Geometry & Topology

Volume 28 (2024)

Thurston's bounded image theorem

CYRIL LECUIRE
KEN'ICHI OHSHIKA

msp

msp

Geometry & Topology **28**:6 (2024) 2971–2999
DOI: 10.2140/gt.2024.28.2971
Published: 21 October 2024

Thurston's bounded image theorem

CYRIL LECUIRE
KEN'ICHI OHSHIKA

Thurston's bounded image theorem is one of the key steps in his proof of the uniformisation theorem for Haken manifolds. Thurston never published its proof, and no proof has previously been known, although a proof of its weaker version, called the bounded orbit theorem, is known. We give a proof of the original bounded image theorem, relying on recent development of Kleinian group theory.

30F40, 57K32

1. Introduction 2971
2. Preliminaries 2974
3. Topological features 2978
4. Convergence, divergence and subsurface projections 2983
5. Unbounded skinning and annuli 2986
6. The proof of Theorem 1.1 2997
References 2998

1 Introduction

From the late 1970s to the early 1980s, Thurston gave lectures on his uniformisation theorem for Haken manifolds [23; 20]. The theorem states that every atoroidal Haken 3–manifold with its (possibly empty) boundary consisting only of incompressible tori admits a complete hyperbolic metric in its interior. His proof is based on an induction making use of a hierarchy for Haken manifolds invented by Waldhausen [24], ie a system of incompressible surfaces cutting the manifold down to balls, together with Maskit's combination theorem; see for instance [10, Section VII].

For simplicity, we now focus on the case of closed atoroidal Haken manifolds. In the last step of the induction, N is a closed atoroidal Haken manifold obtained from a 3–manifold M with nonempty boundary (without torus components) by pasting ∂M to itself by an orientation-reversing involution. The induction hypothesis guarantees the existence of a convex cocompact hyperbolic structure on M. There, Thurston used the so-called bounded image theorem to find a convex compact hyperbolic structure on M,

obtained by quasiconformally deforming the given hyperbolic structure, which can be pasted up along ∂M to give a hyperbolic structure on N.

Let us explain the setting in more detail. Let M be an atoroidal Haken manifold with an even number of boundary components, all of which are incompressible. In the same way as we assumed that N is closed, we assume that no boundary component of M is a torus, for simplicity. Suppose that there is an orientation-reversing involution $\iota\colon \partial M \to \partial M$ taking each component of ∂M to another one. Let N be the closed manifold obtained from M by identifying the points on ∂M with their images under ι. Suppose moreover that N is also atoroidal.

We assume, as the hypothesis of induction, that M admits a convex compact hyperbolic structure; in other words, the interior of M is homeomorphic to $\mathbb{H}^3/\Gamma$ for a convex cocompact Kleinian group Γ. The space of convex compact hyperbolic structures on M, which is nonempty by assumption, modulo isotopy is parametrised by $\mathcal{T}(\partial M)$, as can be seen in the works of Ahlfors, Bers, Kra, Maskit, Marden and Sullivan. From each convex compact hyperbolic structure on M, by taking the covering of M associated with each component S of ∂M, we get a quasi-Fuchsian group isomorphic to $\pi_1(S)$, and by considering the second coordinate of the parametrisation $\mathcal{T}(S)\times\mathcal{T}(\overline{S})$ of the quasi-Fuchsian space, we obtain a map from $\mathcal{T}(\partial M)$ to $\mathcal{T}(\overline{S})$, where $\mathcal{T}(\overline{S})$ denotes the Teichmüller space of S with orientation reversed. By considering this for every component of ∂M, we get a map $\sigma\colon \mathcal{T}(\partial M)\to\mathcal{T}(\bar{\partial} M)$ called the skinning map, where $\mathcal{T}(\bar{\partial} M)$ denotes the product of $\mathcal{T}(\overline{S})$ for the components S of ∂M. Since ι is orientation reversing, it induces a homeomorphism $\iota_*\colon \mathcal{T}(\bar{\partial} M)\to\mathcal{T}(\partial M)$.

Then the bounded image theorem can be stated as follows:

Theorem 1.1 *Suppose that M is a compact (orientable) atoroidal Haken manifold having an even number of boundary components all of which are incompressible and none of which are tori, and assume that M is not homeomorphic to an I–bundle over a closed surface. Assume moreover that M admits a convex compact hyperbolic structure. Suppose that there is an orientation-reversing involution $\iota\colon \partial M\to\partial M$ taking each component of ∂M to another component, and that by pasting each component of ∂M to its image under ι, we get a closed atoroidal manifold N. Then there exists $n\in\mathbb{N}$ depending only on the topological type of M such that the image of $(\iota_*\circ\sigma)^n$ is bounded (precompact) in $\mathcal{T}(\partial M)$.*

There are several expository papers and books on Thurston's uniformisation theorem (see Kapovich [7], Morgan [16] and Otal [18] among others). In all of them, a weaker version of the bounded image theorem called the bounded orbit theorem, which is sufficient for the proof of the uniformisation theorem, was proved and used, instead of this original one.

Up to now, no complete proof of the bounded image theorem as stated above was known. Kent [8] gave a proof of this theorem under the assumption that M is acylindrical, in which case the deformation space of hyperbolic structures on M is compact.

Our purpose here is to give a proof of the original bounded image theorem. Our argument relies on recent progress in Kleinian group theory, in particular the embedding of partial cores in the geometric limit of Brock, Bromberg, Canary and Minsky [4], the relation between the presence of short curves and their relative positions and the behaviour of end invariants of Brock, Bromberg, Canary and Minsky [3], and criteria of convergence/divergence given by Brock, Bromberg, Canary and Lecuire [2].

1.1 Outline

We will find n such that if the image of $(\iota_* \circ \sigma_m)^n$ is unbounded then N contains a nonperipheral incompressible torus, contradicting our assumption. For that purpose we shall use the invariant m introduced in [2].

Given a simple closed curve d on a closed surface S equipped with a hyperbolic metric g, we define

$$m(g, d, \mu) = \max\left\{ \sup_{Y : d \subset \partial Y} d_Y(\mu(g), \mu), \frac{1}{\text{length}_g(d)} \right\},$$

where $\mu(g)$ is a shortest marking for (S, g), μ is a full marking and the supremum of the first term in the maximum is taken over all incompressible subsurfaces Y of S whose boundaries ∂Y contain d; see Definition 4.1 for more details.

It is not difficult to see that in the setting of Theorem 1.1, for a given sequence $\{m_i\}$ in $\mathcal{T}(\partial M)$, if the sequence $\{\sigma(m_i)\}$ is unbounded then there is a simple closed curve d such that $m(\sigma(m_i), d, \mu)$ is unbounded (see Lemma 4.3). The core of our argument consists in showing, with the help of arguments from [2; 3], that in this situation there is a simple closed curve $d' \subset \partial M$ such that $\{m(m_i, d', \mu)\}$ is unbounded and that $d \cup d'$ bounds an essential annulus in M. Using this argument repeatedly, we build (when $\{(\iota_* \circ \sigma)^n(m_i)\}$ is unbounded) an annulus in N which goes through the interior of M (viewed as a subset of N) n times. If n is large enough, this annulus must create an essential torus in N, contradicting the assumption that N is atoroidal.

Although this is the overall logic of the proof, in the following sections we shall present the main steps in a different order. After setting up some preliminary definitions in Section 2 we shall discuss the topological part of the proof in Section 3. First we show that we can add some assumptions on the topology of M which will simplify the arguments later on. Next, we study incompressible surfaces on ∂M which can be extended multiple times through the characteristic submanifold of M when it is viewed as a submanifold of N. This will give us an integer n which appears in Theorem 1.1. In Section 4 we shall discuss the relation between the behaviour of the invariant m defined above and the convergence and divergence of Kleinian groups. In Section 5 we shall prove our key proposition, and obtain the curve d' mentioned above. Finally in Section 6 we shall put these pieces together to prove our main theorem.

Acknowledgements We would like to express our gratitude to the referee for valuable suggestions, which have made it possible to improve our exposition.

2 Preliminaries

2.1 Haken manifolds and characteristic submanifolds

An orientable irreducible compact 3–manifold which contains a nonperipheral incompressible surface is called a *Haken* manifold. We note that a compact irreducible 3–manifold with nonempty boundary is always Haken, except for a 3–ball. We say that a Haken manifold is *atoroidal* when it does not contain a nonperipheral incompressible torus, and *acylindrical* when it does not contain a nonperipheral incompressible annulus. By the torus theorem for Haken manifolds [25; 5; 6], the former condition of the atoroidality is equivalent to the one that every monomorphism from $\mathbb{Z}\times\mathbb{Z}$ into the fundamental group is peripheral, ie is conjugate to the image of the fundamental group of a boundary component.

The Jaco–Shalen–Johannson theory [5; 6] tells us that in a Haken manifold, incompressible tori and incompressible annuli can stay only in a very restricted place. Let us state what the theory says in the case when a Haken manifold M is atoroidal and boundary-irreducible, the latter of which means that ∂M is incompressible.

For an orientable atoroidal Haken boundary-irreducible 3–manifold M, there exists a 3–submanifold X of M, each of whose components is one of the following:

(a) an I–bundle over a surface with negative Euler characteristic whose associated ∂I–bundle coincides with its intersection with ∂M, called a *characteristic I–pair*;

(b) a solid torus Ξ such that $\Xi\cap\partial M$ consists of annuli which are incompressible on both $\partial\Xi$ and ∂M (when $\Xi\cap\partial M$ is connected, it winds around the core curve of Ξ more than once);

(c) a thickened torus $S^1\times S^1\times I$ at least one of whose boundary components lies on a component of ∂M.

The submanifold X satisfies the following condition: every properly embedded essential annulus (an incompressible annulus which is not homotopic into the boundary) is properly isotopic into X, and no component of X is properly isotopic into another component.

This X is unique up to isotopy, and is called the *characteristic submanifold* of M. We note that in the case when M has no torus boundary component, which is the assumption of our main theorem, a component of type (c) does not appear.

We need to consider characteristic submanifolds in a slightly general setting, for pared manifolds. We shall first give a definition of pared manifold.

Definition 2.1 A *pared manifold* is a pair (M, P), where M is a boundary-irreducible Haken 3–manifold, and P is a union of incompressible tori and annuli on ∂M, with the following properties:

(1) Every π_1–injective map from a torus $f: T\to M$ is homotopic to a map into a component of P.

(2) Every π_1–injective map from an annulus $g: A\to M$ with $g(\partial A)\subset P$ is properly homotopic to a map whose image is contained in P.

The subsurface P above is called the *paring locus*.

Let (M, P) be a pared manifold. There exists a submanifold X of M, disjoint from P, each of whose components is of type either (a) or (b) listed in the definition of characteristic manifolds above, and which satisfies the following conditions:

(i) No component of X is a solid torus properly homotopic into P.

(ii) No component of X is properly homotopic into another component of X.

(iii) Every properly embedded essential annulus A in $M \setminus P$ that is not properly homotopic in M into P can be properly homotoped into X.

Such a submanifold is unique up to proper isotopy, and is called the characteristic submanifold of (M, P).

Thurston's celebrated uniformisation theorem for Haken manifolds says that every atoroidal Haken manifold whose boundary consists of incompressible tori admits a hyperbolic structure of finite volume. More generally, he proved that every atoroidal Haken manifold, including the case when it has nontorus boundary components, admits a (minimally parabolic) convex hyperbolic structure of finite volume. The term "convex hyperbolic structure" will be explained in the following subsection.

2.2 Kleinian groups and their deformation spaces

A Kleinian group is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$. We always assume Kleinian groups to be torsion free, and finitely generated except for the case when we talk about geometric limits. For a Kleinian group Γ, we can consider the complete hyperbolic 3–manifold $\mathbb{H}^3/\Gamma$. The *convex core* of $\mathbb{H}^3/\Gamma$ is the smallest convex submanifold that is a deformation retract. The Kleinian group Γ and the corresponding hyperbolic 3–manifold $\mathbb{H}^3/\Gamma$ are said to be *geometrically finite* when the convex core of $\mathbb{H}^3/\Gamma$ has finite volume. In particular, $\mathbb{H}^3/\Gamma$ is said to be *convex compact*, and Γ to be *convex cocompact*, if the convex core is compact. We also say that Γ is *minimally parabolic* when every parabolic element in Γ is contained in a rank-2 parabolic subgroup. Any convex cocompact Kleinian group is automatically minimally parabolic since it does not have parabolic elements.

A 3–manifold M is said to have a hyperbolic structure when $\operatorname{Int} M$ is homeomorphic to $\mathbb{H}^3/\Gamma$ for a Kleinian group Γ, and we regard the pullback of the hyperbolic metric to $\operatorname{Int} M$ as a hyperbolic structure on M. In particular if Γ is taken to be geometrically finite or convex cocompact, we say that M has a geometrically finite or convex compact hyperbolic structure. If M admits a hyperbolic structure, then M must be atoroidal.

The set of hyperbolic structures on M modulo isotopy, which we denote by $\mathrm{AH}(M)$, can be identified with a subset of the set of faithful discrete representations of $\pi_1(M)$ into $\mathrm{PSL}_2(\mathbb{C})$ modulo conjugacy. We put on $\mathrm{AH}(M)$ a topology induced from the weak topology on the representation space. We regard an element of $\mathrm{AH}(M)$ both as a hyperbolic structure on M and as a representation of $\pi_1(M)$ into $\mathrm{PSL}_2(\mathbb{C})$, depending on the situation.

A Kleinian group G is said to be a *quasiconformal deformation* of another Kleinian group Γ if there is a quasiconformal homeomorphism $f:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ such that $G=f\Gamma f^{-1}$ as Möbius transformations on $\widehat{\mathbb{C}}$. When G is a quasiconformal deformation of Γ, there is a diffeomorphism from $\mathbb{H}^3/\Gamma$ to $\mathbb{H}^3/G$ preserving the parabolicity in both directions, which induces an isomorphism between the fundamental groups coinciding with the isomorphism given by the conjugacy $G=f\Gamma f^{-1}$. We note that a quasiconformal deformation of a geometrically finite (resp. convex cocompact, minimally parabolic geometrically finite) group is again geometrically finite (resp. convex cocompact, minimally parabolic geometrically finite).

