) of genus $g - 1$; but the automorphism group of the object of the geometric point of $\tilde{\mathcal{M}}_{g-1,1} \times [\mathbb{A}^1/\mathbb{G}_m]$ is $\underline{\text{Aut}}(\bar{C}, \bar{p}) \times \mathbb{G}_m$. In this case we associate with this object the curve of type (2) obtained by attaching to \bar{p} a rational curve with a cusp.

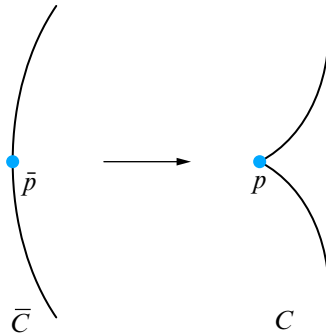


Figure 2: Cuspidal curve of type (1).

The way these two set-theoretic descriptions glue together to a morphism $\tilde{\mathcal{M}}_{g-1,1} \times [\mathbb{A}^1/\mathbb{G}_m] \rightarrow \tilde{\mathcal{C}}_g^c$ is a little subtle. We construct the morphism as follows.

Consider the universal curve $\tilde{\mathcal{C}}_{g-1,1} \rightarrow \tilde{\mathcal{M}}_{g-1,1}$, with its tautological section $\sigma: \tilde{\mathcal{M}}_{g-1,1} \rightarrow \tilde{\mathcal{C}}_{g-1,1}$. Call Σ the image of σ ; then Σ is a closed smooth subscheme of $\tilde{\mathcal{C}}_{g-1,1} \times [\mathbb{A}^1/\mathbb{G}_m]$ of codimension 2. Denote by $\mathcal{D}_{g-1,1}$ the blowup of $\tilde{\mathcal{C}}_{g-1,1} \times [\mathbb{A}^1/\mathbb{G}_m]$ along $\Sigma \times \mathbb{B}\mathbb{G}_m$, and call Σ' the proper transform of Σ in $\mathcal{D}_{g-1,1}$.

Proposition 2.6 *The morphism $\mathcal{D}_{g-1,1} \rightarrow \tilde{\mathcal{M}}_{g-1,1} \times [\mathbb{A}^1/\mathbb{G}_m]$ is an A_2 -curve of genus $g-1$. The composite $\Sigma' \subseteq \mathcal{D}_{g-1,1} \rightarrow \tilde{\mathcal{M}}_{g-1,1} \times [\mathbb{A}^1/\mathbb{G}_m]$ is an isomorphism. The corresponding section $\tilde{\mathcal{M}}_{g-1,1} \times [\mathbb{A}^1/\mathbb{G}_m] \rightarrow \mathcal{D}_{g-1,1}$ has image contained in the smooth locus of the morphism $\mathcal{D}_{g-1,1} \rightarrow \tilde{\mathcal{M}}_{g-1,1} \times [\mathbb{A}^1/\mathbb{G}_m]$.*

Proof Obviously, $\mathcal{D}_{g-1,1}$ is proper and representable over $\tilde{\mathcal{M}}_{g-1,1} \times [\mathbb{A}^1/\mathbb{G}_m]$. Also, it is smooth over k , since both $\tilde{\mathcal{M}}_{g-1,1} \times [\mathbb{A}^1/\mathbb{G}_m]$ and $\Sigma \times \mathbb{G}_m$ are smooth; since the morphism $\mathcal{D}_{g-1,1} \rightarrow \tilde{\mathcal{M}}_{g-1,1} \times [\mathbb{A}^1/\mathbb{G}_m]$ has equidimensional fibers, it is also flat.

The fact that $\Sigma' \rightarrow \tilde{\mathcal{M}}_{g-1,1} \times [\mathbb{A}^1/\mathbb{G}_m]$ is an isomorphism follows immediately from the fact that $\Sigma \times \mathbb{B}\mathbb{G}_m$ is a Cartier divisor in $\Sigma \times [\mathbb{A}^1/\mathbb{G}_m]$.

Now we need to check that the singularities of the geometric fibers of $\mathcal{D}_{g-1,1}$ are at most A_2 , and that the intersection of Σ' with the geometric fibers is contained in the smooth locus. This is a particular case of the following lemma. □

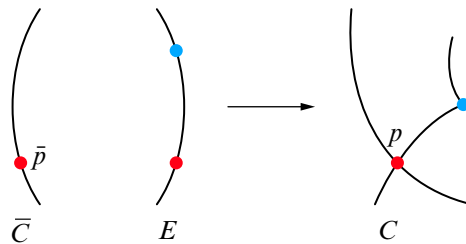


Figure 3: Cuspidal curve of type (2).

Lemma 2.7 *Let $s : \text{Spec } \Omega \rightarrow \tilde{\mathcal{M}}_{g-1,1} \times [\mathbb{A}^1/\mathbb{G}_m]$ be a geometric point. Call $(C, p) \in \tilde{\mathcal{M}}_{g-1,1}$ the marked A_2 -curve corresponding to the composite*

$$\text{Spec } \Omega \xrightarrow{s} \tilde{\mathcal{M}}_{g-1,1} \times [\mathbb{A}^1/\mathbb{G}_m] \xrightarrow{\text{pr}_1} \tilde{\mathcal{M}}_{g-1,1}.$$

If the image of s in $[\mathbb{A}^1/\mathbb{G}_m]$ is in $[\mathbb{A}^1/\mathbb{G}_m] \setminus \mathcal{B}\mathbb{G}_m \simeq \text{Spec } k$, then the fiber of $\mathcal{D}_{g-1,1} \rightarrow \tilde{\mathcal{M}}_{g-1,1} \times [\mathbb{A}^1/\mathbb{G}_m]$ over s is C , and the inverse image of Σ' is $p \in C$.

If the image of s in $[\mathbb{A}^1/\mathbb{G}_m]$ is in $\mathcal{B}\mathbb{G}_m$, then the fiber is C , with a copy of \mathbb{P}^1 glued to p by the point $0 \in \mathbb{P}^1$. The inverse image of Σ' is the point at infinity.

Thus we can apply the pinching construction above and get an A_2 -curve $\hat{\mathcal{D}}_{g-1,1} \rightarrow \tilde{\mathcal{M}}_{g-1,1} \times [\mathbb{A}^1/\mathbb{G}_m]$.

Proposition 2.8 *The morphism $\hat{\mathcal{D}}_{g-1,1} \rightarrow \tilde{\mathcal{M}}_{g-1,1} \times [\mathbb{A}^1/\mathbb{G}_m]$ is a stable A_2 -curve.*

Proof We need to check that the geometric fibers of $\hat{\mathcal{D}}_{g-1,1} \rightarrow \tilde{\mathcal{M}}_{g-1,1} \times [\mathbb{A}^1/\mathbb{G}_m]$ are A_2 -stable; this is clear from Lemma 2.7. □

By construction, $\hat{\mathcal{D}}_{g-1,1} \rightarrow \tilde{\mathcal{M}}_{g-1,1} \times [\mathbb{A}^1/\mathbb{G}_m]$ comes from a stable 1-pointed A_2 -curve $\mathcal{D}_{g-1,1} \rightarrow \tilde{\mathcal{M}}_{g-1,1} \times [\mathbb{A}^1/\mathbb{G}_m]$ by pinching down a section $\tilde{\mathcal{M}}_{g-1,1} \times [\mathbb{A}^1/\mathbb{G}_m] \rightarrow \mathcal{D}_{g-1,1}$ with image $\Sigma' \subseteq \mathcal{D}_{g-1,1}$; from the last statement of Proposition 2.2, we get a factorization $\tilde{\mathcal{M}}_{g-1,1} \times [\mathbb{A}^1/\mathbb{G}_m] \rightarrow \tilde{\mathcal{C}}_g^c \rightarrow \tilde{\mathcal{M}}_g$.

Theorem 2.9 *The morphism $\Pi : \tilde{\mathcal{M}}_{g-1,1} \times [\mathbb{A}^1/\mathbb{G}_m] \rightarrow \tilde{\mathcal{C}}_g^c$ described above is an isomorphism.*

We will use the following:

Lemma 2.10 *Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ an étale morphism of algebraic stacks over k . Assume that the following conditions are satisfied for every algebraically closed extension $k \subseteq \Omega$:*

- (1) *The morphism f induces a bijection between isomorphism classes in $\mathcal{X}(\Omega)$ and in $\mathcal{Y}(\Omega)$.*
- (2) *If $x \in \mathcal{X}(\Omega)$, the morphism $\underline{\text{Aut}}_\Omega(x) \rightarrow \underline{\text{Aut}}_\Omega(f(x))$ induced by f is an isomorphism.*

Then $f : \mathcal{X} \rightarrow \mathcal{Y}$ is an isomorphism.

Proof Let $V \rightarrow \mathcal{Y}$ be a smooth surjective morphism, where V is a scheme. Set $U := \mathcal{X} \times_{\mathcal{Y}} V$; the conditions above are easily seen to imply that the automorphism group schemes of the geometric points of U are trivial, so that U is an algebraic space. The morphism $U \rightarrow V$ is étale, and for each Ω the induced function $U(\Omega) \rightarrow V(\Omega)$ is a bijection. Since $V \rightarrow U$ is étale and surjective, it is an epimorphism of étale sheaves; hence, to show that it is an isomorphism, it is enough to prove that the diagonal $V \rightarrow V \times_U V$ is an isomorphism. But $V \rightarrow U$ is unramified, so V is an open subspace of $V \times_U V$; since it has the same geometric points, the result follows. So $U \rightarrow V$ is an isomorphism; hence, f is an isomorphism, as claimed. □

Proof of Theorem 2.9 By Lemma 2.10, Proposition 2.5 and Lemma 2.7, it is enough to prove that Π is étale.

Let $\tilde{\mathcal{M}}'_{g-1,1}$ be the stack whose sections are 1-pointed A_2 -curves $C \rightarrow S$ such that, if we denote by Σ the image of the section, the invertible sheaf $\omega_{C/S}(2\Sigma)$ is relatively ample on S . The A_2 -curve $\mathcal{D}_{g-1,1} \rightarrow \tilde{\mathcal{M}}_{g-1,1} \times [\mathbb{A}^1/\mathbb{G}_m]$ satisfies this condition; hence, we get a morphism $\Pi': \tilde{\mathcal{M}}_{g-1,1} \times [\mathbb{A}^1/\mathbb{G}_m] \rightarrow \tilde{\mathcal{M}}'_{g-1,1}$.

Also, if $C \rightarrow S$ is in $\tilde{\mathcal{M}}'_{g-1,1}$, the pinched curve $\hat{C} \rightarrow S$ is in $\tilde{\mathcal{M}}_g$; hence, the pinching construction gives a morphism $\Pi'': \tilde{\mathcal{M}}'_{g-1,1} \rightarrow \tilde{\mathcal{C}}_g^c$. The composite

$$\tilde{\mathcal{M}}_{g-1,1} \times [\mathbb{A}^1/\mathbb{G}_m] \xrightarrow{\Pi'} \tilde{\mathcal{M}}'_{g-1,1} \xrightarrow{\Pi''} \tilde{\mathcal{C}}_g^c$$

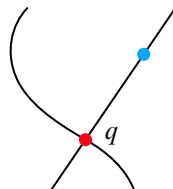
is Π , by construction; so it is enough to show that Π' and Π'' are étale.

For Π'' , this follows from Lemma 2.3. For Π' , the locus of $\tilde{\mathcal{M}}_{g-1,1} \times [\mathbb{A}^1/\mathbb{G}_m]$ where Π' is étale is open, and the only open substack of $\tilde{\mathcal{M}}_{g-1,1} \times [\mathbb{A}^1/\mathbb{G}_m]$ containing $\tilde{\mathcal{M}}_{g-1,1} \times \mathcal{B}\mathbb{G}_m$ is $\tilde{\mathcal{M}}_{g-1,1} \times [\mathbb{A}^1/\mathbb{G}_m]$ itself; hence, it is enough show that Π' is étale along $\tilde{\mathcal{M}}_{g-1,1} \times \mathcal{B}\mathbb{G}_m$.

Denote by $\tilde{\mathcal{M}}_{g-1,1}^o$ the complement $\tilde{\mathcal{M}}'_{g-1,1} \setminus \tilde{\mathcal{M}}_{g-1,1}$, with its reduced scheme structure; this is the scheme-theoretic image of $\tilde{\mathcal{M}}_{g-1,1} \times \mathcal{B}\mathbb{G}_m$ into $\tilde{\mathcal{M}}'_{g-1,1}$.

The morphism $\tilde{\mathcal{M}}_{g-1,1} \times \mathcal{B}\mathbb{G}_m \rightarrow \tilde{\mathcal{M}}_{g-1,1}^o$ can be described as follows. We can interpret $\mathcal{B}\mathbb{G}_m$ as the stack $\mathcal{M}_{0,2}$ of smooth curves $P \rightarrow S$ of genus 0 with two disjoint sections $s_1, s_2: S \rightarrow P$; the \mathbb{G}_m -torsor corresponding to $(P \rightarrow S, s_1, s_2)$ is that associated with the normal bundle to s_1 . Given an object $(C \rightarrow S, s)$ of $\tilde{\mathcal{M}}_{g-1,1}$ and an object $(P \rightarrow S, s_1, s_2)$ of $\mathcal{M}_{0,2}$, we obtain an object $C \sqcup P$ of $\tilde{\mathcal{M}}'_{g-1,1}$ by gluing C with P by identifying s and s_1 , and using s_2 to give the marking. It is easy to check that $\tilde{\mathcal{M}}_{g-1,1} \times \mathcal{B}\mathbb{G}_m \rightarrow \tilde{\mathcal{M}}_{g-1,1}^o$ is an isomorphism.

So, it is enough to show that the scheme-theoretic inverse image of $\tilde{\mathcal{M}}_{g-1,1}^o \subseteq \tilde{\mathcal{M}}_{g-1,1}$ in $\tilde{\mathcal{M}}_{g-1,1} \times [\mathbb{A}^1/\mathbb{G}_m]$ is $\tilde{\mathcal{M}}_{g-1,1} \times \mathcal{B}\mathbb{G}_m$. We can assume that k is algebraically closed; let $c: \text{Spec } k \rightarrow \tilde{\mathcal{M}}_{g-1,1} \times \mathcal{B}\mathbb{G}_m$ be a morphism, and let us show that there is a smooth morphism $U \rightarrow \tilde{\mathcal{M}}_{g-1,1} \times [\mathbb{A}^1/\mathbb{G}_m]$ with a lifting $\text{Spec } k \rightarrow U$ of c such that the pullback of $\tilde{\mathcal{M}}_{g-1,1} \times \mathcal{B}\mathbb{G}_m$ to U coincides with the pullback of $\tilde{\mathcal{M}}_{g-1,1}^o$. Here is a picture of the curve corresponding to the composite $c': \text{Spec } k \rightarrow \tilde{\mathcal{M}}_{g-1,1} \times \mathcal{B}\mathbb{G}_m \rightarrow \tilde{\mathcal{M}}_{g-1,1}^o$; the blue point is the marked point, while the red point, which we call q , is the node where the component containing the marked point, which is isomorphic to \mathbb{P}^1 , meets the rest of the curve:



Let (V_1, q_1) be a versal deformation space of the node q (here $q_1 \in V_1(k)$ is the marked point). Since the universal family over $\tilde{\mathcal{M}}'_{g-1,1}$ is obviously versal, there exists a smooth morphism $V \rightarrow \tilde{\mathcal{M}}'_{g-1,1}$ with a

lifting of c and a smooth morphism $V \rightarrow V_1$ such that the inverse image of $q_1 \in V_1$ in V coincides with the inverse image of $\tilde{\mathcal{M}}_{g-1,1}^o$. Set $U := (\tilde{\mathcal{M}}_{g-1,1} \times \mathbb{G}_a) \times_{\tilde{\mathcal{M}}_{g-1,1}} V$; for the thesis to hold it is enough to show that the inverse image of q_1 along the morphism $V \rightarrow V_1$ is reduced. But this follows immediately from the fact that the curve $\mathcal{D}_{g-1,1}$ is smooth over $\text{Spec } k$. \square

3 The Chow ring of $\bar{\mathcal{M}}_{1,2}$

From now on, the base ring k will be a field of characteristic $\neq 2, 3$.

In this section we compute the integral Chow ring of $\bar{\mathcal{M}}_{1,2}$, the moduli stack of stable genus 1 curves with two markings (Theorem 3.6), and consequently also the rational Chow ring of the coarse moduli space $\bar{M}_{1,2}$ (Corollary 3.7).

These results are achieved in the following way: we first compute the integral Chow ring of $\tilde{\mathcal{C}}_{1,1}$, the universal elliptic stable A_2 -curve (Proposition 3.3), using the patching lemma (Lemma 3.1); then we conclude our computations, leveraging the localization exact sequence induced by the open embedding $\bar{\mathcal{M}}_{1,2} \hookrightarrow \tilde{\mathcal{C}}_{1,1}$.

In this section the reader can already see all the main tools that will be used in the upcoming sections to determine $\text{CH}^*(\bar{\mathcal{M}}_{2,1})$.

We make use of equivariant intersection theory in some points. For a recap of this theory, the reader can consult [Di Lorenzo 2021b, Section 5.1].

3.1 The Chow ring of $\tilde{\mathcal{C}}_{1,1}$

First let us recall one of our key tools, the *patching lemma*:

Lemma 3.1 [Di Lorenzo and Vistoli 2021, Lemma 3.4] *Let X be a smooth variety endowed with the action of a group G , and $Y \xrightarrow{i} X$ a smooth, closed and G -invariant subvariety, with normal bundle \mathcal{N} . Suppose that $c_{\text{top}}^G(\mathcal{N})$ is not a zero-divisor in $\text{CH}_G^*(Y)$. Then the diagram of rings*

$$\begin{CD} \text{CH}_G^*(X) @>i^*>> \text{CH}_G^*(Y) \\ @Vj^*VV @VVqV \\ \text{CH}_G^*(X \setminus Y) @>p>> \text{CH}_G^*(Y)/(c_{\text{top}}^G(\mathcal{N})) \end{CD}$$

is cartesian, where the bottom horizontal arrow p sends the class of a variety V to the equivalence class of i^ξ, where ξ is any element in the set $(j^*)^{-1}([V])$.*

We will call V_i the rank 1 representation of \mathbb{G}_m on which the latter acts with weight i . The notation V_{i_1, \dots, i_r} will stand for the rank r representation $V_{i_1} \oplus \dots \oplus V_{i_r}$.

Let $\mathbf{p}: \tilde{\mathcal{M}}_{1,1} \rightarrow \tilde{\mathcal{C}}_{1,1}$ be the universal section.

Lemma 3.2 *The stack $\tilde{\mathcal{C}}_{1,1} \setminus \text{im}(\mathbf{p})$ is an affine bundle over $[V_{-2,-3}/\mathbb{G}_m]$ and*

$$\text{CH}^*(\tilde{\mathcal{C}}_{1,1} \setminus \text{im}(\mathbf{p})) \simeq \mathbb{Z}[\lambda_1],$$

where λ_1 is the pullback of the first Chern class of the Hodge line bundle along $\tilde{\mathcal{C}}_{1,1} \rightarrow \tilde{\mathcal{M}}_{1,1}$.

Proof Recall that $\tilde{\mathcal{M}}_{1,1}$ is isomorphic to the quotient stack $[V_{-4,-6}/\mathbb{G}_m]$: indeed, the \mathbb{G}_m -torsor associated with the Hodge line bundle is isomorphic to $V_{-4,-6}$ (see [Edidin and Graham 1998, 5.4]). Consider the universal affine Weierstrass curve W inside $V_{-4,-6} \times V_{-2,-3}$ defined by the equation

$$y^2 = x^3 + ax + b,$$

where $(a, b) \in V_{-4,-6}$ and $(x, y) \in V_{-2,-3}$. Observe that this subscheme is \mathbb{G}_m -invariant. The quotient stack $[W/\mathbb{G}_m]$ is isomorphic to $\tilde{\mathcal{C}}_{1,1} \setminus \text{im}(\mathbf{p})$, and it is immediate to check that $W \rightarrow V_{-2,-3}$ is an equivariant affine bundle. Therefore,

$$\text{CH}^*(\tilde{\mathcal{C}}_{1,1} \setminus \text{im}(\mathbf{p})) \simeq \text{CH}^*([V_{-2,-3}/\mathbb{G}_m]) \simeq \text{CH}^*(\mathcal{B}\mathbb{G}_m) \simeq \mathbb{Z}[T],$$

where the isomorphisms are all given by pullback homomorphism. To conclude, observe that we have a commutative diagram of \mathbb{G}_m -quotient stacks

$$\begin{array}{ccc} [W/\mathbb{G}_m] & \longrightarrow & [V_{-2,-3}/\mathbb{G}_m] \\ \downarrow & & \downarrow \\ [V_{-4,-6}/\mathbb{G}_m] & \longrightarrow & [\text{Spec } k/\mathbb{G}_m] \end{array}$$

If we take the induced pullbacks, we obtain a commutative diagram of Chow rings: in particular, the pullback of T along the right vertical and the top horizontal maps is equal to the pullback of T along the other two maps. As the pullback of T along the bottom horizontal arrow is $-\lambda_1$, we get the desired conclusion. □

Proposition 3.3 *We have*

$$\text{CH}^*(\tilde{\mathcal{C}}_{1,1}) \simeq \mathbb{Z}[\lambda_1, \mu_1]/(\mu_1(\lambda_1 + \mu_1)),$$

where λ_1 is the first Chern class of the Hodge line bundle and $\mu_1 := \mathbf{p}_*[\tilde{\mathcal{M}}_{1,1}]$ is the fundamental class of the universal section.