Let M be a compact 3–manifold admitting a minimally parabolic geometrically finite hyperbolic structure m. Let $\mathrm{QH}(M)$ denote the set of all minimally parabolic geometrically finite hyperbolic structures on M modulo isotopy, which is regarded as a subset of $\mathrm{AH}(M)$. Marden [9] showed that every minimally parabolic geometrically finite hyperbolic structure on M is obtained as a quasiconformal deformation of m. Therefore we call $\mathrm{QH}(M)$ the *quasiconformal deformation space*. Furthermore, if ∂M is incompressible, combined with the work of Ahlfors, Bers, Kra, Maskit and Sullivan, there is a parametrisation $q\colon \mathcal{T}(\partial M)\to\mathrm{QH}(M)$, where $\mathcal{T}(\partial M)$ denotes the Teichmüller space of ∂M, ie the direct product of the Teichmüller spaces of the components of ∂M. We shall refer to this map as the *Ahlfors–Bers map*.

When M is homeomorphic to $S\times[0,1]$ for a closed oriented surface S, the deformation spaces $\mathrm{AH}(M)$ and $\mathrm{QH}(M)$ are denoted by $\mathrm{AH}(S)$ and $\mathrm{QF}(S)$, respectively. The quasiconformal deformation space $\mathrm{QF}(S)$ consists of quasi-Fuchsian representations of $\pi_1(S)$, ie quasiconformal deformations of a Fuchsian representation, and is therefore called the *quasi-Fuchsian space*. The Ahlfors–Bers map can be expressed as $qf:\mathcal{T}(S)\times\mathcal{T}(\overline{S})\to\mathrm{QF}(S)$, where the second coordinate $\mathcal{T}(\overline{S})$ denotes the Teichmüller space of S with orientation reversed, which is a more natural parametrisation since the boundary component $S\times\{1\}$ has the opposite orientation from the one given on $S\times\{0\}$ if we identify them with S by dropping the second factor.

Now, let M be an atoroidal Haken 3–manifold with nonempty incompressible boundary which does not contain a torus. Suppose that M has a convex compact hyperbolic metric m, and let S be a component of ∂M. Take a covering of M associated with $\pi_1(S)\subset\pi_1(M)$, and lift the hyperbolic structure m to the hyperbolic structure $\widetilde{m}$ on $S\times[0,1]$. It is known (see [16, Proposition 7.1]) that the lifted structure $\widetilde{m}$ is also convex cocompact, and hence can be regarded as an element of $\mathrm{QF}(S)$. Therefore $\widetilde{m}$ in turn corresponds to a point $(g_S(m),h_S(m))$ in $\mathcal{T}(S)\times\mathcal{T}(\overline{S})$. Let $S_1,\dots,S_k$ be the components of ∂M that are not tori, and consider the point $h_{S_i}(m)\in\mathcal{T}(\overline{S}_i)$ for each $i=1,\dots,k$. We define $\mathcal{T}(\bar{\partial}M)$ to be $\mathcal{T}(\overline{S}_1)\times\cdots\times\mathcal{T}(\overline{S}_k)$. The map taking $g\in\mathcal{T}(\partial M)$ to $\big(h_{S_1}(q(g)),\dots,h_{S_k}(q(g))\big)\in\mathcal{T}(\bar{\partial}M)$ is called the *skinning map*, which we shall denote by σ.

2.3 Curve complexes and projections

Let S be a connected compact orientable surface, possibly with boundary, satisfying $\xi(S)=3g+n\geq 4$, where g denotes the genus and n denotes the number of the boundary components. The *curve complex*

$\mathcal{CC}(S)$ of S with $\xi(S) \geq 5$ is a simplicial complex whose vertices are isotopy classes of nonperipheral noncontractible simple closed curves on S such that $n+1$ vertices span an n–simplex when they are represented by pairwise-disjoint simple closed curves. When $\xi(S) = 4$, we define $\mathcal{CC}(S)$ to be a graph whose vertices are isotopy classes of simple closed curves such that two vertices have smallest possible intersection. When S is an annulus, we define $\mathcal{CC}(S)$ to be a graph whose vertices are isotopy classes (relative to the endpoints) of nonperipheral simple arcs in S such that two vertices are connected when they can be represented by arcs which are disjoint at their interiors. Masur and Minsky [11] proved that $\mathcal{CC}(S)$ is Gromov hyperbolic with respect to the path metric for any S.

A *marking* μ on S consists of a pants decomposition of S, which is denoted by $\mathrm{base}(\mu)$ and whose components are called base curves, and a collection $\boldsymbol{t}(\mu)$ of simple closed curves, called transversals of $\mathrm{base}(\mu)$, such that each component of $\mathrm{base}(\mu)$ intersects at most one among them essentially. For two markings μ and ν on S and a subsurface Y, we define $d_Y(\mu,\nu)$ to be the distance between $\pi_Y(\mathrm{base}(\mu)\cup\boldsymbol{t}(\mu))$ and $\pi_Y(\mathrm{base}(\nu)\cup\boldsymbol{t}(\nu))$, where the projection $\pi_Y\colon \mathcal{CC}(S)\to\mathfrak{P}(\mathcal{CC}(Y))$ is obtained by taking the intersection of curves on S with Y and connecting the endpoints by arcs on $\operatorname{Fr} Y$ when the intersection contains arcs. In [12], a marking defined as such is called clean. We only consider clean markings. A marking is called *full* when every base curve has a transversal. In general, for two sets of simple closed curves a and b and a subsurface Y of S, we define $d_Y(a,b)$ to be the distance in $\mathcal{CC}(Y)$ between $\pi_Y(a)$ and $\pi_Y(b)$ provided that both of them are nonempty. If one of them is empty, the distance is not defined.

For a point m in $\mathcal{T}(S)$, its shortest marking, which is a full marking and is denoted by $\mu(m)$, has a shortest pants decomposition of (S,m) as $\mathrm{base}(\mu(m))$, and $\boldsymbol{t}(\mu(m))$ consisting of shortest transversals, one for each component of $\mathrm{base}(\mu(m))$. When we talk about the distance d_Y between two points in $\mathcal{T}(S)$ or between a point in $\mathcal{T}(S)$ and a marking, we identify points $m\in\mathcal{T}(S)$ with $\mu(m)$.

2.4 Geometric limits and compact cores

Let M be an atoroidal boundary-irreducible Haken 3–manifold. Let $\{\rho_i\}$ be a sequence of faithful discrete representations of $\pi_1(M)$ into $\mathrm{PSL}_2(\mathbb{C})$. We define a *geometric limit* of $\{\rho_i(\pi_1(M))\}$ to be a Kleinian group Γ such that every element γ of Γ is a limit of some sequence $\{g_i\in\rho_i(\pi_1(M))\}$, and every convergent sequence $\{\gamma_{i_j}\in\rho_{i_j}(\pi_1(M))\}$ has its limit in Γ.

Fixing a point $x\in\mathbb{H}^3$, and considering its projections x_i in $\mathbb{H}^3/\rho_i(\pi_1(M))$ and x_∞ in $\mathbb{H}^3/\Gamma$, geometric convergence implies the existence of pointed Gromov–Hausdorff convergence of $\big((\mathbb{H}^3/\rho_i(\pi_1(M)),x_i)\big)$ to $(\mathbb{H}^3/\Gamma,x_\infty)$. This latter convergence means that there exist real numbers r_i going to ∞, K_i converging to 1, and K_i–bi-Lipschitz diffeomorphisms f_i (called *approximate isometries*) between r_i–balls $B_{r_i}\big(\mathbb{H}^3/\rho_i(\pi_1(M)),x_i\big)$ and $B_{K_ir_i}(\mathbb{H}^3/\Gamma,x_\infty)$. Suppose $\{\rho_i\}$ converges to $\rho_\infty\colon\pi_1(M)\to\mathrm{PSL}_2(\mathbb{C})$ as representations and that $\{\rho_i(\pi_1(M))\}$ converges to Γ geometrically. Then $\rho_\infty(\pi_1(M))$ is a subgroup of the geometric limit Γ.

For an open irreducible 3–manifold V with finitely generated fundamental group, a compact 3–dimensional submanifold C in V is called a *compact core* when the inclusion induces an isomorphism between their fundamental groups. The existence of compact cores was proved by Scott [19]. The case which interests us is when V is a hyperbolic 3–manifold.

Let $\mathbb{H}^3/G$ be a hyperbolic 3–manifold associated with a finitely generated torsion free Kleinian group G. By Margulis's lemma, there is a positive constant ε_0 such that the set of points of $\mathbb{H}^3/G$ where the injectivity radii are less than ε_0 consists of a finite disjoint union of tubular neighbourhoods of closed geodesics of length less than ε_0, called *Margulis tubes*, and *cusp neighbourhoods*, each of which is stabilised by a maximal parabolic subgroup of G, and whose quotient by its stabiliser is homeomorphic to $S^1\times\mathbb{R}^2$ when the stabiliser has rank 1 and to $S^1\times S^1\times\mathbb{R}$ when the stabiliser has rank 2. The former cusp neighbourhood is called a $\mathbb{Z}$–cusp neighbourhood, and the latter a torus cusp neighbourhood. The union of the cusp neighbourhoods is called the cuspidal part of $\mathbb{H}^3/G$. The complement of the cuspidal part is called the *noncuspidal part* and is denoted by $(\mathbb{H}^3/G)_0$. Each boundary component of $(\mathbb{H}^3/G)_0$ is either an open annulus or a torus. By the relative compact core theorem of McCullough [13], there is a compact core $C_G\subset(\mathbb{H}^3/G)_0$ such that for each boundary component B of $(\mathbb{H}^3/G)_0$, the intersection $C_G\cap B$ is a core annulus when B is an open annulus and is the entire B when B is a torus. We call such a compact core a *relative compact core* of $(\mathbb{H}^3/G)_0$.

Let $p\colon\mathbb{H}^3/\rho_\infty(\pi_1(M))\to\mathbb{H}^3/\Gamma$ be the covering map associated with the inclusion of $\rho_\infty(\pi_1(M))$ into the geometric limit Γ. Let C be a relative compact core of $\big(\mathbb{H}^3/\rho_\infty(\pi_1(M))\big)_0$. Suppose that $\mathbb{H}^3/\Gamma$ has a torus cusp neighbourhood T. We say that $\mathbb{H}^3/\rho_\infty(\pi_1(M))$ *wraps around* T when $p|_C$ is homotoped to an immersion which goes around T nontrivially, and hence cannot be homotoped to an embedding.

3 Topological features

3.1 Coverings

In this subsection, we shall show that to prove Theorem 1.1, we may assume that all the characteristic I–pairs of M are product bundles.

We consider an atoroidal Haken manifold as given in Theorem 1.1. Let $p\colon\widetilde{M}\to M$ be a finite-sheeted regular covering. Then p induces the covering map between the boundaries $p_\partial\colon\partial\widetilde{M}\to\partial M$. This map induces a proper embedding between Teichmüller spaces $p_\partial^*\colon\mathcal{T}(\partial M)\to\mathcal{T}(\partial\widetilde{M})$ which is obtained by pulling back the conformal structures by p_∂. Also, the involution ι lifts to an orientation-reversing involution $\tilde{\iota}\colon\partial\widetilde{M}\to\partial\widetilde{M}$ taking each component to another one. Since $\widetilde{M}$ is also an atoroidal boundary-irreducible Haken manifold, we can consider the skinning map $\tilde{\sigma}\colon\mathcal{T}(\partial\widetilde{M})\to\mathcal{T}(\bar{\partial}\widetilde{M})$.

Lemma 3.1 *If $(\tilde{\iota}\circ\tilde{\sigma})^n$ has bounded image for some $n\in\mathbb{N}$, then so does $(\iota\circ\sigma)^n$.*

Proof The map p_∂^* properly embeds $\mathcal{T}(\partial M)$ into $\mathcal{T}(\partial\widetilde{M})$. By the definition of the maps $\tilde{\sigma}$ and $\tilde{\iota}$, we have $p_\partial^* \circ (\iota\circ\sigma) = (\tilde{\iota}\circ\tilde{\sigma})\circ p_\partial^*$, and hence $p_\partial^* \circ (\iota\circ\sigma)^n = (\tilde{\iota}\circ\tilde{\sigma})^n \circ p_\partial^*$. Therefore, if the image of $(\tilde{\iota}\circ\tilde{\sigma})^n$ is bounded, the properness of p_∂^* implies that the image of $(\iota\circ\sigma)^n$ must also be bounded. □

This result allows us to work on manifolds with topological features that will make the arguments simpler.

Lemma 3.2 *Let M be an orientable atoroidal Haken manifold with incompressible boundary. Then there is a double covering of M all of whose characteristic I–pairs are product I–bundles.*

Proof Let $\Xi_1,\dots,\Xi_p$ be the characteristic I–pairs of M that are twisted I–bundles. Take their double coverings $\widetilde{\Xi}_1,\dots,\widetilde{\Xi}_p$ corresponding to the orientation double coverings of the base surfaces. For each Ξ_j among $\Xi_1,\dots,\Xi_p$, its frontier components (ie the closures of the components of $\partial\Xi_j \setminus \partial M$) are annuli. Each such annulus has two preimages in $\widetilde{\Xi}_j$, which are taken to each other by the unique nontrivial covering translation.