Proof By definition, the normal bundle of the universal section is equal to $-\psi_1 = -\lambda_1$. We can apply Lemma 3.1, which tells us that we have a cartesian diagram of rings

$$\begin{array}{ccc} \text{CH}^*(\tilde{\mathcal{C}}_{1,1}) & \xrightarrow{\mathbf{p}^*} & \text{CH}^*(\tilde{\mathcal{M}}_{1,1}) \simeq \mathbb{Z}[\lambda_1] \\ j^* \downarrow & & \downarrow \\ \mathbb{Z}[\lambda_1] \simeq \text{CH}^*(\tilde{\mathcal{C}}_{1,1} \setminus \text{im}(\mathbf{p})) & \longrightarrow & \mathbb{Z} \end{array}$$

As the restriction of λ_1 , regarded as an element in $\text{CH}^*(\tilde{\mathcal{C}}_{1,1})$, to $\text{CH}^*(\tilde{\mathcal{M}}_{1,1})$ is equal to λ_1 , we deduce that $\text{CH}^*(\tilde{\mathcal{C}}_{1,1})$ is generated by λ_1 and $\mathbf{p}_*[\tilde{\mathcal{M}}_{1,1}] =: \mu_1$.

The cartesianity of the diagram above implies that the ideal of relations is formed by those polynomials $p(\lambda_1, \mu_1)$ which belong to the kernel of both j^* and p^* .

If $j^*p(\lambda_1, \mu_1) = 0$ then it must be of the form $\mu_1 q(\lambda_1, \mu_1)$. We have

$$p^*(\mu_1 q(\lambda_1, \mu_1)) = -\lambda_1(q(\lambda_1, -\lambda_1)).$$

If this is zero then $q(\lambda_1, \mu_1)$ must be divisible by $\lambda_1 + \mu_1$; thus, the ideal of relations is generated by $\mu_1(\lambda_1 + \mu_1)$. \square

3.2 The Chow ring of $\bar{\mathcal{M}}_{1,2}$

In several points in what follows we will use the following notion (see [Pernice 2022b, Definition 4.1]):

Definition 3.4 If $f: \mathcal{X} \rightarrow \mathcal{Y}$ is a morphism of algebraic stacks, we say that f is a *Chow envelope* if it is proper and representable and, for every extension K/k , the induced functor $f_K: \mathcal{X}(K) \rightarrow \mathcal{Y}(K)$ is essentially surjective.

Proposition 3.5 (1) *Being a Chow envelope is a property that is stable under composition and under base change.*

(2) *If $f: \mathcal{X} \rightarrow \mathcal{Y}$ is a Chow envelope, and \mathcal{Y} is a quotient stack of finite type over k , then*

$$f_*: \mathrm{CH}^{*-c}(\mathcal{X}) \rightarrow \mathrm{CH}^*(\mathcal{Y})$$

is surjective, where c is the relative dimension of f (namely $\dim \mathcal{Y} - \dim \mathcal{X}$).

Proof Part (1) is straightforward.

For part (2), write $\mathcal{Y} = [Y/G]$, where Y is an algebraic space of finite type and $G \rightarrow \mathrm{Spec} k$ an affine algebraic group acting on Y . Recall the basic construction of [Edidin and Graham 1998]. Let N be a positive integer, and $G \rightarrow \mathrm{GL}(V)$ be a finite-dimensional representation with an open subscheme $U \subseteq V$ with $\mathrm{codim}_V(V \setminus U) > N$; then the fppf quotient $(Y \times U)/G$ is an algebraic space, and by definition we have $\mathrm{CH}^i(\mathcal{Y}) = \mathrm{CH}^i((Y \times U)/G)$ for $i \leq N$.

Since the morphism $\mathcal{X} \rightarrow \mathcal{Y}$ is proper and representable, we have a proper G -equivariant proper morphism of algebraic spaces $X \rightarrow Y$ with $[X/G] = \mathcal{X}$, and $\mathrm{CH}^i(\mathcal{X}) = \mathrm{CH}^i((X \times U)/G)$ for $i \leq N$, and we have a cartesian diagram

$$\begin{array}{ccc} (X \times U)/G & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ (Y \times U)/G & \longrightarrow & \mathcal{Y} \end{array}$$

This means that we can assume that \mathcal{X} and \mathcal{Y} are algebraic spaces.

We refer to [Eddidin and Graham 1998, Section 6.1] for basic facts about intersection theory on algebraic spaces. The Chow group of \mathcal{Y} is generated by classes of integral closed subspaces $V \subseteq \mathcal{Y}$; we need to show that, given such a subspace $V \subseteq \mathcal{Y}$, there exists an integral closed subspace $W \subseteq \mathcal{X}$ mapping birationally onto V . The algebraic space V has a dense open subscheme $V' \subseteq V$; set $k(V) := k(V')$. By hypothesis, the composite $\text{Spec } k(V) \rightarrow V' \subseteq V \subseteq \mathcal{Y}$ lifts to $\text{Spec } k(V) \rightarrow \mathcal{X}$. But \mathcal{X} is of finite type, so this means that there exists a nonempty open subscheme $V_1 \subseteq V'$ that lifts to an embedding $V_1 \subseteq \mathcal{X}$. Then we take W to be the closure of V_1 in \mathcal{X} . □

Here is the main result of the section. We identify $\bar{\mathcal{M}}_{1,2}$ with the universal curve over $\bar{\mathcal{M}}_{1,1}$.

Theorem 3.6 *Suppose that the ground field has characteristic $\neq 2, 3$. Then*

$$\text{CH}^*(\bar{\mathcal{M}}_{1,2}) \simeq \mathbb{Z}[\lambda_1, \mu_1]/(\mu_1(\lambda_1 + \mu_1), 24\lambda_1^2),$$

where λ_1 is the first Chern class of the Hodge line bundle and $\mu_1 := p_*[\bar{\mathcal{M}}_{1,1}]$ is the fundamental class of the universal section.

Proof Let $\mathcal{M}_{0,2}$ be the stack of genus 0 smooth curves with two marked points. Up to scalar multiplication, there is only one isomorphism class of 2-pointed smooth genus 0 curves, ie a projective line \mathbb{P}^1 with 0 and ∞ as marked points. This implies that $\mathcal{M}_{0,2} \simeq \mathcal{B}\mathbb{G}_m$ and that the universal marked curve is isomorphic to $[\mathbb{P}^1/\mathbb{G}_m]$, where \mathbb{G}_m acts on \mathbb{P}^1 by scalar multiplication.

Consider the morphism of stacks

$$\mathcal{M}_{0,2} \simeq \mathcal{B}\mathbb{G}_m \rightarrow \tilde{\mathcal{M}}_{1,1}$$

that sends a 2-marked genus 0 smooth curve $(C \rightarrow S, p, \sigma)$ to the cuspidal elliptic curve $(\hat{C} \rightarrow S, p)$, where \hat{C} is obtained from C by pinching σ .

In particular, let $\widehat{\mathbb{P}^1}$ be the cuspidal curve obtained by pinching \mathbb{P}^1 at 0. The scalar action of \mathbb{G}_m on \mathbb{P}^1 descends to an action on $\widehat{\mathbb{P}^1}$ and the normalization map $\mathbb{P}^1 \rightarrow \widehat{\mathbb{P}^1}$ is equivariant with respect to this action; hence, it induces a morphism of quotient stacks $c''' : [\mathbb{P}^1/\mathbb{G}_m] \rightarrow [\widehat{\mathbb{P}^1}/\mathbb{G}_m]$.

Then we have a commutative diagram

$$\begin{array}{ccccc}
 & & & & c' \\
 & & & & \curvearrowright \\
 [\mathbb{P}^1/\mathbb{G}_m] & \xrightarrow{c'''} & [\widehat{\mathbb{P}^1}/\mathbb{G}_m] & \xrightarrow{c''} & \tilde{\mathcal{C}}_{1,1} \\
 & \searrow \rho & \downarrow \pi' & & \downarrow \pi \\
 \mathcal{M}_{0,2} \simeq \mathcal{B}\mathbb{G}_m & & & \xrightarrow{c} & \tilde{\mathcal{M}}_{1,1}
 \end{array}$$

where the square in the diagram is cartesian. Observe that c' is a Chow envelope for the locus of cuspidal curves in $\tilde{\mathcal{C}}_{1,1}$; hence, if we excise the image of

$$(3-1) \quad c'_* : \text{CH}^{*-2}([\mathbb{P}^1/\mathbb{G}_m]) \rightarrow \text{CH}^*(\tilde{\mathcal{C}}_{1,1}) \simeq \mathbb{Z}[\lambda_1, \mu_1],$$

we obtain the Chow ring of $\bar{\mathcal{M}}_{1,2}$.

From the projective bundle formula, we see that $\text{CH}^*([\mathbb{P}^1/\mathbb{G}_m])$ is generated as a $\text{CH}^*(\mathcal{B}\mathbb{G}_m)$ -module by 1 and by the hyperplane section h ; hence, the image of c'_* is generated as an ideal by $c'_*(\rho^*t^i \cdot h)$, where t is the generator of $\text{CH}^*(\mathcal{B}\mathbb{G}_m) \simeq \mathbb{Z}[t]$ and it coincides with $c^*\lambda_1$.

We have that $\rho^*(t^i) = \rho^*c^*(\lambda_1^i) = (c')^*(\pi^*\lambda_1^i)$. This implies that $c'_*\rho^*(t^i h) = c'_*(h \cdot (c')^*(\pi^*\lambda_1^i)) = \pi^*\lambda_1^i \cdot c'_*(h)$; hence, the image of c'_* is generated as an ideal by $c'_*(1)$ and $c'_*(h)$.

The computation of $c'_*(1)$ is straightforward, as we have

$$c'_*(1) = c''_*(\pi')^*(1) = \pi^*c_*(1) = 24\lambda_1^2.$$

Above we are using the facts that c''' is birational, and hence $c'''_*(1) = 1$, and that $c: \mathcal{B}\mathbb{G}_m \rightarrow \tilde{\mathcal{M}}_{1,1} \simeq [V_{-4,-6}/\mathbb{G}_m]$ is the zero section of the vector bundle $[V_{-4,-6}/\mathbb{G}_m] \rightarrow \mathcal{B}\mathbb{G}_m$, and therefore its class coincides with the top Chern class of the \mathbb{G}_m -representation $V_{-4,-6}$, which is equal to $(-4\lambda_1)(-6\lambda_1) = 24\lambda_1^2$.

Let $p': \mathcal{B}\mathbb{G}_m \rightarrow [\mathbb{P}^1/\mathbb{G}_m]$ be the universal section given by the first marking, ie the \mathbb{G}_m -quotient of the \mathbb{G}_m -equivariant map $\text{Spec } k \rightarrow \mathbb{P}^1$ corresponding to the point at infinity. Then $c' \circ p' = p \circ c$ and $p'_*(1) = h$, which readily implies $c'_*(h) = p_*(24\lambda_1^2) = 24\lambda_1^2\mu_1$. □

The rational Chow ring of the coarse moduli space $\overline{\mathcal{M}}_{1,2}$ is isomorphic to the rational Chow ring of $\tilde{\mathcal{M}}_{1,2}$ (see [Vistoli 1989, Proposition 6.1]). Thus, from the theorem above, we also get the following:

Corollary 3.7 *Let $\overline{\mathcal{M}}_{1,2}$ be the coarse moduli space of $\tilde{\mathcal{M}}_{1,2}$. Then, over fields of characteristic $\neq 2, 3$, we have*

$$\text{CH}^*(\overline{\mathcal{M}}_{1,2})_{\mathbb{Q}} \simeq \mathbb{Q}[\lambda_1, \mu_1]/(\mu_1(\lambda_1 + \mu_1), \lambda_1^2).$$

4 The Chow ring of $\tilde{\mathcal{C}}_2$

In this section we determine the integral Chow ring of $\tilde{\mathcal{C}}_2$, the universal stable A_2 -curve of genus 2 (Proposition 4.11). Our strategy resembles the one previously used for computing $\text{CH}^*(\tilde{\mathcal{C}}_{1,1})$.

Notice that a stable A_2 -curve of genus 2 cannot have more than one separating node; furthermore, a small deformation of a cusp cannot be a separating node. Hence, the locus of separating nodes is a closed subset $\tilde{\Theta}_2 \subseteq \tilde{\mathcal{C}}_2$, which is a connected component of the singular locus $\tilde{\mathcal{C}}_2^{\text{sing}} \subseteq \tilde{\mathcal{C}}_2$ of the map $\tilde{\mathcal{C}}_2 \rightarrow \tilde{\mathcal{M}}_2$, with the usual scheme structure given by the first Fitting ideal of $\Omega_{\tilde{\mathcal{C}}_2/\tilde{\mathcal{M}}_2}$. With this structure, $\tilde{\Theta}_2$ is a closed smooth connected substack of $\tilde{\mathcal{C}}_2$.

The composite $\tilde{\Theta}_2 \subseteq \tilde{\mathcal{C}}_2 \rightarrow \tilde{\mathcal{M}}_2$ is proper, representable, unramified, and injective on geometric points; hence, it is a closed embedding. We will denote by $\tilde{\Delta}_1 \subseteq \tilde{\mathcal{M}}_2$ its image; it is the closure in $\tilde{\mathcal{M}}_2$ of the divisor $\Delta_1 \subseteq \overline{\mathcal{M}}_2$.

We will denote by $\tilde{\Theta}_1 \subseteq \tilde{\mathcal{C}}_2$ the inverse image of $\tilde{\Delta}_1$; this is an integral divisor, which is smooth outside of $\tilde{\Theta}_2 \subseteq \tilde{\Theta}_1$.

We will use the following cycle classes:

- (1) $\lambda_1 \in \text{CH}^1(\tilde{\mathcal{M}}_2)$ and $\lambda_2 \in \text{CH}^2(\tilde{\mathcal{M}}_2)$ are, as usual, the Chern classes of the Hodge bundle $\tilde{\mathbb{H}}_2$; we will use the same notation for their pullbacks to $\text{CH}^*(\tilde{\mathcal{C}}_2)$.
- (2) $\psi_1 \in \text{CH}^1(\tilde{\mathcal{C}}_2)$ is the first Chern class of the dualizing sheaf $\omega_{\tilde{\mathcal{C}}_2/\tilde{\mathcal{M}}_2}$.
- (3) $\vartheta_1 \in \text{CH}^1(\tilde{\mathcal{C}}_2)$ and $\vartheta_2 \in \text{CH}^2(\tilde{\mathcal{C}}_2)$ are the classes of $\tilde{\Theta}_1$ and $\tilde{\Theta}_2$, respectively.

If $\mathcal{X} \subseteq \tilde{\mathcal{C}}_2$ is any smooth substack, we will use the same symbols for the pullbacks of these classes to $\text{CH}^*(\mathcal{X})$.

Then we compute $\text{CH}^*(\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_1)$ and $\text{CH}^*(\tilde{\Theta}_1 \setminus \tilde{\Theta}_2)$; we put together these descriptions using the patching lemma (Lemma 3.1) to get the Chow ring of $\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_2$.

We repeat this process once more: we compute $\text{CH}^*(\tilde{\Theta}_2)$ and we apply the patching lemma to finally obtain $\text{CH}^*(\tilde{\mathcal{C}}_2)$.

4.1 The Chow ring of $\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_1$

In this subsection we compute the integral Chow ring of $\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_1$, using its description as a quotient stack and basic equivariant intersection theory.

Let B_2 be the Borel subgroup of lower triangular matrices inside GL_2 . We denote by $\mathbb{A}(6)$ the B_2 -representation on degree 6 binary forms, where the action is defined as

$$A \cdot h(x, z) = \det(A)^2 h(A^{-1}(x, z)).$$

Let $\tilde{\mathbb{A}}(6)$ be the vector space formed by pairs $(h(x, z), s)$ such that $h(0, 1) = s^2$. Then $\tilde{\mathbb{A}}(6)$ can be regarded as a B_2 -representation, where the action is given by

$$A \cdot (h(x, z), s) := (\det(A)^2 h(A^{-1}(x, z)), \det(A) a_{22}^{-3} \cdot s).$$

Observe that the natural map $\pi_6: \tilde{\mathbb{A}}(6) \rightarrow \mathbb{A}(6)$ is a B_2 -equivariant ramified double cover.

Let D_4 be the closed subscheme of $\mathbb{A}(6)$ parametrizing homogeneous binary forms of degree 6 with a root of multiplicity greater than 3 in some field extension, and call \tilde{D}_4 the preimage of D_4 in $\tilde{\mathbb{A}}(6)$. Let us set $U := \tilde{\mathbb{A}}(6) \setminus \tilde{D}_4$; this is a B_2 -invariant open subscheme of $\tilde{\mathbb{A}}(6)$.

Consider the two characters $B_2 \rightarrow \mathbb{G}_m$ defined by $(a_{ij}) \mapsto a_{11}$ and $(a_{ij}) \mapsto a_{22}$; let us denote by ξ and $\eta \in \text{CH}^1(\mathcal{B}B_2)$, respectively, the first Chern classes of these two characters, and by the same symbols their pullbacks to $[U/B_2]$.

Furthermore, if V is the tautological rank 2 representation of $B_2 \subseteq GL_2$, and $c_1 \in CH^1(\mathcal{B}B_2)$ and $c_2 \in CH^2(\mathcal{B}B_2)$ its Chern classes, then $\xi + \eta = c_1$ and $\xi\eta = c_2$.

Proposition 4.1 *We have an equivalence of stacks $\tilde{\mathcal{C}}_2 \simeq \tilde{\mathcal{O}}_1 \simeq [U/B_2]$. Furthermore, under this equivalence, the class $\eta \in CH^1[U/B_2]$ corresponds to $\psi_1 \in CH^1(\tilde{\mathcal{C}}_2)$, while $c_i \in CH^i[U/B_2]$ corresponds to $\lambda_i \in CH^*(\tilde{\mathcal{C}}_2)$.*

The proof of the proposition above is essentially contained in [Pernice 2022b]: there, only smooth curves are considered, but the arguments are virtually identical.

As explained in [Pernice 2022b, Remark 3.1], the B_2 -equivariant Chow ring of a scheme X is isomorphic to its T_2 -equivariant one, where $T_2 \subset B_2$ is the maximal torus. Therefore, what we need is an explicit presentation of $CH_{T_2}^*(U)$.

Remark 4.2 It is easy to check that

$$\tilde{\mathcal{M}}_2 \simeq \tilde{\Delta}_1 \simeq [\mathbb{A}(6) \setminus D_4/GL_2].$$

The morphism $U = \tilde{\mathbb{A}}(6) \setminus \tilde{D}_4 \rightarrow \mathbb{A}(6) \setminus D_4$ is B_2 -equivariant and the induced map

$$\pi: \tilde{\mathcal{C}}_2 \simeq \tilde{\mathcal{O}}_1 \simeq [\tilde{\mathbb{A}}(6) \setminus \tilde{D}_4/B_2] \rightarrow [\mathbb{A}(6) \setminus D_4/GL_2] \simeq \tilde{\mathcal{M}}_2 \simeq \tilde{\Delta}_1$$

is in fact the restriction of the morphism $\tilde{\mathcal{C}}_2 \rightarrow \tilde{\mathcal{M}}_2$.

The localization sequence applied to the closed subscheme \tilde{D}_4 shows that all we need to do is to compute the ideal \tilde{I} given by the image of

$$CH_{T_2}^{*-3}(\tilde{D}_4) \rightarrow CH_{T_2}^*(\tilde{\mathbb{A}}(6)),$$

the pushforward along the equivariant closed embedding $\tilde{D}_4 \hookrightarrow \tilde{\mathbb{A}}(6)$.

If we denote by I the ideal generated by the image of the pushforward along the closed embedding $D_4 \hookrightarrow \mathbb{A}(6)$, clearly we have that $\pi_6^*(I) \subset \tilde{I}$, where $\pi_6: \tilde{\mathbb{A}}(6) \rightarrow \mathbb{A}(6)$ is the aforementioned ramified double cover.

We will prove that the ideal generated by $\pi_6^*(I)$ is in fact equal to \tilde{I} . This works essentially in the same way as in [Pernice 2022b, Theorem 5.7], where, instead of $\tilde{\mathbb{A}}(6) \setminus \tilde{D}_4$, there the author considers the complement of the preimage of the discriminant locus.

We can define a \mathbb{G}_m -torsor

$$\tilde{\mathbb{A}}(6) \setminus 0 \rightarrow \mathbb{P}(2^6, 1),$$

where the \mathbb{G}_m -action on $\tilde{\mathbb{A}}(6)$ is described by

$$\lambda \cdot (h, s) = (\lambda^2 h, \lambda s)$$

for every point (h, s) in $\tilde{\mathbb{A}}(6)$ and for every $\lambda \in \mathbb{G}_m$. (Here, as in what follows, we denote by $\mathbb{P}(2^6, 1)$ the weighted projective stack, that is, the stack quotient of $\mathbb{A}^7 \setminus \{0\}$ by the action of \mathbb{G}_m with weights $(2^6, 1)$.) The T_2 -action descends to an action on $\mathbb{P}(2^6, 1)$. We consider the cartesian diagram

$$\begin{array}{ccc} \tilde{\Delta}_4 & \hookrightarrow & \mathbb{P}(2^6, 1) \\ \phi_6|_{\tilde{\Delta}_4} \downarrow & & \downarrow \phi_6 \\ \Delta_4 & \hookrightarrow & \mathbb{P}^6 \end{array}$$

which is the projectivization of

$$\begin{array}{ccc} \tilde{D}_4 & \hookrightarrow & \tilde{\mathbb{A}}(6) \\ \pi_6|_{\tilde{D}_4} \downarrow & & \downarrow \pi_6 \\ D_4 & \hookrightarrow & \mathbb{A}(6) \end{array}$$

The action of \mathbb{G}_m on the top row is the one described above, while the action on bottom row is the usual one.

Recall that if $X \rightarrow Y$ is a \mathbb{G}_m -torsor with Y a quotient stack and X the complement of the zero section in a line bundle \mathcal{L} on Y , then

$$\text{CH}^*(X) \simeq \text{CH}^*(Y)/(c_1(\mathcal{L})).$$

Let I' be the image of the pushforward along the closed embedding $\Delta_4 \hookrightarrow \mathbb{P}^6$ and let \tilde{I}' be the ideal generated by the image of the pushforward along the closed embedding $\tilde{\Delta}_4 \hookrightarrow \mathbb{P}(2^6, 1)$.

Then, from the formula for the Chow ring of \mathbb{G}_m -torsors, we get that, if \tilde{I}' is equal to the ideal generated by $\phi_6^*(I')$, then the same statement holds in the nonprojective case, ie \tilde{I} is equal to the ideal generated by $\pi_6^*(I)$.

A Chow envelope of Δ_4 is given by

$$\rho: \mathbb{P}^1 \times \mathbb{P}^2 \rightarrow \mathbb{P}^6,$$

where $\rho(f, g) = f^4g$ (as usual \mathbb{P}^n must be regarded as the projective space of binary forms of degree n). The morphism ρ is a Chow envelope because of the following fact: if K is a field and $f \in \Delta_4(K)$, then f has a root of multiplicity 4 and the root must be defined over K because f has degree 6.

To find a Chow envelope of $\tilde{\Delta}_4 \subset \mathbb{P}(2^6, 1)$, one could just consider the cartesian diagram

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{a} & \mathbb{P}(2^6, 1) \\ q \downarrow & & \downarrow \phi_6 \\ \mathbb{P}^1 \times \mathbb{P}^2 & \xrightarrow{\rho} & \mathbb{P}^6 \end{array}$$

The morphism a is in fact a Chow envelope. Nevertheless, we do not know how to describe a Chow envelope for \mathcal{P} . To construct a Chow envelope of $\tilde{\Delta}_4 \subset \mathbb{P}(2^6, 1)$, we define the morphism

$$a: \mathbb{P}^1 \times \mathbb{P}(2^2, 1) \rightarrow \mathbb{P}(2^6, 1)$$

as $a([f], [g, s]) = [(f^4g, f(0, 1)^2s)]$ for every $[f] \in \mathbb{P}^1$ and $[g, s] \in \mathbb{P}(2^2, 1)$. A straightforward computation gives us the diagram

$$\begin{array}{ccccc}
 \mathbb{P}^1 \times \mathbb{P}(2^2, 1) & & & & \\
 \searrow^{\alpha} & & a & & \\
 & \mathcal{P} & \xrightarrow{g} & \mathbb{P}(2^6, 1) & \\
 \text{id} \times \phi_2 \searrow & \downarrow q & & \downarrow \phi_6 & \\
 & \mathbb{P}^1 \times \mathbb{P}^2 & \xrightarrow{\rho} & \mathbb{P}^6 &
 \end{array}$$

where the square is cartesian (here $\phi_n: \mathbb{P}(2^n, 1) \rightarrow \mathbb{P}^n$ sends $(a_0 : \dots : a_n)$ to $(a_0 : \dots : a_n^2)$). One would hope that α is a Chow envelope, but this is not the case. In fact, we have to define the morphism $b: \mathbb{P}(2^3) \rightarrow \mathbb{P}(2^6, 1)$ with the formula $b([g]) = [(x_0^4g, 0)]$ for every $[g] \in \mathbb{P}(2^3)$. This induces another commutative diagram

$$\begin{array}{ccccc}
 \mathbb{P}(2^3) & & & & \\
 \searrow^{\beta} & & b & & \\
 & \mathcal{R} & \hookrightarrow \mathcal{P} & \xrightarrow{g} & \mathbb{P}(2^6, 1) \\
 \varphi_3 \searrow & \downarrow p & \downarrow q & & \downarrow \phi_6 \\
 & \mathbb{P}^2 & \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^2 & \xrightarrow{\rho} & \mathbb{P}^6
 \end{array}$$

where the morphism $\varphi_3: \mathbb{P}(2^3) \rightarrow \mathbb{P}^2$ is the natural μ_2 -gerbe and the inclusion $\mathbb{P}^2 \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^2$ is induced by the rational point $[0 : 1] \in \mathbb{P}^1$.

Lemma 4.3 *In the setting above, a and b are proper representable morphisms of algebraic stacks and the morphism $\alpha \amalg \beta: (\mathbb{P}^1 \times \mathbb{P}(2^2, 1)) \amalg \mathbb{P}(2^3) \rightarrow \mathcal{P}$ is a Chow envelope.*

Proof Properness is clear from the properness of the stacks involved. A straightforward computation shows that a is not only representable (being faithful on every point) but is in fact fully faithful restricted to $U_0 \times \mathbb{P}(2^2, 1)$, where U_0 is the affine open subset of \mathbb{P}^1 where $x_0 \neq 0$. In the same way one can prove that b is fully faithful.

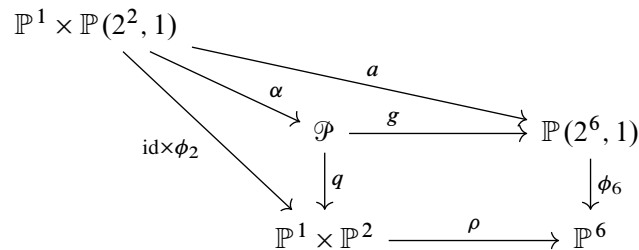
Let us fix a field extension K/k ; we want to prove that $(\alpha \amalg \beta)(K)$ is essentially surjective. Let $([f], [g], [h, s]) \in \mathcal{P}(K)$, ie $h = f^4g$ for some representatives and suppose that $f(0, 1) \neq 0$. Thus, $f(0, 1)^4g(0, 1) = s^2$ and therefore $\alpha(K)([f], [g, s/f(0, 1)^2]) = ([f], [g], [h, s])$. Suppose instead that $f(0, 1) = 0$, ie $h = x_0^4g$ and $s = 0$. Then clearly $b([g]) = [h, 0]$, as desired. \square

Remark 4.4 The previous proof shows us a bit more: it gives us that $\alpha(K)$ is an equivalence on the open $\mathcal{P} \setminus \mathcal{R}$ while $\beta(K)$ is actually an equivalence between $\mathbb{P}(2^3)(K)$ and $\mathcal{R}(K)$ for every K/k field extension. This gives a description of \mathcal{P}_{red} as the fibered product of the two stacks $\mathbb{P}^1 \times \mathbb{P}(2^2, 1)$ and $\mathbb{P}(2^3)$ over $\mathbb{P}(2^2, 1)$.

The previous remark implies $\alpha_*(1) = 1$ and $\beta_*(1) = 1$ at the level of Chow groups.

Proposition 4.5 *The ideal generated by the image of the group homomorphism $a_* \oplus b_*$ in $\text{CH}_{T_2}^*(\mathbb{P}(2^6, 1))$ is equal to the ideal generated by $\phi_6^*(\text{im}(\rho))$.*

Proof First of all, using the projection formula and the explicit description of the Chow rings involved, one can prove that the ideal generated by $a_* \oplus b_*$ is in fact generated by the three cycles $a_*(1)$, $a_*(c_1^{T_2}(\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}(2^2,1)}))$ and $b_*(1)$. For a more detailed discussion about this, see [Pernice 2022b, Section 5]. Consider now the diagram



Therefore, $a_*(1) = g_*\alpha_*(1) = g_*(1) = g_*q^*(1) = \phi_6^*\rho_*(1)$ because the square diagram is cartesian by construction and $\alpha_*(1) = 1$. In the same way we get the result from $b_*(1)$. As far as the last generator is concerned, we get

$$a_*(c_1^{T_2}(\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}(2^2,1)})) = g_*\alpha_*(\text{id} \times \phi_2)^*(c_1^{T_2}(\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2})) = g_*\alpha_*c_1^{T_2}(\alpha^*\mathcal{L}),$$

where \mathcal{L} is the line bundle $q^*(\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2})$. Using the projection formula, we get $\alpha_*c_1^{T_2}(\alpha^*\mathcal{L}) = c_1^{T_2}(\mathcal{L})$ and therefore

$$a_*(c_1^{T_2}(\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}(2^2,1)})) = g_*q^*(c_1^{T_2}(\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2})) = \phi_6^*\rho_*(c_1^{T_2}(\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}(2^2,1)})). \quad \square$$

As a corollary, we finally get the description of $\text{CH}^*(\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_1)$:

Proposition 4.6 $\text{CH}^*(\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_1) \simeq \mathbb{Z}[\psi_1, \lambda_1, \lambda_2]/(\lambda_2 - \psi_1(\lambda_1 - \psi_1), \lambda_1(24\lambda_1^2 - 48\lambda_2), 20\lambda_1^2\lambda_2)$.

Proof The relation $\lambda_2 = \psi_1(\lambda_1 - \psi_1)$ follows from Proposition 4.1 and the computations in the paragraph above it. From Proposition 4.5 we deduce that the image of

$$\text{CH}_{B_2}^{*-3}(\tilde{D}_4) \rightarrow \text{CH}_{B_2}^*(\tilde{\mathbb{A}}(6))$$

is equal to the ideal generated by the pullback of the image of

$$\text{CH}_{GL_2}^{*-3}(D_4) \rightarrow \text{CH}_{GL_2}^*(\mathbb{A}(6)).$$

The latter can be determined with a standard argument of equivariant intersection theory. Consider the projectivization \mathbb{P}^6 of $\mathbb{A}(6)$. If we compute the generators of the image of

$$(4-1) \quad \text{CH}_{GL_2}^{*-3}(\Delta_4) \rightarrow \text{CH}_{GL_2}^*(\mathbb{P}^6),$$

then we are done, because the ideal generated by the pullback to $\text{CH}_{\text{GL}_2}^*(\mathbb{A}(6))$ of this image is easy to compute: we only have to substitute the hyperplane class h with $2\lambda_1$, as in [Vistoli 1998, page 638].

The image of (4-1) is equal to the image of

$$\text{CH}_{\text{GL}_2}^{*-3}(\mathbb{P}^1 \times \mathbb{P}^2) \rightarrow \text{CH}_{\text{GL}_2}^*(\mathbb{P}^6),$$

the pushforward along the proper map $(f, g) \mapsto f^4g$. The latter can be easily computed, either as in [Vistoli 1998, pages 640–643] or via localization formulas. \square

4.2 The Chow ring of $\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_2$

In this subsection our main goal is to compute the integral Chow ring of $\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_2$ (Proposition 4.9) by applying the so-called patching lemma. For this, we are only missing one ingredient, namely an explicit description of the integral Chow ring of $\tilde{\Theta}_1 \setminus \tilde{\Theta}_2$.

Proposition 4.7 $\text{CH}^*(\tilde{\Theta}_1 \setminus \tilde{\Theta}_2) \simeq \mathbb{Z}[\psi_1, \lambda_1, \lambda_2]/(\lambda_2 - \psi_1(\lambda_1 - \psi_1)).$

Proof The objects of the stack $\tilde{\Theta}_1 \setminus \tilde{\Theta}_2$ are of the form $(C \rightarrow S, \sigma)$, where

- $C \rightarrow S$ is a family of stable A_2 -curves of genus 2 whose geometric fibers are étale-locally obtained by gluing two elliptic stable A_2 -curves at their marked points;
- the image of the section $\sigma: S \rightarrow C$ does not contain the separating node in any of the geometric fibers of $C \rightarrow S$.

We can define a morphism

$$(\tilde{\mathcal{C}}_{1,1} \setminus \text{im}(\mathbf{p})) \times \tilde{\mathcal{M}}_{1,1} \rightarrow \tilde{\Theta}_1 \setminus \tilde{\Theta}_2$$

by sending a pair $(C \rightarrow S, p, \sigma), (C' \rightarrow S, p')$ to the family of curves determined by gluing C and C' along p and p' , marked with σ . The morphism misses $\tilde{\Theta}_2$ because $\tilde{\mathcal{C}}_{1,1} \setminus \text{im}(\mathbf{p})$ parametrizes curves in $\tilde{\mathcal{M}}_{1,1}$ together with an extra section which does not coincide with the previous one. This is easily checked to be an isomorphism, with Lemma 2.10.

Lemma 3.2 implies that $(\tilde{\mathcal{C}}_{1,1} \setminus \text{im}(\mathbf{p})) \times \tilde{\mathcal{M}}_{1,1}$ is an affine bundle over $[V_{-2,-3}/\mathbb{G}_m] \times \tilde{\mathcal{M}}_{1,1}$. As $\tilde{\mathcal{M}}_{1,1} \simeq [V_{-4,-6}/\mathbb{G}_m]$, we deduce

$$\text{CH}^*((\tilde{\mathcal{C}}_{1,1} \setminus \text{im}(\mathbf{p})) \times \tilde{\mathcal{M}}_{1,1}) \simeq \text{CH}_{\mathbb{G}_m \times \mathbb{G}_m}^*(V_{-2,-3} \times V_{-4,-6}).$$

This shows that the Chow ring of $(\tilde{\mathcal{C}}_{1,1} \setminus \text{im}(\mathbf{p})) \times \tilde{\mathcal{M}}_{1,1}$ is a ring with two generators, the first coming from $\tilde{\mathcal{C}}_{1,1} \setminus \text{im}(\mathbf{p})$ and the second from $\tilde{\mathcal{M}}_{1,1}$.

A first generator is then the pullback of the first Chern class of the invertible sheaf on $\tilde{\mathcal{C}}_{1,1} \setminus \text{im}(\mathbf{p})$ defined by

$$(C \rightarrow S, p, \sigma) \mapsto \sigma^* \omega_{C/S}.$$

The pullback of the line bundle defined above to $(\tilde{\mathcal{C}}_{1,1} \setminus \text{im}(\mathbf{p})) \times \tilde{\mathcal{M}}_{1,1} \simeq \tilde{\Theta}_1 \setminus \tilde{\Theta}_2$ is isomorphic to the cotangent bundle of the universal section. This implies that a first generator for $\text{CH}^*(\tilde{\Theta}_1 \setminus \tilde{\Theta}_2)$ is ψ_1 , the

first Chern class of the cotangent bundle of the universal section. A second generator is the pullback of the first Chern class of the Hodge line bundle of $\tilde{\mathcal{M}}_{1,1}$, ie

$$(\pi' : C' \rightarrow S, p') \mapsto \pi'_* \omega_{C'/S}.$$

After identifying $(\tilde{\mathcal{C}}_{1,1} \setminus \text{im}(p)) \times \tilde{\mathcal{M}}_{1,1}$ with $\tilde{\Theta}_1 \setminus \tilde{\Theta}_2$, the pullback of the line bundle above gets identified with the line bundle on $\tilde{\Theta}_1 \setminus \tilde{\Theta}_2$ defined by

$$\mathcal{L}: (\pi : C \rightarrow S, \sigma) \mapsto \pi_*(\omega_{C/S}(-\text{im}(\sigma))).$$

Indeed, we have the restriction homomorphism

$$\omega_{C/S}(-\text{im}(\sigma)) \rightarrow \omega_{C/S}(-\text{im}(\sigma))|_{C'} \simeq i_{C'*} \omega_{C'/S}(p) \simeq i_{C'*} \omega_{C'/S}$$

that, when pushed forward along π , gives us the morphism $\mathcal{L}_S \rightarrow \pi'_* \omega_{C'/S}$: for this to be an isomorphism, it is enough to check that it induces an isomorphism of the fibers, ie that, for every geometric point $s \in S$, we have $H^0(C_s, \omega_{C_s}(-\sigma(s))) \simeq H^0(C'_s, \omega_{C'_s})$. Any element in the first group vanishes on the marked component because it vanishes on $\sigma(s)$, from which it follows that the restriction map on the fibers is both injective and surjective; hence, we get the claimed isomorphism $\mathcal{L}_S \simeq \pi'_* \omega_{C'/S}$.

Call s the first Chern class of the line bundle above. Observe that, for every object $(C \rightarrow S, \sigma)$ in $\tilde{\Theta}_1 \setminus \tilde{\Theta}_2$, we have a short exact sequence

$$0 \rightarrow \omega_{C/S}(-\text{im}(\sigma)) \rightarrow \omega_{C/S} \rightarrow \omega_{C/S}|_{\text{im}(\sigma)} \rightarrow 0.$$

A straightforward application of the cohomology and base-change theorem shows that the following sequence, obtained by pushing forward the sequence above along π , is actually a short exact sequence of locally free sheaves:

$$0 \rightarrow \pi_*(\omega_{C/S}(-\text{im}(\sigma))) \rightarrow \pi_* \omega_{C/S} \rightarrow \sigma^* \omega_{C/S} \rightarrow 0.$$

Observe that the vector bundle in the middle is the Hodge vector bundle; the one on the right is the cotangent bundle to the universal section and the one on the left is the line bundle \mathcal{L} that we just defined. This implies that

$$s + \psi_1 = \lambda_1, \quad s\psi_1 = \lambda_2.$$

Putting all together, we deduce that

$$\text{CH}^*(\tilde{\Theta}_1 \setminus \tilde{\Theta}_2) \simeq \mathbb{Z}[\lambda_1, \psi_1]$$

and that $\lambda_2 = \psi_1(\lambda_1 - \psi_1)$. □

Let $\tilde{\Delta}_1$ be the divisor inside $\tilde{\mathcal{M}}_2$ parametrizing curves with one separating node.

Lemma 4.8 $\text{CH}^*(\tilde{\Delta}_1) \simeq \mathbb{Z}[\xi_1, \lambda_1, \lambda_2]/(2\xi_1, \xi_1(\lambda_1 - \xi_1)).$

Proof The objects of the stack $\tilde{\Delta}_1$ are curves $C \rightarrow S$ obtained by gluing two elliptic stable A_2 -curves $(C' \rightarrow S, p')$ and $(C'' \rightarrow S, p'')$ at their marked points. We have an isomorphism

$$\tilde{\Delta}_1 \simeq [(\tilde{\mathcal{M}}_{1,1} \times \tilde{\mathcal{M}}_{1,1})/C_2],$$

where C_2 acts by sending $((C' \rightarrow S, p'), (C'' \rightarrow S, p''))$ to $((C'' \rightarrow S, p''), (C' \rightarrow S, p'))$ (here the quotient stack is in the sense of [Romagny 2005]). If $V_{-4,-6}$ denotes the rank 2 representation of \mathbb{G}_m of weights -4 and -6 , then, from [Edidin and Graham 1998, 5.4], we have

$$(\tilde{\mathcal{M}}_{1,1} \times \tilde{\mathcal{M}}_{1,1})/C_2 \simeq [V_{-4,-6} \times V_{-4,-6}/\mathbb{G}_m^2 \rtimes C_2].$$

The Chow ring of the stack on the left is isomorphic to $CH^*(\mathcal{B}(\mathbb{G}_m^2 \rtimes C_2))$, which can be extracted from the proof of [Di Lorenzo and Vistoli 2021, Proposition 3.1]:

$$CH^*(\mathcal{B}(\mathbb{G}_m^2 \rtimes C_2)) \simeq \mathbb{Z}[\xi_1, c_1, c_2]/(2\xi_1, \xi_1 c_1),$$

where ξ_1 is the first Chern class of the sign representation of C_2 and c_1 and c_2 are the Chern classes of the standard rank 2 representation of $\mathbb{G}_m^2 \rtimes C_2$, ie $V_1 \times V_1$ with C_2 that acts by switching the two factors.