Let C be the closure of a component of $M\setminus(\Xi_1\cup\cdots\cup\Xi_p)$. Let $A_1,\dots,A_k$ be the components of $C\cap(\Xi_1\cup\cdots\cup\Xi_p)$, which are annuli on $\bigcup_{l=1}^{p}\overline{\partial\Xi_l\setminus\partial M}$. We prepare two copies C^+ and C^- of C. Each A_j among $A_1,\dots,A_k$, which is contained some Ξ_i among $\Xi_1,\dots,\Xi_p$, has two lifts A_j^+ and A_j^- in $\widetilde{\Xi}_i$. Now we identify the copy of A_j in C^+ to A_j^+ in $\widetilde{\Xi}_i$ and the one in C^- to A_j^- in $\widetilde{\Xi}_i$ for each annulus among $A_1,\dots,A_k$. We repeat the same procedure for every component C of $M\setminus(\Xi_1\cup\cdots\cup\Xi_p)$, and get a manifold $\widetilde{M}$, which will turn out to be a double cover of M, as shown below.

Define a homeomorphism $t\colon\widetilde{M}\to\widetilde{M}$ to be the covering translation on each $\widetilde{\Xi}_i$ and the map taking $C^\pm$ to $C^\mp$ preserving the identification with C for each of $C^\pm$. It is clear from the definition that this homeomorphism t is a free involution. By taking the quotient of $\widetilde{M}$ under $\langle t\rangle\cong\mathbb{Z}_2$, we get a manifold naturally identified with M. Thus $\widetilde{M}$ is a double cover of M. Since $\widetilde{\Xi}_i$ is a product I–bundle and all of the characteristic I–pairs contained in $C^\pm$ are product bundles by our definition of $\Xi_1,\dots,\Xi_p$, we see that $\widetilde{M}$ is a double cover, as desired. □

For some of our arguments we shall need a stronger assumption than having only product bundles:

Definition 3.3 Let M be a compact orientable Haken 3–manifold with incompressible boundary. We say that M is *strongly untwisted* if and only if

(A) every characteristic I–pair is a product bundle, and

(B) for any characteristic I–pair Ξ and any simple closed curve $d\subset\partial M$, the simple closed curve d can be homotoped on ∂M into at most one component of $\Xi\cap\partial M$.

We are going to construct a cover with the properties (A) and (B). In order to do that, we need to examine how characteristic I–pairs are attached to other components of the characteristic submanifold. In the following proof of Lemma 3.4, it will turn out that there are two situations ((a) and (b) below) where the second condition of "strong untwistedness" breaks down.

Lemma 3.4 *Let M be a compact orientable atoroidal Haken manifold with incompressible boundary. Then there is a finite-sheeted regular covering of M which is strongly untwisted.*

Proof By Lemma 3.2, we have a double covering all of whose characteristic I–pairs are product I–bundles. Therefore we may assume that M satisfies the first condition (A) of "strong untwistedness" and we shall construct a covering satisfying the second condition.

To construct such a covering, let us analyse how this second condition (B) can fail to hold. Let $d \subset \partial M$ be a simple closed curve, and let W be a characteristic I–pair. Let $T \subset M$ be the union of the solid torus and thickened torus in the characteristic submanifold of M. Since no components of $W \cap \partial M$ are annuli, d can be homotoped on ∂M into at most two components of $W \cap \partial M$. Furthermore, if d can be homotoped into two such components, then d lies (up to isotopy on ∂M) on a component T_j of T, and there are two possibilities:

(a) $T_j \cap \partial M$ is an annulus when T_j is a solid torus, and is the union of an annulus and a torus when T_j is a thickened torus, or

(b) d separates two consecutive components of $T_j \cap \overline{M \setminus T}$ both lying (up to isotopy of ∂M) in the same characteristic I–pair.

We shall show that we can take a finite-sheeted covering of M so that neither (a) nor (b) can happen.

First, we consider (a). Let T_j be a component of T such that $T_j \cap \partial M$ is an annulus (and T_j is a solid torus) or the union of an annulus and a torus (when T_j is a thickened torus). This implies that $T_j \cap \overline{M \setminus T_j}$ is connected, and hence is an annulus, which we denote by A. Since T_j is a characteristic solid torus or characteristic thickened torus, the annulus A is essential, and hence is not homotopic to $T_j \cap \partial M$ fixing the boundary. Then we can choose a simple closed curve α, which is not contractible in M, on the component of ∂T_j on which d lies so that both $\alpha \cap A$ and $\alpha \cap \partial M$ are connected, ie arcs. Since $\pi_1(T_j)$ is either $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$, we can take a k–sheeted cyclic covering $\widetilde{T}_j$ of T_j so that α cannot be lifted homeomorphically, whereas the annulus A is homeomorphically lifted. (For instance, in the case when $\pi_1(T_j) \cong \mathbb{Z}$, we choose k which is coprime with the element represented by α.) Then the preimage of the annulus A is k copies of A, which we denote by $A_1, \dots, A_k$. Let C be $\overline{M \setminus T_j}$. We prepare k copies of C, which we denote by $C_1, \dots, C_k$. By pasting C_j along A_j to $\widetilde{T}_j$, we can make a k–sheeted cyclic covering of M in which $\widetilde{T}_j$ does not satisfy (a). If there is another component $T_{j'}$ of T satisfying (a), we repeat the same process for all the k lifts of $T_{j'}$ at the same time. Repeating the process, we get a finite-sheeted covering of M in which there is no characteristic solid torus or a characteristic thickened torus satisfying (a). We use the same symbols M and T for this finite-sheeted covering, abusing the notation.

Now we turn to (b). We choose a colour, labelled by 0, 1 and 2, for each annulus of $T \cap \overline{M \setminus T}$ so that, on ∂T, no two consecutive annuli have the same colour. We take three copies of each component of T and of $\overline{M \setminus T}$, which we name lifts 0, 1 and 2. Consider a component U of T, a component V of $\overline{M \setminus T}$ and an annulus $E \subset U \cap V$ with the colour $k \in \{0, 1, 2\}$. For every $j \in \{0, 1, 2\}$, we glue the lift j of V to the lift $(j + k) \bmod 3$ of U along the appropriate lifts of E. Using the same construction for each

component of $T \cap \overline{M \setminus T}$, we get a triple cover $\widehat{M}$ of M in which any two consecutive components of $\widehat{T} \cap \overline{\widehat{M} \setminus \widehat{T}}$ lie in different components of $\overline{\widehat{M} \setminus \widehat{T}}$. In particular there is no characteristic solid torus or characteristic thickened torus in $\widehat{M}$ for which (b) holds.

Thus, we have shown that by taking a finite-sheeted covering, we can make both situations (a) and (b) disappear, which means, as we saw above, that the covering is strongly untwisted. □

Lemmas 3.1 and 3.4 show that to prove Theorem 1.1, we have only to consider the case when M is strongly untwisted.

3.2 Vertically extendible surfaces

Let M be an atoroidal Haken manifold as in Theorem 1.1. Let X be the characteristic submanifold of M. Assume that every I–bundle in X is a product I–bundle.

Definition 3.5 Given an incompressible subsurface $F \subset \partial M$, we say that F is *one-time vertically extendible* if there is an incompressible surface $F^1 \subset \partial M$ and an essential I–bundle $V_F \subset M$ with $V_F \cap \partial M = F \cup F^1$ and $F^1 \subset \partial X$ up to isotopy. We call F^1 a *first elevation* of F.

It follows from the definition of characteristic submanifold that there is an isotopy which takes V_F into the characteristic submanifold X. From now on, we assume that if F is one-time vertically extendible then $F \subset X$ and $V_F \subset X$.

We note that solid torus components in X may add some complications in the case when F is an annulus. If F is contained in such a component of X, there may be more than one possible first elevation (even up to isotopy) and the I–bundles corresponding to two disjoint annuli may intersect (even up to isotopy).

We now define multiple elevations by induction.

Definition 3.6 Given an incompressible subsurface F in ∂M and $n \geq 2$, we say that F is *n–times vertically extendible* if there is an essential surface $F^1 \subset \partial M$, an essential I–bundle $V_F \subset M$ with $V_F \cap \partial M = F \cup F^1$, and $\iota(F^1)$ is $(n-1)$–times vertically extendible. An $(n-1)^{\text{th}}$ elevation F^n of $\iota(F^1)$ is defined to be an n^{th} elevation of F.

We say that two multicurves $c, d \subset \partial M$ *intersect minimally* if, for all multicurves c' and d' homotopic to c and d, respectively, $\#\{c \cap d\} \leq \#\{c' \cap d'\}$. Let $F, G \subset X \cap \partial M$ be two incompressible surfaces. We say that F and G *intersect minimally* if ∂F intersects ∂G minimally.

Lemma 3.7 *Let $F, G \subset \partial M$ be connected incompressible subsurfaces which intersect minimally and are not disjoint. If F and G are n–times vertically extendible, then so is $F \cup G$.*

Proof If F and G are one-time vertically extendible, as was remarked before, we may assume that $F, G \subset X \cap \partial M$. Since they intersect minimally and are not disjoint, they must lie in the same component H of $X \cap \partial M$ which is not an annulus. Then the component V of X containing H is an I–bundle, which is a product I–bundle by assumption, and can be parametrised as $H \times [0, 1]$.

Then, by moving F and G by isotopies, we have $V_F = F\times[0,1] \subset H\times[0,1]$, $V_G = G\times[0,1] \subset H\times[0,1]$, $F^1 = F\times\{0,1\}\setminus F$ and $G^1 = G\times\{0,1\}\setminus G$. Since F^1 and G^1 lie in the component of $X\cap\partial M$ which does not contain F and G, F^1 and G^1 lie in the same component of $X\cap\partial M$. Therefore $F^1\cup G^1$ lies in $X\cap\partial M$. Thus we have proved that if F and G are one-time vertically extendible then $F\cup G$ is also one-time vertically extendible and $F^1\cup G^1$ is its first elevation.

The case of $n>1$ follows by induction. □

Corollary 3.8 *For any natural number n, there is a (possibly empty) incompressible surface $\Sigma^n \subset X\cap\partial M$ such that each component of Σ^n is n–times vertically extendible, no component of Σ^n can be isotoped on ∂M into another component of Σ^n and every n–times vertically extendible surface can be isotoped into Σ^n.*

Proof If there is no surface that is n–times vertically extendible, we set Σ^n to be $\varnothing$. Otherwise, let $\Sigma\subset X$ be an n–times vertically extendible incompressible surface. If every n–times vertically extendible surface can be isotoped into Σ, we are done, by taking $\Sigma^n=\Sigma$.

Otherwise, there is an n–times vertically extendible surface F which cannot be isotoped into Σ. Moving F by an isotopy we can assume that F intersects Σ minimally. By Lemma 3.7, each connected component of $F\cup\Sigma$ is n–times vertically extendible, and we replace Σ with $\Sigma\cup F$, and call this enlarged surface Σ. We repeat this operation as long as there is an n–times vertically extendible surface which cannot be isotoped into Σ. Every time we add a surface, either we decrease the Euler characteristic of Σ or we add a disjoint annulus which cannot be isotoped into Σ. Hence this process must terminate after finitely many steps. The final resulting surface is Σ^n. □

Since an n–times vertically extendible surface is m–vertically extendible for any $m\le n$, we have $\Sigma^n\subset\Sigma^m$ up to isotopy.

In the next lemma we show that, when N is atoroidal, M cannot contain an n–times extendible surface for sufficiently large n. In the last section, this result will lead us to the constant n of Theorem 1.1.

Lemma 3.9 *There is L depending only on the topological type of ∂M such that if there is an L–times vertically extendible surface, then N is not atoroidal.*

Proof Letting g denote the genus of ∂M, we set $K=3g-3$, which is the number of curves in a pants decomposition of ∂M. Since no components of Σ^n can be isotoped into another component, $\partial\Sigma^n$ has at most $2K$ boundary components. Using this observation, we show in the following claim that Σ^{n+2K} must be a proper subsurface of Σ^n even up to isotopy.

Claim 3.10 *For any $n\in\mathbb{N}$, if Σ^n is nonempty and any component of Σ^n can be isotoped into Σ^{n+2K}, then N cannot be atoroidal.*

Proof Suppose that $\Sigma^n\neq\varnothing$, and that any component of Σ^n can be isotoped into Σ^{n+2K}. Since Σ^{n+j} is contained in Σ^n for any $j\ge 0$ up to isotopy as observed above, and no component of Σ^{n+j} can be isotoped

into another component, $\Sigma^{n+j} = \Sigma^n$ for any $j \leq 2K$ up to isotopy. Let F be a component of Σ^{n+2K} with minimal Euler characteristic, and F^j be its j^{th} elevation. By definition, $\iota(F^j)$ is $(n+2K-j)$–times vertically extendible for any $j \leq 2K$. Therefore $\iota(F^j)$ can be isotoped into Σ^n. Since $\Sigma^n = \Sigma^{n+2K}$ and F has minimal Euler characteristic, $\iota(F^j)$ is a component of Σ^n, up to isotopy. In particular $\partial(\iota(F^j)) \subset \partial\Sigma^n$ up to isotopy.

Let V^j be the I–bundle cobounded by $\iota(F^{j-1})$ and F^j. We note that by definition, F^j and $\iota(F^j)$ are identified in N and that the interior of V^j is embedded in N. Taking the union of the I–bundles V^j in N for $j \leq 2K$, we get a map $F \times [0, 2K] \to N$ such that $F \times \{j\}$ is sent to F^j. Let c be a component of ∂F. The image of the annulus $c \times [0, 2K]$ goes $2K+1$ times through $\partial\Sigma^n$. Since $\partial\Sigma^n$ has at most $2K$ components, there is a component c' of $\partial\Sigma^n$ through which $c \times [0, 2K]$ goes at least twice. The image of the part of this annulus between two such instances forms a torus T embedded in N. Considering the component of ∂M through which T goes, we can construct an infinite cyclic covering of N in which T lifts to an infinite incompressible annulus. It follows that T is incompressible and nonperipheral. Hence N is not atoroidal. □

As mentioned before, $\Sigma^n \subset \Sigma^m$ for any $m \leq n$. Consider monotone increasing indices n_j such that Σ^{n_j+1} is smaller than Σ^{n_j} in the sense that at least one component of Σ^{n_j} cannot be isotoped into Σ^{n_j+1}. Since no component of Σ^n can be isotoped into another component, either $\chi(\Sigma^{n_j+1}) > \chi(\Sigma^{n_j})$ or Σ^{n_j+1} has fewer connected components than Σ^{n_j}. It follows that there are at most K such n_j, namely, there is $J \leq K$ such that for any $n \geq n_J + 1$ we have $\Sigma^n = \Sigma^{n+1}$. By Claim 3.10, if $n_j - n_{j-1} \geq 2K$ for some $j \leq J$ or if $\Sigma^{n_J} \neq \varnothing$, then N is not atoroidal. Since $J \leq K$, we can now conclude the proof just by setting $L = 2K^2$. □

4 Convergence, divergence and subsurface projections

In this section, we shall review the relations between the invariant m mentioned in the introduction and the convergence and divergence of Fuchsian and Kleinian groups.

4.1 Subsurface projections and Fuchsian groups

We first recall the definition of the invariant m from [2], and see how it controls the behaviour of sequences of Fuchsian groups.

Definition 4.1 Let S be a (possibly disconnected) closed surface of genus at least 2 and g a point in its Teichmüller space. Regarding g as a hyperbolic structure on S, we let $\mu(g)$ be a shortest marking for (S, g); see Section 2.3. Although there might be more than one shortest marking, its choice does not matter for our definition and arguments. We fix a full and clean marking μ consisting of a pants decomposition and transversals on S independent of g. For any essential simple closed curve d on S, we define

$$m(g, d, \mu) = \max\left\{ \sup_{Y : d \subset \partial Y} d_Y(\mu(g), \mu), \frac{1}{\text{length}_g(d)} \right\},$$

where the supremum of the first term in the maximum is taken over all incompressible subsurfaces Y of S whose boundaries ∂Y contain d.

It follows from [2, Lemma 5.2] that two curves with unbounded m cannot intersect:

Lemma 4.2 *Let $\{m_j\}$ be a sequence in $\mathcal{T}(S)$, and let c_1 and c_2 be simple closed curve on S. If $m(m_j, c_i, \mu) \to \infty$ as $j \to \infty$ for both $i = 1, 2$, then $i(c_1, c_2) = 0$.*

Proof This is just a special case of [2, Lemma 5.2] for Fuchsian groups. We note that the assumption of bounded projections of end invariants is unnecessary in this special case for which end invariants are empty. □

The invariant m is related to the divergence and convergence of a sequence by the following lemma:

Lemma 4.3 *Let μ be a full marking on S, and let $\{m_i\}$ be a sequence in $\mathcal{T}(S)$. Then every subsequence of $\{m_i\}$ contains a convergent subsequence if and only if $\{d_S(\mu(m_i), \mu)\}$ is bounded for a shortest marking $\mu(m_i)$ of m_i and $\{m(m_i, c, \mu)\}$ is bounded for every essential simple closed curve c on S.*

Proof It follows from classical results on Fenchel–Nielsen coordinates that any subsequence of $\{m_i\}$ contains a converging subsequence if and only if $\{\mu(m_i)\}$ is a finite set and $\{\mathrm{length}_{m_i}(\mu(m_i))\}$ is bounded. By [2, Lemma 2.3], the sequence $\{\mu(m_i)\}$ is infinite if and only if, passing to a subsequence, either $\{d_S(\mu(m_i), \mu)\}$ is unbounded or there is an incompressible subsurface Y such that $d_Y(\mu(m_i), \mu) \to \infty$ (and hence $m(m_i, c, \mu) \to \infty$ for any component c of ∂Y).

On the other hand, if $\{\mu(m_i)\}$ consists of finite elements, then $\mathrm{length}_{m_i}(\mu(m_i))$ is unbounded if and only if, passing to a subsequence, there is a curve c with $\mathrm{length}_{m_i}(c) \to 0$ (and hence $m(m_i, c, \mu) \to \infty$). □

4.2 Relative convergence of Kleinian groups

We shall next establish a necessary condition on the invariant m for algebraic convergence on a submanifold. We start with a fundamental result. Thurston proved in [22] the following, which is the first half of the theorem often referred to as the "broken window only" theorem. We note that the latter half of the broken window only theorem needs some rectification (see [17]) but it is irrelevant to the present paper.

Theorem 4.4 *Let M be an atoroidal Haken 3–manifold and X its characteristic submanifold. Then for any curve γ in $M \setminus X$ and any sequence $\{\rho_i \in \mathrm{AH}(M)\}$, the length of the closed geodesic in $\mathbb{H}^3/\rho_i(\pi_1(M))$ representing the free homotopy class of $\rho_i(\gamma)$ is bounded as $i \to \infty$.*

Using arguments from [2], we establish the following necessary condition for algebraic convergence on a submanifold:

Theorem 4.5 *Let M be an atoroidal boundary-irreducible Haken 3–manifold all of whose characteristic I–pairs are product I–bundles. Let $\{m_i\}$ be a sequence in $\mathcal{T}(\partial M)$, and $\{\rho_i\colon \pi_1(M)\to \mathrm{PSL}_2(\mathbb{C})\}$ a sequence of representations corresponding to $\{q(m_i)\}$. Let $\mu\subset \partial M$ be a full and clean marking, and $W\subset M$ a submanifold with paring locus P which is a union of disjoint nonparallel essential annuli on ∂W.* (See Definition 2.1 for the definition of pared manifolds.) *We assume the following*:

(a) *The closure of $\partial W\setminus \partial M$ is a union of essential annuli contained in P.*

(b) *For any noncontractible simple closed curve c in P, $\mathrm{length}_{\rho_i}(c)$ is bounded as $i\to\infty$.*

(c) *For any essential annulus $E\subset W$ disjoint from P, there is a component c of ∂E such that $\{m(m_i|_S, c, \mu)\}$ is bounded for the component S of ∂M on which c lies.*

(d) *If M is an I–bundle, then $P\neq\varnothing$.*

Then the sequence of the restrictions $\{\rho_i|_{\pi_1(W)}\}$ has a convergent subsequence up to conjugation.

Proof We follow the argument of [2, Proposition 6.1] with some modifications as below. We note that (c) will replace the assumption of "bounding projection without combinatorial wrapped parabolics" imposed there. The condition (a) will allow us to work on the submanifold W rather than the whole manifold. Following [2, Lemma 6.2], we start by constructing a pants decomposition r of ∂W with uniformly bounded length. In the first paragraphs of the proof of [2, Lemma 6.2], the assumption of bounded projection is used to find the first curves in r. In our relative setting, we do not have an equivalent assumption. Instead we use (b), (d) and Theorem 4.4 to find the first curves as below.

Denote by c_i a shortest pants decomposition of ∂M with respect to m_i. Note that

$$\{d_Y(m_i,\mu)=d_Y(\mu(m_i),\mu)\}$$

is bounded for any essential subsurface Y that is not an annulus with its core curve in c_i if and only if $\{d_Y(c_i,\mu)\}$ is bounded. Let X be the characteristic submanifold of the pared manifold (W,P). Consider a multicurve r on $((\overline{W\setminus X})\cap\partial W)\cup P$ which is maximal in the sense that any simple closed curve in $((\overline{W\setminus X})\cap\partial W)\cup P$ either intersects r or is homotopic on ∂W to a component of r. We note that it contains a curve isotopic to each boundary component of $X\cap\partial W$ by the maximality, and is not empty by (d). By (b) and Theorem 4.4, there is L such that $\mathrm{length}_{\rho_i}(r)\le L$.

Next, following the proof of [2, Proposition 6.1], we add curves to r until we get a pants decomposition. Since r is already maximal in $((\overline{W\setminus X})\cap\partial W)\cup P$, we only need to extend it to the union Z of the characteristic I–pairs in X. By assumption, Z is a product I–bundle in the form $\Sigma\times I$ (Σ may be disconnected). We denote by $f\colon Z\to\Sigma$ the projection along the fibres, and for a subsurface $F\subset\Sigma$ and for $j=0,1$, we use the symbol F_j to denote $f^{-1}(F)\cap\Sigma\times\{j\}$. For each component F of Σ that is not a pair of pants, by (c) there is $j\in\{0,1\}$ such that $\{d_{F_j}(c_i,\mu)\}$ is bounded. Let S_j be the component of ∂M containing F_j, and denote by $\theta_i=\rho_i\circ I_*\colon\pi_1(S_j)\to\mathrm{PSL}_2(\mathbb{C})$ the representation induced by the inclusion $I\colon S_j\hookrightarrow M$. The quotient manifold $\mathbb{H}^3/\theta_i(\pi_1(S_j))$ covers $\mathbb{H}^3/\rho_i(\pi_1(M))$, and has end

invariant $m_i|_{S_j}$ on one side. Now, replacing ρ_i with θ_i, we can follow the proof of [2, Lemma 6.2] starting at the penultimate paragraph (the third paragraph on page 836) to find a simple closed curve a_i contained in F as follows. By Theorem 4.4, (a) and (b), each boundary component of F is homotopic to a closed geodesic in $\mathbb{H}^3/\theta_i(\pi_1(S_j))$ with length bounded as $i \to \infty$. We note that (d) implies that F is not the entire S, and hence has nonempty boundary. Then the argument of the proof of Lemma 6.2, which makes use of [2, Lemma 2.11], gives us a simple closed curve a_i in F homotopic to a closed geodesic in $\mathbb{H}^3/\theta_i(\pi_1(S_j))$ with bounded length such that the distance from $f(c_i \cap F_j)$ in $\mathcal{CC}(F)$ is also bounded as $i \to \infty$. Thus we have a constant L' and a sequence of curves $\{a_i\}$ on F such that $\ell_{\rho_i}(a_i) \leq L'$ and $\{d_{F_j}(f^{-1}(a_i) \cap F_j, c_i)\}$ is bounded as $i \to \infty$.

Up to isotopy, $f(r \cap Z)$ consists of boundary components of Σ since Z is contained in X and r lies outside X. We denote $f(r \cap Z)$ by s. If $\{a_i\}$ has a constant subsequence, then we pass to an appropriate subsequence of $\{\rho_i\}$, and add a_i (independent of i) to s. If not, by [2, Lemma 2.3], there is a subsurface $Y \subset F$ with $d_Y(a_i, \mu) \to \infty$, passing to a subsequence. Since $\{d_{F_j}(c_i, \mu)\}$ and $\{d_{F_j}(f^{-1}(a_i) \cap F_j, c_i)\}$ are bounded, Y must be a proper subsurface of F (even up to isotopy). If, passing to a subsequence, there is $k \in \{0, 1\}$ such that $Y_k = f^{-1}(Y) \cap S_k$ is an annulus containing a component of c_i for all i, we add the projection by f of this component of c_i to s. Otherwise, by (c), there exists $k \in \{0, 1\}$ with bounded $\{d_{Y_k}(c_i, \mu)\}$. Hence, passing to a subsequence, $d_{Y_k}(c_i, f^{-1}(a_i) \cap S_k) \to \infty$, and by [14, Theorem B], $\ell_{\rho_n}(\partial Y) \to 0$. In this case, we add ∂Y to s. We repeat the above construction, letting F be a component of $\Sigma \setminus s$, until $\Sigma \setminus s$ becomes a union of annuli and pair of pants. Adding $f^{-1}(s)$ to r, we obtain a pants decomposition of ∂W, which we shall still denote by r, such that $\{\ell_{\rho_n}(r)\}$ is bounded.

Next we attach a transversal with bounded length to each component d of r such that there is an essential annulus E with $d \subset \partial E$ and that $\{m(m_i|_S, d, \mu)\}$ is bounded (where S is the component of ∂M on which d lies). Let d be such a curve. If d is contained in c_i, we replace c_i with a shortest pants decomposition not containing d. Since $\{m(m_i|_S, d, \mu)\}$ is bounded, there is a positive lower bound on $\{\ell_{m_i}(d)\}$, and there is an upper bound on $\{\ell_{m_i}(c_i)\}$ by our definition of c_i. Considering the covering associated with the inclusion $S \hookrightarrow M$, we can use the arguments of [2, proof of Proposition 6.1, after the proof of Lemma 6.2, starting from the fifth paragraph on page 836] to obtain a transversal t_d to d with bounded length $\ell_{\rho_i}(t_d)$.

By (c), the union of r and all the transversals defined above is doubly incompressible in Thurston's sense [22, Section 2]. Then we can deduce from Thurston's relative boundedness theorem [22, Theorem 3.1] that the restriction of $\{\rho_i|_{\pi_1(W)}\}$ has a convergent subsequence. □

5 Unbounded skinning and annuli

The following proposition is the main step of our proof of Theorem 1.1:

Proposition 5.1 *Let M be an orientable atoroidal boundary-irreducible Haken 3–manifold that is strongly untwisted. Let $\{m_i\}$ be a sequence in $\mathcal{T}(\partial M)$, let σ be the skinning map and assume that there is*

a simple closed curve d on ∂M such that $m(\sigma(m_i), d, \mu) \to \infty$ for a full clean marking μ. Then, passing to a subsequence, there is a properly embedded essential annulus $A \subset M$ with $\partial A = d \cup d'$ such that $m(m_i, d', \mu) \to \infty$.

We are going to show that any subsequence of $\{m_i\}$ contains a further subsequence for which the conclusion holds. To simplify the notation we shall use the same subscript i for all subsequences.

5.1 Remarking

Our manifold M is either connected or has two components. In the case when M has two components, by considering the component on which d lies and abusing the symbol M to denote this component, we can assume that M is connected. Recall that, by the assumption throughout this section, M is strongly untwisted. Let $\rho_i : \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$ be a representation corresponding to $q(m_i)$.

As a first step for the proof of Proposition 5.1, we change the markings of M so that the behaviour of the ρ_i can be read more easily from the behaviour of their end invariants.