In the proof of [Di Lorenzo and Vistoli 2021, Proposition 3.1], it is also shown that, after identifying $\tilde{\Delta}_1$ with $[V_{-4,-6} \times V_{-4,-6}/\mathbb{G}_m^2 \rtimes C_2]$, we have

$$\lambda_1 = \xi_1 - c_1, \quad \lambda_2 = c_2.$$

Putting all together, we obtain the claimed expression for the Chow ring of $\tilde{\Delta}_1$. □

Proposition 4.9 *We have*

$$CH^*(\tilde{\mathcal{C}}_2 \setminus \tilde{\mathcal{O}}_2) \simeq \mathbb{Z}[\psi_1, \vartheta_1, \lambda_1, \lambda_2]/I,$$

where I is the ideal of relations generated by

$$\lambda_2 - \psi_1(\lambda_1 - \psi_1), \quad (\lambda_1 + \vartheta_1)(24\lambda_1^2 - 48\lambda_2), \quad 20(\lambda_1 + \vartheta_1)\lambda_1\lambda_2, \quad \vartheta_1(\lambda_1 + \vartheta_1).$$

Proof The proof relies on Lemma 3.1. We have already computed $CH^*(\tilde{\mathcal{C}}_2 \setminus \tilde{\mathcal{O}}_1)$ in Proposition 4.6 and $CH^*(\tilde{\mathcal{O}}_1 \setminus \tilde{\mathcal{O}}_2)$ in Proposition 4.7. Moreover, the normal bundle of $\tilde{\mathcal{O}}_1 \setminus \tilde{\mathcal{O}}_2$ in $\tilde{\mathcal{C}}_2 \setminus \tilde{\mathcal{O}}_2$ is equal to the pullback of the normal bundle of $\tilde{\Delta}_1$ in $\tilde{\mathcal{M}}_2$. The first Chern class of the latter is equal to $\xi_1 - \lambda_1$ (see for instance [Larson 2021, Lemma 6.1]), and the pullback morphism

$$CH^*(\tilde{\Delta}_1) \rightarrow CH^*(\tilde{\mathcal{O}}_1) \rightarrow CH^*(\tilde{\mathcal{O}}_1 \setminus \tilde{\mathcal{O}}_2) \simeq \mathbb{Z}[\lambda_1, \psi_1]$$

sends λ_1 to λ_1 and ξ_1 to 0. Therefore, the first Chern class of the normal bundle of $\tilde{\mathcal{O}}_1 \setminus \tilde{\mathcal{O}}_2$ is $-\lambda_1$ and in particular it is not a zero-divisor.

We can apply Lemma 3.1, which tells us that the following diagram of rings is cartesian:

$$(4-2) \quad \begin{array}{ccc} CH^*(\tilde{\mathcal{C}}_2 \setminus \tilde{\mathcal{O}}_2) & \xrightarrow{j^*} & CH^*(\tilde{\mathcal{C}}_2 \setminus \tilde{\mathcal{O}}_1) \\ i^* \downarrow & & \downarrow \\ CH^*(\tilde{\mathcal{O}}_1 \setminus \tilde{\mathcal{O}}_2) & \longrightarrow & CH^*(\tilde{\mathcal{O}}_1 \setminus \tilde{\mathcal{O}}_2)/(\lambda_1) \end{array}$$

We have seen in Proposition 4.6 that $CH^*(\tilde{\mathcal{C}}_2 \setminus \tilde{\mathcal{O}}_1)$ is generated as a ring by ψ_1 , the first Chern class of the cotangent bundle to the universal section, and λ_1 , the first Chern class of the Hodge line bundle.

Moreover, the Chow ring of $\tilde{\Theta}_1 \setminus \tilde{\Theta}_2$ is generated by ψ_1 and λ_1 , intended as the first Chern classes of the restrictions of the aforementioned vector bundles. This implies that $\text{CH}^*(\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_2)$ is generated by λ_1 , ψ_1 and ϑ_1 , the fundamental class of $\tilde{\Theta}_1 \setminus \tilde{\Theta}_2$ in $\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_2$.

Let $f(\lambda_1, \psi_1, \vartheta_1)$ be a relation in $\text{CH}^*(\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_2)$. As it is sent to zero by j^* in (4-2), it must be of the form $f'(\lambda_1, \psi_1) + \vartheta_1 g(\lambda_1, \psi_1, \vartheta_1)$, where f' belongs to the ideal of relations of $\text{CH}^*(\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_1)$. Therefore,

$$0 = i^* f(\lambda_1, \psi_1, \vartheta_1) = f'(\lambda_1, \psi_1) - \lambda_1 g(\lambda_1, \psi_1, -\lambda_1).$$

As λ_1 is not a zero-divisor, we can reconstruct $g(\lambda_1, \psi_1, -\lambda_1)$ from f' by dividing the latter by λ_1 . This implies that

$$f(\lambda_1, \psi_1, \vartheta_1) = f'(\lambda_1, \psi_1) + \vartheta_1 \frac{f'(\lambda_1, \psi_1)}{\lambda_1} + \vartheta_1 f''(\lambda_1, \psi_1, \vartheta_1),$$

where $f''(\lambda_1, \psi_1, \vartheta_1)$ belongs to the kernel of i^* .

Observe now that, after substituting λ_2 with $\psi_1(\lambda_1 - \psi_1)$, the generators of the ideal of relations of $\text{CH}^*(\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_1)$ are

$$\lambda_1(24\lambda_1^2 - 48\psi_1(\lambda_1 - \psi_1)), \quad 20\lambda_1^2\psi_1(\lambda_1 - \psi_1);$$

hence, the generators of the ideal of relations of $\text{CH}^*(\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_2)$ are

$$(\lambda_1 + \vartheta_1)(24\lambda_1^2 - 48\psi_1(\lambda_1 - \psi_1)), \quad 20(\lambda_1 + \vartheta_1)\lambda_1\psi_1(\lambda_1 - \psi_1)$$

together with the generators of $\ker(i^*)$ times ϑ_1 . An element $f''(\lambda_1, \psi_1, \vartheta_1)$ belongs to $\ker(i^*)$ if and only if $f''(\lambda_1, \psi_1, -\lambda_1) = 0$. This means that f'' must be a multiple of $\lambda_1 + \vartheta_1$. Finally, as the exact sequence

$$0 \rightarrow \pi_* (\omega_{C/S}(-\text{im}(\sigma))) \rightarrow \pi_* \omega_{C/S} \rightarrow \sigma^* \omega_{C/S} \rightarrow 0$$

still holds on $\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_2$, also the relation $\lambda_2 = \psi_1(\lambda_1 - \psi_1)$ holds true, thus concluding the computation. \square

4.3 The Chow ring of $\tilde{\mathcal{C}}_2$

To apply again the patching lemma and finally obtain a description of $\text{CH}^*(\tilde{\mathcal{C}}_2)$ we need to compute the integral Chow ring of $\tilde{\Theta}_2$.

Proposition 4.10 $\text{CH}^*(\tilde{\Theta}_2) \simeq \mathbb{Z}[\xi_1, \lambda_1, \lambda_2]/(2\xi_1, \xi_1(\lambda_1 - \xi_1)).$

Proof The map $\tilde{\mathcal{C}}_2 \rightarrow \tilde{\mathcal{M}}_2$ induces an isomorphism $\tilde{\Theta}_2 \rightarrow \tilde{\Delta}_1$. Using Lemma 4.8, we obtain the claimed expression for the Chow ring of $\tilde{\Theta}_2$. \square

Proposition 4.11 We have

$$\text{CH}^*(\tilde{\mathcal{C}}_2) \simeq \mathbb{Z}[\psi_1, \vartheta_1, \lambda_1, \lambda_2, \vartheta_2]/I,$$

where I is the ideal generated by

$$\begin{aligned} &\lambda_2 - \vartheta_2 - \psi_1(\lambda_1 - \psi_1), \quad (\lambda_1 + \vartheta_1)(24\lambda_1^2 - 48\lambda_2), \quad 20(\lambda_1 + \vartheta_1)\lambda_1\lambda_2, \\ &\vartheta_1(\lambda_1 + \vartheta_1), \quad 2\psi_1\vartheta_2, \quad \vartheta_2(\vartheta_1 + \lambda_1 - \psi_1), \quad \psi_1\vartheta_1\vartheta_2. \end{aligned}$$

Proof From the localization exact sequence of Chow groups, we see that $\text{CH}^*(\tilde{\mathcal{C}}_2)$ is generated by the preimages of the generators of $\text{CH}^*(\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_2)$ as a ring (which we computed in Proposition 4.9) together with the pushforward of the generators of $\text{CH}^*(\tilde{\Theta}_2)$ as a \mathbb{Z} -module (which we computed in Proposition 4.10). The former set of generators is given by $\lambda_1, \lambda_2, \psi_1$ and ϑ_1 : the first two cycles are the Chern classes of the Hodge bundle, which is actually defined on the whole $\tilde{\mathcal{C}}_2$; the fourth one is the fundamental class of $\tilde{\Theta}_1$.

The extension of ψ_1 requires some further explanation. Recall that, on $\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_2$, the cycle ψ_1 is the first Chern class of the cotangent bundle of the universal section, that is,

$$(C \rightarrow S, \sigma) \mapsto \sigma^* \omega_{C/S}.$$

The line bundle above can be extended to a line bundle on the whole $\tilde{\mathcal{C}}_2$ as follows: given a family $C \rightarrow S$ with section σ , let $C' \rightarrow S$ be the blowup of C along the separating node, and let σ' be the proper transform of σ , ie the proper transform of its image. Then σ' is still a section of $C' \rightarrow S$, and $(\sigma')^* \omega_{C'/S}$ is a line bundle on S , which obviously coincides with the definition above when $C \rightarrow S$ belongs to $\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_2$. The cycle ψ_1 is the first Chern class of this line bundle that we just defined.

The second part of the set of generators is formed by the fundamental class ϑ_2 of $\tilde{\Theta}_2$ together with the pushforward of the powers of λ_1, λ_2 and ξ_1 . The projection formula tells us that

$$i_* \lambda_1^j \lambda_2^k = \lambda_1^j \lambda_2^k \cdot i_* 1 = \lambda_1^j \lambda_2^k \vartheta_2, \quad i_*(\lambda_1^j \lambda_2^k \xi_1^\ell) = \lambda_1^j \lambda_2^k \cdot i_*(\xi_1^\ell).$$

Moreover, thanks to the relation $\xi_1^2 = \lambda_1 \xi_1$ in $\text{CH}^*(\tilde{\Theta}_2)$, we have

$$i_*(\xi_1^j) = i_*(\xi_1 \lambda_1^{j-1}) = \lambda_1^{j-1} \cdot i_*(\xi_1).$$

Putting all together, we deduce that $\text{CH}^*(\tilde{\mathcal{C}}_2)$ is generated as a ring by

$$\lambda_1, \quad \lambda_2, \quad \psi_1, \quad \vartheta_1, \quad \vartheta_2, \quad \eta_3 := i_* \xi_1.$$

The next step is to find the relations among the generators. For this, we leverage Lemma 3.1 in order to compute the Chow ring of $\tilde{\mathcal{C}}_2$. We need to determine the top Chern class of the normal bundle of $\tilde{\Theta}_2$ in $\tilde{\mathcal{C}}_2$. By [Di Lorenzo and Vistoli 2021, Proposition A.4], we have

$$\mathcal{N}_{\tilde{\Theta}_2}(C' \cup_\sigma C'' \rightarrow S, \sigma) = TC'_\sigma \oplus TC''_\sigma.$$

Recall that

$$\tilde{\Theta}_2 \simeq (\tilde{\mathcal{M}}_{1,1} \times \tilde{\mathcal{M}}_{1,1})/C_2 \simeq [V_{-4,-6} \times V_{-4,-6}/\mathbb{G}_m^2 \rtimes C_2].$$

If $p: \tilde{\mathcal{M}}_{1,1} \rightarrow \tilde{\mathcal{C}}_{1,1}$ denotes the universal section, the conormal bundle of $\text{im}(p)$ is

$$(C \rightarrow S, p) \mapsto TC_p$$

and its total space is isomorphic to $[V_{-4,-6} \times V_1 / \mathbb{G}_m]$. We deduce that the total space of $\mathcal{N}_{\tilde{\Theta}_2}$ is isomorphic to

$$[(V_{-4,-6} \times V_{-4,-6}) \times (V_1 \times V_1) / \mathbb{G}_m^2 \rtimes C_2],$$

where C_2 acts on $V_1 \times V_1$ by switching the two factors. In other terms, the normal bundle is isomorphic to the pullback from $\mathcal{B}(\mathbb{G}_m^2 \rtimes C_2)$ of the rank 2 vector bundle associated with the standard representation of $\mathbb{G}_m^2 \rtimes C_2$, ie the standard representation of GL_2 , regarded as a representation of $\mathbb{G}_m^2 \rtimes C_2$ via the natural map that sends \mathbb{G}_m^2 to the subtorus of diagonal matrices and the generator of C_2 to the transposition. As already said in the proof of Proposition 4.10, the second Chern class of this vector bundle is equal to λ_2 , which is not a zero-divisor in $\text{CH}^*(\tilde{\Theta}_2)$.

To proceed, we need to know how the pullback morphism $i^*: \text{CH}^*(\tilde{\mathcal{C}}_2) \rightarrow \text{CH}^*(\tilde{\Theta}_2)$ works. The first formula below follows from the fact that the forgetful morphism $\tilde{\Theta}_2 \rightarrow \tilde{\Delta}_1$ is an isomorphism, the other ones are a straightforward consequence of the projection formula:

$$(4-3) \quad i^* \lambda_i = \lambda_i, \quad i^* \eta_3 = \lambda_2 \xi_1, \quad i^* \vartheta_2 = \lambda_2.$$

For the pullback of ψ_1 , let $(C \rightarrow S, \sigma)$ belong to $\tilde{\Theta}_2$. Denote by E the exceptional divisor of the blowup of C along the separating node, and consider the family of genus 0 curves $E \rightarrow S$: this family is marked by the proper transform of σ and by a divisor of degree 2 given by the intersection points of E with the other components. Therefore, there exists an étale double cover $f: S' \rightarrow S$ such that $E' = E \times_S S' \simeq \mathbb{P}_{S'}^1$. In particular, $(\sigma')^* \omega_{E'/S'} \simeq \mathcal{O}_{S'}$; hence, $f^*(\psi_1|_S) = 0$ and $2(\psi_1|_S) = f_* f^*(\psi_1|_S) = 0$. Then we must necessarily have

$$(4-4) \quad i^* \psi_1 = \xi_1.$$

To compute $i^* \vartheta_1$, observe that we have a commutative diagram

$$\begin{array}{ccc} \tilde{\Theta}_2 & \xleftarrow{i} & \tilde{\mathcal{C}}_2 \\ \rho \downarrow & & \downarrow \pi \\ \tilde{\Delta}_1 & \xleftarrow{f} & \tilde{\mathcal{M}}_2 \end{array}$$

This implies that

$$i^* \vartheta_1 = i^* \pi^* \delta_1 = \rho^* f^* \delta_1 = \rho^*(\xi_1 - \lambda_1),$$

and, identifying $\text{CH}^*(\tilde{\Theta}_2)$ with $\text{CH}^*(\tilde{\Delta}_1)$ via ρ^* as in Proposition 4.10, we deduce that

$$i^* \vartheta_1 = \xi_1 - \lambda_1.$$

Observe that

$$\eta_3 = i_* \xi_1 = i_* i^* \psi_1 = \psi_1 \vartheta_2;$$

hence, we do not need η_3 in order to have generators of $\text{CH}^*(\tilde{\mathcal{C}}_2)$ as a ring.

We can apply Lemma 3.1, which tells us that the following commutative diagram of rings is cartesian:

$$(4-5) \quad \begin{array}{ccc} \mathrm{CH}^*(\tilde{\mathcal{C}}_2) & \xrightarrow{j^*} & \mathrm{CH}^*(\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_2) \\ i^* \downarrow & & \downarrow \\ \mathrm{CH}^*(\tilde{\Theta}_2) & \longrightarrow & \mathrm{CH}^*(\tilde{\Theta}_2)/(\lambda_2) \end{array}$$

Therefore, the ideal of relations is given by those polynomials in the generators that are sent to zero both in $\mathrm{CH}^*(\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_2)$ and in $\mathrm{CH}^*(\tilde{\Theta}_2)$. The generators of this ideal are therefore of the form $f = f' + \vartheta_2 g$, where, by Proposition 4.9, we know that f' is 0 or one of

$$(4-6) \quad \lambda_2 - \psi_1(\lambda_1 - \psi_1), \quad (\lambda_1 + \vartheta_1)(24\lambda_1^2 - 48\lambda_2), \quad 20(\lambda_1 + \vartheta_1)\lambda_1\lambda_2, \quad \vartheta_1(\lambda_1 + \vartheta_1).$$

Moreover, we have

$$0 = i^* f = i^* f' + \lambda_2(i^* g) \implies i^* g = \frac{i^*(-f')}{\lambda_2}.$$

If f' is one of the last three elements of (4-6) or zero, then $i^* f' = 0$; hence, g belongs to $\ker(i^*)$. If $f' = \lambda_2 - \psi_1(\lambda_1 - \psi_1)$, then $i^* g = (-1)$. Hence, the ideal of relations is generated by $\ker(i^*)$ times ϑ_2 and

$$\lambda_2 - \psi_1(\lambda_1 - \psi_1) - \vartheta_2, \quad (\lambda_1 + \vartheta_1)(24\lambda_1^2 - 48\lambda_2), \quad 20(\lambda_1 + \vartheta_1)\lambda_1\lambda_2, \quad \vartheta_1(\lambda_1 + \vartheta_1).$$

The determination of the generators of $\ker(i^*)$ is straightforward. Multiplying them by ϑ_2 , we get the set of cycles

$$2\psi_1\vartheta_2, \quad \vartheta_2(\vartheta_1 + \lambda_1 - \psi_1), \quad \psi_1\vartheta_1\vartheta_2.$$

Putting all together, we reach the desired conclusion. □

5 The Chow ring of $\bar{\mathcal{M}}_{2,1}$, abstract characterization

The goal of this section is to find a minimum number of generators for the kernel of the restriction map

$$\mathrm{CH}^*(\tilde{\mathcal{C}}_2) \rightarrow \mathrm{CH}^*(\bar{\mathcal{M}}_{2,1}).$$

In Theorem 5.11, we show that, up to relations coming from the integral Chow ring of $\bar{\mathcal{M}}_2$, this kernel is generated by the fundamental classes of two substacks, together with an additional cycle: the first fundamental class, denoted by $\tilde{\mathcal{C}}_2^{\mathcal{C}}$, is the cuspidal singularity locus of $\tilde{\mathcal{C}}_2$; the second one, dubbed $\tilde{\mathcal{C}}_2^{\mathcal{E}}$, is the substack of cuspidal curves of genus 2 with a separating node and a marked rational component, with its reduced scheme structure. The additional cycle comes also from this second locus.

5.1 Relations from the locus of curves with two cusps

Notice that an A_2 -curve of genus 2 cannot have more than two cusps. Let $\tilde{\mathcal{M}}_2^{c,c}$ denote the closed substack of $\tilde{\mathcal{M}}_2$ parametrizing stable A_2 -curves of genus 2 with two cusps, with its reduced scheme structure. In other words, a section of $\tilde{\mathcal{M}}_2^{c,c}(S)$ is a family of stable A_2 -curves $C \rightarrow S$, which is trivial étale-locally around each node. This is easily seen to be a smooth closed substack of codimension 4.

We set

$$\mathcal{X}^2 := \pi^{-1}(\tilde{\mathcal{M}}_2^{c,c}).$$

We think of \mathcal{X}^2 as the stack of pairs $(C \rightarrow S, \sigma)$ where the geometric fibers of $C \rightarrow S$ have two cusps. The pushforward along the closed embedding of \mathcal{X}^2 in $\tilde{\mathcal{C}}_2$ induces

$$\text{CH}_*(\mathcal{X}^2) \rightarrow \text{CH}_*(\tilde{\mathcal{C}}_2).$$

Our goal is to prove the following:

Proposition 5.1 *Let J denote the ideal generated by the pullback along $\pi : \tilde{\mathcal{C}}_2 \rightarrow \tilde{\mathcal{M}}_2$ of the cuspidal relations. Then the image of $\text{CH}_*(\mathcal{X}^2) \rightarrow \text{CH}_*(\tilde{\mathcal{C}}_2)$ is contained in J .*

Let $\bar{\mathcal{M}}_0$ be the stack of genus 0 curves with at most one node. Edidin and Fulghesu [2008, Proposition 6] give the following presentation of $\bar{\mathcal{M}}_0$: Let E be the standard representation of GL_3 , and set

$$V := \text{Sym}^2 E^\vee \otimes \det(E).$$

We can regard V as the vector space of quadratic ternary forms, endowed with a twisted GL_3 -action. If $U \subset V$ is the invariant open subscheme of quadratic forms of rank > 1 , then

$$\bar{\mathcal{M}}_0 \simeq [U/\text{GL}_3].$$

Let $X \subset V \times \mathbb{P}(E)$ be the closed subscheme of the equation

$$\alpha_{00}X^2 + \alpha_{01}XY + \alpha_{02}XZ + \alpha_{11}Y^2 + \alpha_{12}YZ + \alpha_{22}Z^2 = 0.$$

Set $C := X \cap (U \times \mathbb{P}(E^\vee))$, so that $C \rightarrow U$ is a family of genus 0 curve with at most one node. If $\pi : \bar{\mathcal{C}}_0 \rightarrow \bar{\mathcal{M}}_0$ is the universal curve of genus 0, then

$$\bar{\mathcal{C}}_0 \simeq [C/\text{GL}_3].$$

Lemma 5.2 *Let $h := c_1(\mathcal{O}(1))$ be the hyperplane class of $\mathbb{P}(E)$. Then the pullback homomorphism of $\text{CH}^*(\mathcal{B}\text{GL}_3)$ -modules*

$$\text{CH}^*(\mathcal{B}\text{GL}_3)\langle 1, h, h^2 \rangle \simeq \text{CH}^*(\mathbb{P}(E)) \rightarrow \text{CH}^*(\bar{\mathcal{C}}_0)$$

is surjective.