Lemma 5.2 *Let d be an essential simple closed curve on ∂M, and let $d_1, \dots, d_p$ be disjoint simple closed curves on ∂M representing the homotopy classes of simple closed curves on ∂M homotopic to d in M, where $d_1 = d$. Furthermore, we assume that*

(∗) *$\{m(m_i, d_j, \mu)\}$ is bounded for every $j = 2, \dots, p$.*

Then there is a sequence of orientation-preserving homeomorphisms $\{\psi_i : M \to M\}$ such that, passing to a subsequence, the following hold:

(1) *Let $c \subset \partial M$ be an essential simple closed curve. Then either $\{m(\psi_{i*}(m_i), c, \mu)\}$ is bounded or $m(\psi_{i*}(m_i), c, \mu) \to \infty$.*

(2) *If $A \subset M$ is an essential annulus disjoint from all the d_j such that $m(\psi_{i*}(m_i), \partial_k A, \mu) \to \infty$ for both boundary components $\partial_1 A$ and $\partial_2 A$ of A, then $\ell_{\rho_i \circ \psi_{i*}^{-1}}(\partial A) \to 0$.*

(3) *For every d_j among $d_1, \dots, d_p$ defined above*
   (i) *$\psi_i(d_j) = d_j$ for every i,*
   (ii) *$\{m(\psi_{i*}(m_i), d_j, \mu)\}$ is bounded if and only if $\{m(m_1, d_j, \mu)\}$ is bounded for $j = 1, \dots, p$, and*
   (iii) *$\{m(\sigma \circ \psi_{i*}(m_i), d_j, \mu)\}$ is bounded if and only if $\{m(\sigma(m_i), d_j, \mu)\}$ is bounded.*

Proof We shall first define the homeomorphisms ψ_i, and then verify the desired properties. Let Ξ be a component of the characteristic submanifold X of $M \setminus d$. Suppose first that Ξ is a solid torus. The components of $\overline{\partial \Xi \setminus \partial M}$ are incompressible annuli. We define ψ_i on solid-torus components Ξ of X to be a composition of Dehn twists along these frontier annuli such that:

(a) If Ξ is a solid torus, then $\pi_F\big(\mu(\psi_{i*}(m_i))\big)$ is constant with respect to i for every component F of $\Xi \cap \partial M$ except for at most one.

By (∗), passing to a subsequence, we need not compose Dehn twists along annuli of the frontier components of Ξ to achieve (a) when $\Xi \cap \partial M$ contains an annular neighbourhood of d (up to isotopy), and hence ψ_i, as defined for the moment, also satisfies the following:

(b) For every $j = 1, \dots, p$, we have $\psi_i(d_j) = d_j$ and $\pi_{A_j}\big(\mu(\psi_{i*}(m_i))\big) = \pi_{A_j}(\mu(m_i))$ for an annulus A_j on ∂M whose core curve is d_j.

If Ξ is not a solid torus, Ξ is a product $\Xi = F \times I$. (Recall that we have an assumption that every characteristic I–pair of M is a product bundle. This implies that an I–pair in the characteristic submanifold X of $M \setminus d$ is also a product I–bundle.) Let F_0 be a component of $\Xi \cap \partial M$ which does not contain a curve homotopic on ∂M to d_1 (there is always such a component since M is strongly untwisted). Since the curve complex of F_0 has finitely many orbits under the action of the mapping class group of F_0 (relative to ∂F_0), there is a sequence of orientation-preserving homeomorphisms $g_i \colon F_0 \to F_0$ fixing ∂F_0 such that, passing to a subsequence, $\pi_{F_0}\big(\mu(g_{i*}(m_i))\big)$ is constant. We then define ψ_i on Ξ by extending g_i along the fibres, ie $\psi_i(x,t) = (g_i(x), t)$ for any $(x,t) \in \Xi = F_0 \times I$.

Thus we have the following:

(c) There are $R > 0$ and a component F_0 of $\Xi \cap \partial M$ not containing any curve homotopic on ∂M to d_1 such that $d_Y\big(\mu(\psi_{i*}(m_i)), \mu\big) \le R$ for any incompressible subsurface $Y \subset F_0$.

We note that since Ξ is a component of the characteristic submanifold of $M \setminus d$, if $\partial\Xi$ contains a curve d_j, then it must be peripheral, and hence the action of ψ_i on Ξ does not affect (b).

We repeat the construction above for all the components of the characteristic submanifold X, and we extend the resulting homeomorphisms to a homeomorphism of M which is isotopic to the identity on the complement of the characteristic submanifold.

We now verify properties (1)–(3) for ψ_i thus constructed.

Property (1) can be obtained by passing to a subsequence for any sequence of homeomorphisms.

We next turn to proving property (3). By (∗), taking a subsequence, we may assume that $\pi_F(\mu(m_i))$ is constant whenever F is an annulus containing a curve d_j for $j \neq 1$. We first show the following claim:

Claim 5.3 *For every $j = 1, \dots, p$ and for any sequence of incompressible subsurfaces $Y_i \subset \partial M$ with its boundary containing d_j which are not a pair of pants, $\{d_{Y_i}(\mu, \psi_i(\mu))\}$ is bounded.*

Proof Fix $j = 1, \dots, p$, and consider a sequence of incompressible subsurfaces $Y_i \subset \partial M$ each of which contains d_j in its boundary. If all of the Y_i are annuli after passing to a subsequence, the conclusion follows from (b). From now on, taking a subsequence, we assume that none of the Y_i are annuli.

Assume first that there is a simple closed curve $c \subset \partial M$ intersecting Y_i which lies outside the characteristic submanifold X. Then by our construction of ψ_i, we have $\psi_i(c) = c$, and hence

$$\begin{aligned} d_{Y_i}(\mu, \psi_i(\mu)) &\le d_{Y_i}(\mu, c) + d_{Y_i}(c, \psi_i(\mu)) \le d_{Y_i}(\mu, c) + d_{Y_i}(\psi_i(c), \psi_i(\mu)) \\ &\le d_{Y_i}(\mu, c) + d_{\psi_i^{-1}(Y_i)}(\mu, c) \le 4i(c, \mu) + 2, \end{aligned}$$

where the last inequality is due to Masur and Minsky [11, Lemma 2.1]. Thus we are done in this case.

Otherwise, taking a subsequence, we may assume that Y_i is contained in $\Xi_i \cap \partial M$ for a component Ξ_i of the characteristic submanifold X. Taking a further subsequence, we may assume that $\Xi_i = \Xi$ does not depend on i. Since Y_i is not an annulus, Ξ is a product I–pair $F \times I$. Let F_0 be the component of $\Xi \cap \partial M$ given by (c). Let us denote by Y_i' the projection of Y_i to F_0 along the fibres (setting $Y_i' = Y_i$ if $Y_i \subset F_0$). By our definition of $d_1, \dots, d_p$, the boundary of Y_i' contains some d_k with $k \ge 2$. Then $\{m(m_i, d_k, \mu)\}$ is bounded by $(*)$, and $\psi_i(d_k) = d_k$ by (b). In particular $\{d_{Y_i'}\big(\mu(\psi_i(m_i)), \psi_i(\mu)\big) = d_{\psi_i^{-1}(Y_i')}(\mu(m_i), \mu)\}$ is bounded. On the other hand, by (c), $\big\{d_{Y_i'}\big(\mu(\psi_{i*}(m_i)), \mu\big)\big\}$ is bounded. Thus

$$\big\{d_{Y_i'}(\mu, \psi_i(\mu)) \le d_{Y_i'}\big(\mu, \mu(\psi_{i*}(m_i))\big) + d_{Y_i'}\big(\mu(\psi_{i*}(m_i)), \psi_i(\mu)\big)\big\}$$

is bounded. It follows from the construction of ψ_i that $|d_{Y_i}(\mu, \psi_i(\mu)) - d_{Y_i'}(\mu, \psi_i(\mu))|$ is bounded, and hence $\{d_{Y_i}(\mu, \psi_i(\mu))\}$ is also bounded. □

Now we can show that the sequence $\{\psi_i\}$ satisfies (3) by $(*)$ and the following claim:

Claim 5.4 *For any $j = 1, \dots, p$, the sequence $\{m(\sigma \circ \psi_{i*}(m_i), d_j, \mu)\}$ is bounded if and only if $\{m(\sigma(m_i), d_j, \mu)\}$ is bounded, and $\{m(\psi_{i*}(m_i), d_j, \mu)\}$ is bounded if and only if $\{m(m_i, d_j, \mu)\}$ is bounded.*

Proof Let $\{Y_i \subset \partial M\}$ be a sequence of incompressible subsurfaces with $d_j \subset \partial Y_i$ which are not pairs of pants. Since $d_{Y_i}(m_i, \mu) = d_{\psi_i(Y_i)}\big(\mu(\psi_{i*}(m_i)), \psi_i(\mu)\big)$, the triangle inequalities

$$\begin{aligned} d_{Y_i}(\mu(m_i), \mu) &\le d_{\psi_i(Y_i)}\big(\mu(\psi_{i*}(m_i)), \mu\big) + d_{\psi_i(Y_i)}(\mu, \psi_i(\mu)), \\ d_{\psi_i(Y_i)}\big(\mu(\psi_{i*}(m_i)), \mu\big) &\le d_{\psi_i(Y_i)}\big(\mu(\psi_{i*}(m_i)), \psi_i(\mu)\big) + d_{\psi_i(Y_i)}(\psi_i(\mu), \mu), \end{aligned}$$

lead to

$$\begin{aligned} d_{\psi_i(Y_i)}\big(\mu(\psi_{i*}(m_i)), \mu\big) - d_{\psi_i(Y_i)}(\mu, \psi_i(\mu)) &\le d_{Y_i}(\mu(m_i), \mu) \\ &\le d_{\psi_i(Y_i)}\big(\mu(\psi_{i*}(m_i)), \mu\big) + d_{\psi_i(Y_i)}(\mu, \psi_i(\mu)). \end{aligned}$$

Thus by applying Claim 5.3, $\{d_{Y_i}(\mu(m_i), \mu)\}$ is bounded if and only if $\big\{d_{\psi_i(Y_i)}\big(\mu(\psi_{i*}(m_i)), \mu\big)\big\}$ is bounded.

Since $\psi_i(d_j) = d_j$ by (b), $\operatorname{length}_{m_i}(d_j) = \operatorname{length}_{\psi_{i*}(m_i)}(d_j)$, and we conclude that $\{m(\psi_{i*}(m_i), d_j, \mu)\}$ is bounded if and only if $\{m(m_i, d_j, \mu)\}$ is bounded.

Since σ commutes with ψ_{i*}, the same argument shows that $\{m(\sigma \circ \psi_{i*}(m_i), d, \mu)\}$ is bounded if and only if $\{(\sigma(m_i), d, \mu)\}$ is bounded. □

To conclude the proof of Lemma 5.2, it remains to establish (2). We restate the property as a claim:

Claim 5.5 *Let $A \subset M$ be an essential annulus with its boundary components denoted by $\partial_1 A$ and $\partial_2 A$. Suppose that $m(\psi_{i*}(m_i), \partial_k A, \mu) \to \infty$ for both $k = 1$ and $k = 2$. Then* $\operatorname{length}_{\rho_i \circ \psi_{i*}^{-1}}(\partial_1 A) \to 0$.

Proof Let $a_1, \dots, a_q$ be homotopically distinct simple closed curves on ∂M representing all the homotopy classes (in ∂M) homotopic to $\partial_1 A$ in M. By renumbering them, we can assume $a_k = \partial_k A$ for $k = 1, 2$. If $\operatorname{length}_{\psi_{i*}(m_i)}(a_k) \to 0$ for some $k = 1, \dots, q$, we are done.

For the remaining case, we now assume that there is a positive constant ϵ such that $\operatorname{length}_{\psi_{i*}(m_i)}(a_k) \geq \epsilon$ for every $i \in \mathbb{N}$ and $k = 1, \dots, q$. Then there are a constant L and simple closed curves $c_{k,i}$ for every $i \in \mathbb{N}$ and $k = 1, \dots, q$ such that $c_{k,i}$ intersects a_k essentially and $\operatorname{length}_{\psi_{i*}(m_i)}(c_{k,i}) \leq L$. By making L larger if necessary, we can assume that $\operatorname{length}_{\rho_i \circ \psi_{i*}^{-1}}(c_{k,i}) \leq L$. There is also K_1 such that $d_Y\big(c_{k,i}, \mu(\psi_{i*}(m_i))\big) \leq K_1$ for any j and i and any incompressible subsurface $Y \subset \partial M$ intersecting $c_{k,i}$ that is neither an annulus nor a pair of pants, since by definition the length of $\mu(\psi_{i*}(m_i))$ is also bounded from above by a constant.

Since $m(\psi_{i*}(m_i), a_k, \mu) \to \infty$ and $\ell_{\psi_{i*}(m_i)}(a_k) \geq \epsilon$ for $k = 1, 2$, there are incompressible subsurfaces $Y_{k,i}$ such that $a_k \subset \partial Y_{k,i}$ and $d_{Y_{k,i}}\big(\mu(\psi_{i*}(m_i)), \mu\big) \to \infty$ for $k = 1, 2$. If, passing to a subsequence, $Y_{1,i}$ and $Y_{2,i}$ are both annuli, then, up to homotopy, they lie on the boundary of the same component Ξ of the characteristic submanifold (which is, up to passing to a further subsequence, independent of i). However, the assumption that $m(\psi_{i*}(m_i), a_k, \mu) \to \infty$ contradicts (a) when Ξ is a solid torus, and (c) when Ξ is an I–pair. Therefore we can assume that one of the $Y_{k,i}$ (for $k = 1, 2$), say $Y_{1,i}$, is not an annulus.

Suppose now that $Y_{1,i}$ is not eventually contained in the characteristic submanifold X (up to homotopy), even after passing to a subsequence. By taking a subsequence, we can assume that none of the $Y_{1,i}$ are contained in X. Then there is a simple closed curve $c \subset \partial M$ disjoint from X which intersects $Y_{1,i}$ for all i, by passing to a further subsequence. By Theorem 4.4 there is a constant L such that $\operatorname{length}_{\rho_i \circ \psi_{i*}^{-1}}(c) \leq L$. Since $d_{Y_{1,i}}\big(\mu(\psi_{i*}(m_i)), \mu\big) \to \infty$ by our assumption, we have $d_{Y_{1,i}}(c_{1,i}, c) \to \infty$. Then it follows from [14, Theorem B] that $\operatorname{length}_{\rho_i \circ \psi_{i*}^{-1}}(\partial Y_{1,i}) \to 0$, and hence in particular, $\operatorname{length}_{\rho_i \circ \psi_{i*}^{-1}}(\partial_1 A) \to 0$.

Next suppose that $Y_{1,i}$ eventually lies in X. Taking a subsequence, we can assume that all the surfaces $Y_{1,i}$ lie in the same component Ξ of X. Since $Y_{1,i}$ is not an annulus, Ξ must be an I–bundle, which has the form $\Xi = F \times I$. By (c), there is another surface $Y_{3,i} \subset \Xi \cap \partial M$ such that $Y_{1,i}$ and $Y_{3,i}$ bound an I–bundle compatible with the I–bundle structure of Ξ, and are projected along the fibres of $\Xi = F \times I$ to the same surface Z_i in F and $d_{Y_{3,i}}\big(\mu(\psi_{i*}(m_i)), \mu\big) \leq R$. We note that by our definition of $a_1, \dots, a_q$, there is $k_0 \geq 2$ such that a_{k_0} lies on $\partial Y_{3,i}$. Then since $d_{Y_{3,i}}\big(\mu(\psi_{i*}(m_i)), c_{k_0,i}\big) \leq K_1$, we have $d_{Y_{3,i}}(c_{k_0,i}, \mu) \leq R + K_1$. We shall make use of $\{c_{1,i}\}$ and $\{c_{k_0,i}\}$ to apply [14, Theorem B] as before. Since they do not lie on the same surface, we first need to project them to F. This leads to the following claim:

Claim 5.6 *There are $K > 0$ and two sequences of simple closed curves $\{d_{1,i}\}$ and $\{d_{k_0,i}\}$ on F such that* $\operatorname{length}_{\rho_i \circ \psi_{i*}^{-1}}(d_{k,i}) \leq K$ *for all i and $k = 1, k_0$, and $d_{Z_i}(d_{1,i}, d_{k_0,i}) \to \infty$.*

Proof Let k be either 1 or k_0. If $c_{k,i}$ is contained in Ξ for sufficiently large i, then we let $d_{k,i}$ be the projection of $c_{k,i}$ to F. We also note that in this case $\text{length}_{\rho_i\circ\psi_{i*}^{-1}}(d_{k,i}) \le L$.

Suppose this is not the case. We let S be the component of ∂M containing $c_{k,i}$. Following [14, page 138], we extend the multicurve $B := \text{Fr}(\Xi \cap S)$ to a complete geodesic lamination λ by performing Dehn twists around B infinitely many times to $c_{k,i}$ and adding finitely many isolated leaves spiralling around B. There is a unique pleated surface $h_{k,i}\colon S \to \mathbb{H}^3/\rho_i(\pi_1(S))$ realising λ which induces $\rho_i\circ\psi_{i*}^{-1}$ between the fundamental groups. Let R_λ be the ϵ–thick part of S with respect to the hyperbolic metric induced by $h_{k,i}$. By the efficiency of pleated surfaces [21, Theorem 3.3; 14, Theorem 3.5] there is a constant K_2 such that $\text{length}_{h_{k,i}}(c_{k,i}\cap R_\lambda) \le L + K_2 i(c_{k,i}, B)$ — the relation between the alternation and intersection numbers comes from [14, (4.3)]. In particular, there is an arc $\kappa_{k,i}$ in $c_{k,i}\cap(\Xi\cap S)\cap R_\lambda$ intersecting $Y_{k,i}$ and having length at most $L + K_2$. By Theorem 4.4, the length of each component of B on $h_{k,i}$ is bounded by a constant L' independent of i. By joining one or two copies of $\kappa_{k,i}$ (depending on whether $\kappa_{k,i}$ intersects one or two components of $B\cup\text{Fr}\,R_\lambda$) with arcs on $B\cup\text{Fr}\,R_\lambda$, we can construct in $S\cap\Xi$ a simple closed curve $d_{k,i}$ such that $\text{length}_{h_{k,i}}(d_{k,i}) \le 2(L + K_2 + L' + \epsilon)$. Furthermore, this construction implies that there is a constant K_3 such that $d_Y(d_{k,i}, c_{k,i}) \le K_3$ for any incompressible subsurface $Y \subset S\cap\Xi$ intersecting both $d_{k,i}$ and $c_{k,i}$, and in particular for $Y = Y_{k,i}$. We use the same symbol $d_{k,i}$ to denote the projection of $d_{k,i}$ on F along the fibres of $\Xi = F\times I$.

Thus $\text{length}_{\rho_i\circ\psi_{i*}^{-1}}(d_{k,i}) \le 2(L + K_2 + L' + \epsilon)$, and

$$d_{Z_i}(d_{1,i}, d_{k_0,i}) \ge d_{Y_{1,j}}(c_{1,i}, \mu) - d_{Y_{k_0,i}}(c_{k_0,i}, \mu) - 2K_3 \ge d_{Y_{1,j}}(c_{1,i}, \mu) - R - K_1 - 2K_3 \to \infty. \quad \square$$

Proof of Claim 5.5, continued Set $\vartheta_i = \rho_i\circ\psi_{i*}^{-1}\circ I_*\colon \pi_1(S) \to \text{PSL}_2(\mathbb{C})$ where $I_*\colon \pi_1(S)\to\pi_1(M)$ is the homomorphism induced by the inclusion. Following [14], we denote by $\mathcal{C}_0(\vartheta_i, K)$ the set of simple closed curves on S whose translation lengths with respect to ϑ_i are less than or equal to K. By the claim above, both $d_{1,i}$ and $d_{k_0,i}$ lie in $\mathcal{C}_0(\vartheta_i, K)$ for sufficiently large K, and $d_{Y_{1,i}}(d_{1,i}, d_{k_0,i}) \to \infty$. In particular, $\text{diam}_{Y_{1,i}}(\mathcal{C}_0(\vartheta_i, K)) \to \infty$. It follows from [14, Theorem B] that $\text{length}_{\vartheta_i}(\partial Y_{1,i}) \to 0$. So $\text{length}_{\vartheta_i}(\partial_1 A) \to 0$, and hence $\text{length}_{\rho_i\circ\psi_{i*}^{-1}}(\partial_1 A) \to 0$. $\square$

This also concludes the proof of Lemma 5.2. $\square$

By Claim 5.4, proving Proposition 5.1 for $\{\rho_i\}$ is equivalent to proving it for $\{\rho_i\circ\psi_{i*}^{-1}\}$. Thus we may assume that $\{\rho_i\}$ satisfies the following:

(I) For any simple closed curve $c\subset\partial M$, either $\{m(m_i, c, \mu)\}$ (resp. $\{m(\sigma(m_i), c, \mu)\}$) is bounded or $m(m_i, c, \mu)\to\infty$ (resp. $m(\sigma(m_i), c, \mu)\to\infty$).

(II) If $A\subset M$ is an essential annulus such that $m(m_i, \partial_k A, \mu)\to\infty$ (for $k = 1, 2$) for both boundary components $\partial_1 A$ and $\partial_2 A$ of A, then $\text{length}_{\rho_i}(\partial_1 A)\to 0$.

5.2 End invariants and wrapping

In this subsection, we shall discuss how algebraic limits project to geometric limits and how this is reflected in the behaviour of the end invariants.

Let us now fix the assumptions and notation which will be used in most results of this section.

Setting 5.7 We consider an orientable atoroidal compact boundary-irreducible Haken 3–manifold M without torus boundary components, and a sequence of representations $\rho_i \in QH(M)$ corresponding to Ahlfors–Bers coordinates $m_i \in \mathcal{T}(\partial M)$. We have a noncontractible simple closed curve $d \subset \partial M$, and we denote by $d_1, \dots, d_p \subset \partial M$ simple closed curves representing all homotopy classes of ∂M on ∂M which are homotopic to d in M, with $d = d_1$. We assume that $\ell_{\rho_i}(d) \to 0$.

We also assume that we have a submanifold V_d of M whose frontier consists of incompressible annuli and which has the following three properties:

(i) V_d contains all the curves d_j for $j = 1, \dots, p$, and d_j is not peripheral in $V_d \cap \partial M$ for every $j = 1, \dots, p$.

(ii) The restriction of ρ_i to $\pi_1(V_d)$ converges to a representation $\rho_\infty \colon \pi_1(V_d) \to \mathrm{PSL}_2(\mathbb{C})$.

(iii) If $A \subset V_d$ is an essential annulus disjoint from d with core curve a which is not homotopic to d in M, then $\mathrm{length}_{\rho_i}(a) \to 0$ if and only if A is properly homotopic to the closure of a component of $\partial V_d \setminus \partial M$.

Suppose first $p \geq 2$. If a component of the characteristic submanifold containing d (up to isotopy) is a solid torus, then it contains all of $d_1, \dots, d_p$ up to isotopy. We let T be this characteristic solid torus in this case. If the component is an I–pair, then $p = 2$, and it contains d_2 up to isotopy. In this case, we let T be $A \times [0,1]$ such that $A \times \{0\}$ is an annular neighbourhood of d whereas $A \times \{1\}$ is that of d_2. Since $\mathrm{Fr}\, V_d$ consists of annuli, by (i), T can be assumed to be contained in V_d by moving it by an isotopy in both cases. If $p = 1$, we set $T = \varnothing$.

Given $j = 1, \dots, p$, we denote by F_j the component of $V_d \cap \partial M \setminus \bigcup_{k \neq j} d_k$ containing d_j.

The sequence $\{\rho_i(\pi_1(V_d))\}$ converges geometrically to a Kleinian group Γ containing $\rho_\infty(\pi_1(V_d))$, passing to a subsequence.

In the next section, we shall construct V_d having the properties above, which shows that our argument in the present section really works.

Assuming the existence of V_d for the moment, we now prove that every component of $V_d \setminus T$ has a compact core which is embedded in the geometric limit $\mathbb{H}^3/\Gamma$, making use of the work of [4].

Lemma 5.8 *In Setting 5.7, let W be a submanifold of V_d which is the closure of a component of $V_d \setminus T$. Then there is a relative compact core $C_W \subset \mathbb{H}^3/\rho_\infty(\pi_1(W))$ which is homeomorphic to W and on*

which the restriction of the covering projection $\mathbb{H}^3/\rho_\infty(\pi_1(W)) \to \mathbb{H}^3/\Gamma$ induced by the inclusion is injective. Furthermore, for the closures of two components W_1 and W_2 of $V_d \setminus T$ (in the case when T is nonempty and separates W), the compact cores C_{W_1} and C_{W_2} can be taken so that their images in $\mathbb{H}^3/\Gamma$ are disjoint.

Proof Our conditions in Setting 5.7 imply the assumptions of [4, Proposition 4.4], and applying this proposition we see that there is a compact submanifold of $\mathbb{H}^3/\Gamma$ which lifts to a compact core C_W of $\mathbb{H}^3/\rho_\infty(\pi_1(W))$ such that the restriction of the covering projection $\mathbb{H}^3/\rho_\infty(\pi_1(W)) \to \mathbb{H}^3/\Gamma$ to C_W is injective. Let $\Gamma_W \subset \Gamma$ be the geometric limit of $\{\rho_i(\pi_1(W))\}$. Then the restriction of the covering projection $\mathbb{H}^3/\rho_\infty(\pi_1(W)) \to \mathbb{H}^3/\Gamma_W$ to C_W must also be injective.

By [4, Lemma 4.6], $\rho_\infty(\pi_1(W))$ is either a generalised web group or a degenerate group without accidental parabolic elements. It follows then from [1, Corollary C and Theorem E] that C_W is homeomorphic to W. The last sentence of our lemma also follows from [4, Proposition 4.4]. □

We next show that by performing Dehn twists along embedded annuli bounded by d and d_j for $j = 2, \dots, p$, we can make each F_j embedded in the algebraic limit and mapped injectively in the geometric limit by the covering projection.

In the next lemma and the following, we shall use the expression "the *outward side* of a cusp". We say that an embedding of the surface $F_j \subset \partial V_d$ into the geometric limit $\mathbb{H}^3/\Gamma$ lies on the outward side of a cusp if the cusp lies on the same side of the embedding of F_j as the embeddings of the components of $V_d \setminus T$ intersecting F_j. Otherwise we say that the embedding of F_j lies on the *inward side* of the cusp.

Lemma 5.9 *In Setting 5.7, we denote by D_j the right-hand Dehn twist along an embedded annulus bounded by $d = d_1$ and d_j for $j = 2, \dots, p$. Then for each j, there is a sequence $\{a_i(j)\}$ of integers with the following properties*:

- *The sequence $\{\theta_i = \rho_i \circ D_{j*}^{a_i(j)}|_{\pi_1(F_j)}\}$ converges algebraically to $\theta_\infty : \pi_1(F_j) \to \mathrm{PSL}_2(\mathbb{C})$.*
- *There is an embedding $h_j : F_j \to \mathbb{H}^3/\theta_\infty(\pi_1(F_j))$ inducing θ_∞ such that the restriction of the covering projection $\Pi_{F_j} : \mathbb{H}^3/\theta_\infty(\pi_1(F_j)) \to \mathbb{H}^3/\Gamma$ to $h_j(F_j)$ is an embedding and its image $\Pi_{F_j} \circ h_j(F_j)$ lies on the outward side of the cusp corresponding to $\rho_\infty(d) = \theta_\infty(d)$ when the latter is a rank-2 cusp.*

Proof This is a relative version of [2, Lemma 4.5].