Proof Observe that $X \rightarrow \mathbb{P}(E)$ is an equivariant vector subbundle of $V \times \mathbb{P}(E) \rightarrow \mathbb{P}(E)$, because the equation defining X is linear in the coefficients α_{ij} . Therefore, the induced pullback at the level of equivariant Chow rings (which are the Chow rings of the quotients) is an isomorphism. The pullback along open embeddings is always surjective, and hence so is the pullback along $[X/\text{GL}_3] \rightarrow \bar{\mathcal{C}}_0$. Composing these two homomorphisms, we obtain the claimed surjection: the characterization of $\text{CH}^*(\mathbb{P}(E))$ follows from the usual projective bundle formula. \square

Recall from [Edidin and Fulghesu 2008, Lemma 8] that the sheaf $F := \pi_*(\omega_\pi^\vee)$ is a vector bundle of rank 3 on $\bar{\mathcal{M}}_0$. In particular, we can consider the stack $\mathbb{P}(F)$: its objects are pairs $(C \rightarrow S, D)$ where $C \rightarrow S$ is a family of genus 0 curves with at most one node, and $D \subset C$ is a divisor such that the induced

map $D \rightarrow S$ is finite of degree 2. Moreover, the restriction of D to a singular curve must have degree 1 on each component; in other words, the restriction of D is supported either on two distinct points lying on two distinct components, or on the node.

Consider the cartesian diagram

$$\begin{array}{ccc} \bar{\mathcal{C}}_0 \times_{\bar{\mathcal{M}}_0} \mathbb{P}(F) & \longrightarrow & \mathbb{P}(F) \\ \downarrow & & \downarrow \\ \bar{\mathcal{C}}_0 & \longrightarrow & \bar{\mathcal{M}}_0 \end{array}$$

Then the left vertical map is a projective bundle and we have

$$\mathrm{CH}^*(\bar{\mathcal{C}}_0 \times_{\bar{\mathcal{M}}_0} \mathbb{P}(F)) \simeq \mathrm{CH}^*(\bar{\mathcal{C}}_0)\langle 1, t, t^2 \rangle,$$

where t is the pullback of the hyperplane section of $\mathbb{P}(F)$. Together with Lemma 5.2, this tells us that $\mathrm{CH}^*(\bar{\mathcal{C}}_0 \times_{\bar{\mathcal{M}}_0} \mathbb{P}(F))$ is generated as a $\mathrm{CH}^*(\mathcal{B}\mathrm{GL}_3)$ -module by the pullbacks of $1, h, t, h^2, ht, t^2, h^2t, ht^2$ and h^2t^2 . We have proved the following:

Lemma 5.3 *The Chow ring of $\bar{\mathcal{C}}_0 \times_{\bar{\mathcal{M}}_0} \mathbb{P}(F)$ is generated as a $\mathrm{CH}^*(\mathbb{P}(F))$ -module by the pullbacks of $1, h$ and h^2 .*

Let $\mathbb{P}(F)^{\acute{e}t}$ be the open substack of $\mathbb{P}(F)$ formed by the pairs $(C \rightarrow S, D)$ where $D \rightarrow S$ is étale. The family of genus 0 curves $\bar{\mathcal{C}}_0 \times_{\bar{\mathcal{M}}_0} \mathbb{P}(F)^{\acute{e}t} \rightarrow \mathbb{P}(F)^{\acute{e}t}$ has by construction a divisor \mathcal{D} whose projection onto $\mathbb{P}(F)^{\acute{e}t}$ is étale of degree 2.

We can pinch this family of curves along the divisor \mathcal{D} : the resulting scheme is a family of stable A_2 -curves of genus 2 over $\mathbb{P}(F)^{\acute{e}t}$, with each fiber having two cusps. This in turn induces a map $f: \mathbb{P}(F)^{\acute{e}t} \rightarrow \tilde{\mathcal{M}}_2$ which fits into the commutative diagram

$$(5-1) \quad \begin{array}{ccccc} \bar{\mathcal{C}}_0 \times_{\bar{\mathcal{M}}_0} \mathbb{P}(F)^{\acute{e}t} & \xrightarrow{\mathcal{P}} & f^*\tilde{\mathcal{C}}_2 & \longrightarrow & \tilde{\mathcal{C}}_2 \\ & \searrow & \downarrow & & \downarrow \pi \\ & & \mathbb{P}(F)^{\acute{e}t} & \xrightarrow{f} & \tilde{\mathcal{M}}_2 \end{array}$$

Here \mathcal{P} denotes the pinching morphism, and $f^*\tilde{\mathcal{C}}_2$ is the resulting family of curves. The morphism f sends a pair $(C \rightarrow S, D)$ to the family of curves $\hat{C} \rightarrow S$ obtained by pinching C along the support of D .

Observe that, by construction, the morphism f is an isomorphism onto $\tilde{\mathcal{M}}_2^{c,c}$, the locus of curves with two cusps; hence, $f^*\tilde{\mathcal{C}}_2$ is isomorphic to $\mathcal{L}^2 := \pi^{-1}(\tilde{\mathcal{M}}_2^{c,c})$. As the pinching morphism induces an isomorphism at the level of Chow rings, we deduce the following:

Lemma 5.4 *The induced morphism*

$$\bar{\mathcal{C}}_0 \times_{\bar{\mathcal{M}}_0} \mathbb{P}(F)^{\acute{e}t} \rightarrow \tilde{\mathcal{C}}_2$$

is a Chow envelope for $\mathcal{L}^2 := \pi^(\tilde{\mathcal{M}}_2^{c,c})$.*

By definition, the cycle ψ_1 in $\text{CH}^*(\tilde{\mathcal{C}}_2)$ is the first Chern class of the line bundle

$$(C \rightarrow S, \sigma) \mapsto \sigma^* \omega_{C/S}.$$

Hence, its pullback to $\bar{\mathcal{C}}_0 \times_{\bar{\mathcal{M}}_0} \mathbb{P}(F)^{\text{ét}}$ is the invertible sheaf

$$(C' \rightarrow S, \sigma, D) \mapsto \sigma^* \omega_{C'/S},$$

where C' is the partial normalization of C at the cusps. It is clear that the line bundle above comes from a line bundle \mathcal{L} on $\bar{\mathcal{C}}_0$, which can be characterized as follows: consider the cartesian diagram

$$\begin{array}{ccc} \bar{\mathcal{C}}_0 \times_{\bar{\mathcal{M}}_0} \bar{\mathcal{C}}_0 & \xrightarrow{\text{pr}_2} & \bar{\mathcal{C}}_0 \\ \delta \nearrow \downarrow \text{pr}_1 & & \downarrow \pi \\ \bar{\mathcal{C}}_0 & \xrightarrow{\pi} & \bar{\mathcal{M}}_0 \end{array}$$

where δ is the diagonal embedding. The pair $(\bar{\mathcal{C}}_0 \times_{\bar{\mathcal{M}}_0} \bar{\mathcal{C}}_0 \rightarrow \bar{\mathcal{C}}_0, \delta)$ is the universal marked curve, and the line bundle \mathcal{L} is

$$\delta^* \text{pr}_2^* \omega_\pi = \omega_\pi.$$

Lemma 5.2 tells us that $\text{CH}^*(\bar{\mathcal{C}}_0)$ is generated as a $\text{CH}^*(\bar{\mathcal{M}}_0)$ -module by $1, h$ and h^2 . We claim that $h = -c_1(\mathcal{L})$.

For this, recall that there is a commutative diagram

$$\begin{array}{ccccc} [C/\text{GL}_3] & \hookrightarrow & [U \times \mathbb{P}(E)/\text{GL}_3] & \longrightarrow & [\mathbb{P}(E)/\text{GL}_3] \\ & & \downarrow & & \downarrow \\ & & [U/\text{GL}_3] & \longrightarrow & [\text{Spec } k/\text{GL}_3] \end{array}$$

and that h is obtained by pulling back the hyperplane section of $[\mathbb{P}(E)/\text{GL}_3]$, which coincides with the restriction to $[C/\text{GL}_3]$ of the hyperplane section of $[U \times \mathbb{P}(E)/\text{GL}_3]$, regarded as a projective bundle on $[U/\text{GL}_3]$. In more intrinsic terms, this projective bundle is $\mathbb{P}((\pi_* \omega_\pi)^\vee)$, and the embedding of $\bar{\mathcal{C}}_0$ in $\mathbb{P}(\pi_* \omega_\pi^\vee)$ is induced by the surjection

$$\pi^* \pi_* (\omega_\pi^\vee) \rightarrow \omega_\pi^\vee.$$

Therefore, by the universal property of projective bundles, the restriction of $\mathcal{O}(1)$ to $\bar{\mathcal{C}}_0$ is equal to ω_π^\vee , from which we deduce that $\mathcal{L}^\vee \simeq \mathcal{O}(1)|_{\bar{\mathcal{C}}_0}$. Putting all together, we have proved the following:

Lemma 5.5 *The pullback of ψ_1 along $\bar{\mathcal{C}}_0 \times_{\bar{\mathcal{M}}_0} \mathbb{P}(F)^{\text{ét}} \rightarrow \tilde{\mathcal{C}}_2$ is equal to $-h$.*

We are ready to prove the result stated at the beginning of the subsection.

Proof of Proposition 5.1 It follows from Lemma 5.4 that the image of the pushforward

$$\text{CH}^*(\mathcal{L}^2) \rightarrow \text{CH}^*(\tilde{\mathcal{C}}_2)$$

coincides with the image of

$$\text{CH}^*(\bar{\mathcal{C}}_0 \times_{\bar{\mathcal{M}}_0} \mathbb{P}(F)^{\text{ét}}) \rightarrow \text{CH}^*(\tilde{\mathcal{C}}_2).$$

The diagram (5-1) induces the commutative diagram of Chow groups

$$\begin{array}{ccccc}
 & & & & g_* \\
 & & & & \curvearrowright \\
 & & & & \text{CH}^*(\tilde{\mathcal{C}}_2) \\
 \text{CH}^*(\bar{\mathcal{C}}_0 \times_{\bar{\mathcal{M}}_0} \mathbb{P}(F)^{\acute{e}t}) & \xrightarrow{\cong} & \text{CH}^*(f^*\tilde{\mathcal{C}}_2) & \longrightarrow & \\
 & & \uparrow & & \uparrow \pi^* \\
 & & \text{CH}^*(\mathbb{P}(F)^{\acute{e}t}) & \xrightarrow{f_*} & \text{CH}^*(\tilde{\mathcal{M}}_2)
 \end{array}$$

On one hand, we know from Lemma 5.3 that the Chow ring of $\bar{\mathcal{C}}_0 \times_{\bar{\mathcal{M}}_0} \mathbb{P}(F)^{\acute{e}t}$ is generated as a $\text{CH}^*(\mathbb{P}(F)^{\acute{e}t})$ -module by $1, h, h^2$. On the other hand, Lemma 5.5 says that $g^*\psi_1 = -h$; hence, by the projection formula, the image of g_* is generated as an ideal by $\pi^*(\text{im}(f_*))$.

By construction, $\text{im}(f_*)$ is contained in the ideal of cuspidal relations; hence, its pullback along π is contained in J . □

5.2 Relations from the locus of curves with one cusp

Consider the stack $\bar{\mathcal{M}}_{1,1} \times [\mathbb{A}^1/\mathbb{G}_m]$, where the action of \mathbb{G}_m on \mathbb{A}^1 has weight 1. In Section 2.3, we defined a morphism of stacks

$$c: \tilde{\mathcal{M}}_{1,1} \times [\mathbb{A}^1/\mathbb{G}_m] \rightarrow \tilde{\mathcal{M}}_2$$

that does the following: On one hand, when restricted to the open substack $\tilde{\mathcal{M}}_{1,1} \simeq \tilde{\mathcal{M}}_{1,1} \times [\mathbb{A}^1 \setminus \{0\}/\mathbb{G}_m]$, it sends an elliptic stable A_2 -curve $(C \rightarrow S, p)$ to the stable A_2 -curve of genus 2 $\hat{C} \rightarrow S$ obtained by pinching the section p in C .

On the other hand, we can regard the closed substack $\tilde{\mathcal{M}}_{1,1} \times \mathcal{B}\mathbb{G}_m$ as the stack of pairs

$$((C \rightarrow S, p), (C' \rightarrow S, p')),$$

where $(C \rightarrow S, p)$ is an elliptic stable A_2 -curve and $(C' \rightarrow S, p')$ is a family of elliptic stable A_2 -curves with a cusp. Then the morphism c sends a pair to the stable A_2 -curve of genus 2 obtained by gluing C and C' at the marked points.

Let us recall how to construct this morphism: Let $\text{Bl}(\tilde{\mathcal{C}}_{1,1} \times [\mathbb{A}^1/\mathbb{G}_m])$ be the blowup of $\tilde{\mathcal{C}}_{1,1} \times [\mathbb{A}^1/\mathbb{G}_m]$ along the closed substack

$$p \times 0: \tilde{\mathcal{M}}_{1,1} \times \mathcal{B}\mathbb{G}_m \hookrightarrow \tilde{\mathcal{C}}_{1,1} \times [\mathbb{A}^1/\mathbb{G}_m].$$

The fibers of $\text{Bl}(\tilde{\mathcal{C}}_{1,1} \times [\mathbb{A}^1/\mathbb{G}_m]) \rightarrow \tilde{\mathcal{M}}_{1,1} \times [\mathbb{A}^1/\mathbb{G}_m]$ over the closed substack $\tilde{\mathcal{M}}_{1,1} \times \mathcal{B}\mathbb{G}_m$ have two irreducible components, meeting in one node: the first component is given by a stable A_2 -curve of genus 1; the second is a rational curve.

Moreover, the proper transform of $\text{im}(p \times \text{id})$, once restricted to the fibers over $\tilde{\mathcal{M}}_{1,1} \times \mathcal{B}\mathbb{G}_m$, determines a section whose image lands in the rational curve.

We can apply the pinching construction (Section 2.2) to the curve $\mathrm{Bl}(\tilde{\mathcal{C}}_{1,1} \times [\mathbb{A}^1/\mathbb{G}_m]) \rightarrow \tilde{\mathcal{M}}_{1,1} \times [\mathbb{A}^1/\mathbb{G}_m]$ along the proper transform of $\mathrm{im}(\mathbf{p} \times \mathrm{id})$, so that we get

$$\mathcal{P}: \mathrm{Bl}(\tilde{\mathcal{C}}_{1,1} \times [\mathbb{A}^1/\mathbb{G}_m]) \rightarrow \hat{\mathcal{D}}_{1,1}.$$

By construction, the curve $\hat{\mathcal{D}}_{1,1} \rightarrow \tilde{\mathcal{M}}_{1,1} \times \mathcal{B}\mathbb{G}_m$ is a stable A_2 -curve of genus 2; hence, we obtain the fundamental diagram

$$(5-2) \quad \begin{array}{ccccc} \mathrm{Bl}(\tilde{\mathcal{C}}_{1,1} \times [\mathbb{A}^1/\mathbb{G}_m]) & \xrightarrow{\mathcal{P}} & \hat{\mathcal{D}}_{1,1} \simeq c^* \tilde{\mathcal{C}}_2 & \xrightarrow{c'} & \tilde{\mathcal{C}}_2 \\ \rho \downarrow & \swarrow \mathbf{p} \times \mathrm{id} & \downarrow \pi' & & \downarrow \pi \\ \tilde{\mathcal{C}}_{1,1} \times [\mathbb{A}^1/\mathbb{G}_m] & \xrightarrow{q} & \tilde{\mathcal{M}}_{1,1} \times [\mathbb{A}^1/\mathbb{G}_m] & \xrightarrow{c} & \tilde{\mathcal{M}}_2 \end{array}$$

Remark 5.6 The morphism $c: \tilde{\mathcal{M}}_{1,1} \times [\mathbb{A}^1/\mathbb{G}_m] \rightarrow \tilde{\mathcal{M}}_2$ is proper and surjects onto $\tilde{\mathcal{M}}_2^c$, the substack of stable A_2 -curves with at least a cusp. Moreover, it is injective away from $\tilde{\mathcal{M}}_2^{c,c}$, the substack of curves with two cusps.

Consequently, the morphism $c': c^* \tilde{\mathcal{C}}_2 \rightarrow \tilde{\mathcal{C}}_2$ is proper and surjects onto $\mathcal{X}^1 = \pi^{-1}(\tilde{\mathcal{M}}_2^c)$ and it is injective away from $\mathcal{X}^2 = \pi^{-1}(\tilde{\mathcal{M}}_2^{c,c})$.

Recall that $J \subset \mathrm{CH}^*(\tilde{\mathcal{C}}_2)$ is the ideal generated by the pullback along $\pi: \tilde{\mathcal{C}}_2 \rightarrow \tilde{\mathcal{M}}_2$ of the ideal of cuspidal relations in $\mathrm{CH}^*(\tilde{\mathcal{M}}_2)$.

We also introduced the cycles $[\tilde{\mathcal{C}}_2^c]$ and $[\tilde{\mathcal{C}}_2^E]$: The first one is the fundamental class of the cuspidal locus of the universal curve $\tilde{\mathcal{C}}_2 \rightarrow \tilde{\mathcal{M}}_2$ (not to be confused with the locus of cuspidal curves!). The second one is the fundamental class of the locus of curves with a separating node such that the marked component is a cuspidal curve of geometric genus 0.

Let E be the exceptional divisor of $\mathrm{Bl}(\tilde{\mathcal{C}}_{1,1} \times [\mathbb{A}^1/\mathbb{G}_m])$, so that we have a morphism $\rho: E \rightarrow \tilde{\mathcal{M}}_{1,1} \times \mathcal{B}\mathbb{G}_m$ and a proper morphism $c'': E \rightarrow \tilde{\mathcal{C}}_2$ (there is a little abuse of notation here, as, for the sake of simplicity, we denoted by ρ what should be denoted by $\rho|_E$). Call T the first Chern class of the Hodge line bundle on $\tilde{\mathcal{M}}_{1,1} \times \mathcal{B}\mathbb{G}_m$. Then the main result of this subsection is the following:

Proposition 5.7 *The image of*

$$(5-3) \quad \mathrm{CH}^*(\mathrm{Bl}(\tilde{\mathcal{C}}_{1,1} \times [\mathbb{A}^1/\mathbb{G}_m])) \rightarrow \mathrm{CH}^*(\tilde{\mathcal{C}}_2)$$

is contained in the ideal $(J, [\tilde{\mathcal{C}}_2^c], [\tilde{\mathcal{C}}_2^E], c''_ \rho^* T)$.*

The Chow ring of a blowup is generated by the pullback of the cycles from the base plus the cycles coming from the exceptional divisor. Let $i: E \hookrightarrow \mathrm{Bl}(\tilde{\mathcal{C}}_{1,1} \times [\mathbb{A}^1/\mathbb{G}_m])$ be the exceptional divisor; then the image of (5-3) coincides with the image of

$$(5-4) \quad \mathrm{CH}^*(\tilde{\mathcal{C}}_{1,1} \times [\mathbb{A}^1/\mathbb{G}_m]) \oplus \mathrm{CH}^*(E) \xrightarrow{c'_* \mathcal{P}_* \rho^* + c'_* \mathcal{P}_* i_*} \mathrm{CH}^*(\tilde{\mathcal{C}}_2).$$

Let us separately study the two factors of the homomorphism above.

Proposition 5.8 *The image of $\text{CH}^*(\tilde{\mathcal{C}}_{1,1} \times [\mathbb{A}^1/\mathbb{G}_m]) \rightarrow \text{CH}^*(\tilde{\mathcal{C}}_2)$ is contained in the ideal $(J, [\tilde{\mathcal{C}}_2^c], [\tilde{\mathcal{C}}_2^E])$.*

The same argument of Proposition 3.3 implies that

$$(j \times \text{id})^* q^* : \text{CH}^*(\tilde{\mathcal{M}}_{1,1} \times [\mathbb{A}^1/\mathbb{G}_m]) \rightarrow \text{CH}^*((\tilde{\mathcal{C}}_{1,1} \setminus \text{im}(p)) \times [\mathbb{A}^1/\mathbb{G}_m])$$

is surjective. We deduce that Chow ring of $\tilde{\mathcal{C}}_{1,1} \times [\mathbb{A}^1/\mathbb{G}_m]$ is generated by the cycles coming from $\tilde{\mathcal{M}}_{1,1} \times [\mathbb{A}^1/\mathbb{G}_m]$ together with the ones coming from $\tilde{\mathcal{M}}_{1,1} \times [\mathbb{A}^1/\mathbb{G}_m]$ via

$$(p \times \text{id})_* : \text{CH}^*(\tilde{\mathcal{M}}_{1,1} \times [\mathbb{A}^1/\mathbb{G}_m]) \rightarrow \text{CH}^*(\tilde{\mathcal{C}}_{1,1} \times [\mathbb{A}^1/\mathbb{G}_m]).$$

If a cycle is of the form $q^* \zeta$, the diagram (5-2) tells us that

$$c'_* \mathcal{P}_* \rho^*(q^* \zeta) = c'_*(\pi')^* \zeta = \pi^* c_* \zeta;$$

hence, it is contained in J .