Let W' and W'' be the components of $V_d \setminus T$ intersecting F_j (we set $W' = W''$ if there is only one such component), and set $F_j' = F_j \cap W'$ and $F_j'' = F_j \cap W''$. By Lemma 5.8, there are compact cores $C_{W'} \subset \mathbb{H}^3/\rho_\infty(\pi_1(W'))$ and $C_{W''} \subset \mathbb{H}^3/\rho_\infty(\pi_1(W''))$, homeomorphic to W' and W'', respectively, on which the restrictions of the covering projections to $\mathbb{H}^3/\Gamma$ are injective. The inclusions induce embeddings

$f'\colon F_j' \hookrightarrow \partial C_{W'}$ and $f''\colon F_j'' \hookrightarrow \partial C_{W''}$, which lift to embeddings $g'\colon F_j' \hookrightarrow \mathbb{H}^3/\rho_\infty(\pi_1(F_j))$ and $g''\colon F_j'' \hookrightarrow \mathbb{H}^3/\rho_\infty(\pi_1(F_j))$. The restrictions of the covering projection $\Pi_{F_j}\colon \mathbb{H}^3/\rho_\infty(\pi_1(F_j)) \to \mathbb{H}^3/\Gamma$ to $g'(F_j \cap W')$ and to $g''(F_j \cap W'')$ are embeddings.

If T does not separate F_j, we set $\check{g} = g' = g''$, otherwise, we put g' and g'' together to get an embedding $\check{g}\colon F_j \setminus T \to \mathbb{H}^3/\rho_\infty(\pi_1(F_j))$. Moving $C_{W'}$, $C_{W''}$, f' and f'' by isotopies, we may assume that they send the boundary of $F_j \setminus T$ into the ϵ–thin part. Then for an appropriate choice of ϵ, the map $\check{g}$ sends the boundary of $F_j \setminus T$ to the boundary of the ϵ_1–thin part of $\mathbb{H}^3/\rho_\infty(\pi_1(F_j))$, where ϵ_1 is smaller than the three-dimensional Margulis constant. It is then easy to extend $\check{g}$ to an embedding $g\colon F_j \to \mathbb{H}^3/\rho_\infty(\pi_1(F_j))$ such that $g(T \cap F_j)$ lies on the boundary of the ϵ_2–thin part with $\epsilon_2 \le \epsilon_1$. By Lemma 5.8 and by our construction, the restriction of $\Pi_{F_j} \circ g$ to $F_j \setminus T$, which is $\Pi_{F_j} \circ \hat{g}$, is an embedding, and with an appropriate choice of ϵ, the composition $\Pi_{F_j} \circ g$ maps $F_j \cap T$ to the boundary of the ϵ_0–thin part of $\mathbb{H}^3/\Gamma$.

If $\rho_\infty(d)$ belongs to a rank-1 maximal parabolic subgroup of Γ, then it is easy to change g on $F_j \cap T$ so that $\Pi_{F_j} \circ g$ is an embedding. In this case, we simply take a_i to be 0.

Otherwise, $\rho_\infty(d)$ belongs to a rank-2 maximal parabolic subgroup of Γ. We denote by T_0 the boundary of the corresponding torus cusp-neighbourhood in $\mathbb{H}^3/\Gamma$, ie the boundary of the corresponding component of the ϵ_2–thin part. Let Z be the union of $\Pi_{F_j} \circ g(F_j \setminus T)$ and T_0. Then $\Pi_{F_j} \circ g(F_j)$ is contained in Z by our way of extending $\check{g}$ to g, as described above. As is explained in [2, Lemma 3.1], $\Pi_{F_j} \circ g$ is homotopic to a standard map f_k wrapping k times around T_0 for some $k \in \mathbb{Z}$, and there are two standard embeddings $f_0, f_1\colon F_j \to Z$ such that $f_0(F_j)$ lies on the outward side of the cusp associated with d and $f_1(F_j)$ lies on its inward side, both without wrapping around T_0.

Let $\{q_i\colon B_{r_i}\big(\mathbb{H}^3/\rho_i(\pi_1(M)), x_i\big) \to B_{K_i r_i}(\mathbb{H}^3/\Gamma, x_\infty)\}$ be a sequence of K_i–bi-Lipschitz approximate isometries on the r_i–ball with $r_i \to \infty$ and $K_i \to 1$, given by the geometric convergence as explained in Section 2.4. By [2, Lemma 3.1], there is $s_i \in \mathbb{Z}$ such that $q_i^{-1} \circ f_0$ is homotopic to $q_i^{-1} \circ \Pi_{F_i} \circ g \circ D_j^{s_i}$. The conclusion follows, taking $a_i(j) = s_i$ and setting h_j to be the lift of f_0 to $\mathbb{H}^3/\theta_\infty(\pi_1(F_j))$. □

Next we study how the embedding of a compact core in the geometric limit as above affects the end invariants.

Lemma 5.10 *In Setting 5.7, suppose that there is an embedding $h_j\colon F_j \to \mathbb{H}^3/\rho_\infty(\pi_1(F_j))$ inducing $\rho_\infty|_{\pi_1(F_j)}$ for each $j = 1, \dots, p$ such that the restriction of the covering projection*

$$\Pi_{F_j}\colon \mathbb{H}^3/\rho_\infty(\pi_1(F_j)) \to \mathbb{H}^3/\Gamma$$

to $h_j(F_j)$ is an embedding.

If $\Pi_{F_j}(h_j(F_j))$ lies on the outward side of the cusp associated with $\rho_\infty(d) \in \rho_\infty(\pi_1(M)) \subset \Gamma$, then $\{m(m_i, d_j, \mu)\}$ is bounded whereas $m(\sigma(m_i), d_j, \mu) \to \infty$. If $\Pi_{F_j}(h_j(F_j))$ lies on the inward side of the cusp associated with $\rho_\infty(d)$ then $\{m(\sigma(m_i), d_j, \mu)\}$ is bounded whereas $m(m_i, d_j, \mu) \to \infty$.

Proof Suppose that $\Pi_{F_j}(h_j(F_j))$ lies on the outward side of the cusp associated with $\rho_\infty(d) \in \rho_\infty(\pi_1(M)) \subset \Gamma$. Let $c \subset F_j$ be a simple closed curve intersecting d_j essentially, c^* the closed geodesic homotopic to $\Pi_{F_j}(h_j(c))$, and denote by $q_i\colon B_{r_i}\big(\mathbb{H}^3/\rho_i(\pi_1(M)), x_i\big) \to B_{K_i r_i}(\mathbb{H}^3/\Gamma, x_\infty)$ an approximate isometry associated with the geometric convergence of $\{\rho_i(\pi_1(M))\}$ to Γ as explained in Section 2.4. For i large enough, $q_i^{-1}(c^*)$ is a quasigeodesic lying outside the thin part on the same side as F_j of the Margulis tube associated with $\rho_i(d)$. Let S_j be the component of ∂M containing F_j. In the covering $\mathbb{H}^3/\rho_i(\pi_1(S_j))$ of $\mathbb{H}^3/\rho_i(\pi_1(M))$, the closed geodesic homotopic to $\rho_i(c)$ lies above the Margulis tube associated with $\rho_i(d)$. Therefore, by [3, Theorem 1.3] there is a constant D such that $d_Y(c, \mu(m_i)) \le D$ for any surface $Y \subset S_j$ with $d_j \subset \operatorname{Fr} Y$. Thus for any full marking μ, there is D' such that $d_Y(\mu, \mu(m_i)) \le D'$ for any surface $Y \subset S$ with $d_j \subset \operatorname{Fr} Y$.

To conclude that $\{m(m_i, d_j, \mu)\}$ is bounded, it remains to show that $\operatorname{length}_{m_i}(d_j)$ is bounded away from 0. Assume the contrary, that $\operatorname{length}_{m_i}(d_j) \to 0$ after passing to a subsequence. Then there is an annulus joining the closed geodesic $d_j^* \subset \mathbb{H}^3/\rho_i(\pi_1(S))$ representing $\rho_i(d)$ with $d_j^+ \subset \partial C\big(\mathbb{H}^3/\rho_i(\pi_1(S))\big)$ corresponding to d_j, which lies entirely in the ϵ_i–thin part with $\epsilon_i \to 0$. Since $q_i^{-1}(c^*)$ has bounded length, it cannot intersect such an annulus, whereas $q_i^{-1}(c^*)$ lies in a uniformly bounded neighbourhood of the convex core for large i. Since c^* and $\Pi_{F_j}(h_j(F_j))$ lie on the same side of the cusp associated with $\rho_\infty(d)$, this contradicts the assumption that $\Pi_{F_j}(h_j(F_j))$ lies on the outward side of the cusp associated with $\rho_\infty(d) \in \Gamma$.

Since $\operatorname{length}_{\rho_i}(d) \to 0$ and $\{m(m_i, d_j, \mu)\}$ is bounded, it follows from [15, short curve theorem] that $m(\sigma(m_i), d_j, \mu) \to \infty$.

A quite similar argument also works when $\Pi_{F_j}(h_j(F_j))$ lies on the inward side of the cusp associated with $\rho_\infty(d) \in \Gamma$. □

Corollary 5.11 *In Setting 5.7, assume that $p \ge 2$, and consider $j \le p$ such that $\{m(m_i, d_j, \mu)\}$ is bounded. Then there is an embedding $h\colon F_j \to \mathbb{H}^3/\rho_\infty(\pi_1(F_j))$ inducing ρ_∞ such that the restriction of the covering projection $\Pi_{F_j}\colon \mathbb{H}^3/\rho_\infty(\pi_1(F_j)) \to \mathbb{H}^3/\Gamma$ to $h(F_j)$ is an embedding whose image lies on the outward side of the cusp corresponding to $\rho_\infty(d)$.*

Proof As can be seen in the proof of Lemma 5.9, if $\rho_\infty(d)$ belongs to a rank-1 maximal parabolic subgroup of Γ, then $a_i(j) = 0$ for any i and $\theta_\infty = \rho_\infty$. Therefore, our claim of this corollary follows immediately from Lemmas 5.9 and 5.10.

Otherwise, $\rho_\infty(d)$ belongs to a rank-2 maximal parabolic subgroup of Γ. By Lemma 5.9, there is a sequence of integers $\{a_i(j)\}$ and an embedding $h_j\colon F_j \to \mathbb{H}^3/\theta_\infty(\pi_1(F_j))$ inducing θ_∞ between the fundamental groups such that the restriction of the covering projection $\Pi_{F_j}\colon \mathbb{H}^3/\theta_\infty(\pi_1(F_j)) \to \mathbb{H}^3/\Gamma$ to $h_j(F_j)$ is an embedding and its image $\Pi_{F_j} \circ h_j(F_j)$ lies on the outward side of the cusp corresponding to $\theta_\infty(d) = \rho_\infty(d)$. By Lemma 5.10, $\{m(D_{j*}^{a_i} m_i, d_j, \mu)\}$ is bounded. Since $\{m(m_i, d_j, \mu)\}$ is bounded by assumption, this is possible only when $\{a_i(j)\}$ is bounded. Then we may take $a_i(j) = 0$ for any i in Lemma 5.9 so that $\theta_\infty = \rho_\infty$, and the conclusion follows. □

We now put these results together to get the result which we shall use to prove Proposition 5.1:

Lemma 5.12 *In Setting 5.7, suppose that $\{m(m_i, d_j, \mu)\}$ is bounded for $j \neq 1$. Then there is a relative compact core for $\mathbb{H}^3/\rho_\infty(\pi_1(V_d))$ homeomorphic to V_d on which the restriction of the covering projection $\Pi_d : \mathbb{H}^3/\rho_\infty(\pi_1(V_d)) \to \mathbb{H}^3/\Gamma$ is injective. Further, a cusp neighbourhood corresponding to $\rho_\infty(d)$ intersects the compact core in an annular neighbourhood of d_1.*

Proof By Lemma 5.8, for the components W of $V_d \setminus T$, we have embeddings $g_W : W \to \mathbb{H}^3/\rho_\infty(\pi_1(V_d))$ inducing $\rho_\infty|_{\pi_1(W)}$, on the union of which the restriction of Π_d is injective.

If $p = 1$, then $T = \varnothing$ by definition, and hence $W = V_d$. We can take a cusp neighbourhood corresponding to $\rho_\infty(d)$ intersecting $g_W(W)$ along an annulus in the homotopy class of d. Since $p = 1$, such an annulus is isotopic on ∂V_d to an annular neighbourhood of $d_1 = d$.

Suppose that $p \geq 2$, and assume that $\{m(m_i, d_j, \mu)\}$ is bounded for every $j \neq 1$. Then by Corollary 5.11, for every $j \neq 1$, there is an embedding $g_j : F_j \to \mathbb{H}^3/\rho_\infty(\pi_1(V_d))$ inducing $\rho_\infty|_{\pi_1(F_j)}$ on which the restriction of Π_d is injective. Furthermore, it follows from the construction that g_j and g_W agree on $F_j \cap W$. Putting together the maps g_W for all the components W of $V_d \setminus T$ and the g_j for all $j \neq 1$, we get an embedding $g : V_d \to \mathbb{H}^3/\rho_\infty(\pi_1(V_d))$ inducing $\rho_\infty|_{\pi_1(V_d)}$ on which the restriction of Π_d is injective.

Changing g by an isotopy, we may assume that $g(V_d)$ intersects a cusp neighbourhood C associated with $\rho_\infty(d)$ along an annulus $A \subset g(\partial V_d)$ which is a regular neighbourhood of $g(d_k)$ for some $k = 1, \dots, p$. Then $g(F_k)$ lies on the inward side of C. This is possible only if $\Pi_d \circ g(F_k)$ lies on the inward side of C, for the restriction of Π_d is injective on $g(V_d)$, and hence it cannot wrap around C.