To prove Proposition 5.8, it remains to study the image in $\text{CH}^*(\tilde{\mathcal{C}}_2)$ of the cycles of the form $(p \times \text{id})_*$. This corresponds to the image of the map f in the commutative diagram

$$\begin{array}{ccc} & & \text{CH}^*(\tilde{\mathcal{C}}_2) \\ & \nearrow f & \downarrow \pi \\ \text{CH}^*(\tilde{\mathcal{M}}_{1,1} \times [\mathbb{A}^1/\mathbb{G}_m]) & \xrightarrow{c} & \text{CH}^*(\tilde{\mathcal{M}}_2) \end{array}$$

When restricted to the open substack $[\tilde{\mathcal{M}}_{1,1} \times [\mathbb{A}^1 \setminus \{0\}/\mathbb{G}_m]] \simeq \tilde{\mathcal{M}}_{1,1}$, the map f sends an elliptic stable A_2 -curve $(C \rightarrow S, p)$ to the marked genus 2 curve obtained by creating a cusp along $\text{im}(p)$ and with the marking given by the cusp itself.

Recall that

$$\text{CH}^*(\tilde{\mathcal{M}}_{1,1} \times [\mathbb{A}^1/\mathbb{G}_m]) \simeq \mathbb{Z}[T, S],$$

where T is the dual of the Hodge line bundle on $\tilde{\mathcal{M}}_{1,1}$ and S is the first Chern class of the universal line bundle on $\mathcal{B}\mathbb{G}_m$.

Lemma 5.9 *In the setting above, we have $c^*(\lambda_1) = -S$ and $c^*(\lambda_2) = ST - T^2$, and hence also $f^*(\lambda_1) = -S$ and $f^*(\lambda_2) = ST - T^2$. Moreover, $f^* \psi_1 = T + nS$ for some integer n .*

Proof The pullback along the closed embedding

$$[\{0\}/\mathbb{G}_m] \times [\{0\}/\mathbb{G}_m] \hookrightarrow \tilde{\mathcal{M}}_{1,1} \times [\mathbb{A}^1/\mathbb{G}_m]$$

is an isomorphism of Chow rings; hence, to determine $c^*(\lambda_1)$ and $c^*(\lambda_2)$, we can equivalently compute their restrictions to $[\{0\}/\mathbb{G}_m] \times [\{0\}/\mathbb{G}_m]$.

Consider the composition

$$\{0\} \times \{0\} \rightarrow \{\{0\}/\mathbb{G}_m\} \times \{\{0\}/\mathbb{G}_m\} \hookrightarrow \tilde{\mathcal{M}}_{1,1} \times [\mathbb{A}^1/\mathbb{G}_m] \xrightarrow{c} \tilde{\mathcal{M}}_2,$$

where the first arrow is the universal $\mathbb{G}_m \times \mathbb{G}_m$ -torsor. The pullback of the Hodge bundle on $\tilde{\mathcal{M}}_2$ along this composition, ie the fiber of the Hodge bundle on the closed point $\{0\} \times \{0\} \rightarrow \tilde{\mathcal{M}}_2$, can be regarded as a $\mathbb{G}_m \times \mathbb{G}_m$ -representation of rank 2. The equivariant Chern classes of this representation coincide by construction with $c^*(\lambda_1)$ and $c^*(\lambda_2)$.

Let \hat{C} denote the elliptic stable A_2 -curve represented by the closed point $\{0\} \times \{0\}$; this is the genus 2 curve obtained by gluing together the planar curve $C = \{Y^2Z = X^3\}$ with a projective line pinched in $[1, 0]$. The points that we glue together are, respectively, $[0, 1, 0]$ and $[0, 1]$. What follows is a more detailed construction.

Recall that the family of stable A_2 -curves of genus 2 over $\tilde{\mathcal{M}}_{1,1} \times [\mathbb{A}^1/\mathbb{G}_m]$ inducing the map to $\tilde{\mathcal{M}}_2$ is constructed as follows: Let $\tilde{C}_{1,1} \subset \mathbb{P}(V_{-2,-3,0}) \times V_{-4,-6}$ be the \mathbb{G}_m -invariant family of elliptic curves of equation

$$Y^2Z = X^3 + aXZ^2 + bZ^3,$$

with section given by $p: (a, b) \mapsto ([0, 1, 0], (a, b))$. The quotient stack $[\tilde{C}_{1,1}/\mathbb{G}_m]$ is isomorphic to $\tilde{\mathcal{C}}_{1,1}$, the universal family over $\tilde{\mathcal{M}}_{1,1}$. Consider the blowup of $\tilde{C}_{1,1} \times V_1$ along $p(\mathbb{A}^2) \times \{0\}$, which naturally inherits a $\mathbb{G}_m \times \mathbb{G}_m$ -action. If we pinch this blowup along the proper transform of $p(\mathbb{A}^2) \times \mathbb{A}^1$, we get a family of stable A_2 -curves of genus 2, whose $\mathbb{G}_m \times \mathbb{G}_m$ -quotient is the family on $\tilde{\mathcal{M}}_{1,1} \times [\mathbb{A}^1/\mathbb{G}_m]$ we were looking for.

In particular, the fiber over $\{\{0\}/\mathbb{G}_m\} \times \{\{0\}/\mathbb{G}_m\}$ of this family is given by the $\mathbb{G}_m \times \mathbb{G}_m$ -quotient of $\{Y^2Z = X^3\}$ and the projectivization of the tangent space of $\{[0, 1, 0]\} \times \{0\}$ in $\{Y^2Z = X^3\} \times \mathbb{A}^1$. If u is a coordinate for \mathbb{A}^1 , a basis for this vector space is given by $e_0 = u^\vee$ and $e_1 = (X/Y)^\vee$, on which $\mathbb{G}_m \times \mathbb{G}_m$ acts as

$$(t, s) \cdot (e_0, e_1) = (se_0, te_1),$$

from which it follows that $\mathbb{G}_m \times \mathbb{G}_m$ acts on the projective line as

$$(t, s) \cdot [e_0, e_1] = [se_0, te_1].$$

The fiber of the Hodge bundle on $\{0\} \times \{0\}$ is isomorphic to $H^0(\hat{C}, \omega_{\hat{C}})$. We need to compute the inherited action of $\mathbb{G}_m \times \mathbb{G}_m$ on this vector space. A straightforward computation shows that

$$H^0(\hat{C}, \omega_{\hat{C}}) = H^0(C, \omega_C) \oplus H^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}(2[1, 0]));$$

hence,

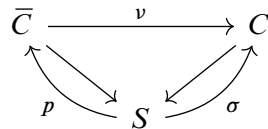
$$\begin{aligned} c_1^{\mathbb{G}_m^2}(H^0(\hat{C}, \omega_{\hat{C}})) &= -T + c_1^{\mathbb{G}_m^2}(H^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}(2[1, 0])), \\ c_2^{\mathbb{G}_m^2}(H^0(\hat{C}, \omega_{\hat{C}})) &= -T \cdot c_1^{\mathbb{G}_m^2}(H^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}(2[1, 0])). \end{aligned}$$

It remains to compute the action of \mathbb{G}_m^2 over $H^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}(2[1, 0]))$. The vector space $H^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}(2[1, 0]))$ is generated by $(x_0/x_1)^2 d(x_1/x_0)$, where $x_0 = u = e_0^\vee$ and $x_1 = X/Y = e_1^\vee$. Therefore, the action is

$$(t, s) \cdot (x_0/x_1)^2 d(x_1/x_0) = s^{-1} t (x_0/x_1)^2 d(x_1/x_0)$$

and the first Chern class of the representation is $T - S$. Putting all together, we get the desired conclusion. We are left with the computation of $f^*\psi_1$.

Let $(C \rightarrow S, \sigma)$ be a stable A_2 -curve of genus 2, with $\text{im}(\sigma)$ contained in the cuspidal locus of $C \rightarrow S$. Suppose moreover that the fibers of C have no separating nodes. Let $\bar{C} \rightarrow S$ be the partial normalization of C along $\text{im}(\sigma)$, and let $p: S \rightarrow \bar{C}$ be the preimage of σ , so that we have



In particular, $\sigma^* \omega_{C/S} \simeq p^* v^* \omega_{C/S} \simeq p^* \omega_{\bar{C}/S}(2p)$, where $\omega_{\bullet/S}$ denotes the dualizing sheaf. As ψ_1 is the first Chern class of the line bundle on $\tilde{\mathcal{C}}_2$ defined as

$$(C \rightarrow S, \sigma) \mapsto \sigma^* \omega_{C/S},$$

we deduce that $f^*\psi_1$, when restricted to the open substack $\tilde{\mathcal{M}}_{1,1} \times [(\mathbb{A}^1 \setminus \{0\})/\mathbb{G}_m] \simeq \tilde{\mathcal{M}}_{1,1}$, is equal to the first Chern class of

$$(\bar{C} \rightarrow S, p) \mapsto p^* \omega_{\bar{C}/S}(2p).$$

This is equal to T , because $p^* \mathcal{O}(-p) \simeq p^* \omega_{\bar{C}/S}$. We have shown in this way that $f^*\psi_1 = T + nS$ for some integer n , as claimed. □

From Lemma 5.9, we deduce that the image of f_* is generated as an ideal in $\text{CH}^*(\tilde{\mathcal{C}}_2)$ by $f_* 1 = [\tilde{\mathcal{C}}_2^c]$: this follows from a straightforward application of the projection formula. In this way we have proved Proposition 5.8.

Proposition 5.10 *Let E be the exceptional divisor of $\text{Bl}(\tilde{\mathcal{C}}_{1,1} \times [\mathbb{A}^1/\mathbb{G}_m])$ and let $\rho: E \rightarrow \tilde{\mathcal{M}}_{1,1} \times \mathcal{B}\mathbb{G}_m$. Then the image of*

$$c_*'' : \text{CH}^*(E) \rightarrow \text{CH}^*(\tilde{\mathcal{C}}_2)$$

is contained in the ideal $(J, [\tilde{\mathcal{C}}_2^c], [\tilde{\mathcal{C}}_2^E], c_''(\rho^* T))$, where T is the first Chern class of the dual of the Hodge line bundle on $\tilde{\mathcal{M}}_{1,1} \times \mathcal{B}\mathbb{G}_m$.*

Proof By construction, the exceptional divisor E is the projectivization of the normal bundle of $\tilde{\mathcal{M}}_{1,1} \times \mathcal{B}\mathbb{G}_m$ in $\tilde{\mathcal{C}}_{1,1} \times [\mathbb{A}^1/\mathbb{G}_m]$; hence, its Chow ring is generated as a $\text{CH}^*(\tilde{\mathcal{M}}_{1,1} \times \mathcal{B}\mathbb{G}_m)$ -module by the powers of the hyperplane class.

There is a commutative diagram

$$\begin{array}{ccccc}
 & & c'' & & \\
 & \curvearrowright & & \curvearrowleft & \\
 E & \xrightarrow{\mathcal{P}} & c^* \tilde{\mathcal{C}}_2 & \xrightarrow{c'} & \tilde{\mathcal{C}}_2 \\
 & \searrow \rho & \downarrow \pi' & & \downarrow \pi \\
 & & \tilde{\mathcal{M}}_{1,1} \times \mathbb{B}\mathbb{G}_m & \xrightarrow{c} & \tilde{\mathcal{M}}_2
 \end{array}$$

which is basically the same as the diagram (5-2) but restricted to $\tilde{\mathcal{M}}_{1,1} \times \mathbb{B}\mathbb{G}_m$. In particular, the square on the right is cartesian.

The pullback of ψ_1 along $c'' = c' \circ \mathcal{P}$ is equal to $nh + m\rho^*\ell$, where h is the hyperplane class and ℓ is the class of a divisor in $\tilde{\mathcal{M}}_{1,1} \times \mathbb{B}\mathbb{G}_m$. We claim that $n = 1$. If so, we can conclude that $\text{im}(c'_*\mathcal{P}_*)$ is contained in the ideal $(J, [\tilde{\mathcal{C}}_2^E], c'_*(\rho^*T))$, where $[\tilde{\mathcal{C}}_2^E] = c''(1)$, as follows: First observe that

$$c''_*(h^i \rho^* \zeta) = c''_*((c''^* \psi_1 - m\rho^* \ell)^i \rho^* \zeta) = \sum_{j=0}^i \psi_1^j \cdot c''_* \rho^*(\eta_j),$$

where η_j is some class in $\text{CH}^*(\tilde{\mathcal{M}}_{1,1} \times \mathbb{B}\mathbb{G}_m)$ for $j = 0, \dots, n$. This shows that the image of c''_* is generated as an ideal by elements of the form $c''_* \rho^* \eta$, where η is in $\text{CH}^*(\tilde{\mathcal{M}}_{1,1} \times \mathbb{B}\mathbb{G}_m)$.

Second, observe that $\text{CH}^*(\tilde{\mathcal{M}}_{1,1} \times \mathbb{B}\mathbb{G}_m)$ is generated as a $\text{CH}^*(\tilde{\mathcal{M}}_2)$ -module by 1 and T : this follows from Lemma 5.9. Therefore, we can write every element $\rho^* \eta$ as

$$\rho^* \eta = (\rho^* T) \cdot (\rho^* c^* \zeta) + (\rho^* c^* \zeta') = (\rho^* T) \cdot (c''^* \pi^* \zeta) + (c''^* \pi^* \zeta').$$

Applying the projection formula, we get

$$c''_*(\rho^* \eta) = (c''_*(\rho^* T)) \cdot \pi^* \zeta + c''_* 1 \cdot \pi^* \zeta';$$

hence, $\text{im}(c''_*)$ is generated as an ideal by $c''_* 1$ and $c''_* \rho^* T$, as claimed.

To check that $n = 1$, first observe that ψ_1 can also be regarded as the first Chern class of the relative dualizing sheaf ω_π of $\pi: \tilde{\mathcal{C}}_2 \rightarrow \tilde{\mathcal{M}}_2$. Pulling back this line bundle to E amounts to the following: first we restrict ω_π to $\tilde{\mathcal{C}}_2^E$ inside $\tilde{\mathcal{C}}_2$, then we pull it back to the relative normalization of $\tilde{\mathcal{C}}_2^E \rightarrow \pi(\tilde{\mathcal{C}}_2^E)$. Finally, we restrict to the irreducible component of the normalization that is equal to E .

The usual formulas for the pullback of the relative dualizing sheaf along a normalization morphism tells us that the pullback of ω_π is equal to $\omega_{E/\tilde{\mathcal{M}}_{1,1} \times \mathbb{B}\mathbb{G}_m}(2\mathbf{q} + \mathbf{r})$, where \mathbf{q} is the preimage in E of the cusp and \mathbf{r} is the preimage of the separating node.

As $\rho: E \rightarrow \tilde{\mathcal{M}}_{1,1} \times \mathbb{B}\mathbb{G}_m$ is a projective bundle with fibers of dimension one, the class of the relative dualizing sheaf is equal to $-2h + \rho^* \ell'$. The class of $\mathcal{O}(2\mathbf{q} + \mathbf{r})$ is equal to $3h + \rho^* \ell''$; hence, the pullback of ψ_1 is equal to $h + \rho^*(\ell' + \ell'')$. □

Putting together Propositions 5.8 and 5.10, we deduce that the image of

$$(5-5) \quad \text{CH}^*(\tilde{\mathcal{C}}_{1,1} \times [\mathbb{A}^1/\mathbb{G}_m]) \oplus \text{CH}^*(E) \xrightarrow{c'_* \mathcal{P}_* \rho^* + c'_* \mathcal{P}_* i_*} \text{CH}^*(\tilde{\mathcal{C}}_2)$$

is contained in the ideal $(J, [\tilde{\mathcal{C}}_2^c], [\tilde{\mathcal{C}}_2^E], c''_* \rho^* T)$. This coincides with the image of the map from the Chow ring of the blowup of $\tilde{\mathcal{C}}_{1,1} \times [\mathbb{A}^1/\mathbb{G}_m]$; hence, we have proved Proposition 5.7.

5.3 Abstract characterization of $\text{CH}^*(\bar{\mathcal{M}}_{2,1})$

The map from the blowup of $\tilde{\mathcal{C}}_{1,1} \times [\mathbb{A}^1/\mathbb{G}_m]$ to $\tilde{\mathcal{C}}_2$ is one-to-one onto the locus of curves with exactly one cusp.

Therefore, the ideal in $\text{CH}^*(\tilde{\mathcal{C}}_2)$ formed by the cycles coming from the locus of cuspidal curves is equal to the sum of the ideal of cycles coming from the locus of curves with two cusps plus the image of (5-5). By Propositions 5.1 and 5.7, both these ideals are contained in $(J, [\tilde{\mathcal{C}}_2^c], [\tilde{\mathcal{C}}_2^E], c''_* \rho^* T)$, thus the latter ideal coincides with the whole ideal of cycles coming from the cuspidal locus.

In this way we have proved the following abstract characterization of the integral Chow ring of $\bar{\mathcal{M}}_{2,1}$:

Theorem 5.11 *Suppose that the ground field has characteristic not 2. Then*

$$\text{CH}^*(\bar{\mathcal{M}}_{2,1}) \simeq \text{CH}^*(\tilde{\mathcal{C}}_2)/(J, [\tilde{\mathcal{C}}_2^c], [\tilde{\mathcal{C}}_2^E], c''_* \rho^* T),$$

where J is the pullback of the ideal of cuspidal relations in $\tilde{\mathcal{M}}_2$.

6 The Chow ring of $\bar{\mathcal{M}}_{2,1}$, concrete computations

In this final section we conclude the computation of $\text{CH}^*(\bar{\mathcal{M}}_{2,1})$ (Theorem 6.11), by determining the last missing pieces: the fundamental class of $\tilde{\mathcal{C}}_2^c$ (Proposition 6.4), and the fundamental class of $\tilde{\mathcal{C}}_2^E$ and $c''_* \rho^* T$ (Proposition 6.10).

6.1 Fundamental class of $\tilde{\mathcal{C}}_2^c$ in $\text{CH}^*(\tilde{\mathcal{C}}_2)$

Recall that $\tilde{\mathcal{C}}_2^c$ is the cuspidal locus of $\tilde{\mathcal{C}}_2$: in other words, the closed substack $\tilde{\mathcal{C}}_2^c$ is the stack of cuspidal, stable A_2 -curves of genus 2 together with a section that lands in the cuspidal locus.

Remark 6.1 The stack $\tilde{\mathcal{C}}_2^c$ is the normalization of $\tilde{\mathcal{M}}_2^c$, the stack of cuspidal stable A_2 -curves of genus 2.

We want to compute the fundamental class of $\tilde{\mathcal{C}}_2^c$. First of all, we determine its restriction to the open stratum $\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_1$.

Lemma 6.2 $[\tilde{\mathcal{C}}_2^c]_{\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_1} = 2\lambda_1 \psi_1 (7\psi_1 - \lambda_1) - 24\psi_1^3 \in \text{CH}^*(\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_1)$.

Proof Recall that $\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_1$ is an open substack of $[\tilde{\mathbb{A}}(6)/\mathbb{B}_2]$: the points in $\tilde{\mathbb{A}}(6)$ can be thought of as pairs (f, s) where f is a binary form of degree 6 and s is a scalar such that $f(0, 1) = s^2$. The action of \mathbb{B}_2 is described in Section 4.1.

Consider the invariant closed subscheme $Z \subset \tilde{\mathbb{A}}(6)$ parametrizing those pairs (f, s) where $(0, 1)$ is a root of multiplicity at least three. The restriction of $[Z/B_2]$ to the appropriate open substack of $[\tilde{\mathbb{A}}(6)/B_2]$ coincides with $\tilde{\mathcal{C}}_2^c$; thus, all we have to do is to compute the B_2 -equivariant fundamental class of Z . Actually, if $\mathbb{G}_m^2 \subset B_2$ is the maximal subtorus of diagonal matrices, we can equivalently compute the \mathbb{G}_m^2 -equivariant class of Z , because the pullback along $[\tilde{\mathbb{A}}(6)/\mathbb{G}_m^2] \rightarrow [\tilde{\mathbb{A}}(6)/B_2]$ induces an isomorphism of Chow rings [Pernice 2022b, Remark 3.1].