By assumption, for every $j \neq 1$, $\{m(m_i, d_j, \mu)\}$ is bounded. It follows then from Corollary 5.11 that $\Pi_d(g(F_j))$ lies on the outward side of C for $j \neq 1$. Hence the only possibility is that A is a regular neighbourhood of $g(d_1)$. □

5.3 Completion of the proof of Proposition 5.1

Proof of Proposition 5.1 If M is an I–bundle, then $m(\sigma(m_i), d, \mu) = m(m_i, d_2, \mu)$ and the conclusion follows. In the other cases, we shall prove the proposition by contradiction. Assume that M is not an I–bundle, that $m(\sigma(m_i), d, \mu) \to \infty$, and that $\{m(m_i, d_j, \mu)\}$ is bounded for every $j = 2, \dots, p$.

By Lemma 5.2, after remarking and passing to a subsequence, we may assume that $\{\rho_i = q(m_i)\}$ satisfies:

(1) For any simple closed curve $c \subset \partial M$, either $\{m(m_i, c, \mu)\}$ (resp. $\{m(\sigma(m_i), c, \mu)\}$) is bounded or $m(m_i, c, \mu) \to \infty$ (resp. $m(\sigma(m_i), c, \mu) \to \infty$).

(2) If $A \subset M$ is an essential annulus such that ∂A does not intersect d (and hence any of d_j) and $m(m_i, \partial_j A, \mu) \to \infty$ for both boundary components $\partial_1 A$ and $\partial_2 A$ of A, then $\text{length}_{\rho_i}(\partial_1 A) \to 0$.

We note that by Lemma 5.2, $\{m(m_i, d_j, \mu)\}$ is bounded for every $j = 2, \dots, p$ and $m(\sigma(m_i), d, \mu) \to \infty$ even after remarking.

Taking a further subsequence we can also assume that for any essential annulus E of M, either $\mathrm{length}_{\rho_i}(\partial E) \to 0$ or $\mathrm{length}_{\rho_i}(\partial E)$ is bounded away from 0. Let $\mathcal{A} = \bigcup_k A_k$ be a maximal family of pairwise-disjoint nonisotopic essential annuli such that

(i) the length of the core curve of each annulus A_k tends to 0 ($\mathrm{length}_{\rho_i}(\partial A_k) \to 0$ for any k),

(ii) $\partial\mathcal{A}$ does not intersect d, and

(iii) no component of $\mathcal{A}$ contains a curve homotopic to d.

Denote by V_d the component of $M \setminus N(\mathcal{A})$ containing d, where $N(\mathcal{A})$ denotes a thin regular neighbourhood of $\mathcal{A}$. Let P be the closure of $V_d \setminus \partial M$, which is a union of annuli. Next we shall control the geometry of V_d and the length of d.

Claim 5.13 *Passing to a subsequence, the restrictions* $\{\rho_i|_{\pi_1(V_d)}\}$ *converge and* $\mathrm{length}_{\rho_i}(d) \to 0$.

Proof Let us first assume that $m(m_i, d_1, \mu) \to \infty$, and verify the hypotheses of Theorem 4.5 with $W = V_d$. The hypothesis (a) follows from the construction of V_d. The hypothesis (b) follows from (2) above. By Lemma 4.2, $\{m(m_i, c, \mu)\}$ is bounded for any simple closed curve c intersecting d. This observation, combined with the assumption that $\{m(m_i, d_j, \mu)\}$ is bounded for any $j = 2, \dots, p$, (2) above and the maximality of $\mathcal{A}$, yields the hypothesis (c). Now by Theorem 4.5 we can take a subsequence in such a way that the restrictions $\{\rho_i|_{\pi_1(V_d)}\}$ converge.

If $\mathrm{length}_{m_i}(d) \to 0$, we are done. Otherwise, since we are assuming that $m(m_i, d, \mu) \to \infty$, there is a sequence of subsurfaces $Y_i \subset \partial M$ such that $d_{Y_i}(m_i, \mu) \to \infty$ and $d \subset \partial Y_i$. Consider a simple closed curve $c \subset V_d \cap \partial M$ intersecting d. Since $\{\rho_i|_{\pi_1(V_d)}\}$ converges, $\{\mathrm{length}_{\rho_i}(c)\}$ is bounded. Then we have $d_{Y_i}(m_i, c) \to \infty$ (for $d_{Y_i}(m_i, \mu) \to \infty$) and it follows from [14, Theorem B] that $\mathrm{length}_{\rho_i}(d) \to 0$.

Suppose that $\{m(m_i, d_1, \mu)\}$ is bounded. Since $\{m(\sigma(m_i), d_1, \mu)\} \to \infty$ by assumption, $\mathrm{length}_{\rho_i}(d) \to 0$ by [15, short curve theorem]. We add to P a thin regular neighbourhood of d on ∂V_d, and we can verify as above that the hypotheses of Theorem 4.5 are satisfied for (V_d, P). □

Now we are in the situation of Setting 5.7, and we use its notation. By Lemma 5.12, $g(F_1)$ lies on the inward side of the cusp corresponding to $\rho_\infty(d)$, and $g(F_j)$ lies on the outward side for every $j = 2, \dots, p$. Then Lemma 5.10 implies that $\{m(\sigma(m_i), d_1, \mu)\}$ is bounded. This contradicts our assumption. □

6 The proof of Theorem 1.1

Now we shall complete the proof of Theorem 1.1. By Lemmas 3.1 and 3.4, we can assume that every M is strongly untwisted. Let L be the number provided by Lemma 3.9, and consider a sequence $\{m_i\}$ such that $\{m_i^{L+1} = (\iota_* \circ \sigma)^{L+1} m_i\}$ has no convergent subsequence. Since M is not an interval bundle, by [4, Theorem 1.1], on each component S of ∂M, there is a simple closed curve a_S such that $\ell_\sigma(a_S)$ is

bounded. It follows that $d_S(\mu(m_i^{L+1}), \mu)$ is bounded. By Lemma 4.3, passing to a subsequence, there is a simple closed curve $d_{L+1} \subset \partial M$ such that $m(m_i^{L+1}, d_{L+1}, \mu) \to \infty$. Then $m(\sigma(m_i^L), \iota(d_{L+1}), \mu) \to \infty$. By Proposition 5.1, passing to a further subsequence, there is an incompressible annulus A_L bounded by $\iota(d_{L+1})$ and another simple closed curve $d_L \subset \partial M$ with $m(m_i^L, d_L, \mu) \to \infty$. Repeating this, we get a family of simple closed curves $\{d_k \mid 0 \leq k \leq L+1\}$ such that $d_k \cup \iota(d_{k+1})$ bounds an incompressible annulus. This means that an annular neighbourhood of $\iota(d_{L+1})$ is L–times vertically extendible, contradicting Lemma 3.9. This completes the proof of Theorem 1.1.

References

[1] **J W Anderson**, **R D Canary**, *Cores of hyperbolic 3–manifolds and limits of Kleinian groups*, Amer. J. Math. 118 (1996) 745–779 MR Zbl

[2] **J Brock**, **K Bromberg**, **R Canary**, **C Lecuire**, *Convergence and divergence of Kleinian surface groups*, J. Topol. 8 (2015) 811–841 MR Zbl

[3] **J F Brock**, **K W Bromberg**, **R D Canary**, **Y N Minsky**, *Convergence properties of end invariants*, Geom. Topol. 17 (2013) 2877–2922 MR Zbl

[4] **J F Brock**, **K W Bromberg**, **R D Canary**, **Y N Minsky**, *Windows, cores and skinning maps*, Ann. Sci. École Norm. Sup. 53 (2020) 173–216 MR Zbl

[5] **W H Jaco**, **P B Shalen**, *Seifert fibered spaces in 3–manifolds*, Mem. Amer. Math. Soc. 220, Amer. Math. Soc., Providence, RI (1979) MR Zbl

[6] **K Johannson**, *Homotopy equivalences of 3–manifolds with boundaries*, Lecture Notes in Math. 761, Springer (1979) MR Zbl

[7] **M Kapovich**, *Hyperbolic manifolds and discrete groups*, Progr. Math. 183, Birkhäuser, Boston (2001) MR Zbl

[8] **R P Kent, IV**, *Skinning maps*, Duke Math. J. 151 (2010) 279–336 MR Zbl

[9] **A Marden**, *The geometry of finitely generated Kleinian groups*, Ann. of Math. 99 (1974) 383–462 MR Zbl

[10] **B Maskit**, *Kleinian groups*, Grundl. Math. Wissen. 287, Springer (1988) MR Zbl

[11] **H A Masur**, **Y N Minsky**, *Geometry of the complex of curves, I: Hyperbolicity*, Invent. Math. 138 (1999) 103–149 MR Zbl

[12] **H A Masur**, **Y N Minsky**, *Geometry of the complex of curves, II: Hierarchical structure*, Geom. Funct. Anal. 10 (2000) 902–974 MR Zbl

[13] **D McCullough**, *Compact submanifolds of 3–manifolds with boundary*, Q. J. Math. 37 (1986) 299–307 MR Zbl

[14] **Y N Minsky**, *Kleinian groups and the complex of curves*, Geom. Topol. 4 (2000) 117–148 MR Zbl

[15] **Y N Minsky**, *The classification of Kleinian surface groups, I: Models and bounds*, Ann. of Math. 171 (2010) 1–107 MR Zbl

[16] **J W Morgan**, *On Thurston's uniformization theorem for three-dimensional manifolds*, from "The Smith conjecture" (J W Morgan, H Bass, editors), Pure Appl. Math. 112, Academic, Orlando, FL (1984) 37–125 MR Zbl

[17] **K Ohshika**, *Degeneration of marked hyperbolic structures in dimensions 2 and 3*, from "Handbook of group actions, III" (L Ji, A Papadopoulos, S-T Yau, editors), Adv. Lect. Math. 40, International, Somerville, MA (2018) 13–35 MR Zbl

[18] **J-P Otal**, *Thurston's hyperbolization of Haken manifolds*, from "Surveys in differential geometry, III" (S-T Yau, editor), International, Boston, MA (1998) 77–194 MR Zbl

[19] **G P Scott**, *Compact submanifolds of 3–manifolds*, J. Lond. Math. Soc. 7 (1973) 246–250 MR Zbl

[20] **W P Thurston**, *Hyperbolic structures on 3–manifolds, I: Deformation of acylindrical manifolds*, Ann. of Math. 124 (1986) 203–246 MR Zbl

[21] **W P Thurston**, *Hyperbolic structures on 3–manifolds, II: Surface groups and 3–manifolds which fiber over the circle*, preprint (1998) arXiv math/9801045 Reprinted in his "Collected works with commentary, II: 3–Manifolds, complexity and geometric group theory" (S P Kerckhoff, B Farb, D Gabai, editors), Amer. Math. Soc., Providence, RI (2022) 79–110

[22] **W P Thurston**, *Hyperbolic structures on 3–manifolds, III: Deformations of 3–manifolds with incompressible boundary*, preprint (1998) arXiv math/9801058 Reprinted in his "Collected works with commentary, II: 3–Manifolds, complexity and geometric group theory" (S P Kerckhoff, B Farb, D Gabai, editors), Amer. Math. Soc., Providence, RI (2022) 111–129

[23] **W P Thurston**, *Collected works of William P Thurston with commentary, IV: The geometry and topology of three-manifolds*, Amer. Math. Soc., Providence, RI (2022) MR Zbl

[24] **F Waldhausen**, *On irreducible 3–manifolds which are sufficiently large*, Ann. of Math. 87 (1968) 56–88 MR Zbl

[25] **F Waldhausen**, *On the determination of some bounded 3–manifolds by their fundamental groups alone*, Proc. Sympos. on Topology and its Appl. (1969) 331–332 Zbl

Laboratoire Emile Picard, Université Paul Sabatier
Toulouse, France

Current address: *ENS de Lyon site Monod, UMPA UMR 5669 CNRS*
Lyon, France

Department of Mathematics, Faculty of Science, Gakushuin University
Tokyo, Japan

cyril.lecuire@math.cnrs.fr, ohshika@math.gakushuin.ac.jp

Proposed: Ian Agol
Seconded: Mladen Bestvina, Dmitri Burago

Received: 18 September 2022
Revised: 14 March 2023

GEOMETRY & TOPOLOGY

msp.org/gt

See inside back cover or msp.org/gt for submission instructions.

The subscription price for 2024 is US $805/year for the electronic version, and $1135/year (+$70, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Geometry & Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Geometry & Topology (ISSN 1465-3060 printed, 1364-0380 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

GT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

GEOMETRY & TOPOLOGY

Volume 28 Issue 6 (pages 2483–2999) 2024

Intersection theory on moduli of disks, open KdV and Virasoro 2483

RAHUL PANDHARIPANDE, JAKE P SOLOMON and RAN J TESSLER

Cluster categories and rational curves 2569

ZHENG HUA and BERNHARD KELLER

Limits of manifolds with a Kato bound on the Ricci curvature 2635

GILLES CARRON, ILARIA MONDELLO and DAVID TEWODROSE

Automorphisms of surfaces over fields of positive characteristic 2747

YIFEI CHEN and CONSTANTIN SHRAMOV

Moduli spaces of residueless meromorphic differentials and the KP hierarchy 2793

ALEXANDR BURYAK, PAOLO ROSSI and DIMITRI ZVONKINE

Homological mirror symmetry for hypersurfaces in $(\mathbb{C}^*)^n$ 2825

MOHAMMED ABOUZAID and DENIS AUROUX

Stable cuspidal curves and the integral Chow ring of $\overline{\mathcal{M}}_{2,1}$ 2915

ANDREA DI LORENZO, MICHELE PERNICE and ANGELO VISTOLI

Thurston's bounded image theorem 2971

CYRIL LECUIRE and KEN'ICHI OHSHIKA