If we write a form f as $f = a_0x_0^6 + a_1x_0^5x_1 + \dots + a_6x_1^6$, where $s^2 = a_6$, we get that (f, s) belongs to Z if and only if $s = a_4 = a_5 = 0$. In other words, the subscheme Z is a complete intersection of the three divisors $\{s = 0\}$, $\{a_4 = 0\}$ and $\{a_5 = 0\}$; hence, its fundamental class is the product of the fundamental classes of those three divisors.

Applying the formula for the equivariant fundamental classes of divisors (see [Eddidin and Fulghesu 2009, Lemma 2.4]), we obtain

$$[Z] = -2\psi_1(\lambda_1 - 3\psi_1)(\lambda_1 - 4\psi_1). \quad \square$$

Consider now the restriction of $\tilde{\mathcal{C}}_2^c$ to $\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_2$. Clearly,

$$[\tilde{\mathcal{C}}_2^c]_{\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_2} = 2\lambda_1\psi_1(7\psi_1 - \lambda_1) - 24\psi_1^3 + \vartheta_1 p_2(\lambda_1, \psi_1),$$

where ϑ_1 is the fundamental class of $\tilde{\Theta}_1$ restricted to $\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_2$. and $p_2(\lambda_1, \psi_1)$ is a homogeneous polynomial of degree 2 (notice that p_2 does not depend on ϑ_1 because of the relation $\vartheta_1(\vartheta_1 + \lambda_1)$ in $\text{CH}^*(\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_2)$; see Proposition 4.9).

Lemma 6.3 *The intersection $\tilde{\mathcal{C}}_2^c \cap (\tilde{\Theta}_1 \setminus \tilde{\Theta}_2)$ is transversal and we have*

$$[\tilde{\mathcal{C}}_2^c \cap (\tilde{\Theta}_1 \setminus \tilde{\Theta}_2)] = -24\psi_1^3 \in \text{CH}^*(\tilde{\Theta}_1 \setminus \tilde{\Theta}_2).$$

Furthermore, we get

$$[\tilde{\mathcal{C}}_2^c]_{\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_2} = 2\psi_1(\lambda_1 + \vartheta_1)(7\psi_1 - \lambda_1) - 24\psi_1^3,$$

where the equality holds in $\text{CH}^*(\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_2)$.

Proof Let $\tilde{\mathcal{C}}_{1,1}$ be the universal elliptic stable A_2 -curve, and let $\mathbf{p}: \tilde{\mathcal{M}}_{1,1} \rightarrow \tilde{\mathcal{C}}_{1,1}$ be the universal section. Recall from Proposition 4.7 that $\tilde{\Theta}_1 \setminus \tilde{\Theta}_2$ is isomorphic to the product $(\tilde{\mathcal{C}}_{1,1} \setminus \text{im}(\mathbf{p})) \times \tilde{\mathcal{M}}_{1,1}$, where the isomorphism is given by taking an elliptic stable A_2 -curve with a second marking and gluing it to another elliptic stable A_2 -curve along the first marking.

The fibered product of stacks $\tilde{\mathcal{C}}_2^c \times_{\tilde{\mathcal{C}}_2} (\tilde{\Theta}_1 \setminus \tilde{\Theta}_2)$ is the substack of curves with a separating node and a marked cusp; hence,

$$(\tilde{\mathcal{C}}_2^c \times_{\tilde{\mathcal{C}}_2} (\tilde{\Theta}_1 \setminus \tilde{\Theta}_2))_{\text{red}} \simeq \mathcal{B}\mathbb{G}_m \times \tilde{\mathcal{M}}_{1,1}.$$

From this we deduce that the dimension of the intersection is the expected one and the intersection is proper. Given a geometric point in $\tilde{\mathcal{C}}_2^c \times_{\tilde{\mathcal{C}}_2} (\tilde{\Theta}_1 \setminus \tilde{\Theta}_2)$, it is enough to prove the transversality of the

intersection in a versal neighborhood of the point. Because both $\tilde{\mathcal{C}}_2^c$ and $\tilde{\Theta}_1 \setminus \tilde{\Theta}_2$ are smooth stacks, we just need to prove that the natural morphism of vector spaces

$$(6-1) \quad T_p \tilde{\mathcal{C}}_2^c \oplus T_p(\tilde{\Theta}_1 \setminus \tilde{\Theta}_2) \rightarrow T_p \tilde{\mathcal{C}}_2$$

is surjective for every geometric point p of $\tilde{\mathcal{C}}_2^c \times_{\tilde{\mathcal{C}}_2} (\tilde{\Theta}_1 \setminus \tilde{\Theta}_2)$ (here $T_p(-)$ denotes the tangent space in the versal neighborhood).

Let us recall some information about the tangent space of $\tilde{\mathcal{C}}_2$ at a geometric point $(C, \sigma) \in \tilde{\mathcal{C}}_2(k)$. In a versal neighborhood of (C, σ) , we have

$$(6-2) \quad T_{(C,\sigma)} \tilde{\mathcal{C}}_2 \simeq \text{im}(d\pi) \oplus (\mathfrak{m}_\sigma/\mathfrak{m}_\sigma^2)^\vee,$$

where $\pi: \tilde{\mathcal{C}}_2 \rightarrow \tilde{\mathcal{M}}_2$ is the forgetful morphism and $d\pi: T_{(C,\sigma)} \tilde{\mathcal{C}}_2 \rightarrow T_C \tilde{\mathcal{M}}_2$ is the induced morphism of tangent spaces (in a versal neighborhood).

Recall that if σ is a smooth point of C , then $d\pi$ is surjective and $\dim(\mathfrak{m}_\sigma/\mathfrak{m}_\sigma^2) = 1$. On the other hand, if σ is a node or a cusp, we have that $\dim(T_C \tilde{\mathcal{M}}_2/\text{im}(d\pi)) = 1$ and $\dim(\mathfrak{m}_\sigma/\mathfrak{m}_\sigma^2) = 2$ (the formula (6-2) actually holds true for every double point singularity).

Let $p = (C, \sigma)$ be a geometric point in the intersection. The vector space $T_p \tilde{\mathcal{C}}_2^c$ parametrizes first-order deformations of (C, σ) that preserve both the cusp and the section: its image in $T_p \tilde{\mathcal{C}}_2 \simeq \text{im}(d\pi) \oplus (\mathfrak{m}_\sigma/\mathfrak{m}_\sigma^2)^\vee$ can be regarded as the subspace of $\text{im}(d\pi)$ classifying first-order deformations of C that preserve the cusp, ie the tangent space $T_C \tilde{\mathcal{M}}_2^c$. We also have the image of $T_{(C,\sigma)}(\tilde{\Theta}_1 \setminus \tilde{\Theta}_2)$ in $T_p \tilde{\mathcal{C}}_2$ can be identified with $(\mathfrak{m}_\sigma/\mathfrak{m}_\sigma^2)^\vee \oplus (T_C \tilde{\Delta}_1 \cap \text{im}(d\pi))$. From this we deduce that (6-1) is surjective, because the sum of the vector subspaces $T_C \tilde{\mathcal{M}}_2^c$ (deformations preserving the cusp) and $T_C(\tilde{\Delta}_1 \cap \text{im}(d\pi))$ (deformations preserving the node) generates $\text{im}(d\pi)$.

We have proved that

$$\tilde{\mathcal{C}}_2^c \times_{\tilde{\mathcal{C}}_2} (\tilde{\Theta}_1 \setminus \tilde{\Theta}_2) \simeq \mathcal{B}\mathbb{G}_m \times \tilde{\mathcal{M}}_{1,1},$$

where the latter can be regarded as a closed substack of $(\tilde{\mathcal{C}}_{1,1} \setminus \text{im}(\mathbf{p})) \times \tilde{\mathcal{M}}_{1,1}$. To complete the statement, we only need to compute the class $[\mathcal{B}\mathbb{G}_m]$ inside the Chow ring of $\tilde{\mathcal{C}}_{1,1} \setminus \text{im}(\mathbf{p})$.

Recall that $\tilde{\mathcal{C}}_{1,1} \setminus \text{im}(\mathbf{p}) \simeq [W/\mathbb{G}_m]$, where

$$W = \{((\alpha, \beta), (x, y)) \in \mathbb{A}^2 \times \mathbb{A}^2 \mid y^2 = x^3 + \alpha x + \beta\}$$

is the universal affine Weierstrass curve and the \mathbb{G}_m -action is described by the formula

$$t \cdot (\alpha, \beta, x, y) = (t^{-4}\alpha, t^{-6}\beta, t^{-2}x, t^{-3}y).$$

The inclusion $\mathcal{B}\mathbb{G}_m \subset \tilde{\mathcal{C}}_{1,1} \setminus \text{im}(\mathbf{p})$ corresponds to the closed embedding

$$[(0, 0, 0, 0)/\mathbb{G}_m] \hookrightarrow [W/\mathbb{G}_m]$$

through the identification above. The \mathbb{G}_m -equivariant class of the origin in $\text{CH}^*_{\mathbb{G}_m}(W)$ can be computed as follows: if $\text{pr}_1, \text{pr}_2: W \rightarrow \mathbb{A}^2$ are the projections respectively on the first and on the second factor of $\mathbb{A}^2 \times \mathbb{A}^2$, then

$$(0, 0, 0, 0) = \text{pr}_1^{-1}\{\alpha = 0\} \cap \text{pr}_2^{-1}\{x = y = 0\}.$$

The equivariant class of the subrepresentation $\{\alpha = 0\} \subset \mathbb{A}^2$ is $-4\psi_1$, because \mathbb{G}_m acts on α by multiplication by t^{-4} . With the same argument, we deduce that the class of $\{x = 0\}$ is $-2\psi_1$ and the class of $\{y = 0\}$ is $-3\psi_1$. Putting all together, we conclude that $[\mathcal{B}\mathbb{G}_m] = -24\psi_1^3$.

Finally, write

$$[\tilde{\mathcal{C}}_2^c]_{\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_2} = p_3(\lambda_1, \psi_1) + \vartheta_1 p_2(\lambda_1, \psi_1),$$

where p_3 and p_2 are homogeneous polynomials of degree 3 and 2, respectively. Lemma 6.2 gives us a formula for the restriction of the cycle above to $\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_1$, which implies that $p_3 = 2\lambda_1\psi_1(7\psi_1 - \lambda_1) - 24\psi_1^3$. On the other hand, we have just computed the restriction of $[\tilde{\mathcal{C}}_2^c]$ to $\text{CH}^*(\tilde{\Theta}_1 \setminus \tilde{\Theta}_2)$, which is $-24\psi_1^3$. From Proposition 4.9 we know that the restriction map sends

$$\lambda_i \mapsto \lambda_i, \quad \psi_1 \mapsto \psi_1, \quad \vartheta_1 \mapsto -\lambda_1.$$

Putting these information together, we get $p_2 = 2\psi_1(7\psi_1 - \lambda_1)$, and this finishes the computation. \square

Finally, we get the description of the fundamental class of $\tilde{\mathcal{C}}_2^c$ inside the Chow ring of $\tilde{\mathcal{C}}_2$:

Proposition 6.4 $[\tilde{\mathcal{C}}_2^c] = 2\psi_1(\lambda_1 + \vartheta_1)(7\psi_1 - \lambda_1) - 24\psi_1^3 \in \text{CH}^*(\tilde{\mathcal{C}}_2).$

Proof First observe that the intersection of $\tilde{\mathcal{C}}_2^c$ with $\tilde{\Theta}_2$ is empty; hence, the restriction of the fundamental class of the first stack to the second is zero.

Lemma 6.3 gives an explicit expression for the restriction of $[\tilde{\mathcal{C}}_2^c]$ to $\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_2$, from which we deduce that

$$[\tilde{\mathcal{C}}_2^c] = 2\psi_1(\lambda_1 + \vartheta_1)(7\psi_1 - \lambda_1) - 24\psi_1^3 + \vartheta_2 p_1(\lambda_1, \psi_1) \in \text{CH}^*(\tilde{\mathcal{C}}_2),$$

where p_1 is a homogeneous polynomial of degree 1. Notice that p_1 does not depend on ϑ_1 because of the relation $\vartheta_2(\vartheta_1 - \lambda_1 + \psi_1)$ inside $\text{CH}^*(\tilde{\mathcal{C}}_2)$ (see Proposition 4.11). Pulling everything back to $\tilde{\Theta}_2$ (see formulas (4-3) and (4-4)) and using the explicit presentation of $\text{CH}^*(\tilde{\Theta}_2)$ given in Proposition 4.10, we get the equation

$$0 = [\tilde{\mathcal{C}}_2^c]_{\tilde{\Theta}_2} = \lambda_2 p_1(\lambda_1, \xi_1) \in \text{CH}^*(\tilde{\Theta}_2),$$

which implies $p_1 = 0$. \square

6.2 Relations coming from $\tilde{\mathcal{C}}_2^E$

Recall that $\tilde{\mathcal{C}}_2^E$ is the closed substack of $\tilde{\mathcal{C}}_2$ whose points are families of 1-pointed, almost stable, genus 2 curves (C, σ) with a separating node satisfying the following property: at least one irreducible component of C is cuspidal and the image of σ belongs to a component with a cusp.

Let us recall the following diagram from the proof of Proposition 5.10: Let $\text{Bl}(\tilde{\mathcal{C}}_{1,1} \times [\mathbb{A}^1/\mathbb{G}_m])$ be the blowup of $\tilde{\mathcal{C}}_{1,1} \times [\mathbb{A}^1/\mathbb{G}_m]$ along $\text{im}(\mathbf{p} \times 0) \simeq \tilde{\mathcal{M}}_{1,1} \times \mathcal{B}\mathbb{G}_m$, and let E be the exceptional divisor. We can form a diagram

$$\begin{array}{ccc} E \simeq \mathbb{P}(N_{\mathbf{p} \times 0}) & \xrightarrow{c''} & \tilde{\mathcal{C}}_2 \\ \rho \downarrow & & \downarrow \pi \\ \tilde{\mathcal{M}}_{1,1} \times \mathcal{B}\mathbb{G}_m & \xrightarrow{c} & \tilde{\mathcal{M}}_2 \end{array}$$

We can identify the Chow ring of $\tilde{\mathcal{M}}_{1,1} \times \mathcal{B}\mathbb{G}_m$ with $\mathbb{Z}[T, S]$, where T is the first Chern class of the dual of the Hodge line bundle. The aim of this subsection is to compute $c''_*[\tilde{\mathcal{C}}_2^E]$ and $c''_*(\rho^* T)$.

Remark 6.5 Let $\pi: \tilde{\mathcal{C}}_2 \rightarrow \tilde{\mathcal{M}}_2$ be the forgetful morphism: the preimage $\pi^{-1}(\tilde{\mathcal{M}}_2^c)$ is the substack of 1-pointed stable A_2 -curves of genus 2 with a cusp. This stack is not irreducible, and one irreducible component is given by $\tilde{\mathcal{C}}_2^E$.

It is clear that $\tilde{\mathcal{C}}_2^E$ has codimension 3 inside $\tilde{\mathcal{C}}_2$; therefore, we have the description

$$(6-3) \quad \begin{aligned} [\tilde{\mathcal{C}}_2^E] &= p_3(\lambda_1, \psi_1) + \vartheta_1 p_2(\lambda_1, \psi_1) + \vartheta_2 p_1(\lambda_1, \psi_1) \in \text{CH}^*(\tilde{\mathcal{C}}_2), \\ c''_*(\rho^* T) &= q_4(\lambda_1, \psi_1) + \vartheta_1 q_3(\lambda_1, \psi_1) + \vartheta_2 q_2(\lambda_1, \psi_1) \in \text{CH}^*(\tilde{\mathcal{C}}_2), \end{aligned}$$

where the p_i and q_i are homogeneous polynomials of degree i . As the intersection $(\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_1) \cap \tilde{\mathcal{C}}_2^E$ is empty, we get $p_3 = q_4 = 0$.

We will adopt the following notation: if $\mathcal{V} \subset \mathcal{X} \subset \mathcal{Y}$ are all closed embeddings of quotient stacks, we denote by $[\mathcal{V}]_{\mathcal{X}}$ (respectively $[\mathcal{V}]_{\mathcal{Y}}$) the fundamental class of \mathcal{V} in $\text{CH}^*(\mathcal{X})$ (respectively in $\text{CH}^*(\mathcal{Y})$).

Lemma 6.6 We have

$$\begin{aligned} [\tilde{\mathcal{C}}_2^E]_{\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_2} &= 24\vartheta_1 \psi_1^2 \in \text{CH}^*(\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_2), \\ (c''_* \rho^* T)_{\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_2} &= 24\vartheta_1 \psi_1^2 (\lambda_1 - \psi_1) \in \text{CH}^*(\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_2). \end{aligned}$$

Proof From (6-3), we know that

$$[\tilde{\mathcal{C}}_2^E]_{\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_2} = \vartheta_1 p_2(\lambda_1, \psi_1), \quad (c''_* \rho^* T)_{\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_2} = \vartheta_1 q_3(\lambda_1, \psi_1).$$

We need to determine p_2 and q_3 .

Using the description $\tilde{\Theta}_1 \setminus \tilde{\Theta}_2$ as the product $(\tilde{\mathcal{C}}_{1,1} \setminus \text{im}(\mathbf{p})) \times \tilde{\mathcal{M}}_{1,1}$, we get that $\tilde{\mathcal{C}}_2^E \simeq \mathcal{V} \times \tilde{\mathcal{M}}_{1,1}$, where $\mathcal{V} \subset (\tilde{\mathcal{C}}_{1,1} \setminus \text{im}(\mathbf{p}))$ is the closed substack classifying cuspidal curves.

Recall that $\tilde{\mathcal{C}}_{1,1} \setminus \text{im}(\mathbf{p}) \simeq [W/\mathbb{G}_m]$, where

$$W = \{(\alpha, \beta, x, y) \mid y^2 = x^3 + \alpha x + \beta\} \subset V_{-4,-6} \times V_{-2,-3}.$$

As before, we use the notation $V_{i,j}$ to indicate the rank 2 \mathbb{G}_m -representation of weights i and j .

Hence, $\mathcal{V} \simeq [V/\mathbb{G}_m]$, where $V \subset W$ is the \mathbb{G}_m -invariant closed subscheme defined by

$$V = \{(\alpha, \beta, x, y) \in W \mid \alpha = 0, \beta = 0\},$$

which is a complete intersection inside W . As \mathbb{G}_m acts on α with weight -4 and on β with weight -6 , we deduce that the equivariant class of V is $24\psi_1^2$.

The codimension 2 closed embedding

$$e: \tilde{\mathcal{C}}_2^E \setminus \tilde{\Theta}_2 \hookrightarrow \tilde{\Theta}_1 \setminus \tilde{\Theta}_2$$

is regular; thus, the excess intersection formula gives us the description

$$j^*[\tilde{\mathcal{C}}_2^E]_{\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_2} = c_1(N_j) \cdot [\tilde{\mathcal{C}}_2^E]_{\tilde{\Theta}_1 \setminus \tilde{\Theta}_2} = c_1(N_j) \cdot 24\psi_1^2,$$

where j is the regular closed embedding $\tilde{\Theta}_1 \setminus \tilde{\Theta}_2 \hookrightarrow \tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_2$ and N_j is the associated normal bundle.

Combining this with (6-3) and the fact that $c_1(N_j)$ is not a zero-divisor (see Proposition 4.9), we get that $p_2(\lambda_1, \psi_1) = 24\psi_1^2$. Observe that $c'': E \rightarrow \tilde{\mathcal{C}}_2$, once restricted over $\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_2$, becomes a closed embedding, and we have a commutative diagram

$$\begin{array}{ccc}
 & & c'' \\
 & \curvearrowright & \\
 E|_{\tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_2} & \xrightarrow{e} & \tilde{\Theta}_1 \setminus \tilde{\Theta}_2 \simeq \tilde{\mathcal{M}}_{1,1} \times (\tilde{\mathcal{C}}_{1,1} \setminus \text{im}(\mathbf{p})) \xrightarrow{j} \tilde{\mathcal{C}}_2 \setminus \tilde{\Theta}_2 \\
 \rho \downarrow & & \downarrow \text{pr}_2 \\
 \tilde{\mathcal{M}}_{1,1} \times \mathcal{B}\mathbb{G}_m & \xrightarrow{\text{pr}_1} & \tilde{\mathcal{M}}_{1,1}
 \end{array}$$

In particular, $T \in \text{CH}^*(\tilde{\mathcal{M}}_{1,1} \times \mathcal{B}\mathbb{G}_m)$ is equal to $\text{pr}_1^*(T)$; hence,

$$c''_* \rho^* T = c''_* \rho^* \text{pr}_1^* T = c''_* e^* \text{pr}_2^* T = j_*(e_* 1 \cdot \text{pr}_2^* T)$$

and therefore $j^* c''_*(\rho^* T) = (e_* 1 \cdot \text{pr}_2^* T) \cdot c_1(N_j)$. We have already computed that $e_* 1 = 24\psi_1^2$. In the proof of Proposition 4.7 we showed that $\text{pr}_2^* T = s = \lambda_1 - \psi_1$, from which we conclude that $j^* c''_*(\rho^* T) = c_1(N_j) 24\psi_1^2 (\lambda_1 - \psi_1)$. This immediately implies our conclusion. \square

The restriction of $\tilde{\mathcal{C}}_2^E$ to $\tilde{\Theta}_2$ requires to be handled with care. Consider the cartesian diagram

$$\begin{array}{ccc}
 \tilde{\mathcal{C}}_2^E \cap \tilde{\Theta}_2 & \xleftarrow{i'} & \tilde{\mathcal{C}}_2^E \\
 \downarrow \text{codim 2} & & \downarrow \text{codim 3} \\
 \tilde{\Theta}_2 & \xleftarrow{\text{codim 2}} & \tilde{\mathcal{C}}_2
 \end{array}$$

where i' is a codimension 1 closed embedding. It can be easily showed that i' is not a regular embedding. In fact, it is a regular embedding away from the locus parametrizing curves with two cusps.

The substack of cuspidal curves in $\tilde{\mathcal{C}}_2$ is the image of the pinching morphism (see Section 2.2)

$$\mathcal{P}: \mathbf{Bl}_{\mathrm{im}(\mathbf{p}) \times \mathcal{B}\mathbb{G}_m}(\tilde{\mathcal{C}}_{1,1} \times [\mathbb{A}^1/\mathbb{G}_m]) \rightarrow \tilde{\mathcal{C}}_2,$$

where $\mathbf{p}: \tilde{\mathcal{M}}_{1,1} \rightarrow \tilde{\mathcal{C}}_{1,1}$ is the universal section and $\mathcal{B}\mathbb{G}_m = [\{0\}/\mathbb{G}_m]$. With this picture in mind, $\tilde{\mathcal{C}}_2^E$ can be regarded as the image of the exceptional divisor of the blowup, ie the image of $\mathbb{P}(N_{\mathbf{p} \times 0})$ via the pinching morphism.

The proper transform of $\mathrm{im}(\mathbf{p}) \times [\mathbb{A}^1/\mathbb{G}_m]$ in the blowup induces a section

$$i'': \tilde{\mathcal{M}}_{1,1} \times \mathcal{B}\mathbb{G}_m \hookrightarrow \mathbb{P}(N_{\mathbf{p} \times 0})$$

of the morphism $\mathbb{P}(N_{\mathbf{p} \times 0}) \rightarrow \tilde{\mathcal{M}}_{1,1} \times \mathcal{B}\mathbb{G}_m$.

Lemma 6.7 *The commutative diagram*

$$(6-4) \quad \begin{array}{ccc} \tilde{\mathcal{M}}_{1,1} \times \mathcal{B}\mathbb{G}_m & \xrightarrow{r} & \tilde{\Theta}_2 \\ \downarrow i'' & & \downarrow i \\ \mathbb{P}(N_{\mathbf{p} \times 0}) & \xrightarrow{c''} & \tilde{\mathcal{C}}_2 \end{array}$$

is cartesian and

$$i^*([\tilde{\mathcal{C}}_2^E]) = r_*(r^*c_1(N_i) - c_1(N_{i''})), \quad i^*(c''_*(\rho^*T)) = r_*(T \cdot (r^*c_1(N_i) - c_1(N_{i''}))),$$

where N_i (respectively $N_{i''}$) is the normal bundle associated with the closed regular embedding i (respectively i'').

Proof The cartesianity is clear from the construction of the pinching morphism (see Section 2.2). The compatibility of the Gysin homomorphism with the pushforward tells us that, for any cycle ζ ,

$$i^*c''_*\zeta = r_*(i^!\zeta).$$

We can apply the excess intersection formula, obtaining

$$i^!\zeta = c_1(r^*N_i/N_{i''}) \cdot i''^*\zeta.$$

Applying these formulas to $\zeta = 1, \rho^*T$ together with the fact that $i''^*\rho^* = \mathrm{id}$, we obtain the desired conclusion. □

We are almost ready to compute explicitly the pullback of $[\tilde{\mathcal{C}}_2^E]$ and $c''_*\rho^*T$. To proceed, we need the following technical result:

Lemma 6.8 *Suppose X, Y and Z are three smooth algebraic stacks with closed embeddings $i: X \hookrightarrow Y$ and $j: Y \hookrightarrow Z$ such that $\dim Z = \dim Y + 1 = \dim X + 2$. Let $Z' := \mathrm{Bl}_X Z$ be the blowup of Z along X . Then we have a section $\sigma: X \hookrightarrow E$ of the natural morphism $E \rightarrow X$, where E is the exceptional divisor, and*

$$N_{X|E} = i^*N_{Y|Z} \otimes N_{X|Y}^\vee.$$

Proof By [Fulton 1998, B.6.10], there is a lifting $j': Y \hookrightarrow Z'$ of j , ie we have a diagram

$$\begin{array}{ccc} Y & \xrightarrow{j'} & Z' \\ & \searrow j & \swarrow \pi \\ & & Z \end{array}$$

where both j' and j are regular closed embedding of codimension 1, and $\pi^*(\mathcal{O}_Z(Y)) = \mathcal{O}_{Z'}(Y) \otimes \mathcal{O}_{Z'}(E)$.

If we consider the two cartesian diagrams

$$\begin{array}{ccccc} X = X \times_Z Y & \xrightarrow{\sigma} & E & \longrightarrow & X \\ \downarrow i & & \downarrow & & \downarrow j \circ i \\ Y & \xrightarrow{j'} & Z' & \xrightarrow{\pi} & Z \end{array}$$

we get

$$N_{X|E} = i^* N_{Y|Z'} = i^* j'^* \mathcal{O}_{Z'}(Y) = i^* j'^* \pi^* \mathcal{O}_Z(Y) \otimes i^* j'^* \mathcal{O}_{Z'}(E)^\vee = i^* N_{Y|Z} \otimes N_{X|Y}^\vee. \quad \square$$

Lemma 6.9 *Let $i: \tilde{\Theta}_2 \hookrightarrow \tilde{\mathcal{C}}_2$ be the closed embedding. Then*

$$i^*[\tilde{\mathcal{C}}_2^E] = -24\lambda_1\lambda_2, \quad i^*(c''_* \rho^* T) = 48\lambda_2^2.$$

Proof We want to apply the formula given by Lemma 6.7. For this, we first need to compute $c_1(N_{i''})$ in the Chow ring of $\tilde{\mathcal{M}}_{1,1} \times \mathcal{B}\mathbb{G}_m$ (with the notation of Lemma 6.7). We write

$$\text{CH}^*(\tilde{\mathcal{M}}_{1,1} \times \mathcal{B}\mathbb{G}_m) \simeq \mathbb{Z}[T, S],$$

where T is the dual of the Hodge line bundle on $\tilde{\mathcal{M}}_{1,1}$ and S is the universal line bundle of weight 1.

To compute $c_1(N_{i''})$, we use Lemma 6.8, where the role of X is played by $\tilde{\mathcal{M}}_{1,1} \times \mathcal{B}\mathbb{G}_m$, the role of Y by $\tilde{\mathcal{C}}_{1,1} \times \mathcal{B}\mathbb{G}_m$ and the role of Z by $\tilde{\mathcal{C}}_2 \times [\mathbb{A}^1/\mathbb{G}_m]$.

The normal bundle of $\tilde{\mathcal{C}}_{1,1} \times \mathcal{B}\mathbb{G}_m$ in $\tilde{\mathcal{C}}_2 \times [\mathbb{A}^1/\mathbb{G}_m]$ is the pullback of the normal bundle of $\mathcal{B}\mathbb{G}_m$ in $[\mathbb{A}^1/\mathbb{G}_m]$ whose class is S . The conormal bundle of $\tilde{\mathcal{M}}_{1,1} \times \mathcal{B}\mathbb{G}_m$ in $\tilde{\mathcal{C}}_{1,1} \times \mathcal{B}\mathbb{G}_m$ is the pullback of the conormal bundle of $\tilde{\mathcal{M}}_{1,1}$ embedded in $\tilde{\mathcal{C}}_{1,1}$ via the universal section; this is well known to be isomorphic to the Hodge line bundle, and hence it is equal to $-T$.

This said, applying Lemma 6.8, we get

$$c_1(N_{i''}) = S - T.$$

We computed the normal bundle of $\tilde{\Theta}_2 \hookrightarrow \tilde{\mathcal{C}}_2$ in the proof of Proposition 4.11: this coincided with the Hodge line bundle twisted by the 2-torsion line bundle on $\tilde{\Theta}_2$; hence, its first Chern class is equal to $\xi_1 - \lambda_1$.

Observe that the pullback of λ_1 along the two maps above is equal to $-S$ (see Lemma 5.9); we deduce that $c_1(r^*N_i) = g^*(\xi_1 - \lambda_1) = S$; hence,

$$c_1(r^*N_i) - c_1(N_{i''}) = T.$$

We can apply Lemma 6.7, which tells us

$$i^*[\tilde{\mathcal{C}}_2^E] = r_*(T), \quad i^*(c''_*\rho^*T) = r_*(T^2).$$

We need to compute the pushforward of this element to $\tilde{\Theta}_2$, which we identify with $\tilde{\Delta}_1$ just as we have done in the proof of Proposition 4.10. Observe that there is a natural factorization

$$\tilde{\mathcal{M}}_{1,1} \times \mathcal{B}\mathbb{G}_m \xrightarrow{g} \tilde{\mathcal{M}}_{1,1} \times \tilde{\mathcal{M}}_{1,1} \xrightarrow{f} (\tilde{\mathcal{M}}_{1,1} \times \tilde{\mathcal{M}}_{1,1})/\mathcal{C}_2.$$

Set

$$\text{CH}^*(\tilde{\mathcal{M}}_{1,1} \times \tilde{\mathcal{M}}_{1,1}) = \mathbb{Z}[T, U],$$

where U is the pullback of the dual of the Hodge line bundle from the second copy of $\tilde{\mathcal{M}}_{1,1}$. To compute the pushforward of T , first observe that $g^*T = T$; hence, $g_*(T) = T \cdot g_*(1)$. The element $g_*(1)$ is the pullback of the fundamental class of $\mathcal{B}\mathbb{G}_m$ in the second copy of $\tilde{\mathcal{M}}_{1,1}$, which is equal to $24U^2$. Similarly, we have $g_*(T^2) = 24U^2T^2$.

Hence, we are reduced to computing $f_*(24U^2T)$ and $f_*(24(UT)^2)$. This is equal to the pushforward of these cycles through the map

$$\mathcal{B}(\mathbb{G}_m \times \mathbb{G}_m) \rightarrow \mathcal{B}(\mathbb{G}_m^{\times 2} \rtimes \mathcal{C}_2),$$

which has been explicitly determined in [Larson 2021, Lemma 7.3; Di Lorenzo and Vistoli 2021, Corollary 3.2]. We get

$$f_*(24U^2T) = -24\lambda_2\lambda_1, \quad f_*(24(UT)^2) = 48\lambda_2^2. \quad \square$$

Proposition 6.10 *We have*

$$\begin{aligned} [\tilde{\mathcal{C}}_2^E] &= 24(\psi_1^2\vartheta_1 - \lambda_1\vartheta_2) \in \text{CH}^*(\tilde{\mathcal{C}}_2), \\ c''_*\rho^*T &= 24(\psi_1^2(\lambda_1 - \psi_1)\vartheta_1 + 2\lambda_2\vartheta_2) \in \text{CH}^*(\tilde{\mathcal{C}}_2). \end{aligned}$$

Proof We know from Lemma 6.6 that $[\tilde{\mathcal{C}}_2^E] = 24\vartheta_1\psi_1^2 + \vartheta_2p_1$. This, combined with Lemma 6.9, gives us

$$-24\lambda_1\lambda_2 = i^*[\tilde{\mathcal{C}}_2^E] = 24(\xi_1 - \lambda_1)\xi_1^2 + \lambda_2p_1,$$

from which we deduce that $p_1 = -24\lambda_1$.

From Lemma 6.6, we also know that $c''_*\rho^*T = 24\vartheta_1\psi_1^2(\lambda_1 - \psi_1) + \vartheta_2q_2$. This, combined with Lemma 6.9, gives us

$$48\lambda_2^2 = i^*(c''_*\rho^*T) = \lambda_2q_2,$$

from which we deduce $q_2 = 48\lambda_2$. □

6.3 Final results

From Theorem 5.11, we know that

$$\mathrm{CH}^*(\bar{\mathcal{M}}_{2,1}) \simeq \mathrm{CH}^*(\tilde{\mathcal{C}}_2)/(J, [\tilde{\mathcal{C}}_2^c], [\tilde{\mathcal{C}}_2^E], c''_*\rho^*T),$$

where J is the ideal generated by the pullback of the relations in $\mathrm{CH}^*(\bar{\mathcal{M}}_2)$. From Proposition 4.11, we know an explicit presentation of $\mathrm{CH}^*(\tilde{\mathcal{C}}_2)$ in terms of generators and relations. The fundamental classes of $\tilde{\mathcal{C}}_2^c$ and $\tilde{\mathcal{C}}_2^E$ together with the cycle $c''_*\rho^*T$ have been computed, respectively, in Propositions 6.4 and 6.10. Putting all together, we derive the following presentation of the integral Chow ring of $\bar{\mathcal{M}}_{2,1}$ in terms of generators and relations:

Theorem 6.11 *Suppose that the characteristic of the base field is not 2 or 3. Then*

$$\mathrm{CH}^*(\bar{\mathcal{M}}_{2,1}) \simeq \mathbb{Z}[\lambda_1, \psi_1, \vartheta_1, \lambda_2, \vartheta_2]/(\alpha_{2,1}, \alpha_{2,2}, \alpha_{2,3}, \beta_{3,1}, \beta_{3,2}, \beta_{3,3}, \beta_{3,4}),$$

where the $\alpha_{2,i}$ have degree 2, the $\beta_{3,j}$ have degree 3 and their explicit expressions are

$$\begin{aligned} \alpha_{2,1} &= \lambda_2 - \vartheta_2 - \psi_1(\lambda_1 - \psi_1), & \beta_{3,1} &= 20\lambda_1\lambda_2 - 4\lambda_2\vartheta_1, \\ \alpha_{2,2} &= 24\lambda_1^2 - 48\lambda_2, & \beta_{3,2} &= 2\psi_1\vartheta_2, \\ \alpha_{2,3} &= \vartheta_1(\lambda_1 + \vartheta_1), & \beta_{3,3} &= \vartheta_2(\vartheta_1 + \lambda_1 - \psi_1), \\ & & \beta_{3,4} &= 2\psi_1(\lambda_1 + \vartheta_1)(7\psi_1 - \lambda_1) - 24\psi_1^3. \end{aligned}$$

Proof The generating relations in the theorem are obtained from the relations in the Chow ring of $\tilde{\mathcal{C}}_2$ together with the relations in the Chow ring of $\bar{\mathcal{M}}_2$, the fundamental classes of $\tilde{\mathcal{C}}_2^c$ and $\tilde{\mathcal{C}}_2^E$ and the cycle $c''_*\rho^*T$; a straightforward computation with Macaulay2 shows that all the relations of degree four as well as $[\tilde{\mathcal{C}}_2^E]$ are redundant, thus giving us the list above. \square

We could have avoided including either λ_2 or ϑ_2 among the generators of the ring, but keeping them allowed us to simplify the explicit expressions of the relations.

As a corollary of the theorem above, we get the following description of the rational Chow ring of the coarse moduli space $\bar{\mathcal{M}}_{2,1}$, whose computation over \mathbb{C} has been done by Faber [1988]:

Corollary 6.12 *Suppose that the characteristic of the base field is not 2 or 3. Then*

$$\mathrm{CH}^*(\bar{\mathcal{M}}_{2,1})_{\mathbb{Q}} \simeq \mathbb{Q}[\lambda_1, \psi_1, \vartheta_1]/(\alpha_{2,3}, \beta'_{3,1}, \beta'_{3,2}, \beta'_{3,3}, \beta'_{3,4}),$$

where $\alpha_{2,3}$ has degree 2, the $\beta_{3,j}$ have degree 3 and their explicit expressions are

$$\begin{aligned} \alpha_{2,3} &= \vartheta_1(\lambda_1 + \vartheta_1), & \beta'_{3,1} &= 10\lambda_1^3 - 2\lambda_1^2\vartheta_1, & \beta'_{3,3} &= (\tfrac{1}{2}\lambda_1^2 - \psi_1(\lambda_1 - \psi_1))(\vartheta_1 + \lambda_1 - \psi_1), \\ & & \beta'_{3,2} &= \psi_1\lambda_1^2 - 2\psi^2(\lambda_1 - \psi_1), & \beta'_{3,4} &= 2\psi_1(\lambda_1 + \vartheta_1)(7\psi_1 - \lambda_1) - 24\psi_1^3. \end{aligned}$$

Proof One can express ϑ_2 in terms of the other generators using the relation α_1 , and we obtain $\vartheta_2 = \lambda_2 - \psi_1(\lambda_1 - \psi_1)$. We also have the relation $\frac{1}{24}\alpha_2 = \frac{1}{2}\lambda_1 - \lambda_2$; hence, λ_1, ψ_1 and ϑ_1 are enough to generate the rational Chow ring. The relations β'_j are obtained from β_j by substituting $2\lambda_2$ with λ_1 and ϑ_2 with $\frac{1}{2}\lambda_1^2 - \psi_1(\lambda_1 - \psi_1)$. \square

6.4 Comparison with Faber’s computation

Faber [1988] computed the rational Chow ring of $\overline{M}_{2,1}$ over \mathbb{C} . Using his notation, let $[(c)]_{\mathcal{Q}}$ be the fundamental class of the locus of banana curves with a component of genus 1 and a marked component of genus 0, and let $[(d)]_{\mathcal{Q}}$ be the fundamental class of the locus of curves with two elliptic tails connected by a marked \mathbb{P}^1 .

Denote by δ_0 the fundamental class of marked curves with a nonseparating node, and let δ_1 be the fundamental class of marked curves with a separating node (this would be ϑ_1 in our notation).

Theorem 6.13 (Faber) *Suppose that the ground field has characteristic zero. Then the rational Chow ring of $\overline{M}_{2,1}$ is given by*

$$\text{CH}^*(\overline{M}_{2,1})_{\mathbb{Q}} \simeq \mathbb{Q}[\psi_1, \delta_0, \delta_1]/(\gamma_{2,1}, \gamma_{3,1}, \gamma_{3,2}, \gamma_{3,3}, \gamma_{3,4}),$$

where the $\gamma_{i,j}$ have degree i and they are given by

$$\begin{aligned} \gamma_{2,1} &= (\delta_0 + 12\delta_1)\delta_1, & \gamma_{3,1} &= 3\delta_0^3 + 11\delta_0^2\delta_1, & \gamma_{3,3} &= \psi_1[(c)]_{\mathcal{Q}}, \\ & & \gamma_{3,2} &= \delta_1(\delta_1^2 + 2\delta_1\psi_1 + 4\psi_1^2), & \gamma_{3,4} &= \psi_1[(d)]_{\mathcal{Q}}. \end{aligned}$$

Two of the three generators picked by Faber coincide with the ones chosen by us, namely ψ_1 and $\delta_1 = \vartheta_1$. The last generator δ_0 is well known to be equal to $10\lambda_1 - 2\vartheta_1$. A straightforward computation with Macaulay2 shows that the ring computed by Faber and the one given in Corollary 6.12 are isomorphic.

Regarding the relations, we have $\gamma_{2,1} = 10\alpha_{2,1}$: this follows by simply substituting δ_0 with $10\lambda_1 - 2\vartheta_1$ (and of course δ_1 with ϑ_1).

We also have that $\gamma_{3,4}$ and $\frac{1}{2}\beta'_{3,2}$ coincide. Indeed, we can rewrite $\frac{1}{2}\beta'_{3,2}$ as $\frac{1}{2}\beta_{3,2} = \psi_1\vartheta_2$, and ϑ_2 is, by definition, the fundamental class of the locus of curves with a marked separating node if we regard $\overline{M}_{2,1}$ as the coarse space of the universal curve over \overline{M}_2 . On the other hand, we can also regard $\overline{M}_{2,1}$ as the coarse moduli space of stable marked curves of genus 2; then ϑ_2 coincides precisely with $[(d)]_{\mathcal{Q}}$, and hence $\psi_1\vartheta_2 = \psi_1[(d)]_{\mathcal{Q}}$.

Moreover, $\gamma_{3,3} = \frac{1}{2}\beta'_{3,4}$. Indeed, $\gamma_{3,3} = \psi_1[(c)]_{\mathcal{Q}}$ and one can compute the class of $[(c)]$ explicitly as the class of the locus of curves with a marked nonseparating node, obtaining

$$[(c)]_{\mathcal{Q}} = (\lambda_1 + \vartheta_1)(7\psi_1 - \lambda_1) - 12\psi_1^2,$$

from which the equality above follows.

The last two relations found by Faber can be expressed in terms of our relations as

$$\gamma_{3,1} = 20((\vartheta_1 - 5\lambda_1)\alpha_{2,3} + 15\beta'_{3,1}), \quad \gamma_{3,2} = (2\psi_1 + \vartheta_1 - \lambda_1)\alpha_{2,3} - \frac{1}{12}\beta'_{3,1} + \frac{17}{6}\beta'_{3,2} + \frac{5}{6}\beta'_{3,3} + \frac{1}{6}\beta'_{3,4}.$$

